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A UNIVERSAL ENVELOPING ALGEBRA FOR COCOMMUTATIVE RACK BIALGEBRAS

ULRICH KRÄHMER AND FRIEDRICH WAGEMANN

ABSTRACT. We construct a bialgebra object in the category of linear maps \mathcal{LM} from a cocommutative rack bialgebra. The construction does extend to some non-cocommutative rack bialgebras, as is illustrated by a concrete example. As a separate result, we show that the Loday complex with adjoint coefficients embeds into the rack bialgebra deformation complex for the rack bialgebra defined by a Leibniz algebra.

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INTRODUCTION

A *shelf* is a set X with a composition $(x, y) \mapsto x \triangleleft y$ which satisfies the self-distributivity relation $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$. More generally, this condition makes sense for coalgebras in braided monoidal categories, as was observed by Carter-Crans-Elhamdadi-Saito [3] and further studied by Lebed [10]. In the present article, we will focus on shelves in the category of vector spaces with the tensor flip as braiding:

Definition 1. A *linear shelf* is a coassociative coalgebra (C, Δ) together with a morphism of coalgebras

$$C \otimes C \rightarrow C, \quad (x, y) \mapsto x \triangleleft y$$

that satisfies

$$(1) \quad (x \triangleleft y) \triangleleft z = (x \triangleleft z_{(1)}) \triangleleft (y \triangleleft z_{(2)}) \quad \forall x, y, z \in C.$$

A counital and coaugmented linear shelf $(C, \Delta, \triangleleft, \epsilon, 1)$ for which

$$x \triangleleft 1 = x, \quad 1 \triangleleft x = \epsilon(x)1, \quad \epsilon(x \triangleleft y) = \epsilon(x)\epsilon(y)$$

holds for all $x, y \in C$ will be called a *rack bialgebra*.

Here and elsewhere, all vector spaces, coalgebras etc will be over a field k , and we use Sweedler's notation $\Delta(z) = z_{(1)} \otimes z_{(2)}$ for coproducts.

If C is spanned by primitive elements (together with the coaugmentation), the definition of a rack bialgebra reduces to that of a Leibniz algebra [11]. Lie racks provide another natural source of examples with a rich structure theory and applications in the deformation quantisation of duals of Leibniz algebras, see [1], [2].

An important step in the theory of Leibniz algebras was the definition of their universal enveloping algebras [12]. Here, we extend this construction to all cocommutative rack bialgebras. The result is a cocommutative bialgebra $U(C)$ in the category of vector spaces, see Theorem 1 below.

One application of universal enveloping algebras is to express cohomology theories as derived functors. Our motivation for the present article was the article [3]. Therein, the authors develop a deformation theory of linear shelves. To this end, they defined cohomology groups $H_{\text{sh}}^n(C, C)$ for $n \leq 3$. However, it remained an open question how to extend this to a fully fledged cohomology theory including an interpretation as derived functors in an abelian category. In [4], this was further studied in the special case where \triangleleft is also associative.

The present article is meant as a first step towards a possible answer to this question. Our initial goal was to apply our results from [9]: therein, we constructed examples of rack bialgebras from Hopf algebras in Loday-Pirashvili's category of linear maps \mathcal{LM} [11]. Therefore, Gerstenhaber-Schack cohomology [7] in the tensor category \mathcal{LM} could be used to define $H_{\text{sh}}^n(C, C)$ if all linear shelves arose in this way.

Let us describe the content of the present article section by section. Section 1 contains preliminaries on coalgebras and points out that counitisation does not provide an equivalence between linear shelves and rack bialgebras. Section 2 recalls from [9] the construction of rack bialgebras from Hopf algebras. Section 3 introduces the notion of a Yetter-Drinfel'd rack which guides the construction of $U(C)$ in the main subsequent Section 4. Therein, we define $U(C)$ and establish its universal property. We also reformulate these results in terms of a bialgebra object in \mathcal{LM} and give some examples. The construction of $U(C)$ does extend also to some non-cocommutative rack bialgebras. This is demonstrated with an explicit example in Section 5.

In Section 6, we describe this example as a deformation of a cocommutative rack bialgebra. Furthermore, the Loday complex with adjoint coefficients is embedded into the deformation complex of the rack bialgebra associated to a Leibniz algebra. The article concludes with an outlook and some open questions.

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1. RACK BIALGEBRAS AND LINEAR SHELVES ARE NOT EQUIVALENT

One might expect that linear shelves and rack bialgebras are related by counitisation. We begin by pointing out that this is not the case, so when developing cohomology or deformation theories, one must be clear which of the two structures one is studying. The construction from [9] inevitably yields rack bialgebras, hence these are the objects we will focus on afterwards.

More precisely, recall that if a coalgebra has a counit

$$\epsilon: C \rightarrow k, \quad \epsilon(c_{(1)})c_{(2)} = \epsilon(c_{(2)})c_{(1)} = c \quad \forall c \in C,$$

and is in addition coaugmented, i.e. has a distinguished group-like element

$$1 \in C, \quad \Delta(1) = 1 \otimes 1,$$

then the vector space $\check{C} := \ker \epsilon$ becomes a coalgebra with coproduct

$$\check{\Delta}(c) := \Delta(c) - 1 \otimes c - c \otimes 1,$$

or, in Sweedler notation,

$$c_{(\check{1})} \otimes c_{(\check{2})} = c_{(1)} \otimes c_{(2)} - 1 \otimes c - c \otimes 1.$$

Furthermore, the map $c \mapsto (\epsilon(c)1, c - \epsilon(c)1)$ splits C canonically into a direct sum $C = k1 \oplus \check{C}$. This shows that $C \mapsto \check{C}$ is an equivalence between the category of counital and coaugmented coalgebras and the category of all coalgebras, with inverse $C \mapsto \hat{C} := k \oplus C$ and $\hat{\Delta}(x) = \Delta(x) + 1 \otimes x + x \otimes 1$.

If C is a rack bialgebra, then \triangleleft does restrict to $\check{C} = \ker \epsilon$, but it is in general *not* self-distributive with respect to $\check{\Delta}$, so \check{C} does *not* become a linear shelf in its own right. Conversely, if C is a linear shelf, then \hat{C} is in general not a rack bialgebra with respect to $\hat{\Delta}$. In fact, we have:

Proposition 1. *The counitisation functor does not lift to an equivalence from the category of linear shelves to the category of rack bialgebras.*

Proof. Consider linear shelves C with vanishing coproduct $\Delta = 0$. The category of these has a zero object, the shelf of vector space dimension $\dim_k(C) = 0$, and a unique simple object, the shelf of dimension $\dim_k(C) = 1$ with $\triangleleft = 0$. For all other objects C there is a morphism of shelves

$C \rightarrow D$ with nonzero kernel and nonzero image $D \neq 0$. Indeed, the quotient vector space $C/\text{im } \triangleleft$ is a linear shelf with respect to $\Delta = \triangleleft = 0$, and the canonical projection $C \rightarrow C/\text{im } \triangleleft$ is a morphism of linear shelves. Its kernel vanishes if and only if $\triangleleft = 0$, that is, if C is a direct sum of 1-dimensional shelves with trivial coproduct and shelf product. In this case, the canonical projection onto any quotient vector space D of dimension $\dim_k(C) - 1$ has the desired properties. If instead $\triangleleft \neq 0$, then we can take the quotient $C \rightarrow D := C/\text{im } \triangleleft$ itself: for $\Delta = 0$, the self-distributivity condition reads $(x \triangleleft y) \triangleleft z = 0$ for all $x, y, z \in C$; in particular, $\text{im } \triangleleft \subseteq \bigcap_{x \in C} \ker(-\triangleleft x) \neq C$, hence $D \neq 0$.

In contrast, the shelf condition on the counitisation \hat{C} of a coalgebra C with vanishing coproduct says precisely that the subspace $C \subset \hat{C}$ is a Leibniz algebra. In particular, this means that among the rack bialgebras with this underlying coalgebra structure, we have the (counitisations of) all simple Lie algebras, and these do not admit any nontrivial proper quotients. \square

However, there are some subclasses of linear shelves which do admit counitisations that can be turned functorially into rack bialgebras. The most important one is obtained from linearised shelves spanned by group-like elements:

Example 1. Assume that $(C, \Delta, \triangleleft)$ is a linear shelf with a vector space basis G consisting of group-like elements. Then $\{1, 1 + x \mid x \in G\}$ is a vector space basis of the counitisation \hat{C} . Now define a new rack product \blacktriangleleft on \hat{C} by

$$(1 + x) \blacktriangleleft (1 + y) := 1 + (x \triangleleft y).$$

The self-distributivity for \blacktriangleleft follows immediately from the selfdistributivity of \triangleleft and thus \hat{C} becomes a rack bialgebra. \triangle

2. FROM HOPF ALGEBRAS TO RACK BIALGEBRAS

Let $(H, \Delta_H, \epsilon_H, \mu_H, 1_H, S_H)$ be a Hopf algebra over k . Then $\ker \epsilon_H$ is a (right right) Yetter-Drinfel'd module with respect to the right adjoint action

$$h \triangleleft h' := S(h'_{(1)})hh'_{(2)}$$

and the right coaction

$$h \mapsto h_{(0)} \otimes h_{(1)} := h_{(1)} \otimes h_{(2)} - 1 \otimes h.$$

If $C \subset \ker \epsilon_H$ is invariant under the adjoint action, then \triangleleft restricts to a product $C \otimes C \rightarrow C$ which satisfies (1). However, $z_{(1)} \otimes z_{(2)}$ is the coproduct in H , and in general, this does not restrict to C . One situation where this approach leads to linear shelves is the following:

Proposition 2. *Let H be a cocommutative Hopf algebra and $C \subset \ker \epsilon_H$ be a Yetter-Drinfel'd submodule. Then $\hat{C} := k1_H \oplus C$ is a rack bialgebra with respect to the adjoint action \triangleleft and the restriction of Δ_H to \hat{C} .*

Proof. As C is a (right) subcomodule, we have $h_{(0)} \otimes h_{(1)} \in C \otimes H$. But $h_{(0)} \otimes h_{(1)} = h_{(1)} \otimes h_{(2)} - 1 \otimes h$, thus we conclude that $h_{(1)} \otimes h_{(2)} = h_{(0)} \otimes h_{(1)} + 1 \otimes h \in C \otimes H$. But by cocommutativity, this implies that $h_{(1)} \otimes h_{(2)} = h_{(2)} \otimes h_{(1)} \in H \otimes C$. In conclusion,

$$h_{(1)} \otimes h_{(2)} \in C \otimes H \cap H \otimes C = C \otimes C,$$

i.e. C is stable under the coproduct of H .

The adjoint action is a morphism of coalgebras thanks to cocommutativity:

$$\begin{aligned} & (a \triangleleft b)_{(1)} \otimes (a \triangleleft b)_{(2)} \\ &= (S(b_{(1)})ab_{(2)})_{(1)} \otimes (S(b_{(1)})ab_{(2)})_{(2)} \\ &= (S(b_{(1)}))_{(1)}a_{(1)}(b_{(2)})_{(1)} \otimes (S(b_{(1)}))_{(2)}a_{(2)}(b_{(2)})_{(2)} \\ &= S(b_{(3)})a_{(1)}b_{(2)} \otimes S(b_{(1)})a_{(2)}b_{(4)} \\ &= S(b_{(1)})a_{(1)}b_{(2)} \otimes S(b_{(3)})a_{(2)}b_{(4)} \\ &= (a_{(1)} \triangleleft b_{(1)}) \otimes (a_{(2)} \triangleleft b_{(2)}), \end{aligned}$$

where we have used cocommutativity in the last step.

As mentioned before, the self-distributivity is the only property which works independently of the cocommutativity of C . Indeed, on the one hand, we have:

$$(a \triangleleft b) \triangleleft c = S(c_{(1)})S(b_{(1)})ab_{(2)}c_{(2)}$$

And on the other hand, we have:

$$\begin{aligned} & (a \triangleleft c_{(1)}) \triangleleft (b \triangleleft c_{(2)}) \\ &= S(S(c_{(3)})_{(1)}b_{(1)}(c_{(4)})_{(1)})(S(c_{(1)})ac_{(2)})(S(c_{(3)})_{(2)}b_{(2)}(c_{(4)})_{(2)}) \\ &= S(S(c_{(4)})b_{(1)}c_{(5)})(S(c_{(1)})ac_{(2)})(S(c_{(3)})b_{(2)}c_{(6)}) \\ &= S(c_{(5)})S(b_{(1)})S^2(c_{(4)})S(c_{(1)})ac_{(2)}S(c_{(3)})b_{(2)}c_{(6)} \\ &= S(c_{(3)})S(b_{(1)})S(c_{(2)})S(c_{(1)})ab_{(2)}c_{(4)} \\ &= S(c_{(1)})S(b_{(1)})ab_{(2)}c_{(2)} \quad \square \end{aligned}$$

Example 2. If X is a shelf in the category of sets [6], then as discussed in Example 1, the counitisation $C = \widehat{kX}$ of its linearisation becomes a rack bialgebra in which all $x \in X$ are group-like. Observe that this construction differs slightly from the construction in [3], Section 3.1. \triangle

Example 3. Given a (right) Leibniz algebra \mathfrak{h} , the k -vector space $C := k \oplus \mathfrak{h}$ becomes a rack bialgebra by extending the bracket $[x, y] =: x \triangleleft y$ to a

shelf product on all of $k \oplus \mathfrak{h}$. More precisely, we endow first of all C with a coproduct requiring that all elements of \mathfrak{h} are primitive, $\Delta(1) = 1 \otimes 1$, $\epsilon(x) = 0$ for all $x \in \mathfrak{h}$ and $\epsilon(1) = 1$. Then put for all $x, y \in \mathfrak{h}$ $x \triangleleft y = [x, y]$, $1 \triangleleft x = \epsilon(x)1$ and $x \triangleleft 1 = x$. This gives a rack bialgebra. \triangle

3. YETTER-DRINFEL'D RACKS

The question arises which rack bialgebras arise as in Proposition 2. Just as the cocommutativity of H was therein a sufficient, but not a necessary assumption, the construction of a bialgebra that we carry out now can also be applied to certain noncocommutative rack bialgebras. Hence we consider the following general setting adapted from [9, Proposition 5.5]:

Definition 2. Let H be a bialgebra. A *Yetter-Drinfel'd rack* over H is a rack bialgebra C together with a right H -module structure $\cdot : C \otimes H \rightarrow C$ rendering C an H -module coalgebra, and a morphism $q : C \rightarrow H$ of counital coaugmented coalgebras such that

$$(2) \quad a \triangleleft b = a \cdot q(b)$$

and

$$(3) \quad h_{(1)}q(a \cdot h_{(2)}) = q(a)h$$

hold for all $h \in H$ and $a, b \in C$.

In the cocommutative setting, we have, as the name suggests:

Proposition 3. *Let H be a cocommutative bialgebra and C be a Yetter-Drinfel'd rack. Then C becomes a Yetter-Drinfel'd module with respect to the coaction $C \rightarrow C \otimes H$,*

$$x \mapsto x_{(0)} \otimes x_{(1)} := (x_{(1)} - \epsilon(x_{(1)})) \otimes q(x_{(2)}) + \epsilon(x) \otimes 1.$$

Proof. The above formula defines a right coaction, as it is constructed using the coproduct, the counit and the morphism of coalgebras q .

Let us check the Yetter-Drinfel'd property. The two sides in the YD-property have three terms. Let us reason term by term. For the first term, we have for $x \in C$ and $h \in H$

$$\begin{aligned} (x \cdot h_{(2)})_{(1)} \otimes h_{(1)}q(x \cdot h_{(2)})_{(2)} &= \\ &= x_{(1)} \cdot h_{(2)} \otimes h_{(1)}q(x_{(2)} \cdot h_{(3)}) = \\ &= x_{(1)} \cdot h_{(1)} \otimes q(x_{(2)})h_{(2)}, \end{aligned}$$

where we were able to apply the above Condition (3) in the last step only thanks to cocommutativity.

For the second term, we have

$$-1 \otimes h_{(1)}q(x \cdot h_{(2)}) = -1 \cdot h_{(1)} \otimes q(x)h_{(2)}$$

thanks to Condition (3) and $1 \cdot a = \epsilon(a)1$. The third term is simply

$$\epsilon(x \cdot h_{(2)})1 \otimes h_{(1)} = \epsilon(x)1 \cdot h_{(1)} \otimes h_{(2)},$$

which is simply true again by $1 \cdot a = \epsilon(a)1$. \square

Remark 1. Note that for elements $h = q(c)$ in the image of q , the H -module coalgebra property $(x \cdot h)_{(1)} \otimes (x \cdot h)_{(2)} = x_{(1)} \cdot h_{(1)} \otimes x_{(2)} \cdot h_{(2)}$ is satisfied automatically by the fact that \triangleleft is a morphism of coalgebras. Hence the H -module coalgebra condition in Definition 2 can be omitted if H is generated as an algebra by $\text{im } q$. \triangle

If H is a Hopf algebra (admits an antipode), then $\ker \epsilon$ is a Yetter-Drinfel'd module with respect to the right adjoint action, and (3) and the coalgebra morphism condition on q are equivalent to $q|_C: C \rightarrow \ker \epsilon$ being a morphism of Yetter-Drinfel'd modules.

4. THE UNIVERSAL ENVELOPING ALGEBRA $U(C)$

Given any rack bialgebra C , let $T = k \oplus \check{C} \oplus \check{C}^{\otimes 2} \oplus \dots$ denote the tensor algebra of $\check{C} = \ker \epsilon$ and $i: C \rightarrow T$ be the canonical inclusion, which is the identity on \check{C} and maps the distinguished group-like $1 \in C$ to the scalar $1 \in k = \check{C}^{\otimes 0}$. As we will also consider the tensor product $T \otimes T$, we denote the product in T by \cdot rather than \otimes . By the universal property of T , the linear map

$$\check{C} \rightarrow T \otimes T, \quad x \mapsto i(x_{(1)}) \otimes i(x_{(2)})$$

extends uniquely to an algebra map $\Delta_T: T \rightarrow T \otimes T$. The coassociativity of the coproduct in C implies that of Δ_T . That is, T becomes a bialgebra and i yields an embedding of counital coaugmented coalgebras $C \rightarrow T$.

Using once more the universal property of T , the rack product

$$\triangleleft: \check{C} \rightarrow \text{End}(C), \quad x \mapsto (y \mapsto y \triangleleft x)$$

extends to an algebra homomorphism $T \rightarrow \text{End}(C)$, so C becomes a right T -module coalgebra such that $x \triangleleft y = x \cdot i(y)$.

However, i does not turn C into a Yetter-Drinfel'd rack over $H = T$, as the commutativity relation (3) is not satisfied in general. Hence we add the relations manually:

Definition 3. For any rack bialgebra C we denote by $U(C)$ the symmetric algebra of C with respect to the vector space braiding

$$\tau: C \otimes C \rightarrow C \otimes C, \quad x \otimes y \mapsto y_{(1)} \otimes x \triangleleft y_{(2)},$$

that is, $U(C) := T/J$ where $T = T(\check{C})$ and

$$J := \langle i(y_{(1)}) \cdot i(x \triangleleft y_{(2)}) - i(x) \cdot i(y) \mid x, y \in C \rangle.$$

We call $U(C)$ the *universal enveloping algebra* of C and denote the canonical map $C \rightarrow U(C)$ by q .

A key observation is that in case C is cocommutative, the coproduct Δ_T descends to $U(C)$:

Lemma 1. *The ideal J is also a coideal in case C is cocommutative.*

Proof. As Δ_T is an algebra map, it is sufficient to prove that the coproduct of a generating element of J belongs to $T \otimes J + J \otimes T$.

$$\begin{aligned} & \Delta_T(i(y_1).i(x \triangleleft y_2) - i(x).i(y)) \\ &= i(y_1).i(x \triangleleft y_3)_{(1)} \otimes i(y_2).i(x \triangleleft y_3)_{(2)} \\ & \quad - i(x_1).i(y_1) \otimes i(x_2).i(y_2) \\ &= i(y_1).i(x_1 \triangleleft y_3) \otimes i(y_2).i(x_2 \triangleleft y_4) \\ & \quad - i(x_1).i(y_1) \otimes i(x_2).i(y_2) \end{aligned}$$

By cocommutativity, this is an element of $T \otimes J + J \otimes T$:

$$\begin{aligned} & i(y_1).i(x_1 \triangleleft y_3) \otimes i(y_2).i(x_2 \triangleleft y_4) \\ & \quad - i(x_1).i(y_1) \otimes i(x_2).i(y_2) \\ &= i(y_3).i(x_1 \triangleleft y_4) \otimes i(y_1).i(x_2 \triangleleft y_2) \\ & \quad - i(x_1).i(y_1) \otimes i(x_2).i(y_2) \\ &= (i(y_3).i(x_1 \triangleleft y_4) - i(x_1).i(y_3)) \otimes i(y_1).i(x_2 \triangleleft y_2) + \\ & \quad + i(x_1).i(y_3) \otimes i(y_1).i(x_2 \triangleleft y_2) \\ & \quad - i(x_1).i(y_1) \otimes i(x_2).i(y_2) \\ &= (i(y_3).i(x_1 \triangleleft y_4) - i(x_1).i(y_3)) \otimes i(y_1).i(x_2 \triangleleft y_2) + \\ & \quad + i(x_1).i(y_1) \otimes (i(y_2).i(x_2 \triangleleft y_3) - i(x_2).i(y_2)) \\ & \in J \otimes T + T \otimes J. \quad \square \end{aligned}$$

Note that the action of T on C passes to an action of $U(C)$ on C , thanks to the self-distributivity of \triangleleft :

Lemma 2. *For all $x, y, z \in C$, we have:*

$$(x \cdot i(y)) \cdot i(z) = (x \cdot i(z_1)) \cdot i(y \triangleleft z_2)$$

Proof. We have

$$\begin{aligned} & (x \cdot i(z_1)) \cdot i(y \triangleleft z_2) \\ &= (x \triangleleft z_1) \triangleleft (y \triangleleft z_2) \\ &= (x \triangleleft y) \triangleleft z \\ &= (x \cdot i(y)) \cdot i(z). \quad \square \end{aligned}$$

We thus arrive at the following theorem, which realises \check{C} as in [9, Proposition 5.5] as a braided Leibniz algebra:

Theorem 1. *The universal enveloping algebra $U(C)$ of a cocommutative rack bialgebra is canonically a bialgebra, and C becomes canonically a $U(C)$ -Yetter-Drinfel'd rack. If furthermore $q_H: C \rightarrow H$ is any Yetter-Drinfel'd rack structure on C , then there exists a unique morphism of bialgebras $u: U(C) \rightarrow H$ such that $u \circ q = q_H$ and hence*

$$(4) \quad x \cdot_{U(C)} s = x \cdot_H u(s)$$

holds for all $x \in C, s \in U(C)$.

Proof. The $U(C)$ -Yetter-Drinfel'd rack structure of C has been established already. By the universal property of the tensor algebra, there exists a unique algebra homomorphism $T(\check{C}) \rightarrow H$ such that

$$c_1 \cdots c_l \mapsto q_H(c_1) \cdots q_H(c_l).$$

This is a morphism of coalgebras on the level of generators, and thus, by multiplicativity, a morphism of coalgebras, i.e. a morphism of bialgebras.

As C is a Yetter-Drinfel'd rack over H , we have

$$q_H(x)q_H(y) = q_H(y_{(1)})q_H(x \triangleleft y_{(2)}), \quad x, y \in \check{C}.$$

Hence the bialgebra map induces a bialgebra morphism

$$u: U(C) \rightarrow H.$$

The equality $u \circ q = q_H$ is true by construction. □

Corollary 1. *If C is a cocommutative rack bialgebra and $U(C)$ is a Hopf algebra, then C arises as in Proposition 2.*

Proof. Indeed, just take $H = T \rtimes U(C)$ - taking the semidirect product is necessary when $q: C \rightarrow U(C)$ is not injective as in Example 5 below. □

We now construct a bialgebra object in Loday-Pirashvili's category \mathcal{LM} out of the Yetter-Drinfel'd rack C . Recall that \mathcal{LM} is the monoidal category of linear maps between vector spaces with the so-called infinitesimal tensor product as monoidal product, see [11] for details. As explained in [9], a bialgebra object in \mathcal{LM} consists of a bialgebra H , an H -tetramodule M and an H -bilinear coderivation $f: M \rightarrow H$. As is well-known [8, Section 13.1.3], any Yetter-Drinfel'd module V over a bialgebra H gives rise to a tetramodule whose underlying vector space is $H \otimes V$. Its actions and coactions are given by

$$g(h \otimes v)g' := hg'_{(1)} \otimes v \cdot g'_{(2)},$$

$$(h \otimes v)_{(-1)} \otimes (h \otimes v)_{(0)} \otimes (h \otimes v)_{(1)} = h_{(1)} \otimes (h_{(2)} \otimes v_{(0)}) \otimes h_{(3)}v_{(1)}.$$

Now we have:

Theorem 2. *If C is a cocommutative rack bialgebra, then*

$$U(C) \otimes \check{C} \rightarrow U(C), \quad x \otimes y \mapsto x.q(y)$$

is canonically a bialgebra object in Loday-Pirashvili's category of linear maps \mathcal{LM} .

Proof. In light of Proposition 3 and Theorem 1, any cocommutative rack bialgebra C becomes a Yetter-Drinfel'd module over $H = U(C)$. Furthermore, the decomposition $C = k \cdot 1 \oplus \check{C}$ is a direct sum of Yetter-Drinfel'd modules. Thus $M := U(C) \otimes \check{C}$ is a Hopf tetramodule. That the linear map given by $s \otimes c \mapsto sq(c)$ is a coderivation and a bimodule map is straightforwardly verified. \square

Example 4. Note that the bialgebra $U(C)$ is not a Hopf algebra in general, i.e. does not necessarily have an antipode. For example, for $C = k \cdot 1 \oplus k \cdot g$ with $\Delta_C(g) = g \otimes g$ and $g \triangleleft g = g$, we obtain for $U(C)$ a polynomial algebra in one group-like generator which does not have an antipode. This rack bialgebra C is the counitisation of the linearisation of the conjugation rack of the trivial group. In general, if $C = \widehat{kX}$ for a rack X as in Example 2, then the group algebra of the associated group of the rack X (see [6] for definitions) is obtained by localisation of $U(C)$ at all group-likes. For the rack bialgebra C with group-like basis $1, g$ and $g \triangleleft g = 1$, $U(C)$ is the bialgebra with one group-like generator g satisfying $g^2 = g$, so $U(C) \cong k \oplus k$ as algebra. If $U(C)$ does admit an antipode, then at least over an algebraically closed field of characteristic 0 it is as a Hopf algebra isomorphic to a semidirect product of a group algebra and a universal enveloping algebra of a Lie algebra (see e.g. [5, Theorem 3.8.2]). Note further that there is an omission in [9, Lemma 4.8] as the last statement only makes sense when H is a Hopf algebra. \triangle

Example 5. If $C = k \cdot 1 \oplus \mathfrak{h}$ is the rack bialgebra associated to a Leibniz algebra \mathfrak{h} as in Example 3, then $U(C)$ is the universal enveloping algebra of the Lie algebra $\mathfrak{h}_{\text{Lie}}$ associated to \mathfrak{h} , that is, the quotient by the Leibniz ideal generated by all squares $[x, x]$. Indeed, the generators of the ideal J are in this case of the form

$$[x, y] + y.x - x.y,$$

and J contains in particular all squares $[x, x]$. The bialgebra object in \mathcal{LM} obtained in Theorem 2 is the universal enveloping algebra of the Lie algebra object $(\mathfrak{h}, \mathfrak{h}_{\text{Lie}})$ as in [11, Definition 4.3]. Thus Theorem 2 extends the construction of the universal enveloping algebra of a Leibniz algebra. \triangle

Example 6. Let \mathfrak{g} be a Lie algebra, $H = U(\mathfrak{g})$ be its universal enveloping algebra, and $C \subset H$ be the image of $k \oplus \mathfrak{g} \oplus \mathfrak{g} \otimes \mathfrak{g}$, that is, the degree 2 part in the PBW filtration. This is a rack bialgebra following the construction

in Proposition 2 (starting with $\check{C} \subset \ker \epsilon_H$). The symmetric algebra $U(C)$ is not H , but $U(\mathfrak{g} \oplus S^2\mathfrak{g})$, where $S^2\mathfrak{g}$ are the symmetric 2-tensors over \mathfrak{g} , viewed as abelian Lie algebra, and the direct sum is a direct sum of Lie algebras. The kernel of $u: U(C) \rightarrow H$ is the ideal generated by $S^2\mathfrak{g}$. \triangle

Remark 2. Before we continue, let us point out that $U(C)$ differs from the Nichols algebra associated to the braided vector space (\check{C}, τ) . The latter can also be defined as a quotient algebra of T , but with homogeneous relations in degrees that can be of arbitrary degree, cf. [13] for a pedagogical introduction and original references. In contrast, the generators of J are in general inhomogeneous involving terms of degree two and one. They are homogeneous if and only if \triangleleft vanishes. In this case τ is the tensor flip and $U(C)$ is the classical symmetric algebra of the vector space \check{C} . This is the only case in which $U(C)$ agrees with the Nichols algebra. \triangle

5. A NON-COCOMMUTATIVE EXAMPLE

Up to now, all examples of rack bialgebras were cocommutative, and this was an essential assumption in our results. Also in our main reference [3], all examples were cocommutative (note that the examples in Lemma 3.8 and Lemma 3.9 therein are isomorphic to each other). In this section, we present a non-cocommutative example of a rack bialgebra that nevertheless admits the structure of a Yetter-Drinfel'd rack and can be constructed from a bialgebra object in \mathcal{LM} .

Proposition 4. *Let $C = \text{Vect}(1, x, y, z, t)$ be the coalgebra in which t, y, z are primitives and $\Delta(x) = 1 \otimes x + x \otimes 1 + y \otimes z$. Then C carries a unique rack bialgebra structure in which $- \triangleleft x, - \triangleleft t: C \rightarrow C$ are zero and*

$$\begin{aligned} x \triangleleft z = t, \quad x \triangleleft y = t, \quad z \triangleleft z = 0, \quad z \triangleleft y = 0, \\ y \triangleleft z = 0, \quad y \triangleleft y = 0, \quad t \triangleleft z = 0, \quad t \triangleleft y = 0. \end{aligned}$$

Proof. The self-distributivity

$$(a \triangleleft b) \triangleleft c = (a \triangleleft c_{(1)}) \triangleleft (b \triangleleft c_{(2)})$$

is clear if one of the three elements is 1. Otherwise, it follows from the fact that $(a \triangleleft b) \triangleleft c = 0$ for $a, b, c \in \check{C}$, as then both sides of the equation vanish.

Now let us check that

$$(a \triangleleft b)_{(1)} \otimes (a \triangleleft b)_{(2)} = (a_{(1)} \triangleleft b_{(1)}) \otimes (a_{(2)} \triangleleft b_{(2)}).$$

For $b = 1$, both sides are equal. For primitive b , the equation reads:

$$(a \triangleleft b)_{(1)} \otimes (a \triangleleft b)_{(2)} = (a_{(1)} \triangleleft b) \otimes a_{(2)} + a_{(1)} \otimes (a_{(2)} \triangleleft b).$$

For $b = t$, both sides are clearly zero. For $b = y$ or $b = z$, the only non-trivial case is $a = x$. We have for $a = x$ and $b = y$ for the LHS:

$$(x \triangleleft y)_{(1)} \otimes (x \triangleleft y)_{(2)} = t_{(1)} \otimes t_{(2)} = 1 \otimes t + t \otimes 1.$$

We have for $a = x$ and $b = y$ for the RHS:

$$\begin{aligned} (x_{(1)} \triangleleft y) \otimes x_{(2)} + x_{(1)} \otimes (x_{(2)} \triangleleft y) &= (1 \triangleleft y) \otimes x + (x \triangleleft y) \otimes 1 \\ &+ 1 \otimes (x \triangleleft y) + x \otimes (1 \triangleleft y) \\ &+ (y \triangleleft y) \otimes z + y \otimes (z \triangleleft y) \\ &= t \otimes 1 + 1 \otimes t. \end{aligned}$$

The case $a = x$ and $b = z$ is similar. The last case is the case $b = x$. The equation reads then:

$$0 = a_{(1)} \otimes (a_{(2)} \triangleleft x) + (a_{(1)} \triangleleft x) \otimes a_{(2)} + (a_{(1)} \triangleleft y) \otimes (a_{(2)} \triangleleft z).$$

The first two terms (and the LHS) are zero, because $- \triangleleft x$ is zero. Concerning the third term, the only case where it is non-zero is when $a_{(1)} \otimes a_{(2)}$ contains a non-zero component proportional to $x \otimes x$. However, there is no a with this property in C . \square

Since C is not cocommutative, Theorem 1 can not be applied to construct a canonical Yetter-Drinfel'd rack structure. However, C is a Yetter-Drinfel'd rack over the coordinate ring of the upper triangular unipotent group in $GL(3)$:

Proposition 5. *Let H be the Hopf algebra whose underlying algebra is the polynomial ring $k[X, Y, Z]$ with the coproduct*

$$\Delta(X) = 1 \otimes X + X \otimes 1 + Y \otimes Z,$$

$$\Delta(Y) = 1 \otimes Y + Y \otimes 1, \quad \Delta(Z) = 1 \otimes Z + Z \otimes 1.$$

Then the rack bialgebra C from Proposition 4 becomes a Yetter-Drinfel'd rack over H with $q(x) = X, q(y) = Y, q(z) = Z, q(t) = 0$.

Proof. The map q is evidently a morphism of coalgebras as $k \cdot t$ is a coideal. As the linear maps $- \triangleleft x, - \triangleleft y, - \triangleleft z: C \rightarrow C$ commute with each other, there is a well-defined right H -module structure on C such that $a \triangleleft b = a \cdot q(b)$. The H -module coalgebra condition $(c \cdot h)_{(1)} \otimes (c \cdot h)_{(2)} = (c_{(1)} \cdot h_{(1)}) \otimes (c_{(2)} \cdot h_{(2)})$ and (3) is verified by direct computation when h is one of the generators X, Y, Z and hence holds for all $h \in H$. \square

Thus C can be constructed as in Corollary 1 inside the Hopf algebra $T(\check{C}) \rtimes H$ despite the fact that it is non-cocommutative, and as well from the corresponding Hopf algebra object in \mathcal{LM} .

6. DEFORMATION COHOMOLOGY

In [3], the authors define cohomology groups controlling the deformation theory of linear shelves. The method is to define the operations (i.e. the co-product and the shelf product) on $C[[t]]$ instead of C , for a formal parameter t , and then to impose self-distributivity and the coalgebra morphism condition on \triangleleft as well as the coassociativity of \triangle . These requirements up to t^{n+1} then give cocycle identities up to t^{n+2} . They are realised in a bicomplex whose differentials are given explicitly up to degree 3, see Section 6 in [3] for details.

On the other hand, in [2] a deformation complex for *cocommutative* rack bialgebras C is defined. Therein, the deformations involve only the shelf product, and not the underlying coalgebra. Cochains are defined to be coderivations with respect to iterates of the shelf product. In [1], [2], left shelves and Leibniz algebras are considered, so we transpose the definitions here to right shelves and Leibniz algebras.

Let C be a rack bialgebra with a cocommutative underlying coalgebra. Then the rack product $\mu(x, y) := x \triangleleft y$ can be iterated to

$$\mu^n(x_1, \dots, x_n) := (\dots (x_1 \triangleleft x_2) \triangleleft \dots) \triangleleft x_n,$$

with the convention that $\mu^1 = \text{id}$ and $\mu^2 = \mu$.

Note that in the following definition we moved the Sweedler notation to the top to avoid confusion with the other indices.

Definition 4. Let C be a rack bialgebra with a cocommutative underlying coalgebra. The deformation complex of C is the graded vector space $C^*(C; C)$ defined in degree n by

$$C^n(C; C) := \text{Coder}(C^{\otimes n}, C, \mu^n)$$

denoting the space of coderivations along μ^n , i.e. of linear maps $\omega : C^{\otimes n} \rightarrow C$ such that

$$\triangle_C \circ \omega = (\omega \otimes \mu^n + \mu^n \otimes \omega) \circ \triangle_{C^{\otimes n}},$$

endowed with the differential $d_C : C^*(C; C) \rightarrow C^{*+1}(C; C)$ defined in degree n by

$$d_C^n := \sum_{i=1}^n (-1)^{i+1} (d_{i,1}^n - d_{i,0}^n) + (-1)^{n+1} d_{n+1}^n$$

where the maps $d_{i,1}^n$ and $d_{i,0}^n$ are defined respectively by

$$d_{i,1}^n \omega(r_1, \dots, r_{n+1}) := \omega(r_1, \dots, r_{i-1}, r_{i+1}^{(1)}, \dots, r_{n+1}^{(1)}) \triangleleft \mu^{n-i+2}(r_i, r_{i+1}^{(2)}, \dots, r_{n+1}^{(2)})$$

and

$$d_{j,0}^n \omega(r_1, \dots, r_{n+1}) := \omega(r_1 \triangleleft r_j^{(1)}, \dots, r_{j-1} \triangleleft r_j^{(j-1)}, r_{j+1}, \dots, r_{n+1})$$

and d_{n+1}^n by

$$\begin{aligned} d_{n+1}^n \omega(r_1, \dots, r_{n+1}) \\ := \mu^n(r_1, r_3^{(1)}, \dots, r_{n+1}^{(1)}) \triangleleft \omega(r_2, r_3^{(2)}, \dots, r_{n+1}^{(2)}) \end{aligned}$$

for all ω in $C^n(C; C)$ and r_1, \dots, r_{n+1} in C .

It is shown in [2] that $d_C^n \circ d_C^{n-1} = 0$ and that d_C^n sends coderivations to coderivations.

Remark 3. Cochains in the cohomology in Section 6 of [3] are maps $C^{\otimes i} \rightarrow C^{\otimes j}$, like in the Gerstenhaber-Schack cohomology of associative bialgebras. Defining *special cochains* as $(\eta_1, 0, \dots, 0)$ where $\eta_1 : C^{\otimes n} \rightarrow C$, one obtains that cocycles in $C^*(C, C)$ as above give rise to special cocycles. Indeed, while the first cocycle identity in [3] is just the cocycle identity with respect to the above coboundary operator d_C^n , the second cocycle identity is the coderivation property and all other identities are trivial. \triangle

Example 7. The rack bialgebra $C = \text{Vect}(1, x, y, z, t)$ defined in Section 5 is a first order deformation in the sense of the cohomology defined in [3] of the cocommutative shelf in coalgebras $C_0 = \text{Vect}(1, x, y, z, t)$ where x, y, z, t are primitives and 1 is group-like. The cocycle associated to the deformation is $\omega : C \rightarrow C^{\otimes 2}$ given by

$$\omega(x) = y \otimes z$$

and is trivial on the other basis elements. Clearly, C is not a deformation in the sense of the cohomology defined in [2] as in this complex, the coproduct is not deformed. \triangle

We come to the main theorem of this section:

Theorem 3. *Consider the rack bialgebra $C = k \oplus \mathfrak{h}$ associated to a (right) Leibniz algebra \mathfrak{h} , see Example 3. The Leibniz cohomology complex with values in the adjoint representation embeds into the deformation complex $(C^*(C; C), d_C^*)$ defined above (by [2]).*

Proof. We extend Leibniz cochains $f : \mathfrak{h}^{\otimes n} \rightarrow \mathfrak{h}$ to cochains in the complex $C^*(C, C)$ with $C = k \cdot 1 \oplus \mathfrak{h}$ by setting them zero on all components in $k \cdot 1 \subset C$. More precisely

$$\omega((\lambda_1, x_1), \dots, (\lambda_n, x_n)) := \text{pr}_{\mathfrak{h}}(f(x_1, \dots, x_n)),$$

for $(\lambda_i, x_i) \in k \cdot 1 \oplus \mathfrak{h}$ for all $i = 1, \dots, n$ and with $\text{pr}_{\mathfrak{h}} : k \cdot 1 \oplus \mathfrak{h} \rightarrow \mathfrak{h}$ the natural projection.

With this definition, it follows that these cochains are coderivations along μ^n , i.e.

$$\Delta_C \circ \omega = (\omega \otimes \mu^n + \mu^n \otimes \omega) \circ \Delta_{C^{\otimes n}}.$$

Indeed, when computing the iterated coproduct $\Delta_{C^{\otimes n}}(r_1, \dots, r_n)$, the elements $r_i \in \mathfrak{h}$ are distributed among the two factors in $C^{\otimes n} \otimes C^{\otimes n}$ and all other components are filled with units. On the LHS, $\omega(r_1, \dots, r_n)$ is primitive by construction, thus we get the two terms $\omega(r_1, \dots, r_n) \otimes 1 + 1 \otimes \omega(r_1, \dots, r_n)$. On the RHS, the only terms which do not vanish are those with all r_i as arguments in ω . This shows the equality.

Now we specify the different parts of the coboundary operator.

$$\begin{aligned} d_{i,1}^n \omega(r_1, \dots, r_{n+1}) &= \\ \omega(r_1, \dots, r_{i-1}, r_{i+1}^{(1)}, \dots, r_{n+1}^{(1)}) &\triangleleft \mu^{n-i+2}(r_i, r_{i+1}^{(2)}, \dots, r_{n+1}^{(2)}) \\ &= [\omega(r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_{n+1}), r_i], \end{aligned}$$

because the only contributing term is the one where all r_j are arguments of ω , i.e. all the units are in μ^{n-i+2} .

$$\begin{aligned} d_{j,0}^n \omega(r_1, \dots, r_{n+1}) &= \omega(r_1 \triangleleft r_j^{(1)}, \dots, r_{j-1} \triangleleft r_j^{(j-1)}, r_{j+1}, \dots, r_{n+1}) \\ &= \omega([r_1, r_j], \dots, r_{j-1}, r_{j+1}, \dots, r_{n+1}) + \dots \\ &\dots + \omega(r_1, \dots, [r_{j-1}, r_j], r_{j+1}, \dots, r_{n+1}), \end{aligned}$$

because only one of the $r_j^{(k)}$ is equal to r_j and all others are equal to 1.

$$\begin{aligned} d_{n+1}^n \omega(r_1, \dots, r_{n+1}) &= \\ &= \mu^n(r_1, r_3^{(1)}, \dots, r_{n+1}^{(1)}) \triangleleft \omega(r_2, r_3^{(2)}, \dots, r_{n+1}^{(2)}) \\ &= [r_1, \omega(r_2, r_3, \dots, r_{n+1})], \end{aligned}$$

because this is the only term where one does not insert 1 into ω . \square

Remark 4. A similar statement is true for the rack bialgebra C associated to a (set-theoretical) shelf, cf Example 2. In fact, the deformation complex in [2] has been constructed as a linearization of the cohomology complex of a shelf. \triangle

7. OUTLOOK AND FURTHER QUESTIONS

One natural direction of further research is to link the three different approaches to deformation theories of rack bialgebras:

- (i) The bicomplex of Carter-Crans-Elhamedadi-Saito [3].
- (ii) The Gerstenhaber-Schack bicomplex in \mathcal{LM} [7].

(iii) The braided Leibniz complex [10].

Our previous article [9] showed how objects in the setting (ii) can be expressed as objects in the setting (iii). The present article provides the link from (i) to (ii) and from (i) to (iii) in the cocommutative case. A deformation theory for cocommutative rack bialgebras alone should however rather be built on the dual version of André-Quillen cohomology (the cohomology theory that controls deformations of commutative algebras) than the Cartier cohomology that underlies Gerstenhaber-Schack cohomology.

In general, Gerstenhaber-Schack cohomology of a bialgebra $f: M \rightarrow H$ in \mathcal{LM} captures more information than the cohomology from [3], as one can deform H , M and f . On the other hand, braided Leibniz cohomology seems to capture less information than [3], because it only indirectly reflects deformations of the coproduct. A full investigation of the relation between the three settings seems a fruitful future research direction.

Two rather concrete questions that arises from this article are:

Question 1. Is there a (necessarily non-cocommutative) rack bialgebra that can not be expressed as a Yetter-Drinfel'd rack over any bialgebra?

Question 2. For which rack bialgebras is $U(C)$ a Hopf algebra?

This matters in the passage from (ii) to (i) as the antipode is in general necessary to turn the coinvariants in a Hopf tetramodule into a right module. One expects that U is then part of an adjoint pair of functors between such Hopf racks and Hopf algebras, at least in the cocommutative setting.

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