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ON THE SEMICLASSICAL SPECTRUM OF THE DIRICHLET-PAULI OPERATOR

J.-M. BARBAROUX, L. LE TREUST, N. RAYMOND, AND E. STOCKMEYER

ABSTRACT. This paper is devoted to semiclassical estimates of the eigenvalues of the Pauli operator on a bounded open set whose boundary carries Dirichlet conditions. Assuming that the magnetic field is positive and a few generic conditions, we establish the simplicity of the eigenvalues and provide accurate asymptotic estimates involving Bergman-Hardy spaces associated with the magnetic field.

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1. INTRODUCTION

In this article we consider the magnetic Pauli operator defined on a bounded and simply-connected domain $\Omega \subset \mathbb{R}^2$ subject to Dirichlet boundary conditions. This operator is the model Hamiltonian of a non-relativistic spin- $\frac{1}{2}$ particle, constraint to move in Ω , interacting with a magnetic field that is perpendicular to the plane.

Formally the Pauli operator acts on two-dimensional spinors and it is given by

$$\mathcal{P}_h = [\sigma \cdot (-ih\nabla - A)]^2,$$

where $h > 0$ is a semiclassical parameter and σ is a two-dimensional vector whose components are the Pauli matrices σ_1 and σ_2 . The magnetic field B enters in the operator through an associated magnetic vector potential $A = (A_1, A_2)$ that satisfies $\partial_1 A_2 - \partial_2 A_1 = B$. Assuming that the magnetic field is positive and few other mild conditions we provide precise asymptotic estimates for the low energy eigenvalues of \mathcal{P}_h in the semiclassical limit (i.e., as $h \rightarrow 0$).

Let us roughly explain our results. Let $\lambda_k(h)$ be the k -th eigenvalue of \mathcal{P}_h counting multiplicity. Assuming that the boundary of Ω is \mathcal{C}^2 , we show that there exist $\alpha > 0$, $\theta_0 \in (0, 1]$ such that the following holds: For all $k \in \mathbb{N}^*$, there exists $C_k > 0$ such that, as $h \rightarrow 0$,

$$\theta_0 C_k h^{-k+1} e^{-2\alpha/h} (1 + o(1)) \leq \lambda_k(h) \leq C_k h^{-k+1} e^{-2\alpha/h} (1 + o(1)).$$

In particular, this result establishes the simplicity of the eigenvalues in this regime. The constants $\alpha > 0$ and C_k are directly related to the magnetic field and the geometry of Ω and C_k is expressed in terms of Bergman and Hardy norms that are naturally associated to the magnetic field. In the case when Ω is a disc and B is radially symmetric we compute C_k explicitly and find that $\theta_0 = 1$. This improves by large the known results about the Dirichlet-Pauli operator [5, 7] (for details see Section 1.3.2).

These results may be reformulated in terms of the large magnetic field limit by a simple scaling argument. Indeed, $\mu_k(b) = b^2 \lambda_k(1/b)$, where $\mu_k(b)$ is the k -th eigenvalue of $[\sigma \cdot (-i\nabla - bA)]^2$.

Our results can also be used to describe the spectrum of the magnetic Laplacian with constant magnetic field B_0 . For instance, when Ω is bounded, strictly convex with a boundary of class $\mathcal{C}^{1,\gamma}$ ($\gamma > 0$), the k -th eigenvalue of $(-ih\nabla - A)^2$ with Dirichlet boundary conditions, denoted by $\mu_k(h)$, satisfies, for some $c, C > 0$ and h small enough,

$$B_0 h + c h^{-k+1} e^{-2\alpha/h} \leq \mu_k(h) \leq B_0 h + C h^{-k+1} e^{-2\alpha/h}.$$

In particular, the first eigenvalues of the magnetic Laplacian are simple in the semiclassical limit. As far as we know, this asymptotic simplicity was not known before and (1.7) is the most accurate known estimate of the magnetic eigenvalues in the case of the constant magnetic field and Dirichlet boundary conditions.

Our study presents a new approach that establishes several connections with various aspects of analysis as Cauchy-Riemann operators, uniformisation, and, to some extent, Toeplitz operators. We may hope that this work will cast a new light on the magnetic Schrödinger operators.

1.1. Setting and main results. Let $\Omega \subset \mathbb{R}^2$ be an open set. All along the paper Ω will satisfy the following assumption.

Assumption 1.1. Ω is bounded and simply connected.

Consider a magnetic field $B \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R})$. An associated vector potential $A : \overline{\Omega} \rightarrow \mathbb{R}^2$ is a function such that

$$B = \partial_1 A_2 - \partial_2 A_1.$$

We will use the following special choice of vector potential.

Definition 1.2. Let ϕ be the unique (smooth) solution of

$$\begin{aligned} \Delta \phi &= B, \text{ on } \Omega, \\ \phi &= 0, \text{ in } \partial\Omega. \end{aligned} \tag{1.1}$$

The vector field $A = (-\partial_2 \phi, \partial_1 \phi)^T := \nabla \phi^\perp$ is a vector potential associated with B .

In this paper, B will be positive (and thus ϕ subharmonic) so that

$$\max_{x \in \overline{\Omega}} \phi = \max_{x \in \partial\Omega} \phi = 0.$$

In particular, the minimum of ϕ will be negative and attained in Ω . Note also that the exterior normal derivative of ϕ , denoted by $\partial_{\mathbf{n}} \phi$, is positive on $\partial\Omega$.

Notation 1. We denote $\langle \cdot, \cdot \rangle$ the \mathbb{C}^n ($n \geq 1$) scalar product (antilinear w.r.t. the left argument), $\langle \cdot, \cdot \rangle_{L^2(U)}$ the L^2 scalar product on the set U , $\|\cdot\|_{L^2(U)}$ the L^2 norm on U and $\|\cdot\|_{L^\infty(U)}$ the L^∞ norm on U .

1.2. The Dirichlet-Pauli operator. This paper is devoted to the Dirichlet-Pauli operator $(\mathcal{P}_h, \text{Dom}(\mathcal{P}_h))$ defined for all $h > 0$ on

$$\text{Dom}(\mathcal{P}_h) := H^2(\Omega; \mathbb{C}^2) \cap H_0^1(\Omega; \mathbb{C}^2),$$

and whose action is given by the second order differential operator

$$\mathcal{P}_h = [\sigma \cdot (\mathbf{p} - A)]^2 = \begin{pmatrix} |\mathbf{p} - A|^2 - hB & 0 \\ 0 & |\mathbf{p} - A|^2 + hB \end{pmatrix} = \begin{pmatrix} \mathcal{L}_h^- & 0 \\ 0 & \mathcal{L}_h^+ \end{pmatrix}. \tag{1.2}$$

Here $\mathbf{p} = -ih\nabla$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\sigma \cdot \mathbf{x} = \sigma_1 \mathbf{x}_1 + \sigma_2 \mathbf{x}_2 + \sigma_3 \mathbf{x}_3$ for $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ and $\sigma \cdot \mathbf{x} = \sigma_1 \mathbf{x}_1 + \sigma_2 \mathbf{x}_2$ for $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$. In terms of quadratic form, we have doing partial integration, for all $u \in \text{Dom}(\mathcal{P}_h)$,

$$\langle u, \mathcal{P}_h u \rangle_{L^2(\Omega)} = \|\sigma \cdot (\mathbf{p} - A)u\|_{L^2(\Omega)}^2 = \|(\mathbf{p} - A)u\|_{L^2(\Omega)}^2 - \langle u, \sigma_3 h B u \rangle_{L^2(\Omega)}. \tag{1.3}$$

Note that we have the following relation, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$,

$$(\sigma \cdot \mathbf{x})(\sigma \cdot \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} 1_2 + i\sigma \cdot (\mathbf{x} \times \mathbf{y}). \tag{1.4}$$

The operator \mathcal{P}_h is selfadjoint and has compact resolvent. This paper is mainly devoted to the investigation of the lower eigenvalues of \mathcal{P}_h . Under the assumption that $B > 0$ on $\overline{\Omega}$, these eigenvalues are the eigenvalues of \mathcal{L}_h^- .

Notation 2. Let $(\lambda_k(h))_{k \in \mathbb{N}^*}$ ($h > 0$) denote the increasing sequence of eigenvalues of the operator \mathcal{P}_h , each one being repeated according to its multiplicity. By the min-max theorem,

$$\lambda_k(h) = \inf_{\substack{V \subset \text{Dom}(\mathcal{P}_h), \\ \dim V = k}} \sup_{u \in V \setminus \{0\}} \frac{\|\sigma \cdot (\mathbf{p} - A)u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2}. \quad (1.5)$$

1.3. Results and relations with the existing literature.

1.3.1. Main theorem.

Notation 3. Let us denote by $\mathcal{H}(\Omega)$ and $\mathcal{H}(\mathbb{C})$ are the sets of holomorphic functions on Ω and \mathbb{C} . We consider the following Bergman space

$$\mathcal{B}^2(\mathbb{C}) = \{u \in \mathcal{H}(\mathbb{C}) : N_{\mathcal{B}}(u) < +\infty\},$$

where

$$N_{\mathcal{B}}(u) = \left(\int_{\mathbb{R}^2} |u(y_1 + iy_2)|^2 e^{-\text{Hess}_{x_{\min}} \phi(y,y)} dy \right)^{1/2}.$$

We also introduce a weighted Hardy space

$$\mathcal{H}^2(\Omega) = \{u \in \mathcal{H}(\Omega) : N_{\mathcal{H}}(u) < +\infty\},$$

where

$$N_{\mathcal{H}}(u) = \left(\int_{\partial\Omega} |u(y_1 + iy_2)|^2 \partial_{\mathbf{n}} \phi dy \right)^{1/2}.$$

Here, $x_{\min} \in \Omega$ and $\text{Hess}_{x_{\min}} \phi \in \mathbb{R}^{2 \times 2}$ are defined in Theorem 1.3 below, $\mathbf{n}(s)$ is the outward pointing unit normal to Ω , and $\partial_{\mathbf{n}} \phi(s)$ is the normal derivative of ϕ on $\partial\Omega$ at $s \in \partial\Omega$. We also define for $P \in \mathcal{H}^2(\Omega)$, $A \subset \mathcal{H}^2(\Omega)$,

$$\text{dist}_{\mathcal{H}}(P, A) = \inf \{N_{\mathcal{H}}(P - Q), \text{ for all } Q \in A\},$$

and for $P \in \mathcal{B}^2(\mathbb{C})$, $A \subset \mathcal{B}^2(\mathbb{C})$,

$$\text{dist}_{\mathcal{B}}(P, A) = \inf \{N_{\mathcal{B}}(P - Q), \text{ for all } Q \in A\}.$$

The main results of this paper are gathered in the following theorem.

Theorem 1.3. *We define*

$$\phi_{\min} = \min_{x \in \Omega} \phi.$$

Assume that Ω is \mathcal{C}^2 , satisfies Assumption 1.1, and

- (a) $B_0 := \inf\{B(x), x \in \Omega\} > 0$,
- (b) *the minimum of ϕ is attained at a unique point x_{\min} ,*
- (c) *the minimum is non-degenerate, i.e., the Hessian matrix $\text{Hess}_{x_{\min}} \phi$ at x_{\min} (or z_{\min} if seen as a complex number) is positive definite.*

Then, there exists $\theta_0 \in (0, 1]$ such that for all fixed $k \in \mathbb{N}^$,*

(i) $\lambda_k(h) \leq C_{\text{sup}}(k)h^{-k+1}e^{2\phi_{\min}/h}(1 + o_{h \rightarrow 0}(1))$, with

$$C_{\text{sup}}(k) = 2 \left(\frac{\text{dist}_{\mathcal{H}}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega))}{\text{dist}_{\mathcal{B}}(z^{k-1}, \mathcal{P}_{k-2})} \right)^2,$$

where $\mathcal{P}_{k-2} = \text{span}(1, \dots, z^{k-2}) \subset \mathcal{B}^2(\mathbb{C})$, $\mathcal{P}_{-1} = \{0\}$ and

$$\mathcal{H}_k^2(\Omega) = \{u \in \mathcal{H}^2(\Omega), u^{(n)}(z_{\min}) = 0, \text{ for } n \in \{0, \dots, k-1\}\}.$$

(ii) $\lambda_k(h) \geq C_{\text{inf}}(k)h^{-k+1}e^{2\phi_{\min}/h}(1 + o_{h \rightarrow 0}(1))$, with

$$C_{\text{inf}}(k) = C_{\text{sup}}(k)\theta_0.$$

Assuming that Ω is the disc of radius 1 centered at 0, and that B is radial, we have

$$C_{\text{sup}}(k) = C_{\text{inf}}(k) = C^{\text{rad}}(k) = \frac{4\Phi(\det \text{Hess}_{x_{\min}}\phi)^{\frac{k}{2}}}{(k-1)!}, \quad (\theta_0 = 1),$$

$$\Phi = \frac{1}{2\pi} \int_{\Omega} B(x)dx = \frac{1}{2\pi} \int_{\Gamma} \partial_{\mathbf{n}}\phi ds.$$

Remark 1.4. Assume that $B = B_0 > 0$ and that Ω is strictly convex, then ϕ has a unique and non-degenerate minimum (see [9, 10] and also [7, Proposition 7.1 and below]). Thus, our assumptions are satisfied in this case.

Remark 1.5. The main properties of the space $\mathcal{H}^2(\Omega)$ can be found in [3, Chapter 10]. Note that whenever $\partial\Omega$ is supposed to be Dini continuous (in particular $\mathcal{C}^{1,\alpha}$ boundaries, with $\alpha > 0$, are allowed), the set $W^{1,\infty}(\Omega) \cap \mathcal{H}^2(\Omega)$ is dense in $\mathcal{H}^2(\Omega)$ (see Lemma B.1). This assumption is in particular needed in the proof of Theorem 1.3(i) (see Remark 3.4)¹. The definition of Dini-continuous functions is recalled in the context of the boundary behavior of conformal maps in [11, Section 3.3]. It is essentially an integrability property of the derivative of a parametrization of $\partial\Omega$. Boundaries of class $\mathcal{C}^{1,\gamma}$ are Dini-continuous.

Remark 1.6. The Cauchy formula [3, Theorem 10.4] and the Cauchy-Schwarz inequality ensure that

$$|u^{(n)}(z_{\min})| \leq \frac{n!}{2\pi\sqrt{\min_{\partial\Omega} \partial_{\mathbf{n}}\phi}} N_{\mathcal{H}}(u) \left(\int_{\partial\Omega} \frac{|dz|}{|z - z_{\min}|^{2(n+1)}} \right)^{1/2},$$

for $n \in \mathbb{N}$ and $u \in \mathcal{H}^2(\Omega)$ (see also the proof of Lemma 3.5). This ensures that $\mathcal{H}_k^2(\Omega)$ is a closed vector subspace of $\mathcal{H}^2(\Omega)$ and that $\text{dist}_{\mathcal{H}}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega)) > 0$ (see [2, Corollary 5.4]).

Remark 1.7. In the case when B is radial on the unit disk $\Omega = D(0, 1)$, we get, using Fourier series, that $(z^n)_{n \geq 0}$ is an orthogonal basis for $N_{\mathcal{B}}$ and $N_{\mathcal{H}}$ which are up to normalization factors, the Szegő polynomials [3, Theorem 10.8]. In particular, $\mathcal{H}_k^2(\Omega)$ is $N_{\mathcal{H}}$ -orthogonal to z^{k-1} so that

$$\text{dist}_{\mathcal{H}}(z^{k-1}, \mathcal{H}_k^2(\Omega))^2 = N_{\mathcal{H}}(z^{k-1})^2 = 2\pi\partial_{\mathbf{n}}\phi = 2\pi\Phi.$$

¹Note also that we do not use here the stronger notion of Smirnov domain in which the set of polynomials in the complex variable is dense in $\mathcal{H}^2(\Omega)$ (see [3, Theorem 10.6]). Starlike domains and domains with analytic boundary are Smirnov domains.

In addition, \mathcal{P}_{k-2} is $N_{\mathcal{B}}$ -orthogonal to z^{k-1} so that

$$\text{dist}_{\mathcal{B}} \left(z^{k-1}, \mathcal{P}_{k-2} \right)^2 = N_{\mathcal{B}}(z^{k-1})^2 = 2\pi \frac{(k-1)!}{2(\det \text{Hess}_{x_{\min}} \phi)^{\frac{k}{2}}},$$

and the radial case part of Theorem 1.3 follows.

Remark 1.8. The proof of the upper bound can easily be extended to the case where Ω is not necessarily simply connected (see Remark 3.4).

Remark 1.9. Theorem 1.3 is concerned with the asymptotics of each eigenvalues $\lambda_k(h)$ of the operator \mathcal{P}_h , ($k \in \mathbb{N}^*$) as $h \rightarrow 0$. In particular, $\lambda_k(h)$ tends to 0 exponentially. Of course, this does not mean that all the eigenvalues go to 0 uniformly with respect to k . For $h > 0$, consider for example

$$\nu_1(h) = \inf_{v \in H_0^1(\Omega; \mathbb{C}) \setminus \{0\}} \frac{\langle u, \mathcal{L}_h^+ u \rangle}{\|v\|_{L^2(\Omega)}^2},$$

the lowest eigenvalue of the operator \mathcal{L}_h^+ . For fixed $h > 0$, there exists $k(h) \in \mathbb{N}^*$ such that $\nu_1(h) = \lambda_{k(h)}(h)$. By (1.2), we have $\nu_1(h) \geq 2B_0 h$ and thus $\nu_1(h)$ does not converge to 0 with exponential speed. Actually, Theorem 1.3 ensures that

$$\lim_{h \rightarrow 0} \text{card}\{j \in \mathbb{N}^*, \lambda_j(h) \leq \nu_1(h)\} = +\infty, \quad \lim_{h \rightarrow 0} k(h) = +\infty.$$

This accumulation of eigenvalues near 0 in the semiclassical is related to the fact that the corresponding eigenfunctions are close to be functions in the Bergman space $\mathcal{B}^2(\mathbb{C})$ which is of infinite dimension.

Remark 1.10. The constant θ_0 does not depend on $k \in \mathbb{N}^*$ and is equal to 1 in the radial case. We conjecture that the upper bounds in Theorem 1.3 (i) are optimal, that is $\theta_0 = 1$ in the general case.

Actually, we can even see from our analysis that there is a class of magnetic fields for which $\theta_0 = 1$. Let Ω be a \mathcal{C}^2 set satisfying Assumption 1.1. We introduce

$$\begin{aligned} \mathcal{B} := \{ & B \in \mathcal{C}^\infty(\overline{D(0,1)}; \mathbb{R}_+^*), \\ & \exists \phi \in H_0^1(D(0,1); \mathbb{R}), \Delta \phi = B \text{ on } D(0,1), \partial_{\mathbf{n}s}^2 \phi = 0 \text{ on } \partial D(0,1) \}, \end{aligned} \quad (1.6)$$

and

$$\mathcal{M}_\Omega := \{G : \overline{\Omega} \rightarrow \overline{D(0,1)} \text{ biholomorphic s.t. } c_1 \leq |G'(\cdot)| \leq c_2, \text{ for some } c_1, c_2 > 0\}.$$

Here, ∂_s denotes the tangential derivative. We define

$$\mathcal{B}_\Omega := \{|G'(z)|^2 B \circ G(z), B \in \mathcal{B}, G \in \mathcal{M}_\Omega\}.$$

Our analysis (see Remarks 5.6 and 5.9) gives

$$\liminf_{h \rightarrow 0} e^{-2\phi_{\min}/h} h^{k-1} \lambda_k(h) \geq C_{\text{sup}}(k),$$

for all $B \in \mathcal{B}_\Omega$.

Using the Riemann mapping theorem, we can deduce the following lower bound for Ω with Dini-continuous boundary. Its proof can be found in Section 5.4.

Corollary 1.11. *Assume that Ω is bounded, simply connected and that $\partial\Omega$ is Dini-continuous. Assume also (a)–(c). Let $k \in \mathbb{N}^*$. Then, there exist $c_k, C_k > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$c_k h^{-k+1} e^{2\phi_{\min}/h} \leq \lambda_k(h) \leq C_k h^{-k+1} e^{2\phi_{\min}/h}.$$

Remark 1.12. Note also that our proof ensures that the constants C_k, c_k can be chosen so that C_k/c_k does not depend on $k \in \mathbb{N}^*$.

Our results can be used to describe the spectrum of the magnetic Laplacian with constant magnetic field (see Remark 1.4).

Corollary 1.13. *Assume that Ω is bounded, strictly convex and that $\partial\Omega$ is Dini-continuous. Assume also that (a)–(c) hold and that B is constant.*

Then, the k -th eigenvalue of $(-ih\nabla - A)^2$ with Dirichlet boundary conditions, denoted by $\mu_k(h)$, satisfies, for some $c, C > 0$ and h small enough,

$$Bh + ch^{-k+1} e^{2\phi_{\min}/h} \leq \mu_k(h) \leq Bh + Ch^{-k+1} e^{2\phi_{\min}/h}. \quad (1.7)$$

In particular, the first eigenvalues of the magnetic Laplacian are simple in the semiclassical limit.

1.3.2. *Relations with the literature.* Let us compare our result with the existing literature.

- i. When $B = 1$, our results improve the bound obtained by Erdős for $\lambda_1(h)$ [5, Theorem 1.1 & Proposition A.1]. In this case and when $\Omega = D(0, 1)$, the asymptotic expansion of the next eigenvalues is considered in [7, Theorem 5.1, c)]. Note that, in this case, $\phi = \frac{|x|^2 - 1}{4}$ and that Theorem 1.3 allows to recover [7, Theorem 5.1, c)] by considering radial magnetic fields.
- ii. In [7] (simply connected case) and [8] (general case), Helffer and Sundqvist have proved, under assumption (a), that

$$\lim_{h \rightarrow 0} h \ln \lambda_1(h) = 2\phi_{\min}.$$

Moreover, under the assumptions (a), (b) and (c), their theorem [7, Theorem 4.2], implies the following upper bound for the *first eigenvalue*

$$\lambda_1(h) \leq 4\Phi \det(\text{Hess}_{x_{\min}} \phi)^{\frac{1}{2}} (1 + o(1)) e^{2\phi_{\min}/h}.$$

They also establish the following lower bound:

$$\forall h > 0, \quad \lambda_1(h) \geq h^2 \lambda_1^{\text{Dir}}(\Omega) e^{2\phi_{\min}/h},$$

where $\lambda_1^{\text{Dir}}(\Omega)$ is the first eigenvalue of the corresponding magnetic Dirichlet Laplacian. This estimate is an improvement of [4, Theorem 2.1].

Corollary 1.11 is an optimal improvement in terms of the order of magnitude of the pre-factor of the exponential. It also improves the existing results by considering the low-energy eigenvalues.

- iii. The problem of estimating the spectrum of the Dirichlet-Pauli operator is closely connected to the spectral analysis of the Witten Laplacian (see for instance [7,

Remark 1.6] and the references therein). For example, in this context, the ground state energy is

$$\min_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{\int_{\Omega} |h \nabla v|^2 e^{-2\phi/h} dx}{\int_{\Omega} e^{-2\phi/h} |v|^2 dx}, \quad (1.8)$$

whereas, in the present paper, we will see that it is

$$\min_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{\int_{\Omega} |h(\partial_{x_1} + i\partial_{x_2})v|^2 e^{-2\phi/h} dx}{\int_{\Omega} e^{-2\phi/h} |v|^2 dx}. \quad (1.9)$$

Considering *real-valued* functions v in (1.9) reduces to (1.8). In this sense, (1.9) gives rise to a “less elliptic” minimization problem.

- iv. The asymptotic simplicity of the magnetic eigenvalues of Corollary 1.13 was not known before, except in the case of constant magnetic field on a disc.

1.4. Some words about the proof and its organization. Let us discuss here the main lines of our strategy.

Our investigation takes a big advantage of the deep connection between the spectral analysis of the Dirichlet-Pauli operator and the Cauchy-Riemann operators on weighted Bergman spaces directly related to the magnetic field (Section 2). This underlying structure was recently exploited in the context of magnetic WKB constructions in [1].

In Section 3, we will provide an upper bound of $\lambda_k(h)$ by inserting test functions in the min-max formulas. The following function was used in [7, Section 4]:

$$u = e^{-\phi/h} - e^{\phi/h}.$$

In the present paper, we will use more appropriate test functions that related to Bergman and Hardy spaces. We will also be led to study a one dimensional variational problem (in the normal variable of $\Gamma = \partial\Omega$) which will be crucial in the proof of the lower bound, see Lemma A.1.

In Section 4, we mainly discuss some ellipticity properties related to the magnetic Cauchy-Riemann operators. Our main result there is Theorem 4.6. These estimates will be used to prove the approximation of the eigenfunctions by their (magnetic) holomorphic component.

Section 5 is devoted to the lower bounds. We will prove a priori estimates on the eigenfunctions associated with $\lambda_k(h)$ (see Proposition 5.4) thank to the ingredients of Section 4. Via the Riemann mapping theorem and a Fourier analysis on the disc, we are able to control the Szegő projection of the eigenfunctions by the Dirichlet-Pauli energy (see Lemma 5.5). This projection can itself be related to the Hardy norm of the eigenfunctions. Ultimately, this analysis reduces to an optimization problem of quadratic forms (Hardy and Bergman quadratic forms) on a space of holomorphic functions and then on a finite dimensional space (see Section 5.3).

2. CHANGE OF GAUGE

The following result allows to remove the magnetic field up to sandwiching the Dirac operator with a suitable matrix.

Proposition 2.1. *We have*

$$e^{\sigma_3\phi/h}\sigma \cdot \mathbf{p} e^{\sigma_3\phi/h} = \sigma \cdot (\mathbf{p} - A). \quad (2.1)$$

The proof follows from the next two lemmas and Definition 1.2 (see also [13, Theorem 7.3]).

Lemma 2.2. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function and A, B be two square matrices such that $AB = -BA$. Then,*

$$Af(B) = f(-B)A.$$

Lemma 2.3 (Change of gauge for the Dirac operator). *Let $\Phi : \Omega \rightarrow \mathbb{R}$ be a regular function. We have*

$$e^{\sigma_3\Phi}(\sigma \cdot \mathbf{p})e^{\sigma_3\Phi} = \sigma \cdot (\mathbf{p} - h\nabla\Phi^\perp)$$

where $\nabla\Phi^\perp$ is defined in Definition 1.2.

Proof. By Lemma 2.2, we have for $k = 1, 2$ that

$$e^{\sigma_3\Phi}\sigma_k = \sigma_k e^{-\sigma_3\Phi}.$$

Thus, by the Leibniz rule,

$$e^{\sigma_3\Phi}(\sigma \cdot \mathbf{p})e^{\sigma_3\Phi} = (\sigma e^{-\sigma_3\Phi} \cdot \mathbf{p})e^{\sigma_3\Phi} = \sigma \cdot (\mathbf{p} - ih\sigma_3\nabla\Phi).$$

It remains to notice that $-i\sigma\sigma_3 = \sigma^\perp := (-\sigma_2, \sigma_1)$ so that

$$e^{\sigma_3\Phi}(\sigma \cdot \mathbf{p})e^{\sigma_3\Phi} = \sigma \cdot \mathbf{p} + h\sigma^\perp \cdot \nabla\Phi = \sigma \cdot \mathbf{p} - h\sigma \cdot \nabla\Phi^\perp.$$

□

We let

$$\partial_z := \frac{\partial_x - i\partial_y}{2}, \quad \partial_{\bar{z}} := \frac{\partial_x + i\partial_y}{2}.$$

We obtain then the following result.

Lemma 2.4. *Let $k \in \mathbb{N}^*$ be such that $\lambda_k(h) < 2B_0h$. Then, we have*

$$\lambda_k(h) = \inf_{\substack{V \subset H_0^1(\Omega; \mathbb{C}), \\ \dim V = k}} \sup_{v \in V \setminus \{0\}} \frac{4 \int_\Omega e^{-2\phi/h} |h\partial_{\bar{z}}v|^2 dx}{\int_\Omega |v|^2 e^{-2\phi/h} dx}. \quad (2.2)$$

We recall that $\lambda_k(h)$ is defined in (1.5).

Proof. By (1.2) and (1.5), we get since $\mathcal{L}_h^+ \geq 2B_0h$,

$$\lambda_k(h) = \inf_{\substack{V \subset H_0^1(\Omega; \mathbb{C}), \\ \dim V = k}} \sup_{u \in V \setminus \{0\}} \frac{\left\| \sigma \cdot (\mathbf{p} - A) \begin{pmatrix} u \\ 0 \end{pmatrix} \right\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2}.$$

Let $u \in H_0^1(\Omega; \mathbb{C})$ and $h > 0$. Letting $u = e^{-\phi/h}v$, we have, by Proposition 2.1,

$$\left\| \sigma \cdot (\mathbf{p} - A) \begin{pmatrix} u \\ 0 \end{pmatrix} \right\|_{L^2(\Omega)}^2 = \left\| e^{\sigma_3\phi/h} \sigma \cdot \mathbf{p} \begin{pmatrix} v \\ 0 \end{pmatrix} \right\|_{L^2(\Omega)}^2 = 4 \int_\Omega e^{-2\phi/h} |h\partial_{\bar{z}}v|^2 dx,$$

and

$$\|u\|_{L^2(\Omega)}^2 = \int_{\Omega} |v|^2 e^{-2\phi/h} dx.$$

□

3. UPPER BOUNDS

This section is devoted to the proof of the following upper bounds.

Proposition 3.1. *Assume that Ω is \mathcal{C}^2 and satisfies Assumption 1.1. For all $k \in \mathbb{N}^*$, we have*

$$\lambda_k(h) \leq C_{\text{sup}}(k) h^{-k+1} e^{2\phi_{\min}/h} (1 + o(1)), \quad (3.1)$$

where $\lambda_k(h)$ and $C_{\text{sup}}(k)$ are defined in (1.5) and in Theorem 1.3 respectively.

3.1. Choice of test functions. Let $k \in \mathbb{N}^*$ and $m \in \mathbb{N}$. By Formula (2.2), we look for a k -dimensional subspace V_h of $H_0^1(\Omega; \mathbb{C})$ such that

$$\sup_{v \in V_h \setminus \{0\}} \frac{4h^2 \int_{\Omega} |\partial_{\bar{z}} v|^2 e^{-2\phi/h} dx}{\int_{\Omega} |v|^2 e^{-2(\phi - \phi_{\min})/h} dx} \leq C_{\text{sup}}(k) h^{-k+1} (1 + o(1)).$$

Using the min-max principle, this would give (3.1). Formula (2.2) suggests to take functions of the form

$$v(x) = \chi(x) w(x),$$

where

- i. w is holomorphic on a neighborhood on Ω ,
- ii. the function $\chi : \bar{\Omega} \rightarrow [0, 1]$ is a Lipschitzian function satisfying the Dirichlet boundary condition and being 1 away from a fixed neighborhood of the boundary.

In particular, there exists $\ell_0 \in (0, \mathbf{d}(x_{\min}, \partial\Omega))$ such that

$$\chi(x) = 1, \quad \text{for all } x \in \Omega \text{ such that } \mathbf{d}(x, \partial\Omega) > \ell_0, \quad (3.2)$$

where \mathbf{d} is the usual Euclidean distance.

Remark 3.2. The most naive test functions set could be the following

$$V_h = \text{span}(\chi_h(z), \dots, \chi_h(x)(z - z_{\min})^{k-1}),$$

where $(\chi_h)_{h \in (0, 1]}$ satisfy (3.2). With this choice, one would get

$$\sup_{v \in V_h \setminus \{0\}} \frac{4h^2 \int_{\Omega} |\partial_{\bar{z}} v|^2 e^{-2\phi/h} dx}{\int_{\Omega} |v|^2 e^{-2(\phi - \phi_{\min})/h} dx} \leq \widetilde{C}_{\text{sup}}(k) h^{-k+1} (1 + o(1)),$$

where

$$\widetilde{C}_{\text{sup}}(k) = 2 \left(\frac{N_{\mathcal{H}}((z - z_{\min})^{k-1})}{\text{dist}_{\mathcal{B}}(z^{k-1}, \mathcal{P}_{k-2})} \right)^2 \geq C_{\text{sup}}(k).$$

Note however that in the radial case $\widetilde{C}_{\text{sup}}(k) = C_{\text{sup}}(k)$. We will rather use functions compatible with the Hardy space structure to get the bound of Proposition 3.1, as explained below.

Notation 4. Let us denote by $(P_n)_{n \in \mathbb{N}}$ the $N_{\mathcal{B}}$ -orthogonal family such that $P_n(Z) = Z^n + \sum_{j=0}^{n-1} b_{n,j} Z^j$. Since P_n is $N_{\mathcal{B}}$ -orthogonal to \mathcal{P}_{n-1} , we have

$$\begin{aligned} \text{dist}_{\mathcal{B}}(Z^n, \mathcal{P}_{n-1}) &= \text{dist}_{\mathcal{B}}(P_n, \mathcal{P}_{n-1}) = \inf\{N_{\mathcal{B}}(P_n - Q), Q \in \mathcal{P}_{n-1}\} \\ &= \inf\{\sqrt{N_{\mathcal{B}}(P_n)^2 + N_{\mathcal{B}}(Q)^2}, Q \in \mathcal{P}_{n-1}\} = N_{\mathcal{B}}(P_n), \text{ for } n \in \mathbb{N}. \end{aligned} \quad (3.3)$$

Let $Q_n \in \mathcal{H}_k^2(\Omega)$ be the unique function such that

$$\text{dist}_{\mathcal{H}}((z - z_{\min})^n, \mathcal{H}_k^2(\Omega)) = N_{\mathcal{H}}((z - z_{\min})^n - Q_n(z)),$$

for $n \in \{0, \dots, k-1\}$, (see Remark 1.6). We recall that $N_{\mathcal{B}}$, $N_{\mathcal{H}}$, \mathcal{P}_{n-1} , and $\mathcal{H}_k^2(\Omega)$ are defined in Section 1.3.1.

Lemma 3.3. *For all $n \in \{0, \dots, k-1\}$, there exists a sequence $(Q_{n,m})_{m \in \mathbb{N}} \subset \mathcal{H}_k^2(\Omega) \cap W^{1,\infty}(\Omega)$ that converges to Q_n in $\mathcal{H}^2(\Omega)$.*

Proof. We can write $Q_n(z) = (z - z_{\min})^{k-1} \tilde{Q}_n(z)$, with $\tilde{Q}_n \in \mathcal{H}^2(\Omega)$. By Lemma B.1, there exists a sequence $(\tilde{Q}_{n,m})_{m \in \mathbb{N}} \subset \mathcal{H}^2(\Omega) \cap W^{1,\infty}(\Omega)$ converging to \tilde{Q}_n in $\mathcal{H}^2(\Omega)$. We have

$$N_{\mathcal{H}}((z - z_{\min})^{k-1}(\tilde{Q}_{n,m} - \tilde{Q}_n)) \leq \left\| (z - z_{\min})^{k-1} \right\|_{L^\infty(\Gamma)} N_{\mathcal{H}}(\tilde{Q}_{n,m} - \tilde{Q}_n),$$

so that the sequence $(Q_{n,m})_{m \in \mathbb{N}} = ((z - z_{\min})^{k-1} \tilde{Q}_{n,m})_{m \in \mathbb{N}} \subset \mathcal{H}_k^2(\Omega)$ converges to Q_n in $\mathcal{H}^2(\Omega)$. \square

Let us now define the k -dimensional vector space $V_{h,k,\text{sup}}$ by

$$\begin{aligned} V_{h,k,\text{sup}} &= \text{span}(w_{0,h}, \dots, w_{k-1,h}), \\ w_{n,h}(z) &= h^{-\frac{1}{2}} P_n \left(\frac{z - z_{\min}}{h^{1/2}} \right) - h^{-\frac{1+n}{2}} Q_{n,m}(z), \text{ for } n \in \{0, \dots, k-1\}. \end{aligned} \quad (3.4)$$

At the end of the proof, m will be sent to $+\infty$. Note that we will not need the uniformity of the semiclassical estimates with respect to m . That is why the parameter m does not appear in our notations.

Remark 3.4. Consider

$$\tilde{w}_{n,h}(z) = h^{-\frac{1}{2}} P_n \left(\frac{z - z_{\min}}{h^{1/2}} \right) - h^{-\frac{1+n}{2}} Q_n(z).$$

Since Q_n belongs to $\mathcal{H}^2(\Omega) \not\subset H^1(\Omega; \mathbb{C})$, the functions $\tilde{w}_{n,h} : x \mapsto \tilde{w}_{n,h}(x_1 + ix_2)$ and $\chi \tilde{w}_{n,h}$ do not belong necessarily to $H^1(\Omega; \mathbb{C})$ and $H_0^1(\Omega; \mathbb{C})$ respectively. That is why we introduced $Q_{n,m}$. Note that to get $H_0^1(\Omega; \mathbb{C})$ test functions, it suffices to impose that χ is compactly supported in Ω . With this strategy, our proof can be adapted to the case where Ω is non necessarily simply connected.

3.2. Estimate of the L^2 -norm. The aim of this section is to prove the following estimate.

Lemma 3.5. *Let $h \in (0, 1]$, $v_h = \chi \sum_{j=0}^{k-1} c_j w_{j,h}$ with $c_0, \dots, c_{k-1} \in \mathbb{C}$, χ satisfying (3.2) and $(w_{j,h})_{j \in \{0, \dots, k-1\}}$ defined in (3.4). We have*

$$\int_{\Omega} |v_h|^2 e^{-2(\phi(x) - \phi_{\min})/h} dx = (1 + o(1)) \sum_{j=0}^{k-1} |c_j|^2 N_{\mathcal{B}}(P_j)^2, \quad (3.5)$$

where $N_{\mathcal{B}}$ is defined in Notation 3 and $o(1)$ does not depend on $c = (c_0, \dots, c_{k-1})$ and χ .

Proof. Let $\alpha \in (\frac{1}{3}, \frac{1}{2})$, $n, n' \in \{0, \dots, k-1\}$.

- i. Let us estimate the weighted scalar products related to P_n for the weighted L^2 -norm. Using the Taylor expansion of ϕ at x_{\min} , we get, for all $x \in D(x_{\min}, h^\alpha)$,

$$\frac{\phi(x) - \phi_{\min}}{h} = \frac{1}{2h} \text{Hess}_{x_{\min}} \phi(x - x_{\min}, x - x_{\min}) + \mathcal{O}(h^{3\alpha-1}). \quad (3.6)$$

By using the change of coordinates

$$\Lambda_h : x \mapsto \frac{x - x_{\min}}{h^{1/2}}, \quad (3.7)$$

we find

$$\begin{aligned} & \int_{D(x_{\min}, h^\alpha)} h^{-1} P_n P_{n'} \left(\frac{x_1 + ix_2 - z_{\min}}{h^{1/2}} \right) e^{-2(\phi(x) - \phi_{\min})/h} dx \\ &= (1 + \mathcal{O}(h^{3\alpha-1})) \int_{D(x_{\min}, h^\alpha)} h^{-1} P_n P_{n'} \left(\frac{x_1 + ix_2 - z_{\min}}{h^{1/2}} \right) e^{-\frac{1}{h} \text{Hess}_{x_{\min}} \phi(x - x_{\min}, x - x_{\min})} dx \\ &= (1 + \mathcal{O}(h^{3\alpha-1})) \int_{D(0, h^{\alpha-1/2})} P_n P_{n'}(y) e^{-\text{Hess}_{x_{\min}} \phi(y, y)} dy \\ &= (1 + \mathcal{O}(h^{3\alpha-1})) \left(\langle P_n, P_{n'} \rangle_{\mathcal{B}} - \int_{\mathbb{C} \setminus D(0, h^{\alpha-1/2})} P_n P_{n'}(y) e^{-\text{Hess}_{x_{\min}} \phi(y, y)} dy \right) \\ &= (1 + \mathcal{O}(h^{3\alpha-1})) \langle P_n, P_{n'} \rangle_{\mathcal{B}} + \mathcal{O}(h^\infty), \end{aligned} \quad (3.8)$$

where the last equality follows from Assumption (c) in Theorem 1.3, and where $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ is the scalar product associated with $N_{\mathcal{B}}$.

We recall Assumptions (b) and (c). Then, by the Taylor expansion of ϕ at x_{\min} , we deduce that

$$\inf_{\Omega \setminus D(x_{\min}, h^\alpha)} \phi \geq \phi_{\min} + \frac{\lambda_{\min}}{2} h^{2\alpha} (1 + \mathcal{O}(h^\alpha)), \quad (3.9)$$

where $\lambda_{\min} > 0$ is the lowest eigenvalue of $\text{Hess}_{x_{\min}} \phi$. Since P_n is of degree n , there exists $C > 0$ such that

$$\sup_{x \in \Omega} \left| h^{-\frac{1}{2}} P_n \left(\frac{x_1 + ix_2 - z_{\min}}{h^{1/2}} \right) \right| \leq Ch^{-\frac{n+1}{2}}.$$

Using this with (3.9), we get

$$\begin{aligned} & \left| \int_{\Omega \setminus D(x_{\min}, h^\alpha)} h^{-1} \chi^2 P_n P_{n'} \left(\frac{x_1 + ix_2 - z_{\min}}{h^{1/2}} \right) e^{-2(\phi(x) - \phi_{\min})/h} dx \right| \\ & \leq Ch^{-\frac{n+1}{2}} h^{-\frac{n'+1}{2}} e^{-\lambda_{\min} h^{2\alpha-1} (1 + \mathcal{O}(h^\alpha))} = \mathcal{O}(h^\infty). \end{aligned} \quad (3.10)$$

From (3.8) and (3.10), we find

$$\begin{aligned} \int_{\Omega} h^{-1} \chi^2 P_n P_{n'} \left(\frac{x_1 + ix_2 - z_{\min}}{h^{1/2}} \right) e^{-2(\phi(x) - \phi_{\min})/h} dx \\ = (1 + \mathcal{O}(h^{3\alpha-1})) \langle P_n, P_{n'} \rangle_{\mathcal{B}} + \mathcal{O}(h^{\infty}). \end{aligned} \quad (3.11)$$

ii. Let us now deal with the weighted scalar products related to the $Q_{n,m}$. Let $u \in \mathcal{H}^2(\Omega)$ and $z_0 \in D(z_{\min}, h^{\alpha})$. By the Cauchy formula (see [3, Theorem 10.4]) and the Cauchy-Schwarz inequality,

$$\begin{aligned} |u^{(k)}(z_0)| &= \frac{k!}{2\pi} \left| \int_{\partial\Omega} \frac{u(z)}{(z - z_0)^{k+1}} dz \right| \\ &\leq \frac{k!}{2\pi \sqrt{\min_{\partial\Omega} \partial_{\mathbf{n}} \phi}} N_{\mathcal{H}}(u) \left(\int_{\partial\Omega} \frac{|dz|}{|z - z_0|^{2(k+1)}} \right)^{1/2} \\ &\leq \frac{k!}{2\pi \sqrt{\min_{\partial\Omega} \partial_{\mathbf{n}} \phi}} N_{\mathcal{H}}(u) \left(\int_{\partial\Omega} \frac{|dz|}{(|z - z_{\min}| - h^{\alpha})^{2(k+1)}} \right)^{1/2} \leq CN_{\mathcal{H}}(u). \end{aligned} \quad (3.12)$$

With the Taylor formula for $u = Q_{n,m}$ at z_{\min} , this gives

$$|Q_{n,m}(z_0)| \leq C |z_0 - z_{\min}|^k N_{\mathcal{H}}(Q_{n,m}).$$

Using (3.6), this implies

$$\begin{aligned} \int_{D(x_{\min}, h^{\alpha})} |h^{-\frac{1+n}{2}} Q_{n,m}(x_1 + ix_2)|^2 e^{-2(\phi(x) - \phi_{\min})/h} dx \\ \leq Ch^{-(1+n)} \int_{D(x_{\min}, h^{\alpha})} |(x_1 + ix_2) - z_{\min}|^{2k} e^{-2(\phi(x) - \phi_{\min})/h} dx \\ \leq Ch^{k-n} N_{\mathcal{B}}(z^k)^2 \leq Ch. \end{aligned} \quad (3.13)$$

Using (3.9) and $Q_{n,m} \in W^{1,\infty}(\Omega) \subset L^2(\Omega)$, we get

$$\begin{aligned} \int_{\Omega \setminus D(x_{\min}, h^{\alpha})} |h^{-\frac{1+n}{2}} \chi Q_{n,m}(x_1 + ix_2)|^2 e^{-2(\phi(x) - \phi_{\min})/h} dx \\ \leq Ch^{-(n+1)} \|Q_{n,m}\|_{L^2(\Omega)}^2 e^{-\lambda_{\min} h^{2\alpha-1} (1 + \mathcal{O}(h^{\alpha}))} = \mathcal{O}(h^{\infty}). \end{aligned} \quad (3.14)$$

With (3.13) and (3.14), we deduce

$$\int_{\Omega} |h^{-\frac{1+n}{2}} \chi Q_{n,m}(x_1 + ix_2)|^2 e^{-2(\phi(x) - \phi_{\min})/h} dx = \mathcal{O}(h). \quad (3.15)$$

With the Cauchy-Schwarz inequality, and (3.15),

$$\int_{\Omega} \chi^2 h^{-\frac{1+n}{2}} Q_{n,m}(x_1 + ix_2) h^{-\frac{1+n'}{2}} \overline{Q_{n',m}(x_1 + ix_2)} e^{-2(\phi(x) - \phi_{\min})/h} dx = \mathcal{O}(h). \quad (3.16)$$

iii. Let us now consider the scalar products involving the P_n and the $Q_{n',m}$. Using (3.11), (3.16), and the Cauchy-Schwarz inequality, we get

$$\int_{\Omega} \chi^2 h^{-\frac{1}{2}} P_n \left(\frac{x_1 + ix_2 - z_{\min}}{h^{1/2}} \right) h^{-\frac{1+n'}{2}} \overline{Q_{n',m}(x_1 + ix_2)} e^{-2(\phi(x) - \phi_{\min})/h} dx = \mathcal{O}(h^{1/2}). \quad (3.17)$$

The conclusion follows by expanding the square in the left-hand-side of (3.5) and by using (3.11), (3.16), (3.17). \square

Remark 3.6. From Lemma 3.5, we deduce that the vectors $\{\chi w_{j,h}, 0 \leq j \leq k-1\}$ are linearly independent for h small enough.

3.3. Estimate of the energy. The aim of this section is to bound from above the energy on an appropriate subspace.

Lemma 3.7. *There exists a family of functions $(\chi_h)_{h \in (0,1]}$ which satisfy (3.2) and such that, for all $w_h \in V_{h,k,\text{sup}}$,*

$$\begin{aligned} 4 \int_{\Omega} h^2 e^{-2\phi/h} |\partial_{\bar{z}}(\chi_h w_h)|^2 dx \\ \leq 2h^{1-k} |c_{k-1}|^2 N_{\mathcal{H}} \left((z - z_{\min})^{k-1} - Q_{k-1,m} \right) + o(1) h^{1-k} \|c\|_{\ell^2}^2. \end{aligned}$$

Here, $o(1)$ does not depend on w_h .

Proof. Let χ be any function satisfying (3.2). We have

$$4 \int_{\Omega} h^2 e^{-2\phi/h} |\partial_{\bar{z}} \chi w_h|^2 dx = h^2 \int_{\Omega} |w_h|^2 e^{-2\phi/h} |\nabla \chi|^2 dx = h^2 \int_{\text{supp} \nabla \chi} |w_h|^2 e^{-2\phi/h} |\nabla \chi|^2 dx,$$

where we have used that $|\nabla \chi|^2 = 4|\partial_{\bar{z}} \chi|^2$ since χ is real.

The proof is now divided into three steps. First, we introduce tubular coordinates near the boundary, then we make an explicit choice of χ , and finally, we control the remainders.

- i. We only need to define χ in a neighborhood of $\Gamma = \partial\Omega$. To do this, we use the tubular coordinates given by the map

$$\eta : \begin{array}{l} \mathbb{R}/(|\Gamma|\mathbb{Z}) \times (0, t_0) \rightarrow \Omega \\ (s, t) \mapsto \gamma(s) - t\mathbf{n}(s) \end{array}$$

for t_0 small enough, γ being a parametrization of Γ with $|\gamma'(s)| = 1$ for all s , and $\mathbf{n}(s)$ the unit outward pointing normal at point $\gamma(s)$ (see e.g. [6, §F]). We let

$$\eta^{-1}(x) = (s(x), t(x)), \text{ for all } x \in \eta \left(\mathbb{R}/(|\Gamma|\mathbb{Z}) \times (0, t_0) \right),$$

the inverse map to η . We let, for all $x \in \Omega$,

$$\chi(x) = \begin{cases} \rho(s(x), \mathbf{d}(x, \partial\Omega)) & \text{if } \mathbf{d}(x, \partial\Omega) \leq \varepsilon, \\ 1 & \text{otherwise.} \end{cases}$$

The parameter $\varepsilon > 0$ and the function ρ are to be determined. We assume that $\rho(s, 0) = 0$ and $\rho(s, t) = 1$ when $t \geq \varepsilon$. We will choose $\varepsilon = o(h^{\frac{1}{2}})$.

Since the metric induced by the change of variable is the Euclidean metric modulo $\mathcal{O}(\varepsilon)$, we get

$$\begin{aligned} h^2 \int_{\text{supp} \nabla \chi} |w_h|^2 e^{-2\phi/h} |\nabla \chi|^2 dx \\ \leq (1 + \mathcal{O}(\varepsilon)) h^2 \int_{\Gamma} \int_0^{\varepsilon} |\tilde{w}_h|^2 e^{-2\tilde{\phi}(s,t)/h} (|\partial_t \rho|^2 + |\partial_s \rho|^2) ds dt, \end{aligned}$$

where $\tilde{w}_h = w_h \circ \eta$ and $\tilde{\phi} = \phi \circ \eta$. Thus, by using the Taylor expansion of $\tilde{\phi}$ at $t = 0$, we get uniformly in $s \in \Gamma$,

$$\tilde{\phi}(s, t) = t\partial_t\tilde{\phi}(s, 0) + \mathcal{O}(t^2) = -t\partial_{\mathbf{n}}\phi(s, 0) + \mathcal{O}(\varepsilon^2),$$

and

$$\begin{aligned} h^2 \int_{\text{supp}\nabla\chi} |w_h|^2 e^{-2\phi/h} |\nabla\chi|^2 dx \\ \leq (1 + \mathcal{O}(\varepsilon + \varepsilon^2/h)) h^2 \int_{\Gamma} \int_0^\varepsilon |\tilde{w}_h|^2 e^{2t\partial_{\mathbf{n}}\phi(s)/h} (|\partial_t\rho|^2 + |\partial_s\rho|^2) ds dt. \end{aligned}$$

Since $Q_{n,m} \in W^{1,\infty}(\Omega)$, we have $\partial_t\tilde{Q}_{n,m} \circ \eta \in L^\infty(\Gamma \times (0, \varepsilon))$ and by using the Taylor expansion of \tilde{w} near $t = 0$, we get

$$\begin{aligned} \tilde{w}_h(s, t) &= \left(\sum_{j=0}^{k-1} c_k w_{n,h} \right) \circ \eta(s, t) = \tilde{w}_h(s, 0) + \int_0^t \partial_t \tilde{w}_h(s, t') dt' \\ &= \tilde{w}_h(s, 0) + \mathcal{O}(\varepsilon) \|c_h\|_{\ell^2}, \end{aligned}$$

where

$$c_h = (h^{-\frac{1}{2}}c_0, \dots, h^{-\frac{k}{2}}c_{k-1}), \quad (3.18)$$

and $\|\cdot\|_{\ell^2}$ is the canonical Euclidian norm on \mathbb{C}^k . Then,

$$\begin{aligned} h^2 \int_{\text{supp}\nabla\chi} |w_h|^2 e^{-2\phi/h} |\nabla\chi|^2 dx \\ \leq (1 + \mathcal{O}(\varepsilon + \varepsilon^2/h)) h^2 \int_{\Gamma} |\tilde{w}_h(s, 0)|^2 \int_0^\varepsilon e^{2t\partial_{\mathbf{n}}\phi(s)/h} (|\partial_t\rho|^2 + |\partial_s\rho|^2) ds dt \\ + Ch^2\varepsilon \|c_h\|_{\ell^2}^2 \int_{\Gamma} \int_0^\varepsilon e^{2t\partial_{\mathbf{n}}\phi(s)/h} (|\partial_t\rho|^2 + |\partial_s\rho|^2) ds dt. \quad (3.19) \end{aligned}$$

ii. Lemma A.1 with $\alpha = 2\partial_{\mathbf{n}}\phi/h > 0$ suggests to consider the trial state defined, for $t \leq \varepsilon$, by

$$\rho(s, t) = \frac{1 - e^{-2t\partial_{\mathbf{n}}\phi(s)/h}}{1 - e^{-2\varepsilon\partial_{\mathbf{n}}\phi(s)/h}},$$

and by 1 otherwise. By Lemma A.1, we get

$$\int_0^\varepsilon e^{2t\partial_{\mathbf{n}}\phi/h} |\partial_t\rho|^2 dt = \frac{2\partial_{\mathbf{n}}\phi/h}{1 - e^{-\varepsilon 2\partial_{\mathbf{n}}\phi/h}},$$

and

$$\begin{aligned} \int_0^\varepsilon e^{2t\partial_{\mathbf{n}}\phi/h} |\partial_s\rho|^2 dt &= |\partial_s\alpha|^2 \int_0^\varepsilon e^{2t\partial_{\mathbf{n}}\phi/h} |\partial_\alpha\rho_{\alpha,\varepsilon}|^2 dt \leq Ch^{-2} (\alpha^{-3} + e^{-\alpha\varepsilon}\varepsilon^2\alpha^{-1}) \\ &\leq C \left(h + e^{-\varepsilon 2\partial_{\mathbf{n}}\phi/h} \varepsilon^2 h^{-1} \right). \end{aligned}$$

We can choose $\varepsilon = h|\log h|$ so that (3.19) becomes

$$h^2 \int_{\text{supp}\nabla\chi} |w_h|^2 e^{-2\phi/h} |\nabla\chi|^2 dx \leq (1 + o(1))h \left(\int_{\Gamma} 2\partial_{\mathbf{n}}\phi |\tilde{w}_h(s, 0)|^2 ds + C\varepsilon \|c_h\|_{\ell^2}^2 \right). \quad (3.20)$$

iii. Let us consider, for all $h \geq 0$,

$$N_h : c \in \mathbb{C}^k \mapsto \left(\int_{\Gamma} \partial_{\mathbf{n}} \phi \left| \sum_{j=0}^{k-1} c_j h^{\frac{1+j}{2}} \tilde{w}_{j,h}(s, 0) \right|^2 ds \right)^{1/2},$$

where we recall that

$$w_{j,h}(z) = h^{-\frac{1}{2}} P_j \left(\frac{z - z_{\min}}{h^{\frac{1}{2}}} \right) - h^{-\frac{j+1}{2}} Q_{j,m}(z).$$

The application $\mathbb{C}^k \times [0, 1] \ni (c, h) \mapsto N_h(c)$ is well defined and continuous (since the degree of P_j is j). Note, in particular, that

$$N_0(c) = \left(\int_{\Gamma} \partial_{\mathbf{n}} \phi \left| \sum_{j=0}^{k-1} c_j [(z - z_{\min})^j - Q_{j,m}(z)] \right|^2 ds \right)^{1/2}.$$

Notice that

$$N_h(c_h)^2 = \int_{\Gamma} \partial_{\mathbf{n}} \phi |\tilde{w}_h(s, 0)|^2 ds = N_{\mathcal{H}}^2(w_h), \quad (3.21)$$

where c_h is defined in (3.18). Since $N_{\mathcal{H}}$ is a norm, and recalling Remark 3.6, we see that the application N_h is a norm when $h \in (0, h_0]$. N_0 is also a norm (as we can see by using the Hardy norm and $Q_{j,m} \in \mathcal{H}_k^2(\Omega)$).

Let us define

$$C_0 = \min_{\substack{h \in [0, h_0] \\ \|c\|_{\ell^2} = 1}} N_h(c) > 0.$$

so that, for all $h \in [0, h_0]$, and all $c \in \mathbb{C}^k$,

$$C_0 \|c\|_{\ell^2} \leq N_h(c). \quad (3.22)$$

Using (3.20), (3.21), and replacing c by c_h in (3.22), we conclude that

$$h^2 \int_{\text{supp} \nabla \chi} |w_h|^2 e^{-2\phi/h} |\nabla \chi|^2 dx \leq 2(1 + o(1)) h N_{\mathcal{H}}(w_h)^2.$$

Let us now estimate $N_{\mathcal{H}}(w_h)$. From the triangle inequality, we get

$$N_{\mathcal{H}}(w_h) \leq |c_{k-1}| N_{\mathcal{H}}(w_{k-1,h}) + \sum_{j=0}^{k-2} |c_j| N_{\mathcal{H}}(w_{j,h}).$$

Then, from degree considerations and the triangle inequality, we get, for $1 \leq j \leq k-2$,

$$N_{\mathcal{H}}(w_{j,h}) = \mathcal{O} \left(h^{\frac{1-k}{2}} \right),$$

and

$$N_{\mathcal{H}}(w_{k-1,h}) = (1 + o(1)) h^{-\frac{k}{2}} N_{\mathcal{H}} \left((z - z_{\min})^{k-1} - Q_{k-1,m} \right).$$

Then,

$$N_{\mathcal{H}}(w_h)^2 \leq |c_{k-1}|^2 h^{-k} N_{\mathcal{H}} \left((z - z_{\min})^{k-1} - Q_{k-1,m} \right)^2 + o(h^{-k}) \|c\|_{\ell^2}^2.$$

This ends the proof. \square

3.4. Proof of Proposition 3.1. Let us define $\tilde{V}_{h,k,\text{sup}} = \{\chi_h w_h, w_h \in V_{h,k,\text{sup}}\}$, where $V_{h,k,\text{sup}}$ is defined in (3.4) and χ_h in Lemma 3.7. By Lemmas 3.5 and 3.7, we get

$$\frac{4 \int_{\Omega} h^2 e^{-2\phi/h} |\partial_{\bar{z}}(w_h \chi_h)|^2 dx}{\int_{\Omega} |w_h \chi_h|^2 e^{-2(\phi-\phi_{\min})/h} dx} \leq 2h^{1-k} \frac{|c_{k-1}|^2 N_{\mathcal{H}} \left((z - z_{\min})^{k-1} - Q_{k-1,m} \right)^2}{\sum_{j=0}^{k-1} |c_j|^2 N_{\mathcal{B}}(P_j)^2} + o(h^{1-k}),$$

for all $w_h = \sum_{j=0}^{k-1} c_j w_{j,h} \in V_{h,k,\text{sup}}$ with $c \in \mathbb{C}^k \setminus \{0\}$. From the min-max principle², it follows

$$\lambda_k(h) \leq 2h^{1-k} N_{\mathcal{H}} \left((z - z_{\min})^{k-1} - Q_{k-1,m} \right)^2 \sup_{c \in \mathbb{C}^k \setminus \{0\}} \frac{|c_{k-1}|^2}{\sum_{j=0}^{k-1} |c_j|^2 N_{\mathcal{B}}(P_j)^2} e^{2\phi_{\min}/h} + o(h^{1-k}).$$

Since

$$\sup_{c \in \mathbb{C}^k \setminus \{0\}} \frac{|c_{k-1}|^2}{\sum_{j=0}^{k-1} |c_j|^2 N_{\mathcal{B}}(P_j)^2} = N_{\mathcal{B}}(P_{k-1})^{-2},$$

we deduce

$$\limsup_{h \rightarrow 0} h^{k-1} e^{-2\phi_{\min}/h} \lambda_k(h) \leq 2 \left(\frac{N_{\mathcal{H}} \left((z - z_{\min})^{k-1} - Q_{k-1,m} \right)}{\text{dist}_{\mathcal{B}}(z^{k-1}, \mathcal{P}_{k-2})} \right)^2.$$

Taking the limit $m \rightarrow +\infty$, it follows

$$\limsup_{h \rightarrow 0} h^{k-1} e^{-2\phi_{\min}/h} \lambda_k(h) \leq C_{\text{sup}}(k).$$

3.5. Computation of $C_{\text{sup}}(k)$ in the radial case. Let $k \in \mathbb{N}^*$. Let us assume that Ω is the disk of radius R centered at 0, and that B is radial. In this case $x_{\min} = 0$, $\partial_{\mathbf{n}}\phi$ is constant and $\text{Hess}_{x_{\min}}\phi = \lambda \text{Id}$ for some $\lambda > 0$.

Thus,

$$\text{dist}_{\mathcal{H}}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega)) = \text{dist}_{\mathcal{H}}(z^{k-1}, \mathcal{H}_k^2(\Omega)) = N_{\mathcal{H}}(z^{k-1})^2 = 2\pi \partial_{\mathbf{n}}\phi R^{2k-1},$$

and we notice that $P_n(z) = z^n$ (see Notation 4) so that

$$\begin{aligned} \text{dist}_{\mathcal{B}}(z^{k-1}, \mathcal{P}_{k-2}) &= N_{\mathcal{B}}(P_{k-1})^2 \\ &= \int_{\mathbb{R}^2} |y|^{2(k-1)} e^{-\text{Hess}_{x_{\min}}\phi(y,y)} dy = 2\pi \int_0^{+\infty} \rho^{2k-1} e^{-\lambda\rho^2} d\rho \\ &= \frac{2\pi}{\lambda^k} \int_0^{+\infty} \rho^{2k-1} e^{-\rho^2} d\rho = \frac{\pi\Gamma(k)}{\lambda^k} = \frac{\pi(k-1)!}{\lambda^k}, \end{aligned}$$

We get

$$C_{\text{sup}}(k) = \frac{4\lambda^k \partial_{\mathbf{n}}\phi R^{2k-1}}{(k-1)!} = \frac{4\Phi\lambda^k R^{2k-2}}{(k-1)!}.$$

²By Remark 3.6, $\dim \tilde{V}_{h,k,\text{sup}} = k$ for h small enough.

4. ON THE MAGNETIC CAUCHY-RIEMANN OPERATORS

In this section, U will denote an open bounded subset of \mathbb{R}^2 . It will be either Ω itself, or a smaller open set.

As we already observed (see (1.2)), the Dirichlet-Pauli operator, considered only as a differential operator, is the square of the magnetic Dirac operator $\sigma \cdot (\mathbf{p} - \mathbf{A})$. It can be written as

$$\sigma \cdot (\mathbf{p} - A) = \begin{pmatrix} 0 & d_{h,A} \\ d_{h,A}^\times & 0 \end{pmatrix} \quad (4.1)$$

where $d_{h,A}$ and $d_{h,A}^\times$ are the magnetic Cauchy-Riemann operators:

$$d_{h,A} = -2ih\partial_z - A_1 + iA_2, \quad d_{h,A}^\times = -2ih\partial_{\bar{z}} - A_1 - iA_2.$$

Let $(d_{h,A}, \text{Dom}(d_{h,A}))$ be the operator acting as $d_{h,A}$ on $\text{Dom}(d_{h,A}) = H_0^1(U; \mathbb{C})$.

4.1. Properties of $d_{1,0}$ and $d_{1,0}^*$. In this part, we study the operators $d_{h,A}$ and $d_{h,A}^*$ in the non-magnetic case $B = 0$ with $h = 1$ in order to get describe their properties in this simplified setting. Various aspects of this section can be related to the spectral analysis of the “zig-zag” operator (see [12]). The next section will be related to the magnetic case that is needed in our study.

Lemma 4.1. *Assume that U is of class \mathcal{C}^2 . The following properties hold.*

- (a) *The operator $(d_{1,0}, \text{Dom}(d_{1,0}))$ is closed with closed range.*
- (b) *The domain of $d_{1,0}^*$ is given by*

$$\begin{aligned} \text{Dom}(d_{1,0}^*) &= \{u \in L^2(U; \mathbb{C}), \partial_{\bar{z}}u \in L^2(U; \mathbb{C})\} \\ &= \{u \in L^2(U; \mathbb{C}), \partial_{\bar{z}}u = 0\} + H^1(U; \mathbb{C}), \end{aligned}$$

and $d_{1,0}^*$ acts as $d_{1,0}^\times$. In particular,

$$\ker(d_{1,0}^*) = \{u \in L^2(U; \mathbb{C}), \partial_{\bar{z}}u = 0\}.$$

- (c) *We have*

$$\ker(d_{1,0}^*)^\perp \cap \text{Dom}(d_{1,0}^*) = \{d_{1,0}w, w \in H_0^1(U; \mathbb{C}) \cap H^2(U; \mathbb{C})\} \subset H^1(U; \mathbb{C}),$$

and there exists $C > 0$ such that, for all $v \in \ker(d_{1,0}^*)^\perp \cap \text{Dom}(d_{1,0}^*)$,

$$\|v\|_{H^1(U)} \leq C \left\| d_{1,0}^* v \right\|_{L^2(U)}.$$

Proof. Let $u \in \text{Dom}(d_{1,0}) = H_0^1(U; \mathbb{C})$. One easily checks that

$$\|d_{1,0}u\|_{L^2(U)}^2 = \|\nabla u\|_{L^2(U)}^2.$$

Hence, the Poincaré inequality ensures that $(d_{1,0}, \text{Dom}(d_{1,0}))$ is a closed operator with closed range. Then, by definition of the domain of the adjoint,

$$\text{Dom}(d_{1,0}^*) \subset \{u \in L^2(U; \mathbb{C}), \partial_{\bar{z}}u \in L^2(U; \mathbb{C})\}.$$

Conversely, if $v \in \{u \in L^2(U; \mathbb{C}), \partial_{\bar{z}}u \in L^2(U; \mathbb{C})\}$, we have, for all $w \in \mathcal{C}_0^\infty(U)$,

$$\langle v, -2i\partial_z w \rangle_{L^2(U)} = \langle -2i\partial_{\bar{z}}v, w \rangle_{L^2(U)}.$$

By density, this equality can be extended to $w \in H_0^1(U; \mathbb{C})$. This shows, by definition, that $v \in \text{Dom}(d_{1,0}^*)$ and $d_{1,0}^*v = -2i\partial_{\bar{z}}v$.

Moreover, we have

$$\begin{aligned} \ker(d_{1,0}^*)^\perp \cap \text{Dom}(d_{1,0}^*) &= \text{ran}(d_{1,0}) \cap \text{Dom}(d_{1,0}^*) \\ &= \{d_{1,0}w, w \in H_0^1(U; \mathbb{C}) \text{ and } -2i\partial_{\bar{z}}(d_{1,0}w) = -\Delta w \in L^2(U; \mathbb{C})\} \\ &= \{d_{1,0}w, w \in H_0^1(U; \mathbb{C}) \cap H^2(U; \mathbb{C})\} \subset H^1(U; \mathbb{C}), \end{aligned}$$

where the last equality follows from the elliptic regularity of the Laplacian. In particular, we get, for all $w \in H_0^1(U; \mathbb{C}) \cap H^2(U; \mathbb{C})$,

$$\|w\|_{H^2(U)} \leq C\|\Delta w\|_{L^2(U)}.$$

Now, take $v \in \ker(d_{1,0}^*)^\perp \cap \text{Dom}(d_{1,0}^*)$. We can write $v = d_{1,0}w$ with $w \in H^2(U; \mathbb{C}) \cap H_0^1(U; \mathbb{C})$. We have $d_{1,0}^*v = -\Delta w$ so that

$$\|v\|_{H^1(U)} \leq C\|d_{1,0}^*v\|_{L^2(U)}.$$

□

4.2. Properties of $d_{h,A}$ and $d_{h,A}^*$. Let us introduce some notations related with the Riemann mapping theorem.

In the following, we gather the main properties related with $d_{h,A}$ and $d_{h,A}^*$. We will use the following lemma.

Lemma 4.2. *For all $u \in \mathcal{C}_0^\infty(U; \mathbb{C})$, we have*

$$\begin{aligned} \|d_{h,A}u\|_{L^2(U)}^2 &= \|(\mathbf{p}-A)u\|_{L^2(U)}^2 + h \int_U B|u|^2 dx \\ \|d_{h,A}^\times u\|_{L^2(U)}^2 &= \|(\mathbf{p}-A)u\|_{L^2(U)}^2 - h \int_U B|u|^2 dx \end{aligned}$$

These formulas can be extended to $u \in H_0^1(U; \mathbb{C})$.

Proof. It follows from integrations by parts and the fact that $d_{h,A}d_{h,A}^\times = (\mathbf{p}-A)^2 - hB$ and $d_{h,A}^\times d_{h,A} = (\mathbf{p}-A)^2 + hB$. The extension to $u \in H_0^1(U; \mathbb{C})$ follows by density. □

Remark 4.3. From Lemma 4.2, we deduce³, that for all $u \in H_0^1(U; \mathbb{C})$,

$$\|(\mathbf{p}-A)u\|_{L^2(U)}^2 \geq \int_U hB|u|^2 dx.$$

Proposition 4.4. *Assume that U is of class \mathcal{C}^2 .*

- (a) *The operator $(d_{h,A}, \text{Dom}(d_{h,A}))$ is closed with closed range.*
- (b) *The adjoint $(d_{h,A}^*, \text{Dom}(d_{h,A}^*))$ acts as $d_{h,A}^\times$ on*

$$\text{Dom}(d_{h,A}^*) = \{u \in L^2(U) : \partial_{\bar{z}}u \in L^2(U)\} = \ker(d_{h,A}^*) + H^1(U; \mathbb{C})$$

and

$$\ker(d_{h,A}^*) = \{e^{-\phi/h}v, v \in L^2(U), \partial_{\bar{z}}v = 0\}.$$

- (c) *We have $\ker(d_{h,A}^*)^\perp \cap \text{Dom}(d_{h,A}^*) = \{d_{h,A}w, w \in H_0^1(U; \mathbb{C}) \cap H^2(U; \mathbb{C})\}$.*

³It may also be found in [6, Lemma 1.4.1].

Here, the notation $d_{h,A,U}$ for $d_{h,A}$ emphasizes the dependence on U .

Notation 5. We denote by $\Pi_{h,A,U}$ (or simply $\Pi_{h,A}$ if there is no ambiguity) the orthogonal projection on $\ker(d_{h,A}^*)$.

Proof. (a) By Lemma 4.2, the graph norm of $d_{h,A}$ and the usual H^1 -norm are equivalent. Thus, the graph of $d_{h,A}$ is a closed subspace of $L^2(U) \times L^2(U)$. From Lemma 4.2 and Remark 4.3, we get, for all $u \in H_0^1(U)$,

$$\|d_{h,A}u\|_{L^2(U)}^2 \geq h \int_U 2B|u|^2 dx.$$

With Assumption (a) and the fact that the operator is closed, the range is closed.

(b) We have

$$\text{Dom}(d_{h,A}^*) = \text{Dom}(d_{1,0}^*),$$

and $d_{h,A}^*$ acts as $d_{h,A}^\times$. By Proposition 2.1 and Lemma 4.1, we deduce

$$\ker(d_{h,A}^*) = \{e^{-\phi/h}v, v \in L^2(U), \partial_{\bar{z}}v = 0\}.$$

(c) As in the proof of Lemma 4.1, we get

$$\begin{aligned} \ker(d_{h,A}^*)^\perp \cap \text{Dom}(d_{h,A}^*) &= \text{ran}(d_{h,A}) \cap \text{Dom}(d_{h,A}^*) \\ &= \{d_{h,A}w, w \in H_0^1(U; \mathbb{C}) \text{ and } d_{h,A}^*d_{h,A}w = (|p-A|^2 + hB)w \in L^2(U; \mathbb{C})\} \\ &= \{d_{h,A}w, w \in H_0^1(U; \mathbb{C}) \text{ and } -\Delta w \in L^2(U; \mathbb{C})\} \\ &= \{d_{h,A}w, w \in H_0^1(U; \mathbb{C}) \cap H^2(U; \mathbb{C})\}. \end{aligned}$$

□

Definition 4.5. We define the self-adjoint operators $(\mathcal{L}_h^\pm, \text{Dom}(\mathcal{L}_h^\pm))$ as the operators acting as

$$\mathcal{L}_h^- = d_{h,A}d_{h,A}^\times = (\mathbf{p}-A)^2 - hB, \quad \mathcal{L}_h^+ = d_{h,A}^\times d_{h,A} = (\mathbf{p}-A)^2 + hB, \quad (4.2)$$

on the respective domains

$$\begin{aligned} \text{Dom}(\mathcal{L}_h^-) &= \{u \in \text{Dom}(d_{h,A}^*), d_{h,A}^*u \in \text{Dom}(d_{h,A})\}, \\ \text{Dom}(\mathcal{L}_h^+) &= \{u \in \text{Dom}(d_{h,A}), d_{h,A}u \in \text{Dom}(d_{h,A}^*)\} \\ &= H_0^1(U; \mathbb{C}) \cap H^2(U; \mathbb{C}). \end{aligned}$$

4.3. Semiclassical elliptic estimates for the magnetic Cauchy-Riemann operator.

Notation 6. By the Riemann mapping theorem, and since $\partial\Omega$ is assumed to be \mathcal{C}^2 , it is Dini-continuous (see [11, Theorem 2.1, and Section 3.3]) and we can consider a biholomorphic function F between $D(0,1)$ and Ω such that $F(\partial D(0,1)) = \partial\Omega$. We write $x = F(y)$. We notice that

$$\partial_{y_1} + i\partial_{y_2} = \overline{F'(y)}(\partial_{x_1} + i\partial_{x_2}),$$

and

$$dx = |F'(y)|^2 dy.$$

By [11, Theorem 3.5], this biholomorphism can be continuously extended to $\overline{D(0,1)}$, and there exist $c_1, c_2 > 0$ such that, for all $y \in \overline{D(0,1)}$,

$$c_1 \leq |F'(y)| \leq c_2.$$

For $\delta \in (0, 1)$, we also let

$$\Omega_\delta = F(D(0, 1 - \delta)).$$

The following theorem is a crucial ingredient in the proof of the lower bound of $\lambda_k(h)$. It is intimately related to the spectral supersymmetry of Dirac operators [13, Theorem 5.5 and Corollary 5.6].

Theorem 4.6. *There exist $\delta_0, h_0 > 0$ and $c > 0$ such that, for all $\delta \in [0, \delta_0]$, for all $h \in (0, h_0)$, and for all $u \in \text{Dom}(d_{h,A,\Omega_\delta}^*) \cap \ker(d_{h,A,\Omega_\delta}^*)^\perp$*

$$\|d_{h,A}^* u\|_{L^2(\Omega_\delta)} \geq \sqrt{2hB_0} \|u\|_{L^2(\Omega_\delta)},$$

$$\|d_{h,A}^* u\|_{L^2(\Omega_\delta)} \geq ch^2 \left(\|\nabla u\|_{L^2(\Omega_\delta)} + \|u\|_{L^2(\partial\Omega_\delta)} \right),$$

where we used Notation 6.

Theorem 4.6 follows from the following two lemmas.

Lemma 4.7. *For all $u \in \text{Dom}(d_{h,A}^*) \cap \ker(d_{h,A}^*)^\perp$, we have*

$$\|d_{h,A}^* u\|_{L^2(U)} \geq \sqrt{2hB_0} \|u\|_{L^2(U)}.$$

Proof. Let $u \in \text{Dom}(d_{h,A}^*) \cap \ker(d_{h,A}^*)^\perp$. By Proposition 4.4, there exists $w \in H_0^1(U; \mathbb{C}) \cap H^2(U; \mathbb{C})$ such that $u = d_{h,A} w$ and $d_{h,A}^* u = \mathcal{L}_h^+ w$. The spectrum of \mathcal{L}_h^+ is a subset of $[2hB_0, +\infty)$ (see Remark 4.3). Thus, we get

$$\|\mathcal{L}_h^+ w\|_{L^2(U)} \geq 2hB_0 \|w\|_{L^2(U)}.$$

By integration by parts and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} 2hB_0 \|d_{h,A} w\|_{L^2(U)}^2 &\leq 2hB_0 \langle w, \mathcal{L}_h^+ w \rangle_{L^2(U)} \leq 2hB_0 \|w\|_{L^2(U)} \|\mathcal{L}_h^+ w\|_{L^2(U)} \\ &\leq \|\mathcal{L}_h^+ w\|_{L^2(U)}^2. \end{aligned}$$

This ensures that

$$\sqrt{2hB_0} \|d_{h,A} w\|_{L^2(U)} \leq \|\mathcal{L}_h^+ w\|_{L^2(U)} = \left\| d_{h,A}^* (d_{h,A} w) \right\|_{L^2(U)}$$

and the conclusion follows. \square

Lemma 4.8. *There exist $\delta_0, h_0 > 0$ and $c > 0$ such that, for all $\delta \in [0, \delta_0]$, for all $h \in (0, h_0)$, and for all $u \in \text{Dom}(d_{h,A,\Omega_\delta}^*) \cap \ker(d_{h,A,\Omega_\delta}^*)^\perp$, we have*

$$\|d_{h,A,\Omega_\delta}^* u\|_{L^2(\Omega_\delta)} \geq ch^2 \|\nabla u\|_{L^2(\Omega_\delta)} + ch^2 \|u\|_{L^2(\partial\Omega_\delta)}.$$

Proof. For notational simplicity, we let $U = \Omega_\delta$ and we write $d_{h,A}$ for $d_{h,A,U}$.

With the same notations as in the proof of Lemma 4.7, we have

$$d_{h,A}^* u = d_{h,A}^* d_{h,A} w = \mathcal{L}_h^+ w, \quad w \in H_0^1(U) \cap H^2(U).$$

i. From Lemma 4.2,

$$\begin{aligned} \|d_{h,A}w\|_{L^2(U)}^2 &= \|(\mathbf{p}-A)w\|_{L^2(U)}^2 + h \int_U B|w|^2 dx \\ &= \langle d_{h,A}^*u, w \rangle_{L^2(U)} \leq \|d_{h,A}^*u\|_{L^2(U)} \|w\|_{L^2(U)}. \end{aligned}$$

Using Assumption (a), there exists $C > 0$ such that

$$h^{\frac{1}{2}} \|(\mathbf{p}-A)w\|_{L^2(U)} + h \|w\|_{L^2(U)} \leq C \|d_{h,A}^*u\|_{L^2(U)},$$

so that, since A is bounded,

$$h^{3/2} \|\nabla w\|_{L^2(U)} \leq C \|d_{h,A}^*u\|_{L^2(U)} + Ch^{1/2} \|w\|_{L^2(U)} \leq Ch^{-1/2} \|d_{h,A}^*u\|_{L^2(U)}.$$

Thus,

$$h^2 \|\nabla w\|_{L^2(U)} + h \|w\|_{L^2(U)} \leq C \|d_{h,A}^*u\|_{L^2(U)}. \quad (4.3)$$

ii. Let us now deal with the derivatives of order two. From the explicit expression of $\mathcal{L}_h^+ w$, we get

$$-h^2 \Delta w = d_{h,A}^*u - 2ihA \cdot \nabla w - |A|^2 w + hBw.$$

Taking the L^2 -norm and using (4.3), we get

$$\begin{aligned} h^2 \|\Delta w\|_{L^2(U)} &\leq \|d_{h,A}^*u\|_{L^2(U)} + \|-2ihA \cdot \nabla w\|_{L^2(U)} + \||A|^2 w\|_{L^2(U)} + \|hBw\|_{L^2(U)} \\ &\leq C(1 + h^{-1}) \|d_{h,A}^*u\|_{L^2(U)}. \end{aligned}$$

Using a standard ellipticity result for the Dirichlet Laplacian, we find

$$h^3 \|w\|_{H^2(U)} + h^2 \|\nabla w\|_{L^2(U)} + h \|w\|_{L^2(U)} \leq C \|d_{h,A}^*u\|_{L^2(U)}. \quad (4.4)$$

The uniformity of the constant with respect to $\delta \in (0, \delta_0)$ can be checked in the classical proof of elliptic regularity. Note that $d_{h,A} = L_1 - iL_2$ where $L_j = -ih\partial_j - A_j$. Using (4.4), we deduce that

$$\|\nabla d_{h,A}w\|_{L^2(U)} \leq Ch \|w\|_{H^2(U)} + C \|w\|_{L^2(U)} + C \|\nabla w\|_{L^2(U)} \leq Ch^{-2} \|d_{h,A}^*u\|_{L^2(U)},$$

and, since $u = d_{h,A}w$,

$$h^2 \|\nabla u\|_{L^2(U)} \leq C \|d_{h,A}^*u\|_{L^2(U)}. \quad (4.5)$$

iii. A classical trace result combined with (4.5) and Lemma 4.7 gives

$$\|u\|_{L^2(\partial U)} \leq C \|u\|_{H^1(U)} \leq Ch^{-2} \|d_{h,A}^*u\|_{L^2(U)},$$

where it can again be checked that C does not depend on $\delta \in (0, \delta_0)$.

□

5. LOWER BOUNDS

The aim of this section is to establish the following proposition.

Proposition 5.1. *Assume that Ω is \mathcal{C}^2 and satisfies Assumption 1.1. There exists a constant $\theta_0 \in (0, 1]$ such that for all $k \in \mathbb{N}^*$,*

$$\liminf_{h \rightarrow 0} e^{-2\phi_{\min}/h} h^{k-1} \lambda_k(h) \geq C_{\sup}(k) \theta_0 = C_{\inf}(k).$$

If $\Omega = D(0, 1)$ and B is radial, we have

$$\liminf_{h \rightarrow 0} e^{-2\phi_{\min}/h} h^{k-1} \lambda_k(h) \geq \frac{4\Phi}{(k-1)!} \det(\text{Hess}_{x_{\min}} \phi)^{\frac{k}{2}}.$$

5.1. Inside approximation by the zero-modes. Let $k \in \mathbb{N}^*$. Let us consider an orthonormal family $(v_{j,h})_{1 \leq j \leq k}$ (for the scalar product of $L^2(e^{-2\phi/h} dx)$) associated with the eigenvalues $(\lambda_j(h))_{1 \leq j \leq k}$. We define

$$\mathcal{E}_h = \text{span}_{1 \leq j \leq k} v_{j,h}.$$

In this section, we will see that the general upper bound proved in the last section implies that all $v_h \in \mathcal{E}_h$ wants to be holomorphic inside Ω .

5.1.1. *Concentration of the groundstate.*

Lemma 5.2. *There exist $C, h_0 > 0$ such that for all $v_h \in \mathcal{E}_h$ and $h \in (0, h_0)$, we have,*

$$\|v_h\|_{L^2(\Omega)}^2 \leq Ch^{-(1+k)} e^{2\phi_{\min}/h} \int_{\Omega} e^{-2\phi/h} |v_h|^2 dx.$$

Proof. We have $\lambda_k(h) = h^{-k+1} \mathcal{O}(e^{2\phi_{\min}/h})$ (see Proposition 3.1). By using the orthogonality of the $v_{j,h}$, one gets

$$\int_{\Omega} e^{-2\phi/h} |h \partial_{\bar{z}} v_h|^2 dx \leq \lambda_k(h) \int_{\Omega} e^{-2\phi/h} |v_h|^2 dx \leq Ch^{-k+1} e^{2\phi_{\min}/h} \int_{\Omega} e^{-2\phi/h} |v_h|^2 dx. \quad (5.1)$$

Now, we use $\phi \leq 0$ to get

$$\int_{\Omega} |\partial_{\bar{z}} v_h|^2 dx \leq Ch^{-(1+k)} e^{2\phi_{\min}/h} \int_{\Omega} e^{-2\phi/h} |v_h|^2 dx.$$

Since v_h satisfies the Dirichlet boundary condition and by integration by parts, we find

$$\int_{\Omega} |\nabla v_h|^2 dx \leq Ch^{-(1+k)} e^{2\phi_{\min}/h} \int_{\Omega} e^{-2\phi/h} |v_h|^2 dx.$$

It remains to use the Poincaré inequality. □

We can now prove a concentration lemma.

Lemma 5.3. *Let $\alpha \in (0, 1/2)$. We have*

$$\lim_{h \rightarrow 0} \sup_{v_h \in \mathcal{E}_h \setminus \{0\}} \left| \frac{\int_{D(x_{\min}, h^\alpha)} e^{-2\phi/h} |v_h(x)|^2 dx}{\int_{\Omega} e^{-2\phi/h} |v_h(x)|^2 dx} - 1 \right| = 0,$$

and

$$\lim_{h \rightarrow 0} \sup_{\delta \in (0, \delta_0]} \sup_{v_h \in \mathcal{E}_h \setminus \{0\}} \left| \frac{\int_{\Omega_\delta} e^{-2\phi/h} |v_h(x)|^2 dx}{\int_{\Omega} e^{-2\phi/h} |v_h(x)|^2 dx} - 1 \right| = 0,$$

where δ_0 is defined in Proposition 4.4.

Proof. Let us remark that the second limit is a consequence of the first one. We have

$$\frac{\int_{D(x_{\min}, h^\alpha)} e^{-2\phi/h} |v_h(x)|^2 dx}{\int_{\Omega} e^{-2\phi/h} |v_h(x)|^2 dx} = 1 - \frac{\int_{\Omega \setminus D(x_{\min}, h^\alpha)} e^{-2\phi/h} |v_h(x)|^2 dx}{\int_{\Omega} e^{-2\phi/h} |v_h(x)|^2 dx}.$$

By (3.9) and Lemma 5.2, we deduce that

$$\begin{aligned} \int_{\Omega \setminus D(x_{\min}, h^\alpha)} e^{-2\phi/h} |v_h(x)|^2 dx &\leq e^{-2\phi_{\min}/h - \lambda_{\min} h^{2\alpha-1}(1+\mathcal{O}(h^\alpha))} \int_{\Omega} |v_h(x)|^2 dx \\ &\leq Ch^{-(1+k)} e^{-\lambda_{\min} h^{2\alpha-1}(1+\mathcal{O}(h^\alpha))} \int_{\Omega} e^{-2\phi/h} |v_h|^2 dx = \mathcal{O}(h^\infty) \int_{\Omega} e^{-2\phi/h} |v_h|^2 dx, \end{aligned}$$

and the conclusion follows. \square

5.1.2. *Interior approximation.* Now that we know that v_h is localized inside Ω , let us explain why it is close to be a holomorphic function.

Notation 7. Let us denote by $\tilde{\Pi}_{h,\delta}$ the orthogonal projection on the kernel of $-i\partial_{\bar{z}}$ (i.e. the Bergman functions on Ω_δ which in defined in Theorem 4.6) for the L^2 -scalar product $\langle \cdot, e^{-2\phi/h} \cdot \rangle_{L^2(\Omega_\delta)}$.

We notice that, if $u = e^{-\phi/h} v$, we have

$$\Pi_{h,A,\Omega_\delta} u = e^{-\phi/h} \tilde{\Pi}_{h,\delta} v.$$

Proposition 5.4. *There exist $C, h_0 > 0$ such that for all $\delta \in (0, \delta_0]$ and $h \in (0, h_0)$, we have for all $v_h \in \mathcal{E}_h$,*

- (a) $\|e^{-\phi/h} (\text{Id} - \tilde{\Pi}_{h,\delta}) v_h\|_{L^2(\Omega_\delta)} \leq Ch^{-\frac{1}{2}} \sqrt{\lambda_k(h)} \|e^{-\phi/h} v_h\|_{L^2(\Omega_\delta)},$
- (b) $\|e^{-\phi/h} (\text{Id} - \tilde{\Pi}_{h,\delta}) v_h\|_{L^2(\partial\Omega_\delta)} \leq Ch^{-2} \sqrt{\lambda_k(h)} \|e^{-\phi/h} v_h\|_{L^2(\Omega_\delta)},$
- (c) $\dim \tilde{\Pi}_{h,\delta} \mathcal{E}_h = k.$

Here, δ_0 is defined in Theorem 4.6.

Proof. For all $v_h \in \mathcal{E}_h$, we have

$$\begin{aligned} 4\|e^{-\phi/h} h \partial_{\bar{z}} v_h\|_{L^2(\Omega_\delta)}^2 &\leq 4\|e^{-\phi/h} h \partial_{\bar{z}} v_h\|_{L^2(\Omega)}^2 \\ &\leq \lambda_k(h) \|e^{-\phi/h} v_h\|_{L^2(\Omega)}^2 \leq (1 + o(1)) \lambda_k(h) \|e^{-\phi/h} v_h\|_{L^2(\Omega_\delta)}^2, \end{aligned}$$

where we used Lemma 5.3 to get the last inequality. With $u_h = e^{-\phi/h} v_h$, we have

$$4\|e^{-\phi/h} h \partial_{\bar{z}} v_h\|_{L^2(\Omega_\delta)}^2 = 4\|e^{-\phi/h} h \partial_{\bar{z}} (\text{Id} - \tilde{\Pi}_{h,\delta}) v_h\|_{L^2(\Omega_\delta)}^2 = \|d_{h,A,\Omega_\delta}^* (\text{Id} - \Pi_{h,A,\Omega_\delta}) u_h\|_{L^2(\Omega_\delta)}^2.$$

Applying Theorem 4.6, we get (a) and (b).

Let $v_h \in \mathcal{E}_h$ be such that $\tilde{\Pi}_{h,\delta} v_h = 0$. Recalling Proposition 3.1, we have

$$\begin{aligned} \|e^{-\phi/h} v_h\|_{L^2(\Omega_\delta)} &\leq \|e^{-\phi/h} (\text{Id} - \tilde{\Pi}_{h,\delta}) v_h\|_{L^2(\Omega_\delta)} + \|e^{-\phi/h} \tilde{\Pi}_{h,\delta} v_h\|_{L^2(\Omega_\delta)} \\ &\leq Ch^{-k/2} e^{\phi_{\min}/h} \|e^{-\phi/h} v_h\|_{L^2(\Omega_\delta)} \end{aligned}$$

and $v_h = 0$ on Ω_δ so that (c) follows. \square

5.2. A reduction to a holomorphic subspace. In the following, we assume that $\delta \in (0, \delta_0)$ and $h \in (0, h_0)$.

Notation 8. We will use the so-called Szegö projection

$$\Pi_+ : L^2(D(0, 1)) \ni \left(\sum_{\mathbf{n} \in \mathbb{Z}} a_n(r) e^{i\mathbf{n}s} \right) \mapsto \left(\sum_{\mathbf{n} \in \mathbb{N}} a_n(r) e^{i\mathbf{n}s} \right) \in L^2(D(0, 1)).$$

Notation 9. We let

$$E = \min_{\Gamma} (\nabla \phi \cdot \mathbf{n}) > 0.$$

Lemma 5.5. *Assume that $\delta/h \rightarrow +\infty$ and $\delta \rightarrow 0$. Then, for all $v_h \in \mathcal{E}_h$,*

$$2hEc_1 \|\Pi_+(v_h \circ F)\|_{L^2(\partial D(0, 1-\delta))}^2 (1 + o(1)) \leq 4h^2 \int_{\Omega} e^{-2\phi/h} |\partial_{\bar{z}} v_h|^2 dx,$$

where F, c_1 are defined in Notation 6.

Proof. i. For all $v \in H_0^1(\Omega)$, we let $\check{v} = v \circ F \in H_0^1(D(0, 1))$ and $\check{\phi} = \phi \circ F$. We get that

$$4h^2 \int_{\Omega} e^{-2\phi/h} |\partial_{\bar{z}} v|^2 dx = 4h^2 \int_{D(0, 1)} e^{-2\check{\phi}/h} |\partial_{\bar{y}} \check{v}|^2 dy.$$

ii. In the tubular coordinates introduced in the proof of Lemma 3.7, the Cauchy-Riemann operator is

$$-\frac{i}{2} (\partial_1 + i\partial_2) = \frac{-i\gamma'}{2} \left(\frac{\partial_s}{1-t} + i\partial_t \right).$$

We write $\check{\psi}(s, t) = \check{\psi}(\eta(s, t))$ for any function $\check{\psi}$ defined on $D(0, 1)$. We have, for all $\check{v} \in H_0^1(D(0, 1))$,

$$\begin{aligned} 4h^2 \int_{D(0, 1)} e^{-2\check{\phi}/h} |\partial_{\bar{z}} \check{v}|^2 dx &\geq 4h^2 \int_{D(0, 1) \setminus D(0, 1-\delta)} e^{-2\check{\phi}/h} |\partial_{\bar{z}} \check{v}|^2 dx \\ &= h^2 \int_0^{2\pi} \int_0^\delta (1-t)^{-1} |((1-t)\partial_t - i\partial_s) \check{v}|^2 e^{-2\check{\phi}/h} ds dt. \end{aligned} \quad (5.2)$$

iii. Let us notice that

$$\partial_{\check{\mathbf{n}}} \check{\phi}(y) = |F'(y)| \partial_{\mathbf{n}} \phi(F(y)), \quad (5.3)$$

where $x \in \partial\Omega$, $\mathbf{n}(x)$ is the outward pointing unit normal to Ω at x , and $\check{\mathbf{n}}(y)$ is outward pointing unit normal to $D(0, 1)$ at y .

By using the Taylor expansion of $\check{\phi}$, there exists $C > 0$ such that, for all $(s, t) \in \Gamma \times (0, \delta)$,

$$-\check{\phi}(s, t) = -t\partial_t \check{\phi}(s, 0) + \mathcal{O}(t^2) \geq (1 - C\delta)\check{E}t,$$

where $0 < \check{E} = c_1 E \leq \min_{\partial D(0, 1)} \partial_{\check{\mathbf{n}}} \check{\phi}$. We have

$$4h^2 \int_{D(0, 1)} e^{-2\check{\phi}/h} |\partial_{\bar{z}} \check{v}|^2 dx \geq h^2 \int_0^{2\pi} \int_0^\delta (1-t)^{-1} |((1-t)\partial_t - i\partial_s) \check{v}|^2 e^{2(1-C\delta)\check{E}t/h} ds dt.$$

Consider the new variable $\tau = -\ln|1-t|$, we get

$$4h^2 \int_{D(0,1)} e^{-2\check{\phi}/h} |\partial_{\bar{z}} \check{v}|^2 dx \geq h^2 \int_0^{2\pi} \int_0^{-\ln|1-\delta|} |(\partial_\tau - i\partial_s)\mathbf{v}(s, \tau)|^2 e^{2(1-C\delta)\check{E}(1-e^{-\tau})/h} ds d\tau,$$

where $\mathbf{v}(s, \tau) = \check{v}(s, 1 - e^{-\tau})$. Since $1 - e^{-\tau} = \tau + \mathcal{O}(\tau^2) = \tau + \mathcal{O}(\delta\tau)$, there exists $\tilde{C} > 0$ such that

$$e^{2(1-C\delta)\check{E}(1-e^{-\tau})/h} \geq e^{2\check{E}(1-\tilde{C}\delta)\tau/h}.$$

Let $\tilde{E} = \check{E}(1 - \tilde{C}\delta)$ and $\tilde{\delta} = -\ln|1 - \delta|$ so that

$$4h^2 \int_{D(0,1)} e^{-2\check{\phi}/h} |\partial_{\bar{z}} \check{v}|^2 dx \geq h^2 \int_0^{2\pi} \int_0^{\tilde{\delta}} |(\partial_\tau - i\partial_s)\mathbf{v}(s, \tau)|^2 e^{2\tilde{E}\tau/h} ds d\tau.$$

iv. Using the Fourier series and the Parseval formula, we get

$$\int_0^{2\pi} \int_0^{\tilde{\delta}} |(\partial_\tau - i\partial_s)\mathbf{v}(s, \tau)|^2 e^{2\tilde{E}\tau/h} ds d\tau = 2\pi \sum_{m \in \mathbb{Z}} \int_0^{\tilde{\delta}} |(\partial_\tau + m)\hat{\mathbf{v}}_m(\tau)|^2 e^{2\tilde{E}\tau/h} d\tau,$$

where

$$\hat{\mathbf{v}}_m(\tau) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ims} \mathbf{v}(s, \tau) ds.$$

Let us consider the quadratic form

$$Q_m(w) = \int_0^{\tilde{\delta}} |(\partial_\tau + m)w|^2 e^{2\tilde{E}\tau/h} d\tau$$

with boundary conditions $w(0) = 0$ and $w(\tilde{\delta}) = 1$.

Let us notice that

$$Q_m(w) = \tilde{Q}_m(\rho) = \int_0^{\tilde{\delta}} |\partial_\tau \rho|^2 e^{2\tau\tilde{E}/h + 2(\tilde{\delta}-\tau)m} d\tau,$$

where $w(\tau) = e^{m(\tilde{\delta}-\tau)}\rho(\tau)$ for all $\tau \in (0, \tilde{\delta})$, $\rho(0) = 0$ and $\rho(\tilde{\delta}) = 1$.

Since $m \mapsto \tilde{Q}_m(\rho)$ is an increasing function, we get that $\tilde{Q}_m(\rho) \geq \tilde{Q}_0(\rho)$ for all $m \geq 0$ and, by Lemma A.1,

$$Q_m(w) \geq \Lambda_0(h),$$

where

$$\Lambda_0(h) = \frac{2\tilde{E}/h}{1 - e^{-2\tilde{\delta}\tilde{E}/h}} \geq 0.$$

We get, by forgetting the negative m , we find

$$4h^2 \int_{\Omega} e^{-2\phi/h} |\partial_{\bar{z}} v|^2 dx \geq 2\pi h^2 \Lambda_0(h) \sum_{m \geq 0} |\hat{v}_m(\delta)|^2 = h^2 \Lambda_0(h) \|\Pi_+(v_h \circ F)\|_{L^2(\partial D(0,1-\delta))}^2.$$

□

Remark 5.6. As we can see in the proof of Lemma 5.5, if $\partial_{\mathbf{n}}\check{\phi}$ is constant equal to \check{E} , we can replace $c_1 E$ by \check{E} .

In the following, we choose $\delta = h^{3/4}$.

Using Proposition 5.4, we show in the following lemma that we can replace v_h by $\tilde{\Pi}_{h,\delta}v_h$ in Lemma 5.5.

Lemma 5.7. *Assume that $\delta = h^{3/4}$ and that $\alpha \in (\frac{1}{3}, \frac{1}{2})$. Then,*

$$2\theta_0 h e^{2\phi_{\min}/h} \left\| \tilde{\Pi}_{h,\delta} v_h \right\|_{\mathcal{H}^2(\Omega_\delta)}^2 (1+o(1)) \leq \lambda_k(h) \left\| e^{-\frac{1}{2h} \text{Hess}_{x_{\min}} \phi(x-x_{\min}, x-x_{\min})} \tilde{\Pi}_{h,\delta} v_h \right\|_{L^2(D(x_{\min}, h^\alpha))}^2,$$

where

$$\theta_0 = \frac{c_1 \min_{\partial\Omega} \partial_{\mathbf{n}} \phi}{c_2 \max_{\partial\Omega} \partial_{\mathbf{n}} \phi} \in (0, 1],$$

and where we used the notation

$$\|w\|_{\mathcal{H}^2(\Omega_\delta)}^2 := \int_{\partial D(0,1-\delta)} |w \circ F|^2 (\partial_{\mathbf{n}} \phi \circ F) |F'| ds.$$

Remark 5.8. Taking $\delta = 0$ in the definition of $\|w\|_{\mathcal{H}^2(\Omega_\delta)}^2$ above, gives

$$\|w\|_{\mathcal{H}^2(\Omega_0)}^2 = \int_{\partial D(0,1)} |w \circ F|^2 (\partial_{\mathbf{n}} \phi \circ F) |F'| ds = \int_{\partial\Omega} |w|^2 \partial_{\mathbf{n}} \phi dy = N_{\mathcal{H}}(w)^2,$$

for $w \in \mathcal{H}^2(\Omega)$.

Proof. i. From Lemma 5.5 and the definition of v_h , we have

$$\begin{aligned} 2Ec_1 h \left\| \Pi_+(v_h \circ F) \right\|_{L^2(\partial D(0,1-\delta))}^2 (1+o(1)) &\leq h^2 \int_{\Omega} e^{-2\phi/h} |\partial_{\bar{z}} v_h|^2 dx \\ &\leq \lambda_k(h) \int_{\Omega} e^{-2\phi/h} |v_h|^2 dx. \end{aligned}$$

Thus, by Lemma 5.3,

$$2Ec_1 h \left\| \Pi_+(v_h \circ F) \right\|_{L^2(\partial D(0,1-\delta))}^2 (1+o(1)) \leq \lambda_k(h) \int_{D(x_{\min}, h^\alpha)} e^{-2\phi/h} |v_h|^2 dx. \quad (5.4)$$

ii. Consider $\check{\Pi}_{h,\delta}$ is the orthogonal projection on $\mathcal{H}(D(0,1-\delta))$ for the scalar product $L^2(e^{-2\check{\phi}/h} dy)$. Let us now replace Π_+ by $\check{\Pi}_{h,\delta}$. Proposition 5.4 ensures that

$$\|e^{-\check{\phi}/h} (\text{Id} - \check{\Pi}_{h,\delta}) v_h \circ F\|_{L^2(\partial D(0,1-\delta))} \leq Ch^{-2} \sqrt{\lambda_k(h)} \|e^{-\phi/h} v_h\|_{L^2(\Omega_\delta)},$$

Using the Taylor expansion of $\check{\phi}$ near the boundary and (5.3), we have, on $\partial D(0,1-\delta)$,

$$e^{-\check{\phi}/h} \geq (1+o(1)) e^{c_1 E h^{-1/4}},$$

so that

$$\|(\text{Id} - \check{\Pi}_{h,\delta}) v_h \circ F\|_{L^2(\partial D(0,1-\delta))} \leq Ch^{-2} \sqrt{\lambda_k(h)} e^{-c_1 E h^{-1/4}} \|e^{-\phi/h} v_h\|_{L^2(\Omega_\delta)}. \quad (5.5)$$

Since Π_+ is a projection and $\check{\Pi}_{h,\delta}$ is valued in the holomorphic functions,

$$\begin{aligned} \|(\text{Id} - \check{\Pi}_{h,\delta}) v_h \circ F\|_{L^2(\partial D(0,1-\delta))} &\geq \|\Pi_+(\text{Id} - \check{\Pi}_{h,\delta}) v_h \circ F\|_{L^2(\partial D(0,1-\delta))} \\ &\geq \|\Pi_+ v_h \circ F - \check{\Pi}_{h,\delta} v_h \circ F\|_{L^2(\partial D(0,1-\delta))} \\ &\geq \left| \|\Pi_+ v_h \circ F\|_{L^2(\partial D(0,1-\delta))} - \|\check{\Pi}_{h,\delta} v_h \circ F\|_{L^2(\partial D(0,1-\delta))} \right|. \end{aligned}$$

Then, with (5.5),

$$\|\Pi_+ v_h \circ F\|_{L^2(\partial D(0,1-\delta))} \geq \|\tilde{\Pi}_{h,\delta} v_h \circ F\|_{L^2(\partial D(0,1-\delta))} - \mathcal{O}(h^\infty) \sqrt{\lambda_k(h)} \|e^{-\phi/h} v_h\|_{L^2(\Omega_\delta)}.$$

By (5.4) and Lemma 5.3,

$$\sqrt{2Ec_1 h} \|\tilde{\Pi}_{h,\delta} v_h \circ F\|_{L^2(\partial D(0,1-\delta))} (1 + o(1)) \leq \sqrt{\lambda_k(h)} \|e^{-\phi/h} v_h\|_{L^2(D(x_{\min}, h^\alpha))}. \quad (5.6)$$

Thus, coming back to Ω_δ (without forgetting the Jacobian of F),

$$\sqrt{2Ec_1/c_2 h} \|\tilde{\Pi}_{h,\delta} v_h\|_{L^2(\partial\Omega_\delta)} (1 + o(1)) \leq \sqrt{\lambda_k(h)} \|e^{-\phi/h} v_h\|_{L^2(D(x_{\min}, h^\alpha))}.$$

Then, by using the (weighted) Hardy norm, we have

$$\sqrt{2\theta_0 h} \|\tilde{\Pi}_{h,\delta} v_h\|_{\mathcal{H}^2(\Omega_\delta)} (1 + o(1)) \leq \sqrt{\lambda_k(h)} \|e^{-\phi/h} v_h\|_{L^2(D(x_{\min}, h^\alpha))}. \quad (5.7)$$

iii. Using Proposition 5.4 and Lemma 5.3, we get

$$\begin{aligned} & \|e^{-\phi/h} v_h\|_{L^2(D(x_{\min}, h^\alpha))} \\ & \leq \|e^{-\phi/h} \tilde{\Pi}_{h,\delta} v_h\|_{L^2(D(x_{\min}, h^\alpha))} + \|e^{-\phi/h} (\text{Id} - \tilde{\Pi}_{h,\delta}) v_h\|_{L^2(D(x_{\min}, h^\alpha))} \\ & \leq \|e^{-\phi/h} \tilde{\Pi}_{h,\delta} v_h\|_{L^2(D(x_{\min}, h^\alpha))} + \|e^{-\phi/h} (\text{Id} - \tilde{\Pi}_{h,\delta}) v_h\|_{L^2(\Omega_\delta)} \\ & \leq \|e^{-\phi/h} \tilde{\Pi}_{h,\delta} v_h\|_{L^2(D(x_{\min}, h^\alpha))} + Ch^{-\frac{1}{2}} \sqrt{\lambda_k(h)} \|e^{-\phi/h} v_h\|_{L^2(D(x_{\min}, h^\alpha))}. \end{aligned}$$

Combing this with (5.7) and Proposition 3.1, we find

$$2\theta_0 h \left\| \tilde{\Pi}_{h,\delta} v_h \right\|_{\mathcal{H}^2(\Omega_\delta)}^2 (1 + o(1)) \leq \lambda_k(h) \|e^{-\phi/h} \tilde{\Pi}_{h,\delta} v_h\|_{L^2(D(x_{\min}, h^\alpha))}^2.$$

iv. Using the Taylor expansion of ϕ at x_{\min} , we get, for all for $x \in D(x_{\min}, h^\alpha)$,

$$\frac{\phi(x) - \phi_{\min}}{h} = \frac{1}{2h} \text{Hess}_{x_{\min}} \phi(x - x_{\min}, x - x_{\min}) + \mathcal{O}(h^{3\alpha-1}),$$

and the conclusion follows. \square

Remark 5.9. We can check in the proof of Lemma 5.7 (see (5.6) and Remark 5.6) that, if $\partial_{\tilde{n}} \check{\phi}$ is constant, the lemma is true with $\theta_0 = 1$.

Remark 5.10. Lemma 5.7 shows in particular that

$$2(1 + o(1))\theta_0 h \tilde{\lambda}_k(h) \leq \lambda_k(h),$$

where

$$\tilde{\lambda}_k(h) = \inf_{\substack{V \subset \mathcal{H}^2(\Omega_\delta) \\ \dim V = k}} \sup_{v \in V \setminus \{0\}} \frac{\|v\|_{\mathcal{H}^2(\Omega_\delta)}^2}{\left\| e^{-\frac{1}{2h} \text{Hess}_{x_{\min}} \phi(x - x_{\min}, x - x_{\min})} v \right\|_{L^2(D(x_{\min}, h^\alpha))}^2}.$$

In the next section, we will essentially provide a lower bound of $\tilde{\lambda}_k(h)$.

5.3. Reduction to a polynomial subspace: Proof of Proposition 5.1. We can now prove Proposition 5.1.

- i. By using (3.12), there exists $C > 0, h_0 > 0$ such that, for all $h \in (0, h_0)$, for all $w \in \mathcal{H}^2(\Omega_\delta)$, all $z_0 \in D(0, h^\alpha)$, and all $n \in \{0, \dots, k\}$,

$$|w^{(n)}(z_0)| \leq C \|w\|_{\mathcal{H}^2(\Omega_\delta)}. \quad (5.8)$$

Let us define, for all $w \in \mathcal{H}^2(\Omega_\delta)$,

$$N_h(w) = \left\| e^{-\frac{1}{2h} \text{Hess}_{x_{\min}} \phi(x-x_{\min}, x-x_{\min})} w \right\|_{L^2(D(x_{\min}, h^\alpha))}.$$

We let $w_h = \tilde{\Pi}_{h,\delta} v_h$. By the Taylor formula, we can write

$$w_h = \text{Tayl}_{k-1} w_h + R_{k-1}(w_h),$$

where

$$\text{Tayl}_{k-1} w_h = \sum_{n=0}^{k-1} \frac{w_h^{(n)}(z_{\min})}{n!} (z - z_{\min})^n,$$

and, for all $z_0 \in D(z_{\min}, h^\alpha)$,

$$|R_{k-1}(w_h)(z_0)| \leq C |z - z_{\min}|^k \sup_{D(z_{\min}, h^\alpha)} |w_h^{(k)}|.$$

With (5.8) and a rescaling, the Taylor remainder satisfies

$$N_h(R_{k-1}(w_h)) \leq Ch^{\frac{k}{2}} h^{\frac{1}{2}} \|w_h\|_{\mathcal{H}^2(\Omega_\delta)}.$$

Thus, by the triangle inequality,

$$N_h(w_h) \leq N_h(\text{Tayl}_{k-1} w_h) + Ch^{\frac{k}{2}} h^{\frac{1}{2}} \|w_h\|_{\mathcal{H}^2(\Omega_\delta)}.$$

Thus, with Lemma 5.7, we get

$$(1+o(1))e^{\phi_{\min}/h} \sqrt{2\theta_0 h} \|w_h\|_{\mathcal{H}^2(\Omega_\delta)} \leq \sqrt{\lambda_k(h)} N_h(\text{Tayl}_{k-1} w_h) + C \sqrt{\lambda_k(h)} h^{\frac{1+k}{2}} \|w_h\|_{\mathcal{H}^2(\Omega_\delta)},$$

so that, thanks to Proposition 3.1,

$$(1+o(1))e^{\phi_{\min}/h} \sqrt{2\theta_0 h} \|w_h\|_{\mathcal{H}^2(\Omega_\delta)} \leq \sqrt{\lambda_k(h)} N_h(\text{Tayl}_{k-1} w_h) \leq \sqrt{\lambda_k(h)} \hat{N}_h(\text{Tayl}_{k-1} w_h), \quad (5.9)$$

with

$$\hat{N}_h(w) = \left\| e^{-\frac{1}{2h} \text{Hess}_{x_{\min}} \phi(x-x_{\min}, x-x_{\min})} w \right\|_{L^2(\mathbb{R}^2)}.$$

This inequality shows in particular that

$$\dim \text{Tayl}_{k-1}(\tilde{\Pi}_{\delta,h} \mathcal{E}_h) = k. \quad (5.10)$$

- ii. Let us recall that

$$\mathcal{H}_k^2(\Omega_\delta) = \{\psi \in \mathcal{H}^2(\Omega_\delta) : \forall n \in \{0, \dots, k-1\}, \psi^{(n)}(x_{\min}) = 0\}.$$

Since $(w_h - \text{Tayl}_{k-1}w_h) \in \mathcal{H}_k^2(\Omega_\delta)$, we have, by the triangle inequality,

$$\begin{aligned} \|w_h\|_{\mathcal{H}^2(\Omega_\delta)} &\geq \left\| \frac{w_h^{(k-1)}(z_{\min})}{(k-1)!} (z - z_{\min})^{k-1} + (w_h - \text{Tayl}_{k-1}w_h) \right\|_{\mathcal{H}^2(\Omega_\delta)} - \|\text{Tayl}_{k-2}w_h\|_{\mathcal{H}^2(\Omega_\delta)} \\ &\geq \frac{|w_h^{(k-1)}(z_{\min})|}{(k-1)!} \text{dist}_{\mathcal{H},\delta}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega_\delta)) - \|\text{Tayl}_{k-2}w_h\|_{\mathcal{H}^2(\Omega_\delta)}, \end{aligned}$$

where

$$\begin{aligned} &\text{dist}_{\mathcal{H},\delta}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega_\delta)) \\ &= \inf \left\{ \left\| (z - z_{\min})^{k-1} - Q(z) \right\|_{\mathcal{H}^2(\Omega_\delta)}, \text{ for all } Q \in \mathcal{H}_k^2(\Omega_\delta) \right\}. \end{aligned}$$

Using again the triangle inequality,

$$\|\text{Tayl}_{k-2}w_h\|_{\mathcal{H}^2(\Omega_\delta)} \leq C \sum_{n=0}^{k-2} |w_h^{(n)}(z_{\min})|.$$

Moreover,

$$\begin{aligned} \sum_{n=0}^{k-2} |w_h^{(n)}(z_{\min})| &\leq h^{-\frac{k-2}{2}} \sum_{n=0}^{k-2} h^{\frac{n}{2}} |w_h^{(n)}(z_{\min})| \leq h^{-\frac{k-2}{2}} \sum_{n=0}^{k-1} h^{\frac{n}{2}} |w_h^{(n)}(z_{\min})| \\ &\leq Ch^{-\frac{k-2}{2}} h^{-\frac{1}{2}} \hat{N}_h(\text{Tayl}_{k-1}w_h), \end{aligned}$$

where we used the rescaling property

$$\hat{N}_h \left(\sum_{n=0}^{k-1} c_n (z - z_{\min})^n \right) = h^{\frac{1}{2}} \hat{N}_1 \left(\sum_{n=0}^{k-1} c_n h^{\frac{n}{2}} (z - z_{\min})^n \right), \quad (5.11)$$

and the equivalence of the norms in finite dimension:

$$\exists C > 0, \forall d \in \mathbb{C}^k, \quad C^{-1} \sum_{n=0}^{k-1} |d_n| \leq \hat{N}_1 \left(\sum_{n=0}^{k-1} d_n (z - z_{\min})^n \right) \leq C \sum_{n=0}^{k-1} |d_n|.$$

We find

$$\begin{aligned} \|w_h\|_{\mathcal{H}^2(\Omega_\delta)} &\geq \frac{|w_h^{(k-1)}(z_{\min})|}{(k-1)!} \text{dist}_{\mathcal{H},\delta}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega_\delta)) \\ &\quad - Ch^{-\frac{k-2}{2}} h^{-\frac{1}{2}} \hat{N}_h(\text{Tayl}_{k-1}w_h), \end{aligned}$$

and thus, by (5.9),

$$\begin{aligned} (1 + o(1)) e^{\phi_{\min}/h} \sqrt{2\theta_0} h \frac{|w_h^{(k-1)}(z_{\min})|}{(k-1)!} \text{dist}_{\mathcal{H},\delta}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega_\delta)) \\ \leq \left(\sqrt{\lambda_k(h)} + Ch^{\frac{2-k}{2}} e^{\phi_{\min}/h} \right) \hat{N}_h(\text{Tayl}_{k-1}w_h). \end{aligned} \quad (5.12)$$

iii. Since we have (5.10), we deduce that

$$(1 + o(1))e^{\phi_{\min}/h} \sqrt{2\theta_0 h} \operatorname{dist}_{\mathcal{H},\delta}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega_\delta)) \sup_{c \in \mathbb{C}^k} \frac{|c_{k-1}|}{\hat{N}_h(\sum_{n=0}^{k-1} c_n (z - z_{\min})^n)} \\ \leq \sqrt{\lambda_k(h)} + Ch^{\frac{2-k}{2}} e^{\phi_{\min}/h}. \quad (5.13)$$

By (5.11), we infer

$$h^{\frac{1}{2}} \sup_{c \in \mathbb{C}^k} \frac{|c_{k-1}|}{\hat{N}_h(\sum_{n=0}^{k-1} c_n (z - z_{\min})^n)} = \sup_{c \in \mathbb{C}^k} \frac{h^{\frac{1-k}{2}} |c_{k-1}|}{\hat{N}_1(\sum_{n=0}^{k-1} c_n (z - z_{\min})^n)}.$$

Since \hat{N}_1 is related to the Bergman norm $N_{\mathcal{B}}$ via a translation, and recalling Notation 4, we get

$$\sup_{c \in \mathbb{C}^k} \frac{|c_{k-1}|}{\hat{N}_1(\sum_{n=0}^{k-1} c_n (z - z_{\min})^n)} = \sup_{c \in \mathbb{C}^k} \frac{|c_{k-1}|}{N_{\mathcal{B}}(\sum_{n=0}^{k-1} c_n z^n)} = \frac{1}{N_{\mathcal{B}}(P_{k-1})}.$$

Thus,

$$(1 + o(1))h^{\frac{1-k}{2}} e^{\phi_{\min}/h} \sqrt{2\theta_0} \frac{\operatorname{dist}_{\mathcal{H},\delta}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega_\delta))}{N_{\mathcal{B}}(P_{k-1})} \leq \sqrt{\lambda_k(h)}. \quad (5.14)$$

iv. Since Ω is regular enough, the Riemann mapping theorem ensures that

$$\lim_{h \rightarrow 0} \operatorname{dist}_{\mathcal{H},\delta}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega_\delta)) = \operatorname{dist}_{\mathcal{H}}((z - z_{\min})^{k-1}, \mathcal{H}_k^2(\Omega)).$$

The conclusion follows.

5.4. Proof of Corollary 1.11. We recall Notation 6 where F , c_1 and c_2 are defined. Let us notice that we can choose F such that $F(0) = x_{\min}$.

For all $v \in H_0^1(\Omega)$, we let $\check{v} = v \circ F \in H_0^1(D(0, 1))$, and we get,

$$\frac{1}{c_2} \frac{\int_{D(0,1)} e^{-2\check{\phi}/h} |\partial_{\check{y}} \check{v}|^2 dy}{\int_{D(0,1)} e^{-2\check{\phi}/h} |\check{v}|^2 dy} \leq \frac{\int_{D(0,1)} e^{-2\check{\phi}/h} |\partial_{\check{y}} \check{v}|^2 dy}{\int_{D(0,1)} e^{-2\check{\phi}/h} |\check{v}|^2 |F'(y)|^2 dy} = \frac{\int_{\Omega} e^{-2\phi/h} |\partial_{\check{z}} v|^2 dx}{\int_{\Omega} e^{-2\phi/h} |v|^2 dx},$$

where $\check{\phi} = \phi \circ F$ has a unique and non-degenerate minimum at $y = 0$ and $\check{\phi}(0) = \phi_{\min}$.

In the same way, we get

$$\frac{\int_{\Omega} e^{-2\phi/h} |\partial_{\check{z}} v|^2 dx}{\int_{\Omega} e^{-2\phi/h} |v|^2 dx} \leq \frac{1}{c_1} \frac{\int_{D(0,1)} e^{-2\check{\phi}/h} |\partial_{\check{y}} \check{v}|^2 dy}{\int_{D(0,1)} e^{-2\check{\phi}/h} |\check{v}|^2 dy}.$$

These inequalities, the min-max principle, and Theorem 1.3 imply Corollary 1.11.

APPENDIX A. A UNIDIMENSIONAL OPTIMIZATION PROBLEM

The goal of this section is to minimize, for each fixed s , the quantity,

$$\int_0^\varepsilon e^{2t\partial_n \phi(s)/h} |\partial_t \rho|^2 dt.$$

This leads to the following lemma.

Lemma A.1. For α and $\varepsilon > 0$, we let $I = (0, \varepsilon)$ and

$$\mathcal{V} = \{\rho \in H^1(I) : \rho(0) = 0, \quad \rho(\varepsilon) = 1\}.$$

Let us consider, for all $\rho \in \mathcal{V}$,

$$F_{\alpha, \varepsilon}(\rho) := \int_0^\varepsilon e^{\alpha \ell} |\rho'(\ell)|^2 d\ell.$$

(a) The minimization problem

$$\inf\{F_{\alpha, \varepsilon}(\rho), \rho \in \mathcal{V}\}$$

has a unique minimizer

$$\rho_{\alpha, \varepsilon}(\ell) = \frac{1 - e^{-\alpha \ell}}{1 - e^{-\alpha \varepsilon}},$$

(b) We have

$$\inf\{F_{\alpha, \varepsilon}(\rho), \rho \in \mathcal{V}\} = \frac{\alpha}{1 - e^{-\alpha \varepsilon}},$$

(c) Let $c_0 > 0$. Assume $1 - e^{-\alpha \varepsilon} \geq c_0$. There exists $C > 0$ such that

$$\int_0^\varepsilon e^{\alpha \ell} |\partial_\alpha \rho_{\alpha, \varepsilon}|^2 d\ell \leq C (\alpha^{-3} + e^{-\alpha \varepsilon} \varepsilon^2 \alpha^{-1}).$$

Proof. i. Since $\alpha > 0$, we have that $F_{\alpha, \varepsilon}(\rho) \geq \int_0^\varepsilon |\rho'(\ell)|^2 d\ell$ for all $\rho \in \mathcal{V}$. There exists $C > 0$ such that, for all $\rho \in \mathcal{V}$,

$$\int_0^\varepsilon |\rho'(\ell)|^2 d\ell \geq C \int_0^\varepsilon |\rho(\ell)|^2 d\ell.$$

This ensures that any minimizing sequence $(\rho_n)_{n \in \mathbb{N}} \subset \mathcal{V}$ is bounded in $H^1(I)$ and any H^1 -weak limit is a minimizer of $\inf\{F_{\alpha, \varepsilon}(\rho), \rho \in \mathcal{V}\}$.

ii. $F_{\alpha, \varepsilon}^{1/2}$ is an euclidian norm on \mathcal{V} so that $F_{\alpha, \varepsilon}$ is strictly convex and the minimizer is unique.

iii. At a minimum ρ , the Euler-Lagrange equation is

$$(e^{\alpha \ell} \rho')' = 0.$$

Thus, there exist $(c, d) \in \mathbb{R}^2$ such that, for all $\ell \in I$,

$$\rho(\ell) = d - c\alpha^{-1}e^{-\alpha \ell},$$

so that, with the boundary conditions we find the function $\rho_{\alpha, \varepsilon}$.

iv. We have

$$\int_0^\varepsilon e^{\alpha \ell} |\rho'(\ell)|^2 d\ell = \alpha^2 (1 - e^{-\alpha \varepsilon})^{-2} \int_0^\varepsilon e^{-\alpha \ell} d\ell = \frac{\alpha}{1 - e^{-\alpha \varepsilon}}.$$

v. We also have

$$\partial_\alpha \rho_{\alpha, \varepsilon}(\ell) = \frac{1}{(1 - e^{-\alpha \varepsilon})^2} \left(\ell e^{-\alpha \ell} (1 - e^{-\alpha \varepsilon}) - (1 - e^{-\alpha \ell}) \varepsilon e^{-\alpha \varepsilon} \right),$$

for $\ell \in (0, \varepsilon)$ and

$$\begin{aligned} \int_0^\varepsilon e^{\alpha\ell} |\partial_\alpha \rho_{\alpha, \varepsilon}|^2 d\ell &\leq \frac{1}{(1 - e^{-\alpha\varepsilon})^4} \left(\int_0^\varepsilon \ell^2 e^{-\alpha\ell} d\ell (1 - e^{-\alpha\varepsilon})^2 + \int_0^\varepsilon e^{-\alpha\ell} d\ell (\varepsilon e^{-\alpha\varepsilon})^2 \right. \\ &\quad \left. + \int_0^\varepsilon e^{\alpha\ell} d\ell (\varepsilon e^{-\alpha\varepsilon})^2 \right) \\ &\leq C (\alpha^{-3} + e^{-\alpha\varepsilon} \varepsilon^2 \alpha^{-1}). \end{aligned}$$

□

APPENDIX B. A DENSITY RESULT

Lemma B.1. *Assume that Ω is bounded, simply connected and that $\partial\Omega$ is Dini-continuous. Then, the set $\mathcal{H}^2(\Omega) \cap W^{1, \infty}(\Omega)$ is dense in $\mathcal{H}^2(\Omega)$.*

Proof. We recall Notation 6. Let $u \in \mathcal{H}^2(\Omega)$. Then, $u \circ F = \sum_{k \geq 0} a_k z^k$ is holomorphic on $D(0, 1)$ and $(a_k)_{k \geq 0} \in \ell^2(\mathbb{N})$. Let $\varepsilon \in (0, 1)$. The function

$$\tilde{u}_\varepsilon : D(0, 1) \ni z \mapsto u \circ F((1 - \varepsilon)z) \in \mathbb{C}$$

is holomorphic on $D(0, 1/(1 - \varepsilon))$. We denote by $u_\varepsilon = \tilde{u}_\varepsilon \circ F^{-1}$. We have

$$\begin{aligned} \|u - u_\varepsilon\|_{\mathcal{H}^2(\Omega)}^2 &:= \int_{\partial\Omega} |u(x) - u_\varepsilon(x)|^2 \partial_{\mathbf{n}} \phi dx \\ &= \int_{\partial D(0, 1)} |u \circ F(y) - u \circ F((1 - \varepsilon)y)|^2 |F'(y)| \partial_{\mathbf{n}} \phi \circ F(y) dy \\ &\leq c_2 \|\partial_{\mathbf{n}} \phi\|_{L^\infty} \int_{\partial D(0, 1)} |u \circ F(y) - u \circ F((1 - \varepsilon)y)|^2 dy \\ &\leq c_2 \|\partial_{\mathbf{n}} \phi\|_{L^\infty} \sum_{k \geq 1} |a_k|^2 |1 - (1 - \varepsilon)^k|^2. \end{aligned}$$

By Lebesgue's theorem, we get that $(u_\varepsilon)_{\varepsilon \in (0, 1)}$ converges to u in $\mathcal{H}^2(\Omega)$. Let us also remark that $(u_\varepsilon)_{\varepsilon \in (0, 1)} \subset W^{1, \infty}(\Omega)$ so that the result follows. □

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