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Weak subsolutions to complex Monge-Ampère equations

By Vincent GUEDJ, Chinh H. LU and Ahmed ZERIAHI

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Abstract. We compare various notions of weak subsolutions to degenerate complex Monge-Ampère equations, showing that they all coincide. This allows us to give an alternative proof of mixed Monge-Ampère inequalities due to Kołodziej and Dinew.

1. Introduction

Let Ω be a domain of \mathbb{C}^n . We consider in this article the notion of subsolution for degenerate complex Monge-Ampère equations in Ω . These are bounded plurisubharmonic functions which satisfy

$$(dd^c\varphi)^n \geq f dV,$$

where dV denotes the Lebesgue measure and $0 \leq f \in L^1(\Omega)$.

This inequality can be interpreted in various senses (pluripotential sense [BT76], viscosity sense [EGZ11], distribution sense [HL13]) and the goal of this note is to show that all point of views do coincide.

Main theorem. *Assume φ is plurisubharmonic and bounded. The following are equivalent :*

- (i) $(dd^c\varphi)^n \geq f dV$ in the pluripotential sense;
- (ii) $(dd^c(\varphi \star \chi_\varepsilon))^n \geq (f^{1/n} \star \chi_\varepsilon)^n dV$ in the classical sense, for all $\varepsilon > 0$;
- (iii) $\Delta_H \varphi \geq f^{1/n}$ in the sense of distributions, for all $H \in \mathcal{H}$.

In a particular case when φ and f are continuous, our main theorem was proved by Błocki (see [Bl96, Theorem 3.10]).

The operator $dd^c = ai\partial\bar{\partial}$ is here normalized so that $dV = (dd^c|z|^2)^n$ is the Euclidean volume form on \mathbb{C}^n . Thus for a smooth function φ ,

$$(dd^c\varphi)^n = \det \left(\frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} \right) dV.$$

We let \mathcal{H} denote the space of Hermitian positive definite matrix H that are normalized by $\det H = 1$, and let Δ_H denote the Laplace operator

$$\Delta_H \varphi := \frac{1}{n} \sum_{j,k=1}^n h_{jk} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}.$$

The functions χ_ε are standard mollifiers, i.e. radial smooth non-negative functions with compact support in the ε -ball centered at the origin, and such that $\int \chi_\varepsilon dV = 1$. It is then classical that the convolutions $\varphi \star \chi_\varepsilon$ are smooth, plurisubharmonic, and decrease to φ as ε decreases to 0.

When f is moreover continuous, one can also interpret the inequality $(dd^c\varphi)^n \geq f dV$ in the viscosity sense, as shown in [EGZ11, Proposition 1.5].

Our main theorem easily implies the following result of Kolodziej [Kol03, Lemma 1.2] (see also [Din09, DL15]) :

Corollary. *Assume $\varphi_1, \dots, \varphi_n$ are bounded plurisubharmonic functions in Ω , such that $(dd^c\varphi_i)^n \geq f_i dV$, where $0 \leq f_i \in L^1(\Omega)$. Then*

$$dd^c\varphi_1 \wedge \dots \wedge dd^c\varphi_n \geq f_1^{1/n} \dots f_n^{1/n} dV.$$

The note is organized as follows. We start by extending Kolodziej's subsolution theorem (see Theorem 2.1), providing a solution to special Monge-Ampère equations that we are going to use in the sequel. We prove our main result in *Section 3.1*. The starting point is an identification of viscosity subsolutions and pluripotential subsolutions obtained in [EGZ11]. We connect these identifications to mixed Monge-Ampère inequalities in *Section 3.2* and propose some generalizations in *Section 3.3*.

2. The subsolution theorem

Let $\Omega \Subset \mathbb{C}^n$ be a bounded hyperconvex domain (in the sequel we only need to deal with the case when Ω is a ball). Let μ be a Borel measure on Ω . If there exists a function $v \in PSH(\Omega) \cap L^\infty(\Omega)$ such that

$$\mu \leq (dd^c v)^n \text{ in } \Omega, \text{ with } \lim_{\Omega \ni z \rightarrow \zeta} v(z) = 0, \forall \zeta \in \partial\Omega,$$

then it was proved by S. Kolodziej [Kol95] that there exists a unique solution $\psi \in PSH(\Omega) \cap L^\infty(\Omega)$ to the equation

$$(dd^c\psi)^n = \mu,$$

such that $\lim_{\Omega \ni z \rightarrow \zeta} \psi(z) = 0$. We need the following generalization:

THEOREM 2.1. *Assume μ is a non pluripolar Borel measure on Ω which has finite total mass. Then there exists a unique function $\varphi \in \mathcal{F}^1(\Omega)$ satisfying*

$$(2.1) \quad (dd^c\varphi)^n = e^\varphi \mu \text{ in } \Omega.$$

Moreover if μ satisfies $\mu \leq (dd^c u)^n$ in Ω , for some bounded negative psh function u , then the solution $\varphi \in PSH(\Omega)$ is bounded with $u \leq \varphi$. In particular, if $\limsup_{z \rightarrow \zeta} u(z) = 0$ for every $\zeta \in \partial\Omega$ then the same property holds for φ .

Before entering into the proof let us recall the definition of Cegrell's finite energy classes. We refer the reader to [Ce98, Ce04] for more details.

A domain Ω is called hyperconvex if there exists a continuous plurisubharmonic exhaustion function $\rho : \Omega \rightarrow [-\infty, 0)$ such that the sublevel sets $\{\rho < -c\}$ are relatively compact in Ω , for all constants $c > 0$.

Let u be a negative plurisubharmonic function in Ω . We recall the following definitions:

- $u \in \mathcal{E}_0(\Omega)$ if u is bounded in Ω , u vanishes on the boundary, i.e. $\lim_{z \rightarrow \partial\Omega} u(z) = 0$, and $\int_{\Omega} (dd^c u)^n < +\infty$.
- $u \in \mathcal{E}(\Omega)$ if for each $z_0 \in \Omega$ there exists an open neighborhood $V_{z_0} \Subset \Omega$ and a decreasing sequence $(u_j) \subset \mathcal{E}_0(\Omega)$ such that u_j converges to u in V_{z_0} and $\sup_j \int_{\Omega} (dd^c u_j)^n < +\infty$.
- $u \in \mathcal{E}^p(\Omega)$, $p > 0$ if there exists a sequence (u_j) in $\mathcal{E}_0(\Omega)$ decreasing to u and satisfying

$$\sup_{j \in \mathbb{N}} \int_{\Omega} (-u_j)^p (dd^c u_j)^n < +\infty.$$

If we ask additionally that $\int_{\Omega} (dd^c u_j)^n$ is uniformly bounded then by definition $u \in \mathcal{F}^p(\Omega)$.

It was proved in [Ce98, Ce04] that the Monge-Ampère operator $(dd^c)^n$ is well-defined for functions in $\mathcal{E}(\Omega)$. Moreover, it was shown in [BGZ09, Theorem A] that if $u \in \mathcal{E}(\Omega)$ then $(dd^c \max(u, -j))^n$ converges in the strong sense of Borel measures in $\Omega \cap \{u > -\infty\}$ to $(dd^c u)^n$.

Theorem 2.1 was proved in [CK06] using a fixed point argument. We provide in this note an alternative proof using the variational method, adapting the techniques developed in Kähler geometry in [BBGZ13] (similar ideas have been used in [ACC12, Lu15]).

Proof of Theorem 2.1. Consider

$$F_{\mu}(\phi) := E_1(\phi) - \int_{\Omega} e^{\phi} d\mu, \quad \phi \in \mathcal{E}^1(\Omega),$$

where

$$E_1(\phi) := \frac{1}{n+1} \int_{\Omega} \phi (dd^c \phi)^n.$$

The Euler-Lagrange equation of F_{μ} can be computed as follows. Fix $\phi \in \mathcal{E}^1(\Omega)$ and assume $(\phi(t))$ is a smooth path in $\mathcal{E}^1(\Omega)$ starting at $\phi(0) = \phi$ with $\dot{\phi}(0) = v \in C(X)$. It follows from Stokes theorem that

$$\frac{d}{dt} E_1(\phi(t))|_{t=0} = \int_{\Omega} v (dd^c \phi)^n,$$

hence

$$\frac{d}{dt} F_{\mu}(\phi(t))|_{t=0} = \int_{\Omega} v (dd^c \phi)^n - \int_{\Omega} v e^{\phi} d\mu.$$

Thus ϕ is a critical point of the functional F_μ if it is a solution to the complex Monge-Ampère equation (2.1). It is thus natural to try and extremize F_μ in order to solve (2.1). We proceed in three steps :

Step 1: Upper semi-continuity of F_μ . Observe first that the functional $J(\phi) := -E_1(\phi) = |E_1(\phi)|$ is a positive proper functional on the space $\mathcal{E}^1(\Omega)$ i.e. its sublevel subsets

$$\mathcal{E}_B^1(\Omega) := \{\phi \in \mathcal{E}^1(\Omega); 0 \leq J(\phi) \leq B\}, \quad B > 0$$

are compact for the L^1 -topology. Moreover the functional E_1 is upper semi-continuous on each compact subset $\mathcal{E}_B^1(\Omega)$ for the L^1 -topology.

The continuity of the functional $L_\mu : \phi \mapsto \int_\Omega e^\phi d\mu$ on each compact subset $\mathcal{E}_B^1(\Omega)$ follows from the following fact due to Cegrell [Ce98], [ACC12, Lemma 4.1] : if $\phi_j \rightarrow \phi$ in $\mathcal{E}_B^1(\Omega)$ then $\phi_j \rightarrow \phi$ μ -a.e., hence by Lebesgue's convergence theorem, $\lim_j L_\mu(\phi_j) = L_\mu(\phi)$ (we use here the fact that μ is non-pluripolar).

This proves that F_μ is upper semi-continuous on each $\mathcal{E}_B^1(\Omega)$.

Step 2: Coercivity of F_μ . Observe that $0 \leq e^\varphi \leq 1$ for $\varphi \in \mathcal{E}^1(\Omega)$, hence

$$F_\mu(\phi) \leq E_1(\phi).$$

We infer that F_μ is J -proper on $\mathcal{E}^1(\Omega)$, i.e.

$$\lim_{J(\phi) \rightarrow +\infty} F_\mu(\phi) = -\infty.$$

This implies that the maximum of F_μ on $\mathcal{E}^1(\Omega)$ is localized at a finite level of energy, i.e. there exists a constant $B > 0$ such that

$$\sup\{F_\mu(\phi); \phi \in \mathcal{E}^1(\Omega)\} = \sup\{F_\mu(\phi); \phi \in \mathcal{E}_B^1(\Omega)\},$$

Since F_μ is upper semi-continuous on the compact set $\mathcal{E}_B^1(\Omega)$, there exists $\phi \in \mathcal{E}_B^1(\Omega)$ which maximizes F_μ on $\mathcal{E}_B^1(\Omega)$ i.e.

$$F_\mu(\phi) = \inf\{F_\mu(\psi); \psi \in \mathcal{E}_B^1(\Omega)\}.$$

Step 3: ϕ is a critical point of F_μ . Fix a continuous test function χ with compact support in Ω and set $\phi(t) := P_\Omega(\phi + t\chi)$ for $t \in \mathbb{R}$, where $P_\Omega(u)$ denotes the plurisubharmonic envelope of u in Ω .

Observe that $\phi(t) \in \mathcal{E}^1(\Omega)$. Indeed let ρ be a continuous psh exhaustion for Ω such that $\rho < -|\chi|$ on the support of χ . Then $\phi + |t|\rho \leq \phi(t)$ for $t \in \mathbb{R}$. Since $\phi + |t|\rho \in \mathcal{E}^1(\Omega)$, it follows that $\phi(t) \in \mathcal{E}^1(\Omega)$. Set

$$h(t) := E_1(\phi(t)) - \int_\Omega e^{\phi + t\chi} d\mu.$$

Then since $\phi(t) \leq \phi + t\chi$, it follows that $h(t) \leq F_\mu(\phi(t)) \leq F_\mu(\phi)$ which means that h achieves its maximum at the point 0.

On the other hand we know by [BBGZ13, ACC12, Lu15] that

$$\frac{d}{dt}h(t) |_{t=0} = \int_{\Omega} \chi(dd^c\phi)^n - \int_{\Omega} \chi e^{\phi} d\mu.$$

Since h achieves its maximum at the point 0, we have $h'(0) = 0$, hence

$$\int_{\Omega} \chi(dd^c\phi)^n = \int_{\Omega} \chi e^{\phi} d\mu.$$

As the test function χ was arbitrary, this means that the function ϕ is a solution of the equation (2.1). As μ has finite total mass we actually have that $\varphi \in \mathcal{F}^1(\Omega)$.

We now prove the uniqueness. If $\psi \in \mathcal{F}^1(\Omega)$ is another solution to (2.1) then it follows from the comparison principle [Ce98, Lemma 4.4] that

$$\begin{aligned} \int_{\{\varphi < \psi\}} e^{\psi} d\mu &= \int_{\{\varphi < \psi\}} (dd^c\psi)^n \leq \int_{\{\varphi < \psi\}} (dd^c\varphi)^n \\ &= \int_{\{\varphi < \psi\}} e^{\varphi} d\mu \leq \int_{\{\varphi < \psi\}} e^{\psi} d\mu. \end{aligned}$$

We infer $\int_{\{\varphi < \psi\}} (e^{\psi} - e^{\varphi}) d\mu = 0$ hence $\psi \leq \varphi$, μ -almost everywhere and $(dd^c\varphi)^n$ -almost everywhere in Ω . For each $\varepsilon > 0$, since $(dd^c\varphi)^n$ vanishes in $\{\varphi \leq \psi - \varepsilon\} \subset \{\varphi < \psi\}$, it follows from [BGZ09, Theorem 2.2] that

$$(dd^c \max(\varphi, \psi - \varepsilon))^n \geq \mathbf{1}_{\{\varphi > \psi - \varepsilon\}} (dd^c\varphi)^n = (dd^c\varphi)^n.$$

It then follows from the comparison principle [Ce98, Theorem 4.5] that $\max(\varphi, \psi - \varepsilon) \leq \varphi$, for all $\varepsilon > 0$, hence $\psi \leq \varphi$. Reversing the role of φ and ψ in the above argument gives $\varphi = \psi$, proving the uniqueness.

Assume finally that $\mu \leq (dd^c u)^n$, where u is a bounded negative psh function in Ω . Then since $\varphi \leq 0$ we have $(dd^c\varphi)^n = e^{\varphi}\mu \leq \mu \leq (dd^c u)^n$. Since u is bounded (in particular it belongs to the domain of definition of the complex Monge-Ampère operator) and $\varphi \in \mathcal{F}^1(\Omega)$ with $(dd^c\varphi)^n$ putting no mass on pluripolar sets, it follows from [BGZ09, Corollary 2.4] that $u \leq \varphi$. In particular φ vanishes on the boundary $\partial\Omega$ if u does so. \square

3. The main result

3.1. Proof of the main result

Given φ a plurisubharmonic function in a domain Ω , we let

$$\varphi_{\varepsilon}(z) = \varphi \star \chi_{\varepsilon}(z) := \int_{\mathbb{C}^n} \varphi(z - \varepsilon w) \chi(w) dV(w) = \int_{\mathbb{C}^n} \varphi(w) \chi_{\varepsilon}(z - w) dV(w)$$

denote the standard regularizations of φ defined in Ω_{ε} for $\varepsilon > 0$ small enough, where $\Omega_{\varepsilon} = \{z \in \Omega, \text{dist}(z, \partial\Omega) > \varepsilon\}$.

Here χ_{ε} are non-negative radial functions with compact support in the ball $\mathbb{B}(\varepsilon)$ of radius ε and such that $\int_{\mathbb{C}^n} \chi_{\varepsilon} dV = 1$, where dV denotes the Euclidean volume form. The first expression shows that φ_{ε} is a (positive) sum of plurisubharmonic functions (hence

itself plurisubharmonic) in Ω_ε , while the second expression shows that φ_ε is smooth in Ω .

3.1.1. The implication (ii) \Rightarrow (i)

The mean value property shows that the φ_ε 's decrease to φ as ε decreases to zero. It follows therefore from Bedford-Taylor's continuity results [BT76, BT82] that (ii) \Rightarrow (i) holds.

3.1.2. The equivalence (ii) \Leftrightarrow (iii)

The starting point of (iii) is the classical interpretation of the determinant as an infimum of traces :

LEMMA 3.1.

$$(\det Q)^{1/n} = \inf\{n^{-1}\text{tr}(HQ) ; H \in \mathcal{H}\}.$$

We first show that (iii) \Rightarrow (ii). Indeed assume that

$$\Delta_H \varphi \geq f^{1/n}$$

for all positive definite Hermitian matrix H normalized by $\det H = 1$. Since Δ_H is a linear operator, we infer

$$\Delta_H(\varphi \star \chi_\varepsilon) \geq f^{1/n} \star \chi_\varepsilon.$$

This inequality holds for all normalized H , hence Lemma 3.1 yields

$$(dd^c(\varphi \star \chi_\varepsilon))^n \geq (f^{1/n} \star \chi_\varepsilon)^n dV,$$

where this inequality holds in the classical (pointwise, differential) sense.

We conversely check that (ii) \Rightarrow (iii). Since $\varphi \star \chi_\varepsilon$ is smooth, Lemma 3.1 shows indeed that

$$\Delta_H(\varphi \star \chi_\varepsilon) \geq f^{1/n} \star \chi_\varepsilon$$

for all normalized $H \in \mathcal{H}$. Letting $\varepsilon \rightarrow 0$ and taking limits in the sense of distributions yields (iii).

3.1.3. The implication (i) \Rightarrow (ii)

We finally focus on the most delicate implication.

Step 1. Assume first that $(dd^c \varphi)^n \geq f dV$, with f continuous. This inequality can be here interpreted equivalently in the pluripotential or in the viscosity sense, as shown in [EGZ11, Proposition 1.5], whose proof moreover shows the equivalence with the property that

$$\Delta_H \varphi \geq f^{1/n} \text{ for all } H \in \mathcal{H}.$$

Thus (i) \Leftrightarrow (iii) in our main theorem, when f is continuous. Since any lower semi-continuous function is the increasing limit of continuous functions, the implication (i) \Rightarrow (iii) immediately extends to the case when f is lower semi-continuous.

It remains to get rid of this extra continuity assumption. We are going to approximate f by continuous densities f_k , use the previous result and stability estimates to conclude.

The approximation process, inspired by [Ber13], is somehow delicate, so we proceed in several steps.

Step 2. Note first that we can assume that f is *bounded*: we can replace f by $\min(f, A) \in L^\infty(\Omega)$ and let eventually A increase to $+\infty$. Since the problem is local, we can work on fixed balls $B' \Subset B$ and use a max construction to modify φ in a neighborhood of the boundary ∂B , making it equal to the defining function of B .

We fix $0 < \delta < 1$ and $j \in \mathbb{N}^*$. Since $f \in L^2(B) \supset L^\infty(B)$, it follows from [CP92, Kol95] that there exists $U_f \in PSH(B) \cap C^0(\bar{B})$ such that

$$(dd^c U_f)^n = f dV, \quad U_f = 0 \text{ in } \partial B.$$

Set $C_j := \sup e^{-j\varphi/n}$ and observe that

$$e^{-j\varphi} \{f dV + \delta (dd^c \varphi)^n\} \leq (dd^c v)^n$$

where $v := C_j(U_f + \varphi)$ is bounded, plurisubharmonic, with $v = 0$ on ∂B .

By Theorem 2.1 there exists a unique bounded plurisubharmonic solution $\varphi_{j,\delta}$ to the Dirichlet problem

$$(3.1) \quad (dd^c \varphi_{j,\delta})^n = e^{j(\varphi_{j,\delta} - \varphi)} \{f dV + \delta (dd^c \varphi)^n\}$$

in B with boundary values 0.

We now observe that $\varphi_{j,\delta}$ uniformly converges to φ , as $j \rightarrow +\infty$, independently of the value of $\delta > 0$:

LEMMA 3.2. *For all $j \geq 1, \delta \in (0, 1), z \in B$,*

$$\varphi(z) - \frac{\log(1+\delta)}{j} \leq \varphi_{j,\delta}(z) \leq \varphi(z) + \frac{(-\log \delta)}{j}.$$

PROOF. It follows from the comparison principle that $\varphi_{j,\delta}$ is the envelope of subsolutions. It thus suffices to find good sub/supersolutions to insure that $\varphi_{j,\delta}$ converges to φ , as $j \rightarrow +\infty$.

Observe that $u = \varphi - \frac{\log(1+\delta)}{j} \leq \varphi$ is plurisubharmonic in B , with boundary values $u|_{\partial B} \leq 0$. Moreover

$$(dd^c u)^n = (dd^c \varphi)^n = e^{j(u-\varphi)}(1+\delta)(dd^c \varphi)^n \geq e^{j(u-\varphi)} \{f dV + \delta (dd^c \varphi)^n\},$$

since $(dd^c \varphi)^n \geq f dV$. Thus u is a subsolution to the Dirichlet problem, showing that $u \leq \varphi_{j,\delta}$.

Set now $v = \varphi + \frac{(-\log \delta)}{j}$. This is a plurisubharmonic function in B such that $v \geq 0$ on ∂B and

$$(dd^c v)^n = (dd^c \varphi)^n = e^{j(v-\varphi)} \delta (dd^c \varphi)^n \leq e^{j(v-\varphi)} \{f dV + \delta (dd^c \varphi)^n\}.$$

Thus v is a supersolution of the Dirichlet problem hence $\varphi_{j,\delta} \leq v$. \square

Step 3. We now approximate f in L^2 by continuous densities $0 \leq f_k$, with $\|f_k - f\|_{L^2} \rightarrow 0$ as $k \rightarrow +\infty$. Extracting a subsequence and relabelling, we can assume that

there exists $g \in L^2(B)$ such that $f_k \leq g$ for all $k \in \mathbb{N}$ and f_k converges almost everywhere to f . Arguing as above we obtain

$$e^{-j\varphi} \{f_k dV + \delta(dd^c \varphi)^n\} \leq (dd^c v)^n,$$

where $v := C_j(U_g + \varphi)$ is bounded, plurisubharmonic, with $v = 0$ in ∂B . By Theorem 2.1, there exists a unique bounded plurisubharmonic solution $\varphi_{j,\delta,k}$ to the Dirichlet problem

$$(dd^c \varphi_{j,\delta,k})^n = e^{j(\varphi_{j,\delta,k} - \varphi)} \{f_k dV + \delta(dd^c \varphi)^n\}$$

in B , with zero boundary values.

The comparison principle shows that for all $k \in \mathbb{N}$,

$$(3.2) \quad C_j(U_g + \varphi) \leq \varphi_{j,\delta,k} \leq 0.$$

Thus $k \mapsto \varphi_{j,\delta,k}$ is uniformly bounded in B . Extracting and relabelling, we can assume that it converges to a plurisubharmonic function $\psi = \psi_{j,\delta}$ in $L^1(\Omega)$ such that

$$(3.3) \quad C_j(U_g + \varphi) \leq \psi_{j,\delta} \leq 0.$$

We claim that $\psi_{j,\delta} = \varphi_{j,\delta}$ in B . To simplify notations we write $u_k := \varphi_{j,\delta,k}$ and $u := \psi_{j,\delta}$. From (3.2) and (3.3), it follows that $u_k = u = 0$ in ∂B . On the other hand let $\tilde{u}_\ell := (\sup_{k \geq \ell} u_k)^*$ for $\ell \in \mathbb{N}$. This is a decreasing sequence of bounded plurisubharmonic functions converging to u in B . We infer for all ℓ ,

$$(dd^c \tilde{u}_\ell)^n \geq e^{\inf_{k \geq \ell} j(u_k - \varphi)} \inf_{k \geq \ell} (f_k dV + \delta(dd^c \varphi)^n).$$

Letting $\ell \rightarrow +\infty$ yields

$$(dd^c u)^n \geq e^{j(u - \varphi)} \{f dV + \delta(dd^c \varphi)^n\},$$

which implies that $u = \psi_{j,\delta}$ is a subsolution to the Dirichlet problem for the equation (3.1). Hence $\psi_{j,\delta} \leq \varphi_{j,\delta}$.

By [Kol95] there exists a bounded plurisubharmonic function ρ_k in B , solution to the Dirichlet problem

$$(dd^c \rho_k)^n = e^{j(\varphi_{j,\delta} - \varphi)} |f - f_k| dV, \text{ with } \rho_k|_{\partial B} = 0,$$

with the uniform bound

$$\|\rho_k\|_{L^\infty(B)} \leq C \|f - f_k\|_{L^2(B)}^{1/n},$$

where $C > 0$ is independent of k . In particular $\rho_k \rightarrow 0$ uniformly in B .

Since $f_k \leq f + |f - f_k|$ and $\rho_k \leq 0$ it follows that $w := \varphi_{j,\delta} + \rho_k$ satisfies

$$(dd^c w)^n = (dd^c(\varphi_{j,\delta} + \rho_k))^n \geq e^{j(w - \varphi)} (f_k dV + \delta(dd^c \varphi)^n).$$

The comparison principle insures $w \leq \varphi_{j,\delta,k}$ hence $\varphi_{j,\delta} \leq \psi_{j,\delta}$ since $\rho_k \rightarrow 0$.

Conclusion. We have thus shown that $\psi_{j,\delta} = \varphi_{j,\delta}$ and $\varphi_{j,\delta,k} \geq \varphi_{j,\delta} + \rho_k$. Lemma 3.2

yields

$$j(\varphi_{j,\delta,k} - \varphi) \geq -\log(1 + \delta) - j\eta_k,$$

where $\eta_k := \|\rho_k\|_{L^\infty(B)} \xrightarrow{k \rightarrow +\infty} 0$.

Since f_k is continuous we can apply *Step 1* to insure that

$$(dd^c \varphi_{j,\delta,k} \star \chi_\varepsilon)^n \geq \frac{e^{-j\eta_k}}{\delta + 1} (f_k^{1/n} \star \chi_\varepsilon)^n dV.$$

We know that $\limsup_{k \rightarrow +\infty} \varphi_{j,\delta,k} \leq \varphi_{j,\delta}$ since $\varphi_{j,\delta,k} \rightarrow \varphi_{j,\delta}$ in $L^1(B)$ as $k \rightarrow +\infty$. Since $\varphi_{j,\delta,k} \geq \varphi_{j,\delta} - \eta_k$ and $\lim_{k \rightarrow +\infty} \eta_k = 0$, it follows from Hartogs lemma that $\varphi_{j,\delta,k} \rightarrow \varphi_{j,\delta}$ in capacity. Letting $k \rightarrow +\infty$ we obtain

$$(dd^c \varphi_{j,\delta} \star \chi_\varepsilon)^n \geq \frac{1}{\delta + 1} (f^{1/n} \star \chi_\varepsilon)^n dV.$$

By Lemma 3.2 ($\varphi_{j,\delta}$) uniformly converges to φ as $j \rightarrow +\infty$, hence

$$(dd^c \varphi \star \chi_\varepsilon)^n \geq \frac{1}{\delta + 1} (f^{1/n} \star \chi_\varepsilon)^n dV.$$

We let finally δ decrease to zero to obtain the desired lower bound (ii).

3.1.4. An extension of the main result

By approximating a given function φ in the Cegrell class $\mathcal{E}(\Omega)$ by the decreasing sequence $\varphi_j := \max(\varphi, -j)$ of bounded plurisubharmonic functions, we let the reader check that the main theorem holds when φ merely belongs to $\mathcal{E}(\Omega)$.

3.2. Mixed inequalities

We now prove the Corollary of the introduction on mixed Monge-Ampère measures of subsolutions, providing an alternative proof of [Kol03, Lemma 1.2]:

PROPOSITION 3.3. *Assume $\varphi_1, \dots, \varphi_n$ are bounded plurisubharmonic functions in Ω , such that $(dd^c \varphi_i)^n \geq f_i dV$, where $0 \leq f_i \in L^1(\Omega)$. Then*

$$dd^c \varphi_1 \wedge \dots \wedge dd^c \varphi_n \geq f_1^{1/n} \dots f_n^{1/n} dV.$$

PROOF. The inequality is classical when the functions φ_i are smooth, and follows from the concavity of $H \mapsto \log \det H$ (see [HJ85, Corollary 7.6.9]).

To treat the general case we replace each φ_i by its convolutions $\varphi_i \star \chi_\varepsilon$. We can always assume that $f_i \in L^\infty(\Omega)$. Our main result insures that

$$(dd^c(\varphi_i \star \chi_\varepsilon))^n \geq (f_i^{1/n} \star \chi_\varepsilon)^n dV,$$

hence

$$dd^c(\varphi_1 \star \chi_\varepsilon) \wedge \dots \wedge dd^c(\varphi_n \star \chi_\varepsilon) \geq (f_1^{1/n} \star \chi_\varepsilon) \dots (f_n^{1/n} \star \chi_\varepsilon) dV.$$

The left hand side converges weakly to $dd^c \varphi_1 \wedge \dots \wedge dd^c \varphi_n$ by Bedford-Taylor's continuity results [BT76, BT82], while $(f_i^{1/n} \star \chi_\varepsilon)$ converges to $(f_i)^{1/n}$ in L^n by Lebesgue

convergence Theorem. Hence

$$(f_1 \star \chi_\varepsilon)^{1/n} \cdots (f_n \star \chi_\varepsilon)^{1/n} \text{ converges to } (f_1)^{1/n} \cdots (f_n)^{1/n}$$

in L^1 . The conclusion follows. \square

We note conversely that these mixed inequalities yield an important implication in our main result. Assume indeed that $(dd^c \varphi)^n \geq f dV$ in the pluripotential sense. Fix $f_1 = f$ and $\varphi_2 = \dots = \varphi_n = \rho_H$, where $\rho_H = \sum h_{j\bar{k}} z_j \bar{z}_k$ with $H \in \mathcal{H}$, so that $(dd^c \varphi_i)^n \geq f_i dV$ with $f_2 = \dots = f_n = 1$. It follows from the mixed inequalities above that

$$\Delta_H \varphi = dd^c \varphi \wedge dd^c \varphi_2 \wedge \cdots \wedge dd^c \varphi_n \geq f^{1/n}.$$

One can alternatively proceed as follows : observe that

$$\begin{aligned} & (dd^c \varphi \star \chi_\varepsilon)^n(z) \\ &= \int dd^c \varphi(z - w_1) \wedge \cdots \wedge dd^c \varphi(z - w_n) \chi_\varepsilon(w_1) \cdots \chi_\varepsilon(w_n) dV(w_1, \dots, w_n) \\ &\geq \int f^{1/n}(z - w_1) \cdots f^{1/n}(z - w_n) dV(z) \chi_\varepsilon(w_1) \cdots \chi_\varepsilon(w_n) dV(w_1, \dots, w_n) \\ &= (f^{1/n} \star \chi_\varepsilon)^n(z) dV(z). \end{aligned}$$

3.3. More general right hand side

There are several ways one can extend our main observation. We note here the following:

THEOREM 3.4. *Assume φ is plurisubharmonic and bounded. Fix $g \in L^1$ and $h : \mathbb{R} \rightarrow \mathbb{R}$ convex. The following are equivalent :*

- (i) $(dd^c \varphi)^n \geq e^{h(\varphi)+g} dV$ in the pluripotential sense;
- (ii) $(dd^c(\varphi \star \chi_\varepsilon))^n \geq e^{h(\varphi \star \chi_\varepsilon)+g \star \chi_\varepsilon} dV$ in the classical sense, for all $\varepsilon > 0$;
- (iii) $\Delta_H \varphi \geq e^{h(\varphi)/n+g/n}$ in the sense of distributions, for all $H \in \mathcal{H}$.

The proof is very close to what we have done above, using the convexity of \exp and h through Jensen's inequality. We leave the details to the reader.

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