



**HAL**  
open science

# Analytic normal forms and inverse problems for unfoldings of 2-dimensional saddle-nodes with analytic center manifold

Chloé Rousseau, Loïc Teyssier

► **To cite this version:**

Chloé Rousseau, Loïc Teyssier. Analytic normal forms and inverse problems for unfoldings of 2-dimensional saddle-nodes with analytic center manifold. *Annales Scientifiques de l'École Normale Supérieure*, 2021, 54 (1), pp.133-233. 10.24033/asens.2457 . hal-01890315

**HAL Id: hal-01890315**

**<https://hal.science/hal-01890315>**

Submitted on 10 Oct 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ANALYTIC NORMAL FORMS AND INVERSE PROBLEMS FOR UNFOLDINGS OF 2-DIMENSIONAL SADDLE-NODES WITH ANALYTIC CENTER MANIFOLD

C. ROUSSEAU<sup>\*</sup> & L. TEYSSIER<sup>†2</sup>

**ABSTRACT.** We give normal forms for generic  $k$ -dimensional parametric families  $(Z_\varepsilon)_\varepsilon$  of germs of holomorphic vector fields near  $0 \in \mathbb{C}^2$  unfolding a saddle-node singularity  $Z_0$ , under the condition that there exists a family of invariant analytic curves unfolding the weak separatrix of  $Z_0$ . These normal forms provide a moduli space for these parametric families. In our former 2008 paper, a modulus of a family was given as the unfolding of the Martinet-Ramis modulus, but the realization part was missing. We solve the realization problem in that partial case and show the equivalence between the two presentations of the moduli space. Finally, we completely characterize the families which have a modulus depending analytically on the parameter. We provide an application of the result in the field of non-linear, parameterized differential Galois theory.

<sup>\*</sup>Département de Mathématiques et de Statistique

Université de Montréal, Canada

<http://www.dms.umontreal.ca/~rousseac>

<sup>†</sup>Unité Mixte Internationale 3457

CNRS & CRM, Université de Montréal, Canada

<http://math.u-strasbg.fr/~teyssier>

## 1. INTRODUCTION

Heuristically, moduli spaces of holomorphic dynamical systems not only encode but also describe qualitatively the dynamics itself, and to some extent allow a better understanding of remarkable dynamical phenomena. This paper is part of a large program aimed at studying the conjugacy classes of dynamical systems in the neighborhood of stationary points (up to local changes of analytic coordinates). Stationary points and their invariant manifolds organize the global dynamics while degenerate stationary points organize the bifurcation diagrams in families of dynamical systems. Stationary points of discrete dynamical systems correspond to fixed-points of the iterated map(s), while for continuous dynamical systems they correspond to singularities in the underlying differential equation(s).

A natural tool for studying conjugacy classes is the use of normal forms. For hyperbolic stationary points (generic situation), the system is locally conjugate to its linear part so that the quotient space of (local) hyperbolic systems is given by the space of linear dynamical systems. However, for most non-hyperbolic stationary

---

*Date:* January 2018.

*2000 Mathematics Subject Classification.* (2010 MSC) 34A26, 34C20, 34C23, 34M35, 37G10, 37L10.

*Key words and phrases.* Normal forms, holomorphic vector fields, saddle-node bifurcation, unfolding of singularities, modulus space, analytic conjugacy.

The first author was supported by NSERC in Canada. The second author was supported by the French National Research Agency grant ANR-13-JS01-0002-01. He would also like to express his gratitude for the welcome and support he received during his stay at the CNRS unit UMI-3457 located in the CRM / Université de Montréal.

points the normalizing changes of coordinates (sending *formally* the system to a normal form) is given by a divergent power series. Divergence is very instructive: it tells us that the dynamics of the original system and that of the normal form are qualitatively different. In that respect, a subclass of singularities that has been thoroughly studied in the beginning of the 80's is that of 1-resonant singularities: these include parabolic fixed-points of germs of 1-dimensional diffeomorphisms, resonant-saddle singularities and saddle-node singularities of 2-dimensional vector fields, as well as non-resonant irregular singular points of linear differential systems. These various resonant dynamical systems share a lot of common properties, among which is the finite-determinacy of their formal normal forms (*e.g.* polynomial expressions in the case of vector fields). Another property they share is that they can be understood as the *coalescence* of special “geometric objects”, either of stationary points or of a singular point with a limit cycle in the case of the Hopf bifurcation at a weak focus.

**1.1. Scope of the paper.** The present work is the follow-up of [40] in which we described a set of functional moduli for unfoldings of codimension  $k$  saddle-node vector fields  $Z = (Z_\varepsilon)_\varepsilon$  depending on a finite-dimensional parameter  $\varepsilon \in (\mathbb{C}^k, 0)$ . Here we focus mainly on the inverse problem and on the question of finding (almost unique) normal forms, as we explain below.

The most basic example of such an unfolding is given by the codimension 1 unfolding (expressed in the canonical basis of  $\mathbb{C}^2$ )

$$(1.1) \quad Z_\varepsilon(x, y) := \begin{bmatrix} x^2 + \varepsilon \\ y \end{bmatrix}, \quad \varepsilon \in \mathbb{C}.$$

Real slices of the phase-portraits are shown in Figure 1.1. The merging (bifurcation) occurs at  $\varepsilon = 0$ : for  $\varepsilon \neq 0$  the system has two stationary points located at  $(\pm\sqrt{-\varepsilon}, 0)$  which collide as  $\varepsilon$  reaches 0.

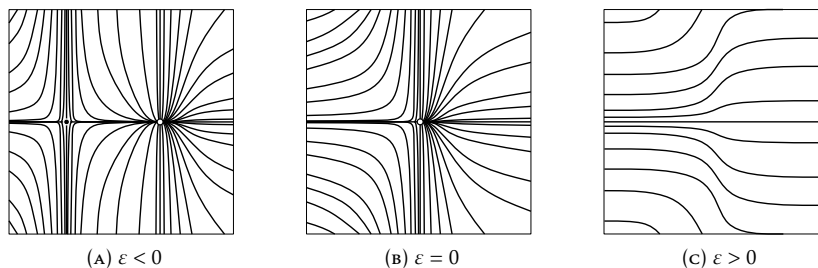


FIGURE 1.1. Typical members of the simplest saddle-node bifurcation.

**1.2. Modulus of classification.** Each merging stationary point organizes the dynamics in its own neighborhood in a rigid way. The local models of these rigid dynamics seldom agree on overlapping areas and in general cannot be glued together. If this incompatibility persists as the confluence happens, then we have divergence of the normalizing series at the limit. In the case of 1- or 2-dimensional resonant systems the normalizing series is  $k$ -summable. The divergence is then

quantified by the *Stokes phenomenon*: there exists a formal normalizing transformation, and a covering of a punctured neighborhood of the singularity by  $2k$  sectors over which there exist unique sectorial normalizing transformations that are Gevrey-asymptotic to the formal normalization. Comparing the normalizing transformations on intersections of consecutive sectors provides a modulus of analytic classification. This modulus takes the form of Stokes matrices for irregular singularities of linear differential systems and functional moduli for singularities of nonlinear dynamical systems (see for instance [19]).

The classification of resonant systems may seem rather mysterious. But if we remember that we are studying the merging of “simple” singularities, then it becomes natural to unfold the situation and study the “multiple” singularity as a limiting case. Indeed, analyzing unfoldings sheds a new light on the “complicated” dynamics of the limiting systems. The idea was suggested by several mathematicians, including V. ARNOLD, A. BOLIBRUCH and J. MARTINET [29]. It was put in practice for unfoldings of saddle-node singularities by A. GLUTSYUK [15] on regions in parameter space over which the confluent singularities are all hyperbolic. The system can be linearized in the neighborhood of each singularity, and the mismatch in the normalizing changes of coordinates tends to the components of the saddle-node’s Martinet-Ramis modulus [30] when the singularities merge. But the tools were still missing for a full classification of unfoldings of multiple singularities, in particular on a full neighborhood in parameter space of the bifurcation value.

The thesis of P. LAVAUERS [23] on parabolic points of diffeomorphisms opened the way for such classifications, for he studied the complementary regions in parameter space. The first classification of generic unfoldings of codimension 1 fixed-point of diffeomorphisms regarded the parabolic point [28], and then the resonant-saddle and saddle-node singularities of differential equations [36, 37]. The first classification of generic unfoldings of codimension  $k$  saddle-nodes was done by the authors [40] using the visionary ideas of A. DOUADY and P. SENTENAC [11, 2] that R. OUDKERK had used on some regions in parameter space in his thesis [33]. Then followed classifications of generic unfoldings of codimension  $k$  parabolic points [35] and of non-resonant irregular singular points of Poincaré rank  $k$  differential systems [18].

In the spirit of this general context we obtained in [40] a (family of) functional data

$$m_\varepsilon = (f_\varepsilon^s, \psi_\varepsilon^s, \psi_\varepsilon^n)_\varepsilon.$$

For  $\varepsilon = 0$  this data coincides with the saddle-node’s modulus [30, 49, 44]. Although the original work of J. MARTINET and J.-P. RAMIS already covered parametric cases, it was then assumed that the (formal) type of the singularity remained constant. On the contrary we were interested in bifurcations, which are deformations where the additional parameters change the type (or number) of singularities. Our main contribution was to reconcile Glutsyuk’s and Lavaurs’s viewpoint and devise a uniform framework valid for a complete neighborhood of the bifurcation value of the parameter. That being said, the very nature of our geometric construction prevented the modulus to be continuous on the whole parameter space. This space needs to be split into a finite number of *cells* whose closures cover a neighborhood

of the bifurcation value, on which the modulus is analytic on  $\varepsilon$  with continuous extension to the closure.

**1.3. The inverse (or realization) problem.** At the time of the first works on the question, identifying the moduli space was still out of reach. Performing this identification is called the inverse problem. It was first solved for codimension 1 parabolic fixed-points and resonant-saddle singularities [9, 38], as well as for the irregular singularities of linear differential systems with Poincaré rank 1 [22]. For codimension  $k$  the realization problem was first solved for unfoldings of non-resonant irregular singular points of Poincaré rank  $k$  [39]. But the realization question is still open for unfoldings of codimension  $k$  parabolic points.

Let us formulate the inverse problem in the case at hands.

**Inverse problem.** *Among all elements of the vector space  $\mathcal{M}$  to which  $\mathfrak{m} = (\mathfrak{m}_\varepsilon)_\varepsilon$  belongs, to identify those coming as moduli of a saddle-node bifurcation.*

The present paper answers completely this challenge in the case of bifurcations with a persistent analytic center manifold. The common feature to that case and the one studied in [39] is that solving the inverse problem ultimately provides unique normal forms (privileged representative in each analytic class).

Having persistent analytic center manifold can be read in the modulus as the condition  $\psi^n = \text{Id}$ . Although any element of the specialization of  $\mathcal{M}$  at  $\varepsilon = 0$  can be realized as the modulus of a saddle-node vector field [30, 44], this property does not hold anymore for bifurcations: the typical element of  $\mathcal{M} \cap \{\psi^n = \text{Id}\}$  can never be realized as a modulus of saddle-node bifurcation. Let us explain how this is so. It is rather easy to get convinced that there is no obstruction to realize any given deformation  $(\mathfrak{m}_\varepsilon)_{\varepsilon \in \text{cl}(\mathcal{E})}$  of a saddle-node's modulus  $\mathfrak{m}_0$  over any given cell  $\mathcal{E}$  in parameter space. By this we mean that for each fixed  $\varepsilon \in \text{cl}(\mathcal{E})$  it is possible to find a holomorphic vector field  $Z_\varepsilon$  on a neighborhood  $\mathcal{U}$  of  $(0,0)$  such that comparisons between its sectorial normalizing maps coincide with  $\mathfrak{m}_\varepsilon$ . Furthermore the dependence  $\varepsilon \mapsto Z_\varepsilon$  has the expected regularity on the cell's closure, and the neighborhood  $\mathcal{U}$  is independent on  $\varepsilon$ . The sole obstacle lies therefore in gluing these cellular realizations together over cellular intersections in order to obtain a genuine analytic parametric family  $Z$  whose modulus agrees with  $\mathfrak{m}$ . Favorable situations can be characterized by a strong criterion imposed on  $\mathfrak{m}$ , called *compatibility condition*. A necessary and sufficient condition is that two realizations over different cells in parameter space be conjugate over the intersection of the two cells, thus allowing correction to a uniform family. One difficulty is to express this condition on the abstractly encoded dynamics  $\mathfrak{m}$  (that is, before performing the cellular realization). The compatibility condition takes the simple form that the abstract holonomy pseudogroups generated by  $\mathfrak{m}$  be conjugate, a condition which can easily be expressed in terms of the modulus. The general case of a bifurcation without analytic center manifold remains open, and we hope to address it in the near future.

**1.4. Summary of the paper's content.** Here we review the content of the present work. For precise statements of our main results, as for more detailed proof techniques, we refer to Section 2. Recall that one can associate two dynamical data to a vector field  $X = A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y}$ :

- the trajectories of  $X$  parametrized by the complex time in the associated flow-system

$$\begin{cases} \dot{x} &= A(x, y) \\ \dot{y} &= B(x, y) \end{cases}'$$

- the underlying foliation  $\mathcal{F}_X$  whose leaves coincide with orbits of  $X$ , obtained by forgetting about a particular parametrization of the trajectories. The foliation really is attached to the underlying non-autonomous differential equation

$$A(x, y)y' = B(x, y)$$

rather than to the vector field itself.

The action of (analytic or formal) changes of variables  $\Psi$  on vector fields  $X$  by conjugacy is obtained as the pullback

$$\Psi^*X := D\Psi^{-1}(X \circ \Psi).$$

The vector fields  $X$  and  $\Psi^*X$  are then (analytically or formally) **conjugate**. When two foliations  $\mathcal{F}_X$  and  $\mathcal{F}_{\tilde{X}}$  are conjugate (when  $X$  is conjugate to a scaling of  $\tilde{X}$  by a non-vanishing function) it is common to say that  $X$  and  $\tilde{X}$  are **orbitally equivalent**. While for unfoldings we also allow parameter changes, we restrict our study to parameter / coordinates changes of the form

$$\Psi : (\varepsilon, x, y) \mapsto (\phi(\varepsilon), \Psi_\varepsilon(x, y)).$$

In this paper we focus on families  $Z = (Z_\varepsilon)_{\varepsilon \in (\mathbb{C}^k, 0)}$  unfolding a codimension  $k$  saddle-node singularity for  $\varepsilon = 0$  and the study of their conjugacy class (*resp.* orbital equivalence class) under local analytic changes of variables and parameter (*resp.* and scaling by non-vanishing functions). Such families can always be brought by a formal change of variables and parameter into the formal normal form<sup>1</sup>

$$u_\varepsilon(x) \left( P_\varepsilon(x) \frac{\partial}{\partial x} + y(1 + \mu_\varepsilon x^k) \frac{\partial}{\partial y} \right),$$

where

$$\begin{aligned} P_\varepsilon(x) &= x^{k+1} + \varepsilon_{k-1}x^{k-1} + \dots + \varepsilon_1x + \varepsilon_0, \quad k \in \mathbb{N} \\ u_\varepsilon(x) &= u_{0,\varepsilon} + u_{1,\varepsilon}x + \dots + u_{k,\varepsilon}x^k, \quad u_{0,\varepsilon} \neq 0 \end{aligned}$$

and  $\varepsilon \mapsto (\mu_\varepsilon, u_{0,\varepsilon}, \dots, u_{k,\varepsilon})$  is holomorphic near 0. A proof of this widely accepted result seems to be missing in the literature, hence we provide one.

The first step in our previous work [40] consisted in preparing the unfolding  $(Z_\varepsilon)_\varepsilon$  by bringing it in a form where the polynomial  $P_\varepsilon$  determines the  $\frac{\partial}{\partial x}$ -component. Formal and analytic equivalences between such forms must consequently preserve the coefficients of  $P_\varepsilon$ , which then become privileged *canonical parameters*. This process eliminates the difficulty of dealing with changes of parameters and allows to work for fixed values of  $\varepsilon$ . Then we established a complete classification. The modulus was composed of two parts: the *formal part* given by

<sup>1</sup>As is customary we write vector fields in the form of derivations, by identifying the canonical basis of  $\mathbb{C}^2$  with  $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ .

the formal normal form above, and the *analytic part* given by an unfolding of the saddle-node's functional modulus. The formal / analytic part of the modulus itself consists in the Martinet-Ramis orbital part (characterizing the vector field up to orbital equivalence) and an additional part classifying the time. For example  $\mu_\varepsilon$  is the formal orbital class while  $u_\varepsilon$  is the formal temporal class.

We completely solve the realization problem for orbital equivalence (*i.e.* for foliations) when each  $Z_\varepsilon$  admit a single analytic invariant manifold passing through every singularity. But we do more: we provide almost unique “normal forms” (the only degree of freedom being linear transformations in  $y$ ), which are polynomial in  $x$  when  $\mu_0 \notin \mathbb{R}_{\leq 0}$ . In that generic situation, an unfolding is orbitally equivalent to an unfolding over  $\mathbb{P}_1(\mathbb{C}) \times (\mathbb{C}, 0)$  of the form

$$P_\varepsilon(x) \frac{\partial}{\partial x} + y \left( 1 + \mu_\varepsilon x^k + \sum_{j=1}^k x^j R_{j,\varepsilon}(y) \right) \frac{\partial}{\partial y},$$

where the  $R_j$  are analytic in both the geometric variable  $y$  and the parameter  $\varepsilon$ . In this generic case the construction is a direct generalization of that of F. LORRAY's [25, Theorems 2 and 4] for  $\varepsilon = 0$  and  $k = 1$ , and only involves tools borrowed from complex geometry. In the non-generic case (when  $\mu_0 \leq 0$ ) we also provide almost unique “normal forms”, which are in some sense global in  $x$ : in this case the foliation is defined on a fiber bundle of negative degree  $-\tau(k+1) < \mu_0$  over  $\mathbb{P}_1(\mathbb{C})$  and is induced by vector fields of the form

$$(1.2) \quad X_\varepsilon(x, y) := P_\varepsilon(x) \frac{\partial}{\partial x} + y \left( 1 + \mu_\varepsilon x^k + \sum_{j=1}^k x^j R_{j,\varepsilon}(P_\varepsilon^\tau(x)y) \right) \frac{\partial}{\partial y}.$$

This result offers a new presentation of the moduli space which has the advantage over that of [40] to be made up of functions analytic in the parameter (it does not require the splitting of the parameter space into cells).

As far as normal forms are concerned, we provide some also for the family of vector fields. This requires normalizing the “temporal part”. The method used is an unfolding of the construction of R. SCHÄPFKE and L. TEYSSIER [42] performed for  $\varepsilon = 0$ . As a by-product we provide an explicit section of the cokernel of the derivation  $X_\varepsilon$  (*i.e.* a linear complement of the image of  $X_\varepsilon$  acting as a Lie derivative on the space of analytic germs).

An important observation is that the normalization we just described does not involve classification moduli in any way (nor does it rely on the analytical classification for that matter), at least in the generic case  $\mu_0 \notin \mathbb{R}_{\leq 0}$ . Therefore it does not answer the inverse (or realization) problem which is posed in terms of classification moduli. This leads us to discuss the compatibility condition.

As we mentioned earlier we can realize any unfolding  $\mathfrak{m} = (\mathfrak{m}_\varepsilon)_{\varepsilon \in \text{cl}(\mathcal{E})}$  of a saddle-node's modulus  $\mathfrak{m}_0$  over a cell  $\mathcal{E}$  in parameter space, but we have such control of the construction that we can guarantee this realization is an unfolding in normal form (1.2), save for the fact that the functions  $\varepsilon \mapsto R_{j,\varepsilon}$  are merely analytic on  $\mathcal{E}$  with continuous extension to the closure. It is possible to express the holonomy group of  $X_\varepsilon$  with respect to the analytic center manifold (the geometrical dynamics) as a representation of an abstract group of words formed with elements of the modulus  $\mathfrak{m}$  (acting in orbits space). The compatibility condition simply states

that the holonomy pseudogroups over the intersection of two neighboring cells are conjugate by a tangent-to-identity mapping. If the condition is satisfied then two cellular realizations are conjugate for values of the parameter in the cells' intersection. Usually when such a situation occurs, we need to apply a conjugacy to the vector fields so that they match in the new coordinates. Here no need for it. Indeed, since the realizations over the different cells are in normal form, they necessarily are conjugate by a linear map. The additional hypothesis in the compatibility condition that the conjugating map is tangent-to-identity allows to conclude that the cellular unfoldings actually agree and therefore define a genuine unfolding analytic in  $\varepsilon \in (\mathbb{C}^k, 0)$ .

Our analysis presents in an effective way the relationship between Rousseau-Teyssier classification moduli and the coefficients of the normal forms, so that numerical, and in some cases symbolic, computations can be performed. Also, we have refined our understanding of the modulus compared to the presentation in [40]. The number of cells is now the optimal number  $C_k = \frac{1}{k+1} \binom{2k}{k}$  (the  $k^{\text{th}}$  Catalan's number) given by the Douady-Sentenac classification [11, 10]. Moreover we have reduced the degrees of freedom: instead of having the modulus given up to conjugacy by linear functions depending both on  $\varepsilon$  and the cell, now the modulus is given up to conjugacy by linear functions depending only on  $\varepsilon$  in an analytic way. This new equivalence relation in the presentation of the modulus was essential in getting the realizations over the different cells to match when the compatibility condition is satisfied.

Last but not least we were able to completely characterize the moduli that depend analytically on the parameter. These only occur when  $k = 1$  and their normal forms are given by Bernoulli unfoldings

$$(1.3) \quad P_\varepsilon(x) \frac{\partial}{\partial x} + y \left( 1 + \mu_\varepsilon x^k + x r_\varepsilon(x) (P_\varepsilon(x)^\tau y)^d \right) \frac{\partial}{\partial y}$$

with  $d \in \mathbb{N}$  and  $d\mu \in \mathbb{Z}$  (in particular  $\mu$  must be a rational constant, which is seldom the case). This proves that the compatibility condition is not trivially satisfied by every element of  $\mathcal{M} \cap \{\psi^n = \text{Id}\}$ . On the contrary, the typical situation is that of moduli which are analytic and bounded only on single cells. This reminds us of the setting of Borel-summable divergent power series, in particular in the case  $k = 1$  where the cells are actual sectors and it can be proved that the moduli are sectorial sums of  $\frac{1}{2}$ -summable power series (as in [9]). When  $k > 1$  the lack of a theory of summation in more than one variable prevents us from reaching similar conclusions, although the moduli are natural candidates for such sums and a general summation theory should probably contain the case we studied here. We reserve such considerations for future works, perhaps using the theory of polynomial summability recently introduced by J. MOZO and R. SCHÄPFKE (this work [32] is still in preparation, we refer to the prior monomial case performed in [5]).

**1.5. Applications.** Our main results can be used to solve problems outside the scope of finding normal forms or addressing the *local* inverse problem. Let us mention two applications, the second of which we develop in Section 2.3.

The first (and most straightforward) one concerns the global inverse problem, also known as non-linear Riemann-Hilbert problem, posed by Y. ILYASHENKO and



S. YAKOVENKO in [19, Chapter IV]. Being given a (germ of a) complex surface  $\mathcal{M}$  seen as the total space of a fiber bundle over a divisor  $\mathbb{P}_1(\mathbb{C}) \subset \mathcal{M}$ , the problem is to characterize the holonomy representations of complex foliations on  $\mathcal{M}$  tangent to (and regular outside) the divisor and transverse to the fibers, except over  $k + 2$  singularities (which are all assumed non-degenerate) where the fibers are invariant by the foliation. Using a sibling of Loray’s technique, they solve it for fiber bundles of degree 0 and  $-1$ , although they only provide details for the former case. Our results open the way to generalizations in several directions:

- allowing saddle-node(s) with central manifold along the divisor and adding to the holonomy representation the components of the modulus of the saddle-nodes, similarly to the generalized linear Riemann-Hilbert problem when irregular singularities are allowed;
- allowing foliations depending analytically on the parameter;
- considering realizations on fiber bundles of negative degree: we obtain here realizations on bundles with degree given by an arbitrary non-positive multiple of  $k + 1$  (see Conjecture 8.5 for a brief discussion of possible improvement to any non-positive degree);
- allowing resonant nodes: in our paper all nodes were linearizable because their Camacho-Sad index was greater than 1. But nodes with smaller Camacho-Sad index pose no additional problem.

We propose to address this matter in the near future.

The other application regards differential Galois theory: heuristically, classification invariants carry Galoisian information (pertaining to the integrability in Liouvillian closed-form). For instance, in the limiting case of a saddle-node singularity it is well-known that Martinet-Ramis moduli play the same role for non-linear equations as Stokes matrices do for linear systems near an irregular singularity. A Galoisian formulation of this fact in terms of Malgrange groupoid [26, 27] can be found in the work of G. CASALE [6]. When the differential equation depends on a parameter  $\varepsilon$ , the recent thesis of D. DAVY describes a form of “semi-continuity” for specializations of its parametrized Malgrange groupoid  $\mathfrak{M}$ . He proves that the size of the groupoid  $\mathfrak{M}_\varepsilon$  is constant if  $\varepsilon$  is generic, more precisely if the parameter does not belong to a (maybe empty) countable union  $\Omega$  of hypersurfaces, while for  $\varepsilon \in \Omega$  the groupoid  $\mathfrak{M}_\varepsilon$  can only get smaller. The present study illustrates and refines this phenomenon.

Consider the extreme case  $P_\varepsilon(x) \frac{\partial}{\partial x} + y(1 + \mu_\varepsilon x^k + \varepsilon R_\varepsilon(x, y)) \frac{\partial}{\partial y}$  for  $R$  arbitrary: the vector field  $X_0$  is surely “not less integrable” (it is the formal normal form) than for  $\varepsilon \neq 0$ . This is actually the only possible kind of degeneracy near the saddle-node bifurcation, for we will establish that  $\Omega \cap (\mathbb{C}^k, 0) \subset \{0\}$ . We obtain the latter property by unfolding a result by M. BERTHIER and F. TOUZET [1], characterizing vector fields admitting local non-trivial Liouvillian first integrals near an elementary singularity. We deduce that normal forms of integrable unfoldings are necessarily a Bernoulli unfolding (1.3). Both proofs are very different in nature, and we obtain a particularly short one by framing the problem for normal forms, revealing the usefulness of their simple expression and of the explicit section of their cokernel.

## 2. STATEMENT OF THE MAIN RESULTS

In all what follows  $\varepsilon$  is the parameter, belonging to some  $(\mathbb{C}^k, 0)$  for  $k \in \mathbb{N}$ , and we study (holomorphic germs of a) parametric family of (germs at  $0 \in \mathbb{C}^2$  of) vector fields  $Z = (Z_\varepsilon)_{\varepsilon \in (\mathbb{C}^k, 0)}$  for which a saddle-node bifurcation occurs at  $\varepsilon = 0$ . That is to say, when  $\varepsilon = 0$  the vector field  $Z_0$  is of **saddle-node** type near the origin of  $\mathbb{C}^2$ :

- 0 is an isolated singularity of  $Z_0$ ,
- the differential at 0 of the vector field has exactly one non-zero eigenvalue (the singularity is elementary degenerate).

The family  $Z = (Z_\varepsilon)_\varepsilon$  is called a holomorphic germ of an **unfolding** of  $Z_0$ . We study in details only “generic” unfoldings, those which possess the “right number” of parameters to encode the bifurcation structure. Roughly speaking we require that for an open and dense set of parameters the vector field  $Z_\varepsilon$  have  $k+1$  distinct non-degenerate singular points. The latter merge into a saddle-node singularity of multiplicity  $k+1$  (codimension  $k$ ) as  $\varepsilon \rightarrow 0$ . Let us make these statements precise.

**Definition 2.1.** An unfolding  $Z$  of a codimension  $k \in \mathbb{N}$  saddle-node  $Z_0$  is **generic** if there exists a biholomorphic change of coordinates and parameter such that, in the new coordinates  $(x, y)$  and new parameter  $\varepsilon$ , the singular points of each  $Z_\varepsilon$  are given by  $P_\varepsilon(x) = y = 0$ , where

$$P_\varepsilon(x) := x^{k+1} + \varepsilon_{k-1}x^{k-1} + \cdots + \varepsilon_1x + \varepsilon_0.$$

*Remark 2.2.* Generic families are essentially *universal*. In particular, the bifurcation diagram of singular points is the elementary catastrophe of codimension  $k$  (in the complex domain).

The analytic unstable manifold of  $Z_0$ , tangent at 0 to the eigenspace associated to the non-zero eigenvalue of its differential, is called the **strong separatrix**. The other eigenspace corresponds to a “formal separatrix”  $\{y = \widehat{s}_0(x)\}$  called the **weak separatrix** (generically divergent [34], always summable in the sense of Borel [17]). We say that a saddle-node is convergent or divergent according to the nature of its weak separatrix.

**Definition 2.3.** We say that the generic unfolding  $Z$  is **purely convergent** when there exists a holomorphic function

$$\begin{aligned} s : (\mathbb{C}^k, 0) \times (\mathbb{C}, 0) &\longrightarrow \mathbb{C} \\ (\varepsilon, x) &\longmapsto s_\varepsilon(x) \end{aligned}$$

such that:

- each graph  $\mathcal{S}_\varepsilon$  of  $s_\varepsilon$  is tangent to  $Z_\varepsilon$  and contains  $\text{Sing}(Z_\varepsilon)$  (the singular set of  $Z_\varepsilon$ , consisting in all zeros of  $Z_\varepsilon$ ),
- $\mathcal{S}_0$  is the weak separatrix of  $Z_0$  (in particular the latter is convergent).

We call  $\text{Convergent}_k$  the set of all such unfoldings.

*Remark 2.4.*

- (1) By applying beforehand the change of variables  $(\varepsilon, x, y) \mapsto (\varepsilon, x, y + s_\varepsilon(x))$  to the unfolding we can always assume that  $\{y = 0\}$  is invariant by  $Z_\varepsilon$  for all  $\varepsilon \in (\mathbb{C}^k, 0)$ .

- (2) There exist unfoldings  $Z$  of a convergent saddle-node  $Z_0$  such that, for all  $\varepsilon$  close enough to 0, no analytic invariant curve  $\mathcal{S}_\varepsilon$  exist. We use the term “purely convergent” to insist that in the present case *every* vector field  $Z_\varepsilon$  for  $\varepsilon \in (\mathbb{C}^k, 0)$  must admit an analytic invariant curve.

**2.1. Normalization of purely convergent unfoldings.** For  $\mathbf{z}$  a finite-dimensional complex multivariable we write  $\mathbb{C}\{\mathbf{z}\}$  the algebra of convergent power series in  $\mathbf{z}$ , naturally identified with the space of germs of a holomorphic function at  $\mathbf{0}$ . We extend this notation in the obvious manner so that  $\mathbb{C}\{\varepsilon, x\}$  is the space of convergent power series in the  $k+1$  complex variables  $\varepsilon_0, \dots, \varepsilon_{k-1}$  and  $x$ .

**2.1.1. Formal classification.** We first give an unfolded version of the well-known Bruno-Dulac-Poincaré normal forms [3, 13, 12]. Here we do not assume that  $Z$  be purely convergent.

**Formal Normalization Theorem.** *Let  $k \in \mathbb{N}$  be given. For  $\varepsilon = (\varepsilon_0, \dots, \varepsilon_{k-1}) \in \mathbb{C}^k$  define the polynomial*

$$P_\varepsilon(x) := x^{k+1} + \sum_{j=0}^{k-1} \varepsilon_j x^j.$$

*Take a generic unfolding  $Z$  of a saddle-node of codimension  $k$ . There exists  $(\mu, u) \in \mathbb{C}\{\varepsilon\} \times \mathbb{C}\{\varepsilon, x\}$  with  $(\varepsilon, x) \mapsto u_\varepsilon(x)$  polynomial in  $x$  of degree at most  $k$  and satisfying  $u_0(0) \neq 0$ , such that  $Z$  is formally conjugate to the **formal normal form***

$$(2.1) \quad \widehat{Z} := u \widehat{X},$$

where

$$(2.2) \quad \widehat{X}_\varepsilon(x, y) := P_\varepsilon(x) \frac{\partial}{\partial x} + y \left(1 + \mu_\varepsilon x^k\right) \frac{\partial}{\partial y}$$

*defines the **formal orbital normal form**. Notice that these vector fields are polynomial in  $(x, y)$  and holomorphic in  $\varepsilon \in (\mathbb{C}^k, 0)$ .*

In general the parameter of the normal form  $\widehat{Z}$  differs from the original parameter of  $Z$ . However the formal change of parameter  $\varepsilon \mapsto \phi(\varepsilon)$  happens to be actually analytic (as proved in [36, Theorem 3.5] and recalled in Theorem 4.1). Moreover such normal forms are essentially unique, in the sense that among all formal conjugacies only some linear changes of variables and parameter preserve the whole family. For example transforming  $x$  into  $\alpha x$  for some nonzero  $\alpha \in \mathbb{C}$  in  $P_\varepsilon \frac{\partial}{\partial x}$  yields the vector field

$$\frac{1}{\alpha} P_\varepsilon(\alpha x) \frac{\partial}{\partial x} = \alpha^k P_{\widetilde{\varepsilon}}(x) \frac{\partial}{\partial x}$$

where  $\widetilde{\varepsilon} := (\varepsilon_j \alpha^{1-j})_{j < k}$ . Therefore by taking  $\alpha^k = 1$  the linear change  $(\widetilde{\varepsilon}, x) \mapsto (\varepsilon, \alpha x)$  transforms  $(\widehat{X}_\varepsilon)_\varepsilon$  into  $(\widehat{X}_{\widetilde{\varepsilon}})_{\widetilde{\varepsilon}}$ . It turns out this is the only degree of freedom for formal changes of parameters (see Section 4), which makes the parameter of the normal form special.

**Definition 2.5.** The parameter of the normal form  $\widehat{Z}$  (modulo the action of  $\mathbb{Z}/k\mathbb{Z}$  on  $(\varepsilon, x)$ ) is called the **canonical parameter** of the original unfolding  $Z$ . In all the

following a representative  $\varepsilon$  of the canonical parameter is always implicitly fixed and forbidden to change.

As a consequence, two formal normal forms with formal invariants  $(\mu, u)$  and  $(\tilde{\mu}, \tilde{u})$  as above are (for fixed canonical parameter  $\varepsilon$ ):

- (1) orbitally formally equivalent if, and only if, they have same **formal orbital invariant**  $\mu = \tilde{\mu}$ ;
- (2) formally conjugate if, and only if, they have same **formal invariant**  $(\mu, u) = (\tilde{\mu}, \tilde{u})$ .

### 2.1.2. Analytical normalization.

**Definition 2.6.** For  $k \in \mathbb{N}$  a positive integer let us introduce the functional space in the complex multivariable  $(\varepsilon, x, v) \in \mathbb{C}^{k+2}$ :

$$\text{Section}_k\{v\} := \left\{ f_\varepsilon(x, v) = v \sum_{j=1}^k f_{\varepsilon, j}(v) x^j : f_{\varepsilon, j}(v) \in \mathbb{C}\{\varepsilon, v\} \right\}.$$

We let  $v$  figure explicitly in the notation  $\text{Section}_k\{v\}$  since this variable (and this variable only) will be subject to further specification.

**Normalization Theorem.** For a given  $k \in \mathbb{N}$  we fix a formal orbital invariant  $\mu \in \mathbb{C}\{\varepsilon\}$  and choose  $\tau \in \mathbb{Z}_{\geq 0}$  such that  $\mu_0 + (k+1)\tau \notin \mathbb{R}_{\leq 0}$ . For every  $Z \in \text{Convergent}_k$  with formal invariant  $(\mu, u)$ , there exist  $Q, R \in \text{Section}_k\{P^\tau y\}$  such that  $Z$  is analytically conjugate to

$$(2.3) \quad \mathcal{Z} := \frac{u}{1 + uQ} \mathcal{X}$$

where

$$(2.4) \quad \mathcal{X} := \widehat{X} + Ry \frac{\partial}{\partial y}.$$

*Remark 2.7.* In case  $\tau = 0$  (which can be enforced whenever the generic condition  $\mu_0 \notin \mathbb{R}_{\leq 0}$  holds) normal forms induce foliations with holomorphic extension to  $\mathbb{P}_1(\mathbb{C}) \times (\mathbb{C}, 0)$ . This is no longer true if  $\tau > 0$  and if  $R$  is not polynomial in the  $y$ -variable.

Specializing the theorem to  $\varepsilon = 0$ , we recover the earlier results [42, 25]. Let us briefly present the unfolded geometric construction of F. LORAY (performed at an orbital level in [25] when  $k = 1$ ) to get the gist of the argument. We define a holomorphic family of abstract foliated complex surfaces  $(\mathcal{M}, \mathcal{F}) = (\mathcal{M}_\varepsilon, \mathcal{F}_\varepsilon)_{\varepsilon \in (\mathbb{C}^k, 0)}$  given by two charts. The first one is a domain  $\mathcal{U}^0 := \{0 \leq |x| < \rho^0\} \times (\mathbb{C}, 0)$  together with some arbitrary convergent unfolding  $Z$ , provided the following non-restrictive properties (see [40]) are fulfilled for all  $\varepsilon \in (\mathbb{C}^k, 0)$ :

- $Z_\varepsilon$  is holomorphic on the domain and has at most  $k+1$  singular points in  $\mathcal{U}^0$  (counted with multiplicity in case of saddle-nodes) each one located within  $\mathcal{U}^0 \cap \{0 \leq |x| < 1/\rho^\infty\}$  for some  $\rho^\infty > 1/\rho_0$ ,
- $Z_\varepsilon$  is transverse to the lines  $\{x = c\}$  whenever  $P_\varepsilon(c) \neq 0$ ,
- $Z_\varepsilon$  leaves  $\{y = 0\}$  invariant.

The other chart is a domain  $\mathcal{U}^\infty := \{1/\rho^\infty < |x| \leq \infty\} \times (\mathbb{C}, 0)$  equipped with a foliation  $\mathcal{F}_\varepsilon^\infty$

- having a single, reduced singularity at  $(\infty, 0)$ ,
- otherwise transverse to the lines  $\{x = \text{cst}\}$ ,
- leaving  $\{y = 0\}$  invariant.

Biholomorphic fibered transitions maps fixing  $\{y = 0\}$  exists on the annulus  $\mathcal{U}^0 \cap \mathcal{U}^\infty$  precisely when  $Z_\varepsilon$  and  $\mathcal{F}_\varepsilon^\infty$  have (up to local conjugacy) mutually inverse holonomy maps above, say, the invariant circle  $\frac{\rho_0 \rho_\infty + 1}{2\rho_\infty} \mathbb{S}^1 \times \{0\}$ . The resulting complex surface  $\mathcal{M}_\varepsilon$  is naturally a holomorphic fibration by discs over the divisor  $\mathcal{L} \simeq \mathbb{P}_1(\mathbb{C})$ . In other words  $\mathcal{M}_\varepsilon$  is a germ of a Hirzebruch surface, classified at an analytic level [21, 47, 14] by the self-intersection  $-\widehat{\tau} \in \mathbb{Z}_{\leq 0}$  of  $\mathcal{L}$  in  $\mathcal{M}_\varepsilon$ . From the compactness of  $\mathcal{L}$  stems the polynomial-in- $x$  nature of the foliation  $\mathcal{F}_\varepsilon$ . Other considerations then allow to recognize that  $\mathcal{F}$  is (globally conjugate to a family of foliations) in normal form (2.3).

Let us explain where  $\mathcal{F}_\varepsilon^\infty$  comes from, and at the same time how the Hirzebruch class  $\widehat{\tau} = (k+1)\tau$  is involved. When the construction of  $(\mathcal{M}, \mathcal{F})$  is possible, the global holomorphic foliation  $\mathcal{F}_\varepsilon$  leaves the compact divisor  $\mathcal{L}$  invariant and Camacho-Sad index formula [4] applies. The sum of indices of  $Z_\varepsilon$  at its  $k+1$  singularities, with respect to  $\mathcal{L}$ , is  $\mu_\varepsilon$  so  $\mathcal{F}_\varepsilon^\infty$  must have index  $-(\mu_\varepsilon + \widehat{\tau})$ . By assumption the singularity at  $(\infty, 0)$  can therefore never be a (saddle-)node. Invoking the realization result of [42, Section 4.4] (more precisely in the chart near  $(\infty, 0)$ ) it is always possible to find a foliation  $\mathcal{F}_\varepsilon^\infty$  with the desired properties. On the contrary when  $\mu_\varepsilon + \widehat{\tau} \leq 0$  then no such  $\mathcal{F}_\varepsilon^\infty$  may exist at all except in very special cases (detailed in [25, Theorem 2]) since, for instance, the holonomy along  $\mathcal{L}$  of a node is always linearizable while the weak holonomy of  $Z_\varepsilon$  has no reason to be linearizable. We discuss this problem in more details while dealing with the non-linear Riemann-Hilbert problem below.

Therefore one can always take  $\tau := 0$  except when  $\mu_0 \leq 0$ , which accounts for the “twist”  $P_\varepsilon(x)^\tau y \sim_{x \rightarrow \infty} x^{\widehat{\tau}} y$  in normal forms (2.3).

2.1.3. *Normal forms uniqueness.* To fully describe the quotient space (moduli space) of  $\text{Convergent}_k$  by analytical conjugacy / orbital equivalence, the Normalization Theorem must be complemented with a description of equivalence classes within the family of normal forms (2.3), leading us to discuss its uniqueness clause.

**Definition 2.8.**

- (1) For  $Z \in \text{Convergent}_k$  we denote

$$(2.5) \quad \mathfrak{n}(Z) := (\mu, u, R, Q)$$

$$(2.6) \quad \mathfrak{o}(Z) := (\mu, R)$$

respectively the **normal invariant** of  $Z$  and its **normal orbital invariant**, where the functional tuples on the right-hand side are given by the Normalization Theorem.

- (2) For  $c \in \mathbb{C}\{\varepsilon\}^\times$  and  $f \in \mathbb{C}\{\varepsilon, x, y\}$  define

$$c^* f := (\varepsilon, x, y) \mapsto f_\varepsilon(x, c_\varepsilon y).$$

We extend component-wise this action of  $\mathbb{C}\{\varepsilon\}^\times$  to tuple of functions such as  $\mathfrak{n}$  and  $\mathfrak{o}$  above.

**Uniqueness Theorem.**

- (1) Two normal forms (2.3) associated to the same fixed  $\tau$  and moduli (2.5)  $\mathfrak{n}$  and  $\tilde{\mathfrak{n}}$  are analytically conjugate (by a change of coordinates fixing the parameter) if, and only if, there exists  $c \in \mathbb{C}\{\varepsilon\}^\times$  such that  $c^*\mathfrak{n} = \tilde{\mathfrak{n}}$ . For any conjugacy  $\Psi : (\varepsilon, x, y) \mapsto (\varepsilon, \Psi_\varepsilon(x, y))$  there exists a unique  $t \in \mathbb{C}\{\varepsilon\}$  such that

$$\Psi = c^*\Phi_Z^t,$$

where  $\Phi_Z^t$  is the local flow of  $Z$  at time  $t \in \mathbb{C}$ . Moreover it is fibered in the  $x$ -variable if, and only if,  $t = 0$ . In that case  $\Psi$  is linear:

$$\Psi = c^*\text{Id} : (\varepsilon, x, y) \mapsto (\varepsilon, x, c_\varepsilon y).$$

- (2) Let  $\mathfrak{v}$  and  $\tilde{\mathfrak{v}}$  be the corresponding orbital invariants. The normal forms are analytically orbitally equivalent (by a change of coordinates fixing the parameter) if, and only if, there exists  $c \in \mathbb{C}\{\varepsilon\}^\times$  such that  $c^*\mathfrak{v} = \tilde{\mathfrak{v}}$ . For any orbital equivalence  $\Psi$  there exists a unique  $F \in \mathbb{C}\{\varepsilon, x, y\}$  such that

$$\Psi = c^*\Phi_Z^F.$$

Moreover  $\Psi$  is fibered in the  $x$ -variable if, and only if,  $F = 0$ . In that case  $\Psi$  is linear.

*Remark 2.9.* In particular normal forms (2.3) are unique when only tangent-to-identity in the  $y$ -variable, fibered in the  $x$ -variable conjugacies are allowed.

Again the proof is largely based on the strategy of F. LORAY introduced in [25], although the actual implementation in the parametric case calls for subtle adaptations. The idea is to extend any local and fibered conjugacy between normal forms to a global conjugacy on a “big” neighborhood of  $\mathcal{L}$ , from which it easily follows that only linear maps can do that.

**2.2. Inverse problem.** For given  $k \in \mathbb{N}$  we can split the parameter space  $(\mathbb{C}^k, 0)$  into  $C_k = \frac{1}{k+1} \binom{2k}{k}$  open cells  $\mathcal{E}_\ell$  such that

$$\bigcup_{\ell} \mathcal{E}_\ell = (\mathbb{C}^k, 0) \setminus \Delta_k,$$

where  $\Delta_k$  is the set of parameters  $\varepsilon$  for which  $P_\varepsilon$  has at least a multiple root ( $\Delta_k$  is the discriminant curve). We recall that we can associate [40] an orbital modulus to a purely convergent unfolding  $Z$

$$\mathfrak{m}(Z) := (\mathfrak{m}_\ell)_{1 \leq \ell \leq C_k}$$

$$\mathfrak{m}_\ell := \left( \phi_\ell^{j,s} \right)_{j \in \mathbb{Z}/k\mathbb{Z}}$$

where for each  $j \in \mathbb{Z}/k\mathbb{Z}$  and each  $\ell$  the map

$$(\varepsilon, h) \in \mathcal{E}_\ell \times (\mathbb{C}, 0) \mapsto \phi_{\ell,\varepsilon}^{j,s}(h)$$

is holomorphic, vanishes along  $\{h = 0\}$  and admits a continuous extension to  $\text{cl}(\mathcal{E}_\ell) \times (\mathbb{C}, 0)$ .

*Remark 2.10.*

- (1) The upper index “s” is purely notational and refers to the fact that the function  $\phi_{\ell,\varepsilon}^{j,s}$  comes from the  $j^{\text{th}}$  “s”addle intersection, where the dynamics behaves very much like a saddle point.

- (2) The diffeomorphisms  $\psi_{\ell,\varepsilon}^{j,s}$ , which unfold the components  $\psi_0^{j,s}$  of the (classical) Martinet-Ramis modulus, are given by  $\psi_{\ell,\varepsilon}^{j,s}(h) = h \exp\left(\frac{2i\pi\mu_\varepsilon}{k} + \phi_{\ell,\varepsilon}^{j,s}(h)\right)$ .

Let us write  $\mathcal{H}_\ell\{h\}$  the vector space of all such functions, so that

$$\mathfrak{m}(Z) \in \prod_{\ell} \mathcal{H}_\ell\{h\}^k.$$

The data  $\mathfrak{m}(Z)$  is a complete orbital invariant for the local analytic classification of purely convergent unfoldings.

2.2.1. *Orbital realization.* The definition of the *compatibility condition* involves notions going beyond the scope of the present summarized statements. We refer to Section 7.3 for a precise definition. Instead let us use the following terminology.

**Definition.** We say that  $(\mu, \mathfrak{m}) \in \mathbb{C}\{\varepsilon\} \times \prod_{\ell} \mathcal{H}_\ell\{h\}^k$  is **realizable** if there exists a generic convergent unfolding  $Z$  with formal orbital class  $\mu$  and orbital modulus  $\mathfrak{m} = \mathfrak{m}(Z)$ .

For the sake of completeness, let us state the following fundamental result even though all material was not properly introduced.

**Orbital Realization Theorem.** *Let  $\mu \in \mathbb{C}\{\varepsilon\}$  be given. A functional data  $\mathfrak{m} \in \prod_{\ell} \mathcal{H}_\ell\{h\}^k$  yields a realizable  $(\mu, \mathfrak{m})$  if, and only if,  $(\mu, \mathfrak{m})$  satisfies the compatibility condition (presented in Definition 7.16).*

Although it is not directly used in the present paper, considerations akin to those from [42] show that the map sending a normal form to its orbital modulus

$$\mathfrak{v} = (\mu, R) \mapsto (\mu, \mathfrak{m})$$

is upper-triangular, in the sense that the  $n^{\text{th}}$ -jet of  $\mathfrak{m}_\ell$  with respect to  $h$  is completely determined by  $\mu$  and the  $n^{\text{th}}$ -jet of  $R$  with respect to  $y$ . In that sense passing from modulus to normal form is a (non-effective) computable process. In the case  $k = 1$  we show how to compute the diagonal entries. More details on this topic can be found in Section 10.1.

2.2.2. *Moduli which are analytic with respect to the parameter.* Our final main result proves that the compatibility condition defines a proper subset of the vector space  $\mathbb{C}\{\varepsilon\} \times \prod_{\ell} \mathcal{H}_\ell\{h\}^k$ .

**Parametrically Analytic Orbital Moduli Theorem.** *Let  $\mu \in \mathbb{C}\{\varepsilon\}$  and  $\mathfrak{m} = (\mathfrak{m}_\ell)_\ell \in \prod_{\ell} \mathcal{H}_\ell\{h\}^k$  be given. Assume  $\mathfrak{m}$  is holomorphic, in the sense that  $\mathfrak{m}_\ell = M|_{\varepsilon_\ell \times (\mathbb{C}, 0)}$  for some  $M \in h\mathbb{C}\{\varepsilon, h\}^k$ . The following conditions are equivalent:*

- (1)  $(\mu, \mathfrak{m})$  satisfies the compatibility condition,
- (2) either  $\mathfrak{m} = 0$ , or  $k = 1$  and there exists  $d \in \mathbb{N}$ ,  $\alpha \in \mathbb{C}\{\varepsilon\} \setminus \{0\}$  such that
  - $d\mu \in \mathbb{Z}$  (in particular  $\mu$  is a rational constant),
  - $M(h) = -\frac{1}{d} \log(1 - \alpha h^d)$ .

If one of the conditions is satisfied and  $\mathfrak{m} \neq 0$ , then an orbital normal form realizing  $(\mathfrak{m}_\ell)_\ell$  is:

$$\mathcal{X}_\varepsilon = \widehat{X}_\varepsilon + r_\varepsilon x P_\varepsilon(x)^{\tau d} y^{d+1} \frac{\partial}{\partial y}$$

for some  $r \in \mathbb{C}\{\varepsilon\} \setminus \{0\}$ . We then speak of **Bernoulli unfoldings** because the underlying non-autonomous differential equation induced by the flow is Bernoulli.

*Remark 2.11.* By letting  $\mathbb{C}\{\varepsilon\}^\times$  act linearly through  $(\varepsilon, x, y) \mapsto (\varepsilon, x, c_\varepsilon y)$  we may normalize further  $r$  to some  $\varepsilon^\kappa$  for  $\kappa \in \mathbb{Z}_{\geq 0}$ . See also Section 10.1.

**2.3. Application: non-linear differential Galois theory.** M. BERTHIER and F. TOUZET [1] proved that the Martinet-Ramis modulus of a convergent saddle-node vector field admitting non-trivial Liouvillian first integrals [48] must be a ramified homography  $h \mapsto \alpha h(1 + \beta h^d)^{-1/d}$ , from which they deduce that the vector field is Bernoulli. (It is indeed straightforward to compute the modulus of a Bernoulli vector field by solving explicitly the underlying differential equation.) Roughly speaking this situation corresponds to differential equations admitting “closed-form” solutions obtained by iteratively taking quadrature (or exponential thereof) of elements of (algebraic extensions of) the base-field (here, meromorphic functions on a poly-disk containing  $P_\varepsilon^{-1}(0) \cap \{y = 0\}$ ).

**Integrability Theorem.** *Let  $(Z_\varepsilon)_\varepsilon$  be a generic, purely convergent saddle-node unfolding and denote by  $\mathcal{L}$  the germ of set consisting in those  $\varepsilon \in (\mathbb{C}^k, 0)$  for which  $Z_\varepsilon$  admits a Liouvillian first integral. The following statements are equivalent.*

- (1) *The locus of integrability  $\mathcal{L}$  is full:  $\mathcal{L} = (\mathbb{C}^k, 0)$ .*
- (2) *Its (analytic) Zariski closure is full:  $\overline{\mathcal{L}}^{\text{Zar}} = (\mathbb{C}^k, 0)$ .*
- (3) *Its orbital normal form  $\mathcal{X}$  is a Bernoulli unfolding.*

*Remark 2.12.*

- (1) The case  $\overline{\mathcal{L}}^{\text{Zar}} \neq (\mathbb{C}^k, 0)$  corresponds to  $\mathcal{L}$  being a germ at 0 of a proper analytic subvariety. Then  $\mathcal{L}$  is the locus of parameters for which the normal form is Bernoulli. For instance in case of the normal form given by  $R_\varepsilon(t) := t^d + L(\varepsilon)t^{d+1}$  we have  $\mathcal{L} = L^{-1}(0)$ , as we discuss after the proof of the theorem.
- (2) In the case  $k = 1$  the second condition is equivalent to any of the following three conditions: the germ  $\mathcal{L}$  accumulates on 0,  $\mathcal{L}$  is infinite,  $\mathcal{L} \neq \{0\}$ .

*Proof.* The property of having a non-trivial Liouvillian first integral is both orbital and invariant by change of analytic coordinates, so we do not lose generality by taking  $Z = \mathcal{X}$  in normal form (2.4). Integrability is equivalent to the existence of a Godbillon-Vey sequence [16] of length at most 2, that is to the existence of two non-zero meromorphic 1-forms  $\omega$  and  $\eta$  for which

$$\begin{aligned} d\eta &= 0 \\ d\omega &= \delta\omega \wedge \eta \quad , \quad \delta \in \{0, 1\} \\ \omega(\mathcal{X}) &= 0. \end{aligned}$$

(The multivalued map  $H := \int \exp(\delta \int \eta) \omega$  is indeed a Liouvillian first integral of  $\mathcal{X}$ , obtained by quadrature of closed 1-forms.) This in turn is (almost) equivalent to solving for meromorphic, transverse  $Y \neq 0$  in the Lie-bracket equation

$$(2.7) \quad [\mathcal{X}, Y] = \delta Y \quad , \quad \delta \in \{0, 1\},$$

since the dual basis  $(\eta, \omega)$  of  $(\mathcal{X}, Y)$  is a Godbillon-Vey sequence and *vice versa*. There is a subtlety here, because  $\mathcal{X}$  may fail to meet this condition while there



could exist an integrating factor  $V$  for which  $V\mathcal{X}$  does. We deliberately ignore this eventuality, because the case  $V \neq 1$  can be deduced from the particular case  $V = 1$  by a direct (albeit cumbersome) adaptation. For the same reason we only deal with the case  $\mu_0 \notin \mathbb{R}_{\leq 0}$ .

The implications (3) $\Rightarrow$ (1) $\Rightarrow$ (2) are clear enough (we particularly refer to Lemma 9.3 for the first one). Let us prove (2) $\Rightarrow$ (3). The strategy is the following: we first show that the vector field is Bernoulli for each  $\varepsilon \in \mathcal{L}$ , then we invoke the analyticity of the normal form and the fact that  $\overline{\mathcal{L}}^{\text{zar}}$  is full to cover a whole neighborhood of 0 in parameter space. Hence, let us fix  $\varepsilon \in \mathcal{L}$  and drop the index  $\varepsilon$  altogether. According to the above discussion one can find  $\delta \in \{0, 1\}$  and a vector field

$$Y = A(x, y) \frac{\partial}{\partial y} + B(x, y) \mathcal{X}$$

solving (2.7) for two functions  $A \neq 0$  and  $B$  meromorphic on a polydisk containing  $P^{-1}(0) \cap \{y = 0\}$ . From (2.7) we deduce the relations

$$(2.8) \quad \begin{cases} \mathcal{X} \cdot B &= \delta B, \\ \mathcal{X} \cdot A &= \left( \delta + 1 + \mu x^k + \frac{\partial y R}{\partial y} \right) A. \end{cases}$$

The second equation tells us that  $\{A = 0\} \cup \{A = \infty\}$  is a union of separatrices of  $\mathcal{X}$ , therefore of the form

$$A(x, y) = y^{d+1} U(x, y) \prod_{P(z)=0} (x-z)^{\ell(z)},$$

for some choice of  $d, \ell(z) \in \mathbb{Z}$  and for some holomorphic and never vanishing function  $U$ . Let us prove that  $R = r(x)y^d$ , from which follows either  $d \in \mathbb{N}$  or  $R = 0$ .

The last equation of (2.8) becomes

$$\mathcal{X} \cdot \log U = \delta - d(1 + \mu x^k) - \sum_{P(z)=0} \ell(z) \frac{P(x)}{x-z} + \left( y \frac{\partial R}{\partial y} - dR \right),$$

because  $\mathcal{X} \cdot \log y = 1 + \mu x^k + R$ . Evaluating this identity at any one of the  $k+1$  points  $(x, y) = (x, 0)$  such that  $P(x) = 0$  yields  $0 = \delta - d(1 + \mu x^k) - \ell(x)P'(x)$ , since on the one hand  $R$  and  $y \frac{\partial R}{\partial y}$  vanish when  $y = 0$  while on the other hand  $\ell(z) \frac{P(x)}{x-z}$  evaluates to 0 if  $z \neq x$  and to  $\ell(x)P'(x)$  otherwise. As a consequence we have equality of the polynomials  $\sum_{P(z)=0} \ell(z) \frac{P(x)}{x-z} = \delta - d(1 + \mu x^k)$  of degree  $k$ . Therefore

$$\mathcal{X} \cdot \log U = y \frac{\partial R}{\partial y} - dR.$$

In the course of Section 6 we show that  $\text{im}(\mathcal{X} \cdot) \cap \text{Section}_k\{y\} = \{0\}$  (see Remark 6.3). Hence, the fact that  $y \frac{\partial R}{\partial y} - dR \in xy\mathbb{C}[x]_{<k}\{y\} = \text{Section}_k\{y\}$  lies in the image of  $\mathcal{X} \cdot$  can only mean  $y \frac{\partial R}{\partial y} - dR = 0$ . From this we deduce at once that

$$R(x, y) = xr(x)y^d, \quad r \in \mathbb{C}[x]_{<k}.$$

The condition that, for a specific  $\varepsilon$ , the vector field  $\mathcal{X}_\varepsilon$  be Bernoulli corresponds to the vanishing of all coefficients in  $R_\varepsilon$  of  $y^n$  but for  $n = d$ . Since  $(\varepsilon, y) \mapsto R_\varepsilon(y)$  is analytic with respect to  $\varepsilon$  and  $\overline{\mathcal{L}}^{\text{zar}} = (\mathbb{C}^k, 0)$ , if  $d$  is independent on  $\varepsilon$  then the

identity principle implies  $R_\varepsilon(x, y) = xr_\varepsilon(x)y^d$  for all  $(\varepsilon, x, y) \in (\mathbb{C}^{k+2}, 0)$ . The fact that  $d$  is indeed independent on  $\varepsilon$  stems from Baire's category theorem.  $\square$

*Remark 2.13.* The proof relies in an essential way on the analyticity of the orbital normal form  $\mathcal{X}$  with respect to  $\varepsilon$  near 0. Compare with the method of proof of [1]: for  $\varepsilon = 0$  the argument is based on the fact that the existence of a Godbillon-Vey sequence forces the Martinet-Ramis modulus to be a ramified homography. This argument works as well for  $\varepsilon \neq 0$ , but we could not have argued on from there since the modulus is in general not analytic at  $\varepsilon = 0$ : although being a ramified homography is an analytic condition, an accumulation of zeros of this relation as  $\varepsilon \rightarrow 0$  could arise without holding for all  $\varepsilon$  (for  $k = 1$ , say). This situation cannot occur, and our shorter argument does not involve the unfolded modulus of classification.

The Galoisian characterization of the existence of Godbillon-Vey sequences of length at most 2 is performed in [6], and for fixed  $\varepsilon$  its length equals the (transverse) rank  $\text{rk}(\mathfrak{M}_\varepsilon)$  of the Galois-Malgrange groupoid  $\mathfrak{M}_\varepsilon$ . This rank takes values in  $\{0, 1, 2, \infty\}$ , integrability corresponding to finite values. For the normal forms (2.4) with  $R_\varepsilon(x, t) = \sum_{n>0} r_{\varepsilon,n}(x)t^n$ , we have

$$\text{rk}(\mathfrak{M}_\varepsilon) = \begin{cases} 1 + \#\{n : r_{\varepsilon,n} \neq 0\} & \text{if it is } \leq 2, \\ \infty & \text{otherwise.} \end{cases}$$

Therefore  $\varepsilon \mapsto \text{rk}(\mathfrak{M}_\varepsilon)$  is lower semi-continuous: accidental values of the rank can only correspond to more integrable systems.

**Example.** Taking into account Remark 2.11, in the case  $k = 1$  and  $R \neq 0$  the vector field  $\mathcal{X}_0$  is “more integrable” (transverse rank 1) than the generic  $\mathcal{X}_\varepsilon$  (transverse rank 2) if and only if the exponent  $\kappa$  is positive.

This is a special instance of a general result on parametrized Galois-Malgrange groupoids obtained very recently by G. CASALE and D. DAVY [7]. They show that for rather general deformations of foliations  $(\mathcal{F}_\varepsilon)_\varepsilon$ , the rank  $\text{rk}(\mathfrak{M}_\varepsilon)$  of the specialization  $\mathfrak{M}_\varepsilon$  of the Galois-Malgrange groupoid of the family is lower semi-continuous in  $\varepsilon$ . Moreover, the locus of discontinuity is contained in a countable union of proper analytic subvarieties. We showed that in the case of purely convergent saddle-node bifurcations, the locus of discontinuity is at most a proper analytic subvariety.

#### 2.4. Structure of the paper.

- We begin with fixing notations and providing precise definitions in Section 3. Readers familiar with complex foliations may skip this section.
- The Formal Normalization Theorem is proved in Section 4.

We first present the generic case (for which one can take  $\tau = 0$ ), since it is easier to highlight the ideas than in the case  $\tau > 0$ .

- The orbital part of the Normalization and Uniqueness Theorems are established in Section 5 when  $\tau = 0$ .
- The temporal part of the Normalization and Uniqueness Theorems are established in Section 6 when  $\tau = 0$ .
- In Section 7 one finds the definition of compatibility condition, and the proof of the Orbital Realization Theorem in the generic case  $\tau = 0$ .

- In Section 8 we prove the Orbital Realization Theorem in the case  $\tau > 0$ . This provides *a posteriori* a proof of the orbital part of the Normalization and Uniqueness Theorems when  $\tau > 0$ .
- In Section 9 we discuss the Bernoulli unfoldings and prove the Parametrically Analytic Orbital Moduli Theorem.
- Finally, in Section 10, we conclude with a few words on computations.

### 3. PRELIMINARIES

#### 3.1. Notations.

##### 3.1.1. General notations.

- We let the set  $\mathbb{N} := \{1, 2, \dots\}$  stand for all positive integers, whereas the set of non-negative integers will be written  $\mathbb{Z}_{\geq 0} = \{0, 1, \dots\}$ .
- For  $n \in \mathbb{N}$  we let  $(\mathbb{C}^n, 0)$  stand for any small enough domain in  $\mathbb{C}^n$  containing 0.
- The domain  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$  is the standard open unit disk.
- The closure of a subset  $A$  of a topological space is written  $\text{cl}(A)$ .
- $k \in \mathbb{N}$  is fixed,  $\varepsilon = (\varepsilon_0, \dots, \varepsilon_{k-1}) \in (\mathbb{C}^k, 0)$  is the parameter and

$$(3.1) \quad P_\varepsilon(x) = x^{k+1} + \sum_{j=0}^{k-1} \varepsilon_j x^j.$$

- The parameter space  $(\mathbb{C}^k, 0)$  is covered by the closure of  $C_k = \frac{1}{k+1} \binom{2k}{k}$  open and contractible cells  $\mathcal{E}_\ell$ .
- The period operator  $\Upsilon = (\Upsilon^j)_{j \in \mathbb{Z}/k\mathbb{Z}}$  is built near Definition 6.10.
- The very nature of constructions involves using more sub- and superscripts than one is generally comfortable with. To alleviate this downside we stick to a single convention: **subscripts are always parameter-related**, while superscripts are in general related to the geometric variables  $(x, y)$  or to indices in power series expansions. Example: we write  $V_{\ell, \varepsilon}^{j, s}$  for the “s”addle part of the  $j^{\text{th}}$  sector in  $x$ -variable, relatively to the parameter  $\varepsilon$  being taken in the  $\ell^{\text{th}}$  parametric cell. In the course of the text we try to drop indices whenever possible.
- **The dependency on the parameter  $\varepsilon$  is implicit in most instances.** For example,  $\mu \in \mathbb{C}\{\varepsilon\}$  stands for the formal orbital modulus while  $\mu_\varepsilon$  stands for the value of  $\mu$  at the particular value of the parameter  $\varepsilon$ . Yet in many places where  $\varepsilon$  is fixed we do use  $\mu$  instead of  $\mu_\varepsilon$  in order to help reducing the notational footprint. This also applies for other parametric objects.

3.1.2. *Functional spaces.* In the following  $\mathcal{R}$  is a commutative ring with a multiplicative action by complex numbers.

- $\mathcal{R}^\times$  is the multiplicative group of its invertible elements.
- $\mathcal{R}[\mathbf{z}]$  is the commutative ring of polynomials in the complex finite-dimensional (multi)variable  $\mathbf{z} = (z_1, \dots, z_n)$  with coefficients in  $\mathcal{R}$ .
- After choosing a binary relation  $<$  among  $\{=, <, \leq, \dots\}$  we let  $\mathcal{R}[\mathbf{z}]_{<d}$  be the subset of  $\mathcal{R}[\mathbf{z}]$  consisting of polynomials  $P$  such that  $\deg P < d$ .
- The projective limit  $\mathcal{R}[[\mathbf{z}]] := \lim_{d \rightarrow \infty} \mathcal{R}[\mathbf{z}]_{\leq d}$  is the ring of formal power series in  $\mathbf{z}$  with coefficients in  $\mathcal{R}$ .

- $\mathbb{C}\{\mathbf{z}\}$  is the algebra of convergent formal power series in the complex multivariable  $\mathbf{z} \in \mathbb{C}^n$ , naturally identified to the set of germs of a holomorphic function near  $0 \in \mathbb{C}^n$ .

*Remark 3.1.* We will mostly use the spaces:

- $\mathbb{C}[[\varepsilon]]$ ,  $\mathbb{C}[[\varepsilon, x]]$  and  $\mathbb{C}[[\varepsilon, x, y]]$
- $\mathbb{C}\{\varepsilon\}$ ,  $\mathbb{C}\{\varepsilon\}^\times$ ,  $\mathbb{C}\{\varepsilon, x\}$  and  $\mathbb{C}\{\varepsilon, x, y\}$
- $\mathbb{C}\{\varepsilon\}[x]$ ,  $\mathbb{C}\{\varepsilon\}[x]_{\leq k}^\times$  and

$$\text{Section}_k\{v\} := xv\mathbb{C}\{\varepsilon, v\}[x]_{<k}.$$

Let  $\mathcal{D} \subset \mathbb{C}^n$  be a domain containing 0 equipped with the affine coordinates  $\mathbf{z} = (z_1, \dots, z_n)$ .

- $\text{Holo}(\mathcal{D})$  is the algebra of complex-valued functions holomorphic on  $\mathcal{D}$ .
- $\text{Holo}_c(\mathcal{D})$  is the Banach subalgebra of  $\text{Holo}(\mathcal{D})$  of all holomorphic functions  $f : \mathcal{D} \rightarrow \mathbb{C}$ , with bounded continuous extension to  $\text{cl}(\mathcal{D})$ , equipped with the norm

$$\|f\|_{\mathcal{D}} := \sup_{\mathbf{z} \in \mathcal{D}} |f(\mathbf{z})|.$$

- $\text{Holo}_c(\mathcal{D})'$  is the Banach space of all holomorphic functions  $f : \mathcal{D} \rightarrow \mathbb{C}$  vanishing on  $\{z_n = 0\}$  with the norm

$$\|f\|'_{\mathcal{D}} := \sup_{\mathbf{z} \in \mathcal{D}} \left| \frac{f(\mathbf{z})}{z_n} \right|.$$

Notice that  $\|f\|'_{\mathcal{D}} \leq \left\| \frac{\partial f}{\partial z_n} \right\|_{\mathcal{D}}$  whenever  $\frac{\partial f}{\partial z_n} \in \text{Holo}_c(\mathcal{D})$  and  $\mathcal{D}$  is convex in the variable  $z_n$ .

- We let

$$\mathcal{H}_\ell\{\mathbf{z}\} := \bigcup_{\mathcal{D}=(\mathbb{C}^n, 0)} \text{Holo}_c(\mathcal{E}_\ell \times \mathcal{D})',$$

where  $\mathcal{E}_\ell$  is a parametric cell.

3.1.3. *Vector fields and Lie derivative.* Let  $Z = \sum_{j=1}^n A_j \frac{\partial}{\partial z_j}$  be a germ of a holomorphic vector field at the origin of  $\mathbb{C}^n$  (or formal vector field at this point).

- If  $f$  is a formal power series or a holomorphic function in  $\mathbf{z} = (z_1, \dots, z_n) \in (\mathbb{C}^n, 0)$ , we denote by  $Z \cdot f$  the directional Lie derivative of  $f$  along  $Z$

$$Z \cdot f := \sum_{j=1}^n A_j \frac{\partial f}{\partial z_j} = Df(Z).$$

The operator is extended component-wise on vectors of power series or functions.

- We define recursively for  $n \in \mathbb{Z}_{\geq 0}$  the  $n^{\text{th}}$  iterate of the Lie derivative, the operator written  $Z \cdot^n$ , by

$$\begin{aligned} Z \cdot^0 &:= \text{Id} \\ Z \cdot^{n+1} &:= Z \cdot (Z \cdot^n). \end{aligned}$$

- The flow of  $Z$  at time  $t$  starting from  $\mathbf{z}$  is the formal  $n$ -tuple of power series  $\Phi_Z^t(\mathbf{z})$  solving the flow-system

$$\frac{\partial \Phi_Z^t(\mathbf{z})}{\partial t} = Z \circ \Phi_Z^t(\mathbf{z}),$$

which is a convergent power series in  $(t, \mathbf{z})$  if, and only if,  $Z$  is holomorphic. At some point we invoke the classical formal identity of Lie

$$(3.2) \quad f \circ \Phi_Z^t = \sum_{n \geq 0} \frac{t^n}{n!} Z \cdot^n f.$$

- Two vector fields  $Z$  and  $\tilde{Z}$  are formally / locally conjugate when there exists a  $n$ -tuple of formal / convergent power series  $\Psi$  with invertible derivative at 0 such that

$$\tilde{Z} \cdot \Psi = Z \circ \Psi.$$

In that case we write  $\tilde{Z} = \Psi^* Z$ .

- Two vector fields  $Z$  and  $\tilde{Z}$  are formally / locally orbitally equivalent when there exists a formal power series / holomorphic function  $U$  with  $U(0) \neq 0$  such that  $UZ$  and  $\tilde{Z}$  are conjugate (in the same convergence class).

### 3.2. Conjugacy and orbital equivalence.

**Definition 3.2.** Two unfoldings  $Z = (Z_\varepsilon)_\varepsilon$  and  $\tilde{Z} = (\tilde{Z}_{\tilde{\varepsilon}})_{\tilde{\varepsilon}}$  are locally **conjugate** (*resp.* **orbitally equivalent**) if there exists a holomorphic mapping

$$\Psi : (\varepsilon, x, y) \mapsto (\phi(\varepsilon), \Psi_\varepsilon(x, y))$$

such that:

- (1)  $\varepsilon \in (\mathbb{C}^k, 0) \mapsto \tilde{\varepsilon} = \phi(\varepsilon)$  has invertible derivative at 0,
- (2) for each  $\varepsilon \in (\mathbb{C}^k, 0)$  the component  $\Psi_\varepsilon$  is a local conjugacy (*resp.* orbital equivalence) between  $Z_\varepsilon$  and  $\tilde{Z}_{\phi(\varepsilon)}$ .

If the above conditions are fulfilled we write

$$\Psi^* Z = \tilde{Z}.$$

We extend in the obvious way the definition for formal conjugacy / orbital equivalence.

*Remark 3.3.* The very first step of any construction performed here consists in recalling the preparation of the generic unfolding  $Z$  (Theorem 4.1). For unfoldings in prepared form (4.1) the parameter  $\varepsilon$  becomes a formal invariant. Hence we only use conjugacies fixing  $\varepsilon$ , that is  $\Psi : (\varepsilon, x, y) \mapsto (\varepsilon, \Psi_\varepsilon(x, y))$ . In that setting one can always deduce  $\Psi$  knowing  $\Psi_\varepsilon$ , therefore when we use the notation  $\Psi$  we generally refer to the map  $(\varepsilon, x, y) \mapsto \Psi_\varepsilon(x, y)$ , except when the context is ambiguous.

**Definition 3.4.** Consider a formal transform  $\Psi : (\varepsilon, x, y) \mapsto (\varepsilon, \Psi_\varepsilon(x, y))$ . We say that  $\Psi$  is **fibred** when  $\Psi_\varepsilon(x, y) = (x, \psi_\varepsilon(x, y))$ .

**Definition 3.5.**  $\Psi$  is a **symmetry** (*resp.* **orbital symmetry**) of  $Z$  when  $\Psi$  is a self-conjugacy (*resp.* orbital self-equivalence) of  $Z$ .

*Remark 3.6.* Hence, to determine the orbital symmetries of  $Z$  it suffices to determine the changes  $\Psi$  such that  $\Psi^*Z = UZ$  for some  $U$  with  $U_0(0,0) \neq 0$ .

#### 4. FORMAL NORMALIZATION

The formal normalization is based on three ingredients, each one corresponding to a step of the construction:

- a preparation *à la* Dulac of unfoldings: for  $\varepsilon = 0$  one recovers Dulac prepared form [12, 13];
- the existence of a formal “family of weak separatrices” which we can straighten to  $\{y = 0\}$ ;
- a variation on Lie’s identity (3.2) already used in [40, 44] to perform the analytic classification of saddle-nodes vector fields and their unfoldings. The formula reduces the problem of finding changes of variables to solving an uncoupled system of cohomological equations.

**4.1. Preparation.** Take  $\theta \in \mathbb{Z}/k\mathbb{Z}$  and set  $\alpha := \exp 2i\pi\theta/k$ . For  $\varepsilon := (\varepsilon_j)_{j < k} \in (\mathbb{C}^k, 0)$  we define

$$\theta^* \varepsilon := (\varepsilon_j \alpha^{j-1})_{j < k}.$$

**Theorem 4.1.** [40, Proposition 3.1 and Theorem 3.5] *Any generic unfolding is analytically conjugate to an unfolding of the form*

$$(4.1) \quad Z = UX$$

$$(4.2) \quad X = \widehat{X} + A \frac{\partial}{\partial y}$$

$$A(x, y) = P(x)a(x) + yR(x, y)$$

where  $\widehat{X}$  and  $P$  are defined in (2.2) and (3.1), while  $a \in \mathbb{C}\{\varepsilon, x\}$ ,  $R \in y\mathbb{C}\{\varepsilon, x, y\}$  and  $U \in \mathbb{C}\{\varepsilon, x, y\}$  with  $U_0(0,0) \neq 0$ . In the particular case of an analytic weak separatrix one can take  $a := 0$ .

Besides if two such prepared forms  $(Z_\varepsilon)_\varepsilon$  and  $(\widetilde{Z}_{\widetilde{\varepsilon}})_{\widetilde{\varepsilon}}$  are formally orbitally equivalent then there exists  $\theta \in \mathbb{Z}/k\mathbb{Z}$  such that  $\widetilde{\varepsilon} = \theta^* \varepsilon$ : the parameter is unique modulo this action and is called **canonical**.

*Remark 4.2.* Although the original result is stated in [40] at an analytic level, the proof that  $\varepsilon$  becomes an invariant *modulo* the action of  $\mathbb{Z}/k\mathbb{Z}$  stems from a formal computation and is therefore valid for formal orbital equivalence too. The idea of the proof is that the parameter completely determines the data of local eigenratios and *vice versa*, which are well-known orbital invariants.

From now on we only deal with unfoldings in prepared form 4.1 and only consider transforms fixing the canonical parameter  $\varepsilon$ .

#### 4.2. Straightening weak separatrices.

**Proposition 4.3.** [20, Proposition 2] *For any unfolding  $X$  in prepared form (4.2) there exists a formal power series*

$$\widehat{\varepsilon} \in \mathbb{C}[[\varepsilon, x]]^P$$

solving the parametric family of differential equations

$$(4.3) \quad P_\varepsilon(x) \frac{d\widehat{s}_\varepsilon}{dx}(x) = \widehat{s}_\varepsilon(1 + \mu_\varepsilon x^k) + A_\varepsilon(x, \widehat{s}_\varepsilon(x)).$$

Performing the transform  $(\varepsilon, x, y) \mapsto (\varepsilon, x, y + \widehat{s}_\varepsilon(x))$  sends  $Z$  to a prepared unfolding

$$(4.4) \quad Z = \widehat{U} \left( \widehat{X} + \widehat{A}y \frac{\partial}{\partial y} \right)$$

for some formal power series  $\widehat{U}$  and  $\widehat{A}$  in  $\mathbb{C}[[\varepsilon, x, y]]$  with

$$\begin{aligned} \widehat{U}(\varepsilon, x, 0) &=: \widehat{U}_\varepsilon(x, 0) = u_\varepsilon(x) + \mathcal{O}(P_\varepsilon(x)) \\ \widehat{A}(\varepsilon, x, 0) &=: \widehat{A}_\varepsilon(x, 0) = \mathcal{O}(P_\varepsilon(x)), \end{aligned}$$

with  $u \in \mathbb{C}[[\varepsilon]][x]$  a polynomial (in  $x$ ) of degree at most  $k$  such that  $u_0(0) \neq 0$ . In the particular case when  $Z$  is purely convergent the latter power series (and the coefficients of  $u$ ) are convergent.

The proof of [20] is done for  $k = 1$  but the general case is similar. It is based first on the following classical lemma, the proof of which is included for the sake of completeness.

**Lemma 4.4.** *Let  $\widetilde{g} \in \mathbb{C}[[x, y]]$  and  $\bar{g} \in x^p \mathbb{C}[[x]]$  be given with  $p \in \mathbb{Z}_{\geq 0}$ , such that either  $p > 0$  or  $\widetilde{g}(x, y) = \mathcal{O}(x)$ . Let  $h \in \mathbb{C}[[x]]$  be such that  $h(0) \neq 0$  and define  $g(x, y) := \bar{g}(x) + y^2 \widetilde{g}(x, y)$ .*

*The differential equation*

$$(4.5) \quad x^{k+1} f'(x) + h(x) f(x) + g(x, f(x)) = 0$$

*has a unique formal solution  $f$ , which moreover belongs to  $x^p \mathbb{C}[[x]]$ .*

*Remark 4.5.* Note that equation (4.5) is nothing else than the differential equation determining the center manifold of the saddle-node vector field

$$x^{k+1} \frac{\partial}{\partial x} - (yh(x) + g(x, y)) \frac{\partial}{\partial y}$$

when  $g$  and  $h$  are holomorphic germs.

*Proof.* Letting  $C := h(0) \neq 0$  and  $g(x, 0) = \bar{g}(x) =: \sum_{m=p}^{\infty} b^m x^m$ ; then substituting  $f(x) =: \sum_{m=p}^{\infty} a^m x^m$  into (4.5) and grouping terms of same degree  $m \geq p$ , we get

$$Ca^m + b^m + F^m(a^p, \dots, a^{m-1}) = 0$$

for some polynomial  $F^m$  depending on the  $m$ -jet of  $g$  and  $h$ . Hence, we can solve uniquely for each  $a^m$ .  $\square$

We then derive Proposition 4.3 from the following technical lemma which we will also use later on.

**Lemma 4.6.** (See [20] for the case  $k = 1$ ) *Let  $\widetilde{g}_\varepsilon \in \mathbb{C}[[\varepsilon, x, y]]$  and  $\bar{g}_\varepsilon \in \mathbb{C}[[\varepsilon, x]]^P$  be given, i.e.  $\bar{g}_\varepsilon(x) = P_\varepsilon(x) \sum_{|m| \geq 0} \bar{g}^m(x) \varepsilon^m$  where each  $\bar{g}^m(x)$  is itself a formal power series in  $x$ , and let  $h_\varepsilon \in \mathbb{C}[[\varepsilon, x]]$  be such that  $h_0(0) \neq 0$ . Define  $g_\varepsilon(x, y) := \bar{g}_\varepsilon(x) + y^2 \widetilde{g}_\varepsilon(x, y)$ .*

*The family of differential equations*

$$(4.6) \quad P_\varepsilon(x) f'_\varepsilon(x) + h_\varepsilon(x) f_\varepsilon(x) + g_\varepsilon(x, f_\varepsilon(x)) = 0$$

*has a unique formal solution  $f$ , which moreover belongs to  $\mathbb{C}[[\varepsilon, x]]^P$ .*

*Proof.* Let  $g_\varepsilon(x, y) =: P_\varepsilon(x) \sum_{|\mathbf{m}| \geq 0} b^{\mathbf{m},0}(x) \varepsilon^{|\mathbf{m}|} + \sum_{|\mathbf{m}| \geq 0} \left( \sum_{n \geq 2} b^{\mathbf{m},n}(x) y^n \right) \varepsilon^{|\mathbf{m}|}$ . Substituting  $f_\varepsilon(x) = P_\varepsilon(x) \sum_{|\mathbf{m}| \geq 0} a^{\mathbf{m}}(x) \varepsilon^{|\mathbf{m}|}$  into (4.6) and setting  $\varepsilon := 0$  we first get

$$x^{k+1} \frac{da^0}{dx}(x) + \left( h^0(x) + kx^k \right) a^0(x) + x^{-(k+1)} g^0(x, x^{k+1} a^0(x)) = 0,$$

which admits a formal solution in  $x^{k+1} \mathbb{C}[[x]]$  by direct application of Lemma 4.4 in the case  $p = 0$ . Likewise, by grouping terms with same  $\varepsilon^{|\mathbf{m}|}$  for  $|\mathbf{m}| \geq 1$  we obtain

$$(4.7) \quad x^{k+1} \frac{da^{\mathbf{m}}}{dx}(x) + \left( h^0(x) + kx^k + \ell^{\mathbf{m}}(x) \right) a^{\mathbf{m}}(x) + \left( b^{\mathbf{m},0}(x) + F^{\mathbf{m}}(x) \right) = 0,$$

where

$$\ell^{\mathbf{m}}(x) = \sum_{n \geq 2} nx^{(n-1)(k+1)} a^0(x)^{n-1} b^{0,n}(x) = \mathcal{O}(x^{k+1}),$$

and  $F^{\mathbf{m}} \in \mathbb{C}[[x]]$  is some formal power series depending polynomially on  $(a^n(x))_{|n| < |\mathbf{m}|}$  and on the  $|\mathbf{m}|$ -jet of  $g$  and  $h$ . By induction on  $|\mathbf{m}|$ , we recursively find formal solutions  $a^{\mathbf{m}} \in \mathbb{C}[[x]]$ , for equation (4.7) has the same type as (4.5) with  $\tilde{g} := 0$ , and hence, has a formal solution given by Lemma 4.4. Uniqueness is straightforward.  $\square$

**4.3. Normalization and cohomological equations.** The tool for proving the Formal Normalization Theorem is the following.

**Proposition 4.7.** [46, 45] *Let  $W$  and  $Y$  be commuting, formal (resp. holomorphic) planar vector fields. Let  $F \in \mathbb{C}[[x, y]]$  (resp. a germ of a holomorphic function) be given with  $F(0, 0) = 0$ . Then  $\Psi := \Phi_Y^F$  is a formal (resp. analytic) change of variables near  $(0, 0)$  and*

$$\Psi^* W = W - \frac{W \cdot F}{1 + Y \cdot F} Y.$$

This tool is used in the following manner.

- First if we could find formal solution  $T$  of the (parametric families of) cohomological equations

$$(4.8) \quad X \cdot T = \frac{1}{U} - \frac{1}{u}$$

for a convenient choice of  $u \in \mathbb{C}\{\varepsilon, x\}^\times$ , then  $uX$  would be formally conjugate to  $Z$  by the tangential change of variables  $T$  given by

$$(4.9) \quad \mathcal{T} := \Phi_{uX}^T.$$

This is the content of the proposition for  $Y := W := uX$  and  $F := T$ .

- From Proposition 4.3 we built the formal, fibered transform  $\mathcal{S}$  given by

$$\mathcal{S} : (x, y) \mapsto (x, y - \widehat{s}(x))$$

such that  $\mathcal{S}^* \left( \widehat{X} + \widehat{A}y \frac{\partial}{\partial y} \right) = X$ .

- Finally, since  $y \frac{\partial}{\partial y}$  commutes with the normal form  $\widehat{X}$ , if we could solve formally the cohomological equation

$$(4.10) \quad \left( \widehat{X} + \widehat{A}y \frac{\partial}{\partial y} \right) \cdot O = -\widehat{A}$$



then  $\widehat{X}$  would be formally conjugate to  $\widehat{X} + \widehat{A}y \frac{\partial}{\partial y}$  by the fibered, transverse change of variables  $\mathcal{O}$  given by

$$(4.11) \quad \mathcal{O} := \Phi_{y \frac{\partial}{\partial y}}^{\mathcal{O}} : (x, y) \mapsto (x, y \exp \mathcal{O}(x, y)).$$

We explain below how those formal power series are built and to which extent they are unique. We consequently obtain a formal conjugacy  $\mathcal{O} \circ \mathcal{S} \circ \mathcal{T}$  between  $\widehat{Z}$  and  $Z$  (notice that  $u$  is left invariant by the fibered  $\mathcal{O} \circ \mathcal{S}$ , so that it also conjugates  $\widehat{Z}$  to  $uX$ ).

**Lemma 4.8.** *Let  $X$  be in the form (4.2) for  $A \in \mathbb{C}[[\varepsilon, x, y]]$ , and take  $G \in \mathbb{C}[[\varepsilon, x, y]]$ . There exists a formal solution  $F \in \mathbb{C}[[\varepsilon, x, y]]$  of the cohomological equation*

$$(4.12) \quad X \cdot F = G$$

*if, and only if,  $G(x, 0)$  belongs to the ideal generated by  $P$ . In that case  $F$  is unique up to the free choice of  $F(0, 0) \in \mathbb{C}[[\varepsilon]]$ .*

*Proof.* Let

$$F(x, y) =: \sum_{n \geq 0} F^n(x) y^n \quad \text{and} \quad G(x, y) =: \sum_{n \geq 0} G^n(x) y^n.$$

We proceed by induction on  $n \geq 0$  by identifying coefficients of powers of  $y$  in (4.12). For each  $n \in \mathbb{Z}_{\geq 0}$  we must therefore solve

$$(4.13) \quad P \frac{\partial F^n}{\partial x} + n(1 + \mu x^k) F^n = G^n + o(n),$$

where  $o(n)$  stands for terms containing  $F^m$  for  $m < n$  only, and are thus already known.

- The case  $n = 0$  outlines the formal obstruction (notice that the choice of  $F^0(0)$  is free).
- For  $n > 0$  no additional obstruction appears and  $F^n$  is uniquely determined. Then Lemma 4.6 provides the unique formal solution of the family of differential equations (4.13).

□

We finally derive the Formal Normalization Theorem by writing

$$U(x, 0) = u(x) + \mathcal{O}(P(x)) \quad , \quad u \in \mathbb{C}\{\varepsilon\}[x]_{\leq k}^{\times},$$

and finding a (unique with  $T(0, 0) = 0$ ) formal solution  $T$  of (4.8) by Lemma 4.8. As for the power series  $O$ , a (unique with  $O(0, 0) = 0$ ) formal solution of (4.10) exists by Proposition 4.3, and Lemma 4.8, for  $X$  given in (4.2).

**Definition 4.9.** Let  $Z$  be an unfolding in prepared form (4.1). We write  $\mathcal{N} := \mathcal{O} \circ \mathcal{S} \circ \mathcal{T}$  the canonical **formal normalization** of  $Z$  satisfying  $\mathcal{N}^* \widehat{Z} = Z$  where  $\mathcal{O}$ ,  $\mathcal{S}$  and  $\mathcal{T}$  are built above.

**4.4. Uniqueness.** Addressing the uniqueness clause in the Formal Normalization Theorem boils down to studying the case of the normal forms, because of the canonical choice of normalization maps  $\mathcal{N}$  done in Definition 4.9.

**Lemma 4.10.** *Let  $\Psi$  be a formal orbital symmetry of the formal normal form  $\widehat{Z}$  (fixing the canonical parameter).*

- (1) There exist unique  $F \in \mathbb{C}[[\varepsilon, x, y]]$  and  $c \in \mathbb{C}[[\varepsilon]]^\times$  such that

$$\Psi = (c^* \text{Id}) \circ \Phi_{\widehat{Z}}^F$$

where  $c^* \text{Id}$  is the linear mapping  $(x, y) \mapsto (x, cy)$ . (The converse statement clearly holds.)

- (2)  $\Psi$  is a symmetry of  $\widehat{Z}$  if, and only if,  $F \in \mathbb{C}[[\varepsilon]]$ .  
 (3)  $\Psi$  is fibered if, and only if,  $F = 0$ .

*Proof.*

- (1) By Remark 3.6 we want to determine  $V \in \mathbb{C}[[\varepsilon, x, y]]^\times$  such that  $\Psi^* \widehat{Z} = V \widehat{Z}$ . Because  $\varepsilon$  is a formal invariant governing the eigenvalues of (the differential of) the vector fields at the singularities,  $\Psi$  cannot change the eigenvalues, so that  $V(x, y) = 1 + O(P(x)) + O(y)$ . According to Lemma 4.8 there exists a (unique) formal solution  $F$  with  $F(0, 0) = 0$  to the cohomological equation

$$\widehat{X} \cdot F = \frac{1}{uV} - \frac{1}{u}.$$

Therefore  $\widehat{\Psi} := \Psi \circ (\Phi_{\widehat{Z}}^F)^{\circ -1}$  induces a symmetry of  $\widehat{Z}$ . Write  $\widehat{\Psi} : (\varepsilon, x, y) \mapsto (\varepsilon, \phi_\varepsilon(x, y), \psi_\varepsilon(x, y))$ . By considering the  $\frac{\partial}{\partial x}$ -component of  $\widehat{Z}$  one obtains the relation

$$(uP) \circ \phi = \widehat{Z} \cdot \phi.$$

Setting  $y := 0$  yields

$$(u_\varepsilon P_\varepsilon) \circ \phi_\varepsilon(x, 0) = u_\varepsilon(x) P_\varepsilon(x) \frac{\partial \phi_\varepsilon}{\partial x}(x, 0)$$

so that

$$\phi_\varepsilon(x, 0) = \Phi_{u_\varepsilon P_\varepsilon \frac{\partial}{\partial x}}^{t_\varepsilon}(x) = \Phi_{\widehat{Z}}^{t_\varepsilon}(x, 0)$$

for some  $t \in \mathbb{C}[[\varepsilon]]$ . Hence we may assume without loss of generality that  $F_\varepsilon(0, 0) = t_\varepsilon$  and  $\phi_\varepsilon(x, 0) = x$ . Writing  $\phi_\varepsilon(x, y) = x + \sum_{n \geq \nu} \phi_\varepsilon^n(x) y^n$  with  $\nu > 0$  we obtain for the term of  $y$ -degree  $\nu$

$$P' \phi^\nu = P \frac{\partial \phi^\nu}{\partial x} + \nu(1 + \mu x^k) \phi^\nu$$

whose unique formal solution is  $\phi^\nu = 0$ , since it is the equation of the weak separatrix of  $P \frac{\partial}{\partial x} + y(\nu(1 + \mu x^k) + P') \frac{\partial}{\partial y}$ . As a matter of consequence  $\phi_\varepsilon(x, y) = x$  and  $\widehat{\Psi}$  is fibered. Lastly, by considering the  $\frac{\partial}{\partial y}$ -component of  $\widehat{Z}$  one obtains the relation

$$(1 + \mu x^k) \psi = \widehat{X} \cdot \psi.$$

Setting  $y := 0$  yields

$$\psi_\varepsilon(x, 0) = 0$$

so that  $\psi_\varepsilon(x, y) = y \exp N_\varepsilon(x, y)$  for some  $N \in \mathbb{C}[[\varepsilon, x, y]]$ . The corresponding cohomological equation reads

$$0 = \widehat{X} \cdot N$$

and only admits  $N \in \mathbb{C}[[\varepsilon]]$  as formal solution (uniqueness clause of Lemma 4.8). We then set  $c := \exp N$ .

(2) and (3) are clear from the previous arguments.  $\square$

We derive the following precise statement. Item (2) plays an essential role in proving the (analytic) Uniqueness Theorem.

**Corollary 4.11.** *Consider two unfoldings  $Z$  and  $\tilde{Z}$  in prepared form (4.1).*

- (1) *Let  $\Psi$  be a formal conjugacy between  $Z$  and  $\tilde{Z}$  (fixing the canonical parameter), namely  $\Psi^*Z = \tilde{Z}$ . Let  $\mathcal{N} = \mathcal{O} \circ \mathcal{T} \circ \mathcal{S}$  and  $\tilde{\mathcal{N}} = \tilde{\mathcal{O}} \circ \tilde{\mathcal{T}} \circ \tilde{\mathcal{S}}$  be the respective canonical tangent-to-identity formal normalizations as in Definition 4.9.*
- (a) *There exists unique  $c \in \mathbb{C}[[\varepsilon]]^\times$  and  $t \in \mathbb{C}[[\varepsilon]]$  such that*

$$\Psi = \mathcal{N}^{\circ-1} \circ (c^* \text{Id}) \circ \tilde{\mathcal{N}} \circ \Phi_{\tilde{Z}}^t.$$

*(The converse statement clearly holds.)*

- (b) *If  $\Psi$  is analytic then so are  $t$  and  $c$ . (The converse statement does not generally hold.)*
- (2) *If  $Z$  and  $\tilde{Z}$  are analytically orbitally equivalent (by an orbital equivalence fixing the canonical parameter) then there exists a fibered analytic orbital equivalence between them*

$$\mathcal{S}^{\circ-1} \circ \mathcal{O}^{\circ-1} \circ (c^* \text{Id}) \circ \tilde{\mathcal{O}} \circ \tilde{\mathcal{S}}$$

*for some  $c \in \mathbb{C}\{\varepsilon\}$ .*

*Remark 4.12.* The partial conclusion “there exists a fibered orbital equivalence” in Claim (2) was proved in [40, Lemma 3.4] by unfolding the homotopy technique of [30, Lemma 2.2.2]. We give here an alternate proof. In the other part of the conclusion, pay attention that  $\mathcal{O} \circ \mathcal{S}$  and  $\tilde{\mathcal{O}} \circ \tilde{\mathcal{S}}$  are only formal power series, but the composition is a convergent power series.

*Proof.*

(1)

- (a) follows from Lemma 4.10: the formal map  $\mathcal{N} \circ \Psi \circ \tilde{\mathcal{N}}^{\circ-1}$  is a symmetry of the normal form  $\tilde{Z}$ , and  $\tilde{\mathcal{N}}$  is a formal conjugacy between  $\tilde{Z}$  and  $\tilde{Z}$ , hence conjugating their flow (as formal power series):

$$\Phi_{\tilde{Z}}^t \circ \tilde{\mathcal{N}} = \tilde{\mathcal{N}} \circ \Phi_{\tilde{Z}}^t.$$

- (b) Here we assume that  $\Psi$  is analytic. Following (a) we have

$$\Psi = \mathcal{T}^{\circ-1} \circ (\mathcal{S}^{\circ-1} \circ \mathcal{O}^{\circ-1} \circ (c^* \text{Id}) \circ \tilde{\mathcal{O}} \circ \tilde{\mathcal{S}} \circ \Phi_{u\tilde{X}}^t) \circ \tilde{\mathcal{T}}.$$

Using both facts that  $\widehat{\Psi} := \mathcal{S}^{\circ-1} \circ \mathcal{O}^{\circ-1} \circ (c^* \text{Id}) \circ \tilde{\mathcal{O}} \circ \tilde{\mathcal{S}}$  is fibered, and that

$$\Phi_{u\tilde{X}}^t(x, y) = \left( \Phi_{uP \frac{\partial}{\partial x}}^t(x), y(\exp t + y\phi(x, y, t)) \right),$$

we first derive

$$\Psi = \mathcal{T}^{\circ-1} \circ \left( \Phi_{uP \frac{\partial}{\partial x}}^t, \psi \right) \circ \tilde{\mathcal{T}}.$$

Because  $T(0,0) = \widetilde{T}(0,0) = 0$ , we have

$$\Psi_\varepsilon(0,0) = (t_\varepsilon \varepsilon_0, \dots)$$

from which we deduce the convergence of  $t$ . We also have the identity

$$\frac{\partial \psi}{\partial y}(0,0) = c \exp t,$$

from which the convergence of  $c$  follows also.

- (2) It is sufficient to assume that  $Z := X$  is analytically conjugate by some  $\Psi$  (fixing the canonical parameter) to  $\widetilde{Z} := \widetilde{U}\widetilde{X}$  for some  $\widetilde{U} \in \mathbb{C}\{\varepsilon, x, y\}^\times$ . In that setting we have  $u = 1$  and  $\mathcal{T} = \text{Id}$ , so that according to (1)

$$\Psi = S^{\circ-1} \circ O^{\circ-1} \circ (c^* \text{Id}) \circ \widetilde{O} \circ \widetilde{S} \circ \widetilde{T} \circ \Phi_Z^t,$$

where  $t \in \mathbb{C}\{\varepsilon\}$ . As a matter of consequence the mapping  $\Phi_Z^t$  is analytic, and so is  $\widehat{\Psi} := \Psi \circ (\Phi_X^t)^{\circ-1}$ . Because  $\widehat{\Psi} \circ \widetilde{T}^{\circ-1}$  is fibered, the  $x$ -component of  $\widehat{\Psi}$  (which is analytic) is equal to the  $x$ -component of  $\widetilde{T}$ . The former is of the form  $(x, y) \mapsto A(x, y, \widetilde{T}(x, y))$  for some holomorphic function  $A \in \mathbb{C}\{\varepsilon, x, y, t\}$  with  $\frac{\partial A}{\partial t} \neq 0$ , and where  $\widetilde{T}$  is the solution of (4.8) for  $U := \widetilde{U}$ . Thus  $\widetilde{T}$  is a convergent power series, and so is  $\widehat{\Psi} \circ \widetilde{T}^{\circ-1}$ .  $\square$

## 5. GEOMETRIC ORBITAL NORMALIZATION

Here we prove the orbital part of the Normalization and Uniqueness Theorems for  $\tau = 0$ . Sections 5.2–5.5 are devoted to the construction of the normal form conjugacy, while its uniqueness is thoroughly studied in Section 5.6. Before going into the details we start with a brief description of the general strategy. Let us call  $\mathbb{D}$  the unit disk.

For fixed

$$0 < \frac{1}{\rho^\infty} < \rho^0$$

we introduce two analytic charts:

- the original coordinates

$$(x, y) \in \mathcal{U}^0 := \rho^0 \mathbb{D} \times (\mathbb{C}, 0),$$

- the coordinates at infinity

$$(u, v) \in \mathcal{U}^\infty := \rho^\infty \mathbb{D} \times (\mathbb{C}, 0)$$

with (involutive) standard transition map on  $\mathbb{C}^\times \times (\mathbb{C}, 0)$

$$(u, v) \mapsto \left( \frac{1}{u}, v \right) = (x, y).$$

For convenience we write  $O^0$  and  $O^\infty$  the respective expression of a holomorphic object  $O$  in the charts  $\mathcal{U}^0$  and  $\mathcal{U}^\infty$  respectively.

Start from an arbitrary  $X^0 \in \text{Convergent}_k$  in prepared form (4.2), with  $A \in y^2 \mathbb{C}\{\varepsilon, x, y\}$  holomorphic and bounded on  $\mathcal{U}^0$ , and such that  $\mu_0 \notin \mathbb{R}_{\leq 0}$ . It is always possible to make this assumption thanks to Theorem 4.1, since  $a = 0$  in that

case. Notice in particular that  $A$  is bounded since we can always take a smaller  $\rho^0$  and decrease similarly the size of the neighborhood of  $\{y = 0\}$ : hence  $\mathcal{U}^0$  can be taken inside a compact set on which  $A$  is defined.

In the following we assume that  $\varepsilon$  is so small that the  $k + 1$  singularities of  $X_\varepsilon^0$  lie in  $\{0 \leq |x| < 1/\rho^\infty\} \times \{0\}$ . The following steps constitute what we refer to as the **unfolded Loray construction**.

**Gluing** We find a vector field family  $X^\infty$  on  $\mathcal{U}^\infty$  whose holonomy over  $\rho^\infty \mathbb{S}^1 \times \{0\}$  is the inverse of  $\mathfrak{h}_\varepsilon$ , the corresponding “weak” holonomy of  $X^0$  (Section 5.2). Therefore foliations induced by each vector field can be glued one to the other over the annulus  $\mathcal{U}^0 \cap \mathcal{U}^\infty$  by an identification of the form

$$(u, v) = \left( \frac{1}{x}, y \exp \phi(x, y) \right)$$

(Section 5.3). This operation results in a family of foliated abstract complex surfaces  $(\mathcal{M}, \mathcal{F})$ .

**Normalizing** We construct a fibered biholomorphic equivalence between  $\mathcal{M}$  and a standard neighborhood of  $\{y = 0\} \simeq \mathbb{P}_1(\mathbb{C})$ , that is a complex surface with charts  $\mathcal{U}^0, \mathcal{U}^\infty$  and transition map *exactly*  $(u, v) = \left( \frac{1}{x}, y \right)$  (Section 5.4). Because  $\mathbb{P}_1(\mathbb{C}) \times \{0\}$  is compact the expression of the new  $X^0$  is polynomial in  $x$  with controlled degree, thus in orbital normal form (2.4) as expected by the Normalization Theorem (Section 5.5).

**Uniqueness** From the special form of the normalized vector field, it can be seen that the closure of the saturation of any small neighborhood of  $(0, 0)$  contains a whole  $\mathbb{P}_1(\mathbb{C}) \times r\mathbb{D}$ . Therefore any local conjugacy between normal forms (which we choose fibered thanks to Corollary 4.11 (2)) can be analytically continued by a construction *à la* Mattei-Moussu on  $\mathbb{P}_1(\mathbb{C}) \times r\mathbb{D}$ . But this manifold has very few fibered automorphisms, allowing to conclude (Section 5.6).

In the unfolded Loray construction, only what happens in the first chart  $(x, y)$  is of a different nature than when  $\varepsilon = 0$ . As seen from the other chart  $(u, v)$ , the only important ingredient for the construction is the “weak” holonomy  $\mathfrak{h}_\varepsilon$  of the unfolding (see Section 5.1 below). Hence the original arguments do not need to be unfolded near  $(\infty, 0)$ , although we must take care that everything remains holomorphic in the parameter. The first two steps of the unfolded Loray construction require external results that need to be parametrically controlled:

- (1) the realization of the weak holonomy  $\mathfrak{h}$  by a foliation near  $(\infty, 0)$ , obtained by the construction of [42];
- (2) the normalization of the transition map between the charts  $(x, y)$  and  $(u, v)$  on the annulus

$$\mathcal{A} := \{1/\rho^0 < |u| < \rho^\infty\} \times (\mathbb{C}, 0),$$

as done in [41].

Both proofs are similar in spirit and only rely on complex (holomorphic) analysis and (what amounts to) a fixed-point method. Parametric holomorphy follows from the explicit integral formulas. Because normalizing transition maps is relatively easy, we prove a parametric version of Savelev’s theorem in Section 5.4. It contains the main steps and ideas upon which are based the respective proofs

of the Normalization Theorem for vector fields (Section 6) and of the Realization Theorem (Section 7). The latter is nothing but an unfolded version of the main result of [42], retrospectively making the present article more self-contained.

**5.1. Weak holonomy.** We name

$$\Pi : (x, y) \mapsto x$$

the projection on the invariant line  $\{y = 0\}$  and let

$$\Sigma \subset \Pi^{-1}(x_*)$$

be a germ of a transverse disc endowed with the coordinate  $y \in (\mathbb{C}, 0)$ . Starting from  $y \in \Sigma$  there exists a unique path

$$\begin{aligned} \gamma_y &: [0, 1] \longrightarrow \mathcal{U}^0 \\ \gamma_y(0) &= (x_*, y) \end{aligned}$$

tangent to  $X_\varepsilon^0$  such that

$$\gamma := \Pi \circ \gamma_y = s \mapsto x_* \exp 2i\pi s.$$

We define  $\mathfrak{h}_\varepsilon(y)$  as the  $y$ -component of the final value  $\gamma_y(1)$ . The **weak holonomy mapping**  $\mathfrak{h}_\varepsilon$  as described is a germ of a biholomorphism near the fixed-point 0 whose linear part is governed by the formal orbital invariant  $\mu$  in the following way:

$$\mathfrak{h}_\varepsilon(y) = y \exp 2i\pi\mu_\varepsilon + o(y) \in \text{Diff}(\Sigma, 0).$$

The analytic dependence of trajectories of  $X_\varepsilon^0$  on the parameter  $\varepsilon$  ensures that  $\mathfrak{h} \in \mathbb{C}\{\varepsilon, y\}$ .

**5.2. Parametric holonomy realization at  $(\infty, 0)$ .**

**Theorem 5.1.** [42, Main Theorem and Section 4.4] *Let  $(\Delta_\eta)_{\eta \in (\mathbb{C}^n, 0)}$  be an analytic family of elements of  $\text{Diff}(\mathbb{C}, 0)$ , that is  $(\eta, v) \mapsto \Delta_\eta(v) \in \mathbb{C}\{\eta, v\}$  and  $\Delta'_0(0) \neq 0$ . Let  $\lambda \in \mathbb{C}\{\eta\}$  be given such that  $\Delta'_\eta(0) = \exp(-2i\pi\lambda_\eta)$  and  $\lambda_0 \notin \mathbb{R}_{\leq 0}$ . There exists an analytic family of vector fields  $(X_\eta^\infty)_{\eta \in (\mathbb{C}^n, 0)}$  of the form*

$$(5.1) \quad X_\eta^\infty(u, v) = -u \frac{\partial}{\partial u} + v \left( \lambda_\eta + u \left( 1 + f_\eta(v) \right) + g_\eta(v) \right) \frac{\partial}{\partial v}, \quad f, g \in v\mathbb{C}\{\eta, v\},$$

holomorphic on the domain  $\mathcal{U}^\infty$  and satisfying for all  $\eta \in (\mathbb{C}^n, 0)$ :

- (1)  $(0, 0)$  is the only singularity of  $X_\eta^\infty$  in  $\mathcal{U}^\infty$ ,
- (2) the holonomy of  $X_\eta^\infty$  above the circle  $w_*\mathbb{S}^1 \times \{0\}$ , computed on a germ of transverse disc  $\{u = u_*\}$  with respect to the projection  $(u, v) \mapsto u$ , is exactly  $\Delta_\eta$ .

*Remark 5.2.* The result of [42] asserts the existence of a vector field of the form (5.1) with  $f := 0$  whose holonomy on  $\Sigma$  is conjugate to  $\Delta$  by some analytic family  $\Psi$ . The conjugacy  $(u, v) \mapsto (u, \Psi(v))$  transforms the vector field into the form (5.1) for different  $f, g$  but its holonomy is exactly  $\Delta$  on  $\Sigma$ .

In the generic case  $\lambda_0 \notin \mathbb{R}$  the theorem is (almost) trivial. All holonomy maps

$$\Delta_\eta : v \mapsto v \exp(-2i\pi\lambda_\eta + \delta_\eta(v)) \quad , \quad \delta_\eta(0) = 0,$$

are hyperbolic and locally analytically linearizable for that matter (Koenig's theorem), the unique tangent-to-identity linearization being given by  $\Psi_\eta : v \mapsto v \exp \psi_\eta(v)$ , where

$$\psi_\eta := \sum_{n=0}^{\infty} \delta_\eta \circ \Delta_\eta^{\circ n}.$$

Local uniform convergence ensures that  $\psi$  is analytic in both  $t$  and  $\eta$ . The fibered mapping  $(u, v) \mapsto (u, \Psi_\eta(v))$  transforms the linear vector field  $-u \frac{\partial}{\partial u} + \lambda v \frac{\partial}{\partial v}$  into a vector field  $X_\eta^\infty$  fulfilling the conclusions (1)-(2) of the theorem (but not of the form (5.1)). However if  $\lambda_0 \in \mathbb{R}$  this construction fails: the linearization domain may shrink to a point (if  $\Delta_0$  is not analytically linearizable). The form (5.1) has the advantage of being valid for all cases, analytically in the parameter. Notice that the presence of the term  $-uv \frac{\partial}{\partial v}$  in (5.1) discards any linear realization even when  $\lambda_0 \notin \mathbb{R}$ .

We define  $\eta := \varepsilon \in (\mathbb{C}^k, 0)$ ,

$$\begin{aligned} \lambda_\varepsilon &:= \mu_\varepsilon \notin \mathbb{R}_{\leq 0}, \\ \Delta_\varepsilon &: v \mapsto \mathfrak{h}_\varepsilon^{\circ -1}(v), \end{aligned}$$

and apply Theorem 5.1, to obtain an analytic family  $X^\infty$  in the chart  $(u, v)$ . In order to stitch the induced foliation with that of  $X_\varepsilon^0$  we need to prepare it by changing slightly the coordinates on  $\mathcal{U}^\infty$ . Let  $\tilde{X}_\varepsilon^0$  be the vector field corresponding to  $X_\varepsilon^0$  in the coordinates  $(u, v) = (\frac{1}{x}, y)$ , that is

$$\begin{aligned} \tilde{X}_\varepsilon^0(u, v) &= -u^2 P_\varepsilon\left(\frac{1}{u}\right) \frac{\partial}{\partial u} + v(\mu_\varepsilon u^{-k} + 1 + \mathcal{O}(v)) \frac{\partial}{\partial v} \\ &= u P_\varepsilon\left(\frac{1}{u}\right) \times \left(-u \frac{\partial}{\partial u} + v(\lambda_\varepsilon + h_\varepsilon(u) + \mathcal{O}(v)) \frac{\partial}{\partial v}\right), \end{aligned}$$

where

$$(5.2) \quad h_\varepsilon : u \mapsto \frac{u^k + \mu_\varepsilon}{u^{k+1} P_\varepsilon\left(\frac{1}{u}\right)} - \mu_\varepsilon \in \text{Holo}\left((\mathbb{C}^k, 0) \times \rho^\infty \mathbb{D}\right)$$

vanishes at 0. Notice indeed that the polynomial  $u^{k+1} P_\varepsilon\left(\frac{1}{u}\right) \in \mathbb{C}[u]_{\leq k+1}$  has its roots outside the closed disc  $\text{cl}(\rho^\infty \mathbb{D})$ , whereas it takes the value 1 at 0. Remark also that the quantity  $u P_\varepsilon\left(\frac{1}{u}\right)$  needs to be factored out in order to recognize a vector field alike to  $X_\varepsilon^\infty$  near  $(\infty, 0)$ . This function is non-vanishing on the annulus  $\mathcal{A}$ . Let  $\tilde{X}_\varepsilon^\infty$  be the vector field corresponding to  $X_\varepsilon^\infty$  through the inverse transform

$$(5.3) \quad (u, v) \mapsto \left(u, v \exp \int_{u_*}^u (h_\varepsilon(z) - z) \frac{dz}{z}\right).$$

By construction we have

$$\tilde{X}_\varepsilon^\infty(u, v) = -u \frac{\partial}{\partial u} + v(\lambda_\varepsilon + h_\varepsilon(u) + \mathcal{O}(v)) \frac{\partial}{\partial v},$$

which glues with  $\widetilde{X}_\varepsilon^0$  through  $(u, v) = \left(\frac{1}{x}, y\right)$  as presented in the next paragraph.

**5.3. Gluing.** Both transformed vector fields  $\widetilde{X}^0$  and  $\widetilde{X}^\infty$  built in the previous section have same holonomy  $\Delta_\varepsilon$  on  $\Sigma$ . We glue the (foliations defined by the) vector fields  $\widetilde{X}_\varepsilon^0$  and  $\widetilde{X}_\varepsilon^\infty$  over the fibered annulus  $\mathcal{A}$  through a fibered map  $\Phi_\varepsilon$  fixing  $\Sigma$  and (classically) obtained by foliated path-lifting, as we explain now. For  $(u, v) \in \mathcal{A}$  we join  $u_*$  to  $u$  in  $\mathcal{A} \cap \{v = 0\}$  by some path  $\gamma$  and define

$$\Phi_\varepsilon(u, v) := \left(u, \mathfrak{h}_{\varepsilon, \gamma}^\infty \circ \left(\mathfrak{h}_{\varepsilon, \gamma}^0\right)^{\circ -1}(v)\right),$$

where  $\mathfrak{h}_{\varepsilon, \gamma}^0$  (resp.  $\mathfrak{h}_{\varepsilon, \gamma}^\infty$ ) is the holonomy map obtained by lifting the path  $\gamma$  through  $\Pi$  in the foliation induced by  $\widetilde{X}_\varepsilon^0$  (resp.  $\widetilde{X}_\varepsilon^\infty$ ). The map  $\Phi_\varepsilon$  is well-defined because when  $\gamma$  is a loop both mappings  $\mathfrak{h}_{\varepsilon, \gamma}^\infty$  and  $\mathfrak{h}_{\varepsilon, \gamma}^0$  coincide with the same corresponding iterate of  $\Delta_\varepsilon$ . Clearly  $\Phi_\varepsilon$  depends analytically on  $\varepsilon \in (\mathbb{C}^k, 0)$  and is a germ of a fibered biholomorphism near  $\mathcal{A} \cap \{v = 0\}$  satisfying

$$\begin{aligned} \Phi^* \widetilde{X}^0 &= uP\left(\frac{1}{u}\right) \widetilde{X}^\infty, \\ \Phi(u, v) &= (u, v \exp \phi(u, v)), \\ \phi(u, 0) &= \phi(u_*, v) = 0. \end{aligned}$$

**5.4. Normalizing.** So far the construction yields an analytic family of complex foliated surfaces, written  $(\mathcal{M}, \mathcal{F})$ , defined by the charts  $(\mathcal{U}^0, \mathcal{F}^0)$  and  $(\mathcal{U}^\infty, \mathcal{F}^\infty)$  with transition map

$$(5.4) \quad (u, v) \mapsto \left(\frac{1}{u}, v \exp \phi(u, v)\right) = (x, y).$$

*Remark 5.3.* The foliation  $\mathcal{F}_\varepsilon$  is transverse to the fibers of the global fibration by discs  $\Pi : \mathcal{M}_\varepsilon \rightarrow \mathcal{L}$  given in the first chart by  $(x, y) \mapsto x$ , except along the  $k + 2$  invariant discs  $\{P_\varepsilon(x) = 0\}$  and  $\{x = \infty\}$ .

Each manifold  $\mathcal{M}_\varepsilon$  is a neighborhood of the invariant divisor  $\mathcal{L} \simeq \mathbb{P}_1(\mathbb{C})$ , corresponding to  $\{y = 0\}$  and  $\{v = 0\}$  in the respective chart, while the natural embedding  $\mathbb{P}_1(\mathbb{C}) \hookrightarrow \mathcal{M}$  has self-intersection 0 according to Camacho-Sad index formula [4] (the singularities near  $(0, 0)$  contribute for a sum of Camacho-Sad indices equal to  $\mu_\varepsilon$  while the singularity at  $(\infty, 0)$  does for  $-\lambda_\varepsilon = -\mu_\varepsilon$ ).

**Definition 5.4.** For  $r > 0$  we define the **standard neighborhood of radius  $r$  of the Riemann sphere**

$$\text{Sphere}(r) := \mathbb{P}_1(\mathbb{C}) \times r\mathbb{D},$$

the complex surface equipped with the (global) affine coordinates

$$(u, v) \in \mathbb{C} \times r\mathbb{D}$$

and transition map on  $\mathbb{C}^\times \times r\mathbb{D}$  given by  $(u, v) = \left(\frac{1}{x}, y\right)$ , i.e. by (5.4) with  $\phi := 0$ . The other chart of  $\text{Sphere}(r)$  is the domain  $(x, y) \in \mathbb{C} \times r\mathbb{D}$ . When speaking of a **standard neighborhood of the sphere** we actually refer to  $\text{Sphere}(r)$  for some  $r > 0$  small enough.



**Theorem 5.5.** *Let  $\mathcal{M}$  be an analytic family of complex surfaces with transition maps (5.4). There exists a standard neighborhood  $\mathcal{V} = \text{Sphere}(r)$  of  $\mathcal{L}$ , for some  $r > 0$ , and an analytic family of fibered holomorphic injective mappings*

$$\Psi : \mathcal{V} \longrightarrow \mathcal{M}$$

agreeing with the identity on  $\mathcal{L}$ .

The rest of the subsection is devoted to the proof of this theorem. We refer to Section (3.1.2) for the definitions of the functional spaces in use. We are looking for  $\Psi$ , or rather its expression in the charts  $\mathcal{U}^0$  and  $\mathcal{U}^\infty$ , in the form

$$\begin{aligned} \Psi^0(x, y) &= (x, y \exp \psi^0(x, y)) \\ \Psi^\infty(u, v) &= (u, v \exp \psi^\infty(u, v)). \end{aligned}$$

The normalization equation becomes a non-linear additive Cousin problem on  $\mathcal{A}$ :

$$\psi^0\left(\frac{1}{u}, v\right) - \psi^\infty(u, v) = \phi \circ \Psi^\infty(u, v).$$

Starting from  $\psi_0^0 := 0$  and  $\psi_0^\infty := 0$  we build iteratively two bounded sequences of holomorphic functions

$$\psi_n^\sharp \in \text{Holo}_c\left(\left(\mathbb{C}^k, 0\right) \times \rho^\sharp \mathbb{D} \times r\mathbb{D}\right), \sharp \in \{0, \infty\}$$

solution of the linearized additive Cousin problem (or discrete cohomological equation)

$$(5.5) \quad \psi_{n+1}^0\left(\frac{1}{u}, v\right) - \psi_{n+1}^\infty(u, v) = \phi(u, v \exp \psi_n^\infty(u, v)).$$

The Cousin problem has explicit solutions given by a Cauchy-Heine transform. From these solutions we obtain *a priori* bounds on the norm of  $\psi_n^\sharp$ , allowing to fix the radius  $r > 0$  beforehand. We let

$$\begin{aligned} \mathcal{U}_r^0 &:= \{(x, y) : |x| < \rho^0, |y| < r\}, \\ \mathcal{U}_r^\infty &:= \{(u, v) : |u| < \rho^\infty, |v| < r\}, \end{aligned}$$

be an atlas for  $\text{Sphere}(r)$ . We postpone the proof of the next main lemma to the end of the section.

**Lemma 5.6.** *Assume that  $\phi \in \text{Holo}_c(\mathcal{A}_\eta)'$  for some domain  $\mathcal{A}_\eta := \left\{ \frac{1}{\rho^0} < |u| < \rho^\infty \right\} \times \eta\mathbb{D}$ . Let  $\psi \in \text{Holo}_c(\mathcal{U}_r^\infty)$  be such that the image of  $\mathcal{A}_r$  by  $(u, v) \mapsto (u, v \exp \psi(u, v))$  is included in  $\mathcal{A}_\eta$ . Define*

$$(5.6) \quad \begin{cases} F^\infty(u, v) &:= \frac{1}{2i\pi} \oint_{\rho^\infty \mathbb{S}^1} \phi(z, v \exp \psi(z, v)) \frac{dz}{z-u}, \\ F^0(x, y) &:= \frac{x}{2i\pi} \oint_{\frac{1}{\rho^0} \mathbb{S}^1} \phi(z, y \exp \psi(z, y)) \frac{dz}{xz-1}. \end{cases}$$

Then the following properties hold.

(1)  $F^0 \in \text{Holo}_c(\mathcal{U}_r^0)$  and  $F^\infty \in \text{Holo}_c(\mathcal{U}_r^\infty)$ . Moreover for  $\sharp \in \{0, \infty\}$

$$(5.7) \quad \left\| F^\sharp \right\|_{\mathcal{U}_r^\sharp} \leq rK \left\| \phi \right\|'_{\mathcal{A}_\eta} \exp \left\| \psi \right\|_{\mathcal{U}_r^\infty}$$

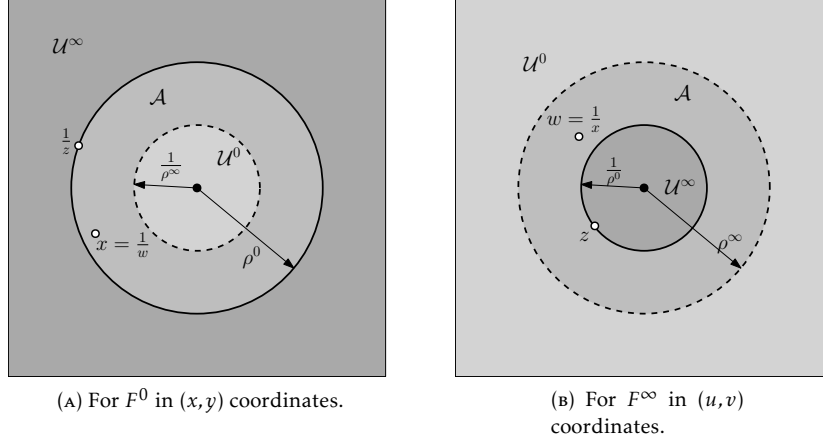


FIGURE 5.1. Integration contours in the respective charts.

where

$$K := \left(1 + \frac{2\rho^0}{\rho^\infty \rho^0 - 1}\right).$$

(2) For all  $(u, v) \in \mathcal{A}_r$  we have

$$(5.8) \quad F^0\left(\frac{1}{u}, v\right) - F^\infty(u, v) = \phi(u, v \exp \psi(u, v)).$$

(3) These are the only holomorphic solutions of the previous equation which are bounded, up to the addition of a function  $v \mapsto f^\infty(v)$  with  $f^\infty \in \text{Holo}_c(r\mathbb{D})$ .

*Remark 5.7.* The integral formula (5.6) shows right away that  $F^\sharp$  depends holomorphically on any extraneous parameter on which  $\phi$  were to depend holomorphically.

It is straightforward to check that fixing some

$$0 < r \leq \eta \exp\left(-\eta K \|\phi\|'_{(\mathbb{C}^k, 0) \times \eta\mathbb{D}}\right)$$

inductively produces well-defined sequences  $(\psi_n^\sharp)_{n \in \mathbb{N}}$  of  $\text{Holo}_c(\mathcal{U}_r^\sharp)$ , for we have the implication

$$\|\psi_{\varepsilon, n}^\infty\|_{\mathcal{U}_r^\infty} < \eta K \|\phi_\varepsilon\|'_{\eta\mathbb{D}} \implies |v \exp \psi_{\varepsilon, n}^\infty(u, v)| < r |\exp \psi_{\varepsilon, n}^\infty(u, v)| < r \exp\left(\eta K \|\phi_\varepsilon\|'_{\eta\mathbb{D}}\right) \leq \eta$$

for all  $(u, v) \in \mathcal{A}_r$ . Using (5.7) with  $\psi := \psi_{\varepsilon, n}^\infty$  finally yields

$$\|\psi_{\varepsilon, n+1}^\infty\|_{\mathcal{U}_r^\infty} < \eta K \|\phi_\varepsilon\|'_{\eta\mathbb{D}}.$$

We establish now that both sequences converge in  $\text{Holo}_c(\mathcal{U}_r^\sharp)$ . The hypothesis  $\phi(u, 0) = 0$  guarantees that  $\psi_{n+1}^\sharp(u, v) = \psi_n^\sharp(u, v) + \mathcal{O}(v^{n+1})$ , hence the bounded

sequence  $(\psi_n^\sharp)_{n \in \mathbb{N}}$  converges for the projective topology on  $\mathbb{C}[[\varepsilon, u]][[v]]$  (for the Krull distance actually). Therefore the sequences converge towards holomorphic and bounded functions

$$\psi^\sharp := \lim_{n \rightarrow \infty} \psi_n^\sharp \in \text{Holo}_c((\mathbb{C}^k, 0) \times \rho^\sharp \mathbb{D} \times r \mathbb{D})$$

according to the next lemma.

**Lemma 5.8.** [42, Lemma 2.10] *Let  $\mathcal{D}$  be a domain in  $\mathbb{C}^m$  and consider a bounded sequence  $(f_p)_{p \in \mathbb{N}}$  of  $\text{Holo}_c(\mathcal{D})$  satisfying the additional property that there exists some point  $\mathbf{z}_0 \in \mathcal{D}$  such that the corresponding sequence of Taylor series  $(T_p)_{p \in \mathbb{N}}$  at  $\mathbf{z}_0$  is convergent in  $\mathbb{C}[[\mathbf{z} - \mathbf{z}_0]]$  (for the projective topology). Then  $(f_p)_p$  converges uniformly on compact sets of  $\mathcal{D}$  towards some  $f_\infty \in \text{Holo}_c(\mathcal{D})$ .*

*Remark 5.9.* The limiting functions  $\psi^\sharp$  are *not* obtained through the use of a fixed-point theorem, although they are a fixed-point of (5.5). The method used here, based on Lemma 5.8, does not use the fact that  $\text{Holo}_c(\mathcal{D})$  is a Banach space, only that it is a Montel space (any bounded subset is sequentially compact). Also the estimate (5.7) obtained in Lemma 5.6 (1) is easier to derive than a sharper estimate aimed at establishing that  $\psi_n^\sharp \mapsto \psi_{n+1}^\sharp$  is a contraction mapping.

5.4.1. *Proof of Lemma 5.6 (2).* This is nothing but Cauchy formula.

5.4.2. *Proof of Lemma 5.6 (1).* Clearly the function  $F^\sharp$  is holomorphic on the domain  $\mathcal{U}_r^\sharp$ . Notice also that modifying slightly the integration path does not change the value of the function, so that  $F^\sharp$  is bounded on  $\mathcal{U}_r^\sharp$ . Let us evaluate its norm.

Set  $\Psi : (u, v) \mapsto (u, v \exp \psi(u, v))$  and define  $\rho > 0$  by  $2\rho := \rho^\infty + \frac{1}{\rho^0}$ . We prove the estimate on  $\|F^\infty\|_{\mathcal{U}_r^\infty}$  and  $\|F^0\|_{\mathcal{U}_r^0}$  in two steps: first we bound  $|F^\infty(u, v)|$  when  $|u| \leq \rho$  (resp.  $\|F^0\|_{\mathcal{U}_r^0}$  when  $|x| \leq \frac{1}{\rho}$ ), then when  $\rho \leq |u| < \rho^\infty$  (resp.  $\frac{1}{\rho} \leq |x| < \rho^0$ ).

- For  $|u| \leq \rho$  and  $|v| < r$  one has

$$|F^\infty(u, v)| \leq \|\phi \circ \Psi\|_{\mathcal{A}_r} \times \frac{1}{2\pi} \oint_{\rho^\infty \mathbb{S}^1} \left| \frac{dz}{z-u} \right|.$$

On the one hand

$$\frac{1}{2\pi} \oint_{\rho^\infty \mathbb{S}^1} \left| \frac{dz}{z-u} \right| \leq \frac{1}{\rho^\infty - \rho} = \frac{2\rho^0}{\rho^\infty \rho^0 - 1} < K,$$

while on the other hand, for all  $(u, v) \in \mathcal{A}_r$ ,

$$|\phi(u, v \exp \psi(u, v))| \leq |v| \|\phi\|'_{\mathcal{A}_r} \exp \|\psi\|_{\mathcal{U}_r^\infty}.$$

Taking both bounds together completes the first step of the proof.

- This gives a corresponding bound for  $F^0$  when  $|x| \leq \frac{1}{\rho}$  since

$$\frac{|x|}{2\pi} \oint_{\frac{1}{\rho^0} \mathbb{S}^1} \left| \frac{dz}{xz-1} \right| \leq \frac{1}{\rho - \frac{1}{\rho^0}} = \frac{2\rho^0}{\rho^\infty \rho^0 - 1}.$$

Taking (5.8) into account, one therefore deduces for  $\frac{1}{\rho^0} < |u| \leq \rho$  the estimate

$$\begin{aligned} |F^\infty(u, v)| &\leq \left| \phi(u, v \exp \psi(u, v)) \right| + \left| F^0\left(\frac{1}{u}, v\right) \right| \\ &\leq |v| \|\phi\|'_{\mathcal{A}_\eta} \exp \|\psi\|_{\mathcal{U}_r^\infty} \left( 1 + \frac{2\rho^0}{\rho^\infty \rho^0 - 1} \right) \end{aligned}$$

as expected.

- The bound for  $F^0$  when  $\frac{1}{\rho} \leq x < \rho^0$  is obtained similarly.

5.4.3. *Proof of Lemma 5.6 (3).* Assume that  $(\widetilde{F}^0, \widetilde{F}^\infty)$  is another pair of solution. Then for all  $(\frac{1}{x}, y) = (u, v) \in \mathcal{A}_r$  we have

$$f^0(x, y) := F^0(x, y) - \widetilde{F}^0(x, y) = F^\infty(u, v) - \widetilde{F}^\infty(u, v) =: f^\infty(u, v),$$

defining a bounded and holomorphic function  $f$  on  $\text{Sphere}(r)$ . The next lemma ends the proof.

**Lemma 5.10.** *If  $f \in \text{Holo}_c(\text{Sphere}(r))$  then  $\frac{\partial f^\infty}{\partial u} = 0$ . In other words there is a natural isometry of Banach spaces*

$$\text{Holo}_c(\text{Sphere}(r)) \simeq \text{Holo}_c(r\mathbb{D}).$$

*Proof.* In the chart  $\mathcal{U}_r^\infty$  expand  $f^\infty$  into a power series  $f^\infty(u, v) = \sum_{n \geq 0} f_n(u) v^n$  convergent on  $\mathbb{C} \times r\mathbb{D}$ . By assumption  $f$  is bounded so that from Cauchy's estimate we get

$$|f_n(u)| \leq \|f\|_{\text{Sphere}(r)} r^{-n}$$

for all  $u \in \mathbb{C}$ . Liouville theorem tells us that each  $f_n$  is constant.  $\square$

**5.5. Normal form recognition (proof of orbital Normalization Theorem).** The aim of this subsection is to shortly prove that the vector field  $\Psi^* X_\varepsilon^0$  resulting from Theorem 5.5 is in normal form (2.4). Because Savelev's normalizing fibered mapping  $\Psi$  agrees with the identity on  $\mathcal{L}$ , each  $\mathcal{F}_\varepsilon$  is induced in the chart  $\mathcal{U}_r^0$  by a holomorphic vector field of the form

$$\begin{aligned} \mathcal{X}_\varepsilon^0(x, y) &:= \Psi^* X_\varepsilon^0 = P_\varepsilon(x) \frac{\partial}{\partial x} + y \left( 1 + \mu_\varepsilon x^k + A_\varepsilon(x, y) \right) \frac{\partial}{\partial y}, \\ A &\in \text{Holo}\left(\left(\mathbb{C}^k, 0\right) \times \rho^0 \mathbb{D} \times r\mathbb{D}\right) \\ A(x, 0) &= 0. \end{aligned}$$

We must prove the following result.

**Lemma 5.11.** *There exists a sequence of polynomials  $a_n \in \mathbb{C}\{\varepsilon\}[x]_{\leq k}$  such that*

$$A(x, y) = \sum_{n=1}^{\infty} a_n(x) y^n$$

on  $\mathcal{U}_r^0$ .

*Proof.* The expansion for  $A$  is valid for  $(x, y) \in \mathcal{U}_r^0$  and  $a_n$  holomorphic in  $x$ . In the other chart  $(u, v) = \left(\frac{1}{x}, y\right)$  the vector field  $\mathcal{X}_\varepsilon^0$  is orbitally equivalent (conjugate after division by  $uP_\varepsilon\left(\frac{1}{u}\right)$ ) to

$$\mathcal{X}_\varepsilon^\infty(u, v) := -u \frac{\partial}{\partial u} + v \left( \lambda_\varepsilon + h_\varepsilon(u) + \frac{1}{u^{k+1}P_\varepsilon\left(\frac{1}{u}\right)} u^k A_\varepsilon\left(\frac{1}{u}, v\right) \right) \frac{\partial}{\partial v}$$

where  $h$  is given by (5.2). This particular vector field must coincide with the holomorphic vector field defining  $\mathcal{F}_\varepsilon$  in the chart  $\mathcal{U}_r^\infty$  after application of (5.3), because every transform used from the start is fibered so that the factor  $uP_\varepsilon\left(\frac{1}{u}\right)$  over  $\mathcal{A}_r$  remains the same and no other function can be factored out. Therefore  $u^k A_\varepsilon\left(\frac{1}{u}, v\right)$  is holomorphic near  $(0, 0)$ , and the conclusion follows.  $\square$

**5.6. Proof of orbital Uniqueness Theorem (2).** Assume that there exists an orbital equivalence between two normal forms  $\mathcal{X}$  and  $\tilde{\mathcal{X}}$ . Those vector fields are in prepared form (4.2) thus they satisfy the hypothesis of the results presented in Section 4, and in particular there exists a fibered analytical conjugacy  $\Psi$  near  $(0, 0)$  between  $\mathcal{X}$  and  $\tilde{\mathcal{X}}$ , according to Corollary 4.11 (2).

Let  $(\text{Sphere}(r), \mathcal{F})$  and  $(\text{Sphere}(r), \tilde{\mathcal{F}})$  be the family of foliated standard neighborhoods of the sphere induced respectively by  $\mathcal{X}$  and  $\tilde{\mathcal{X}}$ . The fibered mappings  $\Psi$  are holomorphic and injective on a domain  $\mathcal{D} \subset \mathcal{U}^0 \subset \text{Sphere}(r)$  containing  $(0, 0)$ . By a foliated path-lifting technique (as before)  $\Psi$  can be analytically continued on the domain

$$\mathcal{U}_\varepsilon := \text{Sat}_{\mathcal{F}_\varepsilon}(\mathcal{D}) \subset \text{Sphere}(r).$$

Using the special form of the normal form  $\mathcal{X}_\varepsilon$  we derive the following lemma in Section 5.6.2.

**Lemma 5.12.** *There exists  $r' > 0$  such that  $\text{Sphere}(r') \setminus \{x = \infty\} \subset \mathcal{U}_\varepsilon$  for all  $\varepsilon \in (\mathbb{C}^k, 0)$ .*

This lemma implies that  $\Psi_\varepsilon$  extends to a fibered, injective and holomorphic mapping  $\text{Sphere}(r') \setminus \{x = \infty\} \rightarrow \text{Sphere}(r)$ . The fact that  $\Psi_\varepsilon$  extends analytically to  $\{x = \infty\}$  uses a variation on the Mattei-Moussu construction. The proof is standard, but we include it for the sake of completeness.

**Lemma 5.13.** [31, Theorem 2] *We consider two germs of a holomorphic vector field  $X$  and  $\tilde{X}$ , both with a singularity at the origin of same eigenratio  $\lambda \notin \mathbb{R}_{\geq 0}$  and in the form*

$$(5.9) \quad x \frac{\partial}{\partial x} + \lambda y (1 + \mathcal{O}(x)) \frac{\partial}{\partial y}.$$

*Fix a germ of a transverse disc  $\Sigma := \{x = x_*, y \in (\mathbb{C}, 0)\}$ , for  $x_*$  small enough, and assume that there exists an injective and holomorphic mapping  $\psi : \Sigma \rightarrow \{x = x_*\}$  conjugating the respective holonomies induced by  $X$  and  $\tilde{X}$ , computed through the fibration  $(x, y) \mapsto x$  by turning around  $\{x = 0\}$ . Then there exists a holomorphic and injective, fibered mapping  $\Psi$  conjugating  $X$  and  $\tilde{X}$  on a connected neighborhood of  $(0, 0)$  containing  $\Sigma$ . We can even require that  $\Psi$  coincides with  $\psi$  on  $\Sigma$ .*

*Proof.* Assume first that  $\lambda < 0$ . We can consider that the holonomies  $\Gamma$  and  $\tilde{\Gamma}$  are defined on  $\Sigma := \{x = x_*\} \times r'\mathbb{D}$  and set  $\Psi(x_*, y) := (x_*, \psi(y))$  on  $\Sigma$ . We then extend  $\Psi$  over the circle  $\{|x| = |x_*|\}$  as a map of the form  $\Psi(x, y) = (x, \psi(x, y))$ , with  $\psi(x_*, y) =$

$\psi(y)$ : the extension is done by the path-lifting technique detailed in Section 5.3.  $\Psi$  is of course well-defined because  $\psi$  conjugates the holonomies. To extend  $\Psi$  to  $\rho\mathbb{D} \times r'\mathbb{D}$ , we use the path-lifting along rays  $\{\arg|x| = \text{cst}\}$ . Starting at  $(x_0, y)$  we lift the ray through  $x_0$  up to  $|x| = \rho$  in the leaf of  $X$ . We apply  $\Psi$  to the resulting point and then lift the ray back in the leaf of  $\tilde{X}$ . The corresponding point is called  $\Psi(x_0, y)$ . We must show that

$$\{x_0\} \times C_1 r' \mathbb{D} \subset \Psi(\{x_0\} \times r' \mathbb{D}) \subset \{x_0\} \times C_2 r' \mathbb{D}$$

for some positive constants  $C_1, C_2$  independent of  $x_0$ . For this purpose we can suppose that the  $O(x)$  part in (5.9) is bounded by  $1/2$  (this is the case if  $\rho$  is sufficiently small). Then

$$|\lambda| |y| \left(1 - \frac{1}{2} |x_0| \exp t\right) < \frac{d|y|}{dt} < |\lambda| |y| \left(1 + \frac{1}{2} |x_0| \exp t\right),$$

yielding by Gronwall inequality that

$$|y(0)| \exp\left(\lambda t - \int_0^t \frac{1}{2} |x_0| \exp t \, dt\right) \leq |y(t)| \leq |y(0)| \exp\left(\lambda t + \int_0^t \frac{1}{2} |x_0| \exp t \, dt\right).$$

The conclusion follows since  $\exp\left(\int_0^t \frac{1}{2} |x_0| \exp t \, dt\right) \in \left[\exp^{-\frac{|x_0|}{2}}, 1\right]$  is bounded and bounded away from 0 for  $t < 0$ .

The previous argument remains valid when  $\lambda$  is not real. It suffices to replace  $|\lambda|$  by  $|\Re(\lambda)|$ .  $\square$

*Remark 5.14.* The proof clearly shows that  $\Psi$  depends analytically on  $\varepsilon$  were  $X$  and  $\tilde{X}$  to depend analytically on  $\varepsilon$ .

The following lemma proved in Section 5.6.1 allows to complete the proof of the Uniqueness Theorem (2) by observing that injective holomorphic mappings on some standard neighborhood of the sphere are of a rather special kind.

**Lemma 5.15.** *Take some analytic family of maps  $\Psi : \text{Sphere}(r') \rightarrow \text{Sphere}(r)$  satisfying the following properties:*

- $\Psi$  is fibered,
- $\Psi_\varepsilon$  is injective and holomorphic on  $\text{Sphere}(r')$  for every  $\varepsilon \in (\mathbb{C}^k, 0)$ .

Then

$$(5.10) \quad \Psi_\varepsilon^0(x, y) = \left(x, y \sum_{n=0}^{\infty} \psi_n y^n\right),$$

where, for all  $n \in \mathbb{Z}_{\geq 0}$ ,

$$\psi_n \in \mathbb{C}\{\varepsilon\}$$

with a common radius of convergence, and  $\psi_0$  does not vanish for  $\varepsilon = 0$ . Conversely, any convergent power series  $\Psi$  as above defines an analytic family satisfying the above properties for some  $r' > 0$  small enough.

As a matter of consequence for every  $\varepsilon \in (\mathbb{C}^k, 0)$  and for any  $(x, y) \in \mathcal{U}_r^0$

$$\Psi_\varepsilon(x, y) = (x, y\psi_\varepsilon(y)) \quad , \quad \psi_\varepsilon(0) \neq 0.$$

To preserve globally orbital normal forms (2.4) is so demanding that  $\psi_\varepsilon$  ends up being constant. Indeed, from

$$\Psi_\varepsilon^* \mathcal{X}_\varepsilon(x, y) = \widetilde{X}(x, y) + y \frac{A_\varepsilon(x, y)}{y\psi'_\varepsilon(y) + \psi_\varepsilon(y)} \frac{\partial}{\partial y} = \widetilde{\mathcal{X}}_\varepsilon(x, y),$$

where

$$A_\varepsilon(x, y) := x\psi_\varepsilon(y)R_\varepsilon(x, y\psi_\varepsilon(y)) - y\psi'_\varepsilon(y)(1 + \mu x^k),$$

we deduce by setting  $x := 0$  that

$$0 = A_\varepsilon(0, y) = -y\psi'_\varepsilon(y)$$

so that  $\psi_\varepsilon$  is constant, for otherwise  $\widetilde{\mathcal{X}}_\varepsilon$  would not be in normal form. In each case we obtain finally

$$\psi_\varepsilon(v) = c_\varepsilon \in \mathbb{C}^\times$$

as expected. The remaining claim is a straightforward consequence of the study performed in Section 4.

5.6.1. *Proof of Lemma 5.15.* The expansion (5.10) is valid on  $\mathcal{U}_r^0$ , provided  $\psi_n$  depend holomorphically on  $x$ . Let us show that  $\psi_n$  is constant. Applying the transition mapping  $(x, y) \mapsto (\frac{1}{x}, y)$  we obtain the expression of  $\Psi$  in the other chart:

$$\Psi^\infty(u, v) = \left( u, v \sum_{n=0}^{\infty} \psi_n\left(\frac{1}{u}\right) v^n \right),$$

holomorphic in  $(u, v) \in \mathcal{U}_r^\infty$ . This implies in particular that each function  $u \mapsto \psi_n(\frac{1}{u})$  must be holomorphic at 0; the conclusion follows. The converse statement is straightforward.

5.6.2. *Proof of Lemma 5.12.* We can find  $\rho, r' > 0$  such that  $\text{cl}(\rho\mathbb{D} \times r'\mathbb{D}) \subset \mathcal{D}$ , where  $\mathcal{D}$  is the domain of  $\Psi$ . We show that, for some convenient choice of  $r'' \leq r'$  every point  $(x_*, y_*) \in \{|y| < r''\}$  can be linked to a point of  $\rho\mathbb{D} \times r'\mathbb{D}$  by a path contained in a leaf of  $\mathcal{F}_\varepsilon^0$ . Only the case  $|x_*| > \rho$  is not trivial. Since the singularity at  $(\infty, 0)$  is neither a node nor a saddle-node, every small germ of a disk  $\{u = u_*\}$  sufficiently close to  $\{u = 0\}$ , which is transverse to the separatrix  $\mathcal{L}$ , saturates a full pointed neighborhood  $(\mathbb{C}, 0)^2 \setminus \{u = 0\} \subset \mathcal{U}_r^\infty$  under  $\mathcal{F}_\varepsilon^\infty$ . Therefore there exists  $r''' > 0$  such that  $\{0 < |u| \leq |u_*|, |v| < r'''\} \subset \mathcal{U}_\varepsilon$ . Because  $\mathcal{L}$  is invariant by  $\mathcal{F}_\varepsilon$  and  $\mathcal{L} \setminus (\{|x| < \rho\} \cup \{|u| < |u_*|\})$  is compact we may reduce  $r'''$  to some  $r''$  in such a way that  $\rho\mathbb{S}^1 \times r''\mathbb{D} \subset \mathcal{U}_\varepsilon$  (flow-box argument), which settles the proof.

## 6. TEMPORAL NORMAL FORMS

This section is devoted to proving the temporal part of the Normalization Theorem and of the Uniqueness Theorem in the case  $\tau = 0$  (which particularly implies  $\mu_0 \notin \mathbb{R}_{\leq 0}$ ). Recall how in Section 4 we obtained formal normal forms. The time-component  $U$  of any unfolding in orbital normal form (2.4)

$$Z = U\mathcal{X}$$

can be written as

$$\frac{1}{U} = C + I,$$

where

$$I \in \text{im}(\mathcal{X} \cdot)$$

$$C \in \text{coker}(\mathcal{X} \cdot) \quad , \quad C(0,0) = \frac{1}{U(0,0)} \neq 0,$$

for a given (arbitrary for now) choice of  $\text{coker}(\mathcal{X} \cdot)$ , an algebraic supplementary in  $\mathbb{C}[[\varepsilon, x, y]]$  to the image  $\text{im}(\mathcal{X} \cdot)$  of the (formal) Lie derivative  $\mathcal{X} \cdot : \mathbb{C}[[\varepsilon, x, y]] \rightarrow \mathbb{C}[[\varepsilon, x, y]]$ . According to the discussion following Proposition 4.7,  $Z$  is (formally) conjugate to  $\frac{1}{C}\mathcal{X}$ .

We have shown in Lemma 4.8 that

$$\mathbb{C}[[\varepsilon, x, y]] = \text{im}(\mathcal{X} \cdot) \oplus \mathbb{C}[[\varepsilon]][x]_{<k},$$

or more precisely that the following sequence of  $\mathbb{C}[[\varepsilon]]$ -linear operators is exact:

$$(6.1) \quad 0 \longrightarrow \mathbb{C}[[\varepsilon]] \longrightarrow \mathbb{C}[[\varepsilon, x, y]] \xrightarrow{\mathcal{X} \cdot} \mathbb{C}[[\varepsilon, x, y]] \xrightarrow{\widehat{\Gamma}} \mathbb{C}[[\varepsilon]][x]_{\leq k} \longrightarrow 0,$$

where  $\widehat{\Gamma}$  maps  $G$  to the remainder of the Euclidean division of its partial function  $x \mapsto G(x, 0)$  by  $P$ . As a consequence we may take

$$\text{coker}(\mathcal{X} \cdot) := \mathbb{C}[[\varepsilon]][x]_{<k},$$

so that  $Z$  is formally conjugate to  $\frac{1}{\widehat{\Gamma}(\frac{1}{U})}\mathcal{X}$ .

*Remark 6.1.* The additional fact that

$$\widehat{\Gamma}\left(\frac{1}{U}\right) = \frac{1}{\widehat{\Gamma}(U)} + O(P)$$

finally implies that  $Z$  is formally conjugate to  $u\mathcal{X}$  where  $u := \widehat{\Gamma}(U)$ , as in the Formal Normalization Theorem. This is because one can write (for  $u_0(0) \neq 0$ )

$$\begin{aligned} \frac{1}{U(x, y)} &= \frac{1}{u(x) + O(P(x)) + O(y)} = \frac{1}{u(x)} \times \frac{1}{1 + O(P(x)) + O(y)} \\ &= \frac{1}{u(x)} + O(P(x)) + O(y). \end{aligned}$$

The previous argument still works for convergent power series, by replacing  $\mathbb{C}[[\varepsilon, x, y]]$  with  $\mathbb{C}\{\varepsilon, x, y\}$ : if we provide an explicit cokernel in  $\mathbb{C}\{\varepsilon, x, y\}$  of  $\mathcal{X} \cdot |_{\mathbb{C}\{\varepsilon, x, y\}}$  then we can describe an explicit family of temporal normal forms.

**Theorem 6.2.** *Assume  $\tau = 0$  (which particularly implies  $\mu_0 \notin \mathbb{R}_{\leq 0}$ ). Let an orbital normal form  $\mathcal{X}$  be given. It acts by directional derivative on the linear space  $\mathbb{C}\{\varepsilon, x, y\}$  in such a way that*

$$\mathbb{C}\{\varepsilon, x, y\} = \text{im}(\mathcal{X} \cdot) \oplus \mathbb{C}\{\varepsilon\}[x]_{\leq k} \oplus \text{Section}_k\{y\}.$$

(We refer to Section 3.1.2 for the definition of the functional spaces.)

*Remark 6.3.* The construction of the cokernel of  $\mathcal{X} \cdot$  is eventually performed for  $\varepsilon$  fixed. Therefore the theorem can also be specialized in the following way: for every  $\varepsilon \in (\mathbb{C}^k, 0)$  such that  $\mu_\varepsilon \notin \mathbb{R}_{\leq 0}$  and every disk  $D \supset P_\varepsilon^{-1}(0)$  not containing any root of  $1 + \mu_\varepsilon x^k$ , we have the  $\mathbb{C}$ -linear decomposition

$$\text{Holo}_c(D)\{y\} = \text{im}(\mathcal{X}_\varepsilon \cdot) \oplus \mathbb{C}[x]_{\leq k} \oplus xy\mathbb{C}[x]_{<k}\{y\}.$$

If  $\mu_\varepsilon \in \mathbb{R}_{\leq 0}$  a section of the cokernel is given by  $xP_\varepsilon^\tau y\mathbb{C}[x]_{<k}\{P_\varepsilon^\tau y\}$ .



The aim of this section is to prove this theorem but, before doing so, let us explain how it helps completing the proofs of the Normalization and Uniqueness Theorems. Every function  $U \in \mathbb{C}\{\varepsilon, x, y\}^\times$  can be written uniquely as

$$U = \frac{u}{1 + uG}$$

where  $\widehat{\mathcal{T}}(G) = 0$ , by simply taking  $u := \widehat{\mathcal{T}}(U)$  as in Remark 6.1. Then Theorem (6.2) allows decomposing  $G$  uniquely as

$$G = Q + I$$

with  $Q \in \text{Section}_k\{y\}$  and  $I \in \mathcal{X} \cdot \mathbb{C}\{\varepsilon, x, y\}$ , so that  $Z$  is analytically conjugate to some  $\frac{u}{1+uQ}\mathcal{X}$ , unique up to the action of linear transforms  $(x, y) \mapsto (x, cy)$  as expected (as follows from Uniqueness Theorem (2) which has been proved in the previous section). This yields Uniqueness Theorem (1).

**6.1. Reduction of the proof.** We must study the obstructions to solve analytically cohomological equations of the form

$$\mathcal{X} \cdot F = G \quad , \quad G \in \mathbb{C}\{\varepsilon, x, y\} \cap \ker \widehat{\mathcal{T}}.$$

First observe that this equation, restricted to the invariant line  $\{y = 0\}$ , is always satisfied by a holomorphic function  $f : x \mapsto F(x, 0)$  solving

$$f'(x) = \frac{G(x, 0)}{P(x)} \in \mathbb{C}\{\varepsilon, x\}.$$

By subtracting  $f$  from  $F$  and  $x \mapsto G(x, 0)$  from  $G$ , we may always assume without loss of generality that

$$G(x, 0) = F(x, 0) = 0,$$

i.e.  $G \in \mathbb{C}\{\varepsilon, x, y\}'$  as defined in Section 3.1.2.

Let

$$\Delta_k := \left\{ \varepsilon \in (\mathbb{C}^k, 0) : \#P_\varepsilon^{-1}(0) \leq k \right\}$$

be a germ at 0 of the discriminant hypersurface of  $P_\varepsilon$ , so that each open set  $(\mathbb{C}^k, 0) \setminus \Delta_k$  consists in generic values of the parameter for which  $P_\varepsilon$  has only simple roots. Proving Theorem 6.2 will require to work in the functional spaces

$$\mathcal{H}_\ell\{\mathbf{z}\} := \bigcup_{\mathcal{D}=(\mathbb{C}^n, 0)} \text{Holo}_c(\mathcal{E}_\ell \times \mathcal{D})' \quad , \quad \mathbf{z} := (z_1, \dots, z_n)$$

for some decomposition  $(\mathcal{E}_\ell)_\ell$  of  $(\mathbb{C}^k, 0) \setminus \Delta_k$  into finitely many (germs of) open cells as explained in Section 6.3. (We recall that the definition of the space  $\text{Holo}_c(\mathcal{D})'$  is given in Section 3.1.2.) We choose these spaces because of the next property.

**Lemma 6.4.**

$$\mathbb{C}\{\varepsilon, \mathbf{z}\}' = \bigcap_\ell \mathcal{H}_\ell\{\mathbf{z}\}.$$

(By the intersection on the right hand side we of course mean the functions who have an extension on the unions of the different domains.)

*Proof.* We certainly have

$$\mathbb{C}\{\varepsilon, \mathbf{z}\}' \subset \bigcap_{\ell} \mathcal{H}_{\ell}\{\mathbf{z}\}.$$

Conversely if  $f \in \bigcap_{\ell} \mathcal{H}_{\ell}\{\mathbf{z}\}$  then  $f$  defines a bounded, holomorphic function on  $((\mathbb{C}^k, 0) \setminus \Delta_k) \times (\mathbb{C}^n, 0)$ , which extends holomorphically to  $(\mathbb{C}^{k+n}, 0)$  according to Riemann's theorem on removable singularities.  $\square$

Working over a fixed cell germ  $\mathcal{E}_{\ell}$  is easy as compared to  $(\mathbb{C}^k, 0)$ .

**Proposition 6.5.** [40] *Let  $\mathcal{E}_{\ell}$  be a parameter cell. There exists  $\Upsilon_{\ell}$ , called the **period operator** over  $\mathcal{E}_{\ell}$ , such that the sequence of  $\text{Holo}_{\mathbb{C}}(\mathcal{E}_{\ell})$ -linear operators is exact:*

$$(6.2) \quad 0 \longrightarrow \text{Holo}_{\mathbb{C}}(\mathcal{E}_{\ell}) \longrightarrow \mathcal{H}_{\ell}\{x, y\} \xrightarrow{\mathcal{X} \cdot} \mathcal{H}_{\ell}\{x, y\} \xrightarrow{\Upsilon_{\ell}} \prod_{\mathbb{Z}/k\mathbb{Z}} \mathcal{H}_{\ell}\{h\}$$

where  $h$  is a one-dimensional variable (meant to take the values of a first integral).

The surjectivity of the period operator  $\Upsilon_{\ell}$  has not been established in the cited reference, but it would have followed from an immediate adaptation of the argument of [44, Lemma 3.4]. Here, though, we prove a stronger result by producing an explicit section to the period operator (Proposition 6.6 to come). The construction of the period operator over  $\mathcal{E}_{\ell}$  is explained in Section 6.2 below. It involves cutting up  $(\mathbb{C}^2, 0) \setminus (P_{\varepsilon}^{-1}(0) \times \{0\})$  into  $k$  open (bounded) spiraling sectors and building sectorial solutions of the cohomological equation. The period operator measures how much solutions on neighboring sectors disagree on intersections. Contrary to what would have made things easier

$$\Upsilon_{\ell}(\mathbb{C}\{\varepsilon, x, y\}') \neq \bigcap_p \prod_{\mathbb{Z}/k\mathbb{Z}} \mathcal{H}_p\{h\} = \prod_{\mathbb{Z}/k\mathbb{Z}} \mathbb{C}\{\varepsilon, h\}',$$

so that  $\Upsilon_{\ell}$  is neither onto nor into the natural candidate  $\prod_{\mathbb{Z}/k\mathbb{Z}} \mathbb{C}\{\varepsilon, h\}'$ . This situation differs drastically from the case  $\varepsilon = 0$ , and can be explained. It turns out that the variable  $h$  in the  $j^{\text{th}}$  factor of  $\prod_{j \in \mathbb{Z}/k\mathbb{Z}} \mathcal{H}_p\{h\}$  stands for values of the canonical first integral of  $\mathcal{X}$  on the  $j^{\text{th}}$  sector (see the discussion preceding Definition 6.10). Different sectorial decompositions for fixed  $\varepsilon$ , corresponding to different cells  $\mathcal{E}_{\ell}$  containing  $\varepsilon$ , lead to incommensurable sectorial dynamics: there is no correspondence between  $h$ -variables coming from different overlapping cells (see also Section 9). Therefore we need to relocate the obstructions in geometrical space  $(x, y)$ , by introducing a well-chosen section  $\mathfrak{S}_{\ell}$  of  $\Upsilon_{\ell}$ .

**Proposition 6.6.** *Let  $\mathcal{E}_{\ell}$  be a parameter cell and assume  $\tau = 0$  (which particularly implies  $\mu_0 \notin \mathbb{R}_{\leq 0}$ ). There exists a linear isomorphism*

$$\mathfrak{S}_{\ell} : \prod_{\mathbb{Z}/k\mathbb{Z}} \mathcal{H}_{\ell}\{h\} \longrightarrow x\mathcal{H}_{\ell}\{y\}[x]_{<k}$$

such that  $\Upsilon_{\ell} \circ \mathfrak{S}_{\ell} = \text{Id}$ . This particularly means we recover a cellular cokernel of  $\mathcal{X} \cdot$  as follows:

$$\mathcal{H}_{\ell}\{x, y\} = (\mathcal{X} \cdot \mathcal{H}_{\ell}\{x, y\}) \oplus x\mathcal{H}_{\ell}\{y\}[x]_{<k}.$$

This proposition is showed later in Section 6.4 using a refinement of the Cauchy-Heine transform, this time on unbounded sectors in the  $x$ -variable. Theorem 6.2 is proved once we establish the next gluing property, as done in Section 6.5.

**Proposition 6.7.** *For every parameter cells  $\mathcal{E}_\ell$  and  $\mathcal{E}_{\bar{\ell}}$  with non-empty intersection we have*

$$\mathfrak{S}_\ell \circ \mathfrak{T}_\ell = \mathfrak{S}_{\bar{\ell}} \circ \mathfrak{T}_{\bar{\ell}}$$

on  $\mathcal{H}_\ell\{x, y\} \cap \mathcal{H}_{\bar{\ell}}\{x, y\}$ .

From Lemma 6.4 we deduce the identity

$$\text{Section}_k\{y\} = \bigcap_{\ell} x\mathcal{H}_\ell\{y\}[x]_{<k},$$

hence the proposition actually provides us with a well-defined, surjective operator

$$(6.3) \quad \begin{aligned} \tilde{\mathfrak{K}} : \mathbb{C}\{\varepsilon, x, y\}' &\longrightarrow \text{Section}_k\{y\} \\ G &\longmapsto \mathfrak{S}_\ell(\mathfrak{T}_\ell(G)), \end{aligned}$$

whose kernel coincides with  $\mathcal{X} \cdot \mathbb{C}\{\varepsilon, x, y\}'$ , i.e. the sequence of  $\mathbb{C}\{\varepsilon\}$ -linear operators

$$(6.4) \quad 0 \longrightarrow \mathbb{C}\{\varepsilon, x, y\}' \xrightarrow{\mathcal{X}} \mathbb{C}\{\varepsilon, x, y\}' \xrightarrow{\tilde{\mathfrak{K}}} \text{Section}_k\{y\} \longrightarrow 0$$

is exact, as required to establish Theorem 6.2.

## 6.2. Cohomological equation and period operator.

**Theorem 6.8.** [40] *For every  $\rho > 0$  there exists:*

- a covering of  $(\mathbb{C}^k, 0) \setminus \Delta_k$  by finitely many open, contractible cells  $(\mathcal{E}_\ell)_\ell$ ,
- for every  $\varepsilon \in \mathcal{E}_\ell$ , a covering of

$$V_\varepsilon := \rho\mathbb{D} \setminus P_\varepsilon^{-1}(0)$$

into  $k$  open, contractible squid sectors

$$V_{\ell, \varepsilon}^j, \quad j \in \mathbb{Z}/k\mathbb{Z},$$

for which the following properties are satisfied. Recall that the closure of a subset  $A$  of a topological space is written  $\text{cl}(A)$ .

- (1) Each map  $\varepsilon \mapsto \text{cl}(V_{\ell, \varepsilon}^j)$  is continuous for the Hausdorff distance on compact sets and

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon \in \mathcal{E}_\ell}} \text{cl}(V_{\ell, \varepsilon}^j) = \text{cl}(V_0^j)$$

coincides with (the closure of) a usual sector of the limiting saddle-node, namely

$$V_0^j := \left\{ x : 0 < |x| < \rho, \arg x \in \left] -\frac{3\pi}{2k} + \eta + j\frac{2\pi}{k}, \frac{3\pi}{2k} - \eta + j\frac{2\pi}{k} \right[ \right\}$$

for some  $\eta \in \left] 0, \frac{\pi}{2k} \right[$ .

(2) We let

$$V_\ell^j := \bigcup_{\varepsilon \in \mathcal{E}_\ell} \{\varepsilon\} \times V_{\ell,\varepsilon}^j.$$

For every  $G \in \text{Holo}_c(\mathcal{E}_\ell \times \rho\mathbb{D} \times (\mathbb{C}, 0))'$  there exists a unique family  $(F_\ell^j)_{j \in \mathbb{Z}/k\mathbb{Z}}$  such that  $F_\ell^j$  is the unique solution of

$$\mathcal{X} \cdot F = G$$

in the space  $\text{Holo}_c(V_\ell^j \times (\mathbb{C}, 0))'$ . Moreover

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon_\ell}} F_{\ell,\varepsilon}^j = F_0^j$$

uniformly on compact sets of  $V_0^j \times (\mathbb{C}, 0)$ , where  $F_0^j$  is the canonical sectorial solution of the limiting cohomological equation [45].

(3) There exists a solution  $F \in \text{Holo}_c(\mathcal{E}_\ell \times \rho\mathbb{D} \times (\mathbb{C}, 0))$  of  $\mathcal{X} \cdot F = G$  if, and only if, for every  $\varepsilon \in \mathcal{E}_\ell$  and  $j \in \mathbb{Z}/k\mathbb{Z}$

$$F_{\ell,\varepsilon}^{j+1} = F_{\ell,\varepsilon}^j$$

on corresponding pairwise intersections of sectors  $V_{\ell,\varepsilon}^j \times (\mathbb{C}, 0)$ .

We provide details regarding how squid sectors and parameter cells are obtained in Section 6.3 below. The way sectorial solutions  $(F_\ell^j)_{j \in \mathbb{Z}/k\mathbb{Z}}$  are built is explained in [40, Section 7]. The third property encodes all we need to know in order to characterize algebraically the obstructions to solve analytically cohomological equations. It is, as usual, eventually a consequence of Riemann's theorem on removable singularities.

*Remark 6.9.*

- (1) A usual saddle-node sector is divided by rays separated by an angle slightly larger than  $\frac{\pi}{k}$ : allowing an extra  $\frac{\pi}{2k}$  on each side yields sectors of opening between  $\frac{\pi}{k}$  and  $\frac{2\pi}{k}$ . However we are in the particular case of a saddle-node with analytic center manifold, meaning that we need twice less sectors to describe the singularity structure. Hence the angle between the dividing rays can be taken as big as  $\frac{2\pi}{k}$ : allowing an extra  $\frac{\pi}{2k}$  on each side yields an opening between  $\frac{2\pi}{k}$  and  $\frac{3\pi}{k}$ .
- (2) A corollary to this theorem is the fact that any generic convergent unfolding is conjugate to its formal normal form over every region  $V_\ell^j \times (\mathbb{C}, 0)$ . In particular each  $\mathcal{X}$  is conjugate over  $V_{\ell,\varepsilon}^j \times (\mathbb{C}, 0)$  to  $\widehat{\mathcal{X}}$  by a fibered mapping

$$(x, y) \mapsto \left( x, y \exp N_{\ell,\varepsilon}^j(x, y) \right)$$

built upon a sectorial solution of

$$\mathcal{X}_\varepsilon \cdot N_{\ell,\varepsilon}^j = -R_\varepsilon$$

as in Proposition 4.7.

- (3) A really important property of the construction: it is performed [40, Section 7] for each *fixed*  $\varepsilon \in \mathcal{E}_\ell$ , the holomorphic / continuous dependence on  $\varepsilon$  of resulting objects being a by-product. This greatly simplifies understanding what happens on overlapping cells. This is also the reason why we omit to include the subscripts  $\ell$  and  $\varepsilon$  in the sequel, whenever doing so does not introduce ambiguity.

The period operator  $\Upsilon_\ell$  is obtained as follows. Fix  $\varepsilon \in \mathcal{E}_\ell$  and  $\rho > 0$  as in the previous theorem. Starting from any  $G \in \text{Holo}(\rho\mathbb{D} \times (\mathbb{C}, 0))'$  we can find a unique collection  $(F^j)_{j \in \mathbb{Z}/k\mathbb{Z}} \in \prod_j \text{Holo}(V^j \times (\mathbb{C}, 0))'$  of bounded functions solving the equation  $\mathcal{X} \cdot F = G$  over a squid sector. On each intersection we have  $\mathcal{X} \cdot F^{j+1} = G = \mathcal{X} \cdot F^j$  so that  $F^{j+1} - F^j$  is a first integral of  $\mathcal{X}$ . Therefore it factors as

$$(6.5) \quad F^{j+1} - F^j = T^j \circ H^j \quad , \quad T^j \in \mathbb{C}\{h\}'$$

where  $H^j = H_{\ell, \varepsilon}^j$  is the **canonical sectorial first integral** with connected fibers

$$(6.6) \quad H^j := \widehat{H}^j \exp N^j,$$

obtained from that of the formal normal form

$$(6.7) \quad \widehat{H}^j(x, y) := y \exp \int^x -\frac{1 + \mu z^k}{P(z)} dz$$

by composition with the sectorial normalization (Remark 6.9). We can fix once and for all a determination of each first integral  $\widehat{H}^j = \widehat{H}_\ell^j$  on  $V_\ell^j$  in such a way that

$$(6.8) \quad \widehat{H}^{j+1} = \widehat{H}^j \exp 2i\pi\mu/k$$

in  $V^{j,s}$ . The linear factor appearing on the right-hand side is here to accommodate the multivaluedness of  $\exp \int^x -\frac{1 + \mu z^k}{P(z)} dz = x^{-\mu} \times \text{holo}(x)$  near  $\infty$ , so that  $\widehat{H}^{j+k} = \widehat{H}^j$ .

**Definition 6.10.** Consider a parameter cell  $\mathcal{E}_\ell$  and  $\rho > 0$  as in Theorem 6.8. For  $G \in \text{Holo}_c(\mathcal{E}_\ell \times \rho\mathbb{D} \times (\mathbb{C}, 0))$  define the **period** of  $G$  with respect to  $\mathcal{X}$  as the  $k$ -tuple

$$\Upsilon_\ell(G) := \frac{1}{2i\pi} (T^j)_{j \in \mathbb{Z}/k\mathbb{Z}} \in \prod_{\mathbb{Z}/k\mathbb{Z}} \mathcal{H}_\ell\{h\}$$

where  $T_\ell^j := T^j$  is build as above in (6.5) for  $G := G_\varepsilon$  and  $\varepsilon \in \mathcal{E}_\ell$ . We define  $\Upsilon_\ell^j(G) := \frac{1}{2i\pi} T^j$  to be the  $j^{\text{th}}$  component of  $\Upsilon_\ell(G)$ .

*Remark 6.11.* Following up on Remark 6.9 (1), it seems that the period of  $R$  must play a special role regarding classification, since it measures the discrepancy between sectorial orbital conjugacies to the formal normal form  $\widehat{\mathcal{X}}$ . It is actually the case that the unfolded Martinet-Ramis modulus is linked to this period through the relationship

$$\psi_\ell^{j,s}(h) = h \exp\left(\frac{2i\pi\mu}{k} + \phi_\ell^{j,s}(h)\right) = h \exp\left(\frac{2i\pi\mu}{k} - \Upsilon_\ell^j(R)(h)\right).$$

A similar formula holds for the temporal modulus, namely  $f_\ell^{j,s} = \Upsilon_\ell^j\left(\frac{1}{U} - 1\right)$ . We refer to [40] for a more detailed discussion regarding these integral representations of the modulus of classification.

We sum up the relevant results needed in the sequel as a corollary to Theorem 6.8.

**Corollary 6.12.** *Pick  $\varepsilon \in (\mathbb{C}^k, 0) \setminus \Delta_k$  and  $\rho > 0$  such that  $P_\varepsilon^{-1}(0) \subset \rho\mathbb{D}$ , as well as some holomorphic function  $G \in \text{Holo}(\rho\mathbb{D} \times (\mathbb{C}, 0))'$ . The following assertions are equivalent.*

- (1) *There exists  $F \in \text{Holo}(\rho\mathbb{D} \times (\mathbb{C}, 0))'$  such that  $\mathcal{X}_\varepsilon \cdot F = G$ .*
- (2) *There exists  $\ell$  with  $\varepsilon \in \mathcal{E}_\ell$  such that*

$$\mathfrak{T}_\ell(G)_\varepsilon = 0.$$

- (3) *For all  $\ell$  with  $\varepsilon \in \mathcal{E}_\ell$  we have*

$$\mathfrak{T}_\ell(G)_\varepsilon = 0.$$

If moreover  $G \in \mathcal{H}_\ell\{x, y\}$  then

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \mathcal{E}_\ell}} \mathfrak{T}_\ell(G)_\varepsilon = \mathfrak{T}(G_0)$$

uniformly on  $(\mathbb{C}, 0)$ , where  $\mathfrak{T} : \mathbb{C}\{x, y\}' \rightarrow \prod_{\mathbb{Z}/k\mathbb{Z}} \mathbb{C}\{h\}'$  is the period operator of the limiting saddle-node [45].

*Proof.* For fixed  $\varepsilon$  and  $\ell$  Theorem 6.8 asserts the equivalence between existence of an analytic solution  $F$  of the cohomological equation  $X_\varepsilon \cdot F = G$  and vanishing of the period  $\mathfrak{T}_\ell(G)_\varepsilon$ . But the analyticity of  $F$  has nothing to do with the way the underlying squid sectors are cut, therefore  $\mathfrak{T}_\ell(G)_\varepsilon = 0$  as soon as  $\varepsilon \in \mathcal{E}_\ell$ .  $\square$

**6.3. Description of (unbounded) squid sectors and parameter cells.** To characterize the dynamics, describe the modulus of analytic classification and more generally build the period operator, we need to work over  $k$  open *squid sectors* in  $x$ -space covering either  $\rho\mathbb{D} \setminus P_\varepsilon^{-1}(0)$  (bounded case) or  $\mathbb{C} \setminus P_\varepsilon^{-1}(0)$  (unbounded case). Since  $\{y = 0\}$  is an analytic center manifold, each sector in this paper is the union of two consecutive sectors described originally in [40]. The cited reference also guarantees that it is sufficient to limit ourselves to the complement of the discriminant hypersurface  $\Delta_k \ni 0$  in parameter space. Although we only reach parameters for which all roots of  $P_\varepsilon$  are simple, the construction passes without difficulty to the limit  $\varepsilon \rightarrow \Delta_k$ . For  $\varepsilon \notin \Delta_k$  the squid sectors are attached to two or three roots. When  $\varepsilon \rightarrow 0$  they converge to the sectors used in the description of the Martinet-Ramis modulus for convergent saddle-nodes.

The singular points depend analytically on  $\varepsilon \in (\mathbb{C}^k, 0) \setminus \Delta_k$ . To obtain a family of squid sectors suiting our needs, we must ensure that the sectors vary continuously as  $\varepsilon$  does. This is however not achievable on a full pointed neighborhood of  $\Delta_k$  in parameter space, for reasons we are about to explain (we particularly refer to Remark 6.16). Even so, we manage to deal with all values of  $\varepsilon$  by covering the space  $(\mathbb{C}^k, 0)$  with the closure of finitely many contractible domains  $(\mathcal{E}_\ell)_\ell$  in  $\varepsilon$ -space, which we call *cells*, on which admissible families of squid sectors exist.

**6.3.1. The dynamics of  $\dot{x} = P_\varepsilon(x)$ .** Let us recall the main features of the vector field  $P_\varepsilon \frac{\partial}{\partial x}$ . When  $P_\varepsilon$  has distinct roots  $x_\varepsilon$ , each singular point  $x_\varepsilon$  has an associated nonzero eigenvalue  $\lambda_\varepsilon = P'_\varepsilon(x_\varepsilon)$ .

- The point  $x_\varepsilon$  is a **radial node** if  $\lambda_\varepsilon \in \mathbb{R}$ . It is attracting (*resp.* repelling) if  $\lambda_\varepsilon < 0$  (*resp.*  $\lambda_\varepsilon > 0$ ).
- The point  $x_\varepsilon$  is a **center** if  $\lambda_\varepsilon \in i\mathbb{R}$ .

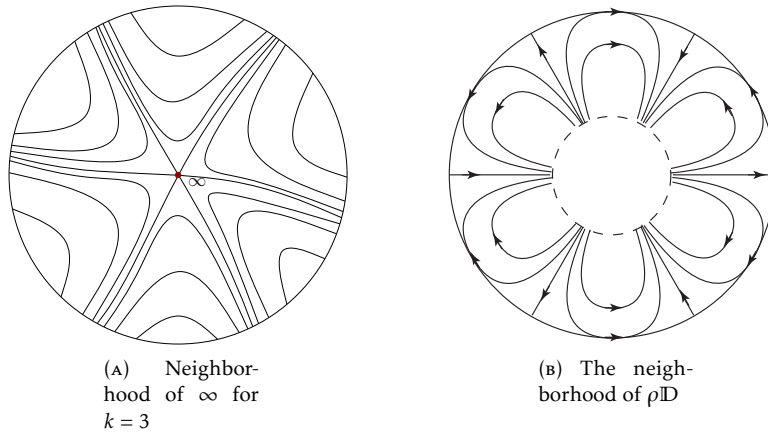


FIGURE 6.1. The separatrices of the pole at  $\infty$  and the petals along the boundary of the disk  $\rho\mathbb{D}$ .

- The point  $x_\varepsilon$  is a **focus** if  $\lambda_\varepsilon \notin \mathbb{R} \cup i\mathbb{R}$ . It is attracting (*resp.* repelling) if  $\Re(\lambda_\varepsilon) < 0$  (*resp.*  $\Re(\lambda_\varepsilon) > 0$ ).

The point  $x = \infty$  serves as an organizing center; indeed, the vector field  $P_\varepsilon \frac{\partial}{\partial x}$  has a pole of order  $k-1$  with  $2k$  separatrices at  $x = \infty$ , alternately attracting and repelling (see Figure 6.1), thus limiting  $2k$  saddle sectors at  $\infty$ . The system is structurally stable in the neighborhood of  $\infty$  for  $\varepsilon$  small. These saddle sectors give a phase portrait resembling  $2k$  petals along the boundary of any (sufficiently large) disk centered at the origin. The relationship between the magnitude of the parameter and the size of the disk will be detailed in Section 6.3.4.

The dynamics is completely determined by the separatrices of  $\infty$ . Because all roots of  $P_\varepsilon$  are simple, only two types of behavior occur.

- For generic values of  $\varepsilon$ , following the separatrices from  $\infty$  (either in backward or forward direction) one lands at repelling ( $t \rightarrow -\infty$ ) or attracting ( $t \rightarrow \infty$ ) singular points  $x_\varepsilon$  of focus or radial node type. In that case, each singular point is attached to at least one separatrix and the system is structurally stable among polynomial systems of degree  $k + 1$ . See Figure 6.2 for a phase portrait with generic  $\varepsilon$ .
- The sets of generic  $\varepsilon$  are separated by bifurcation hypersurfaces of (real) codimension 1. For these non-generic values of  $\varepsilon$  a homoclinic connection occurs between an attracting separatrix and a repelling separatrix of infinity: there is then a real integral curve flowing out from infinity in the  $x$ -plane and flowing back to infinity in finite time. For these bifurcation sets, the singular points  $P_\varepsilon^{-1}(0)$  can be split into two (nonempty) subsets  $I_1$  and  $I_2$  satisfying

$$(6.9) \quad \sum_{x \in I_m} \frac{1}{P'_\varepsilon(x)} \in i\mathbb{R}, \quad m = 1, 2.$$

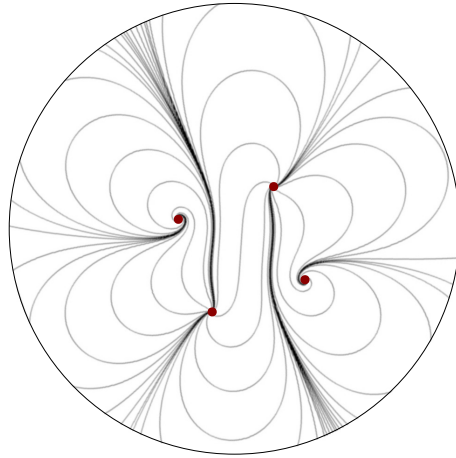


FIGURE 6.2. An example of a structurally stable real foliation induced by a complex polynomial vector field of degree 4 for  $\varepsilon$  in some  $K_\ell$ .

This can be seen by integrating the 1-form  $dt = \frac{dx}{P_\varepsilon(x)}$  along a homoclinic orbit, and evaluating residues. When  $I_m$  is a singleton, the corresponding singular point is a center.

The union of the  $2k$  separatrices of  $\infty$  is called the **separating graph** in [11] (see Figure 6.3(A)). It splits  $\mathbb{C}$  into  $k$  simply connected regions. In each of these regions we can draw a curve  $\gamma_j$  connecting the interior of a saddle sector at  $\infty$  to the interior of another saddle sector (see Figure 6.3(B)). There are exactly  $C_k = \frac{1}{k+1} \binom{2k}{k}$  ways of pairing two by two the saddle sectors of  $\infty$  by non-intersecting curves, thus providing a topological invariant for the vector field (we also refer to [10]).

6.3.2. *Rough description of the cells  $\mathcal{E}_\ell$  in parameter space.* The non-generic values of  $\varepsilon$  form a set of (real) codimension 1 which partitions a convenient neighborhood of 0 in parameter space (to be described slightly later) into  $C_k$  open regions  $K_\ell$ , corresponding to structurally stable vector fields with same topological invariant. In each region  $K_\ell$ , the topology of the phase portrait is completely determined by the topological way of attaching the  $2k$  separatrices to the  $k+1$  singular points. If  $x_\varepsilon$  is a root of  $P_\varepsilon$  (depending continuously on  $\varepsilon$ ) then  $\Re(P'_\varepsilon(x_\varepsilon))$  has a constant sign for all  $\varepsilon \in K_\ell$ . Each cell  $\mathcal{E}_\ell$  in parameter space will be a small enlargement of  $K_\ell$ , so that the cells cover the complement of  $\Delta_k$ .

A. DOUADY and P. SENTENAC have also provided a very clever parametrization of the domains  $K_\ell$ , thus showing that they are simply connected.

**Theorem 6.13.** [11] *Let  $K_\ell$  be a maximal domain corresponding to structurally stable vector fields. Then, there exists a biholomorphism  $\Phi_\ell : K_\ell \rightarrow \mathbb{H}^k$ , where  $\mathbb{H}$  is the upper half-plane. In particular,  $K_\ell$  is contractible. The set  $\Phi_\ell^{-1}((i\mathbb{R}_{\geq 0})^n)$ , which we call the **spine** of  $K_\ell$ , corresponds to polynomial vector fields with real eigenvalues at each singular point.*



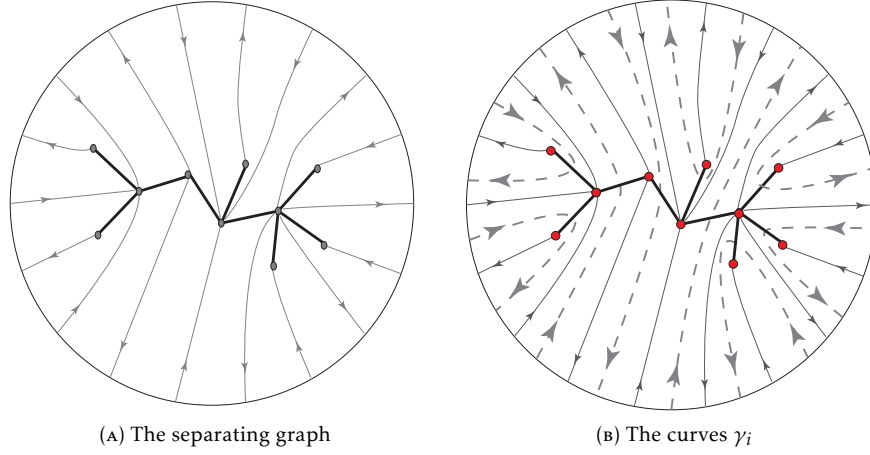


FIGURE 6.3. On the left, the separating graph formed by the separatrices landing at the singular points and flow lines (in bold) between the singular points. On the right, the curves  $\gamma_i$  (in dotted lines) used to calculate the  $\tau_i$ .

The map  $\Phi_\ell$  is defined as follows: let  $(\gamma_\varepsilon^j)_{j \in \{1, \dots, k\}}$  be  $k$  disjoint loops attached to  $\infty$  and pairing the saddle sectors of  $\infty$ , without intersecting the separating graph. Then  $\Phi_\ell(\varepsilon) = (\tau_\varepsilon^1, \dots, \tau_\varepsilon^k)$ , where

$$\tau_\varepsilon^j := \int_{\gamma_\varepsilon^j} dt = \int_{\gamma_\varepsilon^j} \frac{dx}{P_\varepsilon(x)},$$

the orientation of  $\gamma_\varepsilon^j$  being chosen so that  $\text{Im}(\tau_\varepsilon^j) > 0$ .

Since  $\tau_\varepsilon^j = 2i\pi \sum_{x \in I} \frac{1}{P'_\varepsilon(x)}$ , where  $I$  is the set of singular points in a domain bounded by  $\gamma_\varepsilon^j$ , the sum  $\tau_\varepsilon^j$  admits an analytic continuation outside  $K_\ell$ . In particular, when  $\varepsilon$  is a boundary point of  $K_\ell$  for which there is a homoclinic loop through  $\infty$ , some of the  $\tau_\varepsilon^j$  become real.

The cells have a very useful conic structure, induced by a multiplicative action of  $\mathbb{R}_{>0} \ni \lambda$  through linear rescaling

$$(6.10) \quad (\varepsilon_{k-1}, \dots, \varepsilon_0, x, t) \mapsto (\lambda^{-(k-2)}\varepsilon_{k-1}, \dots, \varepsilon_1, \lambda\varepsilon_0, \lambda x, \lambda^{-k}t),$$

as indeed the differential equation  $\dot{x} = P_\varepsilon(x)$  is invariant under this action. The cones we use are of the form

$$\{(\lambda^2\varepsilon_{k-1}, \dots, \lambda^k\varepsilon_1, \lambda^{k+1}\varepsilon_0) : \lambda \in ]0, 1[, \varepsilon \in K\},$$

where  $K$  is a relative domain within a sphere-like real hypersurface. This compact hypersurface takes the form  $\{\|\varepsilon\| = \text{cst}\}$  with

$$(6.11) \quad \|\varepsilon\| := \max\left(|\varepsilon_{k-1}|^{\frac{1}{2}}, \dots, |\varepsilon_1|^{\frac{1}{k}}, |\varepsilon_0|^{\frac{1}{k+1}}\right).$$

*Remark 6.14.* The expression 6.11 does not define a norm because the homogeneity axiom is not satisfied. However, if we take into account that the  $\varepsilon_j$  are the symmetric functions in the roots  $(x_0, \dots, x_k) \in \mathbb{C}^{k+1}$  of  $P_\varepsilon$ , it lifts to a norm on  $\mathbb{C}^{k+1}$ . Thus  $\|\varepsilon\|$  measures the magnitude of the parameter  $\varepsilon$  and the  $\|\bullet\|$ -balls form a fundamental basis of neighborhood of 0. In the following we consider only these parametric neighborhoods.

The regions  $K_\ell$  of structural stability defined above are cones of this form, and so will be their enlargements to cells  $\mathcal{E}_\ell$  covering the complement of  $\Delta_k$ . Also, when considering limits for  $\varepsilon \rightarrow 0$  it will be natural to consider limits for  $\lambda \rightarrow 0$  along orbits of the  $\mathbb{R}_{>0}$ -action

$$(6.12) \quad \left\{ (\lambda^2 \varepsilon_{k-1}, \dots, \lambda^k \varepsilon_1, \lambda^{k+1} \varepsilon_0) : \lambda \in ]0, 1[ \right\}.$$

6.3.3. *Saddle- and node-like singular points, admissible angles.* We want to stress that a singular point  $x_\varepsilon$  of  $\dot{x} = P_\varepsilon(x)$  with non-real eigenvalue  $\lambda = a + ib$  can be both attracting and repelling depending on how we approach it along logarithmic spirals. Making sense of this statement entails complexifying the time. Let us explain how.

- Consider the linear equation  $\dot{x} = \lambda x$ . Its solutions are  $x(t) = x_0 \exp(\lambda t)$ . Now, let us allow complex values of  $t$  along some slanted real line  $t = (c + id)T = T \exp(i\theta)$  in  $\mathbb{C}$ -space for some fixed  $c + id \in \mathbb{S}^1$ , with  $c > 0$  (corresponding to  $\theta \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$ ) and  $T \in \mathbb{R}$ . Then

$$x(t(T)) = x_0 \exp(((ac - bd) + i(ad + bc))T),$$

and  $\lim_{T \rightarrow +\infty} x(t(T)) = 0$  (resp.  $\lim_{T \rightarrow -\infty} x(t(T)) = 0$ ) when  $ac - bd < 0$  (resp.  $ac - bd > 0$ ).

- Since  $b \neq 0$ , it is always possible to find  $c_1 > 0, d_1$  (resp.  $c_2 > 0, d_2$ ) such that  $ac_1 - bd_1 > 0$  (resp.  $ac_2 - bd_2 < 0$ ).
- Note that approaching the singular point along a line  $t = (c + id)T$  in  $t$ -space is the same as approaching it along a real trajectory of the rotated equation  $\frac{dx}{dT} = \lambda \exp(i\theta) \times x$ . Such a trajectory is a logarithmic spiral.
- All these properties hold for the original system too, since the vector field  $P_\varepsilon \frac{\partial}{\partial x}$  is analytically linearizable near the singular point (Poincaré's theorem).

Locally around each root  $x_\varepsilon$  the squid sectors will coincide with domains bounded by asymptotic logarithmic spirals, given by trajectories of rotated vector fields  $\exp(i\theta_\varepsilon) P_\varepsilon \frac{\partial}{\partial x}$ . The angular function  $(\varepsilon, x) \in \mathcal{E}_\ell \times \mathbb{C} \mapsto \theta_\varepsilon(x) \in ]-\frac{\pi}{4}, \frac{\pi}{4}[$  will be piecewise constant and zero outside a neighborhood of  $\partial K_\ell$ , and for  $x$  far from the singular points.

**Definition 6.15.** Let  $\mathcal{E}$  be a domain in the complement of  $\Delta_k$ .

- (1) An **admissible angle** on  $\mathcal{E}$  is a piecewise constant function  $\theta : \mathcal{E} \times \mathbb{C} \rightarrow ]-\frac{\pi}{4}, \frac{\pi}{4}[$  such that for any analytic family of roots  $(x_\varepsilon)_{\varepsilon \in \mathcal{E}}$  of  $(P_\varepsilon)_{\varepsilon \in \mathcal{E}}$ , the function  $\varepsilon \in \mathcal{E} \mapsto \Re(P'_\varepsilon(x_\varepsilon) \exp(i\theta_\varepsilon(x_\varepsilon)))$  keeps a constant sign. In the following we use the notation

$$(6.13) \quad \vartheta := \exp(i\theta).$$

- (2) We say that an analytic family  $(x_\varepsilon)_{\varepsilon \in \mathcal{E}}$  of singular points of  $\dot{x} = P_\varepsilon(x)$  is of **node type** on  $\mathcal{E}$  if there exists an admissible angle such that

$$\Re(P'_\varepsilon(x_\varepsilon) \vartheta_\varepsilon(x_\varepsilon)) > 0 \quad (\forall \varepsilon \in \mathcal{E})$$

and of **saddle type** on  $\mathcal{E}$  if

$$\Re(P'_\varepsilon(x_\varepsilon) \vartheta_\varepsilon(x_\varepsilon)) < 0 \quad (\forall \varepsilon \in \mathcal{E}).$$

We use the notation  $(x_\varepsilon^n)_\varepsilon$  (*resp.*  $(x_\varepsilon^s)_\varepsilon$ ) for a family of roots of node (*resp.* saddle) type on the domain  $\mathcal{E}$ .

*Remark 6.16.* (1) The cells  $\mathcal{E}_\ell$  in parameter space will be small contractible enlargements of the cones  $K_\ell$ , on which there exist admissible angles. Additional constraints will be demanded to these angular functions in order to guarantee that the cells and sectors meet all technical requirements.

- (2) The choice of  $\frac{\pi}{4}$  for an upper bound in the size of an admissible angle  $\theta$  is arbitrary as any bound  $\alpha \in ]0, \frac{\pi}{2}[$  would do. However the larger  $\alpha$ , the smaller the bound  $\rho$  on  $\|\varepsilon\|$ . Indeed we approach each singular point along a trajectory of some vector field  $\vartheta_\varepsilon(x)P_\varepsilon(x)$ . When  $\theta$  is large and the singular points are not far enough from  $r\mathbb{S}^1$ , the trajectory follows wide spirals and may escape  $r\mathbb{D}$  before landing at the singular point. An “absolute” (*i.e.* independent of the bound  $\alpha$ ) necessary condition for the existence of an admissible angle such that  $(x_\varepsilon)_\varepsilon$  has node- (*resp.* saddle-) type on a neighborhood of  $K_\ell$  is that  $P'_\varepsilon(x_\varepsilon) \notin \mathbb{R}_{<0}$  (*resp.*  $P'_\varepsilon(x_\varepsilon) \notin \mathbb{R}_{>0}$ ) for  $\varepsilon \in K_\ell$ . Therefore no admissible angle exists on a full pointed neighborhood of  $\Delta_k$ .
- (3) We can illustrate on the formal normal form why admissible angles are of paramount importance. In the flow system of  $\vartheta\widehat{X}$  for real time

$$\begin{cases} \dot{x} &= \vartheta_\varepsilon(x)P_\varepsilon(x) \\ \dot{y} &= \vartheta_\varepsilon(x)y(1 + \mu_\varepsilon x^k) \end{cases}$$

the variation of the modulus  $\phi := |y|^2 = y\bar{y}$  of a solution follows the law

$$\dot{\phi} = 2\phi \Re(\vartheta_\varepsilon(x)(1 + \mu_\varepsilon x^k)).$$

Close enough to the singularity  $(x_\varepsilon, 0)$  all non-zero solutions therefore accumulate backwards exponentially fast on  $(x_\varepsilon, 0)$  if  $x_\varepsilon = x_\varepsilon^n$  is of node type or, on the contrary, diverge forwards exponentially fast for a saddle type root  $x_\varepsilon^s$ . This behavior mimics that of a node / saddle planar foliation near a point with real residue  $\vartheta_\varepsilon(x_\varepsilon)P'_\varepsilon(x_\varepsilon)$ . This dynamical dichotomy is the cornerstone of the construction of the period operator (the modulus of classification) in [40].

**6.3.4. Size of sectors and of the parameter.** The diameter  $\rho$  of the bounded part of the sectors is such that  $|1 + \mu x^k| > \frac{1}{2}$  when  $|x| < \rho$ . Note that the roots of  $P_\varepsilon$  all lie within  $\sqrt{k}\|\varepsilon\| \text{cl}(\mathbb{D})$ . Indeed it suffices to show that if  $|x| > \sqrt{k}\|\varepsilon\|$ , then  $P_\varepsilon(x) \neq 0$ . On the one hand  $|x^{k+1}| > k^{\frac{k+1}{2}} \|\varepsilon\|^{k+1}$ . On the other hand

$$\left| \sum_{j=0}^{k-1} \varepsilon_j x^j \right| \leq \|\varepsilon\|^{k+1} \sum_{j=0}^{k-1} k^{\frac{j}{2}} \leq \|\varepsilon\|^{k+1} k^{\frac{k+1}{2}}.$$

In fact outside the disk  $\sqrt{k}\|\varepsilon\|\text{cl}(\mathbb{D})$  the trajectories of  $P_\varepsilon \frac{\partial}{\partial x}$  are petals as depicted in Figure 6.1 (B). Set

$$(6.14) \quad \rho_\varepsilon := 2\sqrt{k}\|\varepsilon\|.$$

Then we choose  $\varepsilon$  sufficiently small so that  $\rho_\varepsilon < \frac{\rho}{2}$ . Later in Lemma 6.23 we will further reduce  $\rho$  and  $\varepsilon$  so that

$$(6.15) \quad |\mu x^k| + 2\rho|P''(x)| \leq \frac{3}{4}$$

for  $|x| < \rho$ .

6.3.5. *The ideal construction of sectors.* Let us now choose a cone  $K_\ell$  and describe the corresponding open squid sectors  $(V_{\ell,\varepsilon}^j)_{j \in \mathbb{Z}/k\mathbb{Z}}$  covering  $\rho\mathbb{D} \setminus P_\varepsilon^{-1}(0)$ . On a “large” neighborhood of the spine of  $K_\ell$  (to be made precise below), *i.e.* not too close to the boundary of  $K_\ell$ , they are limited by real trajectories of  $P_\varepsilon \frac{\partial}{\partial x}$  chosen as follows (see also Figure 6.4).

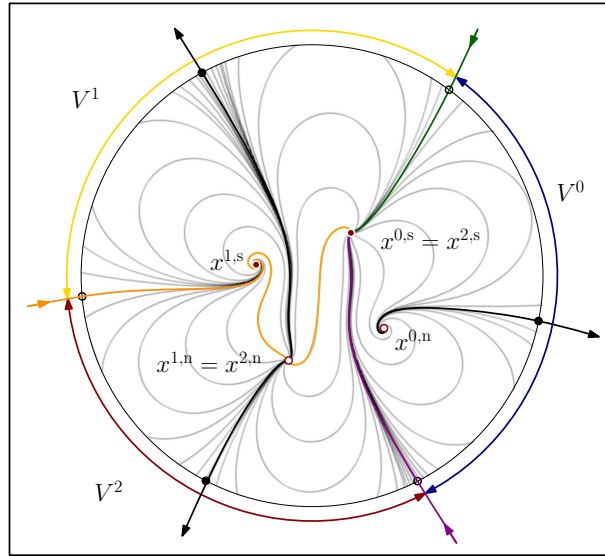


FIGURE 6.4. Curves involved in the ideal decomposition. Stable separatrices at  $\infty$  in black, unstable ones in green, orange and purple.

- (1) The unstable separatrices of  $P_\varepsilon \frac{\partial}{\partial x}$  through  $\infty$  split  $\rho\mathbb{S}^1$  into  $k$  arcs. We enlarge slightly these arcs to an open covering of the circle. Each arc is one piece of the boundary of a sector  $V_\varepsilon^j$ .
- (2) Two other pieces of the boundary of  $V_\varepsilon^j$  are given by the forward trajectories of  $P_\varepsilon \frac{\partial}{\partial x}$  through the endpoints of the arc, which land in singular points  $x^{j-1,s}$  and  $x^{j,s}$  (not necessarily distinct) such that  $\Re(P'_\varepsilon(x^{j,s})) < 0$  (*i.e.* the roots are of saddle type). These trajectories spiral as soon as  $\Im(P'_\varepsilon(x^{j,s})) \neq 0$  (which is the generic situation).

- (3) Suppose  $x^{j,s} \neq x^{j-1,s}$ . For a given boundary arc of  $\rho\mathbb{S}^1$  there exists one stable separatrix through  $\infty$  which cuts it at one point and lands at root  $x^{j,n}$  of node type. This singular point belongs to the boundary of  $V_\varepsilon^j$ . The last two pieces of the boundary are two complete trajectories of  $P_\varepsilon \frac{\partial}{\partial x}$ , one joining  $x^{j,n}$  to  $x^{j-1,s}$  and the other joining  $x^{j,n}$  to  $x^{j,s}$ . These trajectories are chosen in such a way that  $(V_\varepsilon^j)_{j \in \mathbb{Z}/k\mathbb{Z}}$  cover  $\rho\mathbb{D} \setminus P_\varepsilon^{-1}(0)$ .
- (4) When  $x^{j,s} = x^{j-1,s}$ , we introduce two trajectories between  $x^{j,s}$  and  $x^{j,n}$ , thus introducing a self-intersection of  $V_\varepsilon^j$ . This is motivated by the need of dealing with ramified functions near  $x^{j,n}$ . See Figure 6.13 (A).

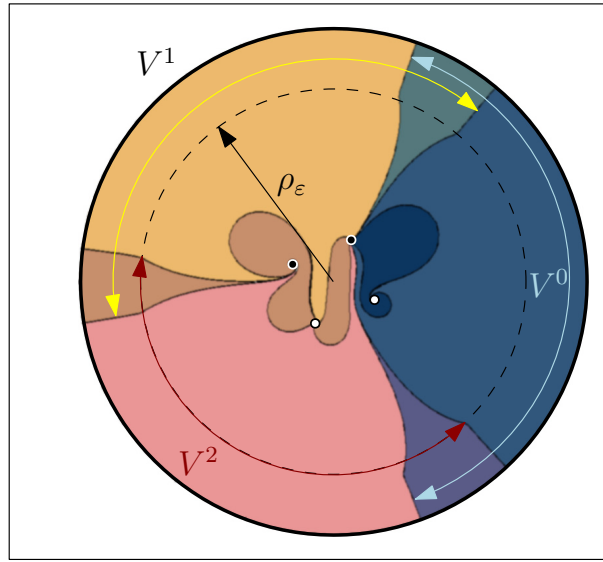


FIGURE 6.5. Decomposition into bounded, overlapping squid sectors induced by the flow depicted in Figures 6.2 and 6.4.

6.3.6. *The problem with the ideal construction of sectors.* Of course the ideal construction will not always work. It can fail for the following reasons. For a set  $I \subset P_\varepsilon^{-1}(0)$  and  $\varepsilon \notin \Delta_k$  define

$$(6.16) \quad \nu_\varepsilon(I) := \sum_{x \in I} \frac{1}{P'_\varepsilon(x)}.$$

- The first one is when  $\varepsilon$  is not generic: the separatrices may form a homoclinic loop preventing them to land at singular points. A homoclinic loop  $\gamma$  through  $\infty$  partitions the set of singular points  $P_\varepsilon^{-1}(0)$  into  $I$  and  $I'$  such that

$$(6.17) \quad \Re(\nu_\varepsilon(I)) = \Re(\nu_\varepsilon(I')) = 0.$$

- When  $\varepsilon$  is close to a hypersurface corresponding to a homoclinic loop, it can also occur that the trajectories through the endpoints of the arc first exit the disk  $\rho\mathbb{D}$  before landing at a singular point.

- When  $\varepsilon$  crosses a hypersurface corresponding to a homoclinic loop, then  $\Re(P'_\varepsilon(x_\varepsilon))$  can change sign, thus preventing the above construction to be continuous in  $\varepsilon \in \mathcal{E}_\ell$ .
- As  $\varepsilon$  approaches 0 (or, more generally,  $\Delta_k$ ) we would like the sectors to converge to usual sectors associated to saddle-node singularities.

6.3.7. *The remedy in the construction of sectors.* The remedy to all these problems is the same. We want to keep the above picture all over the cell  $\mathcal{E}_\ell$  and we want the cells to cover the complement of  $\Delta_k$ . The boundary of  $K_\ell$  is composed of real hypersurfaces corresponding to homoclinic loop bifurcations. On each such hypersurface we have (6.17) for some  $I$ , while on  $K_\ell$  the real part of the corresponding  $v_\varepsilon(I)$  has a fixed sign and so does  $\text{Im}(\tau_j)$ . But we have seen in Section 6.3.3 that this is not an obstruction for having the points remaining of node- or saddle-type: we just need to be sufficiently careful on how we approach them, by adjusting the spiraling of the sectors. In practice, this boils down to replacing the piece of a trajectory of  $P_\varepsilon \frac{\partial}{\partial x}$  inside the disk  $\rho_\varepsilon \mathbb{D}$  by the piece of a trajectory of  $\exp(i\theta) \times P_\varepsilon \frac{\partial}{\partial x}$  for some admissible angle  $\theta$  as in Definition 6.15 (with some additional specifications).

**Proposition 6.17.** *Being given  $\delta \in ]0, \frac{\pi}{4}[$  and  $\rho > 0$ , there exists  $\eta > 0$  such that the following properties hold.*

- (1) Let  $\mathcal{E}_\ell$  be the open set in  $\{\|\varepsilon\| < \eta\} \setminus \Delta_k$  defined by the next conditions:
  - for each homoclinic-loop bifurcation hypersurface on the boundary of  $K_\ell$ , separating the singular points in two nonempty groups  $I \cup I'$  as in (6.17), we have
 
$$\begin{cases} \arg v_\varepsilon(I) \in ]-\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta[ & \text{if } \Re(v_\varepsilon(I)) > 0 \text{ on } K_\ell, \\ \arg v_\varepsilon(I) \in ]\frac{\pi}{2} - \delta, \frac{3\pi}{2} + \delta[ & \text{if } \Re(v_\varepsilon(I)) < 0 \text{ on } K_\ell, \end{cases}$$
  - for the  $\tau_\varepsilon^j$  defined in Theorem 6.13 we have

$$\arg \tau_\varepsilon^j \in ]-\delta, \pi + \delta[ \quad \text{for all } j \in \{1, \dots, k\}.$$

Then  $\mathcal{E}_\ell$  is a conic contractible neighborhood of  $K_\ell$  and  $\bigcup_\ell \mathcal{E}_\ell = \{\|\varepsilon\| < \eta\} \setminus \Delta_k$ .

- (2) There exists an admissible angle  $\theta$  (corresponding to a direction  $\vartheta = \exp(i\theta)$ ) on  $\mathcal{E}_\ell$  such that for each homoclinic-loop bifurcation hypersurface on the boundary of  $K_\ell$ , separating the singular points in two nonempty groups  $I \cup I'$ , we have

$$(6.18) \quad \begin{cases} \arg \sum_{x \in I} \frac{1}{\vartheta_\varepsilon(x) P'_\varepsilon(x)} \in ]-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta[ & \text{if } \Re(v_\varepsilon(I)) > 0 \text{ on } K_\ell, \\ \arg \sum_{x \in I} \frac{1}{\vartheta_\varepsilon(x) P'_\varepsilon(x)} \in ]\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta[ & \text{if } \Re(v_\varepsilon(I)) < 0 \text{ on } K_\ell. \end{cases}$$

- (3) Any trajectory of  $\vartheta_\varepsilon P_\varepsilon \frac{\partial}{\partial x}$ , starting from a point of  $\rho \mathbb{S}^1$ , does not exit the disk  $\rho \mathbb{D}$  before landing at a singular point.

*Proof.*

- (1) This is clear.
- (2) We build the angle  $\theta$  (piece-wise constant in  $x$ ) in such a way that  $\vartheta_\varepsilon = 1$  when  $\varepsilon$  is on the spine of  $K_\ell$ . When we approach a component of  $\partial K_\ell$  corresponding to a homoclinic loop separating the roots of  $P_\varepsilon$  as  $I \cup I'$ , we can rotate the vector field by an angle  $|\theta| \leq 2\delta < \frac{\pi}{4}$  so that  $\arg \sum_{x \in I} \frac{1}{\vartheta_\varepsilon(x) P'_\varepsilon(x)}$  belongs to the given interval.

- (3) A more precise quantitative description of the sectors is needed to show that the magnitude of  $\rho$  (in  $x$ -space) together with the choice of  $\delta$  give constraints on the size  $\eta$  of the  $\|\bullet\|$ -ball in  $\varepsilon$ -space, and that taking  $|\theta|$  large enough is sufficient to secure the conclusion. All this is done in the time coordinate  $t = \int \frac{dx}{P_\varepsilon(x)}$ . We come back to this below in Section 6.3.9.

□

**Definition 6.18.**

- (1) The contractile, conic domain  $\mathcal{E}_\ell$  given by the previous proposition is called a **cell** in parameter space.
- (2) The  $k$  domains in  $x$ -space built like ideal sectors but bounded by trajectories of  $\vartheta_\varepsilon P_\varepsilon \frac{\partial}{\partial x}$  instead of  $P_\varepsilon \frac{\partial}{\partial x}$  are called **squid sectors**.

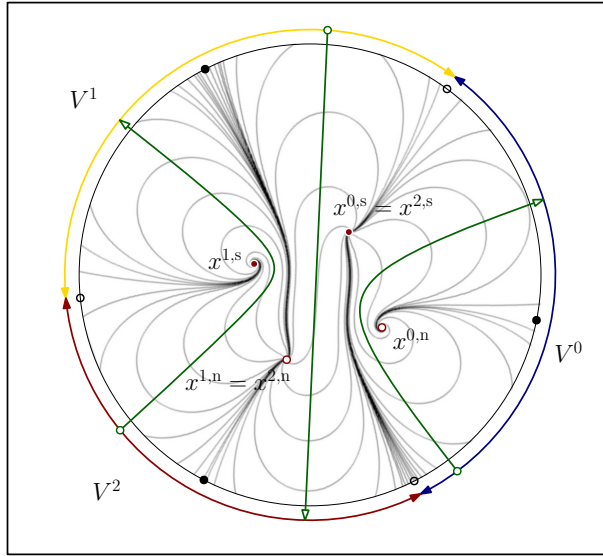


FIGURE 6.6. Construction of the non-crossing permutation  $\sigma$ ;

here  $\sigma = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ .

6.3.8. *Pairing sectors.*

**Definition 6.19.** Recall that a *non-crossing permutation*  $\sigma \in \mathfrak{S}_k$  is a permutation such that if  $p_0, \dots, p_{k-1}$  are circularly ordered points on a circle, there exist pairwise non-intersecting curves within the inscribed disc joining  $p_j$  and  $p_{\sigma(j)}$  for all  $j$ .

- (1) There exists a (non-crossing) permutation  $\sigma = \sigma_\varepsilon$  on  $\{0, \dots, k-1\}$  yielding a pairing of the sector  $V_\varepsilon^j$  with  $V_\varepsilon^{\sigma(j)}$  (see Figure 6.6) in the following way. If the sector  $V_\varepsilon^j$  shares its vertices  $x^{j-1,s}$  and  $x^{j,n}$  with a distinct sector  $V_\varepsilon^{j'}$ , then we define  $\sigma(j) := j'$ . Otherwise we let  $\sigma(j) = j$ .
- (2) The squid sector  $V_\varepsilon^j$  is **introvert** if  $\sigma(j) = j$ , and **extrovert** otherwise (see Figure 6.7).

The permutation  $\sigma$  is a complete topological invariant [11, 2] for structurally stable vector field  $P \frac{\partial}{\partial x}$  (i.e. for generic  $\varepsilon$ ) and any non-crossing permutation can be realized in this way. In particular  $\varepsilon \mapsto \sigma_\varepsilon$  is constant on the conic domains  $K_\ell$ .

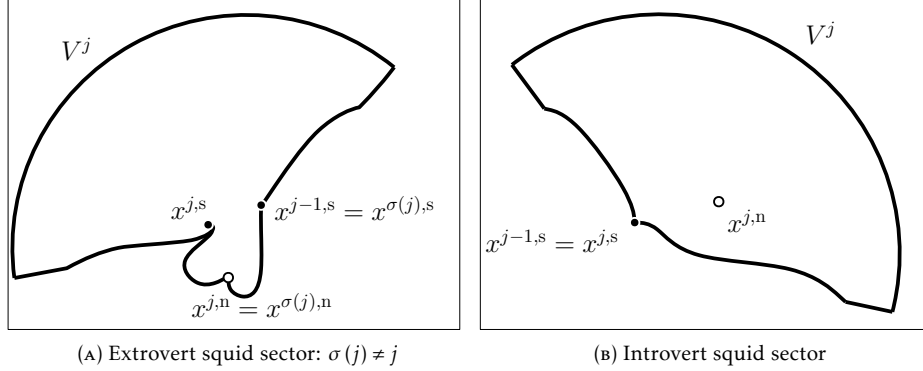


FIGURE 6.7. The two kinds of a bounded squid sector for  $k > 1$ .

6.3.9. *Practical description and quantitative estimates.* Here we end the proof of Proposition 6.17. As discussed earlier, finding an admissible angular function is equivalent to finding suitable piecewise affine real curves in the complex time coordinates. Studying  $\dot{x} = P_\varepsilon(x)$  for complex values of the time  $t$  is the natural point of view taken by [11, 2]. In that setting we could view the whole  $x$ -line as a single complex trajectory of the flow of  $P_\varepsilon \frac{\partial}{\partial x}$ . Although one might consequently try to parametrize points in the  $x$ -variable by values of the time  $t(x) \in \mathbb{C}$  this is too simplistic: the time function is multivalued at  $\infty$ . Nonetheless, the idea is very powerful and fruitful if we limit ourselves to simply connected domains in time space. Let us define the time function by

$$t(x) := \int_{\infty}^x \frac{dz}{P_\varepsilon(z)}.$$

When  $\varepsilon$  is generic we obtain

$$t(x) = \sum_{x_\varepsilon \in P_\varepsilon^{-1}(0)} \frac{1}{P'_\varepsilon(x_\varepsilon)} \log(x - x_\varepsilon).$$

To understand the image of the complement in  $\mathbb{C}$  of  $\rho\mathbb{D}$  under this map, it is first necessary to describe the  $k$ -sheeted Riemann surface over  $\mathbb{C}$ , on which the map  $t$  is defined. Since the integral starts at  $x = \infty$  then  $t(\infty) = 0$ . We have to remember that  $\infty$  is a pole of order  $k - 1$  of the vector field over which the time function has locally the form  $t = -\frac{1}{kx^k}$ , which means that it is a ramification point. Moreover,  $\infty$  can be reached in finite time and, for  $\varepsilon \neq 0$ , there are *periods*, which correspond to the time to go from  $\infty$  to  $\infty$  along a path circling some singular points. This means that the time function is multi-valued and there are an infinite number of ramification points, all corresponding to images of  $\infty$ . The distance between two images on one sheet is a period of a loop around singular points. These periods are all greater than some  $C\|\varepsilon\|$  (see Lemma 6.20 below).



The image of the complement in  $\mathbb{C}$  of  $\rho\mathbb{D}$  under the map  $t$  is therefore, for small  $\varepsilon$ , a union of holes (topological disks) of approximate radius  $\frac{1}{k\rho^k}$  in the  $k$ -sheeted Riemann surface (Figure 6.8) over  $\mathbb{C}$ , with one central hole around 0. The ramifications occur at the images of  $\infty$ . Each hole contains an image of  $\infty$  by the multivalued continuation of  $t$ . A half turn around the central hole corresponds to an angle of  $\frac{\pi}{k}$  on  $\partial\mathbb{D}_\rho$  (or to a saddle sector of  $\infty$ ). Hence one  $\tau_j$  of Theorem 6.13 is associated to each half turn, thus pairing the half turns two by two. Since  $\tau_j$  is a period in  $t$ -space, it is a distance between centers of holes and, on each half sheet, the next hole is obtained by translating the current hole by  $\tau_j$ .

**Lemma 6.20.** *There exists  $C > 0$ , depending only on  $k$ , such that*

$$|\tau_j| > C \|\varepsilon\|^{-k}.$$

*Proof.* It suffices to show that there exists  $C > 0$  such that  $|\tau_j| > C$  when  $\|\varepsilon\| = 1$ , and then to use the rescaling (6.10). This is done as follows. Changing the time  $t \mapsto t' := e^{-i\arg \tau_j} t$ , then  $\tau'_j = e^{-i\arg \tau_j} \tau_j$  is the time along a homoclinic loop between two separatrices of  $\infty$  for the transformed vector field. In Section 6.3.4, all roots have been shown to belong to  $\sqrt{k}\text{cl}(\mathbb{D})$ . The time  $\tau'_j$  is then larger than twice the minimum time to go from  $\infty$  to  $\{|x| = 2k\}$ , and this minimum is positive on the compact set  $\|\varepsilon\| = 1$ .  $\square$

Let us first describe what happens on the spine of the cell. There, holes are aligned vertically (the  $\tau_j$  are pure imaginary) and each sector (which is an ideal sector) corresponds to a horizontal strip as in Figure 6.8. If we want to cover  $\rho\mathbb{D} \setminus \{P_\varepsilon^{-1}(0)\}$  then we should cover a little more than a full turn around one hole. The width of the strip should be a little over  $\frac{\tau_j}{2}$  on the top side and over  $\frac{\tau_{j+1}}{2}$  on the bottom side. When moving to  $t$ -space the singular points have been sent to  $\infty$ , to the left (*resp.* right) for the singular points of node (*resp.* saddle) type. In such a picture we see the connected parts of the intersections of two consecutive sectors that go to the boundary.

The internal intersection parts (that we later call gate parts) can only be seen by using the periodicity of  $t$ . There are similar half strips on the  $\sigma(j)$ -th sheet, with a hole at a distance  $\tau_j$  and on the  $\sigma(j+1)$ -th sheet, with a hole at a distance  $\tau_{j+1}$ . Their translations by the corresponding period  $\tau_j$  and  $-\tau_{j+1}$  brings them on the  $j$ -th sheet where they intersect the initial strip (Figure 6.8).

If we now move away from the spine of  $K_\varepsilon$ , then two things happen.

- On the one hand, the  $\tau_j$  bend. When they approach the real line (horizontal direction), then it is no more possible to pass a horizontal strip because the holes block the way: the remedy is to slant the strip so that it avoids the hole altogether.
- Also, in the  $t$ -space, each singular point  $x_\varepsilon$  turns, since it is located at infinity in the direction of  $-\frac{1}{P'_\varepsilon(x_\varepsilon)}$ . An infinite half-strip in the direction  $\vartheta = \exp(i\theta)$  can only be sent to a sector with vertex at  $x_\varepsilon$  if

$$(6.19) \quad \Re\left(-\frac{\vartheta}{P'_\varepsilon(x_\varepsilon)}\right) > 0,$$

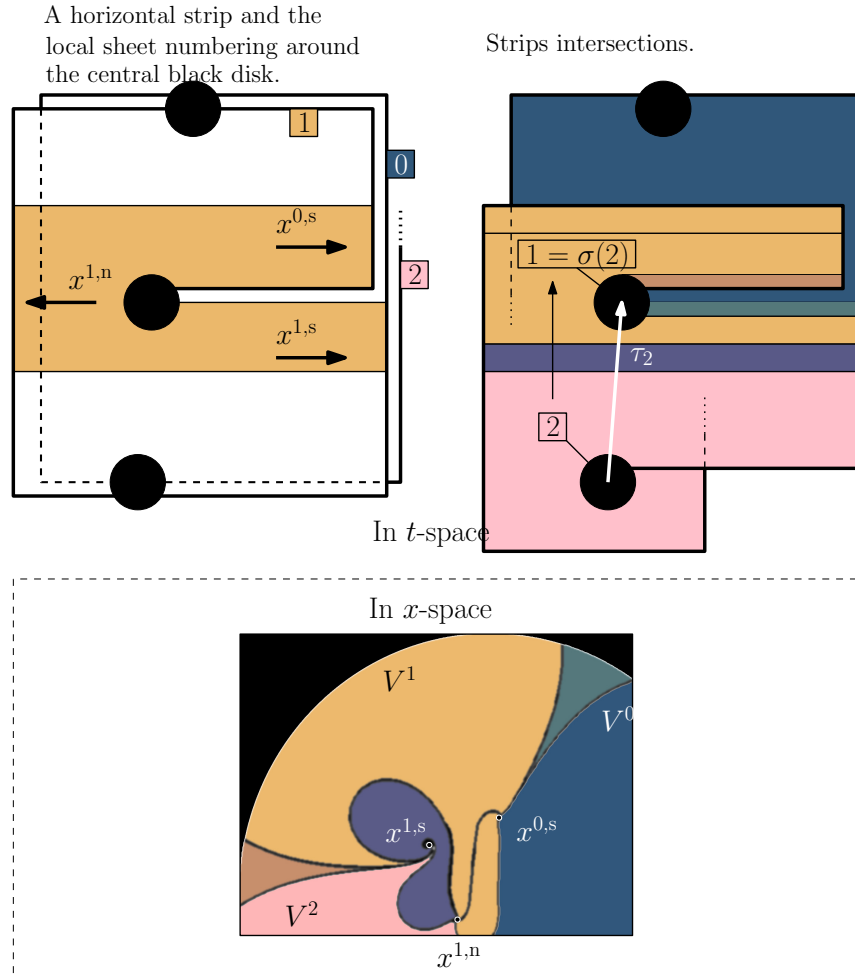


FIGURE 6.8. The images of these horizontal strips in  $t$ -space are sectors  $V_\epsilon^j$  in  $x$ -space.

(corresponding to the scalar product of  $-\frac{1}{P'_\epsilon(x_\epsilon)}$  and  $\vartheta$  being positive). This forces giving an angle to the strip in the infinite end of the half-strip approaching a singular point.

The choice of  $\delta$  in (6.18) guarantees that  $\tau_j$  cannot turn of an angle larger than  $\frac{\pi}{2} + \delta$ . The size of the holes is of the order of  $\frac{1}{k\rho^k}$ , which is very small compared to the  $\tau_j$  and  $\frac{1}{|P'_\epsilon(x_\epsilon)|}$  if  $\epsilon$  is sufficiently small.

Now the strip has three infinite ends, a wide one on the left side attached to a point of node type  $x_\epsilon^{j,n}$ , and two thinner ones attached to  $x_\epsilon^{j-1,s}$  and  $x_\epsilon^{j,s}$ . The slope  $\vartheta_\epsilon$  for each infinite end should be chosen so that (6.19) be satisfied for the singular point corresponding to that end of the strip.

This is how it is done. In the ideal situation the curves  $\gamma_j$ , used to pair the saddle sectors (permutation  $\sigma$ ) and to define the  $\tau_j$ , split the disk into  $k + 1$  regions, each

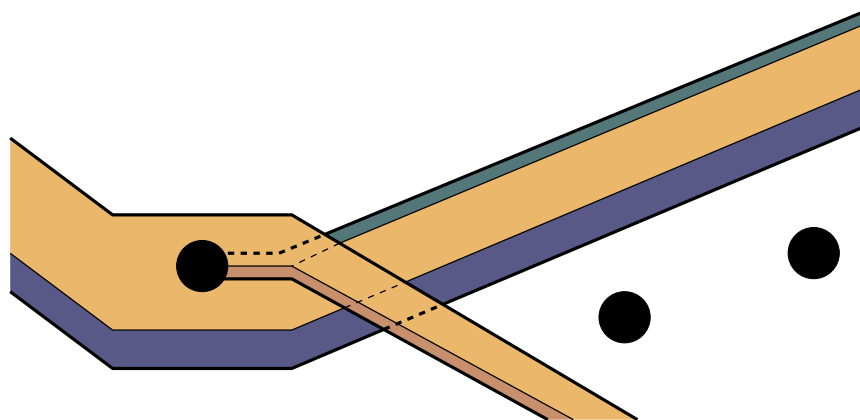


FIGURE 6.9. A slanted strip in  $t$ -space whose image is a sector  $V_\varepsilon^j$  in  $x$ -space.

containing a singular point. When we are no more in the ideal situation, then several of the curves  $\gamma_j$  have disappeared, corresponding to the fact that some strips are either too thin so as to pass a trajectory or have disappeared. Then there remains only a few  $\gamma_j$  dividing the disk in  $m < k + 1$  regions. Each of these regions contains some singular points. In a given region, we have two possibilities:

- (1) either there are several singular points: then they have kept their saddle or node type and are linked by trajectories that form a tree;
- (2) or there is a unique singular point, which is a center or a very widely spiraling focus.

For each  $\gamma_j$  that has disappeared because  $\text{Im}(\tau_j)$  is too small, we bend the strip between the holes while keeping its width a little more than  $\frac{\tau_j}{2}$  (resp.  $\frac{\tau_{j+1}}{2}$ ) (see Figure 6.9). This process restores that part of the strip and forces the bent separatrices to stay inside the disk.

Just before the disappearance of  $\gamma_j$ , each separatrix was attached to a singular point. If the singular point is close to a center as in (2) above, then the bent separatrix will spiral to the singular point: we may add a little more bending so that it does not escape the disk before doing so. In (1) the bent separatrix has no choice but to cross one of the trajectories of the tree between two singular points, one of which is the singularity to which it was attached before. When it does so, we turn to follow a parallel trajectory going to the singular point then bringing back the strip to the horizontal direction. We make the same thing for the three infinite ends of each strip. When doing so, we pay attention to take the same slope at all infinite ends attached to a given singular point.

*Remark 6.21.* When  $\varepsilon \rightarrow 0$  along a curve (6.12) then  $P'_\varepsilon(x_\varepsilon) \rightarrow 0$  and the half-strips are replaced by half-planes. More generally when  $\varepsilon$  tends to a point of  $\Delta_k$ , some half-strips are replaced by half-planes.

6.3.10. *Large (unbounded) squid sectors.* When  $\mu_0 \notin \mathbb{R}_{\leq 0}$ , we will also need a covering of the whole of  $\mathbb{C}$  by  $k$  sectors. For that purpose, we append to the sectors  $V_\varepsilon^j$  an infinite part obtained in the following way: if  $x_1$  and  $x_2$  are the endpoints of the

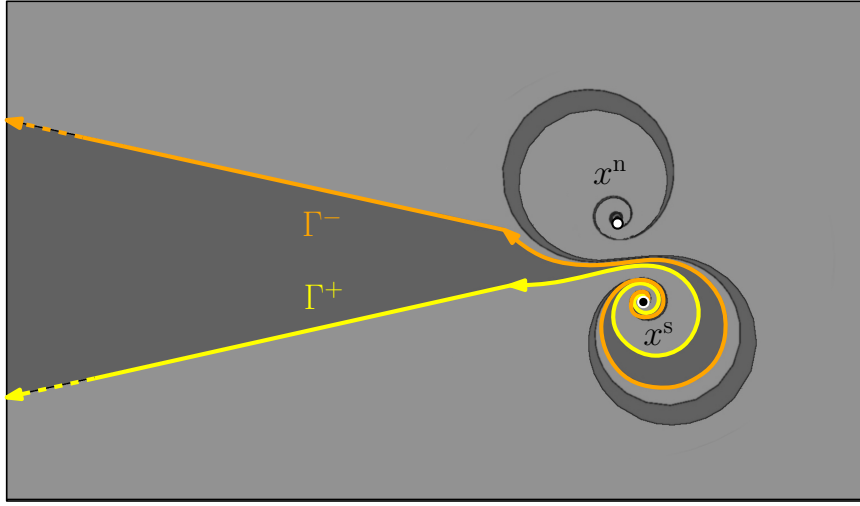


FIGURE 6.10. Unbounded squid sector for  $k = 1$  and  $\Re(\mu) > 0$ . When  $\Re(\mu) \leq 0$  and  $\mu \notin \mathbb{R}$  the shaded area bends to form a geometric spiral near  $\infty$ . See also Figure 6.14 for the case  $k > 1$ .

boundary arc of  $V_\varepsilon^j$  along  $\rho\mathbb{D}$ , then we follow geometric spirals  $x_m \exp((1 + i\nu)\mathbb{R}_{\geq 0})$  for  $m \in \{1, 2\}$  and some  $\nu$  such that

$$\Re(\mu_0) > \nu \Im(\mu_0).$$

If we come back to the representation of the sector in  $t$ -space, this amounts to appending some spiraling sector inside the holes (a neighborhood of  $\infty$  in  $x$ -space is covered by a sector of opening  $2k\pi$  in  $t$ -space).

We still denote by  $V_\varepsilon^j$  the resulting unbounded sectors, since the context will never be ambiguous.

### 6.3.11. Intersections of squid sectors.

**Definition 6.22.** We let  $\Gamma^{j,+}$  (resp.  $\Gamma^{j-1,-}$ ) be the part of the boundary of the unbounded sector  $V_\varepsilon^j$  joining  $x^{j,s}$  (resp.  $x^{j-1,s}$ ) to  $\infty$  with this orientation. The intersection of two squid sectors  $V_\varepsilon^j$  and  $V_\varepsilon^{j'}$  is made of up to three parts in general, and up to four parts when  $k = 2$  (see Figure 6.13).

- If  $j' = j + 1$  (resp.  $j' = j - 1$ ) a (connected) **saddle part**  $V_\varepsilon^{j,s}$  (resp.  $V_\varepsilon^{j-1,s}$ ), bounded by the two curves  $\Gamma^{j,\pm}$  (resp.  $\Gamma^{j-1,\pm}$ ) to the common point  $x^{j,s}$  (resp.  $x^{j-1,s}$ ) of saddle type. When  $k = 1$ , the saddle-part corresponds to a self-intersection.
- If  $j' = \sigma(j)$  a **gate part**  $V_\varepsilon^{j,g}$  included in  $\rho_\varepsilon\mathbb{D}$  and adherent to the two singular points  $x^{j,s}$  and  $x^{j,n}$ . When  $j = \sigma(j)$ , the gate part of an introvert sector corresponds to a self-intersection.
- If  $j' = \sigma(j)$  and  $j = \sigma(j')$  for  $j \neq j'$ , a second gate part  $V_\varepsilon^{j',g}$  adherent to the singular points  $x^{j-1,s} = x^{j',s}$  and  $x^{j,n}$  (Figure 6.13 (B)).

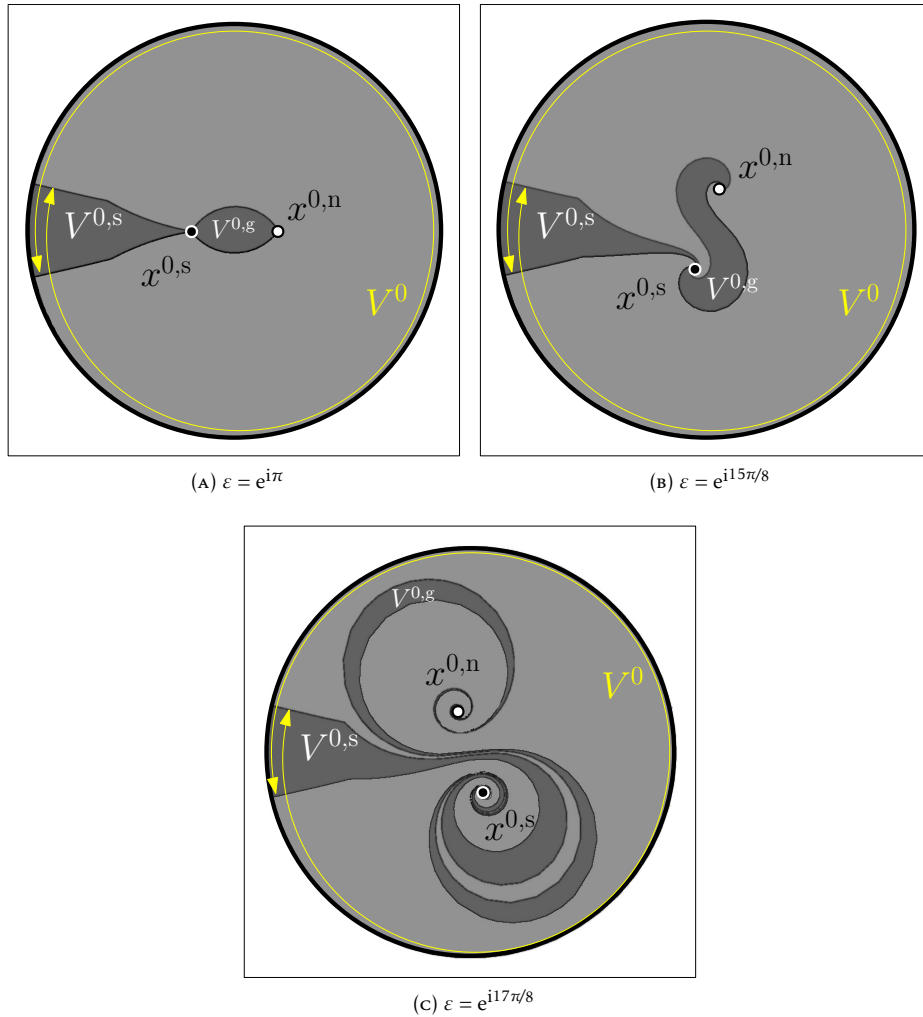


FIGURE 6.11. Squid sectors for different values of  $\varepsilon$  when  $k = 1$ .

6.3.12. *Non-equivalent decompositions.* For the same value of the parameter  $\varepsilon$  in the intersection of two cells (or a cell's self-intersection), the disc  $\rho\mathbb{D}$  is split in non-equivalent ways into bounded squid sectors (see Figures 6.12 and 6.13). By “non-equivalent” we mean that at least one boundary of a squid sector is attached to another root of  $P_\varepsilon$  when passing from one cell to the other.

6.3.13. *Some useful estimates.* We shape the squid sectors in this way because in doing so we gain control on the convergence and on the magnitude of integrals involved in the Cauchy-Heine transform appearing in the next section, in the wake of Remark 6.16. In the following lemma we use the boundary  $\Gamma^{j,\pm}$  of saddle-parts of unbounded sectors as depicted in Figure 6.10.

**Lemma 6.23.** *Assume  $\tau = 0$  (which particularly implies  $\mu_0 \notin \mathbb{R}_{\leq 0}$ ). One can take  $\rho$  and  $\mathcal{E}_\ell$  sufficiently small so that the following properties hold.*

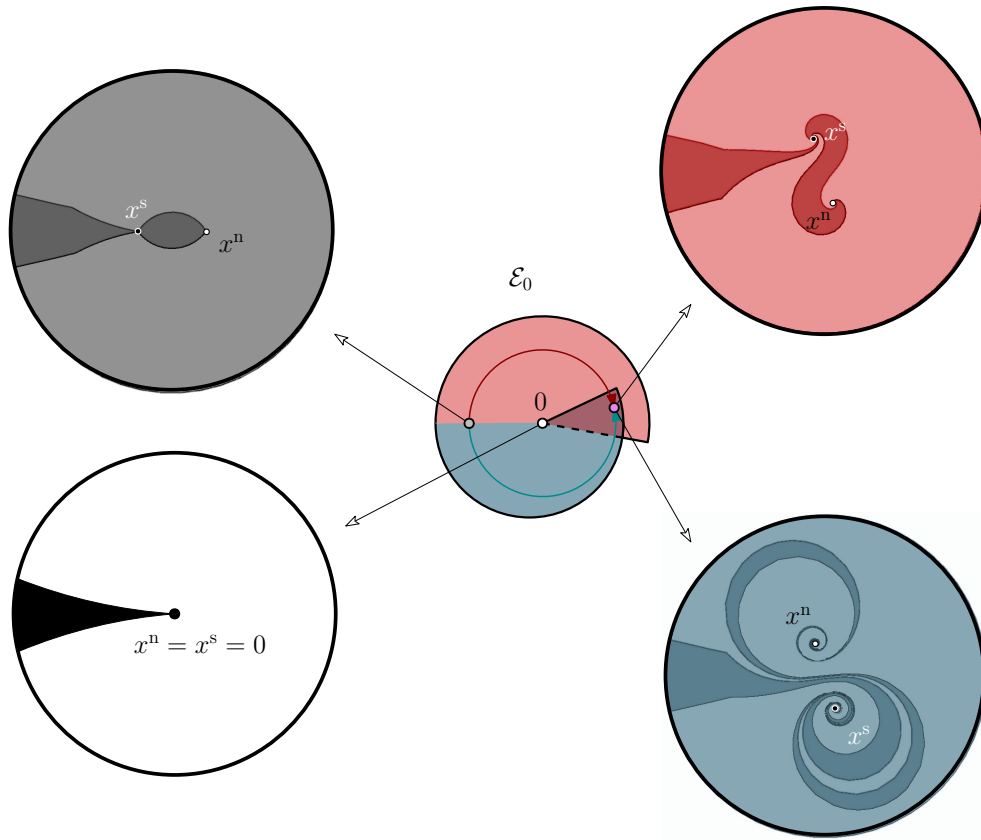


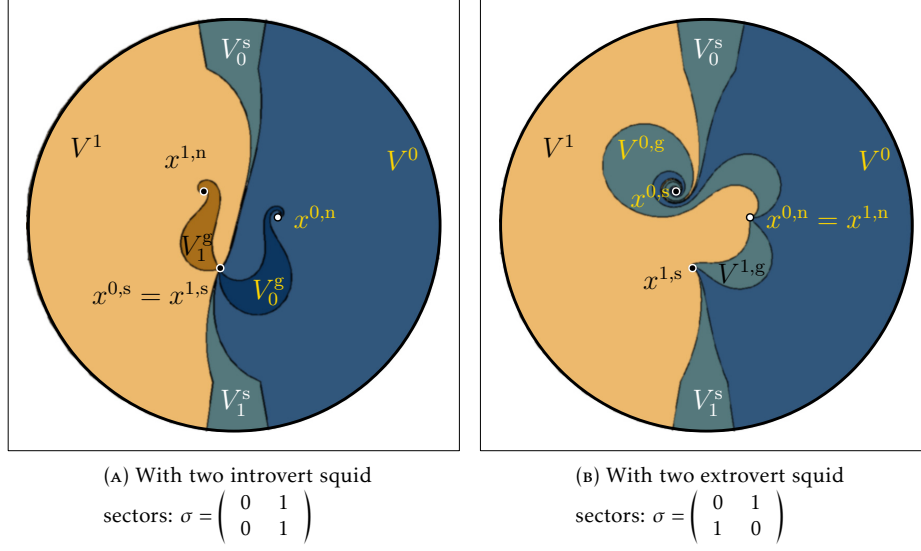
FIGURE 6.12. The single (self-overlapping) cell  $\mathcal{E}$  with diverse configurations when  $k = 1$ . Non-equivalent decompositions are shown on the right. In each picture the location of the node-like singularity  $x^n$  is given by the analytic continuation of the principal determination of  $\sqrt{-\varepsilon}$ .

- (1) For all  $r > 0$  the model first integral (6.7) is bounded on  $V_\varepsilon^{j,s} \times r\mathbb{D}$ , more precisely there exists  $C > 0$  such that

$$(\forall \varepsilon \in \mathcal{E}_\ell) \quad \sup_{V_\varepsilon^{j,s} \times r\mathbb{D}} |\widehat{H}^j| \leq rC.$$

- (2) Also  $\widehat{H}^j$  is  $\frac{dz}{z-x}$ -absolutely integrable over any component  $\Gamma = \Gamma^{j,\pm}$  of the boundary of saddle part intersections (given the outgoing orientation): for all  $x \in V_\varepsilon^j \setminus \Gamma$  and  $y \in \mathbb{C}$  we have

$$\int_\Gamma \widehat{H}^j(z, y) \frac{dz}{z-x} =: yI^j(x) \in \mathbb{C}.$$


 FIGURE 6.13. Non-equivalent decompositions for same  $\varepsilon$  when  $k = 2$ .

(3) There exist a constant  $C > 0$  such that for all  $\varepsilon \in \mathcal{E}_\ell$  and all  $x \in V_\varepsilon^j \setminus \Gamma$

$$|I^j(x)| \leq \frac{C}{|z_* - x_*|},$$

where  $z_* = \Gamma \cap \rho\mathbb{S}^1$  and  $x_*$  is likewise the intersection of  $\rho\mathbb{S}^1$  and the curve passing through  $x$  built in the same way as  $\Gamma$ .

*Proof.* Because  $\widehat{H}^j$  is linear in  $y$  we may only consider the case  $y := 1$ . Let

$$\widehat{h} : x \mapsto \widehat{H}^j(x, 1)$$

be the corresponding partial function. The proof is done in two steps, corresponding to the two different components “inner” (inside  $\rho\mathbb{D}$ ) and “outer” ( $|x| \geq \rho$ ). We parametrize  $\Gamma$  by a piecewise analytic curve  $z : \mathbb{R} \rightarrow \mathbb{C}$  detailed below, such that (with the obvious abuse of notations)

$$\begin{cases} z(-\infty) & = \infty \\ z(0) & = z_\pm^j \in \rho\mathbb{S}^1. \\ z(\infty) & = x^{j,s} \end{cases}$$

In what follows,  $C > 0$  indicates a real constant (independent on  $\varepsilon$ ) whose value varies according to the place where it appears.

(1) We invoke again the variational argument presented in Remark (6.16). Over  $]0, \infty[$  we follow the flow of  $\partial P \frac{\partial}{\partial x}$  and we can indeed estimate the modulus

$$\phi(t) := \left| \widehat{h}(z(t)) \right|,$$

as  $\widehat{h}$  is solution of

$$\frac{d\widehat{h}}{\widehat{h}} = -(1 + \mu z^k) \frac{dz}{P_\varepsilon},$$

so that

$$\frac{\dot{\phi}}{\phi}(t) = -\Re \left( \vartheta (1 + \mu z^k) \right).$$

Since  $\frac{1}{2|\mu_0|} > \rho^k$ , and taking the hypothesis  $|\arg \vartheta| < \frac{\pi}{4}$  into account we obtain

$$\frac{\dot{\phi}}{\phi} \leq -C < 0$$

and

$$(6.20) \quad \left| \widehat{h}(z(t)) \right| \leq \left| \widehat{h}(z(t_\varepsilon)) \right|.$$

Over  $]-\infty, 0[$  we follow the flow of

$$\dot{z} = -(1 + i\nu)z,$$

above which the modulus of  $\widehat{h}$  is governed by

$$\begin{aligned} \frac{\dot{\phi}}{\phi} &= \Re \left( \frac{(1 + \mu z^k)(1 + i\nu)z}{P_\varepsilon(z)} \right) \\ &= \Re \left( \frac{(1 + \mu z^k)(1 + i\nu)}{z^k} \times \frac{z^{k+1}}{P_\varepsilon(z)} \right) \\ &\geq C \Re(\mu + i\nu\mu) =: \alpha > 0 \end{aligned}$$

for  $|z|$  sufficiently large since  $\nu$  is chosen in such a way that  $\Re(\mu + i\nu\mu) > 0$  for all  $\varepsilon \in \mathcal{E}_\rho$ . To conclude the proof we only have to remark that  $\sup_{|z|=\rho} \left| \widehat{h}(z) \right| \geq \left| \widehat{h}(z(0)) \right|$  converges uniformly towards  $\sup_{|z|=\rho} \left| \widehat{H}_0^j(z, 1) \right| < \infty$  as  $\varepsilon \rightarrow 0$ .

(2) and (3) We use the following trick. We work with the integral

$$J^j(x) = \int_{-\infty}^{\infty} \widehat{h}(z(t)) \frac{z'(t)dt}{z(t) - x(t)},$$

where  $t \mapsto x(t)$  is defined similarly as  $t \mapsto z(t)$  except for the fact that it passes through  $x$ , uniquely defining  $x_* = x(0)$ . To conclude we will need to bound away from 0 (uniformly in  $\varepsilon$ ) the quantity  $\left| \frac{z(t) - x}{z(t) - x(t)} \right|$ . But this is clear from the pictures because if  $\text{dist}(x, \Gamma)$  is realized for  $z = z(t)$  then  $x \simeq x(t)$ .

Now, to study  $J^j(x)$  we repeat the above argument but with the function

$$\phi_\#(t) := \left| \frac{\widehat{h}(z(t)) A_\#(z(t))}{z(t) - x(t)} \right|, \quad \# \in \{0, \infty\},$$



where  $A_0(z) := \vartheta P_\varepsilon(z)$  and  $A_\infty(z) := -(1 + i\nu)z$ . The variations of  $\phi_\sharp$  are governed by

$$\frac{\dot{\phi}_\sharp}{\phi_\sharp}(t) = \Re \left( -\frac{1 + \mu z^k}{P_\varepsilon(z(t))} A_\sharp(z(t)) + A'_\sharp(z(t)) - \frac{A_\sharp(z(t) - A_\sharp(x(t)))}{z(t) - x(t)} \right)$$

for  $t$  in the corresponding interval so that  $\dot{z} = A_\sharp(z)$  and  $\dot{x} = A_\sharp(x)$ . In the case  $\sharp = \infty$ , the sum of the last two terms vanishes and then

$$\frac{\dot{\phi}_\infty}{\phi_\infty} \geq C > 0$$

for large  $z$  (hence  $t$  close to  $-\infty$ ) from the choice of  $\nu$ . Let us now deal with the case  $\sharp = 0$ . We have chosen  $\rho > \rho_\varepsilon$  so that

$$\sup_{|z| < \rho} (|\mu z^k| + 2\rho |P''(z)|) \leq \frac{3}{4}.$$

Because for all  $x, z \in \rho\mathbb{D}$

$$|P(x) - P(z) - (x - z)P'(z)| \leq |x - z|^2 \sup_{\rho\mathbb{D}} |P''|$$

we obtain

$$\frac{\dot{\phi}_0}{\phi_0} \leq -C < 0$$

and

$$|\phi_0(t)| \leq |\phi_0(0)| \exp(-Ct)$$

for  $t \geq 0$ .

Therefore the integral

$$\int_{z(0)}^{z(t)} \widehat{h}(z) \frac{dz}{z - x} = \vartheta \int_0^t \widehat{h}(z(t)) \frac{P_\varepsilon(z(t))}{z(t) - x} dt$$

is absolutely convergent as  $t \rightarrow \infty$  and

$$\left| \int_{z(0)}^{z(\infty)} \widehat{h}(z) \frac{dz}{z - x} \right| \leq C |\phi_0(0)|.$$

But  $C |\phi_0(0)| \leq \frac{C}{|x(0) - z(0)|}$  as expected. □

**6.4. Cellular section of the period: proof of Proposition 6.6.** The cellular section  $\mathfrak{S}_\ell$  of the period operator is obtained from a variation on the method introduced in Section 5.4 to normalize the glued abstract manifold by solving a linear Cousin problem. It is an unfolding of the technique used in [42] for  $\varepsilon = 0$ . The initial data is a  $k$ -tuple

$$T = (T^j)_j \in \prod_{z/kz} \mathcal{H}_\ell\{h\}$$

and we seek  $Q \in x\mathcal{H}_\ell\{y\}[x]_{<k}$ , that is

$$Q(x, y) = x \sum_{n>0} Q_n(x) y^n$$

for some polynomial  $Q_n \in \text{Holo}_c(\mathcal{E}_\ell)[x]_{<k}$  in  $x$  of degree less than  $k$ , such that

$$\mathfrak{T}_\ell(Q) = T.$$

We then define the section as

$$\mathfrak{S}_\ell(T) := Q.$$

The construction goes along the following steps. They are performed for fixed  $\varepsilon$  in a fixed  $\mathcal{E}_\ell$ , with explicit control on the parametric regularity. Hence we omit mentioning explicitly the dependence on  $\varepsilon$  and  $\ell$ . For  $r > 0$  define

$$\mathcal{V}_r^j := \{(x, y) \in V^j \times \mathbb{C} : |y| < r\}.$$

We define in a similar fashion the fibered intersections  $\mathcal{V}_r^{j, \sharp}$  for  $\sharp \in \{s, g\}$ .

- Build sectorial, bounded functions  $F^j$  on  $\mathcal{V}_r^j$  such that

$$(6.21) \quad F^{j+1} - F^j = 2i\pi T^j \circ H^j$$

on  $\mathcal{V}_r^{j,s}$ , where  $H^j$  is the  $j^{\text{th}}$  canonical sectorial first integral of  $\mathcal{X}$ , as in (6.6). This is done again by a Cauchy-Heine transform (Section 6.4.1).

- Because of the functional equation (6.21) the identity  $\mathcal{X} \cdot F^{j+1} = \mathcal{X} \cdot F^j$  holds and allows to patch together a holomorphic function  $Q := \mathcal{X} \cdot F^j$  on a whole  $\mathbb{C} \times (\mathbb{C}, 0)$  which, by construction, satisfies

$$\begin{aligned} \mathfrak{T}^j(Q) \circ H^j &= F^{j+1} - F^j \\ &= T^j \circ H^j \end{aligned}$$

(Section 6.4.2).

- Growth control near  $x = \infty$  and a final normalization allows concluding that  $Q \in x\mathbb{C}\{y\}[x]_{<k}$  (Section 6.4.3).

#### 6.4.1. Cauchy-Heine transform.

**Definition 6.24.** In the following we fix a collection  $N = (N^j)_j \in \prod_{j \in \mathbb{Z}/k\mathbb{Z}} \text{Holo}_c(\mathcal{V}_r^j)'$ , which is a  $k$ -tuple of functions with an expansion

$$N^j(x, y) = \sum_{n>0} N^{j,n}(x) y^n$$

uniformly absolutely convergent on every  $\mathcal{V}_{r'}^j$ , for all  $0 \leq r' < r$ , whose norm is given by

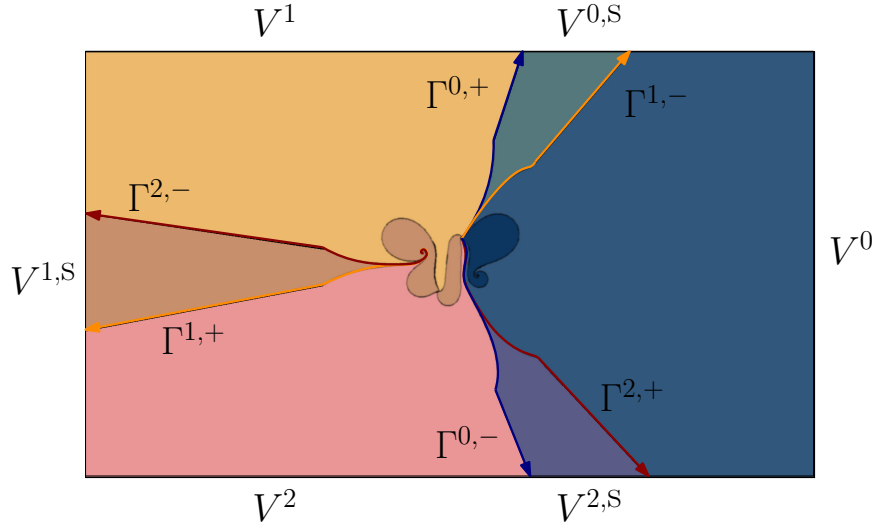
$$\|N\| := \max_j \sup_{\mathcal{V}_{r'}^j} |N^j|.$$

- (1) We define the  $j^{\text{th}}$  **sectorial first integral associated to  $N$**  as the holomorphic function

$$\begin{aligned} H_N^j : \mathcal{V}_r^j &\longrightarrow \mathbb{C} \\ (x, y) &\longmapsto \widehat{H}^j(x, y) \exp N^j(x, y), \end{aligned}$$

where  $\widehat{H}^j$  is the sectorial canonical model first integral (6.7) continued over unbounded squid sectors.

- (2) For a given  $\eta > 0$  we say that  $N$  is  $\eta$ -**adapted** if  $H_N^j(\mathcal{V}_r^{j,s}) \subset \eta\mathbb{D}$ .


 FIGURE 6.14. Unbounded squid sectors and paths of integration ( $k > 1$ ).

Of course we prove in due time (Corollary 7.7) that  $N := N_\varepsilon$ , defined as the collection of sectorial solutions of the normalizing equation  $\mathcal{X}_\varepsilon \cdot N_\varepsilon^j = -R_\varepsilon$ , satisfies the hypothesis of the definition and that  $\sup \left| H_N^j(\mathcal{V}_r^{j,s}) \right| \rightarrow 0$  as  $r \rightarrow 0$  (uniformly in  $\varepsilon \in \mathcal{E}_\ell$ ), mainly because it is already the case for the model first integral (Lemma 6.23 (1)). Therefore, for given  $\eta > 0$ , it will always be possible to find  $r$  (independently on  $\varepsilon$ ) such that  $N$  is  $\eta$ -adapted, allowing us to use the next result, genuinely the key point in building the cellular section of the period.

**Proposition 6.25.** *Assume  $\tau = 0$  (which particularly implies  $\mu_0 \notin \mathbb{R}_{\leq 0}$ ). Let  $\mathcal{E}_\ell$  be a fixed cell as in Section 6.3. For every  $T \in \prod_{j \in \mathbb{Z}/k\mathbb{Z}} \text{Holo}_c(\eta\mathbb{D})'$  holomorphic on a disc of radius  $\eta > 0$ , for every  $\eta$ -adapted collection  $N$ , the  $k$ -tuple of functions*

$$\mathbb{F} = \mathbb{F}(T, N) := (F^j)_j \in \prod_{j \in \mathbb{Z}/k\mathbb{Z}} \text{Holo}_c(\mathcal{V}_r^j)'$$

defined by

$$(6.22) \quad F^j(x, y) := \sum_{p \neq j+1} \int_{\Gamma^{p,-}} \frac{T^{p-1}(H_N^{p-1}(z, y))}{z-x} dz + \int_{\Gamma^{j,+}} \frac{T^j(H_N^j(z, y))}{z-x} dz$$

fulfills the next conclusions. The paths of integration  $\Gamma^{j,\pm}$  bounds the unbounded squid sectors in the following way : The boundary of the saddle part  $V^{j,s}$  of (unbounded) squid sectors is  $\Gamma^{j,+} \cup \Gamma^{j+1,-}$ , as in Figures 6.10 and 6.14, and we set

$$\|T'\| := \max_j \sup_{\eta\mathbb{D}} \left| \frac{dT^j}{dh} \right|.$$

(1) For every  $(x, y) \in \mathcal{V}_r^{j,s}$

$$(6.23) \quad F^{j+1}(x, y) - F^j(x, y) = 2i\pi T^j \left( H_N^j(x, y) \right)$$

while for every  $(x, y) \in \mathcal{V}_r^{\sigma(j),g}$

$$F^j(x, y) = F^{\sigma(j)}(x, y).$$

(When  $k = 1$  we refer to (3) of the following remark for a fuller explanation.)

(2)  $F^j \in \text{Holo}_c(\mathcal{V}^j)$ .

(3) There exists  $K > 0$  independent on  $T, N, r$  and  $\varepsilon$  such that the following estimates hold.

(a)

$$\|F\| \leq rK \|T'\| \exp\|N\|.$$

(b)

$$\left\| y \frac{\partial F}{\partial y} \right\| \leq rK \|T'\| \left\| 1 + y \frac{\partial N}{\partial y} \right\| \exp\|N\|.$$

(c)

$$\left\| x \frac{\partial F}{\partial x} \right\| \leq rK \|T'\| \left\| 1 + x \frac{\partial N}{\partial x} \right\| \exp\|N\|.$$

*Remark 6.26.*

- (1) The absolute convergence of the integrals involved in (6.22) is established in the course of the proof, mainly thanks to the estimates given by Lemma 6.23. Notice also that for fixed  $\varepsilon$  and  $y$  the mapping  $x \mapsto F^j(x, y)$  is holomorphic on  $V^j$  since the squid sector does not contain any of the curves  $\Gamma^{p,-}$  except for  $p = j + 1$ .
- (2) The integral expression (6.22) and Item (3) above clearly show that  $F$ , as a function of  $\varepsilon \in \mathcal{E}_\ell$ , has the same regularity as  $T$ .
- (3) In the case  $k = 1$  the expression (6.22) yields  $F(x, y) = \int_{\Gamma^+} (\cdots) dz$ , which can be analytically continued in the  $x$ -variable on the self-overlapping squid sector (Figure 6.10). As  $x$  reaches  $\Gamma^-$  “from below” the analytic continuation coincides with  $\int_{\Gamma^-} (\cdots) dz$ , because the difference of determination is given by

$$D(x, y) := \int_{\Gamma^+ - \Gamma^-} \frac{T(H_N(z, y))}{z - x} dz,$$

and Cauchy’s formula asserts that  $D(x, y) = 0$  whenever  $x$  is outside the saddle-part  $V^s$  enclosed by  $\Gamma^+ \cup \Gamma^-$ . On the contrary if  $x \in V^s$  then  $D(x, y) = 2i\pi T(H_N(x, y))$ , which is the way to understand (6.23).

*Proof.* This proposition follows the general lines of [42, Theorem 2.5] for  $\varepsilon = 0$ . A simpler instance of the strategy can be found in Lemma 5.6. Except when necessary we drop every sub- and super-scripts.

- (1) This is nothing but Cauchy residue formula. We indeed compute (omitting to include the integrand for the sake of readability)

$$\begin{aligned} F^{j+1}(x, y) - F^j(x, y) &= \int_{\Gamma^{j+1,+}} - \int_{\Gamma^{j,+}} + \sum_{p \neq j+2} \int_{\Gamma^{p,-}} - \sum_{p \neq j+1} \int_{\Gamma^{p,-}} \\ &= \left( \int_{\Gamma^{j+1,-}} - \int_{\Gamma^{j,+}} \right) - \left( \int_{\Gamma^{j+2,-}} - \int_{\Gamma^{j+1,+}} \right). \end{aligned}$$

The candidate singularity in the common integrand  $\frac{T^p(H_N^p(z, y))}{z-x}$  in  $\int_{\Gamma^{p+1,-}} - \int_{\Gamma^{p,+}}$  is  $z = x$ . This happens only when  $x \in V^{p,s}$ . By hypothesis  $x \in V^{j,s}$  hence (6.23) holds.

Actually one needs to use a growing family of compact loops within  $V^{j,s}$  converging toward  $\partial V^{j,s}$ , then to apply Cauchy formula to each one of them and take the limit. The only possible choice for the connected component of  $\mathbb{C} \setminus (\Gamma^{j+1,-} \cup \Gamma^{j,+})$  for which this construction works is  $V^{j,s}$ , since in that sector we can establish tame estimates for the growth of the integrand (see (3) below), and we can also establish untamed estimates outside a neighborhood of  $\text{cl}(V^{j,s})$ .

- (2) Taking for granted that the integrand defining  $F(x, y)$  for  $(x, y) \in \text{cl}(\mathcal{V}_r)$  is bounded from above by a real-analytic, integrable function on  $\partial V^s$ , the analyticity of  $F$  on  $\mathcal{V}_r$  is clear from the definition (6.22). Integration paths used to evaluate  $F$  can be slightly deformed outwards without changing the value of the integral, which shows that  $F$  can be analytically continued to any point  $(x, y)$  with  $x \in \partial V \setminus P_\varepsilon^{-1}(0)$  and  $|y| \leq r$ . Concluding that  $F$  extends as a continuous function on  $\text{cl}(\mathcal{V}_r) \setminus P_\varepsilon^{-1}(0)$  is again a consequence of (6.22) for  $y$  is an extraneous parameter. Dominated convergence of  $F(x, y)$ , continuity on  $\text{cl}(\mathcal{V}_r) \cap P_\varepsilon^{-1}(0)$  and boundedness of  $F$  are established in (3).
- (3) We begin with proving (a). Since, for  $p \in \mathbb{Z}/k\mathbb{Z}$ ,

$$|T^p(h)| \leq |h| \|T'\|,$$

we deduce

$$\left| \frac{T^p(H(z, y))}{z-x} \right| \leq \frac{|\widehat{H}(z, y)|}{|z-x|} \|T'\| \exp \|N\|.$$

We then invoke the estimates derived for the model family in Lemma 6.23, showing dominated convergence for  $F(x, y)$ . In order to bound  $F$  it is sufficient to consider only the problem of bounding  $F$  near a single  $\Gamma := \Gamma^{j,+}$ . A uniform bound  $K$  for the rightmost sum of integrals simply requires bounding uniformly  $\frac{1}{|z_* - x_*|}$  where  $z_*$ ,  $x_* \in \rho\mathbb{S}^1$ . Of course no uniform bound in  $x$  exists when  $x$  tends to  $\Gamma$  (i.e.  $x_*$  tends to  $z_*$ ). To remedy this problem we bisect  $V^s$  with a curve  $\widehat{\Gamma}$  parallel to  $\Gamma$  and passing through the middle of the arc  $\rho\mathbb{S}^1 \cap V^s$ . When  $x$  is taken in the component of  $V^j \setminus \widehat{\Gamma}$  not accumulating on  $\Gamma$  the value of  $\frac{1}{|z_* - x_*|}$  is uniformly bounded. When  $x$  is taken in the other part we use the functional relation (6.23): in that configuration  $x$  is understood as an element of  $V^{j+1}$  far from  $\Gamma^{j+1,-}$  and we are back to the situation we just solved.

A little bit more detailed analysis allows proving that  $x \mapsto F(x, y)$  is Cauchy<sup>2</sup> near  $x^{j,s}$ , so that  $F$  extends continuously to  $\{x^{j,s}\} \times r\mathbb{D}$ . Items (b) and (c) are obtained much in the same way, the details are straightforward adaptations of (a).  $\square$

6.4.2. *Holomorphy of  $Q_\varepsilon$ .* Now all functions  $\mathcal{X}_\varepsilon \cdot F^j$  patch on intersecting squid sectors to define

$$Q \in \text{Holo}\left(\left(\mathbb{C} \setminus P_\varepsilon^{-1}(0)\right) \times r\mathbb{D}\right).$$

If we show that  $Q$  is bounded near each disk  $\{x^{j,s}\} \times r\mathbb{D}$  then Riemann's theorem on removable singularities guarantees the holomorphic extension of  $Q$  to  $\mathbb{C} \times \mathbb{D}$ . But

$$(6.24) \quad \left|Q(x, y)\right| \leq |P_\varepsilon(x)| \left\| \frac{\partial F}{\partial x} \right\| + \left(1 + |\mu||x|^k + |R(x, y)|\right) \left\| y \frac{\partial F}{\partial y} \right\|$$

so that taking Proposition 6.25 (3) into account brings the conclusion.

6.4.3. *Growth control of  $Q_\varepsilon$  near  $x = \infty$ .* In Section 7.2 we prove that the  $k$ -tuple of sectorial solutions  $N$  of the cohomological equation of normalization  $\mathcal{X} \cdot N^j = -R$  satisfy the conditions  $\|x \frac{\partial N}{\partial x}\| \leq \frac{1}{3}$  and  $\|y \frac{\partial N}{\partial y}\| \leq \frac{1}{3}$  if  $r$  is chosen small enough (Corollary 7.7).

**Lemma 6.27.** *For every fixed  $y \in r\mathbb{D}$  the entire function  $x \mapsto Q(x, y)$  is actually a polynomial of degree at most  $k$ , and*

$$(6.25) \quad Q(x, y) = \sum_{n>0} q_n(x) y^n, \quad q_n \in \mathbb{C}[x]_{\leq k}$$

on  $\mathbb{C} \times r\mathbb{D}$ .

*Proof.* Since  $x \mapsto R(x, y)$  is a polynomial of degree at most  $k$ , there exists a constant  $C > 0$  such that  $1 + |\mu||x|^k + |R(x, y)| \leq C|x|^k$  for every  $|x| \geq \rho$ . The bound (6.24) on  $x \mapsto Q(x, y)$  also holds near  $\infty$  so that

$$\left|Q(x, y)\right| \leq \left| \frac{P_\varepsilon(x)}{x} \right\| \left\| x \frac{\partial F^j}{\partial x} \right\| + C|x|^k \left\| y \frac{\partial F}{\partial y} \right\|.$$

From Proposition 6.25(3)(b,c) and the control on  $\|x \frac{\partial N^j}{\partial x}\|, \|y \frac{\partial N^j}{\partial y}\|$  we infer  $\|x \frac{\partial F^j}{\partial x}\|, \|y \frac{\partial F^j}{\partial y}\| < +\infty$ , from which we deduce  $\frac{P_\varepsilon(x)}{x} \left\| x \frac{\partial F^j}{\partial x} \right\| = O(x^k)$  and finally  $Q(x, y) = O(x^k)$  as well.  $\square$

To complete the proof of Proposition 6.6 we need to modify  $Q$  so that  $Q(0, y) = 0$ . In order not to change the period of  $Q$  we can only subtract from  $Q$  a function of the form  $\mathcal{X}_\varepsilon \cdot F$  with  $F$  holomorphic. This is done by setting

$$F(y) := \int_0^y \frac{Q(0, v)}{v} dv,$$

so that  $Q - \mathcal{X}_\varepsilon \cdot F$  vanishes on  $\{x = 0\}$  while still admitting an expansion of the form (6.25).

<sup>2</sup>A function  $f$  from a metric space  $E$  to another one  $F$  is Cauchy at  $a$  if for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $x, y \in B(a, \delta)$  implies  $d(f(x), f(y)) < \varepsilon$ .

**6.5. Stitching cellular sections together: proof of Proposition 6.7.** Fix  $(\mathbb{C}^k, 0) \setminus \Delta_k$  and  $\rho > 0$  not larger than what is allowed in Lemma 6.23, and take  $G \in \text{Holo}_c(\rho\mathbb{D} \times (\mathbb{C}, 0))'$ . We prove now that for any fixed  $\varepsilon \in \mathcal{E}_\ell$ , at most one  $Q \in x\mathbb{C}[x]_{<k}\{y\}'$  exists such that  $G - Q \in \text{im}(\mathcal{X}_\varepsilon \cdot)$ , that is  $\widehat{\mathcal{T}}(G) = \widehat{\mathcal{T}}(Q)$ . This amounts to showing that  $\text{im}(\mathcal{X}_\varepsilon \cdot) \cap x\mathbb{C}[x]_{<k}\{y\}' = \{0\}$  for all fixed  $\varepsilon \in (\mathbb{C}^k, 0) \setminus \Delta_k$ .

Let  $G \in \text{im}(\mathcal{X}_\varepsilon \cdot) \cap x\mathbb{C}[x]_{<k}\{y\}'$  and write

$$G(x, y) = \mathcal{X}_\varepsilon \cdot \sum_{n \geq d} F_n(x) y^n = \sum_{n \geq d} G_n(x) y^n \in \text{Holo}(r\mathbb{D} \times (\mathbb{C}, 0)), \quad d \in \mathbb{N};$$

we claim that  $G_d = 0$ , which is sufficient to establish the result. It turns out that for its part of least degree in  $y$  the cohomological equation only depends on its formal normal form:

$$\widehat{X}_\varepsilon \cdot (y^d F_d(x)) = y^d G_d(x).$$

Such a relation holds if and only if the period of  $y^d G_d$  along the formal normal form vanishes:  $\widehat{\mathcal{T}}(y^d G_d) = 0$ . Therefore we need to prove that

$$\begin{aligned} \widehat{\mathcal{T}} : x\mathbb{C}[x]_{<k} y^d &\longrightarrow \mathbb{C}^k h^d \\ y^d G_d &\longmapsto \widehat{\mathcal{T}}(y^d G_d) \end{aligned}$$

is injective if  $\varepsilon$  is small enough. As recalled in Corollary 6.12 we know that for every  $a \in \mathbb{N}$

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon_\ell}} \widehat{\mathcal{T}}(x^a y^b) = \widehat{\mathcal{T}}_0(x^a y^b),$$

where  $\widehat{\mathcal{T}}_0$  is the period of the model saddle-node  $\widehat{X}_0$ . The auxiliary result [46, Proposition 2] states precisely that  $\widehat{\mathcal{T}}_0$  is invertible, and therefore so is  $\widehat{\mathcal{T}}$  for small  $\varepsilon$  as expected.

## 7. ORBITAL REALIZATION THEOREM

In this section we address the inverse problem for the classification of unfoldings performed in [40], in the special case of convergent unfoldings of formal invariant  $\mu$  with

$$\mu_0 \notin \mathbb{R}_{\leq 0}$$

and  $\tau = 0$ . The residual cases  $\mu_0 \leq 0$  or  $\tau > 0$  are dealt with in Section 8. Also notice that we only carry this study for the orbital part, the case of the temporal realization is explained in [43] when  $k = 1$ . Generalizing this approach for  $k > 1$  using the tools introduced in Section 6 should not be difficult.

We summarize in Section 7.1 how the invariants of classification are built. They unfold Martinet-Ramis's invariants [30] for the limiting saddle-node, obtained as transition maps between sectorial spaces of leaves. Yet the construction can only be carried out analytically on a given parametric cell  $\mathcal{E}_\ell$ , yielding a cellular invariant  $m_\ell \in \prod_{z/kz} \mathcal{H}_\ell\{h\}$  (see Section 6.1 for the definition of the functional spaces  $\mathcal{H}_\ell$  and Section 7.1 for the definition of  $m_\ell$ ). The orbital modulus  $m(X)$  of an unfolding  $X$  consists in the whole collection  $(m_\ell)_\ell$ .

**Definition 7.1.** We say that  $(\mu, \mathfrak{m}) \in \mathbb{C}\{\varepsilon\} \times \prod_{\ell} \mathcal{H}_{\ell}\{h\}^k$  is **realizable** if there exists a generic convergent unfolding  $X$  with formal orbital class  $\mu$  and orbital modulus  $\mathfrak{m} = \mathfrak{m}(X)$ .

In Section 7.2 we prove the next result.

**Theorem 7.2.** Assume  $\tau = 0$  (which particularly implies  $\mu_0 \notin \mathbb{R}_{\leq 0}$ ). Fix a germ at  $0 \in (\mathbb{C}^{k+1}, 0)$  of a cell  $\mathcal{E}_{\ell}$ . Given  $\mathfrak{m}_{\ell} \in \prod_{j \in \mathbb{Z}/k\mathbb{Z}} \mathcal{H}_{\ell}(h)$  and  $\mu$  with  $\mu_0 \notin \mathbb{R}_{\leq 0}$ , there exists a unique  $R_{\ell} \in x\mathcal{H}_{\ell}\{y\}[x]_{<k}$  such that

$$\mathcal{X}_{\ell, \varepsilon} := \widehat{X} + yR_{\ell, \varepsilon} \frac{\partial}{\partial y}$$

has  $\mathfrak{m}_{\ell}$  for transition maps in sectorial space of leaves (i.e. for modulus).

The fact that this “analytical synthesis” gives unique forms of the same kind as those given by Loray’s “geometric” construction bolsters the naturalness of the normal forms presented here. Indeed the next corollary provides an indirect solution of the inverse problem.

**Corollary 7.3.** A couple  $(\mu, \mathfrak{m})$  with  $\mu_0 \notin \mathbb{R}_{\leq 0}$  is realizable if and only if  $R_{\ell, \varepsilon} = R_{\bar{\ell}, \varepsilon}$  for all  $\varepsilon \in \mathcal{E}_{\ell} \cap \mathcal{E}_{\bar{\ell}}$  and all  $(\ell, \bar{\ell})$ .

*Proof.* The equality  $R_{\ell} = R_{\bar{\ell}}$  on  $\mathcal{E}_{\ell} \cap \mathcal{E}_{\bar{\ell}}$  defines a bounded, holomorphic function  $R$  in the parameter  $\varepsilon \in (\mathbb{C}^{k+1}, 0) \setminus \Delta_k$ , which extends holomorphically to a whole neighborhood  $(\mathbb{C}^{k+1}, 0)$  by Riemann’s theorem on removable singularities. The corresponding unfolding  $\mathcal{X}$  has modulus  $\mathfrak{m}(\mathcal{X}) = (\mathfrak{m}_{\ell})_{\ell}$  by construction.

Conversely, the Normalization Theorem tells us that we can as well assume that the vector field is in normal form  $\mathcal{X}$  (2.4), without changing the orbital modulus  $\mathfrak{m} = \mathfrak{m}(\mathcal{X})$ . Moreover, the normalization can be performed by tangent-to-identity mappings in the  $y$ -variable. According to Theorem 7.2,  $R_{\ell}$  is uniquely determined by the component  $\mathfrak{m}_{\ell}$  of  $\mathfrak{m}$ , hence  $R = R_{\ell}$  on  $\mathcal{E}_{\ell}$ .  $\square$

Somehow this characterization is not satisfying since it involves the auxiliary unfolding  $\mathcal{X}_{\ell}$ . In Section 7.3 we present an intrinsic characterization of realizable  $(\mu, \mathfrak{m})$  as a *compatibility condition* imposed on the different dynamics induced by each pair  $(\mu, \mathfrak{m}_{\ell})$  on the sectorial space of leaves (Definition 7.16). Roughly speaking the condition requires that the abstract holonomy groups be conjugate over cells overlaps. In case of an actual unfolding  $X$  (i.e. realizable  $(\mu, \mathfrak{m})$ ) these groups represent in the space of leaves the actual weak holonomy group induced by  $X$  in  $(x, y)$ -space.

**7.1. Classification moduli.** Starting from a generic convergent unfolding  $X$  of codimension  $k$  in prepared form (4.2) with given orbital formal invariant  $\mu$  (with no restriction on  $\mu_0$ ), we can build the following  $k$ -tuple of periods (Definition 6.10) on a germ of a cellular decomposition  $(\mathcal{E}_{\ell})_{1 \leq \ell \leq C_k}$ , called the **orbital modulus** of  $X$ :

$$\begin{aligned} \mathfrak{m}(X) &:= (\mathfrak{m}_{\ell}(X))_{1 \leq \ell \leq C_k}, \\ \mathfrak{m}_{\ell}(X) &:= \left( \phi_{\ell}^{j, s} \right)_{j \in \mathbb{Z}/k\mathbb{Z}}, \\ (7.1) \quad \phi_{\ell}^{j, s} &:= 2i\pi \Upsilon_{\ell}^j(-R) \in \mathcal{H}_{\ell}\{h\}. \end{aligned}$$

We state the main result of [40] in the specific context of convergent unfoldings.



**Definition 7.4.**

- (1) Fix a germ of a cell  $\mathcal{E}_\ell$ . For  $c \in \mathbb{C}\{\varepsilon\}^\times$ ,  $\theta \in \mathbb{Z}/k\mathbb{Z}$  and  $f = (f^j)_{j \in \mathbb{Z}/k\mathbb{Z}} \in \mathcal{H}_\ell\{h\}^k$  define

$$(c, \theta)^* f : (\varepsilon, h) \mapsto \left( f_\varepsilon^{j+\theta}(c_\varepsilon h) \right)$$

and extend component-wise this action to tuples.

- (2) We say that two collections  $\mathfrak{m}, \tilde{\mathfrak{m}} \in \prod_\ell \mathcal{H}_\ell\{h\}^k$  are **equivalent** if there exists  $c \in \mathbb{C}\{\varepsilon\}^\times$  and  $\theta \in \mathbb{Z}/k\mathbb{Z}$  such that

$$(7.2) \quad (c, \theta)^* \mathfrak{m} = \tilde{\mathfrak{m}}.$$

*Remark 7.5.* The presentation of Definition 7.4 is equivalent to that of [30] for  $\varepsilon = 0$ . The transition functions there are simply given by  $\psi^{j,s}(h) = h \exp\left(\frac{2i\pi\mu}{k} + \phi^{j,s}\right)$ . This fact will be explained in more details in Section 7.3.

**Theorem 7.6.** [40] *Two generic, prepared convergent unfoldings  $X$  and  $\tilde{X}$ , in the same formal orbital class  $\mu$  with respective orbital moduli  $\mathfrak{m}(X)$  and  $\mathfrak{m}(\tilde{X})$ , are equivalent by some local analytic diffeomorphism if and only if their respective orbital moduli  $\mathfrak{m}(X)$  and  $\mathfrak{m}(\tilde{X})$  are equivalent. Moreover  $X$  is locally equivalent to its formal normal form  $\widehat{X}$  if and only if  $\mathfrak{m}(X) = 0$ .*

The pair  $(c, \theta)$  involved in the equivalence between moduli has a geometrical interpretation. First set  $\lambda := \exp 2i\pi\theta/k$  and apply the diagonal mapping

$$(\varepsilon_0, \dots, \varepsilon_{k-1}, x) \mapsto (\varepsilon_0 \lambda^{-1}, \dots, \varepsilon_j \lambda^{j-1}, \dots, \varepsilon_{k-1} \lambda^{k-2}, x \lambda)$$

to  $X$  so that the moduli of the new unfolding, still written  $X$ , differs from the original by a shift in the indices  $j$  of offset  $\theta$ , as explained in Section 4.1. According to Corollary 4.11 we may as well restrict our study now to fibered conjugacies  $\Psi$  between  $X$  and  $\tilde{X}$  fixing  $\{y = 0\}$ . Under these assumptions we have

$$\Psi : (\varepsilon, x, y) \mapsto (\varepsilon, x, y(c + o(1))).$$

This very fact explains why  $c$  is independent on the cell  $\mathcal{E}_\ell$  in the equivalence relation (7.2).

**7.2. Parametric normalization: proof of Theorem 7.2.** In this section we solve the inverse problem on a given parametric cell  $\mathcal{E}_\ell$  when  $\mu_0$  is not in  $\mathbb{R}_{\leq 0}$ . Given any collection

$$\mathfrak{m}_\ell := (\phi^{j,s})_j \in \prod_{\mathbb{Z}/k\mathbb{Z}} \mathcal{H}_\ell\{h\}$$

we can fix  $\eta > 0$  such that every  $\phi^{j,s}$  belongs to  $\text{Holo}_c(\mathcal{E}_\ell \times \eta\mathbb{D})'$ . The strategy is to synthesize a  $k$ -tuple of sectorial functions  $(H^j)_j$  whose transition maps over saddle parts are determined by  $\mathfrak{m}_\ell$  as in (7.3) below, then to recognize that they actually are sectorial first-integrals of a holomorphic vector field  $X_\varepsilon$  in normal form.

We repeat the recipe of Theorem 5.5 in order to solve the nonlinear equation

$$(7.3) \quad H^{j+1} = H^j \exp\left(2i\pi\mu/k + \phi^{j,s} \circ H^j\right),$$

by successively solving the linear Cousin problem of Proposition 6.25 in the way we explain now. For  $\varepsilon := 0$  this is precisely the technique of [42].

We want to find a solution of  $N = \frac{1}{2\pi i} \mathcal{F}(\mathfrak{m}_\ell, N)$  with  $N \in \text{Holo}_c(V_\ell^j \times r\mathbb{D})'$ , where  $\mathcal{F}$  is given in (6.22), and we build one through an iterative process. We start from

$$N_0 := (0)_j$$

and build

$$N_{n+1} := \frac{1}{2i\pi} \mathcal{F}(\mathfrak{m}_\ell, N_n)$$

given by Proposition 6.25. The fact that each sequence  $(N_n^j)_n$  converges uniformly to some  $N^j \in \text{Holo}_c(V_\ell^j \times r\mathbb{D})'$  for some  $r > 0$  follows in every other respect the argument presented in the proof of Theorem 5.5, thus we shall not repeat it here.

So far we have built a  $k$ -tuple of bounded, holomorphic functions  $N = (N^j)_j$  satisfying the next properties.

**Corollary 7.7.** *Assume  $\tau = 0$  (which particularly implies  $\mu_0 \notin \mathbb{R}_{\leq 0}$ ). Let*

$$H^j := \widehat{H}^j \exp N^j$$

be the canonical first-integral associated with  $N^j$ .

- (1)  $(H^j)_j$  is a solution of (7.3).
- (2) Up to decrease  $r > 0$  we can assume that:
  - (a)  $N$  is  $\eta$ -adapted (as in Definition 6.24), more precisely:

$$\|H^j\| \leq rC$$

for some constant  $C > 0$ ,

- (b)  $\left| x \frac{\partial N^j}{\partial x} \right| \leq \frac{1}{3}$  and  $\left| y \frac{\partial N^j}{\partial y} \right| \leq \frac{1}{3}$  on  $V^j \times r\mathbb{D}$ .

*Proof.*

- (1) Because  $H^j = \widehat{H}^j \exp N^j$  and  $\widehat{H}^{j+1} = \widehat{H}^j \exp 2i\pi\mu/k$  (see (6.8)) we have

$$\frac{H^{j+1}}{H^j} = \exp(2i\pi\mu/k + N^{j+1} - N^j).$$

Because  $(N^j)_j$  is obtained as the fixed-point of the Cauchy-Heine operator

$$(N^j)_j \mapsto \frac{1}{2i\pi} \mathcal{F}\left((\phi^{j,s})_j, (N^j)_j\right),$$

according to Proposition 6.25 (1) the identity  $N^{j+1} - N^j = \phi^{j,s} \circ H^j$  holds, which validates the claim.

- (2)
  - (a) This is clear thanks to Proposition 6.25.
  - (b) Up to decrease slightly  $\eta$  we can assume that the derivative of each component of  $\mathfrak{m}_\ell$  is bounded on  $\eta\mathbb{D}$ . From the construction of  $N^j$  and Proposition 6.25 (3) we have

$$\frac{\left\| y \frac{\partial N_{n+1}}{\partial y} \right\|}{1 + \left\| y \frac{\partial N_n}{\partial y} \right\|} \leq \frac{rK}{2i\pi} \|\mathfrak{m}'\| \exp \|N_n\| \leq \frac{1}{4},$$

if  $r$  is taken small enough. The conclusion follows by taking the limit  $n \rightarrow \infty$ . The argument for  $x \frac{\partial N}{\partial x}$  is identical.  $\square$

Now define

$$X^j := \widehat{X} + yR^j \frac{\partial}{\partial y}$$

with

$$(7.4) \quad R^j := -\frac{P \frac{\partial N}{\partial x} + y(1 + \mu x^k) \frac{\partial N}{\partial y}}{1 + y \frac{\partial N^j}{\partial y}}.$$

**Lemma 7.8.**

- (1)  $X^j \cdot N^j = -R^j$  or, equivalently,  $X^j \cdot H^j = 0$ .
- (2)  $R^{j+1} = R^j$  on  $\mathcal{V}^j$ 's.

*Proof.* This is formally the same proof as for  $\varepsilon = 0$ : we refer to [42] for details.

- (1) follows from elementary calculations.
- (2) is equivalent to showing  $X^j \cdot H^{j+1} = 0$ . But this condition is met because of (1) and the fact that  $H^{j+1}$  is a function of  $H^j$ , as per (7.3).  $\square$

The lemma indicates that all pieces of  $(R^j)_j$  glue together into a holomorphic function  $R$ . From (7.4) and the estimates on the derivatives of  $N^j$  obtained in Corollary 7.7 we conclude that  $R$  is bounded near the roots of  $P_\varepsilon$  (hence Riemann's theorem on removable singularities applies). The argument of Section 6.4.3 can now be invoked identically with  $Q := R$  to obtain

$$R(x, y) = \sum_{n>0} r_n(x) y^n$$

for some polynomials  $r_n$  in  $x$  of degree at most  $k$ . We can simplify  $R$  further by applying to  $\widehat{X} + Ry \frac{\partial}{\partial y}$  the change of coordinates

$$(x, y) \mapsto (x, y \exp N(y)) \quad , \quad N \in y\mathbb{C}\{y\}$$

where

$$N' = -\frac{R(0, y)}{y(1 + R(0, y))}.$$

The new vector field  $\widehat{X} + \widetilde{R}y \frac{\partial}{\partial y}$  satisfies  $\widetilde{R} \in x\mathcal{H}_\ell\{y\}[x]_{<k}$ , as sought.

*Remark 7.9.* Notice that Lemma 7.8 asserts  $(x, y) \mapsto (x, N_\ell^j(x, y))$  is a fibered normalization of  $\mathcal{X}$  over squid sectors.

**7.3. Compatibility condition.** Here we impose no restriction on  $\mu_0$ .

7.3.1. *Node-leaf coordinates.* To each squid sector  $V_\ell^j$  we attach a unique natural coordinate  $h$  which parametrizes the space of leaves  $\Omega_\ell^j$  over that sector: this coordinate corresponds to values taken by the canonical first-integral  $H_\ell^j$  (with connected fibers) as defined in Corollary 7.7. Moreover,

$$H_\ell^j\left(V_\ell^j \times (\mathbb{C}, 0)\right) = \mathbb{C}.$$

This comes from the fact that the sector's shape adheres to the point in a node-like configuration, forcing the model first integral  $\widehat{H}_\ell^j$  to be surjective: a complete proof of the above statement can be found in [40]. This space of leaves is customarily compactified as the Riemann sphere  $\Omega_\ell^j$  by adding the point  $\infty$  corresponding to the “vertical separatrices”  $\{x = x^{j,n}\}$  of the node-type singularity.

Because we deal with convergent unfoldings, this coordinate is completely determined by the space of leaves of the singular point  $x^{j,n}$  of node type attached to  $V^j$ , with two distinguished leaves corresponding to 0 (along  $\{y = 0\}$ ) and  $\infty$  (along  $\{x = x^{j,n}\}$ ). In particular, it remains the same when we change the point(s) of saddle type  $x^{j,s}$  and  $x^{\sigma(j),s}$  attached to a sector  $V^j$  but leave the point of node type  $x^{j,n}$  unchanged, while passing from one cell to another.

Let us prove briefly the result on which the compatibility condition is built. We recall that  $\rho_\varepsilon$  is the radius of a disk containing all roots of  $P_\varepsilon$ , as defined by (6.14).

**Lemma 7.10.** *For every  $x_* \in V_\ell^j \setminus \rho_\varepsilon \mathbb{D}$  the partial mapping*

$$h_\ell^j : y \mapsto H_\ell^j(x_*, y)$$

*is a local diffeomorphism near 0 whose multiplier at 0 does not depend on  $\ell$ . In particular for any  $\ell$  such that  $\mathcal{E}_\ell \cap \mathcal{E}_{\bar{\ell}} \neq \emptyset$ , the diffeomorphism*

$$\delta := h_{\bar{\ell}}^j \circ \left(h_\ell^j\right)^{\circ - 1}$$

*is tangent-to-identity. Moreover there exists  $\eta_1, \eta_2, r > 0$  such that for all  $\varepsilon \in \text{cl}(\mathcal{E}_\ell \cap \mathcal{E}_{\bar{\ell}})$*

$$\eta_1 \mathbb{D} \subset \delta_\varepsilon(r\mathbb{D}) \subset \eta_2 \mathbb{D}$$

*and  $\delta_\varepsilon$  is injective on  $r\mathbb{D}$ .*

*In the sequel we write this map  $\delta_{\bar{\ell} \leftarrow \ell}$ .*

*Proof.* According to Corollary 7.7 we have

$$H_\ell^j(x_*, y) = y \widehat{H}_\ell^j(x_*, 1) + o(y).$$

Since  $x_*$  lies outside the disk containing the roots of  $P$  the value of  $\widehat{H}_\ell^j(x_*, 1)$ , as fixed by the determination chosen in (6.7), does not depend on  $\ell$  (but it does on  $j$ ). The existence of  $\eta_1, \eta_2, r > 0$  satisfying the expected properties is a consequence of [40, Corollary 8.8] and Lemma 6.23 (1).  $\square$

**Definition 7.11.** For a choice of  $x_*^j \in V^j \setminus \rho_\varepsilon \mathbb{D}$  we call  $h_\ell^j$  the **node-leaf coordinate** of the unfolding  $X_\ell$  above  $x_*$  in the sector  $V_\ell^j$  and relative to the cell  $\mathcal{E}_\ell$ .

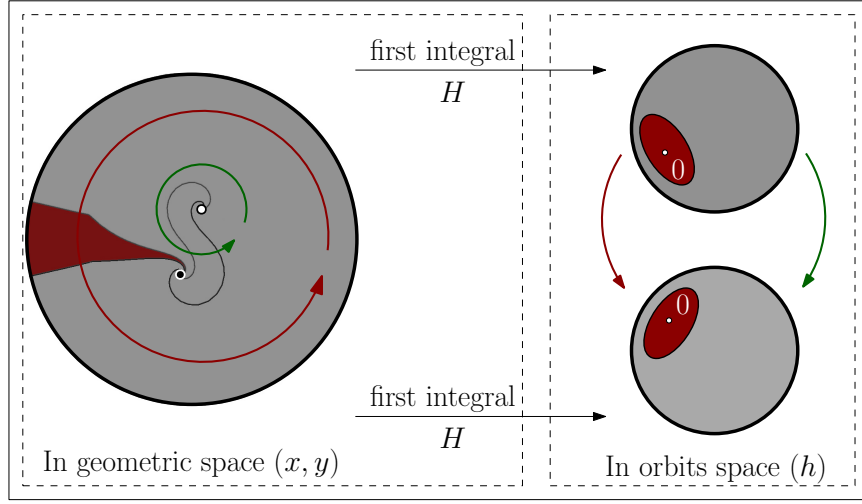


FIGURE 7.1. Passing from geometric to orbits space *via* the sectorial first integral  $H^j$ . Colored arrows show how the change of determination in  $H^j$  takes place as a mapping between two sectorial spaces of leaves: the necklace dynamics.

7.3.2. *Necklace dynamics.* Here we work in a fixed germ of a cell  $\mathcal{E}_\ell$  for fixed  $\varepsilon \in \mathcal{E}_\ell$ ; we drop the  $\ell$  and  $\varepsilon$  indices whenever not confusing. According to the constructions performed in [40], and hinted at by Theorem 7.2, the orbital modulus  $(\mu, \mathfrak{m}(X))$  of a convergent unfolding encodes the way the different node-leaf coordinates glue above the intersection of squid sectors:

$$\begin{cases} H^{j+1} = H^j \exp(2i\pi\mu/k + \phi^{j,s} \circ H^j) & \text{above } V^{j,s}, \\ H^{\sigma(j)} = L_{v_j} \circ H^j & \text{above } V^{j,g}, \end{cases}$$

where

$$L_c : h \mapsto ch \quad , \quad c \neq 0,$$

and  $v^j = v_\ell^j \in \mathbb{C}^\times$  relates to the dynamical invariants  $\mu$  and the residues  $\left(\frac{1}{P_\ell'(x^m)}\right)_m$  at the roots  $(x^m)_{0 \leq m \leq k}$ : indeed, the ramification at the linear level of the first integral at a singular point, given by  $\exp\left(-2i\pi \frac{1+\mu_\varepsilon(x^m)^k}{P_\ell'(x^m)}\right)$ , is equal to the product of all ramifications when crossing sectors while turning around the point, *i.e.* to the product of one factor  $\exp 2i\pi\mu/k$  for each crossed sector  $V^{j,s}$  and one factor  $v_j$  for each crossed sector  $V^{j,g}$ . It is therefore rather natural to consider the germs of diffeomorphisms in node-leaf coordinate

$$(7.5) \quad \begin{aligned} \psi_\ell^{j,s} &: h \mapsto h \exp\left(2i\pi\mu/k + \phi_\ell^{j,s}(h)\right), \\ \psi_\ell^{j,g} &: h \mapsto v_\ell^j h, \end{aligned}$$

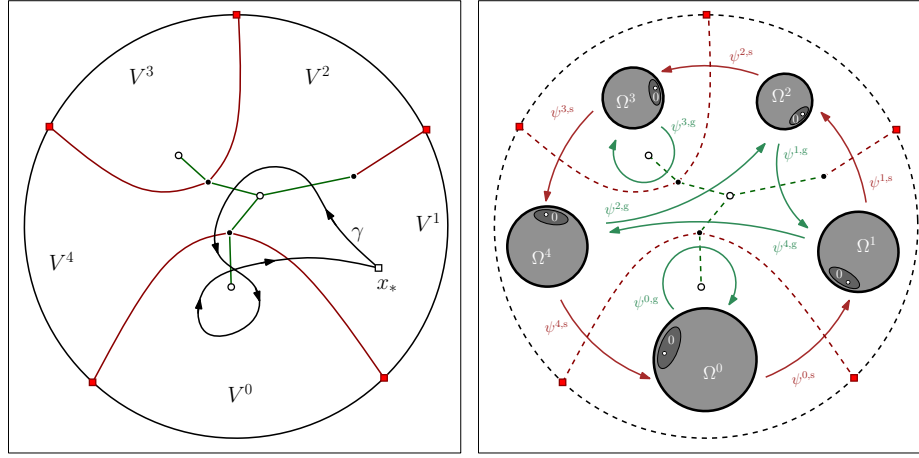


FIGURE 7.2. Schematics of the necklace dynamics and of the corresponding sectorial decomposition for  $k = 5$  and  $\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 4 & 1 & 3 & 2 \end{pmatrix}$ . The loop  $\gamma \in \pi_1(\rho\mathbb{D} \setminus P_\varepsilon^{-1}(0), x_*)$  corresponds to the word  $w(\gamma) = s_0^+ g_0^+ s_4^+ s_4^- g_2^- g_1^-$  in necklace dynamics.

where  $(m_\ell)_\ell = m(X)$  and  $m_\ell = \left(\phi_\ell^{j,s}\right)_j$ . Obviously one can do the same construction starting from any tuple  $m \in \prod_\ell \mathcal{H}_\ell \{h\}^k$ .

*Remark 7.12.* For some value of the parameter  $\varepsilon$  in a given cell  $\mathcal{E}_\ell$ , the saddle mappings  $\psi^s$  are entirely determined by  $\mu$  and  $m$ , while the gate mappings  $\psi^g$  are entirely determined by  $\mu$ .

The dynamics induced by these germs is of interest to us only if it encodes the underlying dynamics of the unfolding (weak holonomy group). A necessary condition is that the latter group does not depend on  $\ell$ , *i.e.* on the peculiar way of slicing the space into sectors which is imposed by our construction. Therefore we only want to consider the “abstract” holonomy representation of  $\pi_1(\rho\mathbb{D} \setminus P_\varepsilon^{-1}(0), x_*)$  in the space of leaves. Let us describe this representation (see Figure 7.2 for an example).

**Definition 7.13.** We fix a base-sector  $V^{j_*}$  and a base-point  $x_* \in V^{j_*} \setminus \rho_\varepsilon \mathbb{D}$ , as well as some  $m_\ell = \left(\phi_\ell^{j,s}\right)_j \in \mathcal{H}_\ell \{h\}^k$ .

- (1) To any loop  $\gamma \in \pi_1(\rho\mathbb{D} \setminus P_\varepsilon^{-1}(0), x_*)$  we associate the multiplicative word  $w_\ell(\gamma)$  in the  $4k$  letters  $\{s_j^\pm, g_j^\pm : j \in \mathbb{Z}/k\mathbb{Z}\}$  obtained by keeping track of bounded squid sectors boundaries crossed successively when traveling along  $\gamma$ . The superscript  $+$  (*resp.*  $-$ ) is given to  $s_j$  according to whether one crosses the saddle boundary from  $V^j$  to  $V^{j+1}$  (*resp.* from  $V^{j+1}$  to  $V^j$ ), “in the same direction” as  $\psi^{j,s}$  (*resp.*  $(\psi^{j,s})^{\circ-1}$ ) applies. For  $g_j$  we take the same convention for gate transitions  $\psi^{j,g}$  and postulate the algebraic relations  $(s_j^\pm)^{-1} = s_j^\mp, (g_j^\pm)^{-1} = g_j^\mp$ .

- (2) To any word  $\mathbf{w} = \prod_n \omega_{j_n}^{\pm}$  we associate the germ

$$\psi_\ell[\mathbf{w}] : h \mapsto \circ_n \left( \psi_\ell^{j_n, \omega} \right)^{\circ \pm 1}.$$

For instance

$$\psi \left[ s_0^+ g_0^+ g_0^+ s_4^+ g_2^- g_1^- \right] = \psi^{0,s} \circ (\psi^{0,g})^{\circ 2} \circ \psi^{4,s} \circ (\psi^{2,g})^{\circ -1} \circ (\psi^{1,g})^{\circ -1}.$$

- (3) We write

$$\mathcal{W}_\ell := \mathbf{w}_\ell \left( \pi_1 \left( \rho \mathbb{D} \setminus P_\varepsilon^{-1}(0), x_* \right) \right)$$

the image group of **admissible words**, that is all the words corresponding to all the encodings (1) of a loop with given base-point  $x_*$  in a disk of given radius  $\rho$  punctured with the roots of  $P_\varepsilon$ .

- (4) Let  $\mathbf{m} = (\mathbf{m}_\ell)_\ell \in \prod_\ell \mathcal{H}_\ell \{h\}$ . The collection of image groups  $\mathcal{G}(\mathbf{m}) = (\mathcal{G}_\ell)_\ell$  of germs of a biholomorphism fixing 0 given by

$$\mathcal{G}_\ell := \psi_\ell[\mathcal{W}_\ell],$$

is called the **necklace dynamics** associated to  $(\mu, \mathbf{m})$  based at the sector  $V^{j_*}$ .

*Remark 7.14.*

- (1) To keep notations light we write  $\psi_\ell[\gamma]$  instead of  $\psi_\ell[\mathbf{w}_\ell(\gamma)]$  for  $\gamma \in \pi_1(\rho \mathbb{D} \setminus P^{-1}(0), x_*)$ . The context will never be ambiguous.
- (2) Obviously the morphisms  $\mathbf{w}_\ell$  and  $\mathbf{w}_{\tilde{\ell}}$  are distinct. The change of cell in  $\mathcal{E}_\ell \cap \mathcal{E}_{\tilde{\ell}}$  can be translated algebraically as a group isomorphism  $\mathcal{W}_\ell \rightarrow \mathcal{W}_{\tilde{\ell}}$ . For instance when  $k = 1$  the isomorphism acts on generators as

$$\begin{cases} g^+ & \mapsto g^- s^+ \\ s^+ g^- & \mapsto g^+ \end{cases}$$

with notations of Figure 7.3.

*Remark 7.15.*

- (1) The groups  $\mathcal{W}_\ell$  and  $\mathcal{G}_\ell$  do not depend on the particular choice of the base-point  $x_* \in V^{j_*}$ , but do depend on the base-sector  $V^{j_*}$ .
- (2) Changing the base-sector from  $V^{j_*}$  to another sector  $V^j$  induces an inner conjugacy between respective necklace dynamics.

### 7.3.3. Compatibility condition.

**Definition 7.16.** Let  $\mathbf{m} \in \prod_\ell \mathcal{H}_\ell \{h\}^k$  and  $\mu \in \mathbb{C} \{\varepsilon\}$ . We say that  $(\mu, \mathbf{m})$  satisfies the **compatibility condition** if the different necklace dynamics (*i.e.* abstract holonomy pseudogroups) combined to form  $\mathcal{G}(\mathbf{m})$  are conjugate, in the sense that there exists  $x_* \in \rho \mathbb{D} \setminus \rho_\varepsilon \mathbb{D}$  in a fixed base sector  $V^{j_*}$  such that for every  $\ell, \tilde{\ell}$  and any connected component  $C$  of  $\mathcal{E}_\ell \cap \mathcal{E}_{\tilde{\ell}} \neq \emptyset$  there exists a (perhaps small) subdomain  $\Lambda \subset C$  such that for all  $\varepsilon \in \Lambda$  there exists  $\delta_{\tilde{\ell} \leftarrow \ell, \varepsilon} \in \text{Diff}(\mathbb{C}, 0)$  satisfying:

- $\delta'_{\tilde{\ell} \leftarrow \ell, \varepsilon}(0) = 1,$

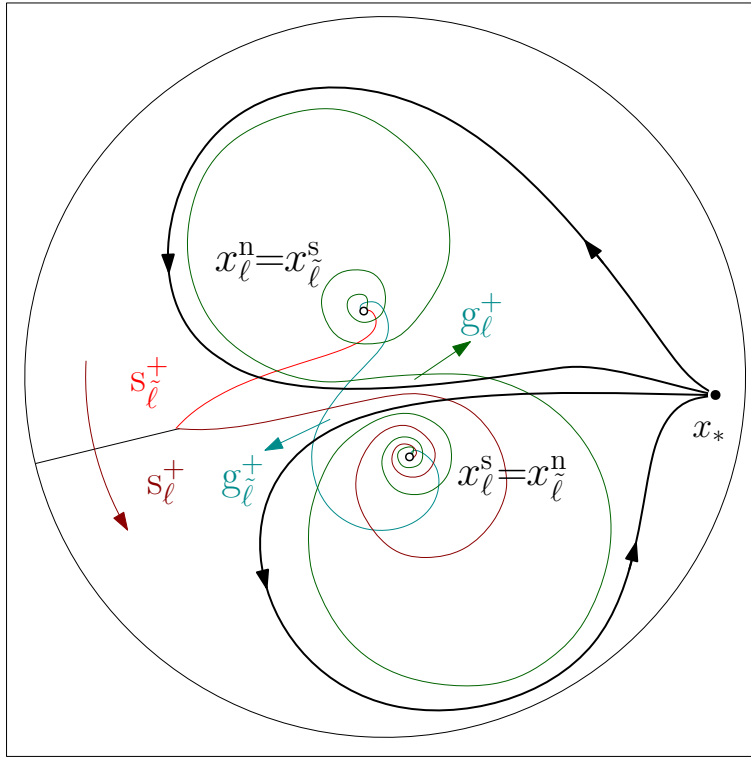


FIGURE 7.3

The generators of the two holonomies on the self intersection of the unique cell  $\mathcal{E}$ .

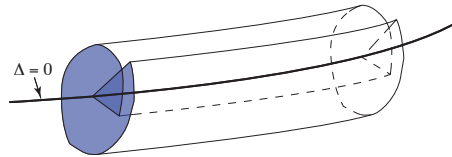


FIGURE 7.4. A cell  $\mathcal{E}_\ell$  having self-intersection around a regular part of  $\Delta_k$ .

- for all  $\gamma \in \pi_1(\rho\mathbb{D} \setminus P^{-1}(0), x_*)$ ,

$$(7.6) \quad \delta_{\ell \leftarrow \ell, \varepsilon}^* \psi_{\ell, \varepsilon}[\gamma] = \psi_{\tilde{\ell}, \varepsilon}[\gamma]$$

where  $\delta^* \psi = \delta^{-1} \circ \psi \circ \delta$  is the usual conjugacy for diffeomorphisms.

*Remark 7.17.* Notice that the compatibility condition also applies when  $\tilde{\ell} = \ell$ , i.e.  $\mathcal{E}_\ell$  is a self-intersecting cell with self-intersection  $\mathcal{E}_\ell^\cap$  around a regular part of  $\Delta_k$  as in Figure 7.4, with the obvious adaptations. To avoid confusion we denote by  $\bar{\varepsilon}$  and  $\tilde{\varepsilon}$  the “distinct points” corresponding to the same parameter  $\varepsilon \in \mathcal{E}_\ell^\cap$  seen from two different overlapping parts of the cell. More generally we decorate objects with corresponding signs, like  $\bar{\psi}$  or  $\tilde{\psi}$  in order to really stand for  $\psi_{\ell, \bar{\varepsilon}}$  and  $\psi_{\ell, \tilde{\varepsilon}}$  respectively.



**Lemma 7.18.** *If  $(\mu, m)$  is realizable then the compatibility condition holds.*

*Proof.* Fix some point  $x_* \in V^0 \setminus \rho_\varepsilon \mathbb{D}$  and take  $\delta_{\bar{\ell} \leftarrow \ell} := h_\ell^0 \circ (h_\ell^0)^{\circ - 1}$  on  $\mathcal{E}_\ell \cap \mathcal{E}_{\bar{\ell}}$  as in Lemma 7.10.  $\square$

*Remark 7.19.*

- (1) Although we do not impose that the mappings  $\delta_{\bar{\ell} \leftarrow \ell}$  exist on the connected component  $C$  of  $\mathcal{E}_\ell \cap \mathcal{E}_{\bar{\ell}}$ , nor depend analytically on  $\varepsilon \in \Lambda \subset C$ , it will be true retrospectively and the dynamical conjugacies  $\delta_{\bar{\ell} \leftarrow \ell}$  are always of the form described in Lemma 7.10. In particular the collection  $(\delta_{\bar{\ell} \leftarrow \ell})_{\ell, \bar{\ell}}$  is a cocycle:

$$\delta_{\ell_2 \leftarrow \ell_1} \circ \delta_{\ell_1 \leftarrow \ell_0} = \delta_{\ell_2 \leftarrow \ell_0}$$

whenever all three mappings are simultaneously defined.

- (2) The compatibility condition could be weakened further. The existence of  $\delta_{\bar{\ell} \leftarrow \ell}$  as above is only needed for  $\varepsilon$  belonging to a set  $\Lambda$  of full analytic Zariski closure, *i.e.* such that if a holomorphic function  $f$  on  $C$  satisfies  $f|_\Lambda = 0$  then  $f = 0$ . The cornerstone of the proof of the Realization Theorem consists indeed in applying Corollary 7.3: it suffices to check whether the identity  $R_\ell - R_{\bar{\ell}} = 0$  holds on every connected component  $C$  of  $\mathcal{E}_\ell \cap \mathcal{E}_{\bar{\ell}}$ .

**7.4. Normal forms stitching: proof of Orbital Realization Theorem when  $\mu_0 \notin \mathbb{R}_{\leq 0}$  and  $\tau = 0$ .** Thanks to Lemma 7.18, only the converse direction of the Realization Theorem still requires a full proof at this stage. Assume then that the compatibility condition holds. Let us fix a base point  $x_*$  in a base sector  $V^{j*}$  and pick  $\varepsilon \in \Lambda \subset C \subset \mathcal{E}_\ell \cap \mathcal{E}_{\bar{\ell}}$  as in Definition 7.16. Recalling Lemma 7.10, the tangent-to-identity mapping

$$(7.7) \quad \Psi : (x_*, y) \mapsto \left( x_*, (h_\ell^0)^{\circ - 1} \circ \delta_{\bar{\ell} \leftarrow \ell} \circ h_\ell^0 \right)$$

conjugates the weak holonomy pseudogroups given by the representation

$$\mathfrak{h}_\ell : \pi_1(\rho \mathbb{D} \setminus P^{-1}(0), x_*) \longrightarrow \text{Diff}(\{x = x_*\}, 0).$$

Let us formulate a direct consequence of the main results of [11] (see [2]) in a manner adapted to our setting.

**Lemma 7.20.** *The map  $\varepsilon \in \mathcal{E}_\ell \mapsto \left( v_\varepsilon^j \right)_{j \in \mathbb{Z}/k\mathbb{Z}}$  is holomorphic and locally injective. In particular there exists a subdomain  $\Lambda' \subset \Lambda$  such that for all  $\varepsilon \in \Lambda'$ , every singular point of  $X_\ell$  and  $X_{\bar{\ell}}$  is hyperbolic.*

Using an extension of the Mattei-Moussu construction for hyperbolic singularities (see below) we can analytically continue  $\Psi$  (defined in (7.7)) on a whole neighborhood of  $\{y = 0\}$  as a fibered equivalence between  $\mathcal{X}_\ell$  and  $\mathcal{X}_{\bar{\ell}}$ . The argument developed in Section 5.6 (to prove uniqueness of the normal form) is performed for fixed  $\varepsilon$ , therefore there exists

$$c \in \mathbb{C}^\times$$

such that

$$R_{\ell,\varepsilon}(x, cy) = R_{\bar{\ell},\varepsilon}^-(x, y).$$

But the conjugacy  $\Psi$  is tangent to the identity in the  $y$ -variable thus  $c = 1$ . Therefore  $R_{\ell,\varepsilon} = R_{\bar{\ell},\varepsilon}^-$  on  $\Lambda$ , thus on  $C$  by analytic continuation. Since this argument can be carried out for any connected component  $C$  of any cellular intersection, Corollary 7.3 yields the conclusion.

*Remark 7.21.* In fact  $\Psi$  itself must be the identity, therefore

$$\delta_{\bar{\ell} \leftarrow \ell} = h_{\ell}^0 \circ (h_{\ell}^0)^{\circ - 1}$$

as in Lemma 7.10.

There only remains a single gap in the above argument, namely that of extending  $\Psi$  near each hyperbolic singularity. Let  $\mathcal{F}_{\ell}$  be the foliation induced by  $\mathcal{X}_{\ell}$  and take a germ  $\Sigma \subset \{x = x_*\}$  of a transverse disk at  $(x_*, 0)$  in such a way that  $\Psi$  is holomorphic and injective on  $\Sigma$ . The union of the saturation  $\text{Sat}_{\mathcal{F}_{\ell}}(\Sigma)$  and the vertical separatrices  $P^{-1}(0)$  is a full neighborhood of  $\{y = 0\}$  since no singular point of  $\mathcal{F}_{\ell}$  is a node. Therefore  $\Psi$  can be extended as a fibered, injective mapping by the usual path-lifting technique except along the separatrices  $P^{-1}(0)$ . Up to divide  $\mathcal{X}_{\ell}$  and  $\mathcal{X}_{\bar{\ell}}$  by a local holomorphic unit near each singularity, we can assume that the hypothesis of Lemma 5.13 are met. This completes the proof of the Realization Theorem when  $\mu_0 \notin \mathbb{R}_{\leq 0}$ .

## 8. GENERAL CASE $\tau > 0$

In this section we fix  $\tau \in \mathbb{N}$  such that

$$\mu_0 + \tau(k+1) \notin \mathbb{R}_{\leq 0}.$$

**8.1. End of proof of (orbital) Normalization, Uniqueness and Realization Theorems.** We explain now how to reduce the case  $\tau > 0$  to the case  $\tau = 0$  already dealt with. We exploit the observation that formally  $\text{Section}_k\{P^{\tau}y\}$  is the pullback of  $\text{Section}_k\{y\}$  by the mapping

$$(8.1) \quad T : (\varepsilon, x, y) \mapsto (\varepsilon, x, P_{\varepsilon}^{\tau}(x)y).$$

Albeit not invertible along the lines  $\{P_{\varepsilon}(x) = 0\}$  (its image is not a neighborhood of  $\{y = 0\}$ ), the mapping  $T$  transforms the model unfolding

$$(8.2) \quad \widehat{X}(x, y) = P_{\varepsilon}(x) \frac{\partial}{\partial x} + y(1 + \mu_{\varepsilon}x^k) \frac{\partial}{\partial y}$$

into

$$\widehat{Y} := T^* \widehat{X} = P_{\varepsilon} \frac{\partial}{\partial x} + (1 + \tau P'_{\varepsilon} + \mu_{\varepsilon}x^k)y \frac{\partial}{\partial y}.$$

Observe that

$$\tau P'(x) + \mu x^k \sim_{\infty} (\tau(k+1) + \mu)x^k,$$

so that involving  $P^{\tau}$  in this way shifts the formal invariant by  $\tau(k+1)$ . Apart from the fact that  $\widehat{Y}$  is not in prepared form (4.2), all the theory developed before for the realization theorem applies in this case too. Let us be more specific. The key property we used intensively was to be able to perform most arguments for fixed

$\varepsilon$ . This was proved sufficient because automorphisms of prepared forms fixing the  $x$ -variable must also fix the canonical parameter  $\varepsilon$ .

**Lemma 8.1.**

(1) *The group of (fibered) symmetries*

$$(\varepsilon, x, y) \mapsto (\eta(\varepsilon), X(\varepsilon, x), Y(\varepsilon, x, y))$$

*of (the unfolding of) vector fields defined by (8.2), is isomorphic to  $\mathbb{Z}/k\mathbb{Z} \times \mathbb{C}^\times$  through the linear representation*

$$(8.3) \quad \zeta_0 : \mathbb{Z}/k\mathbb{Z} \times \mathbb{C}^\times \longrightarrow \mathrm{GL}_{k+2}(\mathbb{C})$$

$$(8.4) \quad (\theta, c) \mapsto ((\varepsilon_0, \dots, \varepsilon_{k-1}, x, y) \mapsto (\alpha \varepsilon_0, \varepsilon_1, \dots, \varepsilon_{k-1} \alpha^{-(k-2)}, \alpha x, cy))$$

*where  $\alpha = \exp 2i\pi\theta/k$ .*

(2) *This statement continues to hold in the more general case of an unfolding*

$$(8.5) \quad P_\varepsilon(x) \frac{\partial}{\partial x} + (1 + Q_\varepsilon(x))y \frac{\partial}{\partial y},$$

*where  $Q_\varepsilon \in \mathbb{C}[x]_{\leq k}$  is a polynomial in  $x$  of degree at most  $k$  and  $Q_\varepsilon(0) = 0$ , save for the fact that the representation  $\zeta_\tau : \mathbb{Z}/k\mathbb{Z} \times \mathbb{C}^\times \rightarrow \mathrm{Diff}(\mathbb{C}^{k+2}, 0)$  has no reason to be linear.*

(3) *In particular, any symmetry tangent to the identity is the identity.*

*Proof.* (1) is shown in [40]. For (2), there exists a diffeomorphism  $\Psi$  of the form  $(\varepsilon, x, y) \mapsto (\eta, X, Y)$  transforming a general formal normal form (8.5) to the standard formal normal form (8.2). Then any symmetry of a general formal normal form is given by  $\Psi^{-1} \circ \zeta_0(\theta, c) \circ \Psi$  for some  $(\theta, c) \in \mathbb{Z}/k\mathbb{Z} \times \mathbb{C}^\times$ . (3) follows.  $\square$

**Remark 8.2.**

- (1) In view of Lemma 8.1, we could have replaced (8.2) by some other (8.5) in all our constructions regarding realization. In such a form, the parameters are again canonical, as long as we consider changes of coordinates tangent to the identity.
- (2) The structure of sectors, and also the decomposition in cells  $\mathcal{E}$ , are determined from  $P_\varepsilon$  alone in (8.2): only the size of the neighborhoods of the origin in  $x$ -space and in parameter space might need to be slightly adjusted when passing from the coordinates  $(x, y)$  to the coordinates  $(x, P^\tau(x)y)$ . Hence, instead of considering (8.2), we could have taken a normal form (8.5) with the same sectors  $V_\ell^j$  and same cells  $\mathcal{E}_\ell$ .

The rest of our argument relies on the next transport result.

**Lemma 8.3.**

- (1)  $(\mu, \mathfrak{m})$  satisfies the compatibility condition if and only if  $(\mu + \tau(k+1), \mathfrak{m})$  does.
- (2) Take  $\mathcal{X}$  in orbital normal form (2.4) with  $\tau := 0$ . Consider the corresponding unfolding

$$\mathcal{Y} := T^*\mathcal{X} = P_\varepsilon \frac{\partial}{\partial x} + y \left( 1 + \tau P'_\varepsilon + \mu x^k + R(x, P^\tau y) \right) \frac{\partial}{\partial y},$$

*for  $T$  as in (8.1). Then  $\mathcal{X}$  and  $\mathcal{Y}$  have same orbital invariant  $\mathfrak{m}(\mathcal{X}) = \mathfrak{m}(\mathcal{Y})$ .*

We postpone the proof till Section 8.1.4. In the meantime we finish establishing the main theorems.

8.1.1. *End of proof of Orbital Realization Theorem.* Let  $(\mu, \mathfrak{m})$  satisfy the compatibility condition and let us prove it is realizable as the orbital modulus of some convergent unfolding. Normalization and Realization Theorems so far hold when  $\tau = 0$  (in particular  $\mu_0 \notin \mathbb{R}_{\leq 0}$ ): in that case  $\mathfrak{m}$  is the modulus of an unfolding in normal form

$$(8.6) \quad P_\varepsilon(x) \frac{\partial}{\partial x} + y \left( 1 + \mu(\varepsilon)x^k + y \sum_{j=1}^k x^j R_j(y) \right) \frac{\partial}{\partial y}.$$

To consider the case  $\tau > 0$ , we need to use the following remark: the whole proof for  $\tau = 0$  would have worked *verbatim* with the formal part and parameters given in some alternate form (8.5). This would have produced a realization of the form

$$(8.7) \quad P_\varepsilon(x) \frac{\partial}{\partial x} + y \left( 1 + Q_\varepsilon(x) + y \sum_{i=j}^k x^j R_j(y) \right) \frac{\partial}{\partial y},$$

with new canonical parameters. Let  $\tau$  be a positive integer such that  $\mu_0 + \tau(k+1) > 0$  and consider the new formal normal form

$$\widehat{Y}(x, y) = P_\varepsilon(x) \frac{\partial}{\partial x} + (1 + \tau P'_\varepsilon(x) + \mu(\varepsilon)x^k)y \frac{\partial}{\partial y}$$

corresponding to  $Q_\varepsilon := \tau P'_\varepsilon + \mu(\varepsilon)x^k$  in (8.5), with formal invariant

$$\widehat{\mu} := \mu + \tau(k+1).$$

But according to Lemma 8.3:

- (1)  $(\widehat{\mu}, \mathfrak{m})$  is compatible,
- (2) it is realized in the form (8.7),
- (3) the change  $(x, y) \mapsto (x, P_\varepsilon^{-\tau}(x)y)$  transforms (8.7) back into an unfolding

$$P_\varepsilon(x) \frac{\partial}{\partial x} + y \left( 1 + \mu(\varepsilon)x^k + \sum_{j=1}^k x^j R_j(P_\varepsilon^\tau(x)y) \right) \frac{\partial}{\partial y},$$

- (4) the latter unfolding is holomorphic on a whole neighborhood of  $(\mathbb{C}^{k+2}, 0)$ , and is therefore a realization of  $(\mu, \mathfrak{m})$ .

8.1.2. *End of proof of Normalization Theorem.* The proof we just finished shows that any realizable  $(\mu, \mathfrak{m})$  can be realized in normal form.

8.1.3. *End of proof of Uniqueness Theorem.* Each vector field  $\mathcal{X}_\varepsilon$  of the unfolding in normal form (2.4) is holomorphic on a domain

$$D(r) := \bigcup_{\varepsilon \in (\mathbb{C}^k, 0)} \{(\varepsilon, x, y) : |x| < \rho, |P_\varepsilon^\tau(x)y| < r\}.$$

Let  $E$  be a neighborhood of 0 in  $\mathbb{C}^{k+2}$  and  $\Psi : E \rightarrow (\mathbb{C}^{k+2}, 0)$  be a local conjugacy between normal forms  $\mathcal{X}$  and  $\widetilde{\mathcal{X}}$ , which can be assumed fibered thanks to Corollary 4.11 (2). We can use the Uniqueness Theorem in the coordinates  $(x, P^\tau(x)y)$  (given by the Uniqueness Theorem for  $\mu_0 \notin \mathbb{R}_{\leq 0}$ , already proved) at the cost of showing that  $T^*\Psi = T \circ \Psi \circ T^{-1}$  is holomorphic and injective on some small neighborhood of  $(0, 0)$  uniformly in  $\varepsilon$ . This is not trivial since the image of  $E \cap \{\varepsilon = \text{cst}\}$

by  $T$  can never be such a uniform neighborhood of  $(0,0)$  if  $E$  is bounded in the  $y$ -variable. But  $T(D(r) \cap \{\varepsilon = \text{cst}\})$  is, so we wish to extend  $\Psi$  to some  $D(r') \subset D(r)$ . The usual path-lifting technique in the foliation  $\mathcal{F}_\varepsilon$  induced by  $\mathcal{X}_\varepsilon$  allows to extend  $\Psi_\varepsilon$  on

$$\mathcal{U}_\varepsilon := \{\varepsilon\} \times \text{Sat}_{\mathcal{F}_\varepsilon}(E) \subset D(r).$$

Using the special form of the normal form  $\mathcal{X}_\varepsilon$  we conclude the proof of the Uniqueness Theorem.

**Lemma 8.4.** *Assume that  $\rho > 0$  is small enough so that  $|\mu_\varepsilon x^k + \tau P'_\varepsilon(x)| < \frac{1}{4}$  for all  $x \in \rho\mathbb{D}$  and all  $\|\varepsilon\|$  small enough. There exists  $r \geq r' > 0$  such that for  $\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(r)$  defined as above one has  $D(r') \subset \bigcup_{\varepsilon \in (\mathbb{C}^k, 0)} \mathcal{U}_\varepsilon \subset D(r)$ .*

*Proof.* For a solution of the flow system

$$\begin{cases} \dot{x} &= -P_\varepsilon(x) \\ \dot{y} &= -y(1 + \mu_\varepsilon x^k + R_\varepsilon(x, y)) \end{cases}$$

with  $t \in \mathbb{R}$  and initial value  $(x_*, y_*)$ , the modulus of  $\phi(t) := |P_\varepsilon^\tau(x(t))y(t)|$  satisfies

$$\dot{\phi} = -\phi \Re \left( 1 + \mu_\varepsilon x^k + R_\varepsilon + \tau P'_\varepsilon \right).$$

Since  $R_\varepsilon(x, 0) = 0$  we can choose  $r$  so small that  $|\mu_\varepsilon x^k + R_\varepsilon + \tau P'_\varepsilon(x)| < \frac{1}{2}$  for all  $(\varepsilon, x, y) \in D(r)$ , and  $\dot{\phi} < -\phi/2$ . Hence starting at  $(x_*, y_*)$  with  $|P_\varepsilon^\tau(x_*)y_*| < r$  and  $|x_*| < \rho$ , the trajectory for positive  $t$  never escapes  $D(r)$ . But  $t \mapsto |y(t)|$  is also exponentially decreasing, therefore we eventually reach a point within  $E$ .

Again, this is the ideal situation, because it may happen that  $x(t)$  exits  $\text{cl}(\rho\mathbb{D})$ . If  $|x(t_0)| = \rho$  then we modify the trajectory  $x$  by solving  $\dot{x} = \pm i P_\varepsilon(x)$  from  $t_0$  on, the sign being chosen so that  $\pm i P_\varepsilon(x(t_0))$  points inside  $\rho\mathbb{S}^1$ , until we reach a point  $x(t_1)$  through which the solution of  $\dot{x} = -P_\varepsilon(x)$  stays in  $\rho\mathbb{D}$  in positive time (*i.e.* accumulate on an attractive singularity). While for  $t \in [t_0, t_1]$  we cannot control the sign of  $\dot{\phi} = \pm \phi \Im(\mu_\varepsilon x^k + R_\varepsilon + \tau P'_\varepsilon)$ , resulting in a probable increase in  $\phi$ , the total amount by which  $\frac{\phi}{r}$  increases is bounded uniformly in  $(x_*, y_*)$  and  $\varepsilon$ . Therefore there exists a radius  $r \geq r' > 0$  for which, if  $(x_*, y_*) \in D(r')$ , the modified trajectory  $t \geq 0 \mapsto (x(t), y(t))$  does not escape from  $D(r)$  and thus eventually enters  $E$ .  $\square$

8.1.4. *Proof of Lemma 8.3.* First, as noted in Remark 8.2, we can choose the same sectors in  $x$  and same cells in the parameter  $\varepsilon$ , possibly after adjusting their diameter. Also, we have chosen to take the linear parts of the  $\psi_\varepsilon^{j,s}$  of the form  $\exp 2i\pi\mu/k$ . This choice is arbitrary. What is needed is that the product of these linear parts be equal to  $\exp 2i\pi\mu$ . Because  $(k+1)\tau \in \mathbb{Z}$ , so that  $\exp 2i\pi\mu = \exp(2\pi i(\mu + (k+1)\tau))$ , we are perfectly entitled to take the same linear parts for  $\mathfrak{m}(\mathcal{X})$  and  $\mathfrak{m}(\mathcal{Y})$ .

The Camacho-Sad index  $\tilde{\lambda}^j$  (*resp.*  $\lambda^j$ ) of the singular point  $(z, 0) \in P^{-1}(0) \times \{0\}$  in  $\mathcal{Y}_\varepsilon$  (*resp.*  $\mathcal{X}_\varepsilon$ ), relatively to the invariant line  $\{y = 0\}$ , is given by

$$\tilde{\lambda}^j = \frac{P'_\varepsilon(z)}{1 + \tau P'_\varepsilon(z) + \mu_\varepsilon z^k}, \quad \lambda^j = \frac{P'_\varepsilon(z)}{1 + \mu_\varepsilon z^k}.$$

Hence,  $\frac{1}{\tilde{\lambda}^j} = \frac{1}{\lambda^j} + \tau$ , yielding  $\exp 2i\pi/\tilde{\lambda}^j = \exp 2i\pi/\lambda^j$ . This means that the gate transition maps are the same for both dynamical necklaces induced by  $(\mu, \mathfrak{m})$  and by

$(\widehat{\mu}, \mathfrak{m})$ . Thus, the holonomies involved in the compatibility condition are the same provided (2) holds. In particular, this means that  $(\widehat{\mu}, \mathfrak{m})$  satisfies the compatibility condition, proving (1).

Show now that  $\mathfrak{m}$  is the analytic part of the modulus of  $\mathcal{Y}$ . It suffices to consider a fixed  $\varepsilon \in \mathcal{E}_\ell$  and a corresponding saddle part  $V^{j,s}$ . Recall how a normalizing map between  $\mathcal{Y}$  and its formal model, as in Remark 6.9, defines the canonical sectorial first integral

$$\widetilde{H}(x, y) = y\widetilde{E}(x)\exp\widetilde{N}^j(x, y),$$

where  $\widetilde{E}(x) = \prod_{j=0}^k (x - x^j)^{-1/\lambda_j}$  is the multiplier in the model first integral of  $\mathcal{Y}_\varepsilon$ . Let  $\psi^{j,s} : h \mapsto h \exp(2i\pi\mu/k + \mathfrak{m}(h))$  be the Martinet-Ramis invariant as in Section 7.1, that is

$$\widetilde{H}^{j+1} = \widetilde{\psi}^{j,s} \circ \widetilde{H}^j.$$

Let us now move to  $\mathcal{X}$ . It is clear that a normalizing map over  $\mathcal{V}^j$  transforming  $\mathcal{X}_\varepsilon$  into its normal form is given by

$$\begin{aligned} (x, y) &\mapsto (x, y \exp N^j(x, y)) \\ N_j(x, y) &= \widetilde{N}_j(x, P_\varepsilon^\tau(x)y). \end{aligned}$$

Moreover, the domain of this map is of the form  $V^j \times \{|P_\varepsilon^\tau(x)y| < r\}$ . Since

$$\prod_{j=0}^k (x - x^j)^{-1/\lambda_j} = \widetilde{E}(x)P^\tau(x)$$

the canonical first integral of  $\mathcal{X}$  has the form

$$H^j(x, y) = E(x)y \exp N_j(x, y) = \widetilde{E}(x)(P^\tau(x)y) \exp \widetilde{N}^j(x, P^\tau(x)y).$$

It follows at once that

$$H^{j+1} = \widetilde{\psi}^{j,s} \circ H^j,$$

yielding the conclusion  $\psi^{j,s} = \widetilde{\psi}^{j,s}$  as expected.

## 8.2. Section of the period operator: end of proof of the Normalization Theorem.

Let  $\mathcal{X}$  be a generic unfolding in orbital normal form (2.4), understood as a derivation. Theorem 6.8 holds regardless of the value of  $\mu_0$  or  $\tau$ . The study performed in Section 6 to establish Theorem 6.2 can be repeated here but for the fact that the canonical section of the period operator needs to be adapted. The mapping defined in (6.3) becomes

$$\begin{aligned} \widetilde{\mathfrak{K}} : \mathbb{C}\{\varepsilon, x, y\}' &\longrightarrow \text{Section}_k\{P^\tau y\} \\ G &\longmapsto \mathfrak{S}_\ell(\mathfrak{T}_\ell(G)) \end{aligned}$$

whose kernel coincides with  $\mathcal{X} \cdot \mathbb{C}\{\varepsilon, x, y\}'$ , *i.e.* the sequence of  $\mathbb{C}\{\varepsilon\}$ -linear operators

$$0 \longrightarrow \mathbb{C}\{\varepsilon, x, y\}' \xrightarrow{\mathcal{X} \cdot} \mathbb{C}\{\varepsilon, x, y\}' \xrightarrow{\widetilde{\mathfrak{K}}} \text{Section}_k\{P^\tau y\} \longrightarrow 0$$

is exact. Up to this modification the temporal part of Realization Theorem is established.

The most obvious reason why one must adapt the target space of the section operator is computational. Proposition 10.5 below recalls the formula for the period of the formal model  $\widehat{X}$  for  $k = 1$ . For  $xy^m \in \text{Section}_k\{y\}$ ,  $m \in \mathbb{N}$ , it may happen that  $\widehat{\mathcal{I}}(xy^m)$  vanishes, exactly when  $m\mu \in \mathbb{Z}_{\leq 0}$ . This situation cannot happen if  $\mu_0 \notin \mathbb{R}_{\leq 0}$ , of course. Pre-composing  $xy^m$  by  $P^\tau(x)y$  yields

$$\widehat{\mathcal{I}}(xP^{m\tau}(x)y^m) = \widehat{\mathcal{I}}(x^{m\tau(k+1)+1}y^m) + \mathcal{O}(\varepsilon),$$

and by hypothesis  $m(\mu_0 + (k+1)\tau) \notin \mathbb{Z}_{\leq 0}$ . As already noticed, the presence of  $P^\tau$  acts as a shift by  $(k+1)\tau$  on powers of  $x$ . Here it guarantees that  $\mathfrak{S}_\ell$  remains invertible. Notice that the map  $\mathfrak{S}_\ell$  needs to undertake the same modification as in (8.1); compare (6.22). We will not go into further details.

**8.3. Alternate normal forms.** The normal forms we propose in the Normalization Theorem are not strictly speaking a generalization of [25, 42], which is what we expected to accomplish in the first place and which we propose as a conjecture.

**Conjecture 8.5.** *Fix  $k \in \mathbb{N}$ , a germ of holomorphic function  $\mu \in \mathbb{C}\{\varepsilon\}$ , and  $\widehat{\tau} \in \mathbb{Z}_{\geq 0}$  such that  $\mu_0 + \widehat{\tau} \notin \mathbb{R}_{\leq 0}$ . Any generic convergent unfolding of a germ of saddle-node holomorphic vector field with the formal invariant  $\mu$  is orbitally conjugate to an unfolding of the form*

$$\widehat{X} + y\widehat{R}\frac{\partial}{\partial y} \quad , \quad \widehat{R} \in x\mathbb{C}[x]_{<k}\{x^{\widehat{\tau}}y\}.$$

Such a form is unique up to conjugacy by linear maps  $(\varepsilon, x, y) \mapsto (\varepsilon, x, c_\varepsilon y)$ ,  $c \in \mathbb{C}\{\varepsilon\}^\times$ .

(A similar conjecture can be stated for the temporal part.) This conjecture is very likely to be true as we almost managed to ascertain both the geometric normalization and the cellular realization in that form. In both questions we encountered difficulties of a technical nature, which can surely be overcome by bringing in tedious estimates.

## 9. BERNOULLI UNFOLDINGS

The primary aim of this section is to establish that the compatibility condition is not trivially satisfied by proving the Parametrically Analytic Orbital Moduli Theorem. The most difficult direction is (1) $\Rightarrow$ (2). The whole proof is geared toward using rigidity results of Abelian finitely generated pseudogroups  $G < \text{Diff}(\mathbb{C}, 0)$ . Let us briefly explain how Abelian pseudogroups come into consideration here. Elements  $\psi_\ell[\gamma]$  and  $\psi_{\bar{\ell}}[\gamma]$  in overlapping cellular necklace dynamics are conjugate by the transition mapping  $\delta_{\ell \leftarrow \bar{\ell}}$  coming from the compatibility condition. The parametric holomorphy of  $m$  forces the equality  $\psi_\ell[\Gamma] = \psi_{\bar{\ell}}[\Gamma]$  for well-chosen loops  $\Gamma$ , from which stems the commutativity relation

$$\psi_\ell[\Gamma] \circ \delta_{\bar{\ell} \leftarrow \ell} = \delta_{\bar{\ell} \leftarrow \ell} \circ \psi_\ell[\Gamma].$$

Such pseudogroups are completely understood and form now a classical topic of complex dynamical systems, we refer for instance to [8, 24]. ‘‘Bernoulli diffeomorphisms’’ (defined below) play a central role in this theory as archetypal examples of solvable and Abelian pseudogroups.

### 9.1. Bernoulli diffeomorphism.

**Definition 9.1.** We say that  $\psi \in \text{Diff}(\mathbb{C}, 0)$  is a **Bernoulli diffeomorphism of index**  $d \in \mathbb{N}$  if there exist  $\alpha, \beta \in \mathbb{C}$  with  $\alpha \neq 0$  such that

$$\psi(h) = \frac{\alpha h}{(1 + \beta h^d)^{1/d}} =: \text{Ber}\left(d, \begin{matrix} \alpha \\ \beta \end{matrix}\right)(h).$$

We define  $\text{Ber}(d)$  the set of all such algebraic functions, regardless of the special values of  $\alpha$  and  $\beta$  (these are in particular germs of analytic diffeomorphisms at the origin). Of course when  $d \neq \tilde{d}$  the intersection  $\text{Ber}(d) \cap \text{Ber}(\tilde{d})$  coincides with the group  $\text{GL}_1(\mathbb{C})$ .

Let us quickly state without proof the next basic property.

**Lemma 9.2.** *The set  $\text{Ber}(d)$  is a group equipped with a semi-direct law. More precisely*

$$\text{Ber}\left(d, \begin{matrix} \alpha \\ \beta \end{matrix}\right) \circ \text{Ber}\left(d, \begin{matrix} \tilde{\alpha} \\ \tilde{\beta} \end{matrix}\right) = \text{Ber}\left(d, \begin{matrix} \alpha \tilde{\alpha} \\ \beta \tilde{\alpha}^d + \tilde{\beta} \end{matrix}\right).$$

The definition of Bernoulli diffeomorphisms is motivated by the following computation.

**Lemma 9.3.** *The necklace dynamics of an unfolding of Bernoulli vector field  $\mathcal{X} = \widehat{X} + y^{d+1}r(x)\frac{\partial}{\partial y}$  consists in Bernoulli diffeomorphisms of index  $d$ . Moreover*

$$m(\mathcal{X}) = -\frac{1}{d} \log\left(1 + 2i\pi d \widehat{\mathcal{T}}(y^d r)\right).$$

*Proof.* As in [46, Section 3.3] one tries and finds an expression for the sectorial first integrals  $H^j$  in the form

$$H^j(x, y) = \frac{\widehat{H}^j(x, y)}{(1 - d f^j(x) y^d)^{1/d}}.$$

Because

$$\begin{aligned} \mathcal{X} \cdot H^j &= \frac{\widehat{H}^j}{(1 - d f^j(x) y^d)^{1/d+1}} \left( (1 - d f^j(x) y^d) y^d r(x) + \mathcal{X} \cdot (f^j(x) y^d) \right) \\ &= \frac{\widehat{H}^j}{(1 - d f^j(x) y^d)^{1/d+1}} \left( y^d r(x) + \widehat{X} \cdot (f^j(x) y^d) \right), \end{aligned}$$

then  $H^j$  is a first integral for  $\mathcal{X}$  if and only if

$$\widehat{X} \cdot (y^d f^j(x)) = -y^d r(x).$$

This equation admits a formal solution (Lemma 4.8) because  $\widehat{X}$  is linear in the  $y$ -variable, and the  $f^j(x) y^d$  are the sectorial solutions of this equation (Theorem 6.8).

In fact  $(x, y) \mapsto \left(x, \frac{y}{(1 - d y^d f^j(x))^{1/d}}\right)$  is the canonical sectorial normalization of  $\mathcal{X}$ .

First notice that by definition of the period operator for the formal model (Definition 6.10) we have for all  $(x, y) \in V^{j,s} \times \mathbb{C}$ :

$$y^d f^{j+1}(x) - y^d f^j(x) = -\widehat{\mathcal{T}}^j(y^d r) \left( \widehat{H}^j(x, y) \right).$$



From the special form of  $H^j$  we deduce

$$\frac{H^{j+1}}{H^j \exp 2i\pi\mu/k} = \left(1 - dy^d \frac{f^{j+1} - f^j}{1 - df^j y^d}\right)^{-1/d} = \exp \left( -\frac{1}{d} \log \left( 1 + d2i\pi \frac{\widehat{\Gamma}^j(y^d r)(\widehat{H}^j)}{(\widehat{H}^j)^d} \times (H^j)^d \right) \right).$$

Because  $\widehat{H}$  is linear in the  $y$ -variable we know that  $\widehat{\Gamma}(y^d r)(h) = \alpha h^d$  for some complex coefficients  $\alpha = \left(\alpha_\varepsilon^j\right)_{j \in \mathbb{Z}/k\mathbb{Z}}$ . Hence  $\frac{\widehat{\Gamma}(y^d r)(\widehat{H})}{\widehat{H}^d} \times H^d = \widehat{\Gamma}(y^d r)(H)$ . The rest follows from (7.3).  $\square$

**9.2. Holomorphic modulus: proof of the Parametrically Analytic Orbital Moduli Theorem.** The direction (2) $\Rightarrow$ (1) is a consequence of Lemma 9.3 above and of Proposition 10.5 below stating that the model period operator  $\widehat{\Gamma}(y^d r)$  is analytic in the parameter when  $k = 1$  and  $d\mu \in \mathbb{Z}$ .

Conversely let us suppose that  $(\mu, m)$  is realizable and that  $m_\ell = \phi|_{\mathcal{E}_\ell \times (\mathbb{C}, 0)}$  for some holomorphic  $k$ -tuple

$$\phi = \left(\phi^j\right)_j \in h\mathbb{C}\{\varepsilon, h\}^k.$$

If  $\phi = 0$  then  $m = m(\widehat{X})$  (Theorem 7.6), so we can as well assume that  $\phi \neq 0$ . We first establish that  $k = 1$  by contraposition, and then present the case  $k = 1$ . That case can be found originally in [43, Proposition 6] for  $\mu = 0$ . We generalize here the result to arbitrary  $\mu$ .

Recall that for  $c \in \mathbb{C}^\times$  we write

$$L_c : h \mapsto ch.$$

9.2.1. *Reduction to the case  $k = 1$ .* Assume then that  $k > 1$  and prove  $\phi = \left(\phi^j\right)_j = 0$ .

For each  $j \in \mathbb{Z}/k\mathbb{Z}$  there exists a cell  $\mathcal{E}_\ell$  for which  $x^{j,s}$  is attached to only one saddle sector  $V^{j,s}$ . Let  $x^{j,n}$  be the node point attached to  $x^{j,s}$  in the boundary of  $V^{j,s}$ . The cell  $\mathcal{E}_\ell$  self-intersects around a regular part of  $\Delta_k$  in such a way that the nature of the points  $x^{j,s}$  and  $x^{j,n}$  is exchanged when seen from one part or the other of the intersection. With the conventions discussed in Remark 7.17, by this we mean

$$\begin{cases} \bar{x}^{j,s} = \bar{x}^{j,n} \\ \bar{x}^{j,n} = \bar{x}^{j,s} \end{cases}$$

We refer to Figures 7.4 and 9.1.

Fix a base-point and base-sector  $x_* \in V^j \setminus \rho_\ell \mathbb{D}$  and take  $\gamma^-, \gamma^+$  two loops based at  $x_*$  of index 1 around respectively  $x_\varepsilon^n$  and  $x_\varepsilon^s$ , and index 0 with respect to the other roots of  $P$  as in Figure 9.1. Let  $\Gamma := \gamma^+ \gamma^-$  be a loop encircling only  $\{x^n, x^s\}$ . The compatibility condition ensures the existence of a tangent-to-identity map

$$\delta := \delta_{\ell \leftarrow \ell}$$

which conjugates the respective necklace dynamics based at  $x_*$ . In particular

$$(9.1) \quad \begin{aligned} \delta^* \bar{\psi}[\gamma^\pm] &= \widetilde{\psi}[\gamma^\pm], \\ \delta^* \bar{\psi}[\Gamma] &= \widetilde{\psi}[\Gamma]. \end{aligned}$$

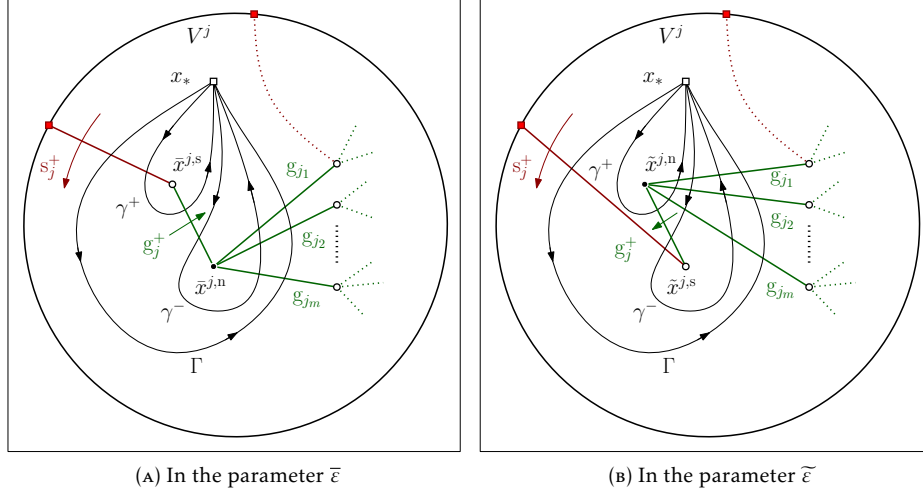


FIGURE 9.1. The construction involved in Lemma 9.4.

**Lemma 9.4.** *We follow the notations of Figure 9.1. Let  $k > 1$ , and let  $m \geq 1$  be the number of singular points different from  $x^{j,s}$  and  $x^{j,n}$ . Each passage of a gate by  $\Gamma$  in the figure yields a linear gate map  $L_{\bar{v}^{j,p}}$  (resp.  $L_{\tilde{v}^{j,p}}$ ) for some  $\bar{v}^{j,p} \in \mathbb{C}^\times$  (resp.  $\tilde{v}^{j,p} \in \mathbb{C}^\times$ ) and  $1 \leq p \leq m$ . We also set  $\psi^{j,g} =: L_{v^j}$ .*

- (1) *The equality  $\bar{\psi}[\Gamma] = \tilde{\psi}[\Gamma]$  holds, defining a germ  $\Delta \in \text{Diff}(\mathbb{C}, 0)$ .*
- (2)

$$\Delta'(0) \exp \frac{-2i\pi\mu}{k} = \prod_{p=1}^m \bar{v}^{j,p} = \prod_{p=1}^m \tilde{v}^{j,p}.$$

(3)

$$\delta \circ \Delta = \Delta \circ \delta.$$

*Proof.* Observe that

$$(9.2) \quad \tilde{\psi}[\gamma^+] = L_{\tilde{c}^j} \quad , \quad \tilde{c} := \prod_{p=1}^m \tilde{v}^{j,p}.$$

The linear part is invariant by conjugacy so that  $\tilde{c} = \bar{v}^j \tilde{v}^j \exp 2i\pi\mu/k$ . Similarly, considering  $\gamma^-$  yields  $\bar{c} = \bar{v}^j \tilde{v}^j \exp 2i\pi\mu/k$ . Hence

$$\bar{c} = \prod_{p=1}^m \bar{v}^{j,p} = \bar{v}^j \tilde{v}^j \exp 2i\pi\mu/k = \tilde{c}.$$

Since  $\tilde{\psi}[\Gamma] = L_{\tilde{c}} \circ \psi^{j,s}$  and  $\bar{\psi}[\Gamma] = L_{\bar{c}} \circ \psi^{j,s}$  the result follows.  $\square$

Recall that the map

$$\varepsilon \in \mathcal{E}_\ell \mapsto \left( v_\varepsilon^j \right)_{j \in \mathbb{Z}/k\mathbb{Z}}$$

is locally injective (Lemma 7.20). In particular  $\Delta'(0)$  is not constant and therefore must take non-rational values on a small subdomain  $\Lambda \subset \mathcal{E}_\ell^\cap$ . It follows that for  $\varepsilon \in \Lambda$  the Abelian group  $\langle \delta, \Delta \rangle < \text{Diff}(\mathbb{C}, 0)$  is non-resonant and therefore formally linearizable [24]. Hence  $\delta = \text{Id}$ .

**Lemma 9.5.** *If  $\delta = \text{Id}$  then  $\phi^j = 0$ .*

*Proof.* According to (9.1),  $\delta$  conjugates  $\psi_{\bar{\varepsilon}}[\gamma^+] = L_{\bar{\nu}^j} \circ \psi^{j,s}$  to  $\psi_{\bar{\varepsilon}}[\gamma^+]$ , but the latter is linear thanks to (9.2), therefore  $\psi^{j,s}$  itself is linear. It can only mean that  $\phi^{j,s} = 0 = \phi^j$  using (7.5), the equality holding on the whole cell  $\mathcal{E}_\ell$  by analytic continuation.  $\square$

Since  $j$  is arbitrary we just established

$$(k > 1) \implies (\phi = 0).$$

9.2.2. *The case  $k = 1$ : end of the proof of the Parametrically Analytic Orbital Moduli Theorem.* Since  $k = 1$  we drop the index  $j = 0$ . We work in the self-intersection  $\mathcal{E}^\cap$  of the single parametric cell, and use the notations and constructions involved just above. In particular Figure 9.1 remains the same except for the fact that there are no gate passage  $j_1, \dots, j_m$  on the right-hand side of the pictures.

Recall that we consider a system with  $m \neq 0$ . Lemma 9.5 forbids  $\delta = \text{Id}$ , thus  $\Delta = \psi^s$  is non-linear ( $\Delta$  was introduced in Lemma 9.4). Then  $\langle \delta, \Delta \rangle$  is an Abelian group. Consequently there exists [30] a formal tangent-to-identity change  $\widehat{\varphi}$  in the variable  $h$ , unique  $d \in \mathbb{N}$ ,  $\lambda \in \mathbb{C}$  and  $t \in \mathbb{C} \setminus \{0\}$  such that, writing  $\widehat{f} := \widehat{\varphi}^* f$  for all  $f \in \text{Diff}(\mathbb{C}, 0)$ ,

$$\begin{aligned} \widehat{\delta} &= \Phi_{Z(d,\lambda)}^1 \\ \widehat{\Delta} &= \alpha \Phi_{Z(d,\lambda)}^t, \quad \alpha \in \mathbb{C}^\times \\ Z(d,\lambda) &= \frac{h^{d+1}}{1 + \lambda h^d} \frac{\partial}{\partial h}. \end{aligned}$$

Commutativity forces the relation

$$\alpha^d = 1.$$

Since  $\alpha = \exp 2i\pi\mu$  this gives  $d\mu \in \mathbb{Z}$  as expected. Observe that for all  $s \in \mathbb{C}$

$$\Phi_{Z(d,0)}^s \in \text{Ber}(d),$$

therefore we aim at showing  $\lambda = 0$ . This is ultimately done by applying the next lemma.

**Lemma 9.6.** [8, Assertions 1.1 to 1.4] *In the following  $\xi$  is a formal diffeomorphism in the variable  $h$  at 0.*

- (1) *Let  $Z, \widetilde{Z}$  be formal vector fields in the variable  $h$  at 0 belonging to  $h^{d+1}\mathbb{C}[[h]]^\times \frac{\partial}{\partial h}$ . If  $\xi^* \Phi_Z^1 = \Phi_{\widetilde{Z}}^1$  then  $\xi^* Z = \widetilde{Z}$  (the converse is trivial).*
- (2) *Assume that  $\xi^* Z(d,\lambda) = aZ(d,\lambda)$  with  $a \neq 1$ . Then  $\lambda = 0$  and  $\xi \in \text{Ber}(d)$  (in particular  $\xi$  is analytic).*

Let us show now that  $\lambda = 0$  and  $\widehat{\varphi} \in \text{Ber}(d)$  itself, forcing  $\Delta = \psi^s \in \text{Ber}(d)$  by application of Lemma 9.2. The key is to exploit the fact (9.1), which can be rewritten as:

$$(9.3) \quad \delta^* \bar{\psi}[g^+ s^+] = \bar{\psi}[g^-] = L_{1/\bar{\nu}}.$$

Indeed, referring to Definition 7.13 for the definition of the letters  $g^\pm$ ,  $s^\pm$  and their image by  $\psi[\bullet]$ , and looking at Figure 9.1, we compare the holonomy maps around the upper singular point. On the left, the singular point is of saddle type and the holonomy map is the composition of  $\bar{\psi}[s^+]$  (crossing the saddle sector in the direction of the arrow) with  $\bar{\psi}[g^+]$  (crossing the gate sector in the direction of the arrow). On the right, the same singular point is of node type. Turning around, it comes to crossing the gate sector in the inverse direction of the arrow. Hence its holonomy map is  $\bar{\psi}[g^-]$ . The last equality in (9.3) follows from the fact that  $\bar{\psi}[g^+] = L_{\bar{\nu}}$ . Note that (9.3) means that  $\delta$  linearizes  $\bar{\psi}[g^+s^+]$ .

Of course the multipliers at the fixed point in (9.3) must be the same. On the left, this multiplier is simply that of  $\bar{\psi}[g^+s^+]$ , since conjugacy by  $\delta$  preserves the multiplier. On the one hand the multiplier at the fixed point of  $\bar{\psi}[s^+]$  is  $\exp 2i\pi\mu$ , according to (7.5) for  $k := 1$ , as indeed  $\bar{\psi}[g^+] = L_{\bar{\nu}}$ . On the other hand  $\bar{\psi}[g^-] = L_{1/\bar{\nu}}$  so that

$$\bar{\nu} \bar{\nu} \exp 2i\pi\mu = 1.$$

We also have  $\bar{\psi}[s^+] = \Delta$ , since it is the holonomy obtained by turning counterclockwise around the two singular points. Hence, replacing in (9.3) yields  $L_{\bar{\nu}} \circ \Delta \circ \delta = \delta \circ L_{1/\bar{\nu}}$ . Composing both sides on the left with  $L_{\bar{\nu}}$  and taking  $\widehat{\varphi}^*$  on both sides yields

$$\widehat{L}_{1/\bar{\nu}}^* \widehat{\delta} = \widehat{L}_{\bar{\nu}} \circ \widehat{\delta} \circ \widehat{L}_{1/\bar{\nu}} = \widehat{L}_{\bar{\nu}} \circ \widehat{\Delta} \circ \widehat{\delta}.$$

For the sake of simplicity we only deal with the case  $\mu \in \mathbb{Z}$ , the general case can be adapted by taking into account that  $\widehat{L}_{\bar{\nu}}^{od} = \text{Id}$ . Under the current hypothesis  $\widehat{L}_{\bar{\nu}} = \text{Id}$ , so that  $\widehat{L}_{1/\bar{\nu}}$  is a formal conjugacy between  $\widehat{\delta} = \Phi_{Z(d,\lambda)}^1$  and  $\widehat{\Delta} \circ \widehat{\delta} = \Phi_{Z(d,\lambda)}^{1+t} = \Phi_{(1+t)Z(d,\lambda)}^1$  for some  $t = t_\epsilon \in \mathbb{C}^\times$ . According to Lemma 9.6 with  $\xi := \widehat{L}_{1/\bar{\nu}}$  and  $a := 1 + t \neq 1$ , we have  $\lambda = 0$  and  $\widehat{L}_{1/\bar{\nu}} \in \text{Ber}(d)$ .

So far  $\widehat{\varphi}$  is a formal linearization of  $\widehat{L}_{1/\bar{\nu}}$  which is tangent-to-identity. For values  $\epsilon$  of the parameter corresponding to  $\bar{\nu} \notin \mathbb{R}$  (say  $\text{Im}(\bar{\nu}) > 0$ ) the fix-point 0 of  $\widehat{L}_{1/\bar{\nu}}$  is hyperbolic: the map  $\widehat{\varphi}$  is locally holomorphic at 0, unique and therefore given by

$$\widehat{\varphi} := \lim_{n \rightarrow \infty} L_{-n/\bar{\nu}} \circ \widehat{L}_{1/\bar{\nu}}^{on}$$

uniformly on a neighborhood of 0. Lemma 9.2 implies that for every  $n \in \mathbb{N}$  we have

$$L_{-n/\bar{\nu}} \circ \widehat{L}_{1/\bar{\nu}}^{on} \in \text{Ber}(d),$$

therefore  $\widehat{\varphi} \in \text{Ber}(d)$  as requested, since the group  $\text{Ber}(d)$  is closed for the topology of local uniform convergence. This completes the proof of the Parametrically Analytic Orbital Moduli Theorem.

## 10. A FEW WORDS ABOUT COMPUTATIONS

All the discussion regarding the actual (symbolic or numeric) computations of normal forms and moduli of saddle-nodes, as presented in [42, Section 4] for saddle-nodes, can be repeated *verbatim* in the case of convergent unfoldings: we will not reproduce it here. We nonetheless present in Section 10.1 a consequence

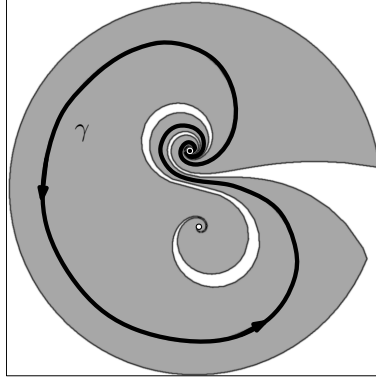


FIGURE 10.1. The (asymptotic) path of integration used to compute the period, which is a cycle when  $k = 1$ .

of one particular result, thus unfolding the main result of [46], which leads us to try and compute the period associated to the formal orbital normal form  $\widehat{X}$  in Section 10.2.

**10.1. Computation of the dominant term of the orbital invariant.** The next lemma holds for a fixed value of  $\varepsilon \in \mathcal{E}_\ell$ .

**Lemma 10.1.** (See [42, Proposition 4.1]) *Let  $r_n \in x\mathbb{C}[x]_{<k}$  be the coefficients of*

$$(10.1) \quad R(x, y) = \sum_{n>0} r_n(x) (P^\varepsilon(x)y)^n$$

*in the normal form  $\mathcal{X}$ . Let  $c_\ell^{j,p}(n, m) \in \mathbb{C}$  be the coefficients of the period*

$$\mathfrak{T}_\ell^j(x^n y^m)(h) = \sum_{p>0} c_\ell^{j,p}(n, m) h^p$$

*relative to  $\mathcal{X}$ . Then we have the following properties.*

**Triangularity:**  $c_\ell^{j,p}(n, m) = 0$ , if  $p < m$  and

$$c_\ell^{j,m}(m, m) h^m = 2i\pi \widehat{\mathfrak{T}}_\ell^j(x^n y^m)(h)$$

*is independent of  $R$ .*

**Algebraicity:** *For  $p > m$ , the coefficient  $c_\ell^{j,p}(n, m)$  depends polynomially in the  $k(p - m)$  variables given by the coefficients of  $r_1, \dots, r_{p-m}$  and vanishes when  $R = 0$ .*

*Proof.* It is exactly the proof done in [42, Proposition 4.1] since exchanging  $x^{k+1}$  for  $P_\varepsilon(x)$  does not modify anything in the actual computation. We give some brief elements of the proof.

Let us drop all indexes and let  $x \mapsto y(x, h)$  be the sectorial solution of the differential equation induced by the vector field  $\mathcal{X}$  with initial value  $H(x_*, y(x_*, h)) = h$  (here  $x_*$  is fixed once and for all in  $V^s$ ). Computing  $\mathfrak{T}(x^n y^m)(h)$  requires to compute the integral  $\int_\gamma x^n y(x, h)^m \frac{dx}{P(x)}$  for an asymptotic path  $\gamma \subset \mathbb{C} \times \{0\}$  joining the two nodes in the closure of the union of consecutive squid sectors (see Figure 10.1).

This integral is absolutely convergent because  $m > 0$  and  $\gamma$  spirals in the right manner (see also Lemma 6.23). Since

$$H(x, y) = \widehat{H}(x, y) \exp N(x, y),$$

with  $\widehat{H}$  linear in the  $y$ -variable, we necessary have

$$(10.2) \quad y(x, h) = \widehat{y}(x, h) + hO(h)$$

where  $\widehat{y}(x, h) = \frac{h}{\widehat{H}(x, 1)}$  is the solution corresponding to the formal model  $\widehat{X}$ . This gives the triangularity. The algebraicity property stems from the fact that the computation can be performed formally in the  $y$ -variable. The sought property is true for the expansion (10.2) (by studying the inverse of the normalizing mapping) because it is true for solutions of cohomological equations  $\mathcal{X} \cdot N = -R$ .  $\square$

We extract from this statement useful consequences.

**Proposition 10.2.**

(1) *The quantity*

$$\begin{aligned} \inf \{n : r_n \neq 0\} &= \inf \left\{ n : (\exists j) \phi_n^j \neq 0 \right\} \\ &=: d \in \overline{\mathbb{N}} \end{aligned}$$

*does not depend on the cell.*

(2) *The valuation  $d$  is infinite if and only if the unfolding is analytically conjugate to its formal normal form.*

(3) *If  $d < \infty$  the dominant term of the invariant is given by the period of the formal model*

$$2i\pi \widehat{\Gamma}(r_d y^d) = \left( h \mapsto \phi_d^j h^d \right)_{j \in \mathbb{Z}/k\mathbb{Z}}.$$

*Remark 10.3.* The value of  $d$  does not depend on the cell but may differ from the value obtained on the boundary  $\Delta_k$ . Yet the analytic continuation principle ensures that

$$\min_{\varepsilon \in \Delta_k} \inf \{n : r_{n, \varepsilon} \neq 0\} \geq d$$

because  $R$  is analytic.

From this proposition we deduce a final normalization ensuring uniqueness.

**Corollary 10.4.** *Assume the generic convergent unfolding  $X$  is not analytically conjugate to its formal normal form  $\widehat{X}$  defined in (2.2). There exists a unique  $(\kappa, j, d) \in \mathbb{Z}_{\geq 0} \times \{1, 2, \dots, k\} \times \mathbb{N}$  such that  $X$  is conjugate to the normal form  $\mathcal{X} = \widehat{X} + R y \frac{\partial}{\partial y}$  as in (10.1) where:*

$$\begin{aligned} r_{\varepsilon, d}(x) &= \varepsilon^\kappa x^j + o(x^j) \\ r_{\varepsilon, n} &= 0 \quad \text{if } n < d. \end{aligned}$$

Notice that in the case  $\kappa > 0$  this normal form may fail to deliver meaningful information at the limit  $\varepsilon \rightarrow 0$ . Take the extreme case  $R_\varepsilon(x, y) = \varepsilon^\kappa x^j y^d$  with  $\kappa > 0$ : for every  $\varepsilon \neq 0$  the vector field  $\mathcal{X}$  is not equivalent to the model  $\widehat{X}_\varepsilon$  but  $\mathcal{X}_0$  is.

**10.2. Formula for the period of formal models.** Unfortunately only the case  $k = 1$  seems tractable enough to obtain closed-form expressions involving the Gamma function. For the case  $k = 2$  one could derive a closed-form formula additionally using generalized hypergeometric functions, which is already stretching a bit far what a “closed-form” is. There is no evidence that similar calculations can be performed for  $k > 2$ .

**Proposition 10.5.** [43, Proposition 8] *Here  $k = 1$ . Let us introduce the double covering  $\varepsilon = -s^2$  in the parameter space. Then for  $m \in \mathbb{N}$  and  $n \in \mathbb{Z}_{\geq 0}$ :*

$$\begin{aligned} \widehat{\mathcal{I}}_s(x^n y^m)(h) &= h^m \times \frac{(-m)^{n+m\mu}}{\Gamma(n+m\mu)} \times t_{s,n,m} \times T_{s,m} \\ t_{s,n,m} &:= \frac{1}{2^n} \sum_{p+q=n} \binom{n}{p} \prod_{j=0}^{p-1} \left(1 - s \left(\mu + \frac{2j}{m}\right)\right) \prod_{j=0}^{q-1} \left(1 + s \left(\mu + \frac{2j}{m}\right)\right) \\ T_{s,m} &:= \frac{\left(-\frac{2s}{m}\right)^{m\mu}}{1+s\mu} \times \frac{\Gamma\left(-\frac{m}{2s} + \frac{m\mu}{2}\right)}{\Gamma\left(-\frac{m}{2s} - \frac{m\mu}{2}\right)}. \end{aligned}$$

This period is holomorphic and bounded in the parameter  $s$  on the sector

$$S := \left\{ 0 < |s| < \frac{1}{2|\mu_0|}, \frac{\pi}{4} < \arg s < \frac{7\pi}{4} \right\}$$

and extends continuously at 0 by

$$\widehat{\mathcal{I}}_0(x^n y^m)(h) = h^m \times \frac{(-m)^{n+m\mu_0}}{\Gamma(n+m\mu_0)}.$$

For given  $s$  small enough, the period is zero if and only if  $n + m\mu_\varepsilon \in \mathbb{Z}_{\leq 0}$ . The period is an even function of  $s$  (i.e. holomorphic in the parameter  $\varepsilon$ ) if and only if  $m\mu \in \mathbb{Z}$ . In that case  $\mu$  is a rational constant.

The result is shown by using the Pochhammer contour integral formula for the Beta function. Indeed an affine change of coordinates sends  $(x-s)^\alpha (x+s)^\beta$  to a multiple of  $(1-z)^\alpha z^\beta$ . The final expression comes from diverse classical properties of the Gamma function. The eventual lack of evenness of the period comes from the term  $T_{s,m}$ . If  $m\mu$  is not an integer then  $T_{s,m}$  is multivalued and has an accumulation of zeros and poles as  $s \rightarrow 0$  outside the sector  $S$ . Only the coincidence of these two infinite sets when  $m\mu \in \mathbb{Z}$  allows the period to be holomorphic through lucky root / pole cancellations.

Since  $T_{s,m}$  is independent on  $n$ , any nonzero period  $\widehat{\mathcal{I}}(y^m g)$  of a germ  $g \in \mathbb{C}\{\varepsilon, x\}$  is holomorphic in  $\varepsilon$  if and only if  $m\mu \in \mathbb{Z}$ . From Lemma 9.3, Theorem 6.2 and the Parametrically Analytic Orbital Moduli Theorem we can generalize this observation.

**Corollary 10.6.** *Let  $G \in \mathbb{C}\{\varepsilon, x, y\}$  with  $G(\bullet, 0) = \mathcal{O}(P)$ . Let us assume that the period  $\widehat{\mathcal{I}}(G)$  is nonzero. Then,  $\widehat{\mathcal{I}}(G)$  is holomorphic in the parameter if and only if all three conditions hold:*

- $k = 1$ ,
- there exists  $d \in \mathbb{N}$  such that  $d\mu \in \mathbb{Z}$ ,

- there exist two germs  $F \in \mathbb{C}\{\varepsilon, x, y\}$  and  $Q \in \text{Section}_1\{P^{d\tau}y^d\} \setminus \{0\}$  such that

$$G = Q + \widehat{X} \cdot F.$$

The fact that the period is never a holomorphic function of the parameter if  $k > 1$  is probably a sign that a “simple” formula for  $\widehat{T}(x^n y^m)$  does not exist.

#### REFERENCES

- [1] Michel Berthier and Frédéric Touzet. Sur l’intégration des équations différentielles holomorphes réduites en dimension deux. *Bol. Soc. Brasil. Mat. (N.S.)*, 30(3):247–286, 1999.
- [2] Bodil Branner and Kealey Dias. Classification of complex polynomial vector fields in one complex variable. *J. Difference Equ. Appl.*, 16(5-6):463–517, 2010.
- [3] Alexander D. Bruno. *Local methods in nonlinear differential equations*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1989. Part I. The local method of nonlinear analysis of differential equations. Part II. The sets of analyticity of a normalizing transformation, Translated from the Russian by William Hovsingh and Courtney S. Coleman, With an introduction by Stephen Wiggins.
- [4] César Camacho and Paulo Sad. Invariant varieties through singularities of holomorphic vector fields. *Ann. of Math. (2)*, 115(3):579–595, 1982.
- [5] Mireille Canalis-Durand, Jorge Mozo-Fernández, and Reinhard Schäfke. Monomial summability and doubly singular differential equations. *J. Differential Equations*, 233(2):485–511, 2007.
- [6] Guy Casale. Feuilletages singuliers de codimension un, groupoïde de Galois et intégrales premières. *Ann. Inst. Fourier (Grenoble)*, 56(3):735–779, 2006.
- [7] Guy Casale and Damien Davy. Specialization of the Galois groupoid. *preprint*, 2017.
- [8] D. Cerveau and R. Moussu. Groupes d’automorphismes de  $(\mathbb{C}, 0)$  et équations différentielles  $yd y + \dots = 0$ . *Bull. Soc. Math. France*, 116(4):459–488 (1989), 1988.
- [9] Colin Christopher and Christiane Rousseau. The moduli space of germs of generic families and analytic diffeomorphisms unfolding a parabolic fixed point. *C. R. Math. Acad. Sci. Paris*, 345(12):695–698, 2007.
- [10] Kealey Dias. Enumerating combinatorial classes of the complex polynomial vector fields in  $\mathbb{C}$ . *Ergodic Theory Dynam. Systems*, 33(2):416–440, 2013.
- [11] A. Douady and P. Sentenac. Champs de vecteurs polynomiaux sur  $\mathbb{C}$ . Preprint 2005.
- [12] Henri Dulac. Recherches sur les points singuliers des équations différentielles. *Journal de l’École Polytechnique*, 2(9):1–125, 1904.
- [13] Henri Dulac. Sur les points singuliers d’une équation différentielle. *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. (3)*, 1:329–379, 1909.
- [14] Wolfgang Fischer and Hans Grauert. Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, 1965:89–94, 1965.
- [15] A. A. Glutsyuk. Confluence of singular points and the nonlinear Stokes phenomenon. *Tr. Mosk. Mat. Obs.*, 62:54–104, 2001.
- [16] Claude Godbillon and Jacques Vey. Un invariant des feuilletages de codimension 1. *C. R. Acad. Sci. Paris Sér. A-B*, 273:A92–A95, 1971.
- [17] Masuo Hukuhara, Tosihusa Kimura, and Tizuko Matuda. *Equations différentielles ordinaires du premier ordre dans le champ complexe*. Publications of the Mathematical Society of Japan, 7. The Mathematical Society of Japan, Tokyo, 1961.
- [18] Jacques Hurtubise, Caroline Lambert, and Christiane Rousseau. Complete system of analytic invariants for unfolded differential linear systems with an irregular singularity of Poincaré rank  $k$ . *Mosc. Math. J.*, 14(2):309–338, 427, 2014.
- [19] Yulij Ilyashenko and Sergei Yakovenko. *Lectures on analytic differential equations*, volume 86 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.
- [20] Martin Klimeš. Confluence of Singularities of Nonlinear Differential Equations via Borel–Laplace Transformations. *J. Dyn. Control Syst.*, 22(2):285–324, 2016.
- [21] K. Kodaira. On stability of compact submanifolds of complex manifolds. *Amer. J. Math.*, 85:79–94, 1963.
- [22] Caroline Lambert and Christiane Rousseau. Moduli space of unfolded differential linear systems with an irregular singularity of Poincaré rank 1. *Mosc. Math. J.*, 13(3):529–550, 553–554, 2013.



- [23] Pierre Lavaurs. *Systèmes dynamiques holomorphes : explosion de points périodiques paraboliques*. PhD thesis, 1989. Thèse de doctorat dirigée par Douady, Adrien Mathématiques Paris 11 1989.
- [24] Frank Loray. Dynamique des groupes d'automorphismes de  $\mathbb{C}, 0$ . *Bol. Soc. Mat. Mexicana* (3), 5(1):1–23, 1999.
- [25] Frank Loray. Versal deformation of the analytic saddle-node. *Astérisque*, (297):167–187, 2004. Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes. II.
- [26] B. Malgrange. On nonlinear differential Galois theory. *Chinese Ann. Math. Ser. B*, 23(2):219–226, 2002. Dedicated to the memory of Jacques-Louis Lions.
- [27] Bernard Malgrange. Le groupoïde de Galois d'un feuilletage. In *Essays on geometry and related topics, Vol. 1, 2*, volume 38 of *Monogr. Enseign. Math.*, pages 465–501. Enseignement Math., Geneva, 2001.
- [28] P. Mardešić, R. Roussarie, and C. Rousseau. Modulus of analytic classification for unfoldings of generic parabolic diffeomorphisms. *Mosc. Math. J.*, 4(2):455–502, 535, 2004.
- [29] Jean Martinet. Remarques sur la bifurcation nœud-col dans le domaine complexe. *Astérisque*, (150-151):131–149, 186, 1987. Singularités d'équations différentielles (Dijon, 1985).
- [30] Jean Martinet and Jean-Pierre Ramis. Problèmes de modules pour des équations différentielles non linéaires du premier ordre. *Inst. Hautes Études Sci. Publ. Math.*, (55):63–164, 1982.
- [31] J.-F. Mattei and R. Moussu. Holonomie et intégrales premières. *Ann. Sci. École Norm. Sup. (4)*, 13(4):469–523, 1980.
- [32] Jorge Mozo-Fernández and Reinhard Schäfke. Asymptotic expansion with respect to an analytic germ, summability and Ramis-Sibuya theorem. (*unpublished material*), 2016.
- [33] Richard Oudkerk. The parabolic implosion: Lavaurs maps and strong convergence for rational maps. In *Value distribution theory and complex dynamics (Hong Kong, 2000)*, volume 303 of *Contemp. Math.*, pages 79–105. Amer. Math. Soc., Providence, RI, 2002.
- [34] G. Remoundos. Contribution à la théorie des singularités des équations différentielles du premier ordre. *Bull. Soc. Math. France*, 36:185–194, 1908.
- [35] C. Rousseau. Analytic moduli for unfoldings of germs of generic analytic diffeomorphisms with a codimension  $k$  parabolic point. *Ergodic Theory Dynam. Systems*, 35(1):274–292, 2015.
- [36] Christiane Rousseau. Modulus of orbital analytic classification for a family unfolding a saddle-node. *Mosc. Math. J.*, 5(1):245–268, 2005.
- [37] Christiane Rousseau. Normal forms for germs of analytic families of planar vector fields unfolding a generic saddle-node or resonant saddle. In *Nonlinear dynamics and evolution equations*, volume 48 of *Fields Inst. Commun.*, pages 227–245. Amer. Math. Soc., Providence, RI, 2006.
- [38] Christiane Rousseau. The moduli space of germs of generic families of analytic diffeomorphisms unfolding of a codimension one resonant diffeomorphism or resonant saddle. *J. Differential Equations*, 248(7):1794–1825, 2010.
- [39] Christiane Rousseau and Jacques Hurtubise. Moduli space for generic unfolded differential linear systems. to appear in *Advances in Mathematics*, 2015.
- [40] Christiane Rousseau and Loïc Teyssier. Analytical moduli for unfoldings of saddle-node vector fields. *Mosc. Math. J.*, 8(3):547–614, 616, 2008.
- [41] V. I. Savel'ev. Zero-type imbedding of a sphere into complex surfaces. *Moscow Univ. Math. Bull.*, 37(4):34–39, 1982.
- [42] R. Schäfke and L. Teyssier. Analytic normal forms for convergent saddle-node vector fields. *Ann. Inst. Fourier*, 2013 (30p) 2013.
- [43] Loïc Teyssier. Coalescing complex planar stationary points. In Toni Bourama, editor, *Interdisciplinary Mathematical Research and Applications: In Honor of Professor Christiane Rousseau*, Proceedings in Mathematics & Statistics. Springer.
- [44] Loïc Teyssier. Analytical classification of singular saddle-node vector fields. *J. Dynam. Control Systems*, 10(4):577–605, 2004.
- [45] Loïc Teyssier. Équation homologique et cycles asymptotiques d'une singularité nœud-col. *Bull. Sci. Math.*, 128(3):167–187, 2004.
- [46] Loïc Teyssier. Examples of non-conjugated holomorphic vector fields and foliations. *J. Differential Equations*, 205(2):390–407, 2004.
- [47] Tetsuo Ueda. On the neighborhood of a compact complex curve with topologically trivial normal bundle. *J. Math. Kyoto Univ.*, 22(4):583–607, 1982/83.

- [48] Marius van der Put and Michael F. Singer. *Galois theory of linear differential equations*, volume 328 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2003.
- [49] Sergeï M. Voronin and Yulia I. Meshcheryakova. Analytic classification of germs of holomorphic vector fields with a degenerate elementary singular point. *Vestnik Chelyab. Univ. Ser. 3 Mat. Mekh. Inform.*, (3(9)):16–41, 2003.

*E-mail address:* rousseac@dms.umontreal.ca -&- teyssier@math.unistra.fr