



HAL
open science

Algebraic deformation quantization of Leibniz algebras

Charles Alexandre, Martin Bordemann, Salim Riviere, Friedrich Wagemann

► **To cite this version:**

Charles Alexandre, Martin Bordemann, Salim Riviere, Friedrich Wagemann. Algebraic deformation quantization of Leibniz algebras. Communications in Algebra, 2018. hal-01889895

HAL Id: hal-01889895

<https://hal.science/hal-01889895v1>

Submitted on 8 Oct 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Algebraic deformation quantization of Leibniz algebras

Charles Alexandre ^{*} Martin Bordemann [†]
Université de Strasbourg Université de Mulhouse

Salim Rivière [‡] Friedrich Wagemann [§]
Université d'Angers Université de Nantes

October 8, 2018

Abstract

In this paper we focus on a certain self-distributive multiplication on coalgebras, which leads to so-called rack bialgebra. We construct canonical rack bialgebras (some kind of enveloping algebras) for any Leibniz algebra.

Our motivation is deformation quantization of Leibniz algebras in the sense of [6]. Namely, the canonical rack bialgebras we have constructed for any Leibniz algebra lead to a simple explicit formula of the rack-star-product on the dual of a Leibniz algebra recently constructed by Dherin and Wagemann in [6]. We clarify this framework setting up a general deformation theory for rack bialgebras and show that the rack-star-product turns out to be a deformation of the trivial rack bialgebra product.

Introduction

The algebraic structures involved in Leibniz deformation quantization

Recall that a *pointed rack* (see [8]) is a pointed set (X, e) together with a binary operation $\triangleright : X \times X \rightarrow X$ such that for all $x \in X$, the map $y \mapsto x \triangleright y$ is bijective and such that for all $x, y, z \in X$, the self-distributivity and unit relations

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z), \quad e \triangleright x = x, \quad \text{and} \quad x \triangleright e = e$$

^{*}Char.Alexandre@gmail.com

[†]Martin.Bordemann@uha.fr

[‡]salim.riviere@univ-angers.fr

[§]wagemann@math.univ-nantes.fr

are satisfied. Observe that racks are not algebras over an operad, but the correct algebraic structure is that of a properad. Therefore the standard deformation theory of algebras over an operad does not apply. Imitating the notion of a Lie group, the smooth version of a pointed rack is called a *Lie rack*.

An important class of examples of racks are the so-called *augmented racks*, see [8]. An augmented rack is the data of a group G , a G -set X and a map $p : X \rightarrow G$ such that for all $x \in X$ and all $g \in G$,

$$p(g \cdot x) = gp(x)g^{-1}.$$

The set X becomes then a rack by setting $x \triangleright y := p(x) \cdot y$. In fact, augmented racks are the Drinfeld center (or the Yetter-Drinfeld modules) in the monoidal category of G -sets over the (set-theoretical) Hopf algebra G , see for example [11]. Any rack may be augmented in many ways, for example by using the canonical morphism to its associated group (see [8]) or to its group of bijections or to its group of automorphisms.

In order to formalize the notion of a rack, one needs the diagonal map $\text{diag}_X : X \rightarrow X \times X$ given by $x \mapsto (x, x)$. Then the self-distributivity relation reads in terms of maps

$$\begin{aligned} \mathbf{m} \circ (\text{id}_M \times \mathbf{m}) \\ = \mathbf{m} \circ (\mathbf{m} \times \mathbf{m}) \circ (\text{id}_M \times \tau_{M,M} \times \text{id}_M) \circ (\text{diag}_M \times \text{id}_M \times \text{id}_M). \end{aligned}$$

Axiomatizing this kind of structure, one may start with a coalgebra C and look for rack operations on this fixed coalgebra, see [4] and [12]. A natural framework where this kind of structure arises is by taking point-distributions over (resp. to) the pointed manifold given by a Lie rack, see [20], [2], [15] or [1]. We dub the arising structure a *rack bialgebra*, see Definition 1.1. We carry out some structure theory for rack bialgebras based on semigroup theory in the article [1].

Lie racks are intimately related to *Leibniz algebras* \mathfrak{h} , i.e. a vector space \mathfrak{h} with a bilinear bracket $[\cdot, \cdot] : \mathfrak{h} \otimes \mathfrak{h} \rightarrow \mathfrak{h}$ such that for all $X, Y, Z \in \mathfrak{h}$, $[X, -]$ acts as a derivation:

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]. \quad (1)$$

Indeed, Kinyon showed in [10] that the tangent space at $e \in H$ of a Lie rack H carries a natural structure of a Leibniz algebra, generalizing the relation between a Lie group and its tangent Lie algebra. Conversely, every (finite dimensional real or complex) Leibniz algebra \mathfrak{h} may be integrated into a Lie rack $R_{\mathfrak{h}}$ (with underlying manifold \mathfrak{h}) using the rack product

$$X \blacktriangleright Y := e^{\text{ad}_X}(Y), \quad (2)$$

noting that the exponential of the inner derivation ad_X for each $X \in \mathfrak{h}$ is an automorphism.

Leibniz deformation quantization

Given a finite-dimensional real Lie algebra $(\mathfrak{g}, [,])$, its dual vector space \mathfrak{g}^* is a smooth manifold which carries a Poisson bracket on its space of smooth functions, defined for all $f, g \in C^\infty(\mathfrak{g}^*)$ and all $\xi \in \mathfrak{g}^*$ by the *Kostant-Kirillov-Souriau formula*

$$\{f, g\}(\xi) := \langle \xi, [df(\xi), dg(\xi)] \rangle.$$

Here $df(\xi)$ and $dg(\xi)$ are linear functionals on \mathfrak{g}^* , identified with elements of \mathfrak{g} .

In the same way, a general finite dimensional Leibniz algebra \mathfrak{h} gives rise to a smooth manifold \mathfrak{h}^* , which carries now some kind of generalized Poisson bracket, namely

$$\{f, g\}(\xi) := -\langle \xi, [df(0), dg(\xi)] \rangle,$$

see [6] for an explanation why we believe that this is the correct bracket to consider. In particular, this generalized Poisson bracket need not be skew-symmetric.

The quantization procedure of this generalized Poisson bracket proposed in [6] works as follows: The cotangent lift of the above rack product

$$X \blacktriangleright Y = e^{\text{ad}_X}(Y)$$

is interpreted as a symplectic micromorphism. The generating function of this micromorphism serves then as a phase function in a Fourier integral operator, whose asymptotic expansion gives rise to a star-product.

One main goal of the present article is to set up a purely algebraic framework in which one may deformation quantize the dual of a Leibniz algebra. The main feature will be to recover –in a rather explicit algebraic manner– the star-product which has been constructed in [6] by analytic methods, see Corollary 2.5. The explicit formula reads:

Let $f, g \in C^\infty(\mathfrak{h}^*)$.

$$(f \triangleright_{\hbar} g)(\alpha) = \sum_{r=0}^{\infty} \frac{\hbar^r}{r!} \sum_{i_1, \dots, i_r=1}^n \frac{\partial^r f}{\partial \alpha_{i_1} \dots \partial \alpha_{i_r}}(0) \left((\widetilde{\text{ad}}_{i_1} \circ \dots \circ \widetilde{\text{ad}}_{i_r})(g) \right)(\alpha), \quad (3)$$

where we have chosen a basis in the finite dimensional Leibniz algebra \mathfrak{h} and the first order differential operators ad_{i_1} are defined by

$$(\widetilde{\text{ad}}_i(f))(\alpha) = \sum_{j,k=1}^n \alpha_k c_{ij}^k \frac{\partial f}{\partial \alpha_j}(\alpha),$$

where c_{ij}^k are the structure constants of \mathfrak{h} w.r.t. this basis. Our first main result is thus Corollary 2.5 where we show that the above \triangleright_{\hbar} is indeed a rack product and that it is equal (up to a sign) to the one constructed in [6] by analytic methods.

In the remaining part of the paper, we answer the natural question in which cohomology theory the first term of the above deformation quantization appears

as a 2-cocycle. For this, we set up a general deformation theory framework in which the above star-product appears as a formal deformation, its infinitesimal term defining a 2-cocycle and thus a second cohomology class. The main result is here Theorem 2.12 where we show that a natural differential d on the adjoint rack bialgebra complex satisfies indeed $d^2 = 0$. This is a combinatorially involved computation. Observe that this cohomology theory is thus well-defined in all degrees, in contrast to the related cohomology theory in [4] for which only degrees up to 3 exist for the moment. One main point in this part of the paper is that our deformation complex for rack bialgebras replaces the (non existing) rack cohomology complex with adjoint coefficients (cf the case of group cohomology for a group G where cohomology with adjoint coefficients in G does not exist, while the Hochschild cohomology of the group algebra $K[G]$ with values in $K[G]$ may play this role).

Acknowledgements: F.W. thanks Université de Haute Alsace for an invitation during which the shape of this research project was defined. Some part of the results of this paper constitute the master thesis of C.A.. We all thank the referee for useful remarks leading to an improvement of the manuscript.

1 Preliminaries

1.1 Rack bialgebras

In the following, let K be an associative commutative unital ring containing all the rational numbers. In the main part of this paper, we will assume $K = \mathbb{R}$ or \mathbb{C} . The symbol \otimes will always denote the tensor product of K -modules over K . For any coalgebra (C, Δ) over K , we shall use Sweedler's notation $\Delta(a) = \sum_{(a)} a^{(1)} \otimes a^{(2)}$ for any $a \in A$. We will feel free to suppress the sum-sign in Sweedler's notation in complicated formulas for typographical reasons.

The following sections will all deal with the following type of *nonassociative bialgebra*: Let $(B, \Delta, \epsilon, \mathbf{1}, \mu)$ be a K -module such that $(B, \Delta, \epsilon, \mathbf{1})$ is a *coassociative counital coaugmented coalgebra* (a C^3 -coalgebra), and such that the linear map $\mu : B \otimes B \rightarrow B$ (the multiplication) is a morphism of C^3 -coalgebras (it satisfies in particular $\mu(\mathbf{1} \otimes \mathbf{1}) = \mathbf{1}$). We shall call this situation a *nonassociative C^3I -bialgebra* (where I stands for $\mathbf{1}$ being an *idempotent* for the multiplication μ). For another nonassociative C^3I -bialgebra $(B', \Delta', \epsilon', \mathbf{1}', \mu')$ a K -linear map $\phi : B \rightarrow B'$ will be called a *morphism of nonassociative C^3I -bialgebras* iff it is a morphism of C^3 -coalgebras and is multiplicative in the usual sense $\phi(\mu(a \otimes b)) = \mu'(\phi(a) \otimes \phi(b))$ for all $a, b \in B$.

Definition 1.1. A **rack bialgebra** $(B, \Delta, \epsilon, \mathbf{1}, \mu)$ is a nonassociative C^3I -bialgebra (where we write for all $a, b \in B$ $\mu(a \otimes b) =: a \triangleright b$) such that the

following identities hold for all $a, b, c \in B$

$$\mathbf{1} \triangleright a = a, \quad (4)$$

$$a \triangleright \mathbf{1} = \epsilon(a)\mathbf{1}, \quad (5)$$

$$a \triangleright (b \triangleright c) = \sum_{(a)} (a^{(1)} \triangleright b) \triangleright (a^{(2)} \triangleright c). \quad (6)$$

The last condition (6) is called the **self-distributivity** condition.

Note that we do not demand that the C^3 -coalgebra B should be cocommutative nor connected. Similar definitions have been proposed in [4] and in [12].

Example 1.1.

Any C^3 coalgebra $(C, \Delta, \epsilon, \mathbf{1})$ carries a *trivial rack bialgebra structure* defined by the left-trivial multiplication

$$a \triangleright_0 b := \epsilon(a)b \quad (7)$$

which in addition is easily seen to be associative and left-unital, but in general not unital. \diamond

Another method of constructing rack bialgebras is *gauging*: Let $(B, \Delta, \epsilon, \mathbf{1}, \mu)$ a rack bialgebra –where we write $\mu(a \otimes b) = a \triangleright b$ for all $a, b \in B$ –, and let $f : B \rightarrow B$ a morphism of C^3 -coalgebras such that for all $a, b \in B$

$$f(a \triangleright b) = a \triangleright (f(b)), \quad (8)$$

i.e. f is μ -equivariant. It is a routine check that $(B, \Delta, \epsilon, \mathbf{1}, \mu_f)$ is a rack bialgebra where for all $a, b \in B$ the multiplication is defined by

$$\mu_f(a \otimes b) := a \triangleright_f b := (f(a)) \triangleright b. \quad (9)$$

We shall call $(B, \Delta, \epsilon, \mathbf{1}, \mu_f)$ the *f-gauge* of $(B, \Delta, \epsilon, \mathbf{1}, \mu)$.

Example 1.2.

Let $(H, \Delta_H, \epsilon_H, \mu_H, \mathbf{1}_H, S)$ be a cocommutative Hopf algebra over K . Then it is easy to see (cf. also the particular case $B = H$ and $\Phi = \text{id}_H$ of Proposition 1.4 for a detailed proof) that the new multiplication $\mu : H \otimes H \rightarrow H$, written $\mu(h \otimes h') = h \triangleright h'$, defined by the usual *adjoint representation*

$$h \triangleright h' := \text{ad}_h(h') := \sum_{(h)} h^{(1)} h'(S(h^{(2)})), \quad (10)$$

equips the C^4 -coalgebra $(H, \Delta_H, \epsilon_H, \mathbf{1}_H)$ with a rack bialgebra structure. \diamond

In general, the adjoint representation does not seem to preserve the coalgebra structure if no cocommutativity is assumed.

Example 1.3.

Recall that a pointed set (X, e) is a *pointed rack* in case there is a binary operation $\triangleright : X \times X \rightarrow X$ such that for all $x \in X$, the map $y \mapsto x \triangleright y$ is bijective and such that for all $x, y, z \in X$, the self-distributivity and unit relations

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z), \quad e \triangleright x = x, \quad \text{and} \quad x \triangleright e = e$$

are satisfied. Then there is a natural rack bialgebra structure on the vector space $K[X]$ which has the elements of X as a basis. $K[X]$ carries the usual coalgebra structure such that all $x \in X$ are set-like: $\Delta(x) = x \otimes x$ for all $x \in X$. The product μ is then induced by the rack product. By functoriality, μ is compatible with Δ and e .

Observe that this construction differs slightly from the construction in [4], Section 3.1. \diamond

Remark 1.2. It is shown in Theorem 4.3 of [4] that for a C^4 -coalgebra B with a self-distributive map $\triangleleft = q : B \otimes B \rightarrow B$ which is a morphism of coalgebras, the map

$$R_q = (\text{id}_B \otimes q) \circ (\tau \otimes \text{id}_B) \circ (\text{id}_B \otimes \Delta)$$

is a solution of the Yang-Baxter equation. We draw our reader's attention to the fact that Carter-Crans-Elhamdadi-Saito work in [4] with right racks, while we work here with left racks. The statement of their theorem works also for left racks, but then one has to take

$$\tilde{R}_q = (\text{id}_B \otimes q) \circ (\Delta \otimes \text{id}_B) \circ \tau.$$

In particular for any rack bialgebra, \tilde{R}_q is a solution of the Yang-Baxter equation.

Example 1.4.

Here we suppose $K = \mathbb{R}$ or \mathbb{C} . Another general construction mechanism for rack bialgebras is exhibited in [1]. Namely, let (M, e, \triangleright) be a Lie rack. In particular, (M, e) is a pointed manifold and it makes sense to associate to it the vector space $\mathcal{E}'_e(M)$ of distributions on M which have their support in $\{e\}$. There is a corresponding functor $F : \mathcal{M}f* \rightarrow \mathbb{K}\mathbf{Vect}$ from the category of pointed manifolds $\mathcal{M}f*$ with values in the category of K -vector spaces $\mathbb{K}\mathbf{Vect}$. Observe that J.-P. Serre [20] used it to define the universal enveloping algebra of a Lie group, see also [2], [15] or [1].

In fact, this framework is a special case of a local multiplication where $0 \times 0 = 0$. Such a multiplication gives rise to a bialgebra of distributions supported at 0. If the multiplication satisfies a certain identity, the bialgebra satisfies the linearized identity. In our case, the rack identity (i.e. self-distributivity) leads to the linearized self-distributivity relation, i.e. condition (6). The general framework is described in [15] Section 3, see also [16].

In our case, the pointed manifold given by a Lie rack yields a rack bialgebra, see [1] where this functor is studied in detail. \diamond

We will also need the following structure:

Definition 1.3. An **augmented rack bialgebra** over K is a quadruple (B, Φ, H, ℓ) consisting of a C^3 -coalgebra $(B, \Delta, \epsilon, \mathbf{1})$, of a cocommutative (!) Hopf algebra $(H, \Delta_H, \epsilon_H, \mathbf{1}_H, \mu_H, S)$, of a morphism of C^3 -coalgebras $\Phi : B \rightarrow H$, and of a left action $\ell : H \otimes B \rightarrow B$ of H on B which is a morphism of C^3 -coalgebras (i.e. B is a H -module-coalgebra) such that for all $h \in H$ and $a \in B$

$$h.\mathbf{1} = \epsilon_H(h)\mathbf{1} \quad (11)$$

$$\Phi(h.a) = \text{ad}_h(\Phi(a)). \quad (12)$$

where ad denotes the usual adjoint representation for Hopf algebras, see e.g. eqn (10).

We shall define a **morphism** $(B, \Phi, H, \ell) \rightarrow (B', \Phi', H', \ell')$ of **augmented rack bialgebras** to be a pair (ϕ, ψ) of K -linear maps where $\phi : (B, \Delta, \epsilon, \mathbf{1}) \rightarrow (B', \Delta', \epsilon', \mathbf{1}')$ is a morphism of C^3 -coalgebras, and $\psi : H \rightarrow H'$ is a morphism of Hopf algebras such that the obvious diagrams commute:

$$\Phi' \circ \phi = \psi \circ \Phi, \quad \text{and} \quad \ell' \circ (\psi \otimes \phi) = \phi \circ \ell \quad (13)$$

An immediate consequence of this definition is the following

Proposition 1.4. Let (B, Φ, H, ℓ) be an augmented rack bialgebra. Then the C^3 -coalgebra $(B, \epsilon, \mathbf{1})$ will become a rack bialgebra by means of the multiplication

$$a \triangleright b := \Phi(a).b \quad (14)$$

for all $a, b \in B$. In particular, each Hopf algebra H becomes an augmented rack bialgebra via $(H, \text{id}_H, H, \text{ad})$. In general, for each augmented rack bialgebra the map $\Phi : B \rightarrow H$ is a morphism of rack bialgebras.

Proof. We check first that \triangleright is a morphism of C^3 -coalgebras $B \otimes B \rightarrow B$: Let $a, b \in B$, then –thanks to the fact that the action ℓ and the maps Φ are coalgebra morphisms–

$$\begin{aligned} \Delta(\mu(a \otimes b)) &= \Delta(a \triangleright b) = \Delta(\Phi(a).b) = \sum_{(\Phi(a)), (b)} \left((\Phi(a)^{(1)}) . b^{(1)} \right) \otimes \left((\Phi(a)^{(2)}) . b^{(2)} \right) \\ &= \sum_{(a), (b)} \left((\Phi(a^{(1)})) . b^{(1)} \right) \otimes \left((\Phi(a^{(2)})) . b^{(2)} \right) \\ &= \sum_{(a), (b)} (a^{(1)} \triangleright b^{(1)}) \otimes (a^{(2)} \triangleright b^{(2)}) \end{aligned}$$

whence μ is a morphism of coalgebras. Clearly

$$\epsilon(a \triangleright b) = \epsilon(\Phi(a).b) = \epsilon_H(\Phi(a))\epsilon(b) = \epsilon(a)\epsilon(b)$$

whence μ preserves counits.

We shall next compute both sides of the self-distributivity identity (6) to get an idea: For all $a, b, c \in B$

$$a \triangleright (b \triangleright c) = \Phi(a).(\Phi(b).c) = (\Phi(a)\Phi(b)).c,$$

and

$$\begin{aligned}
\sum_{(a)} (a^{(1)} \triangleright b) \triangleright (a^{(2)} \triangleright c) &= \sum_{(a)} (\Phi(a^{(1)}).b) \triangleright (\Phi(a^{(2)}).c) \\
&= \sum_{(a)} \left(\Phi(\Phi(a^{(1)}).b) \right) . (\Phi(a^{(2)}).c) \\
&= \sum_{(a)} \left(\Phi(\Phi(a^{(1)}).b) \Phi(a^{(2)}) \right) . c,
\end{aligned}$$

and we compute, using the fact that Φ is a morphism of C^3 -coalgebras,

$$\begin{aligned}
\sum_{(a)} \Phi(\Phi(a^{(1)}).b) \Phi(a^{(2)}) &= \sum_{(a)} \Phi((\Phi(a)^{(1)}).b) (\Phi(a)^{(2)}) \\
&\stackrel{(12)}{=} \sum_{(a)} \left(\text{ad}_{\Phi(a)^{(1)}}(\Phi(b)) \right) (\Phi(a)^{(2)}) \\
&= \sum_{(a)} (\Phi(a)^{(1)} \Phi(b) (S(\Phi(a)^{(2)}))) (\Phi(a)^{(3)}) \\
&= \sum_{(a)} (\Phi(a)^{(1)} \Phi(b) \mathbf{1}_H \epsilon_H(\Phi(a)^{(2)})) \\
&= \Phi(a) \Phi(b),
\end{aligned}$$

which proves the self-distributivity identity. Moreover we have

$$\mathbf{1}_B \triangleright a = \Phi(\mathbf{1}).a = \mathbf{1}_H.a = a,$$

and

$$a \triangleright \mathbf{1} = \Phi(a).\mathbf{1} \stackrel{(11)}{=} \epsilon_H(\Phi(a))\mathbf{1} = \epsilon_B(a)\mathbf{1},$$

whence the C^3 -coalgebra becomes a rack bialgebra. \square

Example 1.5.

Exactly in the same way as a pointed rack gives rise to a rack bialgebra $K[X]$, an augmented pointed rack $p : X \rightarrow G$ gives rise to an augmented rack bialgebra $p : K[X] \rightarrow K[G]$. \diamond

Remark 1.5. Motivated by the fact that the augmented racks $p : X \rightarrow G$ are exactly the Yetter-Drinfeld modules over the (set-theoretical) Hopf algebra G , we may ask whether augmented rack bialgebras are Yetter-Drinfeld modules.

In fact, any cocommutative augmented rack bialgebra (B, Φ, H, ℓ) gives rise to a Yetter-Drinfeld module over the Hopf algebra H . Indeed, B is a left H -module via ℓ , and becomes a left H -comodule via

$$\rho : B \xrightarrow{\Delta_B} B \otimes B \xrightarrow{\Phi \otimes \text{id}_B} H \otimes B.$$

Now, in Sweedler notation, the coaction is denoted for all $b \in B$ by

$$\rho(b) = \sum_{(b)} b_{(-1)} \otimes b_{(0)} \in H \otimes B.$$

Then the *Yetter-Drinfeld compatibility relation* reads

$$\sum_{(h.b)} (h.b)_{(-1)} \otimes (h.b)_{(0)} = \sum_{(b),(h)} h^{(1)} b_{(-1)} S(h^{(3)}) \otimes h^{(2)} b_{(0)}.$$

This relation is true in our case, because ℓ is a morphism of coalgebras and is sent to the adjoint action via Φ .

Conversely, given a Yetter-Drinfeld module C over a Hopf algebra H , together with a linear form $\epsilon_C : C \rightarrow K$ satisfying $\epsilon_C(h.c) = \epsilon_H(h)\epsilon_C(c)$, then define a map $\Phi : C \rightarrow H$ by

$$\Phi := (\text{id}_H \otimes \epsilon_C) \circ \rho.$$

The map Φ intertwines the left action on C and the adjoint action on H thanks to the Yetter-Drinfeld condition.

Now define a rack product for all $x, y \in C$ by

$$x \triangleright y = \Phi(x).y,$$

then we obtain a Yetter-Drinfeld version of self-distributivity

$$x \triangleright (y \triangleright z) = \sum_{(x)} (x_{(-1)}.y) \triangleright (x_{(0)} \triangleright z),$$

as there is no comultiplication on C .

The fact that Φ is a morphism of coalgebras is then replaced by the identity

$$(\text{id}_H \otimes \Phi) \circ \rho = \Delta_H \circ \Phi,$$

which one needs to demand.

Finally, one needs a unit $1_C \in C$ such that for all $h \in H$, $h.1_C = \epsilon_H(h)1_C$, $\epsilon_C(1_C) = 1_K$, $\rho(1_C) = 1_H \otimes 1_C$, and $\Phi(1_C) = 1_H$. This is somehow the closest one can get to a rack bialgebra without having a compatible C^3 coalgebra structure on C .

1.2 (Augmented) rack bialgebras for any Leibniz algebra

In this subsection, we will suppose $\mathbb{Q} \subset K$. Let $(\mathfrak{h}, [,])$ be a *Leibniz algebra over K* , i.e. \mathfrak{h} is a K -module equipped with a K -linear map $[,] : \mathfrak{h} \otimes \mathfrak{h} \rightarrow \mathfrak{h}$ satisfying the (left) Leibniz identity (1).

Recall first that each Lie algebra over K is a Leibniz algebra giving rise to a functor from the category of all Lie algebras to the category of all Leibniz algebras.

Furthermore, recall that each Leibniz algebra has two canonical K -submodules

$$Q(\mathfrak{h}) := \{x \in \mathfrak{h} \mid \exists N \in \mathbb{N} \setminus \{0\}, \exists \lambda_1, \dots, \lambda_N \in K, \exists x_1, \dots, x_N \text{ such that } x = \sum_{r=1}^N \lambda_r [x_r, x_r]\}, \quad (15)$$

$$\mathfrak{z}(\mathfrak{h}) := \{x \in \mathfrak{h} \mid \forall y \in \mathfrak{h} : [x, y] = 0\}. \quad (16)$$

It is well-known and not hard to deduce from the Leibniz identity that both $Q(\mathfrak{h})$ and $\mathfrak{z}(\mathfrak{h})$ are two-sided abelian ideals of $(\mathfrak{h}, [\ , \])$, that $Q(\mathfrak{h}) \subset \mathfrak{z}(\mathfrak{h})$, and that the quotient Leibniz algebras

$$\bar{\mathfrak{h}} := \mathfrak{h}/Q(\mathfrak{h}) \quad \text{and} \quad \mathfrak{g}(\mathfrak{h}) := \mathfrak{h}/\mathfrak{z}(\mathfrak{h}) \quad (17)$$

are Lie algebras. Since the ideal $Q(\mathfrak{h})$ is clearly mapped into the ideal $Q(\mathfrak{h}')$ by any morphism of Leibniz algebras $\mathfrak{h} \rightarrow \mathfrak{h}'$ (which is a priori not the case for $\mathfrak{z}(\mathfrak{h})$!), there is an obvious functor $\mathfrak{h} \rightarrow \bar{\mathfrak{h}}$ from the category of all Leibniz algebras to the category of all Lie algebras.

In order to perform the following constructions of rack bialgebras for any given Leibniz algebra $(\mathfrak{h}, [\ , \])$, choose first a two-sided ideal $\mathfrak{z} \subset \mathfrak{h}$ such that

$$Q(\mathfrak{h}) \subset \mathfrak{z} \subset \mathfrak{z}(\mathfrak{h}), \quad (18)$$

let \mathfrak{g} denote the quotient Lie algebra $\mathfrak{h}/\mathfrak{z}$, and let $p : \mathfrak{h} \rightarrow \mathfrak{g}$ be the natural projection. The data of $\mathfrak{z} \subset \mathfrak{h}$, i.e. of a Leibniz algebra \mathfrak{h} together with an ideal \mathfrak{z} such that $Q(\mathfrak{h}) \subset \mathfrak{z} \subset \mathfrak{z}(\mathfrak{h})$, could be called an *augmented Leibniz algebra*. Thus we are actually associating an augmented rack bialgebra to every augmented Leibniz algebra. In fact, we will see that this augmented rack bialgebra does not depend on the choice of the ideal \mathfrak{z} and therefore refrain from introducing augmented Leibniz algebras in a more formal way.

The Lie algebra \mathfrak{g} naturally acts as derivations on \mathfrak{h} by means of (for all $x, y \in \mathfrak{h}$)

$$p(x).y := [x, y] =: \text{ad}_x(y) \quad (19)$$

because $\mathfrak{z} \subset \mathfrak{z}(\mathfrak{h})$. Note that

$$\mathfrak{h}/\mathfrak{z}(\mathfrak{h}) \cong \{\text{ad}_x \in \text{Hom}_K(\mathfrak{h}, \mathfrak{h}) \mid x \in \mathfrak{h}\}. \quad (20)$$

as Lie algebras.

Consider now the C^5 -coalgebra $(B = S(\mathfrak{h}), \Delta, \epsilon, \mathbf{1})$. Here $S(\mathfrak{h})$ is the symmetric algebra and coalgebra on the vector space \mathfrak{h} , and $S(\mathfrak{h})$ is actually a commutative cocommutative Hopf algebra over K with respect to the symmetric multiplication \bullet . The linear map $p : \mathfrak{h} \rightarrow \mathfrak{g}$ induces a unique morphism of Hopf algebras

$$\tilde{\Phi} = S(p) : S(\mathfrak{h}) \rightarrow S(\mathfrak{g}) \quad (21)$$

satisfying

$$\tilde{\Phi}(x_1 \bullet \dots \bullet x_k) = p(x_1) \bullet \dots \bullet p(x_k) \quad (22)$$

for any nonnegative integer k and $x_1, \dots, x_k \in \mathfrak{h}$. In other words, the association $\mathbb{S} : V \rightarrow \mathbb{S}(V)$ is a functor from the category of all K -modules to the category of all commutative unital C^5 -coalgebras. Consider now the universal enveloping algebra $\mathbb{U}(\mathfrak{g})$ of the Lie algebra \mathfrak{g} . Since $\mathbb{Q} \subset K$ by assumption, the Poincaré-Birkhoff-Witt Theorem (in short: PBW) holds (see e.g. [17, Appendix]). More precisely, the symmetrisation map $\omega : \mathbb{S}(\mathfrak{g}) \rightarrow \mathbb{U}(\mathfrak{g})$, defined by

$$\omega(\mathbf{1}_{\mathbb{S}(\mathfrak{g})}) = \mathbf{1}_{\mathbb{U}(\mathfrak{g})}, \quad \text{and} \quad \omega(\xi_1 \bullet \dots \bullet \xi_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \xi_{\sigma(1)} \cdots \xi_{\sigma(k)}, \quad (23)$$

see e.g. [7, p.80, eqn (3)], is an isomorphism of C^5 -coalgebras (in general not of associative algebras). We now need an action of the Hopf algebra $H = \mathbb{U}(\mathfrak{g})$ on B , and an intertwining map $\Phi : B \rightarrow \mathbb{U}(\mathfrak{g})$. In order to get this, we first look at \mathfrak{g} -modules: The K -module \mathfrak{h} is a \mathfrak{g} -module by means of eqn (19), the Lie algebra \mathfrak{g} is a \mathfrak{g} -module via its adjoint representation, and the linear map $p : \mathfrak{h} \rightarrow \mathfrak{g}$ is a morphism of \mathfrak{g} -modules since p is a morphism of Leibniz algebras. Now $\mathbb{S}(\mathfrak{h})$ and $\mathbb{S}(\mathfrak{g})$ are \mathfrak{g} -modules in the usual way, i.e. for all $k \in \mathbb{N} \setminus \{0\}$, $\xi, \xi_1, \dots, \xi_k \in \mathfrak{g}$, and $x_1, \dots, x_k \in \mathfrak{h}$

$$\xi.(x_1 \bullet \dots \bullet x_k) := \sum_{r=1}^k x_1 \bullet \dots \bullet (\xi.x_r) \bullet \dots \bullet x_k, \quad (24)$$

$$\xi.(\xi_1 \bullet \dots \bullet \xi_k) := \sum_{r=1}^k \xi_1 \bullet \dots \bullet [\xi.\xi_r] \bullet \dots \bullet \xi_k, \quad (25)$$

and of course $\xi.\mathbf{1}_{\mathbb{S}(\mathfrak{h})} = 0$ and $\xi.\mathbf{1}_{\mathbb{S}(\mathfrak{g})} = 0$. Recall that $\mathbb{U}(\mathfrak{g})$ is a \mathfrak{g} -module via the adjoint representation $\text{ad}_\xi(u) = \xi.u = \xi u - u\xi$ (for all $\xi \in \mathfrak{g}$ and all $u \in \mathbb{U}(\mathfrak{g})$). It is easy to see that the map $\tilde{\Phi}$ (22) is a morphism of \mathfrak{g} -modules, and it is well-known that the symmetrization map ω (23) is also a morphism of \mathfrak{g} -modules, see e.g. [7, p.82, Prop. 2.4.10]. Define the K -linear map $\Phi : \mathbb{S}(\mathfrak{h}) \rightarrow \mathbb{U}(\mathfrak{g})$ by the composition

$$\Phi := \omega \circ \tilde{\Phi}. \quad (26)$$

Then Φ is a map of C^5 -coalgebras and a map of \mathfrak{g} -modules. Thanks to the universal property of the universal enveloping algebra, it follows that $\mathbb{S}(\mathfrak{h})$ and $\mathbb{U}(\mathfrak{g})$ are left $\mathbb{U}(\mathfrak{g})$ -modules, via (for all $\xi_1, \dots, \xi_k \in \mathfrak{g}$, and for all $a \in \mathbb{S}(\mathfrak{h})$)

$$(\xi_1 \cdots \xi_k).a = \xi_1.(\xi_2.(\cdots \xi_k.a)\cdots) \quad (27)$$

and the usual adjoint representation (10) (for all $u \in \mathbb{U}(\mathfrak{g})$)

$$\text{ad}_{\xi_1 \cdots \xi_k}(u) = (\text{ad}_{\xi_1} \circ \cdots \circ \text{ad}_{\xi_k})(u), \quad (28)$$

and that Φ intertwines the $\mathbb{U}(\mathfrak{g})$ -action on $C = \mathbb{S}(\mathfrak{h})$ with the adjoint action of $\mathbb{U}(\mathfrak{g})$ on itself.

Finally it is a routine check using the above identities (24) and (10) that $\mathbb{S}(\mathfrak{h})$ becomes a module coalgebra.

We can resume the preceding considerations in the following

Theorem 1.6. *Let $(\mathfrak{h}, [\cdot, \cdot])$ be a Leibniz algebra over K , let \mathfrak{z} be a two-sided ideal of \mathfrak{h} such that $Q(\mathfrak{h}) \subset \mathfrak{z} \subset \mathfrak{z}(\mathfrak{h})$, let \mathfrak{g} denote the quotient Lie algebra $\mathfrak{h}/\mathfrak{z}$ by \mathfrak{g} , and let $p : \mathfrak{h} \rightarrow \mathfrak{g}$ be the canonical projection.*

1. *Then there is a canonical $U(\mathfrak{g})$ -action ℓ on the C^5 -coalgebra $B := S(\mathfrak{h})$ (making it into a module coalgebra leaving invariant $\mathbf{1}$) and a canonical lift of p to a map of C^5 -coalgebras, $\Phi : S(\mathfrak{h}) \rightarrow U(\mathfrak{g})$ such that eqn (12) holds.*

Hence the quadruple $(S(\mathfrak{h}), \Phi, U(\mathfrak{g}), \ell)$ is an augmented rack bialgebra whose associated Leibniz algebra is equal to $(\mathfrak{h}, [\cdot, \cdot])$ (independently of the choice of \mathfrak{z}).

The resulting rack multiplication μ of $S(\mathfrak{h})$ (written $\mu(a \otimes b) = a \triangleright b$) is also independent on the choice of \mathfrak{z} and is explicitly given as follows for all positive integers k, l and $x_1, \dots, x_k, y_1, \dots, y_l \in \mathfrak{h}$:

$$(x_1 \bullet \dots \bullet x_k) \triangleright (y_1 \bullet \dots \bullet y_l) = \frac{1}{k!} \sum_{\sigma \in S_k} (\text{ad}_{x_{\sigma(1)}}^s \circ \dots \circ \text{ad}_{x_{\sigma(k)}}^s)(y_1 \bullet \dots \bullet y_l) \quad (29)$$

where ad_x^s denotes the action of the Lie algebra $\mathfrak{h}/\mathfrak{z}(\mathfrak{h})$ (see eqn (20)) on $S(\mathfrak{h})$ according to eqn (24).

2. *In case $\mathfrak{z} = Q(\mathfrak{h})$, the construction mentioned in 1. is a functor $\mathfrak{h} \rightarrow \text{UAR}^\infty(\mathfrak{h})$ from the category of all Leibniz algebras to the category of all augmented rack bialgebras associating to \mathfrak{h} the rack bialgebra*

$$\text{UAR}^\infty(\mathfrak{h}) := (S(\mathfrak{h}), \Phi, U(\mathfrak{g}), \ell)$$

and to each morphism f of Leibniz algebras the pair $(S(f), U(\bar{f}))$ where \bar{f} is the induced Lie algebra morphism.

3. *For each nonnegative integer k , the above construction restricts to each subcoalgebra of order k , $S(\mathfrak{h})_{(k)} = \bigoplus_{r=0}^k S^r(\mathfrak{h})$, to define an augmented rack bialgebra $(S(\mathfrak{h})_{(k)}, \Phi_{(k)}, U(\mathfrak{g}), \ell|_{U(\mathfrak{g}) \otimes S(\mathfrak{h})_{(k)}})$ which in case $\mathfrak{z} = Q(\mathfrak{h})$ defines a functor $\mathfrak{h} \rightarrow \text{UAR}_{(k)}(\mathfrak{h}) := (S(\mathfrak{h})_{(k)}, \Phi_{(k)}, U(\mathfrak{g}), \ell|_{U(\mathfrak{g}) \otimes S(\mathfrak{h})_{(k)}})$ from the category of all Leibniz algebras to the category of all augmented rack bialgebras.*

Proof. 1. All the statements except the last two ones have already been proven. Note that for all $x, y \in \mathfrak{h}$ we have by definition

$$[x, y] = p(x) \cdot y = x \triangleright y,$$

independently of the chosen ideal \mathfrak{z} . Moreover we compute

$$\begin{aligned} & (x_1 \bullet \dots \bullet x_k) \triangleright (y_1 \bullet \dots \bullet y_l) \\ &= ((\omega \circ \tilde{\Phi})(x_1 \bullet \dots \bullet x_k)) \cdot (y_1 \bullet \dots \bullet y_l) \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} (p(x_{\sigma(1)}) \cdots p(x_{\sigma(k)})) \cdot (y_1 \bullet \dots \bullet y_l), \end{aligned}$$

which gives the desired formula since for all $x \in \mathfrak{h}$ and $a \in \mathbb{S}(\mathfrak{h})$, we have

$$p(x).a = \text{ad}_x^s(a).$$

2. Let $f : \mathfrak{h} \rightarrow \mathfrak{h}'$ be a morphism of Leibniz algebras, and let $\bar{f} : \bar{\mathfrak{h}} \rightarrow \bar{\mathfrak{h}'}$ be the induced morphism of Lie algebras. Hence we get

$$p' \circ f = \bar{f} \circ p \tag{30}$$

where $p' : \mathfrak{h}' \rightarrow \bar{\mathfrak{h}'}$ denotes the corresponding projection modulo $Q(\mathfrak{h}')$. Let $\mathbb{S}(f) : \mathbb{S}(\mathfrak{h}) \rightarrow \mathbb{S}(\mathfrak{h}')$, $\mathbb{S}(\bar{f}) : \mathbb{S}(\bar{\mathfrak{h}}) \rightarrow \mathbb{S}(\bar{\mathfrak{h}'})$, and $\mathbb{U}(\bar{f}) : \mathbb{U}(\bar{\mathfrak{h}}) \rightarrow \mathbb{U}(\bar{\mathfrak{h}'})$ the induced maps of Hopf algebras, i.e. $\mathbb{S}(f)$ (resp. $\mathbb{S}(\bar{f})$) satisfies eqn (22) (with p replaced by f (resp. by \bar{f})), and $\mathbb{U}(\bar{f})$ satisfies

$$\mathbb{U}(\bar{f})(\xi_1 \cdots \xi_k) = \bar{f}(\xi_1) \cdots \bar{f}(\xi_k)$$

for all positive integers k and $\xi_1, \dots, \xi_k \in \bar{\mathfrak{h}}$. If $\omega : \mathbb{S}(\bar{\mathfrak{h}}) \rightarrow \mathbb{U}(\bar{\mathfrak{h}})$ and $\omega' : \mathbb{S}(\bar{\mathfrak{h}'}) \rightarrow \mathbb{U}(\bar{\mathfrak{h}'})$ denote the corresponding symmetrisation maps (23) then it is easy to see from the definitions that

$$\omega' \circ \mathbb{S}(\bar{f}) = \mathbb{U}(\bar{f}) \circ \omega.$$

Equation (30) implies

$$\tilde{\Phi}' \circ \mathbb{S}(f) = \mathbb{S}(p') \circ \mathbb{S}(f) = \mathbb{S}(\bar{f}) \circ \mathbb{S}(p) = \mathbb{S}(\bar{f}) \circ \tilde{\Phi},$$

and composing from the left with ω' yields the equation

$$\Phi' \circ \mathbb{S}(f) = \mathbb{U}(\bar{f}) \circ \Phi. \tag{31}$$

Moreover for all $x, y \in \mathfrak{h}$ we have, since f is a morphism of Leibniz algebras,

$$f(p(x).y) = f([x, y]) = [f(x), f(y)]' = p'(f(x)).f(y) = \bar{f}(p(x)).f(y),$$

hence for all $\xi \in \bar{\mathfrak{h}}$

$$f(\xi.y) = (\bar{f}(\xi)).(f(y)),$$

and upon using eqn (24) we get for all $a \in \mathbb{S}(\mathfrak{h})$

$$\mathbb{S}(f)(\xi.a) = (\bar{f}(\xi)).(\mathbb{S}(f)(a)),$$

showing finally for all $u \in \mathbb{U}(\mathfrak{h})$ and all $a \in \mathbb{S}(\mathfrak{h})$

$$\mathbb{S}(f)(u.a) = (\mathbb{U}(\bar{f})(u)).(\mathbb{S}(f)(a)). \tag{32}$$

Associating to every Leibniz algebra $(\mathfrak{h}, [,])$ the above defined augmented rack bialgebra $(\mathbb{S}(\mathfrak{h}), \Phi, \mathbb{U}(\bar{\mathfrak{h}}), \ell)$, and to every morphism $\psi : \mathfrak{h} \rightarrow \mathfrak{h}'$ of Leibniz algebras the pair of K -linear maps $(\Psi = \mathbb{S}(\psi), \bar{\Psi} = \mathbb{U}(\bar{\psi}))$, we can easily check that Ψ is a morphism of C^5 -coalgebras, $\bar{\Psi}$ is a morphism of Hopf algebras, such

that the two relevant diagrams (13) commute which easily follows from (31) and (32). The rest of the functorial properties is a routine check.

3. By definition, the $U(\mathfrak{g})$ -action on $S(\mathfrak{h})$ (cf. eqs (24) and (27)) leaves invariant each K -submodule $S^r(\mathfrak{h})$ for each nonnegative integer r whence it leaves invariant each subcoalgebra of order k , $S(\mathfrak{h})_{(k)}$. It follows that the construction restricts well. \square

Definition 1.7. The rack bialgebra $UAR^\infty(\mathfrak{h})$ which is by the above theorem canonically associated to each Leibniz algebra \mathfrak{h} is called the universal augmented rack bialgebra.

Remark 1.8. This theorem should be compared to Proposition 3.5 in [4]. In [4], the authors work with the vector space $N := K \oplus \mathfrak{h}$, while we work with the whole symmetric algebra on the Leibniz algebra. In some sense, we extend their Proposition 3.5 “to all orders”, hence the name $UAR^\infty(\mathfrak{h})$. The subset $N = K \oplus \mathfrak{h}$ becomes a sub rack bialgebra denoted by $UAR(\mathfrak{h})$. It turns out that $UAR(\mathfrak{h})$ is already enough to obtain a left-adjoint to the functor of primitives and hence universality, see [1].

The above rack bialgebra $UAR^\infty(\mathfrak{h})$ associated to a Leibniz algebra \mathfrak{h} can be seen as one version of an *enveloping algebra* of the Leibniz algebra \mathfrak{h} . The link to the *universal enveloping algebra of \mathfrak{h}* , $\mathfrak{h} \otimes U(\overline{\mathfrak{h}})$, as a dialgebra (in the sense of of Loday-Pirashvili) has been elucidated in [1].

We shall close the subsection with a geometric explanation of some of the structures appearing here: Let $(\mathfrak{h}, [\ , \])$ be a real finite-dimensional Leibniz algebra. Then for any real number \hbar , there is the following Lie rack structure on the manifold \mathfrak{h} defined by

$$x \blacktriangleright_{\hbar} y := e^{\hbar \text{ad}_x}(y) \quad (33)$$

For later use we note that on the space $\mathfrak{h}[[\hbar]]$ of all formal power series the above formula makes sense if x, y are also formal power series.

Moreover, pick a two-sided ideal $\mathfrak{z} \subset \mathfrak{h}$ with $Q(\mathfrak{h}) \subset \mathfrak{z} \subset \mathfrak{z}(\mathfrak{h})$ so that the quotient algebra $\mathfrak{g} := \mathfrak{h}/\mathfrak{z}$ is a Lie algebra. Let $p : \mathfrak{h} \rightarrow \mathfrak{g}$ be the canonical projection. Let G be the connected simply connected Lie group having Lie algebra \mathfrak{g} . Since \mathfrak{g} acts on \mathfrak{h} as derivations, there is a unique Lie group action ℓ of G on \mathfrak{h} by automorphisms of Leibniz algebras. Consider the smooth map

$$\phi : \mathfrak{h} \rightarrow G : x \mapsto \exp(p(x)). \quad (34)$$

Clearly $\phi(g.x) = g\phi(x)g^{-1}$ for all $x \in \mathfrak{h}$ and $g \in G$ whence $(\mathfrak{h}, \phi, G, \ell)$ is an augmented Lie rack, and it is not hard to see that the Lie rack structure coincides with (33) for $\hbar = 1$. The following theorem is shown in [1]:

Theorem 1.9. *The C^5 -rack bialgebra associated to the augmented Lie rack $(\mathfrak{h}, \phi, G, \ell)$ by means of the functor F , described in Example 1.4, is isomorphic to the universal enveloping algebra of infinite order, $UAR^\infty(\mathfrak{h})$ (Definition 1.7).*

1.3 Quantum racks

In the article [6], the authors are interested in the generalized Poisson manifold given by the linear dual \mathfrak{h}^* of a finite-dimensional Leibniz algebra \mathfrak{h} . Recall that the dual \mathfrak{g}^* of a Lie algebra \mathfrak{g} is a Poisson manifold with the Kostant-Kirillov-Souriau bracket, given for $f, g \in \mathcal{C}^\infty(\mathfrak{g}^*)$ by

$$\{f, g\}(\xi) := \langle \xi, [df, dg] \rangle = \sum_{i,j,k} c_{ij}^k \frac{\partial f}{\partial x_i}(\xi) \frac{\partial g}{\partial x_j}(\xi) X_k,$$

where df and dg are seen as linear functions on \mathfrak{g}^* , i.e. elements of \mathfrak{g} . The c_{ij}^k are the structure constants of the Lie bracket with respect to a certain basis of (X_k) . In the same manner, the dual of a Leibniz algebra \mathfrak{h}^* carries a bracket given for $f, g \in \mathcal{C}^\infty(\mathfrak{h}^*)$ by

$$\{f, g\}(\xi) := -\langle \xi, [df(0), dg] \rangle = -\sum_{i,j,k} c_{ij}^k \frac{\partial f}{\partial x_i}(0) \frac{\partial g}{\partial x_j}(\xi) X_k,$$

i.e. with respect to the above formula, the bracket is here evaluated in $0 \in \mathfrak{h}^*$. The bracket is neither antisymmetric, nor a biderivation, nor does it satisfy some Jacobi/Leibniz identity in general. Nevertheless, [6] shows that this is the natural bracket one encounters when following the standard deformation quantization procedure for the dual of Lie algebras.

Going into details, a finite dimensional Lie algebra \mathfrak{g} admits a local Lie group G giving rise to a (local) multiplication $\mu : G \times G \rightarrow G$. The cotangent lift $T^*\mu$ can be interpreted as a Lagrangian submanifold of the triple product $\overline{T^*G} \times \overline{T^*G} \times T^*G$. It constitutes thus a *symplectic micromorphism* between germs of symplectic manifolds, see [3]. When such a symplectic micromorphism is given by a *generating function* S , the function S can be taken as an ‘‘action’’ in some oscillatory integral giving the deformation quantization of the corresponding Poisson manifold by Fourier Integral Operators. In [6], this quantization scheme is adapted to the dual of Leibniz algebras. The key ingredient is the integration of \mathfrak{h} into a Lie rack where for $x, y \in \mathfrak{h}$, the rack product is given by

$$X \blacktriangleright Y = e^{\text{ad}_X}(Y).$$

We take then $S_{\triangleright}(X, Y, \xi) = \langle \xi, e^{\text{ad}_X}(Y) \rangle$ as a generating function, and the oscillatory integral is written

$$Q_{\triangleright}(f \otimes g)(\xi) = \int_{\mathfrak{g} \times \mathfrak{g}} \widehat{f}(X) \widehat{g}(Y) e^{\frac{i}{\hbar} S_{\triangleright}(X, Y, \xi)} \frac{dX dY}{(2\pi\hbar)^n}.$$

Here n is the dimension of \mathfrak{h} and \widehat{f}, \widehat{g} are the asymptotic Fourier transforms of f and g . The stationary phase series expansion of this integral gives then the corresponding star product. Its first term is the above generalized Poisson bracket.

The main theorem of [6] reads:

Theorem 1.10. *The operation*

$$\triangleright_{\hbar} : \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]] \otimes \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]] \rightarrow \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]]$$

defined by

$$f \triangleright_{\hbar} g := Q_{\triangleright}(f \otimes g)$$

is a quantum rack, i.e.

(1) \triangleright_{\hbar} restricted to $U_{\hbar} = \{E_X := e^{\frac{i}{\hbar}X} \mid X \in \mathfrak{h}\}$ is a rack structure and

$$e^{\frac{i}{\hbar}X} \triangleright_{\hbar} e^{\frac{i}{\hbar}Y} = e^{\frac{i}{\hbar}\text{conj}_*(X,Y)},$$

(2) \triangleright_{\hbar} restricted to $\triangleright_{\hbar} : U_{\hbar} \times \mathcal{C}^\infty(\mathfrak{h}^*) \rightarrow \mathcal{C}^\infty(\mathfrak{h}^*)$ is a rack action and

$$(e^{\frac{i}{\hbar}X} \triangleright_{\hbar} f)(\xi) = (\text{Ad}_{-X}^* f)(\xi).$$

2 Deformation quantization of rack bialgebras

2.1 Algebraic deformation quantization of Leibniz algebras

In this subsection, K denotes the field of real numbers \mathbb{R} or the field of complex numbers \mathbb{C} .

Let $(\mathfrak{h}, [,])$ be a finite dimensional Leibniz algebra of dimension n , and denote by \mathfrak{h}^* its linear dual. In order to make computations more elementary we shall use a fixed basis e_1, \dots, e_n of \mathfrak{h} , but it is a routine check that all the relevant formulas are invariant under a change of basis. Let e^1, \dots, e^n be the corresponding dual basis of \mathfrak{h}^* , i.e. by definition

$$e^i(e_j) = \delta_j^i,$$

for all $i, j = 1, \dots, n$. Furthermore, let c_{ij}^k for $i, j, k = 1, \dots, n$ be the structure constants of the Leibniz algebra \mathfrak{h} with respect to the basis e_1, \dots, e_n , i.e.

$$c_{jk}^i = e^i([e_j, e_k])$$

for all $i, j, k = 1, \dots, n$. We will denote by x, y, z, \dots elements of \mathfrak{h} , while $\alpha, \beta, \gamma, \dots$ will denote elements of \mathfrak{h}^* . Denote by $\alpha_1, \dots, \alpha_n$ the coordinates of $\alpha \in \mathfrak{h}^*$ with respect to the basis e^1, \dots, e^n . For all $x \in \mathfrak{h}$, denote by $\widehat{x} \in \mathcal{C}^\infty(\mathfrak{h}^*, K)$ the linear function given by

$$\widehat{x}(\alpha) := \alpha(x),$$

for all $\alpha \in \mathfrak{h}^*$. In the same vein, let $e^{\widehat{x}}$ be the exponential function given by

$$e^{\widehat{x}}(\alpha) := e^{\alpha(x)} = e^{\widehat{x}(\alpha)},$$

for all $\alpha \in \mathfrak{h}^*$. For all integers $i = 1, \dots, n$, define a first order differential operator $\widetilde{\text{ad}}_i$ on smooth functions $f : \mathfrak{h}^* \rightarrow K$ by

$$(\widetilde{\text{ad}}_i(f))(\alpha) := \sum_{j,k=1}^n \alpha_k c_{ij}^k \frac{\partial f}{\partial \alpha_j}(\alpha).$$

The following star-product formula, where \hbar is a formal parameter (which may be replaced by a real number in situations where the formula is convergent), will render \mathfrak{h}^* a *quantum rack* in the sense of Theorem 1.10, see also [6].

Let $f, g \in \mathcal{C}^\infty(\mathfrak{h}^*, K)$.

$$(f \triangleright_{\hbar} g)(\alpha) := \sum_{r=0}^{\infty} \frac{\hbar^r}{r!} \sum_{i_1, \dots, i_r=1}^n \frac{\partial^r f}{\partial \alpha_{i_1} \dots \partial \alpha_{i_r}}(0) \left((\widetilde{\text{ad}}_{i_1} \circ \dots \circ \widetilde{\text{ad}}_{i_r})(g) \right)(\alpha). \quad (35)$$

Theorem 2.1. *For all $x, y \in \mathfrak{h}$, we have*

$$e^{\widehat{x}} \triangleright_{\hbar} e^{\widehat{y}} = e^{\widehat{x \blacktriangleright_{\hbar} y}},$$

where $\blacktriangleright_{\hbar} : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{h}$ is the Lie rack structure (33) defined by exponentiating the adjoint action of the Leibniz algebra:

$$x \blacktriangleright_{\hbar} y = e^{\hbar \text{ad}_x}(y).$$

Proof. The proof of the theorem relies on the rack bialgebra structure on $\mathbf{S}(\mathfrak{h})$ given by Definition 1.7 and is performed in the following lemmas:

Lemma 2.2. The map “hat” $\widehat{\cdot} : \mathfrak{h} \rightarrow \mathcal{C}^\infty(\mathfrak{h}^*, K)$ which sends $x \in \mathfrak{h}$ to the linear function \widehat{x} extends to an injective morphism of commutative associative unital algebras $\Psi : \mathbf{S}(\mathfrak{h}) \rightarrow \mathcal{C}^\infty(\mathfrak{h}^*, K)$ such that

$$\Psi(x_1 \bullet \dots \bullet x_k) = \widehat{x_1} \dots \widehat{x_k}$$

for all integers k and all $x_1, \dots, x_k \in \mathfrak{h}$.

Proof. This follows immediately from the freeness property of the algebra $\mathbf{S}(\mathfrak{h})$. \square

Lemma 2.3. The morphism Ψ intertwines the adjoint actions $\widetilde{\text{ad}}_i$ and $\text{ad}_{e_i}^s$ (see eqn (29)), i.e. for all $i = 1, \dots, n$, we have

$$\widetilde{\text{ad}}_i(\Psi(a)) = \Psi(\text{ad}_{e_i}^s(a))$$

for all $a \in \mathbf{S}(\mathfrak{h})$.

Proof. Indeed, it is enough to show this for $x \in \mathfrak{h} \subset \mathbf{S}(\mathfrak{h})$ as both adjoint actions are derivations. Now we have for $\alpha \in \mathfrak{h}^*$:

$$\begin{aligned} \Psi(\text{ad}_{e_i}^s(x))(\alpha) &= \widehat{[e_i, x]}(\alpha) = \alpha([e_i, x]) = \sum_{j,k=1}^n \alpha_k e^k([e_i, e_j]) x_j \\ &= \sum_{j,k=1}^n c_{ij}^k \alpha_k \frac{\partial \widehat{x}}{\partial \alpha_j}(\alpha) = \widetilde{\text{ad}}_i(\Psi(a))(\alpha) \end{aligned}$$

□

Lemma 2.4. For all $b \in \mathfrak{S}(\mathfrak{h})$ and all $x_1, \dots, x_r \in \mathfrak{h}$, we have

$$\Psi(x_1 \bullet \dots \bullet x_r) \triangleright_{\hbar} \Psi(b) = \hbar^r \Psi((x_1 \bullet \dots \bullet x_r) \triangleright b),$$

where the left-hand \triangleright is the rack multiplication in the rack bialgebra $\mathfrak{S}(\mathfrak{h})$.

Proof. First of all, note that by linearity it is enough to show this for $x_1, \dots, x_r = e_{i_1}, \dots, e_{i_r}$ with $i_1, \dots, i_r \in \{1, \dots, n\}$. By eqn (29), we have

$$(e_{i_1} \bullet \dots \bullet e_{i_r}) \triangleright b = \frac{1}{k!} \sum_{\sigma \in S_k} (\text{ad}_{i_{\sigma(1)}}^s \circ \dots \circ \text{ad}_{i_{\sigma(r)}}^s)(b).$$

Applying Ψ gives then

$$\begin{aligned} \Psi((e_{i_1} \bullet \dots \bullet e_{i_r}) \triangleright b) &= \frac{1}{k!} \sum_{\sigma \in S_k} \Psi((\text{ad}_{i_{\sigma(1)}}^s \circ \dots \circ \text{ad}_{i_{\sigma(r)}}^s)(b)) \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \widetilde{\text{ad}}_{i_{\sigma(1)}} \circ \dots \circ \widetilde{\text{ad}}_{i_{\sigma(r)}}(\Psi(b)), \end{aligned}$$

by the previous lemma. Now compute

$$\frac{\partial^k \left(\Psi(e_{i_1} \bullet \dots \bullet e_{i_r}) \right)}{\partial \alpha_{j_1} \dots \partial \alpha_{j_k}}(0).$$

This expression is non zero only if $k = r$ and $\{i_1, \dots, i_k\} = \{j_1, \dots, j_k\}$. In this case, the result is 1. One deduces the asserted formula. □

Now we come back to the proof of the theorem. The assertion of the theorem is the equality:

$$e^{\widehat{x}} \triangleright_{\hbar} e^{\widehat{y}} = e^{\widehat{x \blacktriangleright_{\hbar} y}}.$$

Summing up the assertion of the previous lemma (taking $x_1 = \dots = x_r = x$), we obtain:

$$\sum_{r=0}^{\infty} \frac{1}{r!} \Psi(\underbrace{x \bullet \dots \bullet x}_r) \triangleright_{\hbar} \Psi(b) = \Psi \left(\left(\sum_{r=0}^{\infty} \frac{\hbar^r}{r!} (\underbrace{x \bullet \dots \bullet x}_r) \right) \triangleright b \right),$$

and thus (as the rack product in $\mathfrak{S}(\mathfrak{h})$ is given by the adjoint action, using also that Ψ is multiplicative)

$$e^{\widehat{x}} \triangleright_{\hbar} \Psi(b) = \Psi(e^{\hbar \text{ad}_x}(b)). \quad (36)$$

This extends then to the asserted formula using that $e^{\hbar \text{ad}_x}$ is an automorphism of $\mathfrak{S}(\mathfrak{h})$ (because it is the exponential of a derivation). □

Corollary 2.5. The above defined star-product induces the structure of a rack with respect to the product \triangleright_{\hbar} on the set of exponential functions $U_{\mathfrak{h}} = \{E_X := e^{\frac{i}{\hbar}X} \mid X \in \mathfrak{h}\}$ on \mathfrak{h}^* , and this star-product is equal (up to a sign) to the star-product in Theorem 1.10, see also [6].

Proof. Via the formula of the theorem, the self-distributivity property of the rack product \blacktriangleright in the rack bialgebra $\mathcal{S}(\mathfrak{h})$ translates into the self-distributivity property of \triangleright_{\hbar} on the set of exponential functions. Since the star-product defined in [6] is a series of bidifferential operators, and since such a series is uniquely determined by its values on exponential functions, the present star-product coincides with the one found in [6] thanks to the statement of the preceding theorem. \square

Remark 2.6. Observe that the proof of the above theorem contains also an isomorphism of commutative associative unital algebras between $\mathcal{S}(\mathfrak{h})$ and the image of Ψ , i.e. the polynomial algebra generated by the \widehat{x}_i , $i = 1, \dots, n$. Up to factors \hbar^r , Ψ is by construction an isomorphism of rack bialgebras. This mirrors the relation between the universal enveloping algebra $U(\mathfrak{g})$ and the deformation quantization of $\mathcal{S}(\mathfrak{g}^*)$ for a Lie algebra \mathfrak{g} , i.e. between $U(\mathfrak{g})$ and the Gutt star product on \mathfrak{g}^* , see [9], [6].

Remark 2.7. We would like to thank the referee for asking the interesting question of comparing the above star-product formula with the theory of Loday-Pirashvili, see [14] and [15]: recall that they work in the so-called *linear category* \mathcal{LM} whose objects are diagrams $M \xrightarrow{\phi} N$ of vector spaces and linear maps with the obvious commuting squares of linear maps as morphisms. It can be seen as the category of complexes concentrated in degree 1 (M) and 0 (N) with differential ϕ . There is the obvious tensor product of complexes (truncated in degree ≥ 2) equipping \mathcal{LM} with the structure of a symmetric monoidal category (with $0 \rightarrow K$ as unit object and a signless flip as symmetric braiding). In this category, associative algebra objects will be dialgebras, and Lie objects can be seen as Leibniz algebras $\mathfrak{h} \rightarrow \overline{\mathfrak{h}} = \underline{\mathfrak{g}}$, see [14] and [15] for more details. The symmetric algebra generated by $\mathfrak{h} \rightarrow \overline{\mathfrak{h}}$ will be $\mathfrak{h} \otimes \mathcal{S}(\mathfrak{g}) \rightarrow \mathcal{S}(\mathfrak{g})$, and the universal enveloping algebra of $\mathfrak{h} \rightarrow \overline{\mathfrak{h}}$ will be $\mathfrak{h} \otimes U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ where the linear maps are given by $x \otimes g \mapsto p(x) \bullet g$ and $x \otimes u \mapsto p(x)u$, respectively.

For a possible rack-like star-product formula motivated by this category it seems to us not unreasonable to base it on the ‘underlying augmented rack-bialgebras’ since this worked already well for the above rack-star-product (35) on smooth functions on \mathfrak{h}^* . In a slightly more general fashion, let $(B, \Phi_B, U(\mathfrak{g}), \ell)$ be an augmented rack bialgebra for a given Lie algebra \mathfrak{g} . We shall later specialize B to $\mathcal{S}(\mathfrak{h})$ or its rack sub-bialgebra $\mathcal{S}(\mathfrak{h})_{(1)} = K\mathbf{1} \oplus \mathfrak{h}$ mentioned in Theorem 1.6, part 3. Now, $U(\mathfrak{g})$ clearly acts ‘diagonally’ (i.e. using its comultiplication) on the K -module $B \otimes U(\mathfrak{g})$ via ℓ on the first factor B and via the adjoint action on the second factor $U(\mathfrak{g})$. Calling this action ℓ^{\otimes} , and defining $\Phi : B \otimes U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ by $\Phi(b \otimes u) = \Phi_B(b)u$ we easily see that $(B \otimes U(\mathfrak{g}), \Phi, U(\mathfrak{g}), \ell^{\otimes})$ is an augmented

rack bialgebra whose rack multiplication \triangleright' reads

$$(b \otimes u) \triangleright' (c \otimes v) = \sum_{(b),(u)} \ell_{\Phi_B(b^{(1)})u^{(1)}}(c) \otimes \text{ad}_{\Phi_B(b^{(2)})u^{(2)}}(v) \quad (37)$$

where $b, c \in B$ and $u, v \in \mathbf{U}(\mathfrak{g})$.

The interesting cases are $B = \mathbf{S}(\mathfrak{h})$ and its subcoalgebra $B = \mathbf{S}(\mathfrak{h})_{(1)} = K\mathbf{1} \oplus \mathfrak{h}$ for a Leibniz algebra \mathfrak{h} . The latter is important since $(K\mathbf{1} \oplus \mathfrak{h}) \otimes \mathbf{U}(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{h} as the left adjoint functor to the functor associating to any *bar-unital dialgebra* its ‘commutator’-Leibniz algebra, see e.g. [1, Thm.2.8]. Note that the classical universal enveloping algebra of \mathfrak{h} in the sense of Loday-Pirashvili is just the submodule $\mathfrak{h} \otimes \mathbf{U}(\mathfrak{g})$ of $(K\mathbf{1} \oplus \mathfrak{h}) \otimes \mathbf{U}(\mathfrak{g})$, and this is easily seen to be a rack-subalgebra (NOT a subcoalgebra) of $\mathbf{S}(\mathfrak{h}) \otimes \mathbf{U}(\mathfrak{g})$ with respect to the multiplication \triangleright' in the above eqn (37). The primitive part of $\mathbf{S}(\mathfrak{h}) \otimes \mathbf{U}(\mathfrak{g})$ is known to be Leibniz algebra isomorphic to the hemi-semidirect product $\mathfrak{h} \oplus \mathfrak{g}$, see [1, Thm.2.7.4.], hence the rack bialgebra $\mathbf{S}(\mathfrak{h}) \otimes \mathbf{U}(\mathfrak{g})$ is isomorphic to the rack bialgebra $\mathbf{S}(\mathfrak{h} \oplus \mathfrak{g})$. In the finite-dimensional case over $K = \mathbb{R}$ or $K = \mathbb{C}$ we can apply formula (35) to this hemi-semidirect product situation thus giving a star-product formula on the function algebra $\mathcal{C}^\infty(\mathfrak{h}^* \times \mathfrak{g}^*, K)$ reflecting the above multiplication (37), and clearly rack star-products on the subspaces of functions (at most) linear in \mathfrak{h}^* which will then be rack star-product versions corresponding to the universal enveloping algebra $\mathfrak{h} \otimes \mathbf{U}(\mathfrak{g})$ in the sense of Loday-Pirashvili.

Surprisingly, there is a morphism of rack algebras $\Gamma : \mathbf{S}(\mathfrak{h}) \rightarrow B \otimes \mathbf{U}(\mathfrak{g})$ for $B = K\mathbf{1} \oplus \mathfrak{h}$, cf the constructions in [18], [19]: firstly, the following general statement is very easy to check: given any two augmented rack bialgebras (B, Φ_B, H, ℓ) and (C, Φ_C, H, ℓ') over the same cocommutative Hopf algebra H , then any K -linear map $\Gamma : B \rightarrow C$ intertwining the H -actions which satisfies $\Phi_C \circ \Gamma = \Phi_B$ will give a morphism of rack-algebras, i.e. $\Gamma(b \triangleright b') = \Gamma(b) \triangleright \Gamma(b')$ for all $b, b' \in B$. Note that Γ does not have to be a morphism of coalgebras, and will in general NOT be a morphism of rack bialgebras. More concretely, for $\Gamma : B = \mathbf{S}(\mathfrak{h}) \rightarrow C = (K\mathbf{1} \oplus \mathfrak{h}) \otimes \mathbf{U}(\mathfrak{g})$ we make the following ansatz:

$$\Gamma = ((\mathbf{1}\epsilon + \text{pr}) \otimes (F_*(e^{(1)}) \circ \Phi)) \circ \Delta. \quad (38)$$

Here the maps $\Delta, \epsilon, \mathbf{1}$ give the usual augmented coalgebra structure of $\mathbf{S}(\mathfrak{h})$, $\text{pr} : \mathbf{S}(\mathfrak{h}) \rightarrow \mathfrak{h}$ is the canonical projection, and $\Phi : \mathbf{S}(\mathfrak{h}) \rightarrow \mathbf{U}(\mathfrak{g})$ is the above map $\omega \circ \mathbf{S}(p)$, see eqn (26). Next, $*$ denotes the convolution multiplication on $\text{Hom}_K(\mathbf{U}(\mathfrak{g}), \mathbf{U}(\mathfrak{g}))$, and $e^{(1)} : \mathbf{U}(\mathfrak{g}) \rightarrow \mathfrak{g} \subset \mathbf{U}(\mathfrak{g})$ is the *eulerian idempotent of the cocommutative bialgebra* $\mathbf{U}(\mathfrak{g})$, i.e. $e^{(1)} = \text{ln}_* (\mathbf{1}_{\mathbf{U}(\mathfrak{g})} \epsilon_{\mathbf{U}(\mathfrak{g})} + (\text{id}_{\mathbf{U}(\mathfrak{g})} - \mathbf{1}_{\mathbf{U}(\mathfrak{g})} \epsilon_{\mathbf{U}(\mathfrak{g})}))$, see e.g. [13, Ch.4, p.139-141]. Finally, $F(s)$ is a formal series with rational coefficients, and $F_*(e^{(1)}) \in \text{Hom}_K(\mathbf{U}(\mathfrak{g}), \mathbf{U}(\mathfrak{g}))$ denotes the convolution series where s is replaced by $e^{(1)}$. Observe that the bialgebra structures of $\mathbf{S}(\mathfrak{h})$ and of $\mathbf{U}(\mathfrak{g})$ are $\mathbf{U}(\mathfrak{g})$ -module maps (recall that it is the adjoint action on $\mathbf{U}(\mathfrak{g})$) whence all convolutions of these structures are $\mathbf{U}(\mathfrak{g})$ -module maps, hence also $e^{(1)}$ and its convolution series. It follows that Γ is a $\mathbf{U}(\mathfrak{g})$ -module map. Moreover, since $e^{(1)} \circ \omega = \text{pr}_{\mathfrak{g}}$ we get $e^{(1)} \circ \Phi = p \circ \text{pr}$, and since $e^{*e^{(1)}} = \text{id}_{\mathbf{U}(\mathfrak{g})}$, the second

condition $\Phi_C \circ \Gamma = \Phi_B$ gives –using the surjectivity of Φ –

$$(\mathbf{1}_{\mathfrak{U}(\mathfrak{g})}\epsilon_{\mathfrak{U}(\mathfrak{g})} + e^{(1)}) * F_*(e^{(1)}) = \text{id}_{\mathfrak{U}(\mathfrak{g})} = e^{*e^{(1)}}$$

hence with $F(s) = \frac{e^s}{1+s}$ we get the morphism of rack-algebras. Moreover there is a morphism $\Psi : \mathfrak{S}(\mathfrak{h})^+ \rightarrow \mathfrak{h} \otimes \mathfrak{U}(\mathfrak{g})$ given by a similar ansatz: in the above eqn (38) we just replace the expression $(\mathbf{1}\epsilon + \text{pr})$ by pr and the series F by the series $G(s) = \frac{e^s - 1}{s}$.

2.2 General deformation theory for rack bialgebras

In this section, $(R, \Delta, \epsilon, \mu, \mathbf{1})$ is a **cocommutative** rack-bialgebra over a general commutative ring K , and we use the notation $r \triangleright s$ to denote the rack product $\mu(r \otimes s)$ of two elements r and s of R . In this subsection, we will often drop the symbol Σ in Sweedler's notation of (iterated) comultiplications, so that the n -iterated comultiplication of r in R reads

$$r^{(1)} \otimes \dots \otimes r^{(n)} := (\Delta \otimes \text{Id}^{\otimes n-1}) \circ \dots \circ \Delta(r)$$

Let $K_{\hbar} = K[[\hbar]]$ denote the K -algebra of formal power series in the indeterminate \hbar with coefficients in K . If V is a vector space over K , V_{\hbar} stands for $V[[\hbar]]$. Recall that if W is a K -module, a K_{\hbar} -linear morphism from V_{\hbar} to W_{\hbar} is the same as a power series in \hbar with coefficients in $\text{Hom}_K(V, W)$ via the canonical map

$$\text{Hom}_{K_{\hbar}}(V_{\hbar}, W_{\hbar}) \cong \text{Hom}_K(V, W)_{\hbar}.$$

This identification will be used without extra mention in the following.

Definition 2.8. A formal deformation of the rack product μ is a formal power series μ_{\hbar}

$$\mu_{\hbar} := \sum_{n \geq 0} \hbar^n \mu_n$$

in $\text{Hom}_K(R \otimes R, R)_{\hbar}$, such that

1. $\mu_0 = \mu$,
2. $(R_{\hbar}, \Delta, \epsilon, \mu_{\hbar}, \mathbf{1})$ is a rack bialgebra over K_{\hbar} .

Example 2.1.

For a Leibniz algebra \mathfrak{h} , we have introduced in Definition 1.7 a cocommutative augmented rack bialgebra $\text{UAR}^{\infty}(\mathfrak{h})$. Furthermore, the rack star product defined in eqn (35), restricted to $S(\mathfrak{h})_{\hbar}$, is a deformation of the trivial rack product of $S(\mathfrak{h})$ given for all $r, s \in S(\mathfrak{h})$ by

$$r \triangleright s := \epsilon(r)s$$

The self-distributivity relation is shown in a way very similar to the proof of Theorem 2.1, see eqn (36). \diamond

As in the classical setting of deformation theory of associative products, we will relate our deformation theory of rack products to cohomology. For this, let us first examine an introductory example:

Example 2.2.

Let (R, \triangleright) be a rack bialgebra, and suppose there exists a deformation $\triangleright_{\hbar} = \triangleright + \hbar\omega$ of \triangleright . The new rack product \triangleright_{\hbar} should satisfy the self-distributivity identity, i.e. for all $a, b, c \in R$

$$a \triangleright_{\hbar} (b \triangleright_{\hbar} c) = (a^{(1)} \triangleright_{\hbar} b) \triangleright_{\hbar} (a^{(2)} \triangleright_{\hbar} c)$$

To the order \hbar^0 , this is only the self-distributivity relation for \triangleright . But to order \hbar^1 (neglecting order \hbar^2 and higher), we obtain:

$$\omega(a, b \triangleright c) + a \triangleright \omega(b, c) = \omega(a^{(1)} \triangleright b, a^{(2)} \triangleright c) + \omega(a^{(1)}, b) \triangleright (a^{(2)} \triangleright c) + (a^{(1)} \triangleright b) \triangleright \omega(a^{(2)}, c).$$

It will turn out that this is the cocycle condition for ω in the deformation complex which we are going to define. More precisely, we will have

1. $d_{2,0}\omega(a, b, c) = \omega(a, b \triangleright c)$,
2. $d_{1,1}\omega(a, b, c) = a \triangleright \omega(b, c)$,
3. $d_{1,0}\omega(a, b, c) = \omega(a^{(1)} \triangleright b, a^{(2)} \triangleright c)$,
4. $d_3^2\omega(a, b, c) = \omega(a^{(1)}, b) \triangleright (a^{(2)} \triangleright c)$,
5. $d_{2,1}\omega(a, b, c) = (a^{(1)} \triangleright b) \triangleright \omega(a^{(2)}, c)$.

This may perhaps help to understand the general definition of the operators $d_{i,\mu}^n$ for $i = 1, \dots, n$ and $\mu \in \{0, 1\}$ further down.

On the other hand, the requirement that \triangleright_{\hbar} should be a morphism of coalgebras (with respect to the undeformed coproduct Δ of R) means

$$\Delta \circ \triangleright_{\hbar} = (\triangleright_{\hbar} \otimes \triangleright_{\hbar}) \circ \Delta^{[2]}.$$

This reads for $a, b \in R$ to the order \hbar (neglecting higher powers of \hbar) as

$$\omega(a, b)^{(1)} \otimes \omega(a, b)^{(2)} = \omega(a^{(1)}, b^{(1)}) \otimes (a^{(2)} \triangleright b^{(2)}) + (a^{(1)} \triangleright b^{(1)}) \otimes \omega(a^{(2)}, b^{(2)}).$$

This is exactly the requirement that ω is a coderivation along $\triangleright = \mu$, to be defined below. ◇

Recall that R being a rack bialgebra means in particular that $\mu : R^{\otimes 2} \rightarrow R$ is a morphism of coassociative coalgebras. For all positive integer n , let $\mu^n : R^{\otimes n} \rightarrow R$ be the linear map defined inductively by setting

- $\mu^1 := \text{Id} : R \rightarrow R$,
- $\mu^2 := \mu : R^{\otimes 2} \rightarrow R$,
- $\mu^n := \mu \circ (\mu^1 \otimes \mu^{n-1})$, $n \geq 3$,

so that

$$\mu^n(r_1, \dots, r_n) = r_1 \triangleright (r_2 \triangleright (\dots \triangleright (r_{n-1} \triangleright r_n) \dots))$$

for all r_1, \dots, r_n in R .

Proposition 2.9. For all $n \geq 1$, the map μ^n is a morphism of coalgebras satisfying

$$\mu^i(r_1^{(1)}, \dots, r_{i-1}^{(1)}, r_i) \triangleright \mu^{n-1}(r_1^{(2)}, \dots, r_{i-1}^{(2)}, r_{i+1}, \dots, r_n) = \mu^n(r_1, \dots, r_n), \quad (39)$$

$$\mu^n(r_1, \dots, r_{i-1}, r_i^{(1)} \triangleright r_{i+1}, \dots, r_i^{(n+1-i)} \triangleright r_{n+1}) = \mu^{n+1}(r_1, \dots, r_{n+1}) \quad (40)$$

for all positive integers i and n such that $1 \leq i < n$ and for all r_1, \dots, r_n in R .

Proof. • eqn (39): Let us show that the assertion of eqn (39) is true for all n and i with $1 \leq i < n$ by induction over i . Suppose that the induction hypothesis is true and compute

$$\begin{aligned} & \mu^i(r_1^{(1)}, \dots, r_i^{(1)}, r_{i+1}) \triangleright \mu^{n-1}(r_1^{(2)}, \dots, r_i^{(2)}, r_{i+2}, \dots, r_n) \\ & \left(r_1^{(1)} \triangleright \mu^i(r_2^{(1)}, \dots, r_i^{(1)}, r_{i+1}) \right) \triangleright \left(r_1^{(2)} \triangleright \mu^{n-2}(r_2^{(2)}, \dots, r_i^{(2)}, r_{i+2}, \dots, r_n) \right), \end{aligned}$$

which gives, thanks to the self-distributivity relation in the rack algebra R ,

$$\begin{aligned} & r_1 \triangleright \left(\mu^i(r_2^{(1)}, \dots, r_i^{(1)}, r_{i+1}) \triangleright \mu^{n-2}(r_2^{(2)}, \dots, r_i^{(2)}, r_{i+2}, \dots, r_n) \right) \\ & = r_1 \triangleright \mu^{n-1}(r_2, \dots, r_n) = \mu^n(r_1, \dots, r_n), \end{aligned}$$

where we have used the induction hypothesis. This proves the assertion.

- eqn (40): The assertion follows here again from an easy induction using the self-distributivity relation. □

If (C, Δ_C) and (D, Δ_D) are two coassociative coalgebras and $\phi : C \rightarrow D$ is a morphism of coalgebras, we denote by $\text{Coder}(C, V, \phi)$ the vector space of *coderivations from C to V along ϕ* , i.e. the vector space of linear maps $f : C \rightarrow D$ such that

$$\Delta_D \circ f = (f \otimes \phi + \phi \otimes f) \circ \Delta_C$$

Let us note the following permanence property of coderivations along a map under partial convolution which will be useful in the proof of the following theorem. For a coalgebra A , maps $f : A \otimes B \rightarrow V$ and $g : A \otimes C \rightarrow V$ and some product $\triangleright : V \otimes V \rightarrow V$, the *partial convolution* of f and g is the map $f \star_{\text{part}} g : A \otimes B \otimes C \rightarrow V$ defined for all $a \in A$, $b \in B$ and $c \in C$ by

$$(f \star_{\text{part}} g)(a \otimes b \otimes c) := f(a^{(1)} \otimes b) \triangleright g(a^{(2)} \otimes c).$$

Lemma 2.10. Let A, B, C and V be coalgebras, V carrying a product \triangleright which is supposed to be a coalgebra morphism. Let $f : A \otimes B \rightarrow V$ be a coderivation along ϕ and $g : A \otimes C \rightarrow V$ be a coalgebra morphism. Then the partial convolution $f \star_{\text{part}} g$ is a coderivation along $\phi \star_{\text{part}} g$.

Proof. We compute for all $a \in A, b \in B$ and $c \in C$

$$\begin{aligned}
\Delta_V \circ (f \star_{\text{part}} g)(a \otimes b \otimes c) &= \Delta_V(f(a^{(1)} \otimes b) \triangleright g(a^{(2)} \otimes c)) \\
&= (f(a^{(1)} \otimes b))^{(1)} \triangleright (g(a^{(2)} \otimes c))^{(1)} \otimes (f(a^{(1)} \otimes b))^{(2)} \triangleright (g(a^{(2)} \otimes c))^{(2)} \\
&= (f(a^{(1)} \otimes b))^{(1)} \triangleright g(a^{(2)} \otimes c^{(1)}) \otimes (f(a^{(1)} \otimes b))^{(2)} \triangleright g(a^{(3)} \otimes c^{(2)}) \\
&= \phi(a^{(1)} \otimes b^{(1)}) \triangleright g(a^{(2)} \otimes c^{(1)}) \otimes f(a^{(3)} \otimes b^{(2)}) \triangleright g(a^{(4)} \otimes c^{(2)}) + \\
&\quad + f(a^{(1)} \otimes b^{(1)}) \triangleright g(a^{(2)} \otimes c^{(1)}) \otimes \phi(a^{(3)} \otimes b^{(2)}) \triangleright g(a^{(4)} \otimes c^{(2)}) \\
&= (\phi \star_{\text{part}} g)(a^{(1)} \otimes b^{(1)} \otimes c^{(1)}) \otimes (f \star_{\text{part}} g)(a^{(2)} \otimes b^{(2)} \otimes c^{(2)}) + \\
&\quad + (f \star_{\text{part}} g)(a^{(1)} \otimes b^{(1)} \otimes c^{(1)}) \otimes (\phi \star_{\text{part}} g)(a^{(2)} \otimes b^{(2)} \otimes c^{(2)}) \\
&= ((\phi \star_{\text{part}} g) \otimes (f \star_{\text{part}} g) + (f \star_{\text{part}} g) \otimes (\phi \star_{\text{part}} g)) \circ \Delta_{A \otimes B \otimes C}(a \otimes b \otimes c).
\end{aligned}$$

□

Definition 2.11. The **deformation complex of R** is the graded vector space $C^*(R; R)$ defined in degree n by

$$C^n(R; R) := \text{Coder}(R^{\otimes n}, R, \mu^n)$$

endowed with the differential $d_R : C^*(R; R) \rightarrow C^{*+1}(R; R)$ defined in degree n by

$$d_R^n := \sum_{i=1}^n (-1)^{i+1} (d_{i,1}^n - d_{i,0}^n) : + : (-1)^{n+1} d_{n+1}^n$$

where the maps $d_{i,1}^n$ and $d_{i,0}^n$ are defined respectively by

$$d_{i,1}^n \omega(r_1, \dots, r_{n+1}) := \sum_{(r_1), \dots, (r_i)} \mu^i(r_1^{(1)}, \dots, r_{i-1}^{(1)}, r_i) \triangleright \omega(r_1^{(2)}, \dots, r_{i-1}^{(2)}, r_{i+1}, \dots, r_{n+1})$$

and

$$d_{i,0}^n \omega(r_1, \dots, r_{n+1}) := \sum_{(r_i)} \omega(r_1, \dots, r_{i-1}, r_i^{(1)} \triangleright r_{i+1}, \dots, r_i^{(n+1-i)} \triangleright r_{n+1})$$

and d_{n+1}^n by

$$\begin{aligned}
&d_{n+1}^n \omega(r_1, \dots, r_{n+1}) \\
&:= \sum_{(r_1), \dots, (r_{n-1})} \omega(r_1^{(1)}, \dots, r_{n-1}^{(1)}, r_n) \triangleright \mu^n(r_1^{(2)}, \dots, r_{n-1}^{(2)}, r_{n+1})
\end{aligned}$$

for all ω in $C^n(R; R)$ and r_1, \dots, r_{n+1} in R .

Theorem 2.12. d_R is a well defined differential.

Proof. That d_R is well defined means that it sends coderivations to coderivations. It suffices to show that this is already true for all maps $d_{i,1}^n$, $d_{i,0}^n$ and d_{n+1}^n , which is the case. For this, we use Lemma 2.10. Indeed, a cochain $\omega \in C^n(R; R)$ is a coderivation along μ^n . By Proposition 2.9, μ^n is a coalgebra morphism. On the other hand, it is clear from the formula for $d_{i,1}^n$ that $d_{i,1}^n$ is a partial convolution with respect to the first $i - 1$ tensor labels of μ^i and ω . Therefore the Lemma applies to give that the result is a coderivation along the partial convolution of μ^i and μ^n , which is just μ^{n+1} again by Proposition 2.9. This shows that $d_{i,1}^n \omega$ belongs to $\text{Coder}(R^{\otimes n}, R, \mu^{n+1})$ as expected. The maps $d_{i,0}^n$ and d_{n+1}^n can be treated in a similar way.

The fact that d_R squares to zero is related to the so-called *cubical identities* satisfied by the maps $d_{i,1}$ and the maps $d_{i,0}$, namely

$$d_{j,\mu}^{n+1} \circ d_{i,\nu}^n = d_{i+1,\nu}^{n+1} \circ d_{j,\mu}^n \quad \text{for } j \leq i \quad \text{and } \mu, \nu \in \{0, 1\},$$

and auxiliary identities which express the compatibility of the maps $d_{i,1}$ and $d_{i,0}$ with d_{n+1}^n , and an identity involving d_{n+1}^n and d_{n+2}^{n+1} . One could call this kind of object an *augmented cubical vector space*.

We will not show the usual cubical relations, i.e. those which do not refer to the auxiliary coboundary map d_{n+1}^n , because these are well-known to hold for rack cohomology, see [5], Corollary 3.12, and our case is easily adapted from there. One possibility of adaptation (in case one works over the real or complex numbers) is to take a Lie rack, write its rack homology complex (with trivial coefficients in the real or complex numbers), and to apply the functor of point-distributions.

Let us show that the two following extra relations involving the extra face d_{n+1}^n hold:

$$d_{i,\mu}^{n+1} \circ d_{n+1}^n = d_{n+2}^{n+1} \circ d_{i,\mu}^n \quad (41)$$

for all $1 \leq i \leq n$ and μ in $\{0, 1\}$ and

$$d_{n+1,0}^{n+1} \circ d_{n+1}^n = d_{n+2}^{n+1} \circ d_{n+1}^n + d_{n+1,1}^{n+1} \circ d_{n+1}^n \quad (42)$$

Indeed, if ω is a n -cochain and r_1, \dots, r_{n+2} are elements in R , then

$$\begin{aligned} & (d_{i,1}^{n+1} \circ d_{n+1}^n \omega)(r_1, \dots, r_{n+2}) \\ &= \mu^i(r_1^{(1)}, \dots, r_{i-1}^{(1)}, r_i) \triangleright d_{n+1}^n \omega(r_1^{(2)}, \dots, r_{i-1}^{(2)}, r_{i+1}, \dots, r_{n+2}) \\ &= \mu^i(r_1^{(1)}, \dots, r_{i-1}^{(1)}, r_i) \triangleright (\omega(r_1^{(2)}, \dots, r_{i-1}^{(2)}, r_{i+1}^{(1)}, \dots, r_n^{(1)}, r_{n+1})) \triangleright \\ & \quad \triangleright \mu^n(r_1^{(3)}, \dots, r_{i-1}^{(3)}, r_{i+1}^{(2)}, \dots, r_n^{(2)}, r_{n+2}) \end{aligned}$$

By Proposition 2.9 and thanks to the self-distributivity of the rack product, this

equality can be rewritten as

$$\begin{aligned}
& (d_{i,1}^{n+1} \circ d_{n+1}^n \omega)(r_1, \dots, r_{n+2}) \\
&= (\mu^i(r_1^{(1)}, \dots, r_{i-1}^{(1)}, r_i^{(1)}) \triangleright \omega(r_1^{(2)}, \dots, r_{i-1}^{(2)}, r_{i+1}^{(1)}, \dots, r_n^{(1)}, r_{n+1})) \triangleright \\
&\quad \triangleright (\mu^i(r_1^{(3)}, \dots, r_{i-1}^{(3)}, r_i^{(2)}) \triangleright \mu^n(r_1^{(4)}, \dots, r_{i-1}^{(4)}, r_{i+1}^{(2)}, \dots, r_n^{(2)}, r_{n+2})) \\
&= d_{i,1}^n \omega(r_1^{(1)}, \dots, r_n^{(1)}, r_{n+1}) \triangleright \mu^n(r_1^{(2)}, \dots, r_n^{(2)}, r_{n+2}) \\
&= (d_{n+2}^{n+1} \circ d_{i,1}^n \omega)(r_1, \dots, r_{n+2}),
\end{aligned}$$

which proves that Relation (41) holds when $\mu = 1$. The case $\mu = 0$ goes as follows:

$$\begin{aligned}
& (d_{i,0}^{n+1} \circ d_{n+1}^n \omega)(r_1, \dots, r_{n+2}) = d_{n+1}^n \omega(r_1, \dots, r_{i-1}, r_i^{(1)} \triangleright r_{i+1}, \dots, r_i^{(n+2-i)} \triangleright r_{n+2}) \\
&= \omega(r_1^{(1)}, \dots, r_{i-1}^{(1)}, r_i^{(1)} \triangleright r_{i+1}^{(1)}, \dots, r_i^{(n-i)} \triangleright r_n^{(1)}, r_i^{(n+1-i)} \triangleright r_{n+1}) : \triangleright \\
&\quad \triangleright \mu^n(r_1^{(2)}, \dots, r_{i-1}^{(2)}, r_i^{(n+2-i)} \triangleright r_{i+1}^{(2)}, \dots, r_i^{(2n-2i+1)} \triangleright r_n^{(2)}, r_i^{(2n-2i+2)} \triangleright r_{n+2})
\end{aligned}$$

where we have used that the rack product is a morphism of coalgebras. Recall the following equation from Proposition 2.9:

$$\mu^n(s_1, \dots, s_{i-1}, s_i^{(1)} \triangleright s_{i+1}, \dots, s_i^{(n+1-i)} \triangleright s_{n+1}) = \mu^{n+1}(s_1, \dots, s_{n+1})$$

for all s_1, \dots, s_{n+1} in R and $1 \leq i \leq n$. This allows to rewrite the preceding equality as

$$\begin{aligned}
& (d_{i,0}^{n+1} \circ d_{n+1}^n \omega)(r_1, \dots, r_{n+2}) \\
&= \omega(r_1^{(1)}, \dots, r_{i-1}^{(1)}, r_i^{(1)} \triangleright r_{i+1}^{(1)}, \dots, r_i^{(n-i)} \triangleright r_n^{(1)}, r_i^{(n+1-i)} \triangleright r_{n+1}) : \triangleright \\
&\quad \triangleright \mu^{n+1}(r_1^{(2)}, \dots, r_{i-1}^{(2)}, r_i^{(n+2-i)} \triangleright r_{i+1}^{(2)}, \dots, r_n^{(2)}, r_{n+2}) \\
&= d_{i,0}^n : \omega(r_1^{(1)}, \dots, r_n^{(1)}, r_{n+1}) : \triangleright : \mu^{n+1}(r_1^{(2)}, \dots, r_n^{(2)}, r_{n+2}) \\
&= (d_{n+2}^{n+1} \circ d_{i,0}^n \omega)(r_1, \dots, r_{n+2})
\end{aligned}$$

which proves that (41) holds when $\mu = 0$. Relation (42) relies on the fact that cochains are coderivations. Indeed,

$$\begin{aligned}
& (d_{n+1,0}^{n+1} \circ d_{n+1}^n \omega)(r_1, \dots, r_{n+2}) = d_{n+1}^n \omega(r_1, \dots, r_n, r_{n+1} \triangleright r_{n+2}) \\
&= \omega(r_1^{(1)}, \dots, r_{n-1}^{(1)}, r_n) \triangleright \mu^n(r_1^{(2)}, \dots, r_{n-1}^{(2)}, r_{n+1} \triangleright r_{n+2}) \\
&= \omega(r_1^{(1)}, \dots, r_{n-1}^{(1)}, r_n) \triangleright \mu^{n+1}(r_1^{(2)}, \dots, r_{n-1}^{(2)}, r_{n+1}, r_{n+2}) \\
&= \omega(r_1^{(1)}, \dots, r_{n-1}^{(1)}, r_n) \triangleright \left(\mu^n(r_1^{(2)}, \dots, r_{n-1}^{(2)}, r_{n+1}) \triangleright \mu^n(r_1^{(3)}, \dots, r_{n-1}^{(3)}, r_{n+2}) \right)
\end{aligned}$$

where we have used Proposition 2.9 in the last equality. By self-distributivity

of \triangleright and because ω is a coderivation, this gives

$$\begin{aligned}
(d_{n+1,0}^{n+1} \circ d_{n+1}^n \omega)(r_1; \dots, r_{n+2}) &= (\omega(r_1^{(1)}; \dots, r_{n-1}^{(1)}, r_n^{(1)})^{(1)} \triangleright \mu^n(r_1^{(2)}; \dots, r_{n-1}^{(2)}, r_{n+1}^{(2)})) \triangleright \\
&\quad (\omega(r_1^{(1)}; \dots, r_{n-1}^{(1)}, r_n^{(1)})^{(2)} \triangleright \mu^n(r_1^{(3)}; \dots, r_{n-1}^{(3)}, r_{n+2}^{(3)})) \\
&= (\omega(r_1^{(1)}; \dots, r_{n-1}^{(1)}, r_n^{(1)}) \triangleright \mu^n(r_1^{(2)}; \dots, r_{n-1}^{(2)}, r_{n+1}^{(2)})) \triangleright \\
&\quad (\mu^n(r_1^{(3)}; \dots, r_{n-1}^{(3)}, r_n^{(2)}) \triangleright \mu^n(r_1^{(4)}; \dots, r_{n-1}^{(4)}, r_{n+2}^{(4)})) \\
&\quad + (\mu^n(r_1^{(1)}; \dots, r_{n-1}^{(1)}, r_n^{(1)}) \triangleright \mu^n(r_1^{(2)}; \dots, r_{n-1}^{(2)}, r_{n+1}^{(2)})) \triangleright \\
&\quad (\omega(r_1^{(3)}; \dots, r_{n-1}^{(3)}, r_n^{(2)}) \triangleright \mu^n(r_1^{(4)}; \dots, r_{n-1}^{(4)}, r_{n+2}^{(4)}))
\end{aligned}$$

Applying Proposition 2.9 again enables us to rewrite this last equality as

$$\begin{aligned}
(d_{n+1,0}^{n+1} \circ d_{n+1}^n \omega)(r_1; \dots, r_{n+2}) &= (\omega(r_1^{(1)}; \dots, r_n^{(1)}) \triangleright \mu^n(r_1^{(2)}; \dots, r_{n-1}^{(2)}, r_{n+1}^{(2)})) \triangleright \mu^{n+1}(r_1^{(3)}; \dots, r_{n-1}^{(3)}, r_n^{(2)}, r_{n+2}^{(2)}) \\
&\quad + \mu^{n+1}(r_1^{(1)}; \dots, r_n^{(1)}, r_{n+1}^{(1)}) \triangleright (\omega(r_1^{(2)}; \dots, r_n^{(2)}) \triangleright \mu^n(r_1^{(3)}; \dots, r_{n-1}^{(3)}, r_{n+2}^{(3)})) \\
&= d_{n+1}^m \omega(r_1^{(1)}; \dots, r_n^{(1)}, r_{n+1}^{(1)}) \triangleright \mu^{n+1}(r_1^{(2)}; \dots, r_n^{(2)}, r_{n+2}^{(2)}) \\
&\quad + \mu^{n+1}(r_1^{(1)}; \dots, r_n^{(1)}, r_{n+1}^{(1)}) \triangleright d_{n+1}^m \omega(r_1^{(2)}; \dots, r_n^{(2)}, r_{n+2}^{(2)}) \\
&= ((d_{n+2}^{n+1} \circ d_{n+1}^n + d_{n+1,1}^{n+1} \circ d_{n+1}^n) : \omega)(r_1, \dots, r_{n+2})
\end{aligned}$$

which proves (42).

Let us show now how $d_R \circ d_R = 0$ can be deduced from (41), (42) and from the cubical relations. In degree n , we have

$$\begin{aligned}
d_R \circ d_R &= \left(\sum_{i=1}^{n+1} (-1)^{i+1} (d_{i,1}^{n+1} - d_{i,0}^{n+1}) + (-1)^{n+2} d_{n+2}^{n+1} \right) \circ \left(\sum_{i=1}^n (-1)^{i+1} (d_{i,1}^n - d_{i,0}^n) \right. \\
&\quad \left. + (-1)^{n+1} d_{n+1}^n \right) \\
&= \sum_{i=1}^{n+1} \sum_{j=1}^n (-1)^{i+j} (d_{i,1}^{n+1} \circ d_{j,1}^n - d_{i,1}^{n+1} \circ d_{j,0}^n - d_{i,0}^{n+1} \circ d_{j,1}^n + d_{i,0}^{n+1} \circ d_{j,0}^n) \\
&\quad + \sum_{i=1}^n (-1)^{n+i+1} (d_{n+2}^{n+1} \circ d_{i,1}^n - d_{n+2}^{n+1} \circ d_{i,0}^n - d_{i,1}^{n+1} \circ d_{n+1}^n + d_{i,0}^{n+1} \circ d_{n+1}^n) \\
&\quad - d_{n+2}^{n+1} \circ d_{n+1}^n - d_{n+1,1}^{n+1} \circ d_{n+1}^n + d_{n+1,0}^{n+1} \circ d_{n+1}^n
\end{aligned}$$

The first double sum is equal to zero thanks to the cubical relations, the second sum is zero thanks to relation (41). Relation (42) implies that the last one vanishes. This shows that d_R is indeed a differential and concludes the proof of the proposition. \square

Definition 2.13. The cohomology of the deformation complex $(C^*(R; R), d_R)$ is called the adjoint cohomology of the rack bialgebra R and is denoted by $H^*(R; R)$.

Definition 2.14. An **infinitesimal deformation** of the rack product is a deformation of the rack product over the K -algebra of dual numbers $\bar{K}_\hbar := K_\hbar/(\hbar^2)$, i.e. a linear map $\mu_1 : R^{\otimes 2} \rightarrow R$ such that $\bar{R}_\hbar := R \otimes \bar{K}_\hbar$ is a rack bialgebra over \bar{K}_\hbar when equipped with $\mu_0 + \hbar\mu_1$.

Two infinitesimal deformations $\mu_0 + \hbar\mu_1$ and $\mu_0 + \hbar\mu'_1$ are said to be **equivalent** if there exists an automorphism $\phi : \bar{R}_\hbar \rightarrow \bar{R}_\hbar$ of the coalgebra of $(\bar{R}_\hbar, \Delta, \epsilon)$ of the form $\phi := \text{id}_R + \hbar\alpha$ such that

$$\phi \circ (\mu_0 + \hbar\mu_1) = (\mu_0 + \hbar\mu'_1) \circ \phi.$$

As usual, being equivalent is an equivalence relation and one has the following cohomological interpretation of the set of equivalence classes of infinitesimal deformations, denoted $Def(\mu_0, \bar{K}_\hbar)$:

Proposition 2.15.

$$Def(\mu_0, \bar{K}_\hbar) = H^2(R; R)$$

The identificaton is obtained by sending each equivalence class $[\mu_0 + \hbar\mu_1]$ in $Def(\mu_0, \bar{K}_\hbar)$ to the cohomology class $[\mu_1]$ in $H^2(R; R)$.

Proof. One checks easily that the correspondence is well defined (if $\mu_0 + \hbar\mu_1$ is an infinitesimal deformation, then μ_1 is a 2-cocycle, see Example 2.2) and that it is bijective when restricted to equivalence classes. \square

Remark 2.16. (a) The choice of taking coderivations in the deformation complex is explained as follows: The rack product μ is a morphism of coalgebras, and we want to deform it as a morphism of coalgebras with respect to the fixed coalgebra structure we started with. Tangent vectors to μ in $\text{Hom}_{\text{coalg}}(C \otimes C, C)$ are exactly coderivations along μ . This is the first step: Deformations as morphisms of coalgebras. Then as a second step, we look for 1-cocycles, meaning that we determine those morphisms of coalgebras which give rise to rack bialgebra structures. The deformation complex in [4] takes into account also the possibility of deforming the coalgebra structure, and we recover our complex by restriction.

(b) Given a Leibniz algebra \mathfrak{h} , there is a natural restriction map from the cohomology complex with adjoint coefficients of \mathfrak{h} to the deformation complex of its augmented enveloping rack bialgebra $\text{UAR}(\mathfrak{h})$. The induced map in cohomology is not necessarily an isomorphism, as the abelian case shows. Observe that the deformation complex of the rack bialgebra $K[R]$ for a rack R does not contain the complex of rack cohomology for two reasons: First, this latter complex is ill-defined for adjoint coefficients, and second, there are not enough coderivations as all elements are set-like. A way out for this last problem would be to pass to completions.

References

- [1] Alexandre, C., Bordemann, M. Rivière, S., Wagemann, F.: *Structure theory of rack bialgebras*, J. Gen. Lie Theory Appl. **10** (2016), no. 1, Art. ID 1000244, see also [arXiv:1412.5907](https://arxiv.org/abs/1412.5907).
- [2] N. Bourbaki, *Lie groups and Lie algebras*, Chapters 1-3, Springer Verlag Berlin Heidelberg 1989
- [3] Cattaneo, A., Dherin, B., Weinstein, A.: *Symplectic microgeometry II: generating functions*. Bull. Braz. Math. Soc. (N.S.) **42** (2011), no. 4, 507–536
- [4] Carter, J., Crans, A., Elhamdadi, M., Saito, M.: *Cohomology of the Categorical Self-Distributivity* J. Homotopy Relat. Struct. **3** (2008), no. 1, 13–63.
- [5] Covez, S.: *On the conjectural Leibniz cohomology for groups*. J. K-Theory **10** (2012), no. 3, 519–563
- [6] Dherin, B., Wagemann, F.: *Deformation quantization of Leibniz algebras*. Adv. Math. **270** (2015), 21–48.
- [7] Dixmier, J.: *Algèbres enveloppantes*. Gauthier-Villars, Paris 1974.
- [8] Fenn, R., Rourke, C.: *Racks and links in codimension 2* J. Knot Theory Ramifications **1** (1992), no. 4, 343–406
- [9] Gutt, S.: *An explicit *-product on the cotangent bundle of a Lie group*. Lett. Math. Phys. **7** (1983), 249–258.
- [10] Kinyon, M. K. *Leibniz algebras, Lie racks, and digroups*. J. Lie Theory **17** (2007), no. 1, 99–114.
- [11] Krähmer, U., Wagemann, F.: *Racks, Leibniz algebras and Yetter-Drinfel'd modules* Georgian Math. J. **22** (2015), no. 4, 529–542.
- [12] Lebed, V.: *Objets tressés : une étude unificatrice de structures algébriques et une catégorification des tresses virtuelles* doctoral thesis Paris 7, 2012.
- [13] Loday, J. L.: *Cyclic homology*. Springer Verlag, Berlin, 1992.
- [14] Loday, J. L.; Pirashvili, T. *The tensor category of linear maps and Leibniz algebras*. Georgian Math. J. **5** (1998), no. 3, 263–276
- [15] Mostovoy, J.; Pérez-Izquierdo, J. M. *Formal multiplications, bialgebras of distributions and nonassociative Lie theory*. Transform. Groups **15** (2010), no. 3, 625–653
- [16] Pérez-Izquierdo, José M.; Shestakov, Ivan P. *An envelope for Malcev algebras*. J. Algebra **272** (2004), no. 1, 379–393

- [17] Quillen, Daniel: *Rational Homotopy Theory*. Ann. Math. 90 (1969), 205–295.
- [18] Rivière, S. *Sur l'isomorphisme entre cohomologies de Chevalley-Eilenberg et de Hochschild*, thèse de doctorat, Université de Nantes, 2012
- [19] Rivière, S.; Wagemann, F. *On the string Lie algebra*. Algebr. Represent. Theory **18** (2015), no. 4, 1071–1099
- [20] J.-P. Serre: *Lie algebras and Lie groups*. Springer Lecture Notes in Mathematics **1500**, Springer, Berlin, 1964.