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On a Problem of Danzer

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Abstract

Let C be a bounded convex object in \mathbb{R}^d , and P a set of n points lying outside C . Further let c_p, c_q be two integers with $1 \leq c_q \leq c_p \leq n - \lfloor \frac{d}{2} \rfloor$, such that every $c_p + \lfloor \frac{d}{2} \rfloor$ points of P contains a subset of size $c_q + \lfloor \frac{d}{2} \rfloor$ whose convex-hull is disjoint from C . Then our main theorem states the existence of a partition of P into a small number of subsets, each of whose convex-hull is disjoint from C . Our proof is constructive and implies that such a partition can be computed in polynomial time.

In particular, our general theorem implies polynomial bounds for Hadwiger-Debrunner (p, q) numbers for balls in \mathbb{R}^d . For example, it follows from our theorem that when $p > q \geq (1 + \beta) \cdot \frac{d}{2}$ for $\beta > 0$, then any set of balls satisfying the $\text{HD}(p, q)$ property can be hit by $O\left(q^2 p^{1+\frac{1}{\beta}} \log p\right)$ points. This is the first improvement over a nearly 60-year old exponential bound of roughly $O(2^d)$.

Our results also complement the results obtained in a recent work of Keller *et al.* where, apart from improvements to the bound on $\text{HD}(p, q)$ for convex sets in \mathbb{R}^d for various ranges of p and q , a polynomial bound is obtained for regions with low union complexity in the plane.

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1 Introduction

Given a finite set \mathcal{C} of geometric objects in \mathbb{R}^d , we say that \mathcal{C} satisfies the $\text{HD}(p, q)$ property if for any set $\mathcal{C}' \subseteq \mathcal{C}$ of size p , there exists a point in \mathbb{R}^d common to at least q objects of \mathcal{C}' . The goal then is to show that there exists a small set Q of points in \mathbb{R}^d such that each object of \mathcal{C} contains some point of Q ; such a Q is called a hitting set for \mathcal{C} .

These bounds for a set \mathcal{C} of convex sets in \mathbb{R}^d have been studied since the 1950s (see the surveys [7, 8, 15]), and it was only in 1991 that Alon and Kleitman [1], in a breakthrough result, gave an upper-bound that is *independent* of $|\mathcal{C}|$. Unfortunately it depends exponentially on p, q and d . For the case where \mathcal{C} consists of arbitrary convex objects, the current best bounds remain exponential in p, q and d .

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► **Theorem A** ([1, 9]). *Let \mathcal{C} be a finite set of convex objects in \mathbb{R}^d satisfying the HD(p, q) property, where p, q are two integers with $p \geq q \geq d + 1$. Then there exists a hitting set for \mathcal{C} of size*

$$\begin{cases} O\left(p^{d\frac{q-1}{q-d}} \cdot \log^{c'd^3 \log d} p\right), \\ (p-q) + O\left(\left(\frac{p}{q}\right)^d \log^{c'd^3 \log d} \left(\frac{p}{q}\right)\right), & \text{for } q \geq \log p \\ p - q + 2, & \text{for } q \geq p^{1-\frac{1}{d}+\epsilon}, p \geq p(d, \epsilon). \end{cases}$$

where c' is an absolute constant independent of $|\mathcal{C}|, p, q$ and d , and $p(d, \epsilon)$ is a function depending only on d and ϵ .

Consider the basic case where \mathcal{C} is a set of balls in \mathbb{R}^d satisfying the HD(p, q) property. Theorem A implies – ignoring logarithmic factors and for general values of p and q – the existence of a hitting set of size no better than $O(p^d)$. Furthermore, it requires $q \geq d + 1$ – a necessary condition for arbitrary convex objects² but not for balls.

Almost 60 years ago, Danzer [4, 5] considered the HD(p, q) problem for balls. The best bound that we are aware of, derived from the survey of Eckhoff [7] by combining inequalities (4.2), (4.4) and (4.5), is stated below. It is better than the one from Theorem A quantitatively, but also in that it gives a bound requiring only that $q \geq 2$. Further, for a very specific case – namely when $p = q$ and $(d - q)$ is $O(\log d)$ – it succeeds in giving polynomial bounds.

► **Theorem B** ([7]). *Let \mathcal{B} be a finite set of balls in \mathbb{R}^d . If \mathcal{B} satisfies the HD(p, q) property for some $d \geq p \geq q \geq 2$, then there exists a hitting set for \mathcal{B} of size at most*

$$\sqrt{\frac{3\pi}{2}} \cdot 2^{d-q} \cdot \left((p-q) \cdot 2^q \cdot d^{\frac{3}{2}} \cdot g(d) + 4(d-q+2)^{\frac{3}{2}} \cdot g(d-q+2) \right)$$

where $g(x) = \log x + \log \log x + 1$. Ignoring logarithmic terms, the above bound is of the form $\Theta\left((p-q) \cdot 2^d \cdot d^{\frac{3}{2}} + 2^{d-q} \cdot (d-q)^{\frac{3}{2}}\right)$. If $p \neq q$ the first term dominates, otherwise the second term dominates.

Turning towards the lower-bound for the case where \mathcal{C} is a set of unit balls in \mathbb{R}^d , Bourgain and Lindenstrauss [2] proved a lower-bound of 1.0645^d when $p = q = 2$ in \mathbb{R}^d , i.e., one needs at least 1.0645^d points to hit all pairwise intersecting unit balls in \mathbb{R}^d .

Our Result

We consider a more general set up for the HD(p, q) problem, as follows.

Let C be a convex object in \mathbb{R}^d , and P a set of n points lying outside C . For each $p \in P$, let H_p be the set of hyperplanes separating p from C . Let C_p be the set of points in \mathbb{R}^d dual to the hyperplanes in H_p (see [12, Chapter 5.1]), and let $\mathcal{S} = \{C_p : p \in P\}$.

Our goal is to study the HD(p, q) property for \mathcal{S} – namely, that out of every p objects of \mathcal{S} , there exists a point in \mathbb{R}^d common to at least q of them. This is equivalent to the property of C and P that out of every p -sized set $P' \subseteq P$, there exists a hyperplane separating C from a q -sized subset $P'' \subset P'$ – or equivalently, $\text{conv}(P'')$ is disjoint from C .

Our main theorem is the following. For a simpler expression, let c_q, c_p be two positive integers such that $p = c_p + \lfloor \frac{d}{2} \rfloor$ and $q = c_q + \lfloor \frac{d}{2} \rfloor$.

² There are easy examples, e.g. when the convex objects are hyperplanes in \mathbb{R}^d .

► **Theorem 1.** *Let C be a bounded convex object in \mathbb{R}^d and P a set of n points lying outside C . Further let c_p, c_q be two integers, with $1 \leq c_q \leq c_p \leq n - \lfloor \frac{d}{2} \rfloor$, such that for every $c_p + \lfloor \frac{d}{2} \rfloor$ points of P , there exists a subset of size $c_q + \lfloor \frac{d}{2} \rfloor$ whose convex-hull is disjoint from C . Then the points of P can be partitioned into*

$$\lambda_d(c_p, c_q) = K_2 \frac{d}{c_q} \cdot \left(\sqrt{2}K_1\right)^{\frac{d}{c_q}} \cdot ([d/2] + c_q)^2 \cdot ([d/2] + c_p)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}} \cdot \log([d/2] + c_p)$$

sets, each of whose convex-hull is disjoint from C . Here K_1, K_2 are absolute constants independent of n, d, c_p and c_q . Furthermore, such a partition can be computed in polynomial time.

The proof, presented in Section 2, is a combination of three ingredients: the Alon-Kleitman technique [1], bounds on independent sets in hypergraphs [9] and bounds on $(\leq k)$ -sets for half-spaces [3]. It is an extension of the proof in [14] which studied Carathéodory's theorem in this setting.

► **Remark.** The restriction that $q \geq \lfloor \frac{d}{2} \rfloor + 1$ is necessary – as can be seen when P form the vertices of a cyclic polytope in \mathbb{R}^d and C is a slightly shrunk copy of $\text{conv}(P)$.

► **Remark.** Note that when $c_q \geq \beta \cdot \frac{d}{2}$ for any absolute constant $\beta > 0$, the above bound is *polynomial* in the dimension d – it is upper-bounded by $O\left(q^2 p^{1 + \frac{1}{\beta}} \log p\right)$.

► **Remark.** It was shown in [13] that C_p is a convex object in \mathbb{R}^d and thus the bounds of Theorem A apply. As before, Theorem 1 substantially improves upon this, as the bounds following from Theorem A are exponential in d and furthermore, require $q \geq d + 1$.

As an immediate corollary of Theorem 1, we obtain the first improvements to the old bound on the (p, q) problem for balls in \mathbb{R}^d . The bound in Theorem B is exponential in d – except in special cases where $p = q$ and $(d - q)$ is³ $O(\log d)$. On the other hand, our result gives polynomial bounds as long as $q \geq \beta d$ for any constant $\beta > \frac{1}{2}$.

► **Corollary 2** (Hadwiger-Debrunner (p, q) bound for balls in \mathbb{R}^d). *Let \mathcal{B} be collection of balls in \mathbb{R}^d such that for every subset of $c_p + \lfloor \frac{d+1}{2} \rfloor$ balls in \mathcal{B} , some $c_q + \lfloor \frac{d+1}{2} \rfloor$ have a common intersection, where c_p and c_q are integers such that $1 \leq c_q \leq c_p \leq n - \lfloor \frac{d+1}{2} \rfloor$. Then there exists a set X of $\lambda_{d+1}(c_p, c_q)$ points that form a hitting set for the balls in \mathcal{B} . Here $\lambda_{d+1}(\cdot, \cdot)$ is the function defined in the statement of Theorem 1.*

Proof. Observe that one can stereographically ‘lift’ balls in \mathbb{R}^d to caps of a sphere S in \mathbb{R}^{d+1} , where a cap of a sphere is a portion of the sphere contained in a half-space that doesn't contain the center of the sphere. Thus we will prove a slightly more general result where \mathcal{B} consists of caps of a d -dimensional sphere S embedded in \mathbb{R}^{d+1} .

For a point $x \in S$, let h_x denote the hyperplane tangent to S at x . For any point y lying outside S , define the *separating set* of y to be

$$S_y = \{z \in S : h_z \text{ separates } y \text{ and } S\}.$$

Geometrically, S_y is the set of points of S ‘visible’ from y , and form a cap of S . Furthermore, for any cap K of S , there is a unique point w such that $K = S_w$. We denote this point w by $\text{apex}(K)$.

³ Recall that Theorem B assumes $q \leq p \leq d$.

Given the set of caps \mathcal{B} on S , consider the point set

$$\text{apex}(\mathcal{B}) = \{\text{apex}(B) : B \in \mathcal{B}\}.$$

Observe that for a point $x \in S$ and a cap $B \in \mathcal{B}$, $x \in B$ if and only if $x \in S_{\text{apex}(B)}$. As \mathcal{B} satisfies the (p, q) property – namely that for every p -sized subset \mathcal{B}' of \mathcal{B} , there exists a point $x \in S$ lying in some q elements of \mathcal{B}' – we have that for every p -sized subset A' of $\text{apex}(\mathcal{B})$, there exists a point $x \in S$ lying in the separating set of some q points of A' . In other words, h_x separates these q points from S .

Applying Theorem 1 with $C = S$ and $P = \text{apex}(\mathcal{B})$ in dimension $d + 1$, we conclude that P can be partitioned into a family Ξ of $\lambda_{d+1}(c_p, c_q)$ sets, each of whose convex hull is disjoint from S . Consider a set $P' \in \Xi$. Since the convex hull of P' is disjoint from S , we can find a hyperplane h_x tangent to S at x such that h_x separates P' from S . This implies that all the caps in \mathcal{B} corresponding to the points in P' contain the point x . Thus for each set of Ξ we obtain a point which is contained in all the caps corresponding to the points in that set. These $|X| = \lambda_{d+1}(c_p, c_q)$ points form the required set X . ◀

Our results complement the recent results of Keller, Smorodinsky and Tardos [9, 10] who obtain polynomial bounds for regions of low union complexity in the plane.

2 Proof of Theorem 1

Given a set P of points in \mathbb{R}^d and an integer $k \geq 1$, a set $P' \subseteq P$ is called a k -set of P if $|P'| = k$ and if there exists a half-space h in \mathbb{R}^d such that $P' = P \cap h$. A set $P' \subseteq P$ is called a $(\leq k)$ -set if P' is a l -set for some $l \leq k$. The next well-known theorem gives an upper-bound on the number of $(\leq k)$ -sets in a point set (see [17]).

► **Theorem 3** (Clarkson-Shor [3]). *For any integer $k \geq \lfloor \frac{d}{2} \rfloor + 1$, the number of $(\leq k)$ -sets of any set of n points in \mathbb{R}^d is at most*

$$\kappa_d(n, k) = 2 \binom{K_1}{\lceil d/2 \rceil}^{\lceil d/2 \rceil} \binom{n}{\lfloor d/2 \rfloor} (k + \lceil d/2 \rceil)^{\lceil d/2 \rceil} \leq \kappa'_d(k) \cdot n^{\lfloor d/2 \rfloor}, \tag{1}$$

where $\kappa'_d(k) = 2K_1^d \lfloor d/2 \rfloor^{-\lfloor d/2 \rfloor} \left(1 + \frac{k}{\lceil d/2 \rceil}\right)^{\lceil d/2 \rceil}$ and $K_1 \geq e$ is an absolute constant independent of n, d and k .

► **Definition 4** (Depth). Given a set P of n points in \mathbb{R}^d and any set $Q \subseteq P$, define the *depth* of Q with respect to P , denoted $\text{depth}_P(Q)$, to be the minimum number of points of P contained in any half-space containing Q .

For two parameters $l \geq k \geq 2$, let $\tau_d(n, k, l)$ denote the maximum number of subsets of size k and depth at most l with respect to P in any set P of n points in \mathbb{R}^d :

$$\tau_d(n, k, l) = \max_{\substack{P \subseteq \mathbb{R}^d \\ |P|=n}} |\{Q \subseteq P : |Q| = k \text{ and } \text{depth}_P(Q) \leq l\}|.$$

The following statement is easily implied by an application of the Clarkson-Shor technique [3] (e.g., see [16]).

► **Theorem 5.** *For parameters $l \geq k \geq \lfloor \frac{d}{2} \rfloor + 1$,*

$$\tau_d(n, k, l) \leq e \cdot \kappa_d(n, k) \cdot l^{k - \lfloor d/2 \rfloor},$$

where the function $\kappa(\cdot, \cdot)$ is as defined in Equation (1).

Proof. Let P be any set of n points in \mathbb{R}^d . Let t be the number of sets of P of size k and depth at most l . Pick each element of P independently with probability $\rho = \frac{1}{l}$ to get a random sample R . The expected number of k -sets in R satisfies

$$\begin{aligned} \rho^k \cdot (1 - \rho)^{l-k} \cdot t &\leq \mathbb{E}[\text{number of } k\text{-sets in } R] \\ &\leq 2 \left(\frac{K_1}{\lceil d/2 \rceil} \right)^{\lceil \frac{d}{2} \rceil} \mathbb{E} \left[\binom{|R|}{\lceil \frac{d}{2} \rceil} \right] \left(k + \left\lceil \frac{d}{2} \right\rceil \right)^{\lceil \frac{d}{2} \rceil} \\ &= 2 \left(\frac{K_1}{\lceil d/2 \rceil} \right)^{\lceil \frac{d}{2} \rceil} \binom{n}{\lceil \frac{d}{2} \rceil} \rho^{\lceil \frac{d}{2} \rceil} \left(k + \left\lceil \frac{d}{2} \right\rceil \right)^{\lceil \frac{d}{2} \rceil} \\ &= \kappa_d(n, k) \cdot \rho^{\lceil \frac{d}{2} \rceil} \\ \implies t &\leq \frac{\kappa_d(n, k) \cdot \rho^{\lceil \frac{d}{2} \rceil}}{\rho^k \cdot (1 - \rho)^{l-k}} \leq e \cdot \kappa_d(n, k) \cdot l^{k - \lfloor d/2 \rfloor}, \end{aligned}$$

as $(1 - \frac{1}{l})^{-(l-k)} \leq e$ for any $l \geq k \geq 2$. ◀

► **Lemma 6.** Let C be a bounded convex object in \mathbb{R}^d , and P a set of n points lying outside C . Let $p \geq q \geq \lfloor \frac{d}{2} \rfloor + 1$ be parameters such that for every subset $Q \subseteq P$ of size p , there exists a set $Q' \subset Q$ of size q such that Q' can be separated from C by a hyperplane. Then there exists a hyperplane separating at least

$$(2qp^{q-1} \cdot e \kappa'_d(q))^{\frac{1}{\lfloor d/2 \rfloor - q}}$$

fraction of the points of P from C .

Proof. From [6, 9], it follows that the number of distinct q -tuples of P that can be separated from C by a hyperplane is, assuming that $n \geq 2p$, at least

$$\frac{n-p+1}{n-q+1} \frac{\binom{n}{q}}{\binom{p-1}{q-1}} \geq \frac{n^q}{2qp^{q-1}}.$$

Let l be the maximum depth (Definition 4) of any of these separable q -tuples. The number of such tuples is therefore at most $\tau_d(n, q, l)$. Thus by Theorem 5 we must have

$$\frac{n^q}{2qp^{q-1}} \leq \tau_d(n, q, l) \leq e \kappa_d(n, q) l^{q - \lfloor d/2 \rfloor}.$$

Re-arranging the terms and from inequality (1), we get

$$\begin{aligned} l &\geq \left(\frac{n^q}{2qp^{q-1} \cdot e \kappa_d(n, q)} \right)^{\frac{1}{q - \lfloor d/2 \rfloor}} \geq \left(\frac{n^q}{2qp^{q-1} \cdot e \kappa'_d(q) n^{\lfloor \frac{d}{2} \rfloor}} \right)^{\frac{1}{q - \lfloor d/2 \rfloor}} \\ &= n \cdot (2qp^{q-1} \cdot e \kappa'_d(q))^{\frac{1}{\lfloor d/2 \rfloor - q}}. \end{aligned}$$

Thus one of the separable q -tuples, say $P' \subseteq P$, must have depth at least l ; in other words, the hyperplane separating P' from C must contain at least l points of P . This is the required hyperplane. ◀

We now prove a weighted version of the above statement.

► **Corollary 7.** *Let C be a bounded convex object in \mathbb{R}^d , and P a weighted set of n points lying outside C , where the weight of each point $p \in P$ is a non-negative rational number. Let $p \geq q \geq \lfloor \frac{d}{2} \rfloor + 1$ be parameters such that for every subset $Q \subseteq P$ of size p , there exists a set $Q' \subset Q$ of size q such that Q' can be separated from C by a hyperplane. Then there exists a hyperplane separating a set of points whose weight is at least*

$$\alpha_d(p, q) = (2e \kappa'_d(q) q^q p^{q-1})^{\frac{1}{\lfloor d/2 \rfloor - q}}$$

fraction of the total weight of the points in P .

Proof. By appropriately scaling all the rational weights, we may assume that each weight is a non-negative integer and we replace a point with weight m by m unweighted copies of the point. Let P' be the new set of points. Observe that any set S of pq points in P' either contains q copies of some point in P or it contains p distinct points from P . In either case, there is hyperplane separating q points of S from C . Thus, we can apply Lemma 6 to the point set P' with the parameter p in the lemma replaced by pq . The result follows. ◀

Finally we return to the proof of the main theorem.

► **Theorem 1.** *Let C be a bounded convex object in \mathbb{R}^d and P a set of n points lying outside C . Further let c_p, c_q be two integers, with $1 \leq c_q \leq c_p \leq n - \lfloor \frac{d}{2} \rfloor$, such that for every $c_p + \lfloor \frac{d}{2} \rfloor$ points of P , there exists a subset of size $c_q + \lfloor \frac{d}{2} \rfloor$ whose convex-hull is disjoint from C . Then the points of P can be partitioned into*

$$\lambda_d(c_p, c_q) = K_2 \frac{d}{c_q} \cdot (\sqrt{2}K_1)^{\frac{d}{c_q}} \cdot (\lfloor d/2 \rfloor + c_q)^2 \cdot (\lfloor d/2 \rfloor + c_p)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}} \cdot \log(\lfloor d/2 \rfloor + c_p)$$

sets, each of whose convex-hull is disjoint from C . Here K_1, K_2 are absolute constants independent of n, d, c_p and c_q . Furthermore, such a partition can be computed in polynomial time.

Proof. Let $p = c_p + \lfloor d/2 \rfloor$ and $q = c_q + \lfloor d/2 \rfloor$. Let \mathcal{H} be the set of all hyperplanes separating a distinct subset of points of P from C . As the number of subsets of P is finite, one can assume that \mathcal{H} is also finite. Consider the following linear program on $|\mathcal{H}|$ variables $\{u_h \geq 0 : h \in \mathcal{H}\}$:

$$\min \sum_{h \in \mathcal{H}} u_h, \quad \text{such that} \quad \forall r \in P: \sum_{\substack{h \in \mathcal{H} \\ h \text{ separates } r \text{ from } C}} u_h \geq 1. \quad (2)$$

The LP-dual to the above program, on $|P|$ variables $\{w_r \geq 0 : r \in P\}$, is:

$$\max \sum_{p \in P} w_p, \quad \text{such that} \quad \forall h \in \mathcal{H}: \sum_{\substack{r \in P \\ h \text{ separates } r \text{ from } C}} w_r \leq 1. \quad (3)$$

Consider an optimal solution w^* of the dual linear program and interpret w_r^* as the weight of each $r \in P$. Since the weights are rational, by Corollary 7, there exists a hyperplane $h \in \mathcal{H}$ separating a subset of P of combined weight at least $\epsilon = \alpha_d(p, q)$ fraction of the total weight of all the points. Since the total weight of the points in any half-space is constrained to be at most 1 by the linear program, the total weight of all the points of P must be at most $\frac{1}{\epsilon}$. In other words, the optimal value of linear program (3) is at most $\frac{1}{\epsilon}$. Since the optimal values of both linear programs are equal due to strong duality, the optimal value of linear program (2) is also at most $\frac{1}{\epsilon}$.

Let u^* be the optimal solution of linear program (2). If we interpret u_h as the weight of the hyperplane h , the constraints of the program imply that each point is separated by a set of hyperplanes in \mathcal{H} whose combined weight is at least 1 out of a total weight of at most $\frac{1}{\epsilon}$ – in other words, at least ϵ -th fraction of the total weight of \mathcal{H} . By associating with each hyperplane the half-space bounded by it and not containing C , and using the ϵ -net theorem for half-spaces in \mathbb{R}^d (see [11]), there exists a set of $O\left(\frac{d}{\epsilon} \log \frac{1}{\epsilon}\right)$ hyperplanes which together separate all points of P from C . Recalling that

$$\frac{1}{\epsilon} = \frac{1}{\alpha_d(p, q)} = (2e \kappa'_d(q) q^q p^{q-1})^{\frac{1}{q - \lfloor d/2 \rfloor}} = (2e \kappa'_d(q) q^q p^{q-1})^{\frac{1}{c_q}}.$$

and that $\kappa'_d(q) = 2K_1^d \lfloor d/2 \rfloor^{-\lfloor d/2 \rfloor} \left(1 + \frac{q}{\lfloor d/2 \rfloor}\right)^{\lfloor d/2 \rfloor}$, we get

$$\begin{aligned} \frac{1}{\epsilon} &= \left(4K_1^d e^{\lfloor d/2 \rfloor - \lfloor d/2 \rfloor} \left(1 + \frac{q}{\lfloor d/2 \rfloor}\right)^{\lfloor d/2 \rfloor} q^q p^{q-1}\right)^{\frac{1}{c_q}} \\ &\leq \left(4K_1^{d+1} \lfloor d/2 \rfloor^{-d} (c_q + d)^{\lfloor d/2 \rfloor} q^q p^{q-1}\right)^{\frac{1}{c_q}} \quad (\text{using } e \leq K_1 \text{ and } q = c_q + \lfloor d/2 \rfloor) \\ &\leq \left(4K_1^{d+1} \lfloor d/2 \rfloor^{-d} (c_q + d)^{\lfloor d/2 \rfloor} q^{c_q + \lfloor d/2 \rfloor} p^{c_q + \lfloor d/2 \rfloor - 1}\right)^{\frac{1}{c_q}} \\ &= O\left(K_1^{\frac{d}{c_q}} \lfloor d/2 \rfloor^{-\frac{d}{c_q}} (c_q + d)^{\frac{\lfloor d/2 \rfloor}{c_q}} (c_q + \lfloor d/2 \rfloor)^{1 + \frac{\lfloor d/2 \rfloor}{c_q}} (c_p + \lfloor d/2 \rfloor)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}}\right) \\ &= O\left(K_1^{\frac{d}{c_q}} d^{2 + \frac{\lfloor d/2 \rfloor - 1}{c_q}} \left(1 + \frac{c_q}{d}\right)^{\frac{\lfloor d/2 \rfloor}{c_q}} \left(1 + \frac{c_q}{\lfloor d/2 \rfloor}\right)^{1 + \frac{\lfloor d/2 \rfloor}{c_q}} \left(1 + \frac{c_p}{\lfloor d/2 \rfloor}\right)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}}\right) \\ &= O\left(K_1^{\frac{d}{c_q}} d^{2 + \frac{\lfloor d/2 \rfloor - 1}{c_q}} e^{\frac{c_q}{d} \cdot \frac{\lfloor d/2 \rfloor}{c_q}} \left(1 + \frac{c_q}{\lfloor d/2 \rfloor}\right) e^{\frac{c_q}{\lfloor d/2 \rfloor} \cdot \frac{\lfloor d/2 \rfloor}{c_q}} \left(1 + \frac{c_p}{\lfloor d/2 \rfloor}\right)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}}\right) \\ &= O\left(K_1^{\frac{d}{c_q}} d^{2 + \frac{\lfloor d/2 \rfloor - 1}{c_q}} \left(1 + \frac{c_q}{\lfloor d/2 \rfloor}\right) \left(1 + \frac{c_p}{\lfloor d/2 \rfloor}\right)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}}\right) \\ &= O\left(K_1^{\frac{d}{c_q}} 2^{\frac{d}{2c_q}} (\lfloor d/2 \rfloor + c_q) (\lfloor d/2 \rfloor + c_p)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}}\right) \\ &= O\left(\left(\sqrt{2}K_1\right)^{\frac{d}{c_q}} (\lfloor d/2 \rfloor + c_q) (\lfloor d/2 \rfloor + c_p)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}}\right). \end{aligned}$$

The Big-Oh notation here does not hide dependencies on d – namely we do not treat d as a constant. From the above it follows that

$$\log \frac{1}{\epsilon} = O\left(c_q^{-1} (\lfloor d/2 \rfloor + c_q) \log (\lfloor d/2 \rfloor + c_p)\right).$$

Thus, $\frac{d}{\epsilon} \log \frac{1}{\epsilon}$ is

$$O\left(d \cdot \left(\left(\sqrt{2}K_1\right)^{\frac{d}{c_q}} (\lfloor d/2 \rfloor + c_q) (\lfloor d/2 \rfloor + c_p)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}}\right) \cdot \left(c_q^{-1} (\lfloor d/2 \rfloor + c_q) \log (\lfloor d/2 \rfloor + c_p)\right)\right)$$

which simplifies to

$$O\left(\frac{d}{c_q} \left(\sqrt{2}K_1\right)^{\frac{d}{c_q}} (\lfloor d/2 \rfloor + c_q)^2 (\lfloor d/2 \rfloor + c_p)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}} \log(\lfloor d/2 \rfloor + c_p)\right).$$

Since linear programs can be solved in polynomial time and epsilon nets can be computed in polynomial time, the partition of P into the above number of sets can be achieved in polynomial time. The theorem follows. \blacktriangleleft

References

- 1 N. Alon and D. Kleitman. Piercing convex sets and the Hadwiger–Debrunner (p, q) -problem. *Adv. Math.*, 96(1):103–112, 1992.
- 2 J. Bourgain and J. Lindenstrauss. On covering a set in R^n by balls of the same diameter. In *Geometric Aspects of Functional Analysis*, pages 138–144. Springer Berlin Heidelberg, 1991.
- 3 K. Clarkson and P. Shor. Applications of random sampling in computational geometry, II. *Discrete & Computational Geometry*, 4(5):387–421, 1989.
- 4 L. Danzer. *Über zwei Lagerungsprobleme; Abwandlungen einer Vermutung von T. Gallai*. PhD thesis, Techn. Hochschule Munchen, 1960.
- 5 L. Danzer. Über durchschnittseigenschaften n-dimensionaler kugelfamilien. *J. Reine Angew. Math.*, 208:181–203, 1961.
- 6 D. de Caen. Extension of a theorem of Moon and Moser on complete subgraphs. *Ars Combin.*, 16:5–10, 1983.
- 7 J. Eckhoff. A survey of the Hadwiger-Debrunner (p, q) -problem. In *Discrete and Computational Geometry: The Goodman-Pollack Festschrift*, pages 347–377. Springer, 2003.
- 8 A. Holmsen and R. Wenger. Helly-type theorems and geometric transversals. In J. E. Goodman, J. O’Rourke, and C. D. Tóth, editors, *Handbook of Discrete and Computational Geometry*, pages 91–123. CRC Press LLC, 2017.
- 9 C. Keller, S. Smorodinsky, and G. Tardos. Improved bounds on the Hadwiger-Debrunner numbers. *Israel J. of Math.*, to appear, 2017.
- 10 C. Keller, S. Smorodinsky, and G. Tardos. On max-clique for intersection graphs of sets and the hadwiger-debrunner numbers. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2254–2263, 2017.
- 11 A. Kupavskii, N. H. Mustafa, and J. Pach. Near-optimal lower bounds for ϵ -nets for half-spaces and low complexity set systems. In Martin Loeb, Jaroslav Nešetřil, and Robin Thomas, editors, *A Journey Through Discrete Mathematics: A Tribute to Jiří Matoušek*, pages 527–541. Springer International Publishing, 2017.
- 12 J. Matoušek. *Lectures in Discrete Geometry*. Springer-Verlag, New York, NY, 2002.
- 13 N. H. Mustafa and S. Ray. Weak ϵ -nets have a basis of size $O(1/\epsilon \log 1/\epsilon)$. *Comp. Geom. Theory and Appl.*, 40(1):84–91, 2008.
- 14 N. H. Mustafa and S. Ray. An optimal generalization of the colorful Carathéodory theorem. *Discrete Mathematics*, 339(4):1300–1305, 2016.
- 15 N. H. Mustafa and K. Varadarajan. Epsilon-approximations and epsilon-nets. In J. E. Goodman, J. O’Rourke, and C. D. Tóth, editors, *Handbook of Discrete and Computational Geometry*, pages 1241–1268. CRC Press LLC, 2017.
- 16 S. Smorodinsky, M. Sulovský, and U. Wagner. On center regions and balls containing many points. In *Proceedings of the 14th annual International Conference on Computing and Combinatorics, COCOON ’08*, pages 363–373, 2008.
- 17 U. Wagner. k -sets and k -facets. In J.E. Goodman, J. Pach, and R. Pollack, editors, *Surveys on Discrete and Computational Geometry: Twenty Years Later*, Contemporary Mathematics, pages 231–255. American Mathematical Society, 2008.