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# On a Problem of Danzer

Nabil H. Mustafa<sup>1</sup>

Université Paris-Est, Laboratoire d'Informatique Gaspard-Monge, Equipe A3SI, ESIEE Paris  
mustafan@esiee.fr

Saurabh Ray

Department of Computer Science, NYU Abu Dhabi, United Arab Emirates  
saurabh.ray@nyu.edu

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## Abstract

Let  $C$  be a bounded convex object in  $\mathbb{R}^d$ , and  $P$  a set of  $n$  points lying outside  $C$ . Further let  $c_p, c_q$  be two integers with  $1 \leq c_q \leq c_p \leq n - \lfloor \frac{d}{2} \rfloor$ , such that every  $c_p + \lfloor \frac{d}{2} \rfloor$  points of  $P$  contains a subset of size  $c_q + \lfloor \frac{d}{2} \rfloor$  whose convex-hull is disjoint from  $C$ . Then our main theorem states the existence of a partition of  $P$  into a small number of subsets, each of whose convex-hull is disjoint from  $C$ . Our proof is constructive and implies that such a partition can be computed in polynomial time.

In particular, our general theorem implies polynomial bounds for Hadwiger-Debrunner  $(p, q)$  numbers for balls in  $\mathbb{R}^d$ . For example, it follows from our theorem that when  $p > q \geq (1 + \beta) \cdot \frac{d}{2}$  for  $\beta > 0$ , then any set of balls satisfying the  $\text{HD}(p, q)$  property can be hit by  $O\left(q^2 p^{1 + \frac{1}{\beta}} \log p\right)$  points. This is the first improvement over a nearly 60-year old exponential bound of roughly  $O(2^d)$ .

Our results also complement the results obtained in a recent work of Keller *et al.* where, apart from improvements to the bound on  $\text{HD}(p, q)$  for convex sets in  $\mathbb{R}^d$  for various ranges of  $p$  and  $q$ , a polynomial bound is obtained for regions with low union complexity in the plane.

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## 1 Introduction

Given a finite set  $\mathcal{C}$  of geometric objects in  $\mathbb{R}^d$ , we say that  $\mathcal{C}$  satisfies the  $\text{HD}(p, q)$  property if for any set  $\mathcal{C}' \subseteq \mathcal{C}$  of size  $p$ , there exists a point in  $\mathbb{R}^d$  common to at least  $q$  objects of  $\mathcal{C}'$ . The goal then is to show that there exists a small set  $Q$  of points in  $\mathbb{R}^d$  such that each object of  $\mathcal{C}$  contains some point of  $Q$ ; such a  $Q$  is called a hitting set for  $\mathcal{C}$ .

These bounds for a set  $\mathcal{C}$  of convex sets in  $\mathbb{R}^d$  have been studied since the 1950s (see the surveys [7, 8, 15]), and it was only in 1991 that Alon and Kleitman [1], in a breakthrough result, gave an upper-bound that is *independent* of  $|\mathcal{C}|$ . Unfortunately it depends exponentially on  $p, q$  and  $d$ . For the case where  $\mathcal{C}$  consists of arbitrary convex objects, the current best bounds remain exponential in  $p, q$  and  $d$ .

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► **Theorem A** ([1, 9]). *Let  $\mathcal{C}$  be a finite set of convex objects in  $\mathbb{R}^d$  satisfying the HD( $p, q$ ) property, where  $p, q$  are two integers with  $p \geq q \geq d + 1$ . Then there exists a hitting set for  $\mathcal{C}$  of size*

$$\begin{cases} O\left(p^{d\frac{q-1}{q-d}} \cdot \log^{c'd^3 \log d} p\right), \\ (p-q) + O\left(\left(\frac{p}{q}\right)^d \log^{c'd^3 \log d} \left(\frac{p}{q}\right)\right), & \text{for } q \geq \log p \\ p - q + 2, & \text{for } q \geq p^{1-\frac{1}{d}+\epsilon}, p \geq p(d, \epsilon). \end{cases}$$

where  $c'$  is an absolute constant independent of  $|\mathcal{C}|, p, q$  and  $d$ , and  $p(d, \epsilon)$  is a function depending only on  $d$  and  $\epsilon$ .

Consider the basic case where  $\mathcal{C}$  is a set of balls in  $\mathbb{R}^d$  satisfying the HD( $p, q$ ) property. Theorem A implies – ignoring logarithmic factors and for general values of  $p$  and  $q$  – the existence of a hitting set of size no better than  $O(p^d)$ . Furthermore, it requires  $q \geq d + 1$  – a necessary condition for arbitrary convex objects<sup>2</sup> but not for balls.

Almost 60 years ago, Danzer [4, 5] considered the HD( $p, q$ ) problem for balls. The best bound that we are aware of, derived from the survey of Eckhoff [7] by combining inequalities (4.2), (4.4) and (4.5), is stated below. It is better than the one from Theorem A quantitatively, but also in that it gives a bound requiring only that  $q \geq 2$ . Further, for a very specific case – namely when  $p = q$  and  $(d - q)$  is  $O(\log d)$  – it succeeds in giving polynomial bounds.

► **Theorem B** ([7]). *Let  $\mathcal{B}$  be a finite set of balls in  $\mathbb{R}^d$ . If  $\mathcal{B}$  satisfies the HD( $p, q$ ) property for some  $d \geq p \geq q \geq 2$ , then there exists a hitting set for  $\mathcal{B}$  of size at most*

$$\sqrt{\frac{3\pi}{2}} \cdot 2^{d-q} \cdot \left( (p-q) \cdot 2^q \cdot d^{\frac{3}{2}} \cdot g(d) + 4(d-q+2)^{\frac{3}{2}} \cdot g(d-q+2) \right)$$

where  $g(x) = \log x + \log \log x + 1$ . Ignoring logarithmic terms, the above bound is of the form  $\Theta\left((p-q) \cdot 2^d \cdot d^{\frac{3}{2}} + 2^{d-q} \cdot (d-q)^{\frac{3}{2}}\right)$ . If  $p \neq q$  the first term dominates, otherwise the second term dominates.

Turning towards the lower-bound for the case where  $\mathcal{C}$  is a set of unit balls in  $\mathbb{R}^d$ , Bourgain and Lindenstrauss [2] proved a lower-bound of  $1.0645^d$  when  $p = q = 2$  in  $\mathbb{R}^d$ , i.e., one needs at least  $1.0645^d$  points to hit all pairwise intersecting unit balls in  $\mathbb{R}^d$ .

### Our Result

We consider a more general set up for the HD( $p, q$ ) problem, as follows.

Let  $C$  be a convex object in  $\mathbb{R}^d$ , and  $P$  a set of  $n$  points lying outside  $C$ . For each  $p \in P$ , let  $H_p$  be the set of hyperplanes separating  $p$  from  $C$ . Let  $C_p$  be the set of points in  $\mathbb{R}^d$  dual to the hyperplanes in  $H_p$  (see [12, Chapter 5.1]), and let  $\mathcal{S} = \{C_p : p \in P\}$ .

Our goal is to study the HD( $p, q$ ) property for  $\mathcal{S}$  – namely, that out of every  $p$  objects of  $\mathcal{S}$ , there exists a point in  $\mathbb{R}^d$  common to at least  $q$  of them. This is equivalent to the property of  $C$  and  $P$  that out of every  $p$ -sized set  $P' \subseteq P$ , there exists a hyperplane separating  $C$  from a  $q$ -sized subset  $P'' \subset P'$  – or equivalently,  $\text{conv}(P'')$  is disjoint from  $C$ .

Our main theorem is the following. For a simpler expression, let  $c_q, c_p$  be two positive integers such that  $p = c_p + \lfloor \frac{d}{2} \rfloor$  and  $q = c_q + \lfloor \frac{d}{2} \rfloor$ .

<sup>2</sup> There are easy examples, e.g. when the convex objects are hyperplanes in  $\mathbb{R}^d$ .

► **Theorem 1.** *Let  $C$  be a bounded convex object in  $\mathbb{R}^d$  and  $P$  a set of  $n$  points lying outside  $C$ . Further let  $c_p, c_q$  be two integers, with  $1 \leq c_q \leq c_p \leq n - \lfloor \frac{d}{2} \rfloor$ , such that for every  $c_p + \lfloor \frac{d}{2} \rfloor$  points of  $P$ , there exists a subset of size  $c_q + \lfloor \frac{d}{2} \rfloor$  whose convex-hull is disjoint from  $C$ . Then the points of  $P$  can be partitioned into*

$$\lambda_d(c_p, c_q) = K_2 \frac{d}{c_q} \cdot \left(\sqrt{2}K_1\right)^{\frac{d}{c_q}} \cdot ([d/2] + c_q)^2 \cdot ([d/2] + c_p)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}} \cdot \log([d/2] + c_p)$$

*sets, each of whose convex-hull is disjoint from  $C$ . Here  $K_1, K_2$  are absolute constants independent of  $n, d, c_p$  and  $c_q$ . Furthermore, such a partition can be computed in polynomial time.*

The proof, presented in Section 2, is a combination of three ingredients: the Alon-Kleitman technique [1], bounds on independent sets in hypergraphs [9] and bounds on  $(\leq k)$ -sets for half-spaces [3]. It is an extension of the proof in [14] which studied Carathéodory’s theorem in this setting.

► **Remark.** The restriction that  $q \geq \lfloor \frac{d}{2} \rfloor + 1$  is necessary – as can be seen when  $P$  form the vertices of a cyclic polytope in  $\mathbb{R}^d$  and  $C$  is a slightly shrunk copy of  $\text{conv}(P)$ .

► **Remark.** Note that when  $c_q \geq \beta \cdot \frac{d}{2}$  for any absolute constant  $\beta > 0$ , the above bound is *polynomial* in the dimension  $d$  – it is upper-bounded by  $O\left(q^2 p^{1 + \frac{1}{\beta}} \log p\right)$ .

► **Remark.** It was shown in [13] that  $C_p$  is a convex object in  $\mathbb{R}^d$  and thus the bounds of Theorem A apply. As before, Theorem 1 substantially improves upon this, as the bounds following from Theorem A are exponential in  $d$  and furthermore, require  $q \geq d + 1$ .

As an immediate corollary of Theorem 1, we obtain the first improvements to the old bound on the  $(p, q)$  problem for balls in  $\mathbb{R}^d$ . The bound in Theorem B is exponential in  $d$  – except in special cases where  $p = q$  and  $(d - q)$  is<sup>3</sup>  $O(\log d)$ . On the other hand, our result gives polynomial bounds as long as  $q \geq \beta d$  for any constant  $\beta > \frac{1}{2}$ .

► **Corollary 2** (Hadwiger-Debrunner  $(p, q)$  bound for balls in  $\mathbb{R}^d$ ). *Let  $\mathcal{B}$  be collection of balls in  $\mathbb{R}^d$  such that for every subset of  $c_p + \lfloor \frac{d+1}{2} \rfloor$  balls in  $\mathcal{B}$ , some  $c_q + \lfloor \frac{d+1}{2} \rfloor$  have a common intersection, where  $c_p$  and  $c_q$  are integers such that  $1 \leq c_q \leq c_p \leq n - \lfloor \frac{d+1}{2} \rfloor$ . Then there exists a set  $X$  of  $\lambda_{d+1}(c_p, c_q)$  points that form a hitting set for the balls in  $\mathcal{B}$ . Here  $\lambda_{d+1}(\cdot, \cdot)$  is the function defined in the statement of Theorem 1.*

**Proof.** Observe that one can stereographically ‘lift’ balls in  $\mathbb{R}^d$  to caps of a sphere  $S$  in  $\mathbb{R}^{d+1}$ , where a cap of a sphere is a portion of the sphere contained in a half-space that doesn’t contain the center of the sphere. Thus we will prove a slightly more general result where  $\mathcal{B}$  consists of caps of a  $d$ -dimensional sphere  $S$  embedded in  $\mathbb{R}^{d+1}$ .

For a point  $x \in S$ , let  $h_x$  denote the hyperplane tangent to  $S$  at  $x$ . For any point  $y$  lying outside  $S$ , define the *separating set* of  $y$  to be

$$S_y = \{z \in S : h_z \text{ separates } y \text{ and } S\}.$$

Geometrically,  $S_y$  is the set of points of  $S$  ‘visible’ from  $y$ , and form a cap of  $S$ . Furthermore, for any cap  $K$  of  $S$ , there is a unique point  $w$  such that  $K = S_w$ . We denote this point  $w$  by  $\text{apex}(K)$ .

<sup>3</sup> Recall that Theorem B assumes  $q \leq p \leq d$ .

Given the set of caps  $\mathcal{B}$  on  $S$ , consider the point set

$$\text{apex}(\mathcal{B}) = \{\text{apex}(B) : B \in \mathcal{B}\}.$$

Observe that for a point  $x \in S$  and a cap  $B \in \mathcal{B}$ ,  $x \in B$  if and only if  $x \in S_{\text{apex}(B)}$ . As  $\mathcal{B}$  satisfies the  $(p, q)$  property – namely that for every  $p$ -sized subset  $\mathcal{B}'$  of  $\mathcal{B}$ , there exists a point  $x \in S$  lying in some  $q$  elements of  $\mathcal{B}'$  – we have that for every  $p$ -sized subset  $A'$  of  $\text{apex}(\mathcal{B})$ , there exists a point  $x \in S$  lying in the separating set of some  $q$  points of  $A'$ . In other words,  $h_x$  separates these  $q$  points from  $S$ .

Applying Theorem 1 with  $C = S$  and  $P = \text{apex}(\mathcal{B})$  in dimension  $d + 1$ , we conclude that  $P$  can be partitioned into a family  $\Xi$  of  $\lambda_{d+1}(c_p, c_q)$  sets, each of whose convex hull is disjoint from  $S$ . Consider a set  $P' \in \Xi$ . Since the convex hull of  $P'$  is disjoint from  $S$ , we can find a hyperplane  $h_x$  tangent to  $S$  at  $x$  such that  $h_x$  separates  $P'$  from  $S$ . This implies that all the caps in  $\mathcal{B}$  corresponding to the points in  $P'$  contain the point  $x$ . Thus for each set of  $\Xi$  we obtain a point which is contained in all the caps corresponding to the points in that set. These  $|X| = \lambda_{d+1}(c_p, c_q)$  points form the required set  $X$ . ◀

Our results complement the recent results of Keller, Smorodinsky and Tardos [9, 10] who obtain polynomial bounds for regions of low union complexity in the plane.

## 2 Proof of Theorem 1

Given a set  $P$  of points in  $\mathbb{R}^d$  and an integer  $k \geq 1$ , a set  $P' \subseteq P$  is called a  $k$ -set of  $P$  if  $|P'| = k$  and if there exists a half-space  $h$  in  $\mathbb{R}^d$  such that  $P' = P \cap h$ . A set  $P' \subseteq P$  is called a  $(\leq k)$ -set if  $P'$  is a  $l$ -set for some  $l \leq k$ . The next well-known theorem gives an upper-bound on the number of  $(\leq k)$ -sets in a point set (see [17]).

► **Theorem 3** (Clarkson-Shor [3]). *For any integer  $k \geq \lfloor \frac{d}{2} \rfloor + 1$ , the number of  $(\leq k)$ -sets of any set of  $n$  points in  $\mathbb{R}^d$  is at most*

$$\kappa_d(n, k) = 2 \binom{K_1}{\lceil d/2 \rceil}^{\lceil d/2 \rceil} \binom{n}{\lfloor d/2 \rfloor} (k + \lceil d/2 \rceil)^{\lceil d/2 \rceil} \leq \kappa'_d(k) \cdot n^{\lfloor d/2 \rfloor}, \tag{1}$$

where  $\kappa'_d(k) = 2K_1^d \lfloor d/2 \rfloor^{-\lfloor d/2 \rfloor} \left(1 + \frac{k}{\lceil d/2 \rceil}\right)^{\lceil d/2 \rceil}$  and  $K_1 \geq e$  is an absolute constant independent of  $n, d$  and  $k$ .

► **Definition 4** (Depth). Given a set  $P$  of  $n$  points in  $\mathbb{R}^d$  and any set  $Q \subseteq P$ , define the *depth* of  $Q$  with respect to  $P$ , denoted  $\text{depth}_P(Q)$ , to be the minimum number of points of  $P$  contained in any half-space containing  $Q$ .

For two parameters  $l \geq k \geq 2$ , let  $\tau_d(n, k, l)$  denote the maximum number of subsets of size  $k$  and depth at most  $l$  with respect to  $P$  in any set  $P$  of  $n$  points in  $\mathbb{R}^d$ :

$$\tau_d(n, k, l) = \max_{\substack{P \subseteq \mathbb{R}^d \\ |P|=n}} |\{Q \subseteq P : |Q| = k \text{ and } \text{depth}_P(Q) \leq l\}|.$$

The following statement is easily implied by an application of the Clarkson-Shor technique [3] (e.g., see [16]).

► **Theorem 5.** *For parameters  $l \geq k \geq \lfloor \frac{d}{2} \rfloor + 1$ ,*

$$\tau_d(n, k, l) \leq e \cdot \kappa_d(n, k) \cdot l^{k - \lfloor d/2 \rfloor},$$

where the function  $\kappa(\cdot, \cdot)$  is as defined in Equation (1).

**Proof.** Let  $P$  be any set of  $n$  points in  $\mathbb{R}^d$ . Let  $t$  be the number of sets of  $P$  of size  $k$  and depth at most  $l$ . Pick each element of  $P$  independently with probability  $\rho = \frac{1}{l}$  to get a random sample  $R$ . The expected number of  $k$ -sets in  $R$  satisfies

$$\begin{aligned} \rho^k \cdot (1 - \rho)^{l-k} \cdot t &\leq \mathbb{E}[\text{number of } k\text{-sets in } R] \\ &\leq 2 \left( \frac{K_1}{\lceil d/2 \rceil} \right)^{\lceil \frac{d}{2} \rceil} \mathbb{E} \left[ \binom{|R|}{\lceil \frac{d}{2} \rceil} \right] \left( k + \left\lceil \frac{d}{2} \right\rceil \right)^{\lceil \frac{d}{2} \rceil} \\ &= 2 \left( \frac{K_1}{\lceil d/2 \rceil} \right)^{\lceil \frac{d}{2} \rceil} \binom{n}{\lceil \frac{d}{2} \rceil} \rho^{\lceil \frac{d}{2} \rceil} \left( k + \left\lceil \frac{d}{2} \right\rceil \right)^{\lceil \frac{d}{2} \rceil} \\ &= \kappa_d(n, k) \cdot \rho^{\lceil \frac{d}{2} \rceil} \\ \implies t &\leq \frac{\kappa_d(n, k) \cdot \rho^{\lceil \frac{d}{2} \rceil}}{\rho^k \cdot (1 - \rho)^{l-k}} \leq e \cdot \kappa_d(n, k) \cdot l^{k - \lfloor d/2 \rfloor}, \end{aligned}$$

as  $(1 - \frac{1}{l})^{-(l-k)} \leq e$  for any  $l \geq k \geq 2$ . ◀

► **Lemma 6.** Let  $C$  be a bounded convex object in  $\mathbb{R}^d$ , and  $P$  a set of  $n$  points lying outside  $C$ . Let  $p \geq q \geq \lfloor \frac{d}{2} \rfloor + 1$  be parameters such that for every subset  $Q \subseteq P$  of size  $p$ , there exists a set  $Q' \subset Q$  of size  $q$  such that  $Q'$  can be separated from  $C$  by a hyperplane. Then there exists a hyperplane separating at least

$$(2qp^{q-1} \cdot e \kappa'_d(q))^{\frac{1}{\lfloor d/2 \rfloor - q}}$$

fraction of the points of  $P$  from  $C$ .

**Proof.** From [6, 9], it follows that the number of distinct  $q$ -tuples of  $P$  that can be separated from  $C$  by a hyperplane is, assuming that  $n \geq 2p$ , at least

$$\frac{n-p+1}{n-q+1} \frac{\binom{n}{q}}{\binom{p-1}{q-1}} \geq \frac{n^q}{2qp^{q-1}}.$$

Let  $l$  be the maximum depth (Definition 4) of any of these separable  $q$ -tuples. The number of such tuples is therefore at most  $\tau_d(n, q, l)$ . Thus by Theorem 5 we must have

$$\frac{n^q}{2qp^{q-1}} \leq \tau_d(n, q, l) \leq e \kappa_d(n, q) l^{q - \lfloor d/2 \rfloor}.$$

Re-arranging the terms and from inequality (1), we get

$$\begin{aligned} l &\geq \left( \frac{n^q}{2qp^{q-1} \cdot e \kappa_d(n, q)} \right)^{\frac{1}{q - \lfloor d/2 \rfloor}} \geq \left( \frac{n^q}{2qp^{q-1} \cdot e \kappa'_d(q) n^{\lfloor \frac{d}{2} \rfloor}} \right)^{\frac{1}{q - \lfloor d/2 \rfloor}} \\ &= n \cdot (2qp^{q-1} \cdot e \kappa'_d(q))^{\frac{1}{\lfloor d/2 \rfloor - q}}. \end{aligned}$$

Thus one of the separable  $q$ -tuples, say  $P' \subseteq P$ , must have depth at least  $l$ ; in other words, the hyperplane separating  $P'$  from  $C$  must contain at least  $l$  points of  $P$ . This is the required hyperplane. ◀

We now prove a weighted version of the above statement.

► **Corollary 7.** *Let  $C$  be a bounded convex object in  $\mathbb{R}^d$ , and  $P$  a weighted set of  $n$  points lying outside  $C$ , where the weight of each point  $p \in P$  is a non-negative rational number. Let  $p \geq q \geq \lfloor \frac{d}{2} \rfloor + 1$  be parameters such that for every subset  $Q \subseteq P$  of size  $p$ , there exists a set  $Q' \subset Q$  of size  $q$  such that  $Q'$  can be separated from  $C$  by a hyperplane. Then there exists a hyperplane separating a set of points whose weight is at least*

$$\alpha_d(p, q) = (2e \kappa'_d(q) q^q p^{q-1})^{\frac{1}{\lfloor d/2 \rfloor - q}}$$

*fraction of the total weight of the points in  $P$ .*

**Proof.** By appropriately scaling all the rational weights, we may assume that each weight is a non-negative integer and we replace a point with weight  $m$  by  $m$  unweighted copies of the point. Let  $P'$  be the new set of points. Observe that any set  $S$  of  $pq$  points in  $P'$  either contains  $q$  copies of some point in  $P$  or it contains  $p$  distinct points from  $P$ . In either case, there is hyperplane separating  $q$  points of  $S$  from  $C$ . Thus, we can apply Lemma 6 to the point set  $P'$  with the parameter  $p$  in the lemma replaced by  $pq$ . The result follows. ◀

Finally we return to the proof of the main theorem.

► **Theorem 1.** *Let  $C$  be a bounded convex object in  $\mathbb{R}^d$  and  $P$  a set of  $n$  points lying outside  $C$ . Further let  $c_p, c_q$  be two integers, with  $1 \leq c_q \leq c_p \leq n - \lfloor \frac{d}{2} \rfloor$ , such that for every  $c_p + \lfloor \frac{d}{2} \rfloor$  points of  $P$ , there exists a subset of size  $c_q + \lfloor \frac{d}{2} \rfloor$  whose convex-hull is disjoint from  $C$ . Then the points of  $P$  can be partitioned into*

$$\lambda_d(c_p, c_q) = K_2 \frac{d}{c_q} \cdot (\sqrt{2}K_1)^{\frac{d}{c_q}} \cdot (\lfloor d/2 \rfloor + c_q)^2 \cdot (\lfloor d/2 \rfloor + c_p)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}} \cdot \log(\lfloor d/2 \rfloor + c_p)$$

*sets, each of whose convex-hull is disjoint from  $C$ . Here  $K_1, K_2$  are absolute constants independent of  $n, d, c_p$  and  $c_q$ . Furthermore, such a partition can be computed in polynomial time.*

**Proof.** Let  $p = c_p + \lfloor d/2 \rfloor$  and  $q = c_q + \lfloor d/2 \rfloor$ . Let  $\mathcal{H}$  be the set of all hyperplanes separating a distinct subset of points of  $P$  from  $C$ . As the number of subsets of  $P$  is finite, one can assume that  $\mathcal{H}$  is also finite. Consider the following linear program on  $|\mathcal{H}|$  variables  $\{u_h \geq 0 : h \in \mathcal{H}\}$ :

$$\min \sum_{h \in \mathcal{H}} u_h, \quad \text{such that} \quad \forall r \in P: \sum_{\substack{h \in \mathcal{H} \\ h \text{ separates } r \text{ from } C}} u_h \geq 1. \quad (2)$$

The LP-dual to the above program, on  $|P|$  variables  $\{w_r \geq 0 : r \in P\}$ , is:

$$\max \sum_{p \in P} w_p, \quad \text{such that} \quad \forall h \in \mathcal{H}: \sum_{\substack{r \in P \\ h \text{ separates } r \text{ from } C}} w_r \leq 1. \quad (3)$$

Consider an optimal solution  $w^*$  of the dual linear program and interpret  $w_r^*$  as the weight of each  $r \in P$ . Since the weights are rational, by Corollary 7, there exists a hyperplane  $h \in \mathcal{H}$  separating a subset of  $P$  of combined weight at least  $\epsilon = \alpha_d(p, q)$  fraction of the total weight of all the points. Since the total weight of the points in any half-space is constrained to be at most 1 by the linear program, the total weight of all the points of  $P$  must be at most  $\frac{1}{\epsilon}$ . In other words, the optimal value of linear program (3) is at most  $\frac{1}{\epsilon}$ . Since the optimal values of both linear programs are equal due to strong duality, the optimal value of linear program (2) is also at most  $\frac{1}{\epsilon}$ .

Let  $u^*$  be the optimal solution of linear program (2). If we interpret  $u_h$  as the weight of the hyperplane  $h$ , the constraints of the program imply that each point is separated by a set of hyperplanes in  $\mathcal{H}$  whose combined weight is at least 1 out of a total weight of at most  $\frac{1}{\epsilon}$  – in other words, at least  $\epsilon$ -th fraction of the total weight of  $\mathcal{H}$ . By associating with each hyperplane the half-space bounded by it and not containing  $C$ , and using the  $\epsilon$ -net theorem for half-spaces in  $\mathbb{R}^d$  (see [11]), there exists a set of  $O\left(\frac{d}{\epsilon} \log \frac{1}{\epsilon}\right)$  hyperplanes which together separate all points of  $P$  from  $C$ . Recalling that

$$\frac{1}{\epsilon} = \frac{1}{\alpha_d(p, q)} = (2e \kappa'_d(q) q^q p^{q-1})^{\frac{1}{q - \lfloor d/2 \rfloor}} = (2e \kappa'_d(q) q^q p^{q-1})^{\frac{1}{c_q}}.$$

and that  $\kappa'_d(q) = 2K_1^d \lfloor d/2 \rfloor^{-\lfloor d/2 \rfloor} \left(1 + \frac{q}{\lfloor d/2 \rfloor}\right)^{\lfloor d/2 \rfloor}$ , we get

$$\begin{aligned} \frac{1}{\epsilon} &= \left(4K_1^d e^{\lfloor d/2 \rfloor - \lfloor d/2 \rfloor} \left(1 + \frac{q}{\lfloor d/2 \rfloor}\right)^{\lfloor d/2 \rfloor} q^q p^{q-1}\right)^{\frac{1}{c_q}} \\ &\leq \left(4K_1^{d+1} \lfloor d/2 \rfloor^{-d} (c_q + d)^{\lfloor d/2 \rfloor} q^q p^{q-1}\right)^{\frac{1}{c_q}} \quad (\text{using } e \leq K_1 \text{ and } q = c_q + \lfloor d/2 \rfloor) \\ &\leq \left(4K_1^{d+1} \lfloor d/2 \rfloor^{-d} (c_q + d)^{\lfloor d/2 \rfloor} q^{c_q + \lfloor d/2 \rfloor} p^{c_q + \lfloor d/2 \rfloor - 1}\right)^{\frac{1}{c_q}} \\ &= O\left(K_1^{\frac{d}{c_q}} \lfloor d/2 \rfloor^{-\frac{d}{c_q}} (c_q + d)^{\frac{\lfloor d/2 \rfloor}{c_q}} (c_q + \lfloor d/2 \rfloor)^{1 + \frac{\lfloor d/2 \rfloor}{c_q}} (c_p + \lfloor d/2 \rfloor)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}}\right) \\ &= O\left(K_1^{\frac{d}{c_q}} d^{2 + \frac{\lfloor d/2 \rfloor - 1}{c_q}} \left(1 + \frac{c_q}{d}\right)^{\frac{\lfloor d/2 \rfloor}{c_q}} \left(1 + \frac{c_q}{\lfloor d/2 \rfloor}\right)^{1 + \frac{\lfloor d/2 \rfloor}{c_q}} \left(1 + \frac{c_p}{\lfloor d/2 \rfloor}\right)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}}\right) \\ &= O\left(K_1^{\frac{d}{c_q}} d^{2 + \frac{\lfloor d/2 \rfloor - 1}{c_q}} e^{\frac{c_q}{d} \cdot \frac{\lfloor d/2 \rfloor}{c_q}} \left(1 + \frac{c_q}{\lfloor d/2 \rfloor}\right) e^{\frac{c_q}{\lfloor d/2 \rfloor} \cdot \frac{\lfloor d/2 \rfloor}{c_q}} \left(1 + \frac{c_p}{\lfloor d/2 \rfloor}\right)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}}\right) \\ &= O\left(K_1^{\frac{d}{c_q}} d^{2 + \frac{\lfloor d/2 \rfloor - 1}{c_q}} \left(1 + \frac{c_q}{\lfloor d/2 \rfloor}\right) \left(1 + \frac{c_p}{\lfloor d/2 \rfloor}\right)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}}\right) \\ &= O\left(K_1^{\frac{d}{c_q}} 2^{\frac{d}{2c_q}} (\lfloor d/2 \rfloor + c_q) (\lfloor d/2 \rfloor + c_p)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}}\right) \\ &= O\left(\left(\sqrt{2}K_1\right)^{\frac{d}{c_q}} (\lfloor d/2 \rfloor + c_q) (\lfloor d/2 \rfloor + c_p)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}}\right). \end{aligned}$$

The Big-Oh notation here does not hide dependencies on  $d$  – namely we do not treat  $d$  as a constant. From the above it follows that

$$\log \frac{1}{\epsilon} = O\left(c_q^{-1} (\lfloor d/2 \rfloor + c_q) \log (\lfloor d/2 \rfloor + c_p)\right).$$

Thus,  $\frac{d}{\epsilon} \log \frac{1}{\epsilon}$  is

$$O\left(d \cdot \left(\left(\sqrt{2}K_1\right)^{\frac{d}{c_q}} (\lfloor d/2 \rfloor + c_q) (\lfloor d/2 \rfloor + c_p)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}}\right) \cdot \left(c_q^{-1} (\lfloor d/2 \rfloor + c_q) \log (\lfloor d/2 \rfloor + c_p)\right)\right)$$

which simplifies to

$$O\left(\frac{d}{c_q} \left(\sqrt{2}K_1\right)^{\frac{d}{c_q}} (\lfloor d/2 \rfloor + c_q)^2 (\lfloor d/2 \rfloor + c_p)^{1 + \frac{\lfloor d/2 \rfloor - 1}{c_q}} \log(\lfloor d/2 \rfloor + c_p)\right).$$

Since linear programs can be solved in polynomial time and epsilon nets can be computed in polynomial time, the partition of  $P$  into the above number of sets can be achieved in polynomial time. The theorem follows.  $\blacktriangleleft$



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