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# Inertial drag on a sphere settling in a stratified fluid

R. Mehaddi<sup>1</sup>, F. Candelier<sup>2</sup> and B. Mehlig<sup>3</sup>

<sup>1</sup> Université de Lorraine, UMR CNRS 7563, LEMTA (Laboratoire d'Énergétique et Mécanique Théorique et Appliqué ), F-54500 Vandoeuvre-Les-Nancy, France

<sup>2</sup> Université d'Aix-Marseille, UMR CNRS 7343, IUSTI (Institut Universitaire des Systèmes Thermiques et Industriels) F-13013 Marseille, France

<sup>3</sup> Department of Physics, Gothenburg University, SE-41296 Gothenburg, Sweden

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We compute the drag force on a sphere settling slowly in a quiescent, linearly stratified fluid. Stratification can significantly enhance the drag experienced by the settling particle. The magnitude of this effect depends on whether fluid-density transport around the settling particle is due to diffusion, to advection by the disturbance flow caused by the particle, or due to both. It therefore matters how efficiently the fluid disturbance is convected away from the particle by fluid-inertial terms. When these terms dominate, the Oseen drag force must be recovered. We compute by perturbation theory how the Oseen drag is modified by diffusion and stratification. Our results are in good agreement with recent direct-numerical simulation studies of the problem.

**Key words:**

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## 1. Introduction

The settling of small solid particles in either gaseous or liquid flows with density stratification is a topic of great interest in fluid dynamics. Such multi-phase flows are widely encountered in nature. An example is the settling of small aggregates of organic matter, ‘marine snow’, in the ocean (Van Aartrijk & Clercx 2010; Guasto *et al.* 2012; Sozza *et al.* 2016; De Pietro *et al.* 2015). Here density stratification is due to either salt-concentration or temperature gradients. A second example is the dynamics of pollutants in the atmosphere which is often stably-stratified turbulence (Jacobson 2005; Leelóssy *et al.* 2014). It is important to understand the settling speed of particles in such flows, because it determines their residence times and deposition rates. Also, density-stratified fluids occur in industrial processes that involve heated fluids (Linden 1999), or the mixing of fluids of different densities (Turner 1979).

To compute, from first principles, the drag force experienced by the particles as they fall through the flow is a difficult problem, in particular if there is turbulence. Many authors have therefore concentrated on particle dynamics in quiescent stratified fluids. Settling through a sharp density interface was measured in experiments (Srdić-Mitrović *et al.* 1999; Abaid *et al.* 2004; Camassa *et al.* 2009) and in simulation (Blanchette & Shapiro 2012; Ardekani *et al.* 2017; Pierson & Magnaudet 2018). Direct-numerical simulation (DNS) studies of particles settling in quiescent, linearly stratified fluids were reported by Torres *et al.* (2000); Yick *et al.* (2009); Ardekani & Stocker (2010). Stratification can significantly slow down the settling particle by enhancing the drag it experiences

(Yick *et al.* 2009). The reason is that buoyancy differences due to the stratification tend to prevent the vertical motion of the fluid that the particle stirs up as it settles. As a consequence, the disturbance flow remains confined around the particle (Ardekani & Stocker 2010). How much the particle is slowed down depends on the mechanisms that govern the dynamics of the fluid density: diffusion of concentration or temperature, or their advection by the disturbance flow, or a combination of diffusion and advection.

Which of these mechanisms is most important depends on the physical system in question. In salt water, for example, the diffusion coefficient of salt is much smaller than the kinematic viscosity of the fluid. Therefore salt water is often considered non diffusive. But when temperature comes into play this may not be a good approximation, because the diffusion coefficient of temperature in water is roughly of the same order as the kinematic viscosity. This is even more important in gases where the temperature diffusion coefficient may exceed the kinematic viscosity of the fluid (Salazar 2003).

The nature of the disturbance flow caused by the settling particle depends on how efficiently the fluid disturbance is convected away. This is an inertial effect. So stratification, diffusion, and convective fluid inertia compete to determine the drag force on the particle. When the convective fluid-inertia terms dominate – so that stratification and diffusion do not matter – the Oseen drag force must be recovered. The question is how the Oseen drag on the settling sphere is modified by diffusion and stratification.

To answer this question we compute the drag force on a sphere settling slowly in a quiescent linearly stratified fluid. The density gradient points in the direction of gravity, so that the heavier fluid is at the bottom. The importance of convective fluid inertia is measured by the particle Reynolds number,  $Re$ . The relative importance of advection and diffusion is characterised by the Péclet number  $Pe$ . The importance of stratification is often quantified by the viscous Richardson number  $Ri$ , the ratio of buoyancy and viscous forces (Yick *et al.* 2009). Recent direct-numerical simulation studies of the problem (Yick *et al.* 2009; Zhang *et al.* 2017) explored how the drag depends on the importance of diffusivity versus advection, and upon the degree of density stratification. Our goal is to explain their results by perturbation theory, assuming that both  $Re$  and  $Ri$  are small but finite.

Chadwick & Zvirin (1974*b*, *a*) analysed this question, but for a sphere moving horizontally in a quiescent non-diffusive stratified fluid, along surfaces of constant fluid density. Here we study the settling problem, where the particle settles vertically along the fluid-density gradient, so that it crosses the surfaces of constant density. The two problems are quite different: when the particle moves horizontally, the streamlines of the flow tend to encircle the sphere in the horizontal plane. When the sphere moves vertically, by contrast, light fluid is pushed down into regions of larger fluid density, giving rise to complex disturbance-flow patterns (Ardekani & Stocker 2010).

Neglecting effects of convective fluid inertia, the difference between horizontal and vertical motion was compared earlier. When density transport is entirely diffusive, the additional drag due to stratification is five times larger in the vertical than in the horizontal direction (Candelier *et al.* 2014). When density advection dominates, the vertical drag is seven times larger than the horizontal one (Zvirin & Chadwick 1975).

Despite these qualitative and quantitative physical differences, the horizontal and vertical problems share an important mathematical property: regular perturbation expansions fail to describe the effects of convective fluid inertia and buoyancy due to stratification even if these perturbations are weak. Therefore so-called ‘singular-perturbation’ methods are required to solve the problem. We use the standard method of asymptotic matching (Saffman 1965), where inner and outer solutions of the disturbance problem are matched, describing the disturbance flow close to and far from the particle.

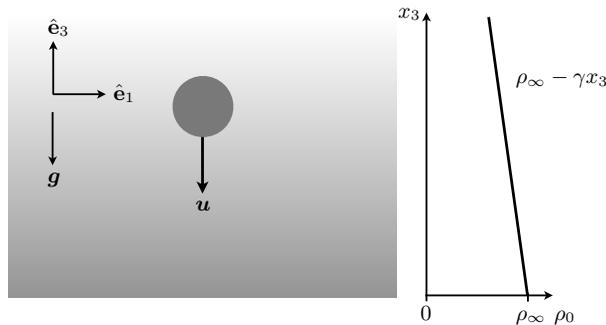


FIGURE 1. Settling of a sphere in a linearly stratified fluid with ambient density  $\rho_0 = \rho_\infty - \gamma x_3$  where gravity  $\mathbf{g}$  points in the negative  $\hat{\mathbf{e}}_3$ -direction (we denote the basis in the lab frame by  $\hat{\mathbf{e}}_1$ ,  $\hat{\mathbf{e}}_2$ , and  $\hat{\mathbf{e}}_3$ , and  $x_3$  is the third component of the coordinate vector  $\mathbf{x}$  in the lab frame). The particle settles with velocity  $\mathbf{u}$  along gravity.

We parametrise the effect of convective inertia and stratification in terms of length scales: the particle radius  $a$ , the Oseen length  $\ell_o = a/\text{Re}$ , and the stratification length  $\ell_s = (\nu\kappa/N^2)^{1/4}$  (Ardekani & Stocker 2010). Here  $\nu$  is the kinematic fluid viscosity,  $\kappa$  is the diffusivity, and  $N$  is the Brunt-Vaisala frequency. Diffusive changes of the fluid density are either due to concentration or temperature diffusion. These different physical effects can be treated in the same model if changes in the fluid density are linearly related to concentration and temperature changes (Ardekani *et al.* 2017). This is assured if these changes are small. In the case of temperature diffusion, the importance of the diffusivity  $\kappa$  is characterised by the Prandtl number  $\text{Pr} = \nu/\kappa = \text{Pe}/\text{Re}$ . For concentration diffusion, this dimensionless number is usually referred to as the Schmidt number  $\text{Sc} = \nu/\kappa$  (Candelier *et al.* 2014; Doostmohammadi *et al.* 2014; Zhang *et al.* 2017). To simplify the notation we do not distinguish between these two conventions here. In the following we refer to this dimensionless parameter as the Prandtl number.

We obtain a uniformly valid perturbation theory to first order in  $\epsilon = a/\ell_s$  and show that analysing the results in terms of the dimensionless parameter  $\ell_s/\ell_o$  reveals three distinct regimes where density diffusion, density advection, and convective fluid inertia dominate. Fluid inertia begins to matter when  $\ell_s/\ell_o$  is of the order of or larger than  $\text{Pr}^{-1/4}$ . This condition is met in recent (DNS) of the problem (Yick *et al.* 2009; Zhang *et al.* 2017), and our results are in good agreement with the simulations at small  $\text{Re}$  and small (but not too small) values of  $\epsilon$ . When  $\epsilon$  is so small that stratification effects are almost entirely negligible, we ascribe deviations between our theory and the DNS of Zhang *et al.* (2017) to finite-size effects: while our theory applies to an unbounded system, the DNS were performed for a bounded system. The comparison shows that finite-size effects are larger in the homogeneous system.

## 2. Formulation of the problem

We consider a spherical particle of radius  $a$  and of material density  $\rho_p$  settling with velocity  $\mathbf{u}$  in a quiescent stratified fluid. The diffusivity of the stratifying agent (salt or temperature) is denoted by  $\kappa$ , and the kinematic viscosity of the fluid is denoted by  $\nu$ . The ambient density of the fluid is assumed to vary linearly with height  $x_3$

$$\rho_0 = \rho_\infty - \gamma x_3, \quad (2.1)$$

where  $\gamma$  is the density gradient, and  $\rho_\infty$  is a reference density. The problem is illustrated in Fig. 1. The settling particle causes disturbances in the fluid, in the local density  $\rho$ , and in the pressure  $p$ . To determine these disturbances one must solve the Navier-Stokes equation for the fluid velocity and the diffusion-advection equation for the density. Following Maxey & Riley (1983) we express these equations in a frame of reference that translates with the particle position  $\mathbf{x}_p(t)$ ,

$$\rho \left[ \frac{\partial \mathbf{w}}{\partial t} \Big|_{\mathbf{r}} + (\mathbf{w} \cdot \nabla) \mathbf{w} \right] = -\nabla p + \mu \Delta \mathbf{w} + \rho \mathbf{g} - \rho \frac{\partial \mathbf{u}}{\partial t} \quad \text{and} \quad \nabla \cdot \mathbf{w} = 0, \quad (2.2a)$$

$$\frac{\partial \rho}{\partial t} + (\mathbf{w} \cdot \nabla) \rho = \kappa \Delta \rho. \quad (2.2b)$$

$$\mathbf{w} = \mathbf{0}, \partial_r \rho = 0 \quad \text{at} \quad r = a \quad \text{and} \quad \mathbf{w} \rightarrow -\mathbf{u}, \rho \rightarrow \rho_0 \quad \text{as} \quad r \rightarrow \infty. \quad (2.2c)$$

Here  $\mathbf{r} \equiv \mathbf{x} - \mathbf{x}_p$ , the spatial gradients  $\nabla$  are w.r.t.  $\mathbf{r}$ , and the fluid velocity is a function of  $\mathbf{r}$ ,  $\mathbf{w}(\mathbf{r}, t)$ . The boundary condition for  $\rho$  on the surface of the particle is derived from the surface condition  $\nabla \rho \cdot \mathbf{n} = 0$ . This means that the particle surface is impermeable.

We define the disturbance velocity, density, and pressure as:

$$\mathbf{w}' = \mathbf{w} + \mathbf{u}, \quad \rho' = \rho - \rho_0 \quad \text{and} \quad p' = p - p_0. \quad (2.3)$$

Here  $p_0$  is the hydrostatic pressure,  $\rho_0$  is the ambient density at height  $x_3$ , and  $-\mathbf{u}$  is the undisturbed fluid velocity in the frame translating with the particle.

We de-dimensionalise the problem in the usual fashion (Alias & Page 2017), using the particle radius  $a$  for lengths, the typical time  $\tau$  of variation of the disturbance velocity induced by the boundary condition (2.4c) for times, the terminal Stokes velocity in a homogeneous fluid  $u_t = [2a^2/(9\nu)](\rho_p/\rho_\infty - 1)g$  for the fluid velocity,  $\rho_\infty \nu u_t/a$  for the pressure, and  $\gamma a$  for the density. All equations below are written in these dimensionless variables.

We further assume that quadratic combinations of the density and pressure disturbances are negligible, and that  $\gamma x_3/\rho_\infty \ll 1$  in the region of interest. This allows us to ignore density gradients except when multiplied by the gravitational acceleration (Gray & Giorgini 1976). This ‘Boussinesq’ approximation was used in the DNS of the problem by Yick *et al.* (2009) and Zhang *et al.* (2017) that we compare with in Section §5, see also (Doostmohammadi *et al.* 2014). Under these assumptions the non-dimensional equations for the disturbance velocity, density and pressure take the form

$$\text{ReSl} \frac{\partial \mathbf{w}'}{\partial t} + \text{Re} [(\mathbf{w}' \cdot \nabla) \mathbf{w}' - (\mathbf{u} \cdot \nabla) \mathbf{w}'] + \text{Ri} \rho' \hat{\mathbf{e}}_3 = -\nabla p' + \Delta \mathbf{w}' \quad \text{and} \quad \nabla \cdot \mathbf{w}' = 0, \quad (2.4a)$$

$$\text{PeSl} \frac{\partial \rho'}{\partial t} + \text{Pe} [(\mathbf{w}' \cdot \nabla) \rho' - (\mathbf{u} \cdot \nabla) \rho' - \mathbf{w}' \cdot \hat{\mathbf{e}}_3] = \Delta \rho', \quad (2.4b)$$

$$\mathbf{w}' = \mathbf{u}, \partial_r \rho' = \cos \theta \quad \text{at} \quad r = 1 \quad \text{and} \quad \mathbf{w}' \rightarrow \mathbf{0}, \rho' \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty. \quad (2.4c)$$

Here  $\theta$  is the angle between the outward unit normal  $\mathbf{n}$  of the sphere and the vertical direction  $\hat{\mathbf{e}}_3$ . The dimensionless parameters in Eqs. (2.4a) and (2.4b) are the particle Reynolds number, the Péclet number, the Richardson number, and the Strouhal number:

$$\text{Re} = a u_t / \nu, \quad \text{Pe} = a u_t / \kappa, \quad \text{Ri} = a^3 N^2 / (u_t \nu), \quad \text{and} \quad \text{Sl} = a / (u_t \tau). \quad (2.5)$$

Here  $N$  is the Brunt-Vaisala frequency

$$N = \sqrt{g \gamma / \rho_\infty}, \quad (2.6)$$

the frequency at which a perturbation describing a vertically displaced parcel of fluid

oscillates in a stably-stratified environment (Mowbray & Rarity 1967). The Brunt-Vaisala frequency is an important physical parameter in problems involving stratified fluids, such as fountains (Bloomfield & Kerr 1998; Mehaddi *et al.* 2012), plumes (McDougall 1978; Woods 2010; Mehaddi *et al.* 2013) or gravity currents (Maxworthy *et al.* 2002), to name but a few.

We wrote Eqs. (2.4) in a frame of reference translating with the particle. It is convenient to use such moving coordinates, because the problem simplifies. For a sphere settling in a homogeneous flow, for example, the problem becomes steady. After a short transient, the buoyancy force balances the viscous drag so that the sphere reaches a steady terminal velocity. In a stratified fluid the situation is slightly more complicated. The problem appears to remain unsteady, because the settling sphere experiences a time-dependent buoyancy force since the ambient density  $\rho_0$  varies as a function of height  $x_3$ . We now show under which circumstances this residual time-dependence is negligible. The time scale  $\tau$  of variation of the disturbance velocity can be estimated as  $\tau^{-1} \sim \dot{u}_t/u_t$  where now  $u_t \sim (2/9)(a^2g/\nu)[\rho_p/\rho_0(t) - 1]$ , and  $\rho_0(t)$  is the ambient density evaluated at the time-dependent particle position. We conclude that

$$\text{Sl} \sim \text{Ri} \frac{\rho_p}{\rho_\infty} \quad (2.7)$$

In this paper we obtain the drag force on the sphere assuming that convective fluid inertia and density stratification matter, but that they are weak enough so that their effects can be treated in perturbation theory. We assume that

$$0 < \text{Re} \ll 1 \quad \text{and} \quad 0 < \text{Ri} \ll 1 \quad (2.8)$$

In this limit the unsteady terms in Eqs. (2.4a) and (2.4b) are negligible, so that we can consider the steady problem.

### 3. Earlier results for $\text{Re} = 0$

For  $\text{Re} = 0$  the drag on a sphere settling in a stratified fluid was studied theoretically by Zvirin & Chadwick (1975) and Candelier *et al.* (2014). These authors made different assumptions concerning the relative importance of advection and diffusion in Eq. (2.4b). Zvirin & Chadwick (1975) assumed that advection is more important than diffusion. When advection dominates, the density disturbance  $\rho'$  scales as  $z/r$  near the particle (Chadwick & Zvirin 1974b), in the ‘inner region’ of the problem. As a consequence, the buoyancy term in Eq. (2.4a) balances the viscous Laplacian term at

$$r \sim \text{Ri}^{-1/3}. \quad (3.1)$$

At this distance inner and outer solutions of the disturbance problem must be matched. This implies that advection is more important than diffusion in Eq. (2.4b) if  $\text{Pe} > \text{Ri}^{1/3}$ . Second, at  $r \sim \text{Ri}^{-1/3}$  the dominant convective inertial term in Eq. (2.4a) is estimated as  $\text{Re}(\mathbf{u} \cdot \nabla)\mathbf{w}'^{(0)} \sim \text{Re} \text{Ri}^{2/3}$ , where  $\mathbf{w}'^{(0)} \sim 1/r$  is the solution of the disturbance problem in the Stokes limit. So convective inertial terms are negligible if  $\text{Re} \ll \text{Ri}^{1/3}$ . Under these conditions,

$$\text{Pe} > \text{Ri}^{1/3} \quad \text{and} \quad \text{Re} \ll \text{Ri}^{1/3}, \quad (3.2)$$

Zvirin & Chadwick (1975) derived the following expression for the drag force

$$f_3 = -6\pi u_3 [1 + B(\text{Ri}^{1/3}/\text{Pe}) \text{Ri}^{1/3}]. \quad (3.3)$$

Here  $B(\cdot)$  is a function given in integral form. In the limit of a non-diffusive fluid,  $\text{Pe} \rightarrow \infty$ , the above expression simplifies to:

$$f_3 = -6\pi u_3(1 + 1.060 \text{ Ri}^{1/3}). \quad (3.4)$$

Now consider the opposite limit, where the diffusive term in Eq. (2.4b) dominates over the advective term. In this case Candelier *et al.* (2014) showed that the spatial dependence of the disturbance density  $\rho'$  is of the form  $\rho' \sim \text{Pe} r$  in the inner region, so that the buoyancy term in Eq. (2.4a) balances the Laplacian viscous term at

$$r \sim \epsilon^{-1} \quad \text{with} \quad \epsilon = a/\ell_s. \quad (3.5)$$

Here  $\ell_s$  is the stratification length (Ardekani & Stocker 2010)

$$\ell_s = (\nu\kappa/N^2)^{1/4}. \quad (3.6)$$

It characterises the effect of stratification on the particle dynamics. Under the condition

$$\text{Pe} \ll \epsilon \ll 1 \quad (3.7)$$

Candelier *et al.* (2014) found

$$f_3 = -6\pi u_3(1 + 0.6621\epsilon). \quad (3.8)$$

Using  $\text{Ri} = \epsilon^4/\text{Pe}$ , we see that the condition  $\text{Pe} \ll \epsilon$  corresponds to  $\text{Pe} \ll \text{Ri}^{1/3}$ . Comparing with the condition (3.2) it seems that the results (3.4) and (3.8) apply in the opposite limits of large and small Péclet numbers. Below we show, however, that the two approaches are in fact equivalent, although they seem to apply in distinct limits.

## 4. Method

We consider the same problem as Candelier *et al.* (2014), but we do not neglect the fluid-inertia terms and the effect of advection of the fluid density by the disturbance flow. The relative importance of stratification and inertial effects is determined by the magnitude of the length scales  $\ell_s$  and  $\ell_o$  in relation to the particle size  $a$ . Therefore we use  $\epsilon = a/\ell_s$  [Eq. (3.5)] and  $\ell_s/\ell_o$  as dimensionless parameters. The third parameter is the Prandtl number. In summary, we solve Eqs. (2.4) to first order in the parameter  $\epsilon$  using the method of asymptotic matching (Saffman 1965). Inner and outer solutions of the disturbance problem are matched at  $r \sim \epsilon^{-1}$  in the limit

$$\epsilon \ll 1, \quad \ell_s/\ell_o \ll \epsilon^{-1}, \quad \text{and Pr arbitrary but fixed.} \quad (4.1)$$

In this way we obtain an expression for drag force that is valid regardless of whether diffusion or advection dominates: our solution is valid in both limits considered by Candelier *et al.* (2014) and Zvirin & Chadwick (1975), as well as uniformly in between.

Previous arguments, summarised in §3, appeal to different behaviours of the density disturbance to show that the non-linear convective terms  $\text{Re}(\mathbf{w}'^{(0)} \cdot \nabla)\mathbf{w}'^{(0)}$  and  $\text{Pe}(\mathbf{w}'^{(0)} \cdot \nabla)\rho'$  in Eq. (2.4) can be disregarded. A weakness of these arguments is that the limits of large and small  $\text{Pe}$  are considered separately. This is not necessary in our formulation.

A general property of the method of asymptotic matching is that it is the magnitude of the different terms in the matching region that matters: all terms that are sub-leading in this region can be entirely neglected. As already mentioned in §3, when  $\text{Re}$  and  $\text{Ri}$  are small, the disturbance velocity close to the particle is well approximated by the Stokes solution  $\mathbf{w}'^{(0)} \sim 1/r$ . Assuming this dependence, we can estimate the magnitude of the non-linear convective term  $\text{Re}(\mathbf{w}'^{(0)} \cdot \nabla)\mathbf{w}'^{(0)}$  in the matching region. Setting  $r \sim \epsilon^{-1}$  we

conclude that  $\text{Re}(\mathbf{w}'^{(0)} \cdot \nabla) \mathbf{w}'^{(0)}$  is small in this region compared with all other terms in Eq. (2.4a), when  $\epsilon$  is small. The orders of magnitude in Eq. (2.4b) are more difficult to determine because the  $\mathbf{r}$ -dependence of the density disturbance is not known unless  $\text{Pe}$  is either small (Candelier *et al.* 2014) or large (Zvirin & Chadwick 1975). However, since  $\mathbf{w}'^{(0)} \sim \epsilon$  in the matching region, we can conclude that the non-linear term  $\text{Pe}(\mathbf{w}'^{(0)} \cdot \nabla) \rho'$  is negligible compared with  $\text{Pe}(\mathbf{u} \cdot \nabla) \rho'$ . As a result, Eqs. (2.4) take the form:

$$-\epsilon \frac{\ell_s}{\ell_o} (\mathbf{u} \cdot \nabla) \mathbf{w}' = -\nabla p' - \epsilon^4 \tilde{\rho} \hat{\mathbf{e}}_3 + \Delta \mathbf{w}' \quad \text{and} \quad \nabla \cdot \mathbf{w}' = 0, \quad (4.2a)$$

$$-\epsilon \text{Pr} \frac{\ell_s}{\ell_o} (\mathbf{u} \cdot \nabla) \tilde{\rho} - \mathbf{w}' \cdot \hat{\mathbf{e}}_3 = \Delta \tilde{\rho}, \quad (4.2b)$$

with boundary conditions corresponding to (2.4c), and  $\rho' = \text{Pe} \tilde{\rho}$ . The inner solution of Eqs. (4.2) is obtained by a regular perturbation expansion in  $\epsilon$ . To obtain the outer solution one replaces the boundary condition on the particle surface by a singular source term (Saffman 1965). To lowest order in  $\epsilon$  this term takes the form  $6\pi\mathbf{u} \delta(\mathbf{r})$ , corresponding to the leading-order force exerted by the sphere on the fluid. Higher  $\epsilon$ -orders in the source term contribute to the force at order  $O(\epsilon^2)$ , and need not be considered to order  $\epsilon$ . Since the non-linear convective terms are negligible, Eq. (4.2) is linear, so that the outer solution can be obtained by Fourier transform, for arbitrary values of  $\epsilon$ . We define:

$$\hat{f}(\mathbf{k}) = \int d\mathbf{x} f(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} \quad \text{and} \quad f(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^3} \hat{f}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (4.3)$$

We expand the Fourier transform  $\hat{\mathbf{w}}'_{\text{out}}(\mathbf{k})$  of the outer solution in  $\epsilon$ , in terms of generalised functions (Candelier *et al.* 2013; Meibohm *et al.* 2016):

$$\hat{\mathbf{w}}'_{\text{out}} = \hat{\mathcal{T}}^{(0)} + \epsilon \hat{\mathcal{T}}^{(1)} + \epsilon^2 \hat{\mathcal{T}}^{(2)} + \dots \quad (4.4)$$

This method differs slightly from the standard approach (Saffman 1965) that formulates the outer problem in terms of strained coordinates  $\bar{\mathbf{r}} = \epsilon \mathbf{r}$ . The advantage of the present approach is that it does not refer to any particular matching length scale – for instance the length scale at which the Laplacian is balanced by the buoyancy term in Eq. (4.2a). The only requirement is that  $\epsilon$  is small. For certain cases this approach is equivalent to using the reciprocal theorem to compute inertial corrections (Meibohm *et al.* 2016).

The first two terms in the expansion (4.4) are obtained from  $\hat{\mathbf{w}}'_{\text{out}}$  as:

$$\hat{\mathcal{T}}^{(0)} = \lim_{\epsilon \rightarrow 0} \hat{\mathbf{w}}'_{\text{out}} \quad \text{and} \quad \hat{\mathcal{T}}^{(1)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\hat{\mathbf{w}}'_{\text{out}} - \hat{\mathcal{T}}^{(0)}). \quad (4.5)$$

The first term,  $\hat{\mathcal{T}}^{(0)}(\mathbf{k})$ , is the Fourier transform of the solution of the outer problem at  $\epsilon = 0$ . The next term in the expansion takes the form

$$\hat{\mathcal{T}}^{(1)}(\mathbf{k}) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^3} \left[ \hat{\mathbf{w}}'_{\text{out}} \left( \frac{\mathbf{k}}{\epsilon} \right) - \hat{\mathcal{T}}^{(0)} \left( \frac{\mathbf{k}}{\epsilon} \right) \right]. \quad (4.6)$$

We evaluate this limit using the homogeneity properties of the functions  $\hat{\mathbf{w}}'_{\text{out}}(\mathbf{k})$  and  $\hat{\mathcal{T}}^{(0)}(\mathbf{k})$  (Candelier *et al.* 2013; Meibohm *et al.* 2016). This yields

$$\hat{\mathcal{T}}^{(1)}(\mathbf{k}) = \delta(\mathbf{k}) \int d\mathbf{k}' \left[ \hat{\mathbf{w}}'_{\text{out}}|_{\epsilon=1}(\mathbf{k}') - \hat{\mathcal{T}}^{(0)}(\mathbf{k}') \right]. \quad (4.7)$$

Here  $\delta$  is the three-dimensional Dirac delta function. The functions  $\hat{\mathcal{T}}^{(0)}(\mathbf{k})$  and  $\hat{\mathcal{T}}^{(1)}(\mathbf{k})$  are readily transformed back from  $\mathbf{k}$ - to obtain the outer solution in configuration space.

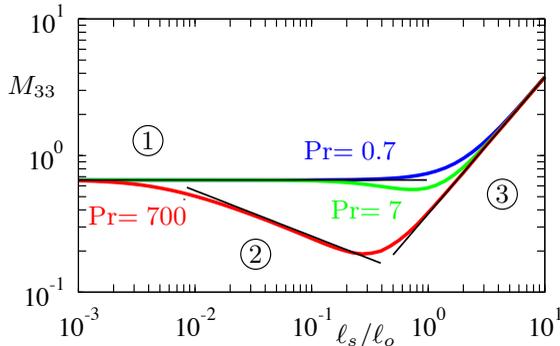


FIGURE 2. Shows Eq. (5.3b) as a function of  $\ell_s/\ell_o$  for different Pr. Also shown are the three different regimes in Eq. (5.5), black solid lines.

In particular,  $\mathcal{T}^{(1)}(\mathbf{r})$  is found to be  $\mathbf{r}$ -independent. Since  $\mathcal{T}^{(1)}(\mathbf{r})$  is constant, the problem to order  $\epsilon$  is equivalent to determining the force on a particle kept fixed in a uniform flow (Saffman 1965; Meibohm *et al.* 2016). It follows that the drag force is given by

$$\mathbf{f} = -6\pi \left\{ \mathbf{u} + \frac{\epsilon}{8\pi^3} \int d\mathbf{k}' [\hat{\mathbf{w}}'_{\text{out}}(\mathbf{k}')|_{\epsilon=1} - \hat{\mathcal{T}}^{(0)}(\mathbf{k}')] \right\}. \quad (4.8)$$

We note that the force is determined entirely by the solution of the outer problem, as first shown by Saffman (1965) for the lift force on a small sphere in a shear flow.

## 5. Results

For  $\epsilon = 1$  the Fourier transforms  $\hat{\mathbf{w}}'_{\text{out}}$  and  $\hat{\rho}'_{\text{out}}$  of the outer solution read:

$$\begin{pmatrix} \hat{\mathbf{w}}'_{\text{out}} \\ \hat{\rho}'_{\text{out}} \end{pmatrix} = -6\pi k^2 \left[ \frac{\ell_s}{\ell_o} (\mathbf{i}\mathbf{k} \cdot \mathbf{u}) \mathbb{I} + \mathbb{A} \right]^{-1} \cdot \mathbb{G} \cdot \begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix}. \quad (5.1)$$

Here  $\mathbb{I}$  is the  $4 \times 4$  unit tensor, and

$$\mathbb{A} = \begin{pmatrix} -k^2 & 0 & 0 & -\frac{k_1 k_3}{k^2} \\ 0 & -k^2 & 0 & -\frac{k_2 k_3}{k^2} \\ 0 & 0 & -k^2 & -\frac{(k^2 - k_3^2)}{k^2} \\ 0 & 0 & \frac{1}{\text{Pr}} & -\frac{k^2}{\text{Pr}} \end{pmatrix}, \quad \mathbb{G} = \begin{pmatrix} \frac{k^2 - k_1^2}{k^4} & -\frac{k_1 k_2}{k^4} & -\frac{k_1 k_3}{k^4} & 0 \\ -\frac{k_2 k_1}{k^4} & \frac{k^2 - k_2^2}{k^4} & -\frac{k_2 k_3}{k^4} & 0 \\ -\frac{k_3 k_1}{k^4} & -\frac{k_3 k_2}{k^4} & \frac{k^2 - k_3^2}{k^4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (5.2)$$

We set  $\mathbf{u} = u_3 \hat{\mathbf{e}}_3$  in Eq. (4.8) to find the drag force on the settling sphere:

$$f_3 = -6\pi u_3 (1 + \epsilon M_{33}), \quad (5.3a)$$

$$M_{33} = \frac{3}{2\pi} \int_0^\infty dk \int_0^\pi d\theta \frac{\sin(\theta)^3 \left\{ 1 - \left[ \text{Pr} \left( \frac{\ell_s}{\ell_o} \right)^2 k^2 + 1 \right] \cos(\theta)^2 - i \cos(\theta) \frac{\ell_s}{\ell_o} k^3 \right\}}{\left[ \text{Pr} \left( \frac{\ell_s}{\ell_o} \right)^2 k^2 + 1 \right] \cos(\theta)^2 + i \frac{\ell_s}{\ell_o} k^3 (\text{Pr} + 1) \cos(\theta) - k^4 - 1}. \quad (5.3b)$$

The imaginary part in Eq. (5.3b) vanishes upon integration.

Fig. 2 shows how  $M_{33}$  depends on the ratio  $\ell_s/\ell_o$  for different values of Pr, namely, 0.7 (temperature-stratified gas), 7 (temperature-stratified water at 20 °C) and 700 (salt-stratified water). When the ratio  $\ell_s/\ell_o$  is very small, the curves collapse onto a horizontal

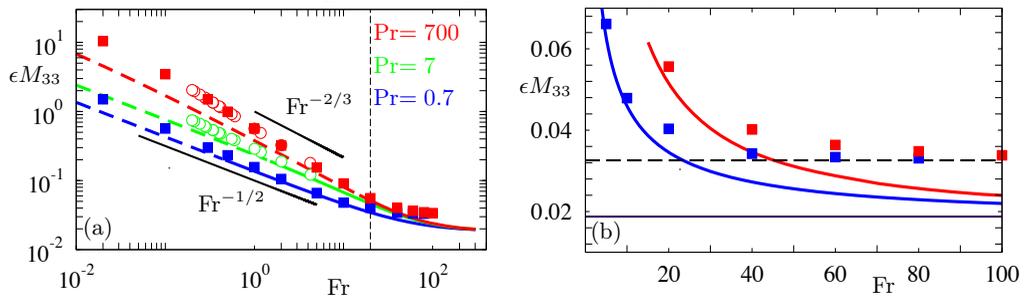


FIGURE 3. (a) Comparison between Eq. (5.3b) and DNS results for  $\text{Re}=0.05$  by Yick *et al.* (2009) for  $\text{Pr}=7$  ( $\circ$ ),  $\text{Pr}=700$  ( $\circ$ ), and by Zhang *et al.* (2017) for  $\text{Pr}=0.7$  ( $\blacksquare$ ) and  $\text{Pr}=700$  ( $\blacksquare$ ). Coloured solid lines show Eq. (5.3b) for  $\epsilon < \epsilon_0$ , dashed lines for  $\epsilon > \epsilon_0$ , for a small value of  $\epsilon_0$ . We take  $\epsilon_0 = 0.3$ . Also shown are power laws in  $\text{Fr}$ , black solid lines. The vertical dashed line corresponds to  $\text{Fr}=1/\text{Re}$ . (b) Same data as in panel (a) but now with linear axes to emphasise the deviations between theory and DNS at large  $\text{Fr}$ . The black dashed line shows the drag for the finite homogeneous system obtained by Zhang *et al.* (2017). The solid black line shows the theoretical value for the homogeneous infinite system.

line, Eq. (3.8). In this limit diffusion dominates. When  $\ell_s/\ell_o$  reaches  $\text{Pr}^{-1}$ , a second regime emerges: diffusion and advection in Eq. (2.4b) become of the same order, resulting in a change in the behaviour of the density disturbance. As a result,  $M_{33}$  in Fig. 2 turns downwards as  $\ell_s/\ell_o$  increases (this does not mean that the drag  $-6\pi(1 + \epsilon M_{33})$  increases at fixed particle radius  $a$  and Reynolds number  $\text{Re}$ , because  $\epsilon = a/\ell_s$ ). A further transition occurs at  $\ell_s/\ell_o \sim 1/\text{Pr}^{1/4}$ , caused by the formation of an Oseen wake behind the particle (Lovalenti & Brady 1993). When  $\ell_s/\ell_o \gg 1/\text{Pr}^{1/4}$  the curves approach  $M_{33} \approx (3/8)(\ell_s/\ell_o)$ , the Oseen correction (Oseen 1910; Proudman & Pearson 1957; Lovalenti & Brady 1993). In this regime stratification and diffusion do not matter, the settling particle experiences the fluid as if it were homogeneous. For small  $\text{Pr}$ , only the first and third regimes are realised.

Eq. (5.3b) is uniformly valid in the limit (4.1), regardless of the value of  $(\ell_s/\ell_o)\text{Pr} = (\text{Ri}^{1/3}/\text{Pe})^{-3/4}$ . It is not necessary to assume that  $\text{Pe} \ll \text{Ri}^{1/3}$ , the expression holds also when  $\text{Pe} \gg \text{Ri}^{1/3}$ . In particular, we can see that Eq. (5.3b) reduces to Eq. (3.3) when convective inertia is negligible, by taking the limit  $\ell_s/\ell_o \rightarrow 0$  at fixed  $(\ell_s/\ell_o)\text{Pr}$ :

$$\lim_{\ell_s/\ell_o \rightarrow 0} M_{33} = \frac{3\text{Ri}^{1/3}}{\pi\epsilon} \int_0^\infty dk \int_0^{\frac{\pi}{2}} d\theta \frac{\sin(\theta)^5 (\sin(\theta)^2 + (\text{Ri}^{1/3}/\text{Pe})k^4)}{(\sin(\theta)^2 + (\text{Ri}^{1/3}/\text{Pe})k^4)^2 + \cos(\theta)^2 k^6}. \quad (5.4)$$

This is precisely the function  $B(\cdot)$  in Eq. (3.3) [Eq. (29) in Zvirin & Chadwick (1975)], computed assuming that convective inertia is negligible, and that  $\text{Pe} > \text{Ri}^{1/3}$ . Since our solution is uniformly valid, we can conclude that Eq. (3.3) must be valid also for  $\text{Pe} \ll \text{Ri}^{1/3}$ , well outside the region of validity stated by Zvirin & Chadwick (1975). Closer inspection of their calculation shows that it corresponds to asymptotic matching at  $r \sim \text{Ri}^{-1/3}$  in the limit  $\text{Ri} \rightarrow 0$  keeping  $\text{Ri}^{1/3}/\text{Pe}$  constant. The two different matching scales  $r \sim \text{Ri}^{-1/3}$  and  $r \sim \epsilon^{-1}$  are equivalent in the limits stated, because the ratio of matching scales  $\text{Ri}^{1/3}/\epsilon = (\text{Ri}^{1/3}/\text{Pe})(\text{Pe}/\epsilon) = (\text{Ri}^{1/3}/\text{Pe})[(\ell_s/\ell_o)\text{Pr}]$  remains constant.

In summary, Eq. (5.3b) is a uniform approximation comprising three distinct regimes

$$f_3 \sim -6\pi u_3 \begin{cases} 1+0.6621\epsilon & \text{for } \ell_s/\ell_o \ll \text{Pr}^{-1} & \text{regime ① (diffusion),} \\ 1+1.060 \text{ Ri}^{1/3} & \text{for } \text{Pr}^{-1} \ll \ell_s/\ell_o \ll \text{Pr}^{-1/4} & \text{regime ② (advection),} \\ 1+\frac{3}{8} \text{ Re} & \text{for } \ell_s/\ell_o \gg \text{Pr}^{-1/4} & \text{regime ③ (fluid inertia).} \end{cases} \quad (5.5)$$

The different regimes are shown in Fig. 2. In the limit of small Pr, the advective regime disappears, as mentioned above.

We now compare the full result, Eq. (5.3b), with DNS by Zhang *et al.* (2017) and Yick *et al.* (2009), at their smallest Re. These authors use slightly different dimensionless parameters, namely Re, Pr, and the Froude number Fr

$$\text{Fr} = u_t/(aN). \quad (5.6)$$

In terms of Fr, the dimensionless parameters  $\epsilon$ ,  $\ell_s/\ell_o$ , and Ri are given by:

$$\epsilon = (\text{Re}/\text{Fr})^{1/2} \text{Pr}^{1/4}, \quad \ell_s/\ell_o = (\text{Re Fr})^{1/2}/\text{Pr}^{1/4}, \quad \text{and} \quad \text{Ri}^{1/3} = \text{Re}^{1/3}/\text{Fr}^{2/3}. \quad (5.7)$$

Zhang *et al.* (2017) and Yick *et al.* (2009) computed the drag coefficient  $C_D^S$  of the stratified system. In Fig. 3(a) we plot their result for  $C_D^S/C_D^{\text{Stokes}} - 1$  versus Fr, and compare it with our result for  $\epsilon M_{33}$ . Here  $C_D^{\text{Stokes}} = 12/\text{Re}$  is the Stokes drag coefficient for an unbounded system. Since Eq. (5.3b) was obtained for small  $\epsilon$ , we plot it as a solid line when  $\epsilon < 0.3$ , and dashed for  $\epsilon > 0.3$ . For  $1 < \text{Fr} < 10$  the data for Pr = 0.7 are in the diffusive regime, where the correction to the drag scales as  $\text{Fr}^{-1/2}$ . For Pr = 700, the data approach the advection regime where the theory predicts that the drag correction scales as  $\text{Ri}^{1/3} \propto \text{Fr}^{-2/3}$ .

Fig. 3(a) shows that the DNS yield a larger drag coefficient than our theory when Fr is small. The likely reason is that the non-linear convective terms matter in this regime, because  $\epsilon$  is not small enough (dashed lines in Fig. 3(a) indicate that  $\epsilon > 0.3$ , as mentioned above). For Pr = 700, for example,  $\epsilon$  becomes larger than 0.3 when Fr falls below 14.7 at Re = 0.05.

When do convective fluid-inertia effects dominate? The condition  $\ell_s/\ell_o = 1/\text{Pr}^{1/4}$  corresponds to  $\text{Fr} = 1/\text{Re}$ , independent of Prandtl number. For Re = 0.05 – the smallest value used in the DNS – this crossover occurs at  $\text{Fr} = 20$ , indicated by the vertical black dashed line in Fig. 3(a). Eq. (5.3b) allows us to determine the relative importance of convective fluid inertia at this value of Fr. For Pr = 0.7 the correction is substantial, 13.5 %. For larger Péclet numbers the correction is smaller, 1.4% at Pr = 7, and 2 % at Pr = 700. That the correction is largest for small Pr can be inferred from Fig. 3(a).

We observe deviations between DNS and theory not only at small values of Fr, discussed above, but also at large values of Fr [Fig. 3(b)]. These deviations at large Fr may be due to finite-size effects. In this limit, the homogeneous Oseen correction dominates, and it is known to be quite sensitive to the size of the simulation domain. Yick *et al.* (2009) chose an elliptical simulation domain, with a smallest size  $L$  that gives  $L/(2a) = 40$ . The domain used by Zhang *et al.* (2017) was spherical and larger [diameter/(2a) = 80], but even in that case a theory for cylindrical domains (Happel & Brenner 1983) indicates that the drag correction is expected to be larger than the Oseen expression  $\frac{3}{8}\text{Re}$ . This is consistent with Fig. 3(b). We also see that the asymptotic value for the homogeneous fluid is reached more quickly in the finite system. Finite-size effects matter less for smaller Fr, because the wake is smaller, of order  $\ell_s$ .

## 6. Conclusions

We calculated how convective fluid inertia modifies the drag on a sphere slowly settling in a density-stratified fluid, at small Richardson and Reynolds numbers. Plotting the results as a function of the dimensionless parameter  $\ell_s/\ell_o$  reveals three distinct regimes, Eq. (5.5). In the first regime, the drag is determined by diffusion of the disturbance density. In the second regime, advection of the disturbance density determines the drag. In the third regime, convection of the disturbance density by fluid-inertia terms dominates. Our main result, Eq. (5.3b), is uniformly valid, independently of whether the density dynamics is diffusive or advective. This allowed us to show that a result by Zvirin & Chadwick (1975) is more generally valid than the authors stated.

We compared with recent DNS at small Re and found that convective fluid-inertia effects matter for the larger Froude numbers simulated. Since finite-size effects appear to be important at large Fr and small Re, it would be of interest to take these corrections into account in the theory.

The results derived in this paper were obtained in the steady limit. But when a particle is released from above the water surface and plunges into the fluid with a given velocity, then unsteady effects must matter, at least initially. DNS of the problem (Doostmohammadi *et al.* 2014) at Re of order unity reveal unsteady effects that depend on the dimensionless numbers of the problem in intricate ways.

A further motivation for considering unsteady effects concerns the unsteady swimming of micro-organism in stratified fluids (Doostmohammadi *et al.* 2012; Jephson & Carlsson 2009; Bergström & Strömberg 1997). In oceans or in lakes the surface layers are known to shelter substantial biological activity. For very small organisms (much smaller than 1mm in size in typical ocean conditions) the dynamics of swimming microorganisms is well understood. Buoyancy (Franks & Jaffe 2008), density or drag asymmetries of the body (Roberts 1970; Jonsson 1989; Kessler 1985), and turbulence (Durham *et al.* 2013; Gustavsson *et al.* 2016) determine the spatial distribution of these organisms, their encounter rates, and thus their population ecology (Guasto *et al.* 2012). For larger organisms less is known. The problem becomes considerably more difficult because inertial effects begin to matter (Wang & Ardekani 2012*b,a*). The method described here allows to take inertial effects into account in perturbation theory. Finally, an important problem is how fluid shears affect the dynamics of motile microorganisms. The approach described by Candelier *et al.* (2018) makes it possible to address this question.

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