



**HAL**  
open science

# An elastoplastic model for saturated freezing soils based on thermo-poromechanics

Enlong Liu, Yuanming Lai, Henry Wong, Jili Feng

► **To cite this version:**

Enlong Liu, Yuanming Lai, Henry Wong, Jili Feng. An elastoplastic model for saturated freezing soils based on thermo-poromechanics. *International Journal of Plasticity*, 2018, 107, pp.246-285. 10.1016/j.ijplas.2018.04.007 . hal-01888351

**HAL Id: hal-01888351**

**<https://hal.science/hal-01888351>**

Submitted on 1 Dec 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# An elastoplastic model for saturated freezing soils based on thermo-poromechanics

Enlong Liu<sup>1,2</sup>, Yuanming Lai<sup>1,\*</sup>, Henry Wong<sup>3</sup>, Jili Feng<sup>4</sup>

1 State Key Laboratory of Frozen Soil Engineering, Northwest Institute of Eco-Environment and Resources, Chinese Academy of Sciences, Lanzhou 730000, China; 2 State Key Laboratory of Hydraulics and Natural River Engineering, College of Water Resource and Hydropower, Sichuan University, Chengdu 610065, China; 3. LTDS (UMR CNRS 5513), ENTPE, Vaulx-en-Velin 69120, France; 4. School of Mechanics and Civil Engineering, China University of Mining and Technology (Beijing), Beijing 100083, P. R. China. Corresponding author: Yuanming Lai. Contact Tel: +86 931 4967024 and Email: ymlai@lzb.ac.cn.

## Abstract:

An elastoplastic theory for saturated freezing soils is presented on the basis of thermoporomechanics. A saturated freezing soil considered as an open system and both Eulerian and Lagrangian formulations considering the phase transition between ice crystals and unfrozen water are given for mass conservation, momentum balance, kinetic energy theorem, first and second thermodynamics, the Clausius-Duhem inequality and conduction laws for fluid mass and heat. Using the Lagrangian saturation and considering solid-fluid interface interactions, a constitutive model for poro-elastoplastic saturated freezing soils is formulated based on the irreversible process. For isotropic linear thermo-poro-elasticity and ideal plasticity, the stress strain relationship for saturated freezing soils considering the influence of temperature and interface energy is proposed. In addition, for hardening plasticity, the general stress strain relationship is formulated under the conditions that the associated or non-associated flow rule is assumed, and a corresponding constitutive model is presented to model the cryogenic triaxial compression of saturated frozen soils. The constitutive theory proposed here provides a potential basis for modelling thermo-hydro-mechanical coupling interactions of saturated soils during the freezing process.

**Keywords:** Thermoporomechanics; Saturated freezing soils; Constitutive model; Elastoplastic theory;

25 Thermodynamics

## 26 **1. Introduction**

27 Frozen soils are compound materials consisting of solid mineral particles, ice crystals, liquid water  
28 (free water and tightly bound water), and gaseous inclusions (water vapor and air), whose mechanical  
29 and deformation features are distinct from other geological materials, such as soil, concrete, and rock at  
30 room temperature (Tsyтовich, 1985; Andersland and Ladanyi, 2004). When the temperature falls below  
31 the freezing point, the in-pore water can be transformed to ice crystals. These ice crystals may  
32 gradually grow, accompanied by water migration to these frozen zones, while the interfaces between  
33 ice crystals and unfrozen water will move (Xu et al., 2010). This process may eventually lead to frost  
34 heave due to large ground displacements and deformations, which cannot be adequately described  
35 using infinitesimal deformation theory widely applied in existing research (Na et al., 2017). There are  
36 approximately  $2.15 \times 10^6 \text{ km}^2$  of permafrost regions and  $5.14 \times 10^6 \text{ km}^2$  of seasonally frozen  
37 regions in Western and Northern China (Zhou et al., 2000). With the increase in railway and oil line  
38 construction in permafrost and seasonally frozen regions, studies on the thermo-hydro-mechanical  
39 coupling interactions related to frost heave problems have become increasingly important. However,  
40 many existing studies on these problems are based on traditional theory of frozen soil mechanics  
41 (Thomas et al., 2009), in which the frozen soils are usually assimilated to deformable solid while the  
42 physical interactions of ice crystals and unfrozen water are ignored (Lai et al., 2013). In fact, frozen  
43 soils are porous media, in which the interactions of soil particle, ice crystals and unfrozen water have  
44 great influences on their deformation and mechanical properties (Coussy, 2004). Compared with other  
45 porous media, like hydrated cement at low temperature, soils during the freezing process can exhibit  
46 irreversible plastic deformations, which dominates their strength and mechanical behaviour (Lai et al.,

47 2009). Some poroelastic models for porous media exposed to freezing temperatures have been  
48 proposed (Coussy and Monteir, 2008), but few of them account for plastic deformations during the  
49 freezing process. Therefore, formulating a general elasto-plastic theory for saturated freezing soils  
50 based on thermoporomechanics is necessary. The presentation of such a general theory is the objective  
51 of this paper.

52 During the freezing process of soils, coupled thermo-hydro-mechanical interactions induce important  
53 changes on the distribution of stresses, strains, displacements and moisture contents, in particular due  
54 to formation of ice crystals and movement of interfaces between soil particles, ice crystals and  
55 unfrozen water. A number of experimental investigations have been carried out and theoretical models  
56 were developed to study the coupled thermo-hydro-mechanical behaviour of soils during the freezing  
57 process (Na et al., 2017). Lai et al. (1998) first derived a mathematical model on the coupled problem  
58 of temperature, seepage and stress fields accounting for phase change between liquid water and ice  
59 crystals. Their model was implemented in a finite element program to analyze the temperature and  
60 stress fields around tunnels in cold regions. Neaupane et al. (1999) proposed a numerical model of  
61 thermal-mechanical-fluid flow coupling system to simulate the laboratory freezing and thawing  
62 experiments on rocks, in which an anisotropic elastic stress-strain constitutive model was used and a  
63 two-dimensional numerical modelling performed. Later, this work (Neaupane and Yamabe, 2001) was  
64 extended to consider a nonlinear elasto-plastic simulation of freezing and thawing of rock. Li et al.  
65 (2000) established a heat-moisture-deformation (HMD) coupled model to simulate the frozen-soil  
66 foundation based on the equilibrium, continuity and energy principles of the multi-phase porous  
67 medium, which was also used to analyze the freezing and thawing processes of soil foundation. Yang et  
68 al. (2006) proposed an analysis model that couples the water freezing, temperature and stress fields,

69 which was also applied to an underground excavation problem of a corridor where ground freezing was  
70 used, and the numerical predictions were compared to field measurements. Boukpeti (2008) derived a  
71 solution to the problem of freezing of a poro-elastic material and analyzed in the case of  
72 one-dimensional deformation within the framework of thermo-poroelasticity of freezing materials, with  
73 special attention to the propagation of the freezing front boundary. Nishimura et al. (2009) presented a  
74 fully coupled thermo-hydro-mechanical finite element simulation of freezing and thawing of  
75 water-saturated soils, accounting explicitly for thermal, hydraulic and mechanical process as well as  
76 their interactions. Lu et al. (2011) proposed a model for saturated porous media undergoing phase  
77 transition using mixture theory and phase field theory. Zhou and Li (2012) proposed the concept of  
78 "separating void ratio" as a criterion for the formation of ice lenses, and established a coupled model of  
79 water, heat and stress transfer for saturated freezing soils. Zhou and Meschke (2013) proposed a  
80 three-phase finite element soil model based on the theory of poromechanics, in which solid particles,  
81 liquid water and ice crystals were considered as separate phases while temperature, liquid pressure, and  
82 solid displacement were the primary variables. In their model, Clapeyron's equation was applied in  
83 freezing soils to describe the relationship between temperature, water pressure and ice pressure when  
84 ice and water coexist in phase equilibrium. Sheng et al. (2014) proposed a simple quantitative model to  
85 simulate the pumping-enhanced frost heave, and the numerical results demonstrated that the proposed  
86 mechanism could indeed provide a rational explanation for the otherwise unexpected frost heave.  
87 Zhang (2014) developed a constitutive model for frozen soils based on the revised Cam-clay model,  
88 and implemented the model into a finite element system with a thermal-hydro-mechanical framework  
89 to simulate the multi-physical processes during freeze-thaw cycles. Na and Sun (2017) proposed a  
90 stabilized thermo-hydro-mechanical finite element model to investigate the freeze-thaw action of

91 frozen porous media in the domain of finite deformations. Mixture theory was used in their work and  
92 frozen soils were idealized as a three-phase composite consisting of solid grains, unfrozen water and  
93 ice crystals. Besides achievements of thermo-hydro-mechanical coupled interactions in the freezing  
94 process, there are many constitutive models proposed to describe the stress-strain behaviour of frozen  
95 soils and geological materials. These constitutive models for frozen soils, geological materials and  
96 other materials are based on elastic theory (Loria et al., 2017), plasticity (Khan et al., 1991;  
97 Muraleetharan et al., 2009; Kamrin, 2010; Zhang et al., 2012; Darabi et al., 2012; Yao et al., 2009,  
98 2013, 2015; Xu et al., 2017; Liu et al., 2017), micromechanical modelling (Steinhauser et al., 2009;  
99 Collard et al., 2010; Zhu et al., 2010; Nicot et al., 2012; Yang et al., 2015; Shen and Shao, 2016),  
100 damage mechanics (Lai et al., 2010; de Sciarra, 2012) and thermodynamics ( Al-Rub and Darabi, 2012;  
101 Henann and Kamrin, 2014; Krairi and Doghri, 2014; Lai et al., 2016; Zhang, 2017).

102 Even though some researchers proposed multi-physics theories to consider the coupling process of  
103 unsaturated soils with varying temperatures above zero (Lei et al., 2014; Sciarra, 2016), to the best of  
104 our knowledge, few studies have been performed on the elasto-plastic theory for saturated freezing  
105 soils within a rigorous theoretical framework based on well-proofed thermodynamic principles and  
106 fundamental physical laws. We will explore this in the present paper. In the following, firstly the  
107 equations of mass and momentum balance and the kinetic theorem for saturated frozen soils in Eulerian  
108 and Lagrangian formulations considering phase transition are presented, followed by the physical laws  
109 governing the behaviours of various components of a saturated frozen soil. Lastly, the constitutive  
110 framework for poro-elasto-plastic saturated frozen soils is proposed and verified with cryogenic triaxial  
111 compression test of saturated frozen soils.

112 **2. Mass balance, momentum balance and the kinetic theorem for saturated**  
113 **frozen soils**

114 A typical elementary volume of a saturated frozen soil is schematically represented in Fig. 1, which  
 115 is treated as a porous medium with the superimposition of three continua, the soil skeleton, ice crystals  
 116 and unfrozen water. Both the soil skeleton and the fluids including ice crystals and unfrozen water  
 117 coincide with one another and the same geometrical point defined by position vector. At time  $t = 0$ , in  
 118 its initial configuration (or "reference configuration"), the position of the soil particle in this elementary  
 119 volume is represented by  $\mathbf{X}$ , or  $X_i \mathbf{e}_i$  relative to a cartesian coordinate system. At time  $t$ , the soil  
 120 particles will move and deform and lie in the current configuration, represented by  $\mathbf{x}$ , or  $x_i(X_j, t)$ . In  
 121 other words:

$$122 \quad \mathbf{X} = X_i \mathbf{e}_i \quad \text{and} \quad \mathbf{x} = x_i(X_j, t) \mathbf{e}_i. \quad (1)$$

123 The deformation gradient  $\mathbf{F}$  is expressed as follows,

$$124 \quad \mathbf{F} = \nabla_{\mathbf{x}} \mathbf{x} \quad \text{or} \quad F_{ij} = \frac{\partial x_i}{\partial X_j}. \quad (2)$$

125 The displacement vector is  $\boldsymbol{\xi}(\mathbf{X}, t)$ , and we have

$$126 \quad \mathbf{x} = \mathbf{X} + \boldsymbol{\xi}; \quad \mathbf{F} = \mathbf{1} + \nabla_{\mathbf{x}} \boldsymbol{\xi}; \quad \text{and} \quad F_{ij} = \delta_{ij} + \frac{\partial \xi_i}{\partial X_j}, \quad (3)$$

127 in which  $\delta_{ij}$  is Kronecker delta and  $\mathbf{1}$  unit tensor.

128

129

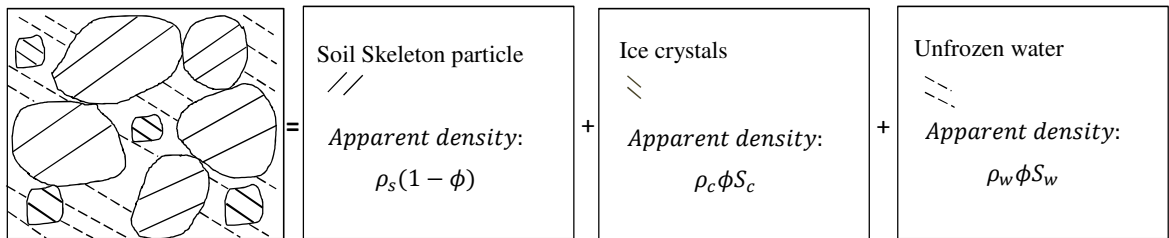
130

131

132

133

134



135

Fig. 1 Macroscopic model of the REV of saturated frozen soils

136 The volume of the undeformed element at the reference configuration  $d\Omega_0$  and the volume of the  
 137 deformed element in the current configuration  $d\Omega$  are related to each other via the Jacobian  $J$  (Lai et  
 138 al., 2010),

$$139 \quad d\Omega = \det \mathbf{F} \cdot d\Omega_0 = J d\Omega_0. \quad (4)$$

140 The infinitesimal representative elementary volume (REV) of a porous continuum, as shown in Fig. 1,

141 consists of solid soil grains and a connected porous volume filled by ice crystals and unfrozen liquid  
 142 water. Both the Eulerian porosity  $n$  and the Lagrangian porosity  $\phi$  refer to the entirety of this  
 143 connected volume (ice and water):

$$144 \quad \phi d\Omega_0 = nd\Omega; \quad \phi = Jn. \quad (5)$$

145 In the undeformed reference configuration, the total porous volume is  $\phi_0 d\Omega_0$  with  $\phi_0 = n_0$ , and  
 146  $\phi_0, n_0$  are Lagrangian porosity and Eulerian porosity at undeformed reference configuration,  
 147 respectively. At current time  $t$ , the porous volume is  $\phi d\Omega_0$ .

148 For a REV of saturated frozen soil shown in Fig. 1, the overall Lagrangian porosity  $\phi$  can be split  
 149 into two partial porosities,  $\phi_c$  (ice crystals) and  $\phi_w$  (unfrozen water):

$$150 \quad \phi = \phi_c + \phi_w. \quad (6)$$

151 The current partial porosities  $\phi_\alpha (\alpha=c, w)$  may be written in the form as follows,

$$152 \quad \phi_\alpha = \phi_0 S_\alpha + \varphi_\alpha S_c + S_w = 1, \quad (7)$$

153 where  $S_\alpha$  is the Lagrangian saturation degree of phase  $\alpha$ . The coefficients  $S_c$  et  $S_w$  reflect changes  
 154 due to phase transformation and subsequent translation of the ice-water interface, whereas the  
 155 coefficients  $\varphi_c$  et  $\varphi_w$  reflect the volume change due to deformation of the porous space. The overall  
 156 change of the porous volume can only due to deformation of porous space, hence:

$$157 \quad \phi - \phi_0 = \varphi_c + \varphi_w. \quad (8)$$

158 Based on (5) and (6), we can also define the Eulerian partial porosities by:

$$159 \quad \phi_\alpha = Jn_\alpha, \quad (9)$$

160 so that  $n_\alpha d\Omega = \phi_\alpha d\Omega_0$  represents the volume of  $\alpha$ -phase in the current elementary volume  $d\Omega$ .

161 Similarly to (6), we have:

$$162 \quad n = n_c + n_w, \quad (10)$$



163 The Eulerian saturation degrees are defined by:

$$164 \quad n_c = n s_c, \quad n_w = n s_w, \quad s_c + s_w = 1, \quad (11)$$

165 in which  $n_\alpha (\alpha = c, w)$  is Eulerian partial porosity of phase  $\alpha$ , and  $s_\alpha (\alpha = c, w)$  is Eulerian  
166 saturation degree of phase  $\alpha$ .

## 167 2.1 Mass conservation

168 In the following, the mass-balance equations in the absence of phase change is presented first and  
169 then that considering phase change are given. For both cases, the Eulerian formulation is presented,  
170 followed by the Lagrangian formulation.

171 Let  $\rho_s$  and  $\rho_\alpha (\alpha = c, w)$  be the intrinsic mass densities of the solid matrix and the in-pore  
172 components, so that  $\rho_s(1-n)d\Omega$  and  $\rho_\alpha n s_\alpha d\Omega$  are respectively the mass of solid grains and that of  
173 the  $\alpha$ -component currently contained in the material volume  $d\Omega$ . Accordingly, the macroscopic (or  
174 apparent) mass densities of the skeleton and  $\alpha$ -component are respectively  $\rho_s(1-n)$  and  $\rho_\alpha n s_\alpha$ .  
175 When phase change is absent, both for the skeleton and the in-phase components, the mass balance  
176 equations take the following form:

$$177 \quad \frac{d^s}{dt} \int_\Omega \rho_s(1-n) d\Omega = 0, \quad \text{and} \quad \frac{d^s}{dt} (\rho_s(1-n) d\Omega) = 0, \quad (12)$$

$$178 \quad \frac{d^\alpha}{dt} \int_\Omega \rho_\alpha n s_\alpha d\Omega = 0, \quad \text{and} \quad \frac{d^\alpha}{dt} (\rho_\alpha n s_\alpha d\Omega) = 0. \quad (13)$$

179 Therefore, the Eulerian continuity equations can be expressed as follows,

$$180 \quad \frac{\partial(\rho_s(1-n))}{\partial t} + \nabla_x \cdot (\rho_s(1-n)\mathbf{V}^s) = 0, \quad (14)$$

$$181 \quad \text{and} \quad \frac{\partial(\rho_\alpha n s_\alpha)}{\partial t} + \nabla_x \cdot (\rho_\alpha n s_\alpha \mathbf{V}^\alpha) = 0, \quad (15)$$

182 in which  $\mathbf{V}^s$  and  $\mathbf{V}^\alpha (\alpha = c, w)$  are the velocities of solid grains and in-pore phases (ice crystals and  
183 unfrozen water) respectively, and can be expressed as follows,

$$184 \quad \mathbf{V}^\pi(\mathbf{x}, t) = \frac{d^\pi \mathbf{x}}{dt} \quad (\pi = s, c, w). \quad (16)$$

185 where  $\mathbf{x}$  refers to the common position of all the particles ( $\pi = s, c, w$ ) at the current time.

186 For the Lagrangian formulation, let  $m_\alpha$  be the Lagrangian fluid mass content related to fluid  $\alpha$ , we  
187 have:

$$188 \quad m_\alpha = \rho_\alpha \phi_\alpha = \rho_\alpha \phi S_\alpha. \quad (17)$$

189 Following the derivation of saturated porous medium (Coussy, 1989), we can obtain the Lagrangian  
190 formulation of the mass conservation of in-pore phases (ice crystals or unfrozen water) as follows:

$$191 \quad \frac{dm_\alpha}{dt} + \nabla_x \cdot \mathbf{M}_\alpha = 0, \text{ and } \frac{\partial m_\alpha}{\partial t} + \frac{\partial M_{\alpha i}}{\partial x_i} = 0, \quad (18)$$

192 where  $\mathbf{M}_\alpha(X_i, t)$  is the Lagrangian flux attached to the initial configuration and linked to the Eulerian  
193 mass flux  $\mathbf{w}_\alpha$  through the relations as follows,

$$194 \quad \mathbf{w}_\alpha \cdot \mathbf{n} da = \mathbf{M}_\alpha \cdot \mathbf{N} dA, \quad (19)$$

$$195 \quad \mathbf{M}_\alpha = J \mathbf{F}^{-1} \cdot \mathbf{w}_\alpha, \text{ and } M_{\alpha i} = J \frac{\partial x_i}{\partial X_j} \cdot w_{\alpha j}, \quad (20)$$

$$196 \quad \nabla_x \cdot \mathbf{w}_\alpha d\Omega = \nabla_X \cdot \mathbf{M}_\alpha d\Omega_0, \text{ and } J \frac{\partial w_{\alpha i}}{\partial x_i} = \frac{\partial M_{\alpha i}}{\partial X_i}. \quad (21)$$

197 Note that the liquid water mass flux writes:  $\mathbf{w}_w(\mathbf{x}, t) = \rho_w \boldsymbol{\vartheta}$  with the filtration vector  $\boldsymbol{\vartheta} =$   
198  $n(\mathbf{V}^w - \mathbf{V}^s)$  whereas  $\mathbf{M}_w = \mathbf{w}_w = 0$  since ice crystals are attached to the solid skeleton with  
199  $\mathbf{V}^c = \mathbf{V}^s$ .  $\mathbf{N}$  is the unit normal of surface  $dA$  in initial configuration, and  $\mathbf{n}$  the unit normal of  
200 surface  $da$  in current configuration.

201 The Lagrangian approach to the mass balance of the soil skeleton is as follows,

$$202 \quad \rho_s(1 - n)d\Omega = \rho_s^0(1 - n_0)d\Omega_0, \quad (22)$$

203 in which  $\rho_s^0$  is the initial matrix mass density, and  $n_0 = \phi_0$  the initial porosity. Therefore, we can  
204 obtain the mass balance equation of the soil skeleton as follows,

$$205 \quad m_s = J\rho_s(1 - n) = m_s^0 = \rho_s^0(1 - \phi_0). \quad (23)$$

206 Note that phase change is not accounted for in the above derivation. The case where phase change

207 occurs for the open system is now considered. Assuming that phase change only concerns liquid water  
 208 and ice crystals, the mass balance equation derived here-above for the solid matrix still applies. On the  
 209 other hand, the mass balance equation for unfrozen water and ice crystals accounting for phase change  
 210 now write as follows:

$$211 \quad \frac{d^c}{dt} \int_{\Omega} \rho_c n_c d\Omega = \int_{\Omega} \Lambda_{w \rightarrow c} d\Omega, \quad (24)$$

$$212 \quad \frac{d^w}{dt} \int_{\Omega} \rho_w n_w d\Omega = \int_{\Omega} \Lambda_{c \rightarrow w} d\Omega, \quad (25)$$

213 where  $\Lambda_{w \rightarrow c}$  stands for the mass of water transforming into ice crystals per unit overall current volume  
 214 and per unit time. The overall mass conservation requires that  $\Lambda_{w \rightarrow c} = -\Lambda_{c \rightarrow w}$ .

215 For any quantity  $\Xi$ , when the particle derivative applies to its volume integral, we have the  
 216 following theorem:

$$217 \quad \frac{d^\pi}{dt} \int_{\Omega} \Xi d\Omega = \int_{\Omega} \frac{d^\pi}{dt} (\Xi d\Omega), \quad (26)$$

$$218 \quad \text{so that} \quad \frac{d^\pi}{dt} \int_{\Omega} \Xi d\Omega = \int_{\Omega} \left( \frac{\partial \Xi}{\partial t} + \nabla_x \cdot (\Xi \mathbf{V}^\pi) \right) d\Omega. \quad (27)$$

219 On account of equation (26), the equations (24) and (25) can be rewritten as follows,

$$220 \quad \frac{d^c}{dt} (\rho_c n_c d\Omega) = -\Lambda_{c \rightarrow w} d\Omega, \text{ and } \frac{d^w}{dt} (\rho_w n_w d\Omega) = \Lambda_{c \rightarrow w} d\Omega. \quad (28)$$

221 Using equation (27), we obtain the mass balance equations for the ice crystals and unfrozen water in  
 222 Eulerian formulation as follows:

$$223 \quad \frac{\partial(\rho_c n_c)}{\partial t} + \nabla_x \cdot (\rho_c n_c \mathbf{V}^c) = -\Lambda_{c \rightarrow w}, \quad (29)$$

$$224 \quad \frac{\partial(\rho_w n_w)}{\partial t} + \nabla_x \cdot (\rho_w n_w \mathbf{V}^w) = \Lambda_{c \rightarrow w}. \quad (30)$$

225 Following the similar derivation of the previous Eulerian fluid continuity equations, the Lagrangian  
 226 formulation of the continuum equations for ice crystals and unfrozen water can be obtained as follows,

$$227 \quad \frac{dm_c}{dt} + \nabla_X \cdot \mathbf{M}_c = -\overline{\omega}_{c \rightarrow w}, \quad (31)$$

$$228 \quad \frac{dm_w}{dt} + \nabla_X \cdot \mathbf{M}_w = \overline{\omega}_{c \rightarrow w}, \quad (32)$$

229 in which  $m_\alpha = J\rho_\alpha n_\alpha = \rho_\alpha \phi_\alpha$ ; and  $\varpi_{\alpha\rightarrow\beta} = J\Lambda_{\alpha\rightarrow\beta}$ .

## 230 2.2 Momentum balance

231 With respect to purely mechanical considerations, the saturated frozen medium may be considered as  
 232 the superposition of the ice crystals and unfrozen water in mechanical interaction. Therefore, the  
 233 development of Coussy (1989, 1995) can be extended to saturated frozen soils to formulate the  
 234 momentum balance. Using the definition of material derivatives, the variation of the linear momentum  
 235 can now be expressed in terms of acceleration fields as follows,

$$236 \frac{d^s}{dt} \int_{\Omega} \rho_s (1-n) \mathbf{V}^s d\Omega + \sum_{\alpha=c,w} \frac{d^\alpha}{dt} \int_{\Omega} \rho_\alpha n_\alpha \mathbf{V}^\alpha d\Omega = \int_{\Omega} (\rho_s (1-n) \boldsymbol{\gamma}^s + \sum_{\alpha=c,w} \rho_\alpha n_\alpha \boldsymbol{\gamma}^\alpha) d\Omega +$$

$$237 \int_{\Omega} \Lambda_{c\rightarrow w} (\mathbf{V}^w - \mathbf{V}^c) d\Omega, \quad (33)$$

238 where  $\rho_s (1-n) \mathbf{V}^s$  and  $\rho_\alpha n_\alpha \mathbf{V}^\alpha$  ( $\alpha = c, w$ ) represent the linear momentum of the soil skeleton, ice  
 239 crystals and unfrozen water, respectively;  $\boldsymbol{\gamma}^s$  and  $\boldsymbol{\gamma}^\alpha$  are the accelerations of soil skeleton, ice crystals  
 240 and unfrozen water, written as  $\boldsymbol{\gamma}^\pi(\mathbf{x}, t) = \frac{d^\pi \mathbf{V}}{dt}$  ( $\pi = s, c, w$ );  $\Lambda_{c\rightarrow w} (\mathbf{V}^w - \mathbf{V}^c)$  accounts for the  
 241 variation in linear momentum due to the mass rate exchanged between ice crystals and unfrozen water.

242 On the other hand, the rate of change of linear momentum of all matters inside a material volume  $\Omega$  is  
 243 equal to the sum of all external forces acting on this matter (Lai et al., 2010). Therefore, the momentum  
 244 balance with respect to all matters inside a generic porous domain  $\Omega$  writes:

$$245 \frac{d^s}{dt} \int_{\Omega} \rho_s (1-n) \mathbf{V}^s d\Omega + \sum_{\alpha=c,w} \frac{d^\alpha}{dt} \int_{\Omega} \rho_\alpha n_\alpha \mathbf{V}^\alpha d\Omega = \int_{\Omega} \rho \mathbf{g}(\mathbf{x}, t) d\Omega + \int_{\partial\Omega} \boldsymbol{\sigma} \cdot \mathbf{n} da, \quad (34)$$

246 where  $\rho = \rho_s (1-n) + \sum_{\alpha=c,w} \rho_\alpha n_\alpha$  is the total apparent mass density,  $\mathbf{g}$  is the gravity and  $\boldsymbol{\sigma} \cdot \mathbf{n}$  is  
 247 the surface force acting on the boundary of the material domain  $\Omega$ .

248 Combining Equations (33) and (34), we obtain the local equation of motion in Eulerian formulation  
 249 as follows (Appendix I),

$$250 \nabla_x \cdot \boldsymbol{\sigma} + \rho_s (1-n) (\mathbf{g} - \boldsymbol{\gamma}^s) + \sum_{\alpha=c,w} \rho_\alpha n_\alpha (\mathbf{g} - \boldsymbol{\gamma}^\alpha) - \Lambda_{c\rightarrow w} (\mathbf{V}^w - \mathbf{V}^c) = 0. \quad (35)$$

251 To get the Lagrangian description, we introduce the Piola-Kirchhoff stress tensor  $\boldsymbol{\pi}$  linked to the  
 252 Cauchy stress  $\boldsymbol{\sigma}$  as follows,

$$253 \quad \boldsymbol{\pi} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot {}^t\mathbf{F}^{-1}. \quad (36)$$

254 A material surface element in the initial configuration  $dA$  and in the current configuration  $da$  (both  
 255 contain the same set of solid skeleton particles) are linked to each other via the following equation:

$$256 \quad \mathbf{F} \cdot \boldsymbol{\pi} \cdot \mathbf{N}dA = \boldsymbol{\sigma} \cdot \mathbf{n}da. \quad (37)$$

257 Considering the relation  $\nabla_X \cdot \mathbf{V}d\Omega_0 = \nabla_x \cdot \mathbf{v}d\Omega$ , the following expression can be obtained,

$$258 \quad \nabla_X \cdot (\mathbf{F} \cdot \boldsymbol{\pi})d\Omega_0 = \nabla_x \cdot \boldsymbol{\sigma}d\Omega. \quad (38)$$

259 Therefore, the local equation of motion (35) in Eulerian formulation can be transformed to that in  
 260 Lagrangian formulation as follows,

$$261 \quad \nabla_X \cdot (\mathbf{F} \cdot \boldsymbol{\pi}) + m_s^0(\mathbf{g} - \boldsymbol{\gamma}^s) + \sum_{\alpha=c,w} m_\alpha(\mathbf{g} - \boldsymbol{\gamma}^\alpha) - \overline{\omega}_{c \rightarrow w}(\mathbf{V}^w - \mathbf{V}^c) = 0. \quad (39)$$

### 262 2.3 Kinetic energy theorem

263 In the following, the kinetic energy theorem in Eulerian formulation is presented first. Favoring the  
 264 motion of the soil skeleton by introducing the relative vector of fluid mass,  $\mathbf{w}_\alpha = \rho_\alpha n_\alpha(\mathbf{V}^\alpha - \mathbf{V}^s)$  and  
 265 the fluid pressure  $p_\alpha$ . The work rate of the external body and surface forces of the RVE for saturated  
 266 frozen soils in Eulerian form is as follows,

$$267 \quad P_{f,T}(\mathbf{V}^s, \mathbf{V}^\alpha) = \int_\Omega (\rho_s(1-n)\mathbf{g} \cdot \mathbf{V}^s + \mathbf{g} \cdot \sum_{\alpha=c,w} \rho_\alpha n_\alpha \mathbf{V}^\alpha) d\Omega + \int_{\partial\Omega} (\mathbf{T}^s \cdot \mathbf{V}^s + \sum_{\alpha=c,w} (\mathbf{T}^\alpha \cdot \mathbf{V}^\alpha)) da, \quad (40)$$

268 in which  $\mathbf{T}^s$  is the traction vector of soil solid grain, and  $\mathbf{T}^\alpha$  the traction vector of fluid phase  $\alpha$ .

269 The above equation can be rewritten as follows (referring to Appendix II),

$$270 \quad P_{f,T}(\mathbf{V}^s, \mathbf{V}^\alpha) = \int_\Omega (\rho \mathbf{g} \cdot \mathbf{V}^s + \mathbf{g} \cdot \sum_{\alpha=c,w} \mathbf{w}^\alpha) d\Omega + \int_{\partial\Omega} (\mathbf{T} \cdot \mathbf{V}^s - \sum_{\alpha=c,w} \left(\frac{p_\alpha}{\rho_\alpha} \mathbf{w}^\alpha\right) \cdot \mathbf{n}) da, \quad (41)$$

271 in which  $\mathbf{T} = \mathbf{T}^s + \sum_{\alpha=c,w} \mathbf{T}^\alpha$  is the total traction vector.

272 For the RVE of saturated frozen soils, the kinetic energy associated with soil matrix and fluids

273 consisting of ice crystals and unfrozen water can be expressed as follows:

$$274 \quad K_s = \frac{1}{2} \int_{\Omega} \rho_s (1 - n) (\mathbf{V}^s)^2 d\Omega \quad \text{and} \quad \sum_{\alpha=c,w} K_{\alpha} = \sum_{\alpha=c,w} \frac{1}{2} \int_{\Omega} \rho_{\alpha} n_{\alpha} (\mathbf{V}^{\alpha})^2 d\Omega. \quad (42)$$

275 The above expression (42) of the kinetic energy does not account for the tortuosity effect (Biot, 1956).

276 Therefore, the particle derivative of the kinetic energy of saturated frozen soils can be rewritten as

277 follows (Appendix II),

$$278 \quad \frac{d^s K_s}{dt} + \sum_{\alpha=c,w} \frac{d^{\alpha} K_{\alpha}}{dt} = \int_{\Omega} (\rho_s (1 - n) \boldsymbol{\gamma}^s + \sum_{\alpha=c,w} \rho_{\alpha} n_{\alpha} \boldsymbol{\gamma}^{\alpha}) \cdot \mathbf{V}^s d\Omega + \int_{\Omega} \sum_{\alpha=c,w} \boldsymbol{\gamma}^{\alpha} \cdot \mathbf{w}^{\alpha} d\Omega +$$

$$279 \quad \int_{\Omega} \frac{1}{2} \Lambda_{c \rightarrow w} ((\mathbf{V}^w)^2 - (\mathbf{V}^c)^2) d\Omega \quad . \quad (43)$$

280 Multiplying Equation (35) by  $\mathbf{V}^s$  and integrating over the volume  $\Omega$  while using Equation (43), we

281 finally extend the kinetic energy theorem in Eulerian formulation as follows (Appendix II),

$$282 \quad P_{f,T}(\mathbf{V}^s, \mathbf{V}^{\alpha}) = P_{\text{def}}(\mathbf{V}^s, \mathbf{V}^{\alpha}) + \frac{d^s K_s}{dt} + \sum_{\alpha=c,w} \frac{d^{\alpha} K_{\alpha}}{dt} - \int_{\Omega} \frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] d\Omega, \quad (44)$$

283 where

$$284 \quad P_{\text{def}}(\mathbf{V}^s, \mathbf{V}^{\alpha}) = \int_{\Omega} \boldsymbol{\sigma} : \mathbf{d}^s d\Omega - \int_{\Omega} \sum_{\alpha=c,w} \left[ \nabla_x \cdot \left( \frac{p_{\alpha}}{\rho_{\alpha}} \mathbf{w}^{\alpha} \right) - (\mathbf{g} - \boldsymbol{\gamma}^{\alpha}) \cdot \mathbf{w}^{\alpha} \right] d\Omega. \quad (45)$$

285 The strain work rate  $P_{\text{def}}(\mathbf{V}^s, \mathbf{V}^{\alpha})$  is not a particle derivative and so the kinetic energy theorem cannot

286 be interpreted as a conservation law. This theorem only expresses a balance of all the mechanical

287 energies involved, without specifying the physical transformations affecting them.

288 In a similar way, the Kinetic energy theorem in Lagrangian formulation can be derived. The work

289 rate of the external body and surface forces of the RVE for saturated frozen soils in Lagrangian form is

290 as follows,

$$291 \quad \mathcal{P}_{f,T}(\mathbf{V}^s, \mathbf{V}^{\alpha}) = \int_{\Omega_0} \left( \rho_0 \mathbf{g} \cdot \mathbf{V}^s + \mathbf{g} \cdot \frac{\mathbf{F}}{J} \cdot \sum_{\alpha=c,w} \mathbf{M}_{\alpha} \right) d\Omega_0 + \int_{\partial\Omega_0} \left( \mathbf{T} \cdot \mathbf{V}^s - \sum_{\alpha=c,w} \frac{\phi_{\alpha} p_{\alpha} \mathbf{F}}{m_{\alpha} J} \cdot \mathbf{M}_{\alpha} \cdot \mathbf{N} \right) dA. \quad (46)$$

292 The kinetic energy associated with soil matrix and fluids consisting of ice crystals and unfrozen water

293 can be expressed in Lagrangian formulation as follows,

$$294 \quad \mathcal{K}_s = \frac{1}{2} \int_{\Omega_0} \rho_s^0 (1 - \phi) (\mathbf{V}^s)^2 d\Omega_0, \quad \text{and} \quad \sum_{\alpha=c,w} \mathcal{K}_{\alpha} = \sum_{\alpha=c,w} \frac{1}{2} \int_{\Omega_0} m_{\alpha} (\mathbf{V}^{\alpha})^2 d\Omega_0. \quad (47)$$

295 The strain work rate of the RVE for saturated frozen soils can be expressed in Lagrangian formulation  
 296 as follows,

$$297 \quad \mathcal{P}_{\text{def}}(\mathbf{V}^s, \mathbf{V}^\alpha) = \int_{\Omega_0} \boldsymbol{\pi} : \frac{d\boldsymbol{\Delta}}{dt} d\Omega_0 - \sum_{\alpha=c,w} \int_{\Omega_0} \left[ \nabla_X \cdot \left( \frac{\phi_{\alpha p \alpha}}{m^\alpha} \cdot \mathbf{M}_\alpha \right) - (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \frac{\mathbf{F}}{J} \cdot \mathbf{M}_\alpha \right] d\Omega_0, \quad (48)$$

298 in which the Green-Lagrange strain tensor  $\boldsymbol{\Delta}$  is defined as,

$$299 \quad \boldsymbol{\Delta} = \frac{1}{2} ( {}^t\mathbf{F} \cdot \mathbf{F} - \mathbf{1} ), \text{ and } \mathbf{d}^s = {}^t\mathbf{F}^{-1} \cdot \frac{d\boldsymbol{\Delta}}{dt} \cdot \mathbf{F}^{-1}. \quad (49)$$

300 Therefore, the kinetic energy theorem in Lagrangian formulation can be obtained as follows,

$$301 \quad \mathcal{P}_{\text{f,T}}(\mathbf{V}^s, \mathbf{V}^\alpha) = \mathcal{P}_{\text{def}}(\mathbf{V}^s, \mathbf{V}^\alpha) + \frac{d^s \mathcal{K}_s}{dt} + \sum_{\alpha=c,w} \frac{d^\alpha \mathcal{K}_\alpha}{dt} - \int_{\Omega_0} \frac{1}{2} \varpi_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] d\Omega_0. \quad (50)$$

### 302 **3. Thermoporoelasticity of saturated frozen soils**

303 In this section, the thermodynamics of classic saturated porous media will be extended to saturated  
 304 frozen soils taking into account the phenomenon of phase transformation (between liquid water and ice  
 305 crystal). According to Coussy (1995), the extension of thermostatics to thermodynamics of continua  
 306 can be obtained on the basis of the postulate of local state with regard to both time and space. With  
 307 regard to time, the postulate of local state stipulates that the current state of internal energy of a  
 308 homogeneous system in any (slow enough) evolution can be considered as crossing different  
 309 equilibrium states such that at any time its state variables satisfy the state equations characterizing  
 310 equilibrium states. With regard to space, the postulate of local state addresses the thermodynamics of a  
 311 heterogeneous continuum by assuming that the material volume  $\Omega$  can be considered as an ensemble  
 312 of juxtaposed sub-systems or elementary material volumes  $d\Omega$  exchanging heat and mechanical work  
 313 between them, each of them being in thermodynamic equilibrium hence verifies again the state  
 314 equations. Note that an elementary material volume here is composed of the soil skeleton and of  
 315 saturating "fluids" (we will use the term "fluid" as a convenient short-hand terminology when referring  
 316 to the in-pore phases) of ice crystals and unfrozen water. Consequently, elementary thermodynamic

317 systems and the continua they form are open systems. As a consequence, the laws of thermodynamics  
 318 can be applied in an integral form to extensive quantities such as the energy, mass, momentum and  
 319 entropy. With the help of the local state postulate, it will be seen how to extend the thermodynamics of  
 320 closed continua to open continua composed of saturated frozen soils.

### 321 **3.1 The First law of thermodynamics principle**

#### 322 **3.1.1 Energy conservation**

323 Based on the postulate of local state, the first law of thermodynamics states that the time rate of the  
 324 energy attached to the whole matter currently contained within the volume of saturated frozen soils is  
 325 equal to the sum of the work rate  $P_{i,T}(\mathbf{V}^s, \mathbf{V}^\alpha)$  of the external forces acting upon this matter, and of the  
 326 rate  $Q^0$  of external heat supply. When applied to a generic material volume of saturated frozen soils,  
 327 this leads to the energy balance equation as follows,

$$328 \frac{d^s}{dt} \int_{\Omega} \rho_s (1 - n) \left( e_s + \frac{1}{2} (\mathbf{V}^s)^2 \right) d\Omega + \sum_{\alpha=c,w} \frac{d^\alpha}{dt} \int_{\Omega} \rho_\alpha n_\alpha \left( e_\alpha + \frac{1}{2} (\mathbf{V}^\alpha)^2 \right) d\Omega = P_{i,T}(\mathbf{V}^s, \mathbf{V}^\alpha) + Q^0, \quad (51)$$

329 in which  $e_s$  stands for the specific (i.e. per unit mass) internal energy of the soil matrix and  $e_\alpha$  that of  
 330 the fluid phases. The total internal energy  $e$  per unit overall current volume can be expressed as  
 331 follows,

$$332 e = \rho_s (1 - n) e_s + \sum_{\alpha=c,w} \rho_\alpha n_\alpha e_\alpha. \quad (52)$$

333 The internal energy  $e$  is a volume density, and not a density per mass unit as it is usually the case in  
 334 mechanics of closed continua. The particular choice of the volume density is more convenient here:  
 335 when following  $d\Omega$  in the skeleton movement,  $d\Omega$  will exchange fluid mass with its neighbours.  
 336 Thus, the quantity  $e d\Omega$  always corresponds to the same set of solid particles but to different sets of  
 337 fluid particles at different times.

338 The rate of external heat supply can be written as follows,



$$339 \quad Q^0 = \int_{\partial\Omega} J_Q(x, n, t) da + \int_{\Omega} \mathbf{r}_Q(x, t) d\Omega, \quad (53)$$

340 in which  $\mathbf{r}_Q$  is a volume density of the heat provided to  $\Omega$  by an external heat sources, and  $J_Q$  is a  
341 surface rate of heat supply by conduction.

### 342 3.1.2 The energy equation

343 Use of the kinetic energy theorem of Equation (44) allows us to rewrite Equation (51) in the form as

344 follows,

$$345 \quad \frac{d^s}{dt} \int_{\Omega} \rho_s (1 - n) e_s d\Omega + \sum_{\alpha=c,w} \frac{d^\alpha}{dt} \int_{\Omega} \rho_\alpha n_\alpha e_\alpha d\Omega = P_{\text{def}}(\mathbf{V}^s, \mathbf{V}^\alpha) - \int_{\Omega} \frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 -$$

$$346 \quad (\mathbf{V}^c - \mathbf{V}^s)^2] d\Omega + Q^0$$

$$347 \quad (54)$$

348 Combining Equation (52), the left side of Equation (54) can be rewritten as follows (Appendix III),

$$349 \quad \frac{d^s}{dt} \int_{\Omega} \rho_s (1 - n) e_s d\Omega + \sum_{\alpha=c,w} \frac{d^\alpha}{dt} \int_{\Omega} \rho_\alpha n_\alpha e_\alpha d\Omega = \int_{\Omega} \left[ \frac{d^s e}{dt} + e \nabla_x \cdot \mathbf{V}^s + \sum_{\alpha=c,w} \nabla_x \cdot (e_\alpha \mathbf{w}^\alpha) \right] d\Omega. \quad (55)$$

350 Expression (45) for the strain work rate  $P_{\text{def}}(\mathbf{V}^s, \mathbf{V}^\alpha)$ , together with Equations (53)-(55), yields:

$$351 \quad \int_{\Omega} \left[ \frac{d^s e}{dt} + e \nabla_x \cdot \mathbf{V}^s + \sum_{\alpha=c,w} \nabla_x \cdot (e_\alpha \mathbf{w}^\alpha) \right] d\Omega = \int_{\Omega} \boldsymbol{\sigma} : \mathbf{d}^s d\Omega - \int_{\Omega} \sum_{\alpha=c,w} \left[ \nabla_x \cdot \left( \frac{p_\alpha}{\rho_\alpha} \mathbf{w}^\alpha \right) - (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \right.$$

$$352 \quad \left. \mathbf{w}^\alpha \right] d\Omega - \int_{\Omega} \frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] d\Omega + Q^0 \quad (56)$$

353 The above equation (56) can be rewritten as follows,

$$354 \quad \int_{\Omega} \left[ \frac{d^s e}{dt} + e \nabla_x \cdot \mathbf{V}^s + \sum_{\alpha=c,w} \nabla_x \cdot \left( e_\alpha \mathbf{w}^\alpha + \frac{p_\alpha}{\rho_\alpha} \mathbf{w}^\alpha \right) - \boldsymbol{\sigma} : \mathbf{d}^s - (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{w}^\alpha \right] d\Omega + \int_{\Omega} \frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w -$$

$$355 \quad \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] d\Omega = \int_{\partial\Omega} J_Q(x, n, t) da + \int_{\Omega} \mathbf{r}_Q(x, t) d\Omega, \quad (57)$$

356 The surface rate  $J_Q$  relies linearly on  $\mathbf{n}$ , expressed as follows (Lai et al., 2010),

$$357 \quad J_Q = -\mathbf{q} \cdot \mathbf{n}. \quad (58)$$

358 where  $\mathbf{q}$  is an outgoing heat flow vector.

359 Substitution of Eq. (58) into Eq. (57) provides the Euler energy equation as follows,

$$360 \quad \int_{\Omega} \left[ \frac{d^s e}{dt} + e \nabla_x \cdot \mathbf{V}^s + \sum_{\alpha=c,w} \nabla_x \cdot \left( e_\alpha \mathbf{w}^\alpha + \frac{p_\alpha}{\rho_\alpha} \mathbf{w}^\alpha \right) - \boldsymbol{\sigma} : \mathbf{d}^s - (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{w}^\alpha \right] d\Omega + \int_{\Omega} \frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w -$$

$$361 \quad (\mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2]d\Omega = \int_{\partial\Omega} (-\mathbf{q} \cdot \mathbf{n})da + \int_{\Omega} \mathbf{r}_Q(x, t)d\Omega \quad .(59)$$

362 Using the theorem of divergence, we can rewrite the above equation (59) as follows,

$$363 \quad \frac{d^s e}{dt} + e \nabla_x \cdot \mathbf{V}^s = \boldsymbol{\sigma} : \mathbf{d}^s - \sum_{\alpha=c,w} \nabla_x \cdot (h_\alpha \mathbf{w}^\alpha) - \nabla_x \cdot \mathbf{q} + \sum_{\alpha=c,w} (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot (\mathbf{w}^\alpha) + \mathbf{r}_Q - \frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w -$$

$$364 \quad (\mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] \quad , (60)$$

365 in which the fluid-specific enthalpy writes  $h_\alpha = e_\alpha + \frac{p_\alpha}{\rho_\alpha}$  ( $\alpha = c, w$ ).

366 With the aim of transforming the energy equation (60) to the initial configuration of the soil skeleton,

367 let  $E$ ,  $\mathbf{Q}$  and  $\mathbf{R}_Q$  be respectively the overall Lagrangian densities of internal energy per unit of initial

368 volume  $d\Omega_0$ , the Lagrangian heat flow vector and volume rate density of the heat provided to  $\Omega_0$ ,

369 such that:

$$370 \quad E d\Omega_0 = e d\Omega, \quad \mathbf{Q} \cdot \mathbf{N} dA = \mathbf{q} \cdot \mathbf{n} da, \quad \text{and} \quad \mathbf{R}_Q d\Omega_0 = \mathbf{r}_Q d\Omega. \quad (61)$$

371 Considering the following equations (Coussy, 1995),

$$372 \quad \left( \frac{d^s e}{dt} + e \nabla_x \cdot \mathbf{V}^s \right) d\Omega = \frac{d^s}{dt} (e d\Omega) = \frac{d^s}{dt} (E d\Omega_0), \quad (\boldsymbol{\sigma} : \mathbf{d}^s) d\Omega = \boldsymbol{\pi} : \frac{d\Delta}{dt} d\Omega_0, \quad \nabla_x \cdot \mathbf{q} d\Omega = \nabla_X \cdot \mathbf{Q} d\Omega_0,$$

$$373 \quad \text{And} \quad \nabla_x \cdot (h_\alpha \mathbf{w}^\alpha) d\Omega = \nabla_X \cdot (h_\alpha \mathbf{M}_\alpha) d\Omega_0 \quad (\alpha = c, w), \quad (62)$$

374 and combining Equations (60), (61) and transport formulae derived in the previous sections, we can

375 have the Lagrangian energy equation as follows,

$$376 \quad \frac{dE}{dt} = \boldsymbol{\pi} : \frac{d\Delta}{dt} - \sum_{\alpha=c,w} \nabla_X \cdot (h_\alpha \mathbf{M}_\alpha) - \nabla_X \cdot \mathbf{Q} + \sum_{\alpha=c,w} ((\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{F}) \cdot \mathbf{M}_\alpha + \mathbf{R}_Q - \frac{1}{2} \varpi_{c \rightarrow w} [(\mathbf{V}^w -$$

$$377 \quad (\mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] \quad . \quad (63)$$

## 378 3.2 The Second law of thermodynamics principle

### 379 3.2.1 The Clausius-Duhem inequality in Eulerian and Lagrangian formulations

380 There exists an extensive thermodynamic function, called entropy. According to the second principle

381 of thermodynamics, during any process, the increase of entropy  $\theta$  attached to any material subsystem

382  $\Omega$  must be superior or at least equal to the rate of entropy externally supplied to it. Assuming that the

383 same temperature holds for the soil particle (or matrix), ice crystals and unfrozen water, the above  
 384 entropy balance admits the following mathematical form:

$$385 \quad \frac{d^s}{dt} \int_{\Omega} \rho_s (1-n) \theta_s d\Omega + \sum_{\alpha=c,w} \frac{d^\alpha}{dt} \int_{\Omega} \rho_\alpha n_\alpha \theta_\alpha d\Omega \geq \int_{\partial\Omega} -\frac{\mathbf{q}\mathbf{n}}{T} da + \int_{\Omega} \frac{r_Q}{T} d\Omega, \quad (64)$$

386 where  $\theta_s$ ,  $\theta_c$  and  $\theta_w$  stand for the specific entropy of the soil matrix, ice crystals and unfrozen water,  
 387 respectively.

388 Using the theorem on material derivative, the left hand side of the above inequality can be written as  
 389 follows (Appendix III),

$$390 \quad \frac{d^s}{dt} \int_{\Omega} \rho_s (1-n) \theta_s d\Omega + \sum_{\alpha=c,w} \frac{d^\alpha}{dt} \int_{\Omega} \rho_\alpha n_\alpha \theta_\alpha d\Omega = \int_{\Omega} \left[ \frac{d^s \theta}{dt} + \theta \nabla_x \cdot \mathbf{V}^s + \sum_{\alpha=c,w} \nabla_x \cdot (\theta_\alpha \mathbf{w}^\alpha) \right] d\Omega, \quad (65)$$

391 where  $\theta$  is the total entropy per unit of overall current volume:

$$392 \quad \theta = \rho_s (1-n) \theta_s + \sum_{\alpha=c,w} \rho_\alpha n_\alpha \theta_\alpha. \quad (66)$$

393 Substituting Equation (65) into Equation (64) and using theorem of divergence, we can obtain after  
 394 some simplification:

$$395 \quad \int_{\Omega} \left[ \frac{d^s \theta}{dt} + \theta \nabla_x \cdot \mathbf{V}^s + \sum_{\alpha=c,w} \nabla_x \cdot (\theta_\alpha \mathbf{w}^\alpha) \right] d\Omega \geq \int_{\Omega} -\nabla_x \cdot \frac{\mathbf{q}}{T} d\Omega + \int_{\Omega} \frac{r_Q}{T} d\Omega. \quad (67)$$

396 Therefore, the following can be obtained,

$$397 \quad \frac{d^s \theta}{dt} + \theta \nabla_x \cdot \mathbf{V}^s + \sum_{\alpha=c,w} \nabla_x \cdot (\theta_\alpha \mathbf{w}^\alpha) + \nabla_x \cdot \frac{\mathbf{q}}{T} - \frac{r_Q}{T} \geq 0. \quad (68)$$

398 Introducing the Helmholtz free energy  $\psi = e - T\theta$ , and combining Equation (60), we are led to the  
 399 fundamental Clausius-Duhem inequality in Eulerian form (Appendix III),

$$400 \quad \boldsymbol{\sigma} : \mathbf{d}^s - \sum_{\alpha=c,w} (g_\alpha \nabla_x \cdot \mathbf{w}^\alpha) - \theta \frac{dT}{dt} - \frac{d\psi}{dt} - \psi \nabla_x \cdot \mathbf{V}^s - \mathbf{w}^\alpha \cdot \sum_{\alpha=c,w} (\nabla_x g_\alpha + \theta_\alpha \nabla_x T - (\mathbf{g} - \boldsymbol{\gamma}^\alpha)) -$$

$$401 \quad \frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] - \frac{q}{T} \cdot \nabla_x T \geq 0 \quad (69)$$

402 To obtain the corresponding Lagrangian formulation, we introduce Lagrangian entropy density  $\Theta$   
 403 defined by

$$404 \quad \Theta d\Omega_0 = \theta d\Omega. \quad (70)$$

405 Therefore, Inequality (68) can be written in the Lagrangian formulation as follows (Appendix III),

$$406 \quad \frac{d\Theta}{dt} + \sum_{\alpha=c,w} \nabla_X \cdot (\Theta_\alpha \mathbf{M}_\alpha) + \nabla_X \cdot \frac{\mathbf{Q}}{T} - \frac{R_Q}{T} \geq 0. \quad (71)$$

407 Let  $\Psi$  be the Lagrangian free energy density which verifies:

$$408 \quad \Psi d\Omega_0 = \psi d\Omega, \text{ and } \Psi = E - T\Theta. \quad (72)$$

409 By recalling the mass conservation (31) and (32), the Lagrangian energy equation (63), Inequality (71)

410 can be written as follows (Appendix III),

$$411 \quad \boldsymbol{\pi} : \frac{d\Delta}{dt} + \sum_{\alpha=c,w} g_\alpha \frac{dm_\alpha}{dt} - \Theta \frac{dT}{dt} - \frac{d\Psi}{dt} - (g_c - g_w) \varpi_{c \rightarrow w} - \frac{1}{2} \varpi_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] -$$

$$412 \quad \sum_{\alpha=c,w} (\nabla_X g_\alpha + \Theta_\alpha \nabla_X T - (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{F}) \cdot \mathbf{M}_\alpha - \frac{\mathbf{Q}}{T} \cdot \nabla_X T \geq 0 \quad (73)$$

413 The above is the Lagrangian formulation of the Clausius-Duhem inequality.

### 414 3.2.2 Identification of dissipation and Thermal equation

415 The left hand side of Inequality (73), which will be noted as  $\Phi$ , is the total dissipation per unit of

416 initial volume  $d\Omega_0$ . The second law requires the dissipation and the associated internal entropy

417 produce  $\Phi/T$  to be non-negative. Following Coussy (2004), the total dissipation  $\Phi$  can be

418 decomposed as the sum of different contributions of different physical origins while each one being

419 non-negative as follows,

$$420 \quad \Phi = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_{\rightarrow}, \quad (74)$$

$$421 \quad \text{in which } \Phi_1 = \boldsymbol{\pi} : \frac{d\Delta}{dt} + \sum_{\alpha} g_\alpha \frac{dm_\alpha}{dt} - \Theta \frac{dT}{dt} - \frac{d\Psi}{dt}, \quad (75)$$

$$422 \quad \Phi_2 = -\frac{\mathbf{Q}}{T} \cdot \nabla_X T, \quad (76)$$

$$423 \quad \Phi_3 = -\sum_{\alpha=c,w} (\nabla_X g_\alpha + \Theta_\alpha \nabla_X T - (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{F}) \cdot \mathbf{M}_\alpha, \quad (77)$$

$$424 \quad \Phi_{\rightarrow} = -(g_c - g_w) \varpi_{c \rightarrow w} - \frac{1}{2} \varpi_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2]. \quad (78)$$

425  $\Phi_1$  is the intrinsic dissipation associated with the open system  $d\Omega_0$ , following the movement of the

426 soil skeleton. On account of the local state postulate, all the quantities which appear in expression (75)

427 of  $\Phi_1$  depend only on the state variables which characterize the free energy  $\Psi d\Omega_0$  of the open  
 428 elementary system  $d\Omega_0$ . Thus, the terminology of intrinsic dissipation for  $\Phi_1$  is due to its dependence  
 429 with regard to the other open elementary systems which constitute the continuum of saturated frozen  
 430 soils. For the same reason,  $\Phi_1/T$  is called the intrinsic entropy in internal production. The other  
 431 contributions  $\Phi_2$ ,  $\Phi_3$  and  $\Phi_{\rightarrow}$  are, respectively, the thermal dissipation associated with heat  
 432 conduction, the dissipation associated with mass transport of in-pore phases, and the dissipation  
 433 associated with phase change.

434 From Equation (72), we have

$$435 \quad \frac{dE}{dt} = \frac{d\psi}{dt} + T \frac{d\Theta}{dt} + \Theta \frac{dT}{dt}. \quad (79)$$

436 Combining energy equation (63), we can have (Appendix IV)

$$437 \quad T \left( \frac{d\Theta}{dt} + \sum_{\alpha=c,w} \nabla_X \cdot (\Theta_\alpha \mathbf{M}_\alpha) \right) = \mathbf{R}_Q - \nabla_X \cdot \mathbf{Q} + \Phi_1 + \Phi_3 + \Phi_{\rightarrow} = \mathbf{R}_Q - \nabla_X \cdot \mathbf{Q} + \Phi_M + \Phi_{\rightarrow}, \quad (80)$$

438 in which  $\Phi_M = \Phi_1 + \Phi_3$  is the mechanical dissipation due to irreversible matter movements. The  
 439 equation (80) is called the Lagrangian thermal equation.

440 Substituting Equation (76) into Equation (80), we have

$$441 \quad \frac{d\Theta}{dt} d\Omega_0 = \left[ -\sum_{\alpha=c,w} \nabla_X \cdot (\Theta_\alpha \mathbf{M}_\alpha) + \left( \frac{\mathbf{R}_Q}{T} - \nabla_X \cdot \frac{\mathbf{Q}}{T} \right) + \frac{\Phi}{T} \right] d\Omega_0 = \left[ -\sum_{\alpha=c,w} \nabla_X \cdot (\Theta_\alpha \mathbf{M}_\alpha) \right] d\Omega_0 +$$

$$442 \quad \left[ \left( \frac{\mathbf{R}_Q}{T} - \nabla_X \cdot \frac{\mathbf{Q}}{T} \right) \right] d\Omega_0 + \frac{\Phi}{T} d\Omega_0$$

$$443 \quad (81)$$

444 Considering that  $\frac{d}{dt}(\Theta d\Omega_0) = \Theta \frac{d}{dt}(d\Omega_0) + d\Omega_0 \frac{d}{dt}(\Theta) = \frac{d\Theta}{dt} d\Omega_0$ , so we have

$$445 \quad \frac{d(\Theta d\Omega_0)}{dt} = -\left[ \sum_{\alpha} \nabla_X \cdot (\Theta_\alpha \mathbf{M}_\alpha) \right] d\Omega_0 + \left( \frac{\mathbf{R}_Q}{T} - \nabla_X \cdot \frac{\mathbf{Q}}{T} \right) d\Omega_0 + \frac{\Phi}{T} d\Omega_0. \quad (82)$$

446 Hence, the thermal equation (80), or equivalently Equation (82), corresponds to a balance in entropy

447 for the elementary open system  $d\Omega_0$ . The term  $d(\Theta d\Omega_0)$  is the entropy variation which could be

448 recorded, during the time interval  $dt$ , by an observer attached to the open system  $d\Omega_0$  following the

449 movement of the soil skeleton. Equation (82) stipulates that this variation must be equal to the external  
 450 supply plus the internal entropy production within the system, during the same time interval.

451 The above is the Lagrangian formulation of the thermal equation. Let  $\varphi_i$  be the Eulerian dissipation  
 452 volume densities defined by

$$453 \quad \varphi_i d\Omega = \Phi_i d\Omega_0 \quad \text{and} \quad J\varphi_i = \Phi_i . \quad (83)$$

454 The dissipations in Eulerian formulation are as follows,

$$455 \quad \varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_{\rightarrow}, \quad (84)$$

456 and

$$457 \quad \varphi_1 = \boldsymbol{\sigma} : \mathbf{d}^s - \sum_{\alpha} (g_{\alpha} \nabla_x \cdot \mathbf{w}^{\alpha}) - \theta \frac{dT}{dt} - \frac{d\psi}{dt} - \psi \nabla_x \cdot \mathbf{V}^s , \quad (85)$$

$$458 \quad \varphi_2 = -\frac{q}{T} \cdot \nabla_x T , \quad (86)$$

$$459 \quad \varphi_3 = -\mathbf{w}^{\alpha} \cdot \sum_{\alpha} (\nabla_x g_{\alpha} + \theta_{\alpha} \nabla_x T - (\mathbf{g} - \boldsymbol{\gamma}^{\alpha})) , \quad (87)$$

$$460 \quad \varphi_{\rightarrow} = -\frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2]. \quad (88)$$

461 From the definition that  $\psi = e - T\theta$ , we can obtain

$$462 \quad de = d\psi + Td\theta + \theta dT \quad \text{and} \quad \frac{de}{dt} = \frac{d\psi}{dt} + T \frac{d\theta}{dt} + \theta \frac{dT}{dt}. \quad (89)$$

463 From Equation (80), we can deduce the following Eulerian thermal balance equation (Appendix IV),

$$464 \quad T \frac{d\theta}{dt} + T\theta \nabla_x \cdot \mathbf{V}^s + T \sum_{\alpha} \nabla_x \cdot (\theta_{\alpha} \mathbf{w}^{\alpha}) = \mathbf{r}_q - \nabla_x \cdot \mathbf{q} + \varphi_1 + \varphi_3 + \varphi_{\rightarrow}. \quad (90)$$

### 465 3.3 Heat conduction and mass transport for in-pore phases

#### 466 3.3.1 Darcy's Law

467 For the fluids ( $\alpha = c, w$ ), their fluid-specific free enthalpy can be expressed as follows,

$$468 \quad g_{\alpha} = g_{\alpha}(p, T): \frac{1}{\rho_{\alpha}} = \frac{\partial g_{\alpha}}{\partial p_{\alpha}}; \theta_{\alpha} = -\frac{\partial g_{\alpha}}{\partial T}. \quad (91)$$

469  $\varphi_3$  in Equation (91) denotes the dissipation associated with fluid mass transport or conduction through  
 470 the porous medium. From Equation (91), we have

$$471 \quad \nabla_x g_{\alpha} = \frac{\partial g_{\alpha}}{\partial p} \nabla_x p_{\alpha} + \frac{\partial g_{\alpha}}{\partial T} \nabla_x T. \quad (92)$$

472 Substituting Equation (92) into Equation (87), we have

$$473 \quad \varphi_3 = -\mathbf{w}^\alpha \cdot \sum_\alpha (\nabla_x g_\alpha + \theta_\alpha \nabla_x T - (\mathbf{g} - \boldsymbol{\gamma}^\alpha)) = \sum_\alpha \frac{\mathbf{w}^\alpha}{\rho_\alpha} \cdot (-\nabla_x p_\alpha + \rho_\alpha (\mathbf{g} - \boldsymbol{\gamma}^\alpha)) \geq 0. \quad (93)$$

474 For the ice crystals or unfrozen water, the dissipation can be assumed to be non-negative, expressed as

475 follows,

$$476 \quad \frac{\mathbf{w}^\alpha}{\rho_\alpha} \cdot (-\nabla_x p_\alpha + \rho_\alpha (\mathbf{g} - \boldsymbol{\gamma}^\alpha)) = \mathbf{v}_\alpha \cdot \boldsymbol{\ell}_\alpha \geq 0, \quad (94)$$

477 in which  $\mathbf{v}_\alpha = \frac{\mathbf{w}^\alpha}{\rho_\alpha}$  and  $\boldsymbol{\ell}_\alpha = -\nabla_x p_\alpha + \rho_\alpha (\mathbf{g} - \boldsymbol{\gamma}^\alpha)$ .

478 At the quasistatic limit where the inertia forces can be neglected, and in the absence of body forces,

479 Inequality (93) implies that the fluid mass transport takes place from high to low fluid pressures. Again,

480 the assumption of the normality of the associated dissipative mechanism leads one to postulate the

481 existence of a dissipation potential  $D_{3\alpha} \left( \frac{\mathbf{w}^\alpha}{\rho_\alpha} \right)$  such that (Lemaitre et al., 1994)

$$482 \quad -\nabla_x p_\alpha + \rho_\alpha (\mathbf{g} - \boldsymbol{\gamma}^\alpha) = \frac{\partial D_{3\alpha}}{\partial \frac{\mathbf{w}^\alpha}{\rho_\alpha}}. \quad (95)$$

483 If the potential  $D_{3\alpha} \left( \frac{\mathbf{w}^\alpha}{\rho_\alpha} \right)$  is chosen as a positively defined quadratic function as follows,

$$484 \quad D_{3\alpha} \left( \frac{\mathbf{w}^\alpha}{\rho_\alpha} \right) = \frac{1}{2} \frac{\mathbf{w}^\alpha}{\rho_\alpha} \cdot \mathbf{k}_\alpha^{-1} \cdot \frac{\mathbf{w}^\alpha}{\rho_\alpha}, \quad (96)$$

485 where  $\mathbf{k}_\alpha$  is a symmetric tensor which corresponds to a positively defined bilinear form, Eqs. (95) and

486 (96) yield the linear mass fluid conduction law or Darcy's law as follows,

$$487 \quad \frac{\mathbf{w}^\alpha}{\rho_\alpha} = \mathbf{k}_\alpha \cdot (-\nabla_x p_\alpha + \rho_\alpha (\mathbf{g} - \boldsymbol{\gamma}^\alpha)), \quad (97)$$

488 where  $\mathbf{k}_\alpha$  is then identified as the permeability tensor, relative to the current configuration.

489 For the Lagrangian equations, corresponding to Equations (93)-(97), we can have the following

490 equations,

$$491 \quad \Phi_3 = -\sum_\alpha (\nabla_x g_\alpha + \theta_\alpha \nabla_x T - (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{F}) \cdot M^\alpha = -\sum_\alpha \frac{M^\alpha}{\rho_\alpha} (\nabla_x p_\alpha - (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{F}) \geq 0, \quad (98)$$

492 which is assumed to be non-negative, expressed as follows,

$$493 \quad \frac{M^\alpha}{\rho_\alpha} (\nabla_x p_\alpha - (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{F}) = \mathbf{v}_\alpha \cdot \boldsymbol{\mathcal{L}}_\alpha \geq 0, \quad (99)$$

494 in which  $\mathbf{v}_\alpha = \frac{M_\alpha}{\rho_\alpha}$  and  $\mathcal{L}_\alpha = \nabla_X p_\alpha - (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{F}$ . (100)

495 Choosing the dissipation potential  $D_{3\alpha} \left( \frac{M_\alpha}{\rho_\alpha} \right)$  as follows,

$$496 \quad D_{3\alpha} \left( \frac{M_\alpha}{\rho_\alpha} \right) = \frac{1}{2} \frac{M_\alpha}{\rho_\alpha} \cdot \mathbf{K}_\alpha^{-1} \cdot \frac{M_\alpha}{\rho_\alpha}, \quad (101)$$

497 so we have

$$498 \quad \nabla_X p - (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{F} = \frac{\partial D_{3\alpha}}{\partial \frac{M_\alpha}{\rho_\alpha}}, \quad (102)$$

$$499 \quad \text{and} \quad \frac{M_\alpha}{\rho_\alpha} = \mathbf{K}_\alpha \cdot (\nabla_X p_\alpha - (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{F}), \quad (103)$$

500 where  $\mathbf{K}_\alpha$  is the convective transportee of  $\mathbf{k}_\alpha$ , with  $\mathbf{K}_\alpha = J\mathbf{F}^{-1} \cdot \mathbf{k}_\alpha \cdot {}^t\mathbf{F}^{-1}$ .

### 501 3.3.2 Fourier's Law

502 The decoupling hypothesis (84) requires the non-negativity of thermal dissipation  $\varphi_2$ , which reads

$$503 \quad \varphi_2 = -\frac{\mathbf{q}}{T} \cdot \nabla_x T = -\mathbf{L}_{q/T} \cdot \frac{\mathbf{q}}{T} \geq 0 \quad \text{and} \quad \mathbf{L}_{q/T} = -\frac{\mathbf{q}}{T}. \quad (104)$$

504 Equations (104) associates, through the thermal dissipation, the entropy efflux vector  $\mathbf{q}/T$  and the  
505 thermodynamic force  $-\nabla_x T$ . According to the above inequality, the heat flows spontaneously from  
506 high to low temperatures. A simple law that verifies automatically Inequality (104) is the Fourier law,

$$507 \quad \mathbf{q} = -\boldsymbol{\kappa} \cdot \nabla_x T \quad (105)$$

508 where  $\boldsymbol{\kappa}$  is identified as the thermal conductivity tensor, relative to the current configuration, which  
509 must be symmetric and definitie positive.

510 In a Lagrangian approach, we have

$$511 \quad \mathbf{Q} = -\mathbf{K} \cdot \nabla_X T, \quad (106)$$

512 where  $\mathbf{K}$  is the convective transportee of  $\boldsymbol{\kappa}$ , with  $\mathbf{K} = J\mathbf{F}^{-1} \cdot \boldsymbol{\kappa} \cdot {}^t\mathbf{F}^{-1}$ .

## 513 4 Poro-elastoplastic constitutive model for saturated frozen soils

### 514 4.1 Solid-fluid interface energy and the Lagrangian saturation

515 In the following, the infinitesimal transformations and quasistatic deformations are assumed, and



516 Lagrangian and Eulerian formulations are indifferent.

517 When the acceleration is ignored, the internal dissipation  $\phi_{int} = \phi_1 + \phi_{\rightarrow}$  can be simplified as  
 518 follows,

$$519 \quad \phi_{int} = \sigma: \frac{d\varepsilon}{dt} + \sum_{\alpha} g_{\alpha} \frac{dm_{\alpha}}{dt} - \theta \frac{dT}{dt} - \frac{d\psi}{dt} - (g_c - g_m)m_{c \rightarrow w}. \quad (107)$$

520 In Equation (107),  $g_{\alpha} dm_{\alpha} = (\psi_{\alpha} + p_{\alpha}/\rho_{\alpha}) dm_{\alpha}$  accounts for the free energy supplied to the open  
 521 system of volume  $d\Omega$  due to the fluid mass  $dm_{\alpha}$  supplied to the system.  $\psi_{\alpha} dm_{\alpha}$  is the free energy  
 522 carried by this fluid mass supply while  $p_{\alpha}/\rho_{\alpha} dm_{\alpha}$  is the mechanical work required to inject this  
 523 additional mass into the elementary volume  $d\Omega$ .  $\varepsilon$  is the strain tensor. However, the energy required  
 524 to insert the fluid mass  $dm_{\alpha}$  into the volume  $d\Omega$  does not always reduce to the mechanical work  
 525  $p_{\alpha} dm_{\alpha}/\rho_{\alpha}$ . For instance, non-local interaction forces can exist between the saturation solution and the  
 526 solid walls of the porous space (Coussy, 2004). Let  $\mu_{\alpha}$  be the specific chemical potential of the  
 527 saturating solution so that  $\mu_{\alpha} dm_{\alpha}$  is the free energy supply associated with the introduction of mass  
 528  $dm_{\alpha}$ , accounting for the non-local interaction forces at work. Instead of Equation (107), we now write:

$$529 \quad \phi_{int} = \sigma: \frac{d\varepsilon}{dt} + \sum_{\alpha} \mu_{\alpha} \frac{dm_{\alpha}}{dt} - \theta \frac{dT}{dt} - \frac{d\psi}{dt} - (\mu_c - \mu_m)m_{c \rightarrow w}. \quad (108)$$

530 The last term in Equation (108) accounts for the dissipation possibly occurring during phase  
 531 transformation. A zero dissipation of the phase transformation implies thermodynamic equilibrium  
 532 between the co-existing phases of the same components, hence the equality of their chemical potentials,  
 533  $\mu_c = \mu_w$ . For the ice crystals and unfrozen water, the following relationship can be obtained when  
 534 considering the simplifying assumption of  $\mu_{\alpha} = g_{\alpha}$ ,

$$535 \quad d\mu_{\alpha} = \frac{dp_{\alpha}}{\rho_{\alpha}} - \theta_{\alpha} dT. \quad (109)$$

536 Now let  $\psi_s$  and  $\theta_s$  be the skeleton free energy and entropy densities defined by

$$537 \quad \psi_s = \Psi - \sum_{\alpha} m_{\alpha} \psi_{\alpha}, \quad (110)$$

538 and 
$$\theta_s = \theta - \sum_{\alpha} m_{\alpha} \theta_{\alpha}. \quad (111)$$

539 Combining Equations (109)-(111) and  $m_{\alpha} = \rho_{\alpha} \phi_{\alpha}$ , we get the Clausius-Duhem inequality stating the  
540 non-negativity of the dissipation related to the skeleton only, that is (Appendix V),

541 
$$\sigma: \frac{d\varepsilon}{dt} + \sum_{\alpha=c,w} p_{\alpha} \frac{d\phi_{\alpha}}{dt} - \theta_s \frac{dT}{dt} - \frac{d\psi_s}{dt} \geq 0. \quad (112)$$

542 Since only the bulk phases have been removed, the ‘skeleton’ does still include the moving interfaces  
543 between the different constituents. Advancement of constituent interfaces, and, consequently, changes  
544 in the partial porosities  $\phi_{\alpha}$ , result from two different processes. The first is the growth of ice crystals,  
545 accompanied by the creation of new interfaces between ice crystals, liquid water, and the solid matrix.  
546 The second process is related only to the skeleton deformation, hence change of pore volume, due to  
547 pressures exerted by ice crystals and liquid water on the internal walls of the porous network. The  
548 simultaneous action of these two processes is reflected in Equation (7) on the volumetric evolutions of  
549 in-pore phases.

550 Consistently the currently (Lagrangian) overall porosity  $\phi$ , which eventually concerns the skeleton  
551 only, is given by

552 
$$\phi = \phi_c + \phi_w = \phi_0 S_c + \varphi_c + \phi_0 S_w + \varphi_w = \phi_0 + \varphi_c + \varphi_w. \quad (113)$$

553 Being associated with the creation/modification of interfaces between constituents, the saturation  
554 degrees  $S_{\alpha}$  vary from 0 to 1, and, thereby, undergo finite changes. In contrast, reflecting the  
555 infinitesimal deformation of the porous network, the partial porosities  $\varphi_{\alpha}$  undergo infinitesimal  
556 changes only. Confining our attention to infinitesimal changes of the liquid density  $\rho_w$ , the saturation  
557 degree of unfrozen water  $S_w$  can conceptually be maintained constant, by preventing any migration of  
558 liquid water inside the porous space and maintaining the temperature constant too in order to prevent  
559 any liquid-crystal transformation. Insofar as  $S_w = 1 - S_c$  is held constant, and henceforth controlled,  
560 the subsequent changes of  $\phi_c$  and  $\phi_w$  reduce to the changes of  $\varphi_c$  and  $\varphi_w$ , respectively.

561 In general, the Lagrangian saturation degrees  $S_w$  and  $S_c$  do not remain constant with time. The  
562 reference state may be conveniently chosen as fully saturated by one of them, the unfrozen water, for  
563 instance, so that we have  $S_w = 1$ . The associated initial pore-water pressure is generally equal to the  
564 atmospheric pressure taken as the reference datum. If the ice crystals grow hence the saturation degree  
565 of ice crystals  $p_c$  increases, the pressure difference  $p_c - p_w$  also increases in consequence. The

566 remaining unfrozen water is simultaneously expelled (due to volume expansion during phase  
567 transformation) and  $S_w$  decreases in consequence according to the last of relations (113). During this  
568 process, the porous network simultaneously deforms.

569 Substituting (113) into (112), we obtain,

$$570 \quad \sigma_{ij}d\varepsilon_{ij} + \sum_{\alpha} p_{\alpha}d\varphi_{\alpha} - \phi_0(p_c - p_w)dS_w - \theta_s dT - d\psi_s \geq 0. \quad (114)$$

571 In Expression (114), the first three items represent the strain work supplied to the porous solid between  
572 time  $t$  and  $t + dt$ . The fourth term identifies with the work done against the interfacial forces in order  
573 to create new interfaces between ice crystals and the solid matrix which accompanies the propagation  
574 of liquid water front inside the porous volume, resulting in the change of ice volume  $-\phi_0 dS_w =$   
575  $\phi_0 dS_c$ . The free energy of soil skeleton, which includes soil matrix and interfaces, thereby changes.  
576 This change is induced not only by the deformation of the porous solid, but also by the  
577 creation/destruction of interfaces between the ice crystal, the liquid water, and the solid matrix. Owing  
578 to its extensive character, the skeleton free energy  $\psi_s$  can be expressed as follows,

$$579 \quad \psi_s = W_s(\varepsilon_{ij}, \varphi_c, \varphi_w, T; \varepsilon_{ij}^p, \varphi_c^p, \varphi_w^p, \chi_j) + \phi U(S_w, \phi, T), \quad (115)$$

580 in which  $\varepsilon_{ij}^p, \varphi_c^p, \varphi_w^p$  are plastic strain, plastic porosities of ice and unfrozen water, respectively. The  
581 first term stands for the free energy of the solid matrix. The second represents the contribution from the  
582 interfaces and depends on the current partition of the porous volume between unfrozen water, ice  
583 crystals and soil matrix.  $W_s$  can also be decomposed into two parts, one of which is the elastic energy  
584 and the other is the locked energy due to irreversible (mechanical) processes. This locked energy is  
585 assumed to only depend on a hardening variable  $\chi_j$  for simplicity. The term  $\phi U(S_w, \phi, T)$  accounts  
586 for the current values of the interfaces between ice crystals, unfrozen water and soil matrix. Here we  
587 assume that the interface energy does not significantly vary as the porous solid deforms. Accordingly,  
588  $U$  is assumed independent of  $\varepsilon_{ij}$ . Moreover, it does not depend separately upon  $\varphi_c$  and  $\varphi_w$ , but  
589 additively through  $\phi$ .

#### 590 4.2 Thermo-poro-elasticity for saturated frozen soils

591 Under linear elastic behaviour, previous expression of the free energy  $\psi_s$  is reduced to the  
592 following form:

$$593 \quad \psi_s = W_s(\varepsilon_{ij}, \varphi_c, \varphi_w, T) + \phi U(S_w, \phi, T). \quad (116)$$

594 Therefore, we have,

$$595 \quad d\Psi_s = \frac{\partial W_s}{\partial \varepsilon_{ij}} d\varepsilon_{ij} + \frac{\partial W_s}{\partial \varphi_c} d\varphi_c + \frac{\partial W_s}{\partial \varphi_w} d\varphi_w + \left( \frac{\partial W_s}{\partial T} + \phi \frac{\partial U}{\partial T} \right) dT + U d\phi + \phi \left( \frac{\partial U}{\partial S_w} dS_w + \frac{\partial U}{\partial \phi} d\phi \right). \quad (117)$$

596 Substituting Equation (117) into Equation (114), we deduce that:

$$597 \quad \left( \sigma_{ij} - \frac{\partial W_s}{\partial \varepsilon_{ij}} \right) d\varepsilon_{ij} + \left( p_c - \frac{\partial W_s}{\partial \varphi_c} \right) d\varphi_c + \left( p_w - \frac{\partial W_s}{\partial \varphi_w} \right) d\varphi_w - \left( \theta_s + \frac{\partial W_s}{\partial T} + \phi \frac{\partial U}{\partial T} \right) dT - \left( \phi_0 (p_c - p_w) + \right. \\ 598 \quad \left. \phi \frac{\partial U}{\partial S_w} \right) dS_w - \left( U + \frac{\partial U}{\partial \phi} \right) d\phi \geq 0. \quad (118)$$

599 From Equation (113), we have  $d\phi = d\varphi_c + d\varphi_w$ , so we have:

$$600 \quad \left( \sigma_{ij} - \frac{\partial W_s}{\partial \varepsilon_{ij}} \right) d\varepsilon_{ij} + \left( p_c - \frac{\partial W_s}{\partial \varphi_c} - \frac{\partial U}{\partial \phi} \right) d\varphi_c + \left( p_w - \frac{\partial W_s}{\partial \varphi_w} - \frac{\partial U}{\partial \phi} \right) d\varphi_w - \left( \theta_s + \frac{\partial W_s}{\partial T} + \phi \frac{\partial U}{\partial T} \right) dT - \\ 601 \quad \left( \phi_0 (p_c - p_w) + \phi \frac{\partial U}{\partial S_w} \right) dS_w - U d\phi \geq 0. \quad (119)$$

602 Conceptually, the skeleton and pore space deformations as well as temperature can vary  
603 independently of water saturation. Restricting our attentions momentarily to processes at constant water  
604 saturation, the inequality sign becomes equality for elastic behaviour and we are led to the following  
605 state equations:

$$606 \quad \sigma_{ij} = \frac{\partial W_s}{\partial \varepsilon_{ij}}, p_c = \frac{\partial W_s}{\partial \varphi_c} + \frac{\partial U}{\partial \phi}, p_w = \frac{\partial W_s}{\partial \varphi_w} + \frac{\partial U}{\partial \phi}, \text{ and } \theta_s = -\frac{\partial W_s}{\partial T} - \phi \frac{\partial U}{\partial T}. \quad (120)$$

607 The entropy  $\theta_s$  for the soil skeleton consists of two parts, in which  $\theta_m = -\frac{\partial W_s}{\partial T}$  is the entropy of  
608 solid matrix and  $S_{int} = -\phi \frac{\partial U}{\partial T}$  is the interface entropy.

609 Substituting relations (120) into Equation (119), we get:

$$610 \quad \Phi_{hys} = -\left( \phi_0 (p_c - p_w) + \phi \frac{\partial U}{\partial S_w} \right) dS_w - U d\phi \geq 0, \quad (121)$$

611 in which  $\Phi_{hys}$  represents the hysteretic dissipation of freezing-thaw cycles. For simplicity, this  
612 hysteretic phenomenon is neglected here and left to a further study. Under such conditions, inequality  
613 (121) becomes an equality, leading after accounting for the assumption of small strains  $(\phi - \phi_0)/$   
614  $\phi_0 \ll 1$  to the following state equation:

$$615 \quad p_{cap} = (p_c - p_w) = -\frac{dU}{dS_w}. \quad (122)$$

616 The thermodynamic equilibrium between ice crystals and unfrozen water requires equality of their  
617 chemical potentials. Neglecting higher order terms, this condition implies:

$$618 \quad p_{cap} = (p_c - p_w) = \Sigma_m (T_m - T), \quad (123)$$

619 where  $T_m$  and  $\Sigma_m$  stand respectively for the melting temperature and the melting entropy. The  
620 water-ice equilibrium equation (123) and state equation (122) combine to reveal the existence of a

621 thermodynamic state function, linking bijectively the liquid saturation degree  $S_w$  to the current  
622 temperature  $T$  as follows,

$$623 \quad S_w = \wp(T_m - T). \quad (124)$$

624 If the interfacial energy  $U$  is assumed to take the following form (Coussy, 2004):

$$625 \quad U = \phi^{-1/3} \Gamma(S_w, T) \quad ; \quad \frac{\partial(\phi U)}{\partial \phi} = \frac{2}{3} U. \quad (125)$$

626 The ice pressure  $p_c$  and water pressure  $p_w$  then satisfy the following state equations:

$$627 \quad \bar{p}_c = \frac{\partial W_s}{\partial \varphi_c}, \quad \bar{p}_w = \frac{\partial W_s}{\partial \varphi_w}, \quad (126)$$

628 where  $\bar{p}_\alpha = p_\alpha - \frac{2}{3} U$  ( $\alpha = c, w$ ) represents the in-pore fluid pressures effectively transmitted to the  
629 solid skeleton. The term  $-\frac{2}{3} U$  accounts for the capillary forces arising from the interfaces.

630 The state equations obtained previously for thermo-poro-elasticity of saturated frozen soils can be  
631 summarized as follows,

$$632 \quad \sigma_{ij} = \frac{\partial W_s}{\partial \varepsilon_{ij}}, \quad \bar{p}_c = \frac{\partial W_s}{\partial \varphi_c}, \quad \bar{p}_w = \frac{\partial W_s}{\partial \varphi_w} \quad \text{and} \quad \theta_m = -\frac{\partial W_s}{\partial T}. \quad (127)$$

633 Introducing the Legendre-Fenchel transform  $W_s^*$  of  $W_s$  with respect to  $\varphi_j$ :

$$634 \quad W_s^* = W_s - \bar{p}_c \varphi_c - \bar{p}_w \varphi_w. \quad (128)$$

635 State equations (127) can be partially inverted in the form as follows,

$$636 \quad \sigma_{ij} = \frac{\partial W_s^*}{\partial \varepsilon_{ij}}, \quad \varphi_c = \frac{\partial W_s^*}{\partial \bar{p}_c}, \quad \varphi_w = \frac{\partial W_s^*}{\partial \bar{p}_w}, \quad \text{and} \quad \theta_m = -\frac{\partial W_s^*}{\partial T}. \quad (129)$$

637 We know that in linear poro-elasticity  $W_s^*$  is a quadratic form of its arguments, namely  $\varepsilon_{ij}$ ,  $\bar{p}_c$ ,  $\bar{p}_w$   
638 and  $T_m - T$ . For isotropic linear thermo-poro-elasticity,  $W_s^*$  has the following form,

$$639 \quad W_s^* = \frac{1}{2} (K - 2G/3) (\varepsilon_{kk})^2 + G \varepsilon_{ij} \varepsilon_{ji} - 3\alpha K \varepsilon_{kk} (T - T_m) - \sum_{J,K=c,w} \left( b_J \bar{p}_J \varepsilon_{kk} - 3\alpha_J \bar{p}_J (T - T_m) + \right. \\ 640 \quad \left. \frac{\bar{p}_J \bar{p}_K}{2N_{JK}} \right) - \frac{c_T}{2T_m} (T - T_m)^2. \quad (130)$$

641 And thus, we have

$$642 \quad \sigma_{ij} = (K - 2G/3) (\varepsilon_{kk}) \delta_{ij} + 2G (\varepsilon_{ij}) - b_c \bar{p}_c \delta_{ij} - b_w \bar{p}_w \delta_{ij} - 3\alpha K (T - T_m) \delta_{ij}, \quad (131)$$

$$643 \quad \varphi_c = b_c \varepsilon_{kk} + \frac{\bar{p}_c}{N_{CC}} + \frac{\bar{p}_w}{N_{CW}} - 3\alpha_c (T - T_m), \quad (132)$$

$$644 \quad \varphi_w = b_w \varepsilon_{kk} + \frac{\bar{p}_c}{N_{CW}} + \frac{\bar{p}_w}{N_{WW}} - 3\alpha_w (T - T_m), \quad (133)$$

$$645 \quad \theta_m = 3\alpha K \varepsilon_{kk} - 3\alpha_c \bar{p}_c - 3\alpha_w \bar{p}_w + \frac{c_T}{T_m} (T - T_m), \quad (134)$$

646 in which  $\varepsilon_{kk}$  is the volumetric dilation;  $K, G$  and  $\alpha$  are the bulk modulus, the shear modulus and the

647 thermal volumetric dilation coefficient of the solid skeleton, respectively, which are the ones related to  
 648 the empty porous materials subject to conditions  $\bar{p}_c = \bar{p}_w = 0$ .  $b_J$  and  $N_{JK}$  are the generalized Biot  
 649 coefficients and the generalized Biot coupling moduli (Biot, 1941), with  $N_{JK} = N_{KJ}$  owing to the  
 650 Maxwell's symmetry relations.  $\alpha_J$  and  $C_T$  are the thermal dilation of the pore volume occupied by  
 651 phase  $J$  and the heat capacity of the soil matrix.

### 652 4.3 Thermo-poro-elasto-plasticity for saturated frozen soils

653 When plastic deformations occur, mechanical dissipation will be generated. The skeleton free energy  
 654  $\Psi_s$  can be decomposed additively into the recoverable elastic free energy of soil skeleton, the locked  
 655 (or frozen) energy and the free energy of interfaces:

$$656 \quad \Psi_s = W_s(\varepsilon_{ij}, \varphi_c, \varphi_w, T; \varepsilon_{ij}^p, \varphi_c^p, \varphi_w^p, \chi_J) + \phi U(S_w, \phi, T). \quad (135)$$

657 Therefore, we have,

$$658 \quad d\Psi_s = \frac{\partial W_s}{\partial \varepsilon_{ij}} d\varepsilon_{ij} + \frac{\partial W_s}{\partial \varphi_c} d\varphi_c + \frac{\partial W_s}{\partial \varphi_w} d\varphi_w + \frac{\partial W_s}{\partial T} dT + \frac{\partial W_s}{\partial \varepsilon_{ij}^p} d\varepsilon_{ij}^p + \frac{\partial W_s}{\partial \varphi_c^p} d\varphi_c^p + \frac{\partial W_s}{\partial \varphi_w^p} d\varphi_w^p + \frac{\partial W_s}{\partial \chi_J} d\chi_J + U d\phi +$$

$$659 \quad \phi \left( \frac{\partial U}{\partial S_w} dS_w + \frac{\partial U}{\partial \phi} d\phi + \frac{\partial U}{\partial T} dT \right). \quad (136)$$

660 Substituting Equation (136) into Inequality (114) and accounting for the identity  $d\phi = d\varphi_c + d\varphi_w$ ,  
 661 we have:

$$662 \quad \left( \sigma_{ij} - \frac{\partial W_s}{\partial \varepsilon_{ij}} \right) d\varepsilon_{ij} + \left( p_c - \frac{\partial W_s}{\partial \varphi_c} - \frac{\partial U}{\partial \phi} \right) d\varphi_c + \left( p_w - \frac{\partial W_s}{\partial \varphi_w} - \frac{\partial U}{\partial \phi} \right) d\varphi_w - \left( S_s + \frac{\partial W_s}{\partial T} + \phi \frac{\partial U}{\partial T} \right) dT -$$

$$663 \quad \left[ \phi_0 (p_c - p_w) + \phi \frac{\partial U}{\partial S_w} \right] dS_w - U d\phi - \frac{\partial W_s}{\partial \varepsilon_{ij}^p} d\varepsilon_{ij}^p - \frac{\partial W_s}{\partial \varphi_c^p} d\varphi_c^p - \frac{\partial W_s}{\partial \varphi_w^p} d\varphi_w^p - \frac{\partial W_s}{\partial \chi_J} d\chi_J \geq 0$$

$$664 \quad (137)$$

665 In view of the state equations (120) and (121), we deduce the following form of the plastic dissipation:

$$666 \quad -\frac{\partial W_s}{\partial \varepsilon_{ij}^p} d\varepsilon_{ij}^p - \frac{\partial W_s}{\partial \varphi_c^p} d\varphi_c^p - \frac{\partial W_s}{\partial \varphi_w^p} d\varphi_w^p - \frac{\partial W_s}{\partial \chi_J} d\chi_J \geq 0. \quad (138)$$

667 In the following, we will discuss ideal plasticity and hardening plasticity for saturated frozen soils,  
 668 respectively.

#### 669 4.3.1 Ideal plasticity

670 For ideal plasticity, it can be defined by the absence of hardening variable  $\chi_J$ . Hence, no variables  
 671 are  $\chi_J$  involved in the expression of  $W_s^*$ . From equation (130) a suitable expression for  $W_s^*$  under  
 672 ideal plasticity can be expressed as follows,

$$673 \quad W_s^* = \frac{1}{2} \left( K - \frac{2G}{3} \right) (\varepsilon_{kk} - \varepsilon_{kk}^p)^2 + G (\varepsilon_{ij} - \varepsilon_{ij}^p) (\varepsilon_{ji} - \varepsilon_{ji}^p) - 3\alpha K (\varepsilon_{kk} - \varepsilon_{kk}^p) (T - T_m)$$

$$674 \quad -\sum_{J,K=c,w} \left( b_J \bar{p}_J (\varepsilon_{kk} - \varepsilon_{kk}^p) - 3\alpha_J \bar{p}_J (T - T_m) + \frac{\bar{p}_J \bar{p}_K}{2N_{JK}} \right) - \frac{c_T}{2T_m} (T - T_m)^2. \quad (139)$$

675 So, we have

$$676 \quad \sigma_{ij} = \left( K - \frac{2G}{3} \right) (\varepsilon_{kk} - \varepsilon_{kk}^p) \delta_{ij} + 2G (\varepsilon_{ij} - \varepsilon_{ij}^p) - b_c \bar{p}_c \delta_{ij} - b_w \bar{p}_w \delta_{ij} - 3\alpha K (T - T_m) \delta_{ij}, \quad (140)$$

$$677 \quad \varphi_c = b_c (\varepsilon_{kk} - \varepsilon_{kk}^p) + \frac{\bar{p}_c}{N_{CC}} + \frac{\bar{p}_w}{N_{CW}} - 3\alpha_c (T - T_m) + \varphi_c^p, \quad (141)$$

$$678 \quad \varphi_w = b_w (\varepsilon_{kk} - \varepsilon_{kk}^p) + \frac{\bar{p}_c}{N_{CW}} + \frac{\bar{p}_w}{N_{WW}} - 3\alpha_w (T - T_m) + \varphi_w^p, \quad (142)$$

$$679 \quad \theta_m = 3\alpha K (\varepsilon_{kk} - \varepsilon_{kk}^p) - 3\alpha_c \bar{p}_c - 3\alpha_w \bar{p}_w + \frac{c_T}{T_m} (T - T_m). \quad (143)$$

680 Therefore, the plastic dissipation of (138) can be rewritten as follows,

$$681 \quad \sigma_{ij} d\varepsilon_{ij}^p + \bar{p}_c d\varphi_c^p + \bar{p}_w d\varphi_w^p \geq 0, \quad (144)$$

682 and the following relationships have been used,  $\bar{p}_c = -\frac{\partial W_s}{\partial \varphi_c^p} = \frac{\partial W_s}{\partial \varphi_c}$ ,  $\bar{p}_w = -\frac{\partial W_s}{\partial \varphi_w^p} = \frac{\partial W_s}{\partial \varphi_w}$ ,  $\sigma_{ij} =$

$$683 \quad -\frac{\partial W_s}{\partial \varepsilon_{ij}^p} = \frac{\partial W_s}{\partial \varepsilon_{ij}}.$$

684 Equation (144) shows that the thermodynamical forces associated with the plastic strain, plastic  
685 porosities of ice crystal and unfrozen water are, respectively,  $\sigma_{ij}$ ,  $\bar{p}_c$  and  $\bar{p}_w$ . Now standard principles  
686 of ideal plasticity can be applied, namely the maximal plastic work principle introduced by Hill (1950).  
687 More precisely, let us note by

$$688 \quad f(\sigma_{ij}, \bar{p}_c, \bar{p}_w) \leq 0, \quad (145)$$

689 the equation delimiting in the stress-pressure space the elastic domain which is assumed to be  
690 independent of the temperature. Let  $(\sigma_{ij}, \bar{p}_c, \bar{p}_w)$  be the current stress-pressure state satisfying (145)  
691 and  $(\sigma_{ij}^*, \bar{p}_c^*, \bar{p}_w^*)$  another couple satisfying (145). Then the maximal plastic work principle implies

$$692 \quad (\sigma_{ij} - \sigma_{ij}^*) d\varepsilon_{ij}^p + (\bar{p}_c - \bar{p}_c^*) d\varphi_c^p + (\bar{p}_w - \bar{p}_w^*) d\varphi_w^p \geq 0. \quad (146)$$

693 From (146), assuming for simplicity the smoothness of the yield locus  $f(\sigma_{ij}, \bar{p}_c, \bar{p}_w) = 0$  (no vertex  
694 effects), it can be standardly shown (Hill, 1950) that  $f(\sigma_{ij}, \bar{p}_c, \bar{p}_w)$  must be a convex function and  
695 that

$$696 \quad \text{if } f(\sigma_{ij}, \bar{p}_c, \bar{p}_w) = 0 \text{ and } df = (\partial f / \partial \sigma_{ij}) d\sigma_{ij} + (\partial f / \partial \bar{p}_c) d\bar{p}_c + (\partial f / \partial \bar{p}_w) d\bar{p}_w = 0, \quad (147)$$

$$697 \quad \text{then: } d\varepsilon_{ij}^p = d\lambda (\partial f / \partial \sigma_{ij}), \quad d\varphi_c^p = d\lambda (\partial f / \partial \bar{p}_c) \text{ and } d\varphi_w^p = d\lambda (\partial f / \partial \bar{p}_w), \quad d\lambda \geq 0,$$

698 while in the other cases,  $f < 0$  or  $f = 0$  and  $df < 0$  (respectively, elastic state or local unloading state)  
699  $d\varepsilon_{ij}^p = 0$ ,  $d\varphi_c^p = 0$  and  $d\varphi_w^p = 0$ . Equation (147) implies that  $f$  can be considered as an associated  
700 plastic potential, the flow rule given by (147) being associated with the plastic yield locus  $f$ .

701 **4.3.2 Hardening plasticity**

702 For hardening plasticity at isothermal conditions, the hardening variables  $\chi_J$  are now present in  
703 the expression of  $W_s^*$ . Therefore, the plastic dissipation of (138) can be rewritten as follows,

$$704 \quad \sigma_{ij}d\varepsilon_{ij}^p + \bar{p}_c d\varphi_c^p + \bar{p}_w d\varphi_w^p - \frac{\partial W_s}{\partial \chi_J} d\chi_J \geq 0. \quad (148)$$

705 To introduce hardening plasticity, we will follow the presentation of Halphen and Nguyen (1974)  
706 which generalizes the maximal plastic work principle. We assume that  $W_s$  is a strictly (smooth  
707 hardening) convex (positive hardening) function and we note

$$708 \quad \xi_J = -\frac{\partial W_s}{\partial \chi_J}, \quad (\sigma_{ij}d\varepsilon_{ij}^p + \bar{p}_c d\varphi_c^p + \bar{p}_w d\varphi_w^p + \xi_J d\chi_J \geq 0). \quad (149)$$

709 Furthermore, let us introduce a strictly (smooth plasticity) convex function  $f(\sigma_{ij}, \bar{p}_c, \bar{p}_w, \xi_J)$ , the  
710 current elastic domain being given by  $f(\sigma_{ij}, \bar{p}_c, \bar{p}_w, \xi_J) \leq 0$ . The generalization of the results stated  
711 for ideal plasticity yields:

$$712 \quad d\varepsilon_{ij}^p = d\lambda(\partial f / \partial \sigma_{ij}), \quad d\varphi_c^p = d\lambda(\partial f / \partial \bar{p}_c), \quad d\varphi_w^p = d\lambda(\partial f / \partial \bar{p}_w), \quad d\chi_J = d\lambda(\partial f / \partial \xi_J), \quad (150)$$

713  $d\lambda \geq 0$  if  $f = 0$  and  $df = 0$ , and  $d\lambda = 0$  if  $f < 0$  or  $df < 0$ .

714 Now, following Halphen and Nguyen (1974) one can retrieve the standard theory of hardening  
715 plasticity. From (135) and (144), one obtains that:

$$716 \quad \text{if } f = 0 \text{ and } df_\zeta > 0, \quad d\varepsilon_{ij}^p = (1/\mathcal{H})(\partial f / \partial \sigma_{ij})df_\zeta,$$

$$717 \quad d\varphi_c^p = (1/\mathcal{H})(\partial f / \partial \bar{p}_c)df_\zeta,$$

$$718 \quad d\varphi_w^p = (1/\mathcal{H})(\partial f / \partial \bar{p}_w)df_\zeta, \quad (151)$$

$$719 \quad \text{and } d\chi_J = (1/\mathcal{H})(\partial f / \partial \xi_J)df_\zeta;$$

$$720 \quad \text{if } f < 0 \text{ or } f = 0 \text{ and } df_\zeta \leq 0, \quad d\varepsilon_{ij}^p = d\varphi_c^p = d\varphi_w^p = d\chi_J = 0, \quad (152)$$

721 where  $\mathcal{H} = -\frac{\partial f}{\partial \xi_J} \frac{\partial \xi_J}{\partial \chi_J} \frac{d\chi_J}{d\lambda} = \frac{\partial f}{\partial \xi_J} \cdot \left( \frac{\partial^2 W_s}{\partial \chi_J^2} \right) \cdot \frac{\partial f}{\partial \xi_J}$  and  $df_\zeta = (\partial f / \partial \sigma_{ij})d\sigma_{ij} + (\partial f / \partial \bar{p}_c)d\bar{p}_c +$   
722  $(\partial f / \partial \bar{p}_w)d\bar{p}_w$ .  $\mathcal{H}$  is the hardening modulus of the phenomenological theory of hardening plasticity.

723 For saturated frozen soils, they may exhibit the property that their experimental results do not  
724 confirm the validity of the maximal work principle. In this case normality does not hold, with that the  
725 plastic strain rate is not normal to the yield locus and  $f$  cannot be considered as a plastic potential.

726 Nevertheless, we can standardly introduce a non-associated plastic potential  $h(\sigma_{ij}, \bar{p}_c, \bar{p}_w, \xi_J)$ , such  
727 that the flow rule under this condition is as follows,

$$728 \quad \text{if } f(\sigma_{ij}, \bar{p}_c, \bar{p}_w, \xi_J) = 0 \text{ and } df_\zeta \geq 0,$$

$$729 \quad \text{then} \quad d\varepsilon_{ij}^p = (1/\mathbb{H})(\partial h / \partial \sigma_{ij})df_\zeta,$$



$$730 \quad d\varphi_c^p = (1/\mathbb{H})(\partial h/\partial \bar{p}_c)df_\zeta, \quad (153)$$

$$731 \quad d\varphi_w^p = (1/\mathbb{H})(\partial h/\partial \bar{p}_w)df_\zeta,$$

$$732 \quad \text{and } d\chi_J = (1/\mathbb{H})(\partial h/\partial \xi_J)df_\zeta;$$

$$733 \quad \text{if } f < 0 \text{ or } f = 0 \text{ and } df_\zeta < 0, \quad d\varepsilon_{ij}^p = d\varphi_c^p = d\varphi_w^p = d\chi_J = 0, \quad (154)$$

734 where  $\mathbb{H}$  is the hardening modulus, and  $\mathbb{H} = -\frac{\partial f}{\partial \xi_J} \frac{\partial \xi_J}{\partial \chi_J} \frac{d\chi_J}{d\lambda} = (\partial f/\partial \xi_J) \cdot \frac{\partial^2 w_s}{\partial \chi^2_J} \cdot \frac{\partial h}{\partial \xi_J}$ . When the  
 735 principle of maximum plastic work does not apply, the plastic strains do not achieve a maximum  
 736 production of entropy. Accordingly, the positivity of the plastic work is not guaranteed and has to be  
 737 checked.

#### 738 4.4 Modelling the cryogenic triaxial compression tests of frozen soils

##### 739 4.4.1 Effective stress in cryogenic triaxial stress state

740 For saturated frozen soils under cryogenic triaxial compression conditions, the stress and strain can  
 741 be described by  $\sigma_m = \frac{1}{3}(\sigma_a + 2\sigma_c)$ ,  $\sigma_s = \left(\frac{3}{2}s_{ij}s_{ij}\right)^{1/2} = \sigma_a - \sigma_c$ ,  $\varepsilon_s = \left(\frac{2}{3}e_{ij}e_{ij}\right)^{1/2} = \frac{2}{3}(\varepsilon_a - \varepsilon_c)$ ,  
 742 and  $\varepsilon_v = \frac{1}{3}(\varepsilon_a + 2\varepsilon_c)$ , in which  $\sigma_a$ ,  $\sigma_c$ ,  $\sigma_m$  and  $\sigma_s$  are the axial stress, confining pressure, mean  
 743 stress and generalized shear stress, respectively;  $e_{ij}$ ,  $s_{ij}$  are the strain deviator tensor and stress deviator  
 744 tensor, respectively;  $\varepsilon_a$ ,  $\varepsilon_c$ ,  $\varepsilon_s$  are the axial strain, radial strain, and shear strain, respectively.

745 When performing the laboratory test on saturated frozen soils using cryogenic triaxial apparatus, we  
 746 can only measure the strain ( $\varepsilon_a$ ,  $\varepsilon_v$ ), axial stress  $\sigma_a$  and confining pressure  $\sigma_c$ . Under these conditions,  
 747 the strains measured for the samples are the same as those of soil skeleton, but the stresses ( $\sigma_a, \sigma_c$ )  
 748 measured are the total stresses applied on the surface of samples. Therefore, considering the theory  
 749 presented in the paper, the yield criterion should be expressed as stress undergone by soil skeleton,  
 750 which can be expressed in terms of stress, water pressure, ice pressure. The soil skeleton deforms  
 751 subjected to the combining actions of external load, pore pressures of water pressure and ice pressure.  
 752 When formulating the elasto-plastic constitutive model, we have to make some assumptions here. It is  
 753 assumed that solid soil grains are plastic incompressible, meaning that plasticity are solely due to  
 754 irreversible sliding between un-deformable solid soil grains, and implies that  $d\varepsilon_v^p = -d\varphi_c^p - d\varphi_w^p$ . By  
 755 introducing a coefficient  $\chi$  ranging from 0 to 1, we can also assume that

$$756 \quad d\varphi_c^p = -(1 - \chi)d\varepsilon_v^p; \quad d\varphi_w^p = -\chi d\varepsilon_v^p. \quad (155)$$

757 This factor  $\chi$  is not easy to be determined, varying with  $S_\alpha$  ( $\alpha = c, w$ ), which needs further study.

758 The dissipation (148) can be expressed in triaxial conditions as follows (Appendix VI),

759 
$$D = -(\sigma_m - \bar{p}_c)d\varphi_c^p - (\sigma_m - \bar{p}_w)d\varphi_w^p + \sigma_s d\varepsilon_s^p + \xi_J d\chi_J \geq 0. \quad (156)$$

760 Substituting (155) into (156), we have

761 
$$D = [\sigma_m - (1 - \chi)\bar{p}_c - \chi\bar{p}_w]d\varepsilon_v^p + \sigma_s d\varepsilon_s^p + \xi_J d\chi_J \geq 0. \quad (157)$$

762 Let  $\sigma_m^E = \sigma_m - (1 - \chi)\bar{p}_c - \chi\bar{p}_w$ , which is the effective stress combining the influence of total stress,  
763 water pressure and ice crystal pressure, we can have

764 
$$D = \sigma_m^E d\varepsilon_v^p + \sigma_s d\varepsilon_s^p + \xi_J d\chi_J \geq 0. \quad (158)$$

765 Let  $p_{por}$  express the combining pore pressure of water pressure and ice pressure, we have

766 
$$p_{por} = (1 - \chi)\bar{p}_c + \chi\bar{p}_w, \quad (159-1)$$

767 and 
$$\sigma_m^E = \sigma_m - p_{por}. \quad (159-2)$$

768 When  $\chi$  is selected as saturation of ice crystals  $S_w$ ,  $p_{por}$  is the similar expression used by other  
769 researchers (Nishimura et al., 2009; Zhou and Meschke, 2013; Shen et al. 2014), but the influence of  
770 interface energy is considered in the paper. Under the assumption of incompressibility of solid soil  
771 grain forming soil skeleton,  $\sigma_m^E$  is the effective stress controlling the deformation and strength of soil  
772 skeleton of saturated frozen soils, which plays the similar role as Terzaghi's effective stress for  
773 saturated soils or Bishop's effective stress for unsaturated soils under unfreezing states.

774 Therefore, under triaxial stress conditions for saturated frozen soils, the current elastic domain can  
775 be given by

776 
$$f(\sigma_m^E, \sigma_s, \xi_J) \leq 0, \quad (160)$$

777 in which  $f$  is the loading function. Combining (151) and (152) for associated plastic criterion or (153)  
778 and (154) for non-associated plastic criterion, the stress-strain relationship for saturated frozen soils can  
779 be formulated when  $f(\sigma_m^E, \sigma_s, \xi_J)$  or  $h(\sigma_m^E, \sigma_s, \xi_J)$  are determined.

#### 780 4.4.2 Formulation of constitutive equations

781 The frozen soils usually behave strain softening at lower confining pressure and gradually transfer to  
782 strain hardening with increasing confining pressure, and the volumetric strain contract first, followed  
783 by dilatancy at lower confining pressure, but contract all the time at higher confining pressure. For  
784 example, the stress-strain curves for frozen sand with 1.0 MPa confining pressure at  $-1^\circ\text{C}$  to  $-10^\circ\text{C}$   
785 was published by Xu et al. (2016) and those for saturated frozen saline sandy soils at  $-6^\circ\text{C}$  by Lai et al.  
786 (2016). Therefore, when formulating the special constitutive model for frozen soils, these mechanical  
787 features should be taken into account.

788 In order to formulate a constitutive model reflecting the main mechanical features of frozen soils

789 with strain softening accompanied by volumetric contraction followed by dilatancy at lower confining  
 790 pressure, and strain hardening accompanied by volumetric contraction at higher confining pressure, the  
 791 yield function  $f$  used is as follows,

$$792 \quad f(\sigma_m^E, \sigma_s, \xi_\alpha, \xi_\beta) = \frac{\sigma_m^E}{1 - (\eta/\xi_\alpha)^z} - \xi_\beta = 0, \quad (161)$$

793 where  $\eta = \sigma_s/\sigma_m^E$ ,  $\xi_\alpha(\varepsilon_s^p)$ ,  $\xi_\beta(\varepsilon_v^p)$  are hardening parameters, and  $z$  is a material parameter.

794 These hardening parameters,  $\xi_\alpha(\varepsilon_s^p)$  and  $\xi_\beta(\varepsilon_v^p)$ , are expressed as follows,

$$795 \quad \xi_\alpha(\varepsilon_s^p) = \alpha_m \left( 1.0 - \kappa \exp\left(-\frac{\varepsilon_s^p}{\gamma_1}\right) \right), \quad (162)$$

$$796 \quad \xi_\beta(\varepsilon_v^p) = \sigma_{mr}^E \exp(\beta \varepsilon_v^p), \quad (163)$$

797 in which  $\alpha_m, \kappa, \gamma_1, \beta$  are material parameters, and  $\sigma_{mr}^E$  is a reference pressure.

798 The plastic potential function  $h$  has the same form as yield function, expressed as follows

$$799 \quad h = \frac{\sigma_m^E}{1 - (\eta/\xi_{\alpha 1})^{z_1}} - \xi_{\beta 1} = 0, \quad (164)$$

$$800 \quad \text{with} \quad \xi_{\alpha 1}(\varepsilon_s^p) = \alpha_{m1} \left( 1.0 - \kappa_1 \exp\left(-\frac{\varepsilon_s^p}{\gamma_2}\right) \right), \quad (165)$$

$$801 \quad \xi_{\beta 1}(\varepsilon_v^p) = \sigma_{mr}^E \exp(\beta_1 \varepsilon_v^p), \quad (166)$$

802 in which  $z_1, \alpha_{m1}, \kappa_1, \gamma_2, \beta_1$  are material parameters.

803 The incremental  $\varepsilon_v^p$  and  $\varepsilon_s^p$  can be obtained as follows,

$$804 \quad d\varepsilon_v^p = d\lambda \frac{\partial h}{\partial \sigma_m^E}, \quad (167-1)$$

$$805 \quad d\varepsilon_s^p = d\lambda \frac{\partial h}{\partial \sigma_s}. \quad (167-2)$$

806 From consistency conditions,  $d\lambda$  can be expressed as follows,

$$807 \quad d\lambda = \frac{1}{H} \left( \frac{\partial f}{\partial \sigma_m^E} d\sigma_m^E + \frac{\partial f}{\partial \sigma_s} d\sigma_s \right), \quad (166)$$

808 in which

$$809 \quad H = -\frac{\partial f}{\partial \xi_\alpha} \frac{\partial \xi_\alpha}{\partial \chi_\alpha} \frac{d\chi_\alpha}{d\lambda} - \frac{\partial f}{\partial \xi_\beta} \frac{\partial \xi_\beta}{\partial \chi_\beta} \frac{d\chi_\beta}{d\lambda}. \quad (167)$$

810 Substituting  $d\chi_\alpha = d\varepsilon_s^p = d\lambda \frac{\partial h}{\partial \sigma_s}$  and  $d\chi_\beta = d\varepsilon_v^p = d\lambda \frac{\partial h}{\partial \sigma_m^E}$  into (167), we have

$$811 \quad H = -\frac{\partial f}{\partial \xi_\alpha} \frac{\partial \xi_\alpha}{\partial \chi_\alpha} \frac{\partial h}{\partial \sigma_s} - \frac{\partial f}{\partial \xi_\beta} \frac{\partial \xi_\beta}{\partial \chi_\beta} \frac{\partial h}{\partial \sigma_m^E}. \quad (168)$$

812 Therefore, the incremental form of volumetric strain and shear strain can be expressed as follows

813 (Appendix VI),

$$814 \quad d\varepsilon_v = d\varepsilon_v^e + d\varepsilon_v^p = A_1 d\sigma_m^E + B_1 d\sigma_s, \quad (169-1)$$

$$d\varepsilon_s = d\varepsilon_s^e + d\varepsilon_s^p = A_2 d\sigma_m^E + B_2 d\sigma_s, \quad (169-2)$$

where  $A_1 = \frac{1}{K} + \frac{1}{H} \frac{\partial f}{\partial \sigma_m^E} \frac{\partial h}{\partial \sigma_m^E}$ ,  $B_1 = \frac{1}{H} \frac{\partial f}{\partial \sigma_s} \frac{\partial h}{\partial \sigma_m^E}$ ,  $A_2 = \frac{1}{H} \frac{\partial f}{\partial \sigma_m^E} \frac{\partial h}{\partial \sigma_s}$ ,  $B_2 = \frac{1}{3G} + \frac{1}{H} \frac{\partial f}{\partial \sigma_s} \frac{\partial h}{\partial \sigma_s}$ . The non-associated plastic potential function used here can assure the dissipation (158)  $D \geq 0$ .

The approaches for determining the model parameters are described as follows.  $K$  and  $G$  are elastic parameters, which can be determined by unloading tests or use of initial loading stage within a small axial strain range, and these elastic parameters can be expressed as function of stress level and temperature.  $S_c, S_w, \Sigma_m, U, T_m, k_s, b_\alpha, \phi_0$  can be determined for particular frozen soils under cryogenic triaxial tests. The other parameters in the model are related to yield function, plastic potential, and hardening parameters, in which  $z$  and  $z_1$  denote the shape of yield surface and plastic potential surface,  $\alpha_m$  or  $\alpha_{m1}$ ,  $\kappa$ ,  $\kappa_1, \gamma_1$  or  $\gamma_2$  denote the evolution of hardening parameters of  $\xi_\alpha$  and  $\xi_{\alpha1}$ , and  $\sigma_{mr}^E, \beta$  or  $\beta_1$  denote the evolution of hardening parameters of  $\xi_\beta$  and  $\xi_{\beta1}$ , and these parameters of parameters do not have clear physical meanings can be determined by trial and error approach. The sensitivity analyses of these parameters are done in Section 4.4.4, from which we know the influences of their variations on stress-strain relationships of frozen soils.

#### 4.4.3 Validation of the proposed constitutive model

Considering the thermodynamic equilibrium between ice crystals and unfrozen water, we can express equation (123) as follows,

$$p_c - p_w = \bar{p}_c - \bar{p}_w = \Sigma_m (T_m - T). \quad (170)$$

It is obvious that at constant  $T$ , the differences between  $p_c, p_w$  (or  $\bar{p}_c, \bar{p}_w$ ) are constant for special freezing materials with determined values of melting entropy  $\Sigma_m$  and melting temperature  $T_m$ .

For the frozen soils tested under cryogenic triaxial compression conditions (Xu et al., 2016; Lai et al., 2016), the water is prevented from escaping from the frozen soil sample upon loading, and the total mass of water and ice crystals  $\rho_c \phi_c + \rho_w \phi_w$  remains constantly equal to the initial mass of water and ice crystals at the start of loading. Therefore, we have

$$d(\rho_c \phi_c + \rho_w \phi_w) = d(\rho_c \phi_c + \rho_w \phi_w) = 0. \quad (171)$$

Combining Equations (132), (133) and (171) with neglecting the variation of densities of ice crystal and unfrozen water within the small range of pressure at isothermal conditions, we can have the relation between  $d\bar{p}_w$  (or  $d\bar{p}_c$ ) with  $d\varepsilon_v$  (or  $d\sigma_m^E$ ) as follows,

$$(b_c + b_w)d\varepsilon_v + \left(\frac{1}{N_{CC}} + \frac{1}{N_{CW}}\right)d\bar{p}_c + \left(\frac{1}{N_{CW}} + \frac{1}{N_{WW}}\right)d\bar{p}_w = 0. \quad (172)$$

845 From the tested results, we can have the values of  $\varepsilon_v$  and  $\varepsilon_s$  for different confining pressures, and  
 846 thus combining (170) and (172) we can obtain  $\bar{p}_c$  and  $\bar{p}_w$ . Therefore, in the process of loading we  
 847 can have  $p_{por}$  and  $\sigma_m^E$  by use of (159-1) and (159-2).

848 Fig. 2 (a) and (b) present the comparison of tested and simulated results of frozen sand soil at  $-1^\circ\text{C}$ ,  
 849  $-5^\circ\text{C}$  and  $-10^\circ\text{C}$  with 1MPa confining pressure. Under triaxial compression conditions, the generalized  
 850 shear stress equals the deviatoric stress. The triaxial compression tests were performed on frozen sand  
 851 under 1MPa confining pressure at temperatures of  $-1^\circ\text{C}$ ,  $-5^\circ\text{C}$  and  $-10^\circ\text{C}$  (Xu et al., 2016). The  
 852 parameters used are as follows (Lee et al., 2002; Coussy, 2004, 2008; Xu et al., 2016):

$$853 \quad K = K_0 p_a \left(\frac{\sigma_m^E}{p_a}\right)^{m_1}, \quad G = G_0 p_a \left(\frac{\sigma_m^E}{p_a}\right)^{m_2}, \quad K_0 = 1666.13|T|^{0.5721} \text{ MPa},$$

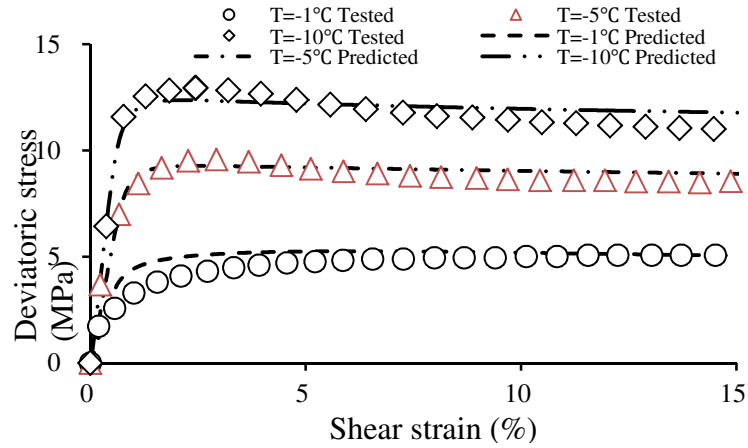
854  $G_0 = 1741.48e^{0.1094|T|} \text{ MPa}$ ,  $m_1 = 0.0404|T|^{0.6584}$ ,  $m_2 = 0.06$ , and  $p_a$  is standard atmospheric  
 855 pressure;  $S_c = 0.95$ ,  $S_w = 0.05$ ,  $\Sigma_m = 1.2 \text{ MPa}\cdot\text{K}^{-1}$ ,  $U = 5.21 \text{ MPa}$ ,  $T_m = 273 \text{ K}$ ,  $k_s = 42400 \text{ MPa}$ ,

$$856 \quad b_0 = 1 - K/k_s, \quad b_\alpha = b_0 S_\alpha (\alpha = c, w), \quad \frac{1}{N_{CC}} + \frac{1}{N_{CW}} = \frac{b_c - \phi_0 S_c}{k_s}, \quad \frac{1}{N_{WW}} + \frac{1}{N_{WC}} = \frac{b_w - \phi_0 S_w}{k_s}, \quad \phi_0 = 0.465;$$

$$857 \quad \alpha_m = \alpha_{m1} = 3.56, \quad \kappa = -0.066 \ln|T| + 0.4281, \quad \gamma_1 = 0.0389T + 0.541, \quad \sigma_{mr}^E = 66.649e^{0.0398|T|},$$

$$858 \quad \beta = \beta_1 = 445.3 \ln|T| + 69.259, \quad z = z_1 = -0.591 \ln|T| + 2.754, \quad \kappa_1 = 0.5348|T|^{-0.176}, \quad \gamma_2 =$$

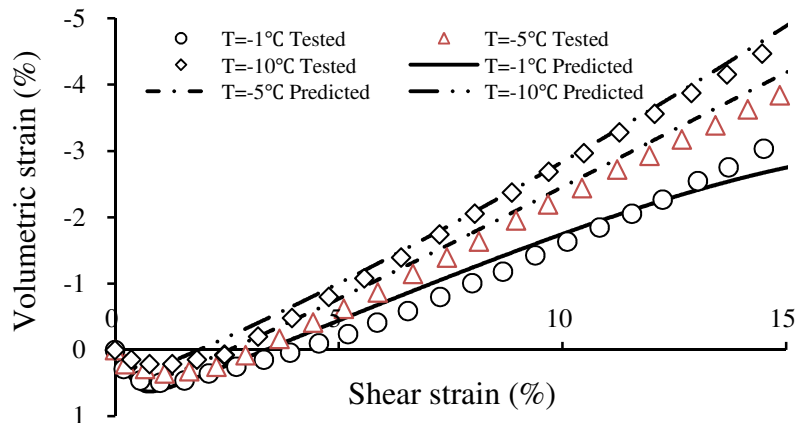
859  $-0.0198T + 0.3779$ . From Fig. 2, we can see that the proposed constitutive model can simulate the  
 860 stress-strain properties of frozen sand at different temperatures, and with decreasing temperature the  
 861 soil samples behave strain hardening transforming to strain softening slightly; the lower the  
 862 temperature, the smaller the volumetric compaction and the higher the temperature, the smaller the  
 863 volumetric dilatancy at failure. Fig. 2 (c) presents the predicted curves of volumetric strain and pore  
 864 pressure in the process of shear loading, indicating that the pore pressure increases gradually in the  
 865 whole loading process.



866

867

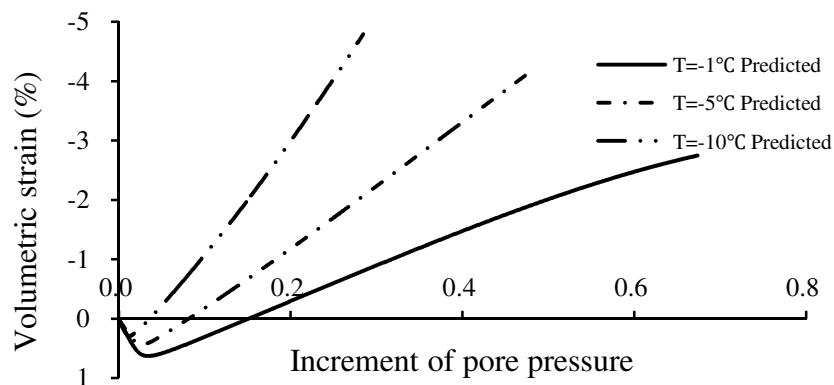
(a) Deviatoric stress-shear strain curves



868

869

(b) Volumetric strain-shear strain curves



870

871

(c) Predicted volumetric strain- pore pressure curves

872 Fig. 2 Comparisons of tested and simulated results of frozen sand soil at  $-1^{\circ}\text{C}$ ,  $-5^{\circ}\text{C}$  and  $-10^{\circ}\text{C}$  with

873

1MPa confining pressure

874 Fig. 3 (a) and (b) show another comparison of tested and simulated results of saline frozen sand soil

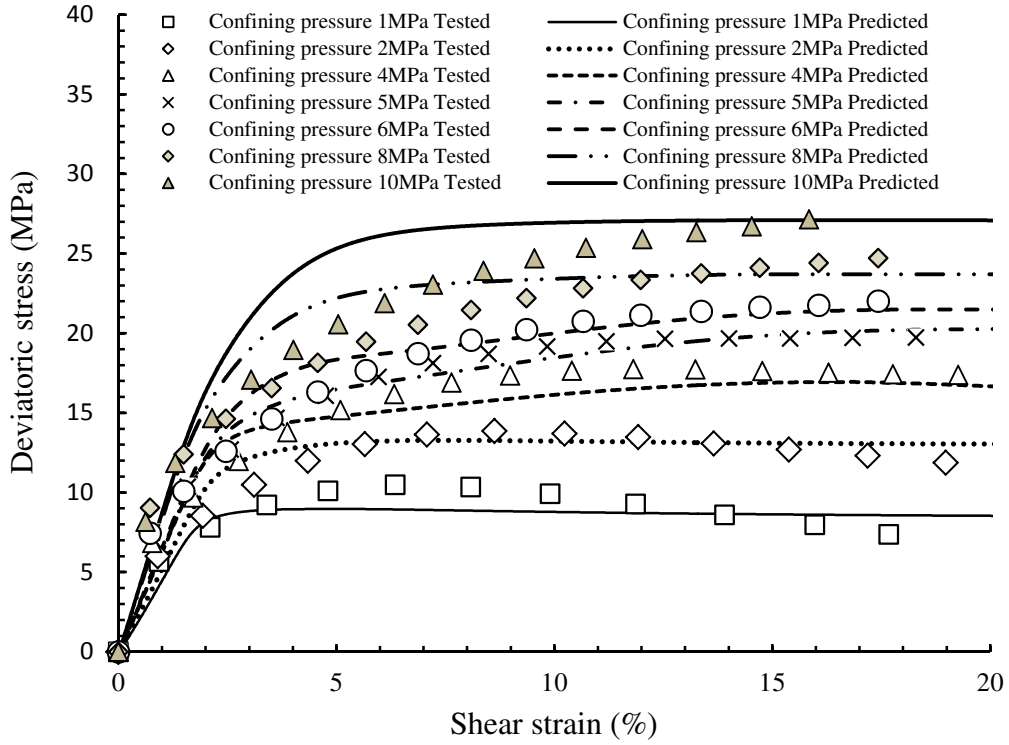
875 at  $-6^{\circ}\text{C}$  with 1.0MPa to 10.0 MPa confining pressures (Lai et al., 2016). The parameters used are as

876 follows (Lee et al., 2002; Coussy, 2004, 2008; Lai et al., 2016):  $K = K_0 p_a \left( \frac{\sigma_m^E}{p_a} \right)^{m_1}$ ,  $G = G_0 p_a \left( \frac{\sigma_m^E}{p_a} \right)^{m_2}$ ,  
877  $K_0 = 200 \text{ MPa}$ ,  $G_0 = 2733.2 \eta^{-1.387} \text{ MPa}$ ,  $m_1 = 0.4051 \ln \eta - 0.1304$ ,  $m_2 = -\frac{0.2991}{\eta} + 0.3127$ ;  
878  $S_c = 0.951$ ,  $S_w = 0.049$ ,  $\Sigma_m = 1.2 \text{ MPa.K}^{-1}$ ,  $U = 5.21 \text{ MPa}$ ,  $T_m = 273 \text{ K}$ ,  $k_s = 42400 \text{ MPa}$ ,  
879  $b_0 = 1 - K/k_s$ ,  $b_\alpha = b_0 S_\alpha (\alpha = c, w)$ ,  $\frac{1}{N_{CC}} + \frac{1}{N_{CW}} = \frac{b_c - \phi_0 S_c}{k_s}$ ,  $\frac{1}{N_{WW}} + \frac{1}{N_{WC}} = \frac{b_w - \phi_0 S_w}{k_s}$ ,  $\phi_0 = 0.4652$ ;  
880  $\alpha_m = \alpha_{m1} = 1.9925 \ln \eta + 1.389$ ,  $\kappa = 0.4081 \left( \frac{1}{\eta} \right)^{0.7037}$ ,  $\gamma_1 = 0.1007 \eta - 0.1418$ ,  
881  $\sigma_{mr}^E = 5.5997 e^{1.2587 \eta}$ ,  $\beta = \beta_1 = 832.1 \ln \eta - 117.32$ ,  $z = z_1 = 2.5$ ,  $\kappa_1 = 1.5199 \ln \eta + 1.5757$ ,  
882  $\gamma_2 = \frac{1}{-5.8072 \eta + 15.909}$ , and  $\eta = \sigma_s / \sigma_m$  is the stress ratio at peak stress state for samples with strain  
883 softening or failure state for samples with strain hardening. From the compared results, it is obvious  
884 that the stress-strain features of frozen saline sands can be duplicated by the constitutive model  
885 proposed here at different confining pressures with  $-6^\circ\text{C}$ . The samples behave slight strain softening  
886 and volumetric compaction first followed by dilatancy with increasing axial strain at lower confining  
887 pressures, and strain hardening and volumetric compaction all the time at higher confining pressures,  
888 which can be simulated relatively well by the proposed constitutive model. Fig. 3 (c) presents the  
889 predicted curves of volumetric strain and pore pressure in the process of shear loading, which indicates  
890 that the higher the confining pressure, the bigger the pore pressure generated at failure for the frozen  
891 samples.

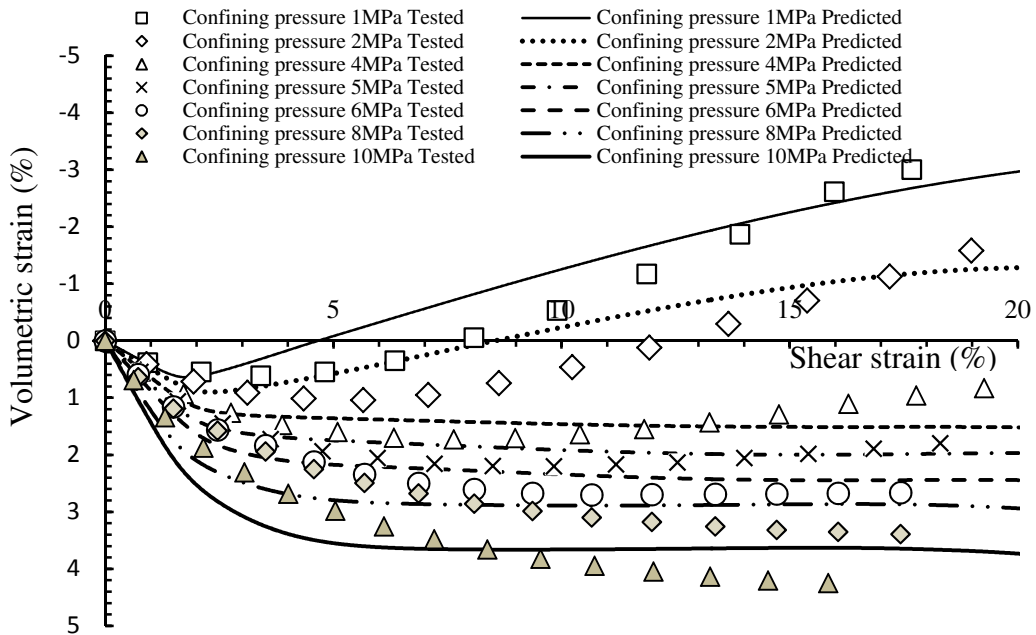
#### 892 4.4.4 Analysis of parameter sensitivity and discussions

893 Figs. 4-11 present the simulated results with varying values of model parameters, including  
894  $\alpha_m (= \alpha_{m1})$ ,  $\kappa$ ,  $\gamma_1$ ,  $\sigma_{mr}^E$ ,  $\beta (= \beta_1)$ ,  $z (= z_1)$ ,  $\kappa_1$  and  $\gamma_2$ , with 2 MPa confining pressure at  $-6^\circ\text{C}$ .  
895 From these simulated results, we can find that the main mechanical features of frozen soils can be  
896 duplicated. For example, with increasing  $\alpha_m (= \alpha_{m1})$ ,  $\kappa$ ,  $\sigma_{mr}^E$ ,  $z (= z_1)$ ,  $\kappa_1$ , and  $\gamma_2$ , the samples  
897 behave strain softening slightly to strain hardening; while with increasing  $\gamma_1$  and  $\beta (= \beta_1)$ , the  
898 samples behave strain hardening slightly to strain softening. For the volumetric strain, with increasing  
899  $\alpha_m (= \alpha_{m1})$ ,  $\sigma_{mr}^E$ ,  $\beta (= \beta_1)$ ,  $z (= z_1)$ ,  $\kappa_1$ , and  $\gamma_2$ , the samples compact more heavily; with increasing  
900  $\gamma_1$ , the samples dilate more greatly, and the bigger  $\kappa$ , the smaller the samples contract and dilate. All  
901 the stress paths of  $\sigma_m^E - \sigma_s$  with varying these parameters, the samples exhibit the similar pattern with  
902 increasing pore pressure during the process of shear loading.

903



(a) Deviatoric stress-shear strain curves

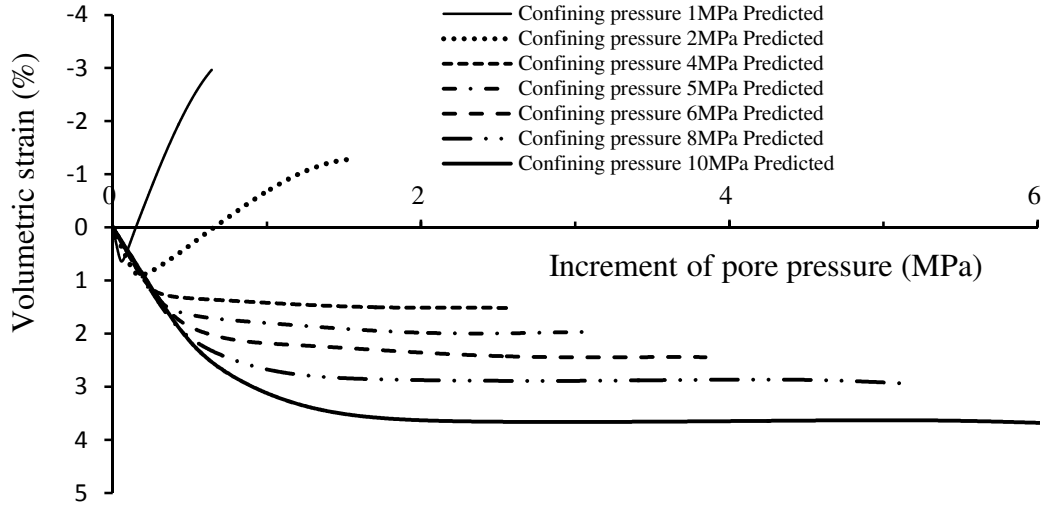


(b) Volumetric strain-shear strain curves

904  
905  
906

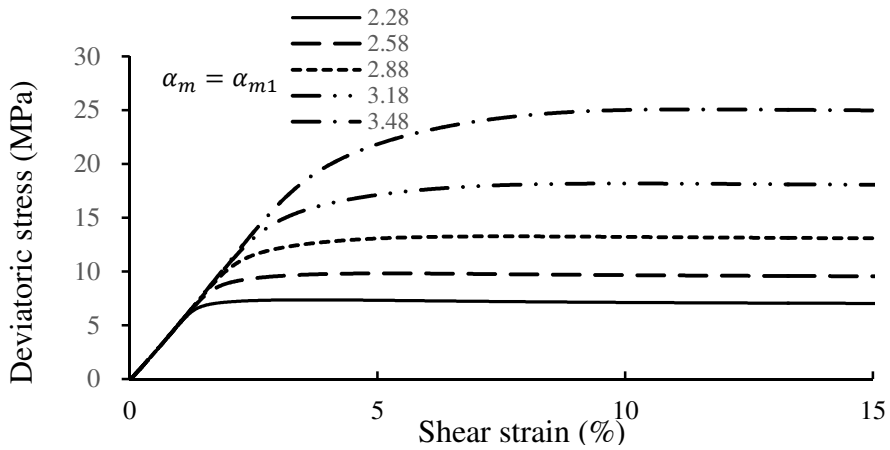
907  
908



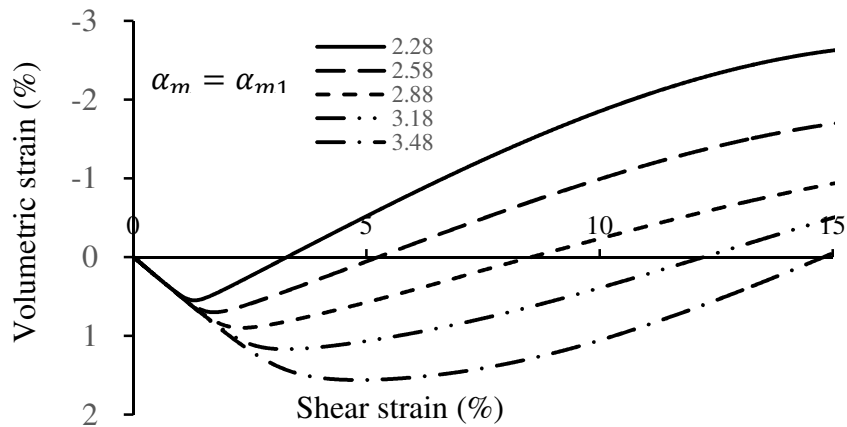


(c) Predicted volumetric strain- pore pressure curves

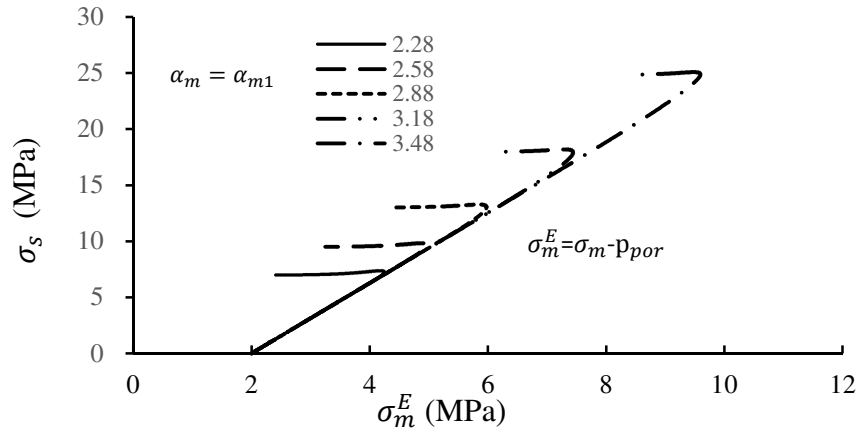
Fig. 3 Comparisons of tested and simulated results of frozen saline sand soil at -6°C with 1MPa-10MPa confining pressures



(a) Deviatoric stress-shear strain curves



(b) Volumetric strain-shear strain curves

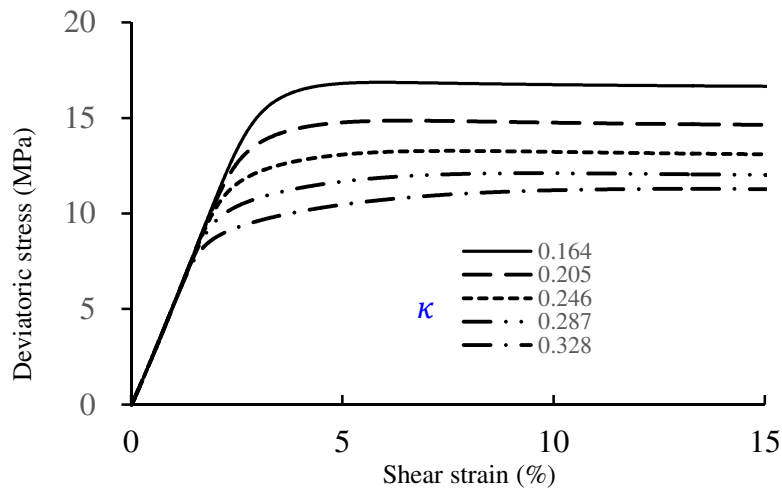


918

919

(c) Stress path of  $\sigma_m^E$ - $\sigma_s$ 

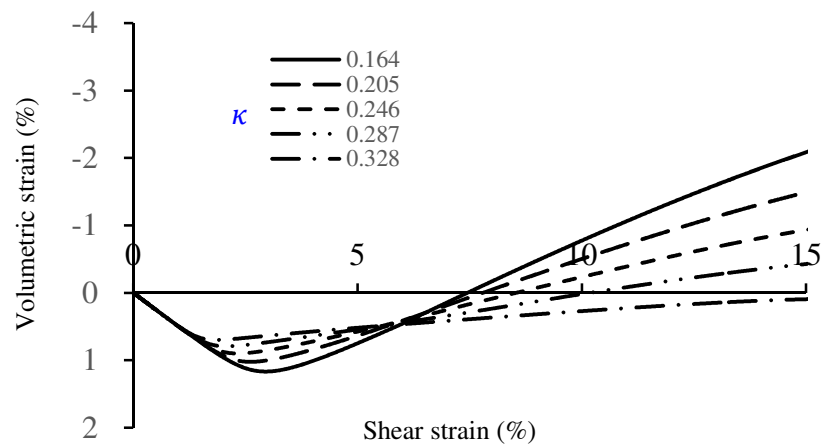
920

Fig. 4 Simulated results with varying  $\alpha_m = \alpha_{m1}$  at 2.0MPa confining pressures

921

922

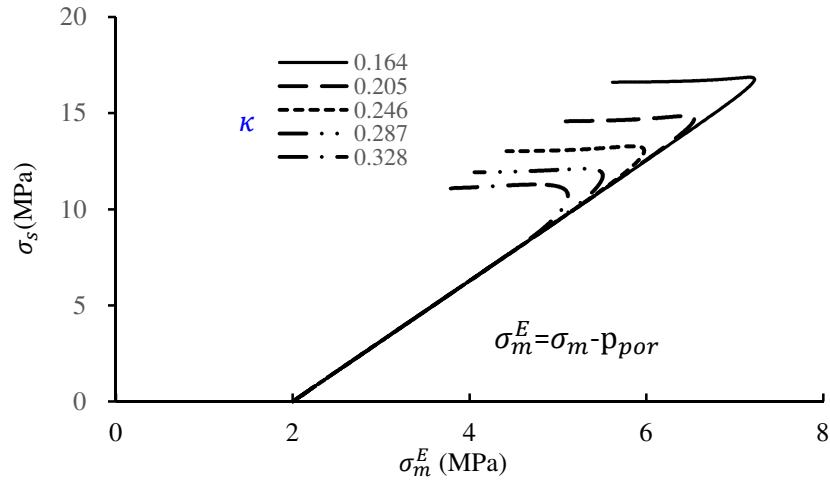
(a) Deviatoric stress-shear strain curves



923

924

(b) Volumetric strain-shear strain curves

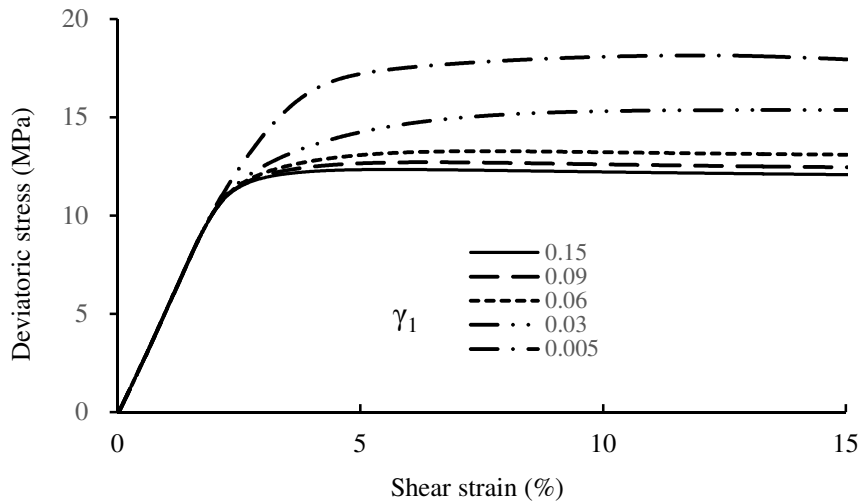


925

926

(c) Stress path of  $\sigma_m^E - \sigma_s$ 

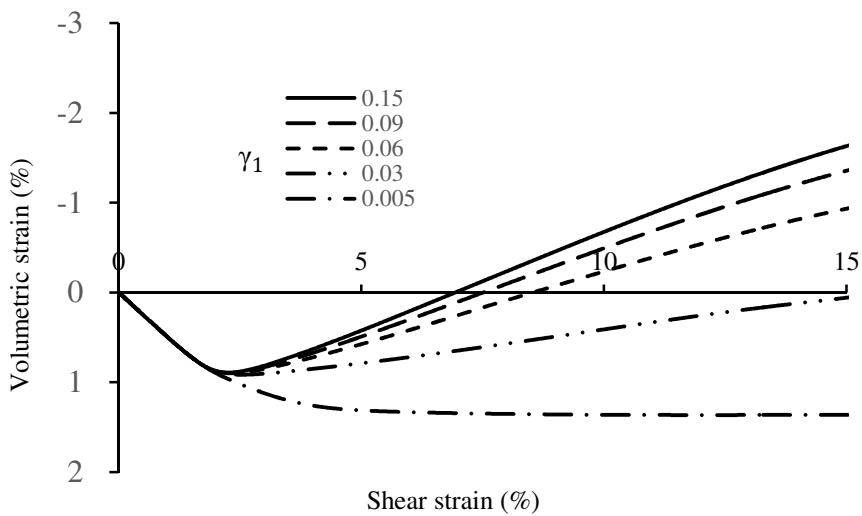
927

Fig. 5 Simulated results with varying  $\kappa$  at 2.0MPa confining pressures

928

929

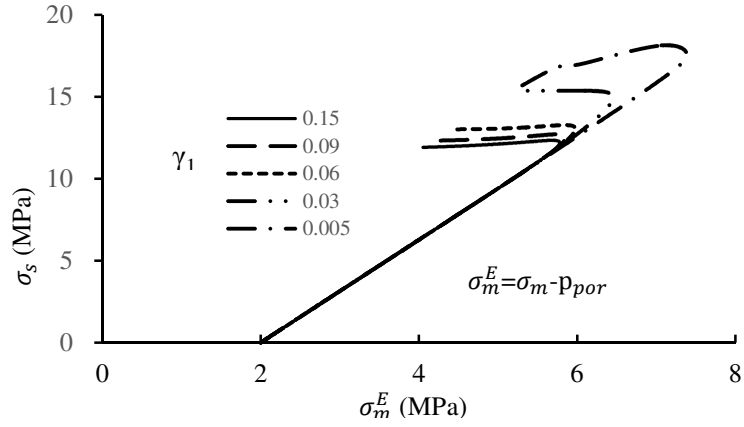
(a) Deviatoric stress-shear strain curves



930

931

(b) Volumetric strain-shear strain curves

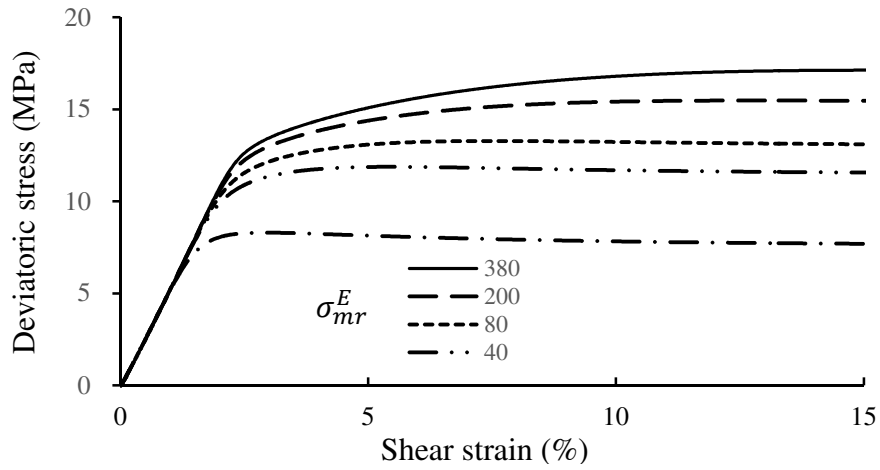


932

933

(c) Stress path of  $\sigma_m^E - \sigma_s$ 

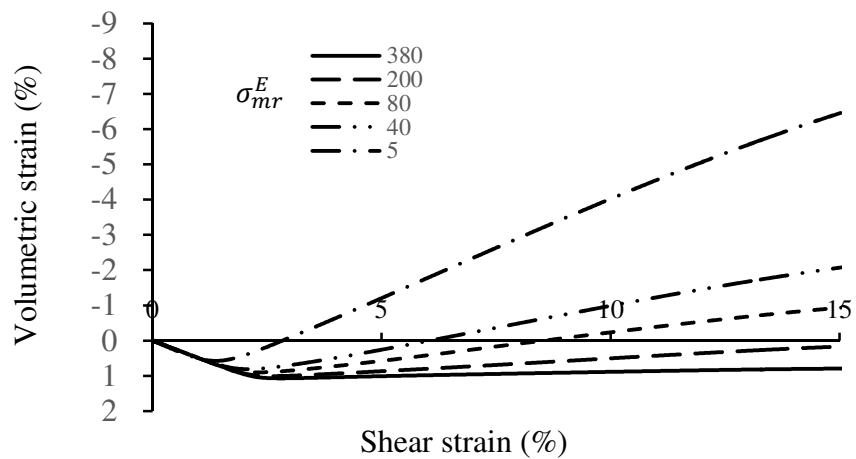
934

Fig. 6 Simulated results with varying  $\gamma_1$  at 2.0MPa confining pressures

935

936

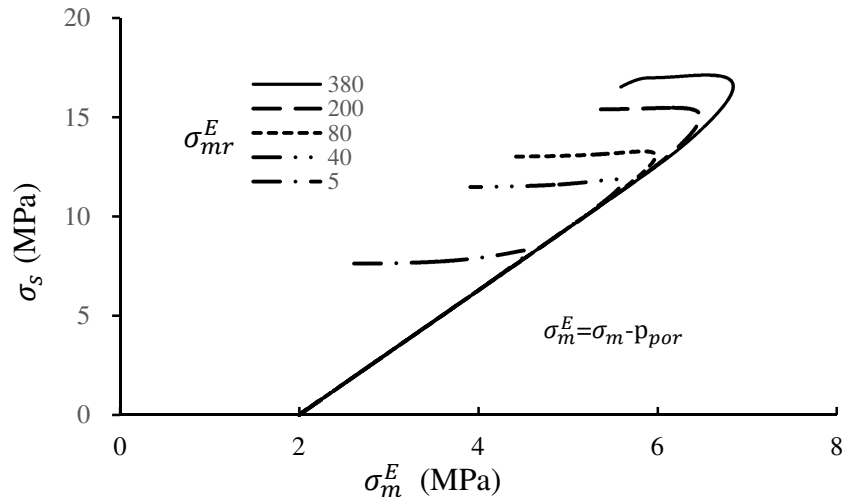
(a) Deviatoric stress-shear strain curves



937

938

(b) Volumetric strain-shear strain curves

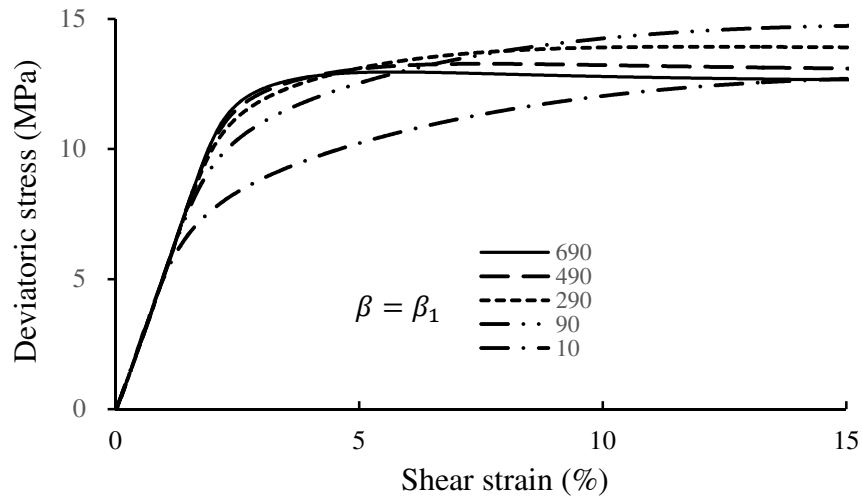


939

940

(c) Stress path of  $\sigma_m^E$ - $\sigma_s$ 

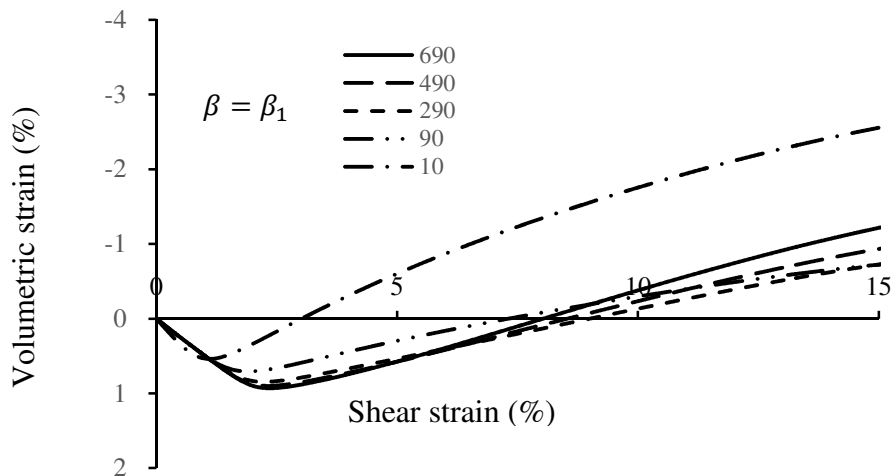
941

Fig. 7 Simulated results with varying  $\sigma_{mr}^E$  at 2.0MPa confining pressures

942

943

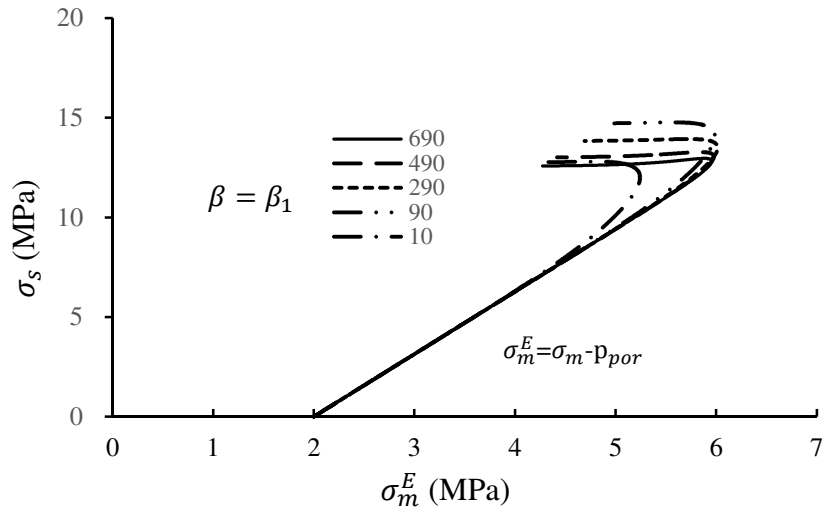
(a) Deviatoric stress-shear strain curves



944

945

(b) Volumetric strain-shear strain curve



946

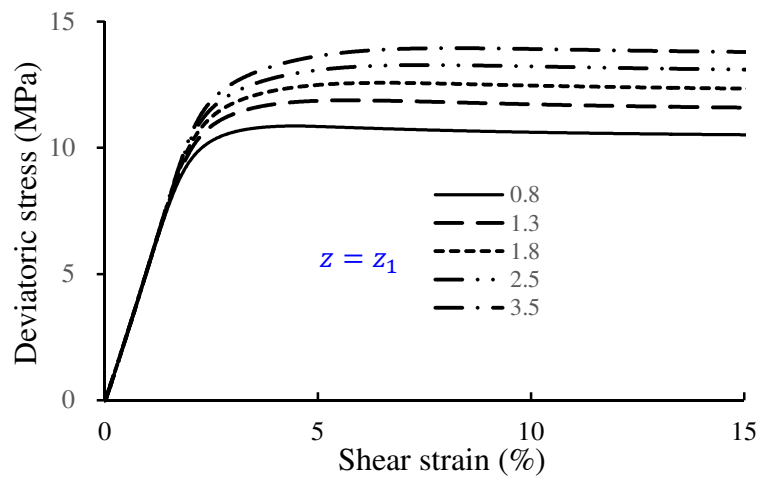
947

(c) Stress path of  $\sigma_m^E - \sigma_s$ 

948

Fig. 8 Simulated results with varying  $\beta = \beta_1$  at 2.0MPa confining pressures

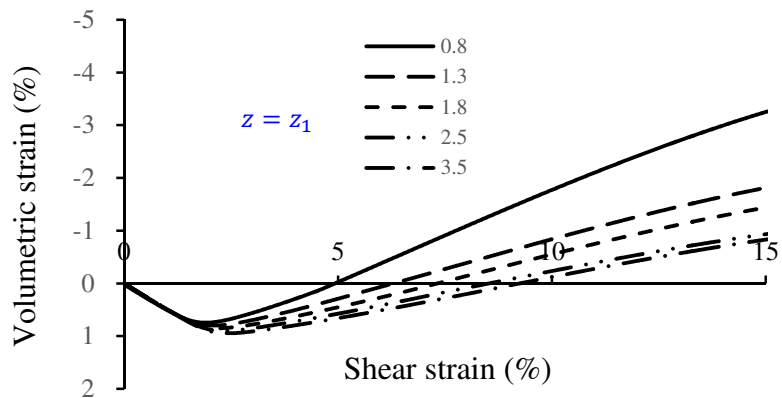
949



950

951

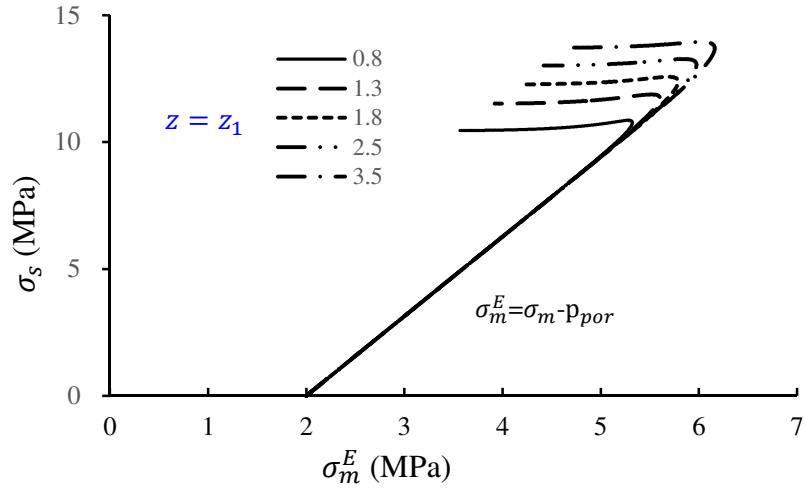
(a) Deviatoric stress-shear strain curves



952

953

(b) Volumetric strain-shear strain curves

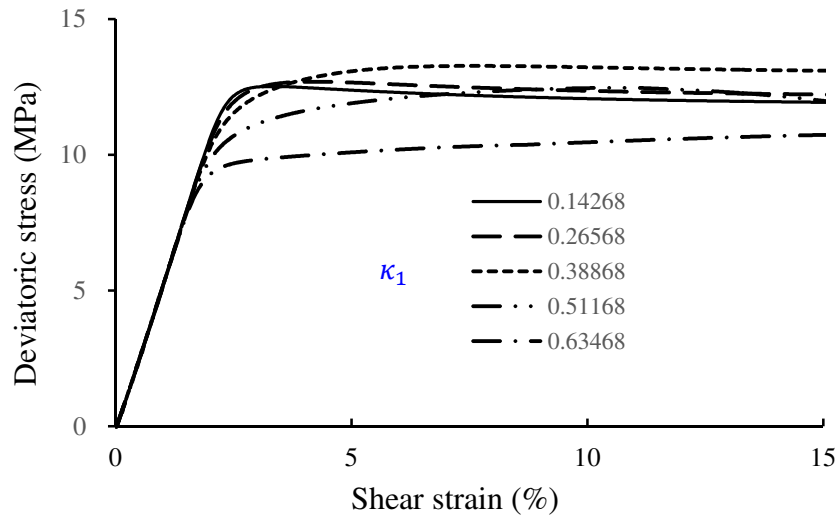


954

955

(c) Stress path of  $\sigma_m^E - \sigma_s$ 

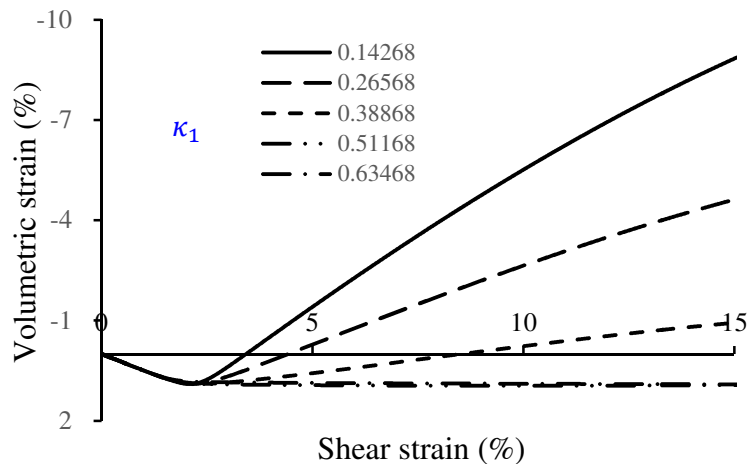
956

Fig. 9 Simulated results with varying  $z = z_1$  at 2.0MPa confining pressures

957

958

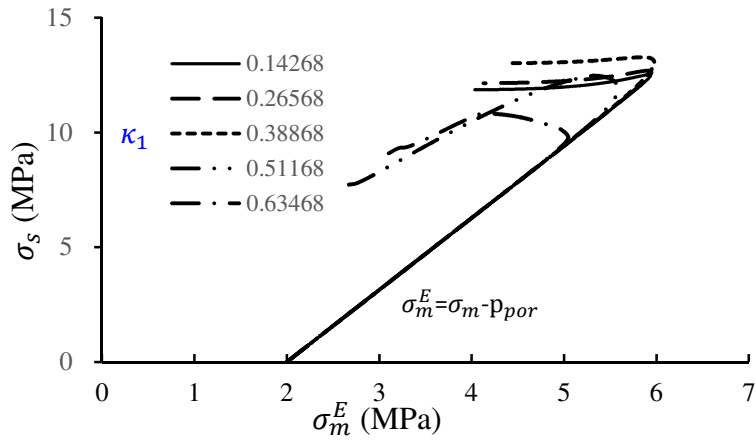
(a) Deviatoric stress-shear strain curves



959

960

(b) Volumetric strain-shear strain curves



961

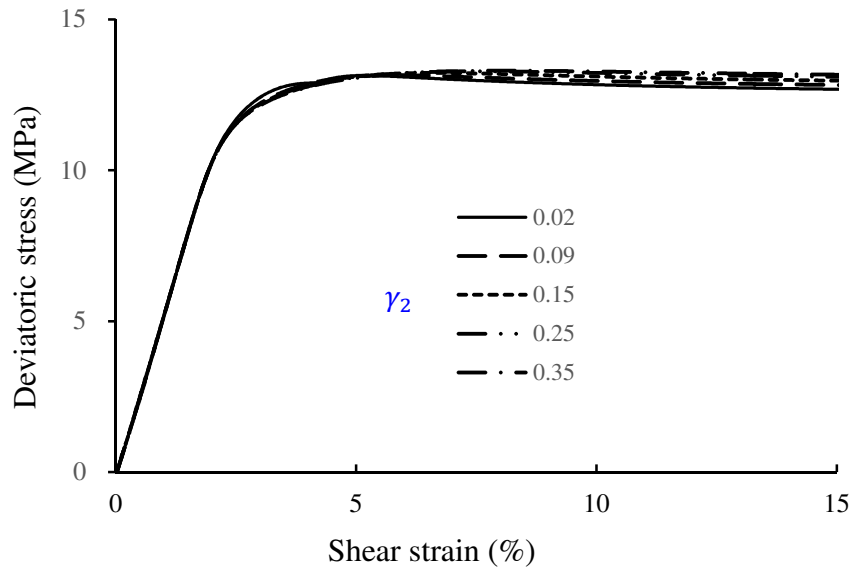
962

(c) Stress path of  $\sigma_m^E - \sigma_s$ 

963

Fig. 10 Simulated results with varying  $\kappa_1$  at 2.0MPa confining pressures

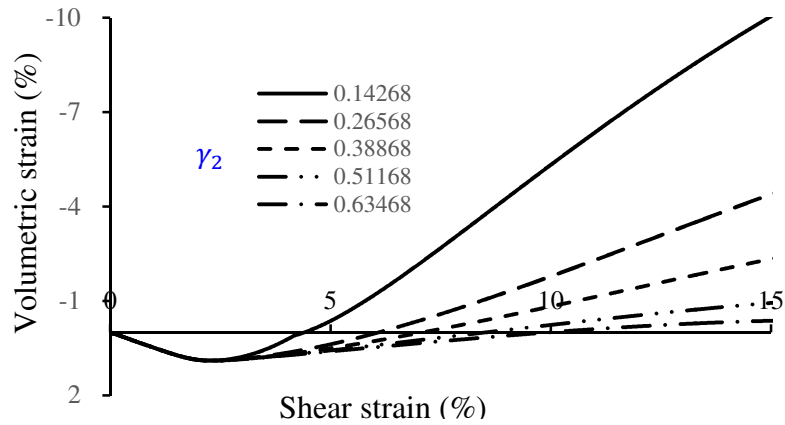
964



965

966

(a) Deviatoric stress-shear strain curves

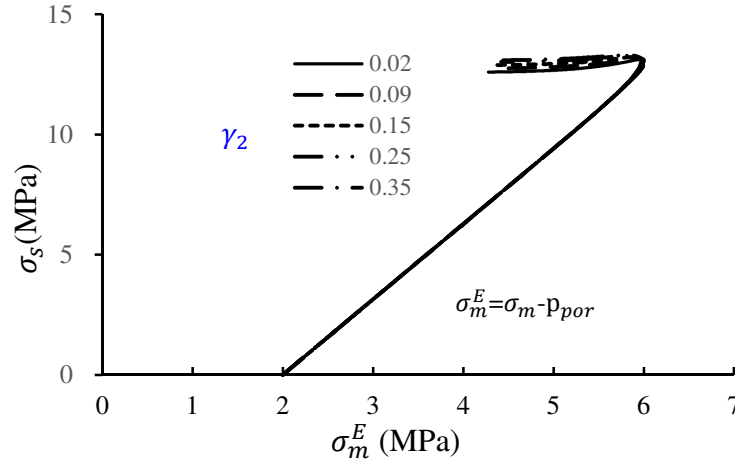


967

968

(b) Volumetric strain-shear strain curves



(c) Stress path of  $\sigma_m^E - \sigma_s$ Fig. 11 Simulated results with varying  $\gamma_2$  at 2.0MPa confining pressures

For geological materials, there are some constitutive models formulated with associated plastic flow rule, such as Cam-clay model for remoulded clay, while some others formulated with non-associated plastic flow rule, such as revised Lade-Duncan model for sand. For frozen soils, we can also use the associated plastic flow rule or non-associated plastic flow rule. In the simulated examples, we use here the non-associated flow rule for frozen soils, with the same form of yield function and plastic potential functions.

For the method proposed here, the stress states of  $f(\sigma_{ij}, \bar{p}_c, \bar{p}_w, \xi_J)$  satisfies the maximal plastic work principle, which means the yield locus  $f = 0$  (no vertex effects) must be a convex function. Therefore,  $d\varepsilon_{ij}^p$ ,  $d\varphi_c^p$  and  $d\varphi_w^p$  in Equations (151) or (153) in Section 4.3.2 can be obtained following the method proposed by Hill (1950). The incremental directions of  $d\varepsilon_{ij}^p$ ,  $d\varphi_c^p$  and  $d\varphi_w^p$  are determined by  $\partial h / \partial \sigma_{ij}$ ,  $\partial h / \partial \bar{p}_c$  and  $\partial h / \partial \bar{p}_w$ , respectively, which means a non-associated plastic flow rule is used. When the associated or non-associated plastic flow rule is used, the plastic dissipation inequality (158) must be satisfied. For frozen soils tested, we can select a yield function  $f$  and a plastic potential function  $h$  and a free energy function, and then determine the corresponding parameters when deriving the incremental generalized stress strain relationships. We supplement a constitutive model with a non-associated plastic rule to simulate the relationships of deviatoric stress-shear strain and volumetric strain-shear strain under cryogenic triaxial loading conditions in Section 4.4.3. When formulating the elasto-plastic constitutive model, we have to make some assumptions. For different soil samples, there are different plastic criteria assumed by different researchers within the corresponding theoretical framework (Khan et al., 1991; Muraleetharan et al.,

992 2009; Kamrin, 2010; Zhang et al., 2012; Darabi et al., 2012; Yao et al., 2009, 2015; Xu et al., 2017).  
 993 Although the constitutive model proposed here can present the simulated results agreeing relatively  
 994 well with the test results, as shown in Figs. 2 and Fig.3 for frozen soils under triaxial compression  
 995 conditions, further studies have to be done in the future to give less model parameters or easily  
 996 methods to determine these parameters.

## 997 5. Conclusions

998 On the basis of thermoporoelasticity, an elastoplastic model for saturated frozen soils are  
 999 formulated in the paper. The saturated frozen soils are described as an open thermodynamics  
 1000 continuum. When phase change occurs, the equations of conservation of mass, momentum, and energy,  
 1001 and the Clausius-Duhem inequality are presented in both Lagrangian and Eulerian formulations. By  
 1002 considering the solid-fluid interface energy and the lagrangian saturation, the constitutive relationships  
 1003 in both elastic and ideal plasticity are given, and for hardening plasticity, the general form of  
 1004 stress-strain relationship is also presented and a hardening elastic-plastic constitutive is also proposed  
 1005 to simulate the cryogenic triaxial compression properties of saturated frozen soils. The theory  
 1006 framework put forward here is an investigation on the extending of poroelasticity to poroelastoplasticity  
 1007 considering phase change between liquid water and ice crystals during the freezing process, which can  
 1008 be applied to study the frost heave of freezing ground in further numerical study.

### 1009 Appendix I: Deriving the local equation of motion in Eulerian formulation.

1010 From Equation (33) and Equation (34), we have

$$1011 \frac{d^s}{dt} \int_{\Omega} \rho_s (1-n) \mathbf{V}^s d\Omega + \sum_{\alpha=c,w} \frac{d^\alpha}{dt} \int_{\Omega} \rho_\alpha n_\alpha \mathbf{V}^\alpha d\Omega = \int_{\Omega} (\rho_s (1-n) \boldsymbol{\gamma}^s + \sum_{\alpha=c,w} \rho_\alpha n_\alpha \boldsymbol{\gamma}^\alpha) d\Omega +$$

$$1012 \int_{\Omega} \Lambda_{c \rightarrow w} (\mathbf{V}^w - \mathbf{V}^c) d\Omega = \int_{\Omega} \rho \mathbf{g}(\mathbf{x}, t) d\Omega + \int_{\partial\Omega} \mathbf{T}(\mathbf{x}, t, \mathbf{n}) da. \quad (\text{I-1})$$

1013 Considering that  $\mathbf{T}(\mathbf{x}, t, \mathbf{n}) = \boldsymbol{\sigma} \cdot \mathbf{n}$  and  $\rho = \rho_s (1-n) + \sum_{\alpha=c,w} \rho_\alpha n_\alpha$ , so we have

$$1014 \int_{\Omega} \rho \mathbf{g}(\mathbf{x}, t) d\Omega + \int_{\partial\Omega} \mathbf{T}(\mathbf{x}, t, \mathbf{n}) da = \int_{\Omega} \rho \mathbf{g}(\mathbf{x}, t) d\Omega + \int_{\partial\Omega} \boldsymbol{\sigma} \cdot \mathbf{n} da = \int_{\Omega} (\rho_s (1-n) + \sum_{\alpha=c,w} \rho_\alpha n_\alpha) \mathbf{g} d\Omega +$$

$$1015 \int_{\Omega} \nabla \cdot \boldsymbol{\sigma} d\Omega. \quad (\text{I-2})$$

1016 Combining (I-1) and (I-2), we can have the following,

$$1017 \int_{\Omega} (\rho_s (1-n) \boldsymbol{\gamma}^s + \sum_{\alpha=c,w} \rho_\alpha n_\alpha \boldsymbol{\gamma}^\alpha) d\Omega + \int_{\Omega} \Lambda_{c \rightarrow w} (\mathbf{V}^w - \mathbf{V}^c) d\Omega$$

$$1018 = \int_{\Omega} (\rho_s (1-n) + \sum_{\alpha=c,w} \rho_\alpha n_\alpha) \mathbf{g} d\Omega + \int_{\Omega} \nabla_x \cdot \boldsymbol{\sigma} d\Omega \quad (\text{I-3})$$

1019 Therefore, the local equation (35) can be obtained as follows,

$$1020 \quad \nabla_x \cdot \boldsymbol{\sigma} + \rho_s(1-n)(\mathbf{g} - \boldsymbol{\gamma}^s) + \sum_{\alpha=c,w} \rho_\alpha n_\alpha (\mathbf{g} - \boldsymbol{\gamma}^\alpha) - \Lambda_{c \rightarrow w}(\mathbf{V}^w - \mathbf{V}^c) = 0. \quad (\text{I-4}) \text{ or } (35)$$

1021 **Appendix II: Deriving kinetic energy theorem.**

1022 From  $\mathbf{w}_\alpha = \rho_\alpha n_\alpha (\mathbf{V}^\alpha - \mathbf{V}^s)$ , we can have

$$1023 \quad \mathbf{V}^\alpha = \frac{\mathbf{w}_\alpha}{\rho_\alpha n_\alpha} + \mathbf{V}^s. \quad (\text{II-1})$$

1024 Use of Equation (II-1), the following can be obtained,

$$1025 \quad \rho_s(1-n)\mathbf{g} \cdot \mathbf{V}^s + \mathbf{g} \cdot \sum_{\alpha=c,w} \rho_\alpha n_\alpha \mathbf{V}^\alpha = \rho_s(1-n)\mathbf{g} \cdot \mathbf{V}^s + \mathbf{g} \cdot \sum_{\alpha=c,w} \rho_\alpha n_\alpha \left( \frac{\mathbf{w}_\alpha}{\rho_\alpha n_\alpha} + \mathbf{V}^s \right) = \rho_s(1-n) \\ 1026 \quad n)\mathbf{g} \cdot \mathbf{V}^s + \mathbf{g} \cdot \sum_{\alpha=c,w} \rho_\alpha n_\alpha \mathbf{V}^s + \mathbf{g} \cdot \sum_{\alpha=c,w} \rho_\alpha n_\alpha \mathbf{V}^s \quad (\text{II-2})$$

1027 The total apparent mass density  $\rho = \rho_s(1-n) + \sum_{\alpha=c,w} \rho_\alpha n_\alpha$ , therefore, (II-2) can be rewritten as  
1028 follows,

$$1029 \quad \rho_s(1-n)\mathbf{g} \cdot \mathbf{V}^s + \mathbf{g} \cdot \sum_{\alpha=c,w} \rho_\alpha n_\alpha \mathbf{V}^\alpha = \rho \mathbf{g} \cdot \mathbf{V}^s + \mathbf{g} \cdot \sum_{\alpha=c,w} \mathbf{w}_\alpha. \quad (\text{II-3})$$

$$1030 \quad \mathbf{T}^s \cdot \mathbf{V}^s + \sum_{\alpha=c,w} (\mathbf{T}^\alpha \cdot \mathbf{V}^\alpha) = \mathbf{T}^s \cdot \mathbf{V}^s + \sum_{\alpha=c,w} \left( \mathbf{T}^\alpha \cdot \left( \frac{\mathbf{w}_\alpha}{\rho_\alpha n_\alpha} + \mathbf{V}^s \right) \right) = \mathbf{T}^s \cdot \mathbf{V}^s + \sum_{\alpha=c,w} \mathbf{T}^\alpha \cdot \mathbf{V}^s +$$

$$1031 \quad \sum_{\alpha=c,w} \mathbf{T}^\alpha \cdot \frac{\mathbf{w}_\alpha}{\rho_\alpha n_\alpha} \quad (\text{II-4})$$

1032 Applying the momentum balance separately to the soil skeleton and to the fluids (the ice crystals and  
1033 unfrozen water), we can obtain the separate existence of a partial volumetric stress  $\boldsymbol{\sigma}^s$  related to the  
1034 soil skeleton and a partial volumetric stress tensor  $\boldsymbol{\sigma}^\alpha (\alpha = c, w)$  related to the fluids, such as:

$$1035 \quad \mathbf{T}^s(\mathbf{x}, t, \mathbf{n}) = (1-n)\boldsymbol{\sigma}^s \cdot \mathbf{n}; \text{ and } \mathbf{T}^\alpha(\mathbf{x}, t, \mathbf{n}) = n_\alpha \boldsymbol{\sigma}^\alpha \cdot \mathbf{n} \quad (\alpha = c, w), \quad (\text{II-5})$$

1036 in which  $n$ ,  $n_\alpha (\alpha = c, w)$  are the porosities,  $\mathbf{n}$  is the normal vector of the surface  $da$ . The intrinsic  
1037 averaged stress within the fluids can be addressed through a spherical tensor as follows,

$$1038 \quad \boldsymbol{\sigma}^\alpha = -p_\alpha \mathbf{1}. \quad (\text{II-6})$$

1039 Substituting (II-5) and (II-6) into (II-4), we can have

$$1040 \quad \mathbf{T}^s \cdot \mathbf{V}^s + \sum_{\alpha=c,w} (\mathbf{T}^\alpha \cdot \mathbf{V}^\alpha) = \mathbf{T}^s \cdot \mathbf{V}^s + \sum_{\alpha=c,w} \mathbf{T}^\alpha \cdot \mathbf{V}^s + \sum_{\alpha=c,w} (-n_\alpha p_\alpha) \mathbf{n} \cdot \frac{\mathbf{w}_\alpha}{\rho_\alpha n_\alpha} = \mathbf{T} \cdot \mathbf{V}^s -$$

$$1041 \quad \sum_{\alpha=c,w} \frac{\mathbf{w}_\alpha}{\rho_\alpha} \cdot \mathbf{n} \quad (\text{II-7})$$

1042 Therefore, substituting Equations (II-4) and (II-7) into Equation (40), we can have

$$1043 \quad P_{f,T}(\mathbf{V}^s, \mathbf{V}^\alpha) = \int_\Omega (\rho \mathbf{g} \cdot \mathbf{V}^s + \mathbf{g} \cdot \sum_{\alpha=c,w} \mathbf{w}_\alpha) d\Omega + \int_{\partial\Omega} \left( \mathbf{T} \cdot \mathbf{V}^s - \sum_{\alpha=c,w} \left( \frac{p_\alpha}{\rho_\alpha} \mathbf{w}_\alpha \right) \cdot \mathbf{n} \right) da. (\text{II-8}) \text{ or } (41)$$

1044 From the kinetic energy of soil skeleton  $K_s$  in Equation (42), we can obtain its particle derivative as  
1045 follows,

$$\begin{aligned}
1046 \quad \frac{d^s K_s}{dt} &= \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho_s (1-n) (\mathbf{V}^s)^2 d\Omega = \frac{1}{2} \int_{\Omega} \frac{d}{dt} (\rho_s (1-n) (\mathbf{V}^s)^2) d\Omega = \frac{1}{2} \int_{\Omega} \left[ (\mathbf{V}^s)^2 \frac{d}{dt} (\rho_s (1-n)) d\Omega + \right. \\
1047 \quad & \left. 2\rho_s (1-n) d\Omega \frac{d\mathbf{V}^s}{dt} \cdot \mathbf{V}^s \right] = \int_{\Omega} [\rho_s (1-n) \boldsymbol{\gamma}^s \cdot \mathbf{V}^s] d\Omega \quad , \text{(II-9)}
\end{aligned}$$

1048 in which the mass balance of soil skeleton is used.

1049 In a similar way, from the kinetic energy of ice crystals and unfrozen water of saturated frozen soils,  
1050 we can also obtain the particle derivative as follows,

$$\begin{aligned}
1051 \quad \frac{d^\alpha}{dt} \sum_{\alpha=c,w} K_\alpha &= \sum_{\alpha=c,w} \frac{d^\alpha}{dt} \left\{ \frac{1}{2} \int_{\Omega} \rho_\alpha n_\alpha (\mathbf{V}^\alpha)^2 d\Omega \right\} = \sum_{\alpha=c,w} \frac{1}{2} \left\{ \frac{d^\alpha}{dt} \left( \int_{\Omega} \rho_\alpha n_\alpha (\mathbf{V}^\alpha)^2 d\Omega \right) \right\} = \\
1052 \quad & \sum_{\alpha=c,w} \frac{1}{2} \int_{\Omega} \left\{ \left[ (\mathbf{V}^\alpha)^2 \frac{d}{dt} (\rho_\alpha n_\alpha) d\Omega \right] + 2\rho_\alpha n_\alpha d\Omega \frac{d\mathbf{V}^\alpha}{dt} \cdot \mathbf{V}^\alpha \right\} \quad . \text{(II-10)}
\end{aligned}$$

1053 Combining Equations (28) and (II-1), we can rewrite the above equation (II-10) as follows,

$$\begin{aligned}
1054 \quad \frac{d^\alpha}{dt} \sum_{\alpha=c,w} K_\alpha &= \\
1055 \quad & \int_{\Omega} \frac{1}{2} \Lambda_{c \rightarrow w} ((\mathbf{V}^w)^2 - (\mathbf{V}^c)^2) d\Omega + \sum_{\alpha=c,w} \int_{\Omega} \left( \rho_\alpha n_\alpha \boldsymbol{\gamma}^\alpha \cdot \left( \frac{\mathbf{w}_\alpha}{\rho_\alpha n_\alpha} + \mathbf{V}^s \right) \right) d\Omega = \int_{\Omega} \frac{1}{2} \Lambda_{c \rightarrow w} ((\mathbf{V}^w)^2 - \\
1056 \quad & (\mathbf{V}^c)^2) d\Omega + \sum_{\alpha=c,w} \int_{\Omega} (\rho_\alpha n_\alpha \boldsymbol{\gamma}^\alpha \cdot \mathbf{V}^s + \boldsymbol{\gamma}^\alpha \cdot \mathbf{w}_\alpha) d\Omega. \quad \text{(II-11)}
\end{aligned}$$

1057 Combining (II-9) and (II-10), we can obtain the particle derivative of the kinetic energy of saturated  
1058 frozen soils as follows,

$$\begin{aligned}
1059 \quad \frac{d^s K_s}{dt} + \sum_{\alpha=c,w} \frac{d^\alpha K_\alpha}{dt} &= \int_{\Omega} (\rho_s (1-n) \boldsymbol{\gamma}^s + \sum_{\alpha=c,w} \rho_\alpha n_\alpha \boldsymbol{\gamma}^\alpha) \cdot \mathbf{V}^s d\Omega + \int_{\Omega} \sum_{\alpha=c,w} \boldsymbol{\gamma}^\alpha \cdot \mathbf{w}^\alpha d\Omega + \\
1060 \quad & \int_{\Omega} \frac{1}{2} \Lambda_{c \rightarrow w} ((\mathbf{V}^w)^2 - (\mathbf{V}^c)^2) d\Omega \quad . \text{(II-12) or (43)}
\end{aligned}$$

1061 From Equation (16), the Eulerian strain rate tensor  $\mathbf{d}^\pi$  is defined as follows,

$$1062 \quad \mathbf{d}^\pi = \frac{1}{2} (\nabla_x \mathbf{V}^\pi + {}^t \nabla_x \mathbf{V}^\pi) \quad (\pi = s, c, w) \quad . \quad \text{(II-13)}$$

1063 From Equations (41) and (43), we can have

$$\begin{aligned}
1064 \quad P_{f,T}(\mathbf{V}^s, \mathbf{V}^\alpha) - \frac{d^s K_s}{dt} - \sum_{\alpha=c,w} \frac{d^\alpha K_\alpha}{dt} &= \int_{\Omega} (\rho \mathbf{g} \cdot \mathbf{V}^s + \mathbf{g} \cdot \sum_{\alpha=c,w} \rho_\alpha n_\alpha \mathbf{w}^\alpha) d\Omega + \int_{\partial\Omega} (\mathbf{T} \cdot \mathbf{V}^s - \sum_{\alpha=c,w} \left( \frac{p_\alpha}{\rho_\alpha} \mathbf{w}^\alpha \right) \cdot \\
1065 \quad & \mathbf{n}) da - \int_{\Omega} (\rho_s (1-n) \boldsymbol{\gamma}^s + \sum_{\alpha=c,w} \rho_\alpha n_\alpha \boldsymbol{\gamma}^\alpha) \cdot \mathbf{V}^s d\Omega - \int_{\Omega} \sum_{\alpha=c,w} \boldsymbol{\gamma}^\alpha \cdot \mathbf{w}^\alpha d\Omega - \int_{\Omega} \frac{1}{2} \Lambda_{c \rightarrow w} ((\mathbf{V}^w)^2 - \\
1066 \quad & (\mathbf{V}^c)^2) d\Omega \quad . \text{(II-14)}
\end{aligned}$$

1067 Using the divergence theorem, we have

$$\begin{aligned}
1068 \quad \int_{\partial\Omega} (\mathbf{T} \cdot \mathbf{V}^s - \sum_{\alpha=c,w} \left( \frac{p_\alpha}{\rho_\alpha} \mathbf{w}^\alpha \right) \cdot \mathbf{n}) da &= \int_{\partial\Omega} (\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{V}^s - \sum_{\alpha=c,w} \left( \frac{p_\alpha}{\rho_\alpha} \mathbf{w}^\alpha \right) \cdot \mathbf{n}) da = \int_{\Omega} (\boldsymbol{\sigma} : \mathbf{d}^s + \mathbf{V}^s \cdot \\
1069 \quad & (\nabla_x \cdot \boldsymbol{\sigma}) - \sum_{\alpha=c,w} \nabla_x \cdot \left( \frac{p_\alpha}{\rho_\alpha} \mathbf{w}^\alpha \right)) d\Omega \quad . \text{(II-15)}
\end{aligned}$$

1070 Substituting (II-15) into (II-14), we can obtain the following expression,

$$\begin{aligned}
1071 \quad & P_{f,T}(\mathbf{V}^s, \mathbf{V}^\alpha) - \frac{d^s K_s}{dt} - \sum_{\alpha=c,w} \frac{d^\alpha K_\alpha}{dt} = \int_\Omega (\rho \mathbf{g} \cdot \mathbf{V}^s + \mathbf{g} \cdot \sum_{\alpha=c,w} \rho_\alpha \mathbf{w}^\alpha) d\Omega + \int_\Omega (\boldsymbol{\sigma} : \mathbf{d}^s + \mathbf{V}^s \cdot (\nabla_x \cdot \boldsymbol{\sigma}) - \\
1072 \quad & \sum_{\alpha=c,w} \nabla_x \cdot \left( \frac{p_\alpha}{\rho_\alpha} \mathbf{w}^\alpha \right)) d\Omega - \int_\Omega (\rho_s (1-n) \boldsymbol{\gamma}^s + \sum_{\alpha=c,w} \rho_\alpha n_\alpha \boldsymbol{\gamma}^\alpha) \cdot \mathbf{V}^s d\Omega - \int_\Omega \sum_{\alpha=c,w} \boldsymbol{\gamma}^\alpha \cdot \mathbf{w}^\alpha d\Omega - \\
1073 \quad & \int_\Omega \frac{1}{2} \Lambda_{c \rightarrow w} ((\mathbf{V}^w)^2 - (\mathbf{V}^c)^2) d\Omega \quad .(II-16)
\end{aligned}$$

1074 Considering that

$$\begin{aligned}
1075 \quad & \int_\Omega (\rho \mathbf{g} \cdot \mathbf{V}^s) d\Omega + \int_\Omega (\mathbf{V}^s \cdot (\nabla_x \cdot \boldsymbol{\sigma})) d\Omega - \int_\Omega (\rho_s (1-n) \boldsymbol{\gamma}^s + \sum_{\alpha=c,w} \rho_\alpha n_\alpha \boldsymbol{\gamma}^\alpha) \cdot \mathbf{V}^s d\Omega = \int_\Omega [(\rho_s (1- \\
1076 \quad & n) + \sum_{\alpha=c,w} \rho_\alpha n_\alpha) (\mathbf{g} \cdot \mathbf{V}^s)] d\Omega + \int_\Omega (\mathbf{V}^s \cdot (\nabla_x \cdot \boldsymbol{\sigma})) d\Omega - \int_\Omega (\rho_s (1-n) \boldsymbol{\gamma}^s + \sum_{\alpha=c,w} \rho_\alpha n_\alpha \boldsymbol{\gamma}^\alpha) \cdot \\
1077 \quad & \mathbf{V}^s d\Omega = \int_\Omega ((\nabla_x \cdot \boldsymbol{\sigma}) \cdot \mathbf{V}^s) d\Omega + \int_\Omega [(\rho_s (1-n) (\mathbf{g} - \boldsymbol{\gamma}^s)) \cdot \mathbf{V}^s] d\Omega + \int_\Omega (\sum_{\alpha=c,w} \rho_\alpha n_\alpha (\mathbf{g} - \boldsymbol{\gamma}^s)) \cdot \\
1078 \quad & \mathbf{V}^s d\Omega = \int_\Omega \Lambda_{c \rightarrow w} (\mathbf{V}^w - \mathbf{V}^c) \cdot \mathbf{V}^s d\Omega \quad . (II-17)
\end{aligned}$$

1079 Expression (II-16) can be rewritten as follows,

$$\begin{aligned}
1080 \quad & P_{f,T}(\mathbf{V}^s, \mathbf{V}^\alpha) - \frac{d^s K_s}{dt} - \sum_{\alpha=c,w} \frac{d^\alpha K_\alpha}{dt} = \int_\Omega \Lambda_{c \rightarrow w} (\mathbf{V}^w - \mathbf{V}^c) \cdot \mathbf{V}^s d\Omega - \int_\Omega \frac{1}{2} \Lambda_{c \rightarrow w} ((\mathbf{V}^w)^2 - (\mathbf{V}^c)^2) d\Omega + \\
1081 \quad & \int_\Omega (\boldsymbol{\sigma} : \mathbf{d}^s) d\Omega - \int_\Omega \sum_{\alpha=c,w} \nabla_x \cdot \left( \frac{p_\alpha}{\rho_\alpha} \mathbf{w}^\alpha \right) d\Omega + \int_\Omega \sum_{\alpha=c,w} (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{w}^\alpha d\Omega = \int_\Omega (\boldsymbol{\sigma} : \mathbf{d}^s) d\Omega - \\
1082 \quad & \int_\Omega \sum_{\alpha=c,w} \left[ \nabla_x \cdot \left( \frac{p_\alpha}{\rho_\alpha} \mathbf{w}^\alpha \right) - (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{w}^\alpha \right] d\Omega - \int_\Omega \frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] d\Omega \quad .(II-18)
\end{aligned}$$

1083 Therefore, the kinetic energy theorem in Eulerian form can be obtained as follows,

$$1084 \quad P_{f,T}(\mathbf{V}^s, \mathbf{V}^\alpha) = P_{\text{def}}(\mathbf{V}^s, \mathbf{V}^\alpha) + \frac{d^s K_s}{dt} + \sum_{\alpha=c,w} \frac{d^\alpha K_\alpha}{dt} - \int_\Omega \frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] d\Omega,$$

$$1085 \quad .(II-19) \text{ or } (44)$$

$$1086 \quad \text{and } P_{\text{def}}(\mathbf{V}^s, \mathbf{V}^\alpha) = \int_\Omega \boldsymbol{\sigma} : \mathbf{d}^s d\Omega - \int_\Omega \sum_{\alpha=c,w} \left[ \nabla_x \cdot \left( \frac{p_\alpha}{\rho_\alpha} \mathbf{w}^\alpha \right) - (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{w}^\alpha \right] d\Omega. \quad (II-20) \text{ or } (45)$$

### 1087 Appendix III: Deriving the energy equation and Clausius-Duhem inequality.

1088 From Equation (54), we have

$$1089 \quad \frac{d^s}{dt} \int_\Omega \rho_s (1-n) e_s d\Omega + \sum_{\alpha=c,w} \frac{d^\alpha}{dt} \int_\Omega \rho_\alpha n_\alpha e_\alpha d\Omega = \int_\Omega \left[ \frac{d^s (\rho_s (1-n) e_s)}{dt} + \rho_s (1-n) e_s \nabla_x \cdot \mathbf{V}^s \right] d\Omega +$$

$$1090 \quad \sum_{\alpha=c,w} \int_\Omega \left[ \frac{d^\alpha (\rho_\alpha n_\alpha e_\alpha)}{dt} + \rho_\alpha n_\alpha e_\alpha \nabla_x \cdot \mathbf{V}^\alpha \right] d\Omega \quad .(III-1)$$

1091 Considering that the particulate derivative of a field respect to the soil skeleton as follows (Coussy,  
1092 1995),

$$1093 \quad \frac{d^\alpha (\rho_\alpha n_\alpha e_\alpha)}{dt} = \frac{d^s (\rho_\alpha n_\alpha e_\alpha)}{dt} + \nabla_x (\rho_\alpha n_\alpha e_\alpha) \cdot (\mathbf{V}^\alpha - \mathbf{V}^s), \quad (III-2)$$

1094 and Equation (II-1), Expression (III-1) can be rewritten as follows,



1116 Substituting  $d\psi = de - Td\theta - \theta dT$  into the above inequality (III-10), we can have

$$1117 \quad \boldsymbol{\sigma} : \mathbf{d}^s + \sum_{\alpha=c,w} (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{w}^\alpha + T \frac{d^s \theta}{dt} - \left( \frac{d\psi}{dt} + T \frac{d\theta}{dt} + \theta \frac{dT}{dt} \right) - \psi \nabla_x \cdot \mathbf{V}^s + T \sum_{\alpha=c,w} \nabla_x \cdot (\theta_\alpha \mathbf{w}^\alpha) -$$

$$1118 \quad \sum_{\alpha=c,w} \nabla_x \cdot (h_\alpha \mathbf{w}^\alpha) - \frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] - \frac{q}{T} \cdot \nabla_x T \geq 0 \quad ,(\text{III-11})$$

1119 Considering that the fluid-specific free enthalpy  $g_\alpha = h_\alpha - T\theta_\alpha$  ( $\alpha = c, w$ ), and

1120  $\nabla_x \cdot (\theta_\alpha \mathbf{w}^\alpha) = \mathbf{w}^\alpha \cdot \nabla_x \theta_\alpha + \theta_\alpha \nabla_x \cdot \mathbf{w}^\alpha$ , we can have

$$1121 \quad T \sum_{\alpha=c,w} \nabla_x \cdot (\theta_\alpha \mathbf{w}^\alpha) - \sum_{\alpha=c,w} \nabla_x \cdot (h_\alpha \mathbf{w}^\alpha) = -\mathbf{w}^\alpha \cdot \sum_{\alpha=c,w} (\nabla_x g_\alpha + \theta_\alpha \nabla_x T) - \sum_{\alpha=c,w} (g_\alpha \nabla_x \cdot \mathbf{w}^\alpha). (\text{III-12})$$

1122 Substituting (III-12) into (III-11), we obtain so-called Clausius-Duhem inequality in Eulerian form as  
1123 follows,

$$1124 \quad \boldsymbol{\sigma} : \mathbf{d}^s - \sum_{\alpha=c,w} (g_\alpha \nabla_x \cdot \mathbf{w}^\alpha) - \theta \frac{dT}{dt} - \frac{d\psi}{dt} - \psi \nabla_x \cdot \mathbf{V}^s - \mathbf{w}^\alpha \cdot \sum_{\alpha=c,w} (\nabla_x g_\alpha + \theta_\alpha \nabla_x T - (\mathbf{g} - \boldsymbol{\gamma}^\alpha)) -$$

$$1125 \quad \frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] - \frac{q}{T} \cdot \nabla_x T \geq 0 \quad .(\text{III-13}) \text{ or } (69)$$

1126 The time rate of the entropy associated with the matter contained in  $\Omega_0$  of saturated frozen soils can  
1127 be expressed as follows,

$$1128 \quad \frac{D\Theta}{Dt} = \frac{D}{Dt} \int_{\Omega_0} \Theta d\Omega_0 = \int_{\Omega_0} \frac{d\Theta}{dt} d\Omega_0 + \sum_{\alpha=c,w} \int_{\Omega_0} \nabla_x \cdot (\Theta_\alpha \mathbf{M}_\alpha) d\Omega_0. \quad (\text{III-14})$$

1129 Therefore, the second law of thermodynamics in Lagrangian formulation can be expressed as follows,

$$1130 \quad \int_{\Omega_0} \frac{d\Theta}{dt} d\Omega_0 + \sum_{\alpha=c,w} \int_{\Omega_0} \nabla_x \cdot (\Theta_\alpha \mathbf{M}_\alpha) d\Omega_0 \geq - \int_{\partial\Omega_0} \frac{\mathbf{Q} \cdot \mathbf{N}}{T} dA + \int_{\Omega_0} \frac{\mathbf{R}\mathbf{Q}}{T} d\Omega_0. \quad (\text{III-15})$$

1131 By using the theorem of divergence, inequality (III-15) can be written in the Lagrangian formulation as  
1132 follows,

$$1133 \quad \int_{\Omega_0} \frac{d\Theta}{dt} d\Omega_0 + \sum_{\alpha=c,w} \int_{\Omega_0} \nabla_x \cdot (\Theta_\alpha \mathbf{M}_\alpha) d\Omega_0 \geq - \int_{\Omega_0} \nabla_x \cdot \frac{\mathbf{Q}}{T} d\Omega_0 + \int_{\Omega_0} \frac{\mathbf{R}\mathbf{Q}}{T} d\Omega_0. \quad (\text{III-16})$$

1134 Therefore, the following can be obtained,

$$1135 \quad \frac{d\Theta}{dt} + \sum_{\alpha=c,w} \nabla_x \cdot (\Theta_\alpha \mathbf{M}_\alpha) + \nabla_x \cdot \frac{\mathbf{Q}}{T} - \frac{\mathbf{R}\mathbf{Q}}{T} \geq 0. \quad (\text{III-17}) \text{ or } (71)$$

1136 Substituting  $\nabla_x \cdot \frac{\mathbf{Q}}{T} = \frac{1}{T} \nabla_x \cdot \mathbf{Q} - \frac{\mathbf{Q}}{T^2} \cdot \nabla_x T$  into Inequality (71), we can have

$$1137 \quad \frac{d\Theta}{dt} + \sum_{\alpha=c,w} \nabla_x \cdot (\Theta_\alpha \mathbf{M}_\alpha) + \frac{1}{T} \nabla_x \cdot \mathbf{Q} - \frac{\mathbf{Q}}{T^2} \cdot \nabla_x T - \frac{\mathbf{R}\mathbf{Q}}{T} \geq 0, \quad (\text{III-18})$$

1138 which can be rewritten as follows,

$$1139 \quad T \frac{d\Theta}{dt} + T \sum_{\alpha=c,w} \nabla_x \cdot (\Theta_\alpha \mathbf{M}_\alpha) + \nabla_x \cdot \mathbf{Q} - \frac{\mathbf{Q}}{T} \cdot \nabla_x T - \mathbf{R}\mathbf{Q} \geq 0. \quad (\text{III-19})$$

1140 Combining Equations (63) and (III-19), we have

$$1141 \quad T \frac{d\Theta}{dt} + T \sum_{\alpha=c,w} \nabla_x \cdot (\Theta_\alpha \mathbf{M}_\alpha) - \frac{dE}{dt} + \boldsymbol{\pi} : \frac{d\Delta}{dt} - \sum_{\alpha=c,w} \nabla_x \cdot h_\alpha \mathbf{M}_\alpha - \nabla_x \cdot \mathbf{Q} + \sum_{\alpha=c,w} ((\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{F}) \cdot$$

$$1142 \quad \mathbf{M}_\alpha - \frac{1}{2} m_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] - \frac{Q}{T} \cdot \nabla_X T \geq 0. \quad (\text{III-20})$$

1143 Considering that  $h_\alpha = g_\alpha + T\theta_\alpha$  ( $\alpha = c, w$ ), we have

$$1144 \quad T \sum_{\alpha=c,w} \nabla_X \cdot (\theta_\alpha \mathbf{M}_\alpha) - \sum_{\alpha=c,w} \nabla_X \cdot h_\alpha \mathbf{M}_\alpha = T \sum_{\alpha=c,w} (\mathbf{M}_\alpha \cdot \nabla_X \theta_\alpha) + T \sum_{\alpha=c,w} \theta_\alpha \nabla_X \cdot \mathbf{M}_\alpha -$$

$$1145 \quad \sum_{\alpha=c,w} \nabla_X \cdot (g_\alpha + T\theta_\alpha) \mathbf{M}_\alpha = -\mathbf{M}_\alpha \cdot \sum_{\alpha=c,w} (\nabla_X g_\alpha + \theta_\alpha \nabla_X T) - \sum_{\alpha=c,w} g_\alpha \nabla_X \cdot \mathbf{M}_\alpha. \quad (\text{III-21})$$

1146 Substituting Equation (III-21) into Equation (III-20), we can obtain

$$1147 \quad \boldsymbol{\pi}: \frac{d\Delta}{dt} + \sum_{\alpha=c,w} ((\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{F}) \cdot \mathbf{M}_\alpha + T \frac{d\theta}{dt} - \frac{dE}{dt} - \mathbf{M}_\alpha \cdot \sum_{\alpha=c,w} (\nabla_X g_\alpha + \theta_\alpha \nabla_X T) - \sum_{\alpha=c,w} g_\alpha \nabla_X \cdot$$

$$1148 \quad \mathbf{M}_\alpha - \frac{1}{2} m_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] - \frac{Q}{T} \cdot \nabla_X T \geq 0. \quad (\text{III-22})$$

1149 Substituting  $d\Psi = dE - Td\theta - \theta dT$  into (III-22), and with the mass conservation (31) and (32), the

1150 Lagrangian formulation of the Clausius-Duhem inequality (III-22) can be written as follows,

$$1151 \quad \boldsymbol{\pi}: \frac{d\Delta}{dt} + \sum_\alpha g_\alpha \frac{dm_\alpha}{dt} - \theta \frac{dT}{dt} - \frac{d\Psi}{dt} - (g_c - g_m) \bar{\omega}_{c \rightarrow w} - \frac{1}{2} \bar{\omega}_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] -$$

$$1152 \quad \sum_\alpha (\nabla_X g_\alpha + \theta_\alpha \nabla_X T - (\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{F}) \cdot \mathbf{M}_\alpha - \frac{Q}{T} \cdot \nabla_X T \geq 0 \quad .(\text{III-23}) \text{ or } (73)$$

#### 1153 Appendix IV: Deriving the thermal balance equation.

1154 Combining Equations (63) and (79), we can have

$$1155 \quad \frac{dE}{dt} = \frac{dw}{dt} + T \frac{d\theta}{dt} + \theta \frac{dT}{dt} = \boldsymbol{\pi}: \frac{d\Delta}{dt} - \sum_\alpha \nabla_X \cdot (h_\alpha \mathbf{M}_\alpha) - \nabla_X \cdot \mathbf{Q} + \sum_\alpha ((\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{F}) \cdot \mathbf{M}_\alpha + \mathbf{R} -$$

$$1156 \quad \frac{1}{2} \bar{\omega}_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] \quad .(\text{IV-1})$$

1157 Considering that

$$1158 \quad -\sum_\alpha \nabla_X \cdot (h_\alpha \mathbf{M}_\alpha) = -\sum_\alpha \nabla_X \cdot (g_\alpha + T\theta_\alpha) \mathbf{M}_\alpha = -\sum_\alpha \mathbf{M}_\alpha \cdot \nabla_X g_\alpha - \sum_\alpha g_\alpha \nabla_X \cdot \mathbf{M}_\alpha - \sum_\alpha \mathbf{M}_\alpha \cdot$$

$$1159 \quad \nabla_X (T\theta_\alpha) - \sum_\alpha T\theta_\alpha \nabla_X \cdot \mathbf{M}_\alpha \quad .(\text{IV-2})$$

1160 Substituting Equation (IV-2) into (IV-1), we can obtain

$$1161 \quad \frac{dw}{dt} + T \frac{d\theta}{dt} + \theta \frac{dT}{dt} = \boldsymbol{\pi}: \frac{d\Delta}{dt} - \sum_\alpha \mathbf{M}_\alpha \cdot \nabla_X g_\alpha - \sum_\alpha g_\alpha \nabla_X \cdot \mathbf{M}_\alpha - \sum_\alpha \mathbf{M}_\alpha \cdot \nabla_X (T\theta_\alpha) - \sum_\alpha T\theta_\alpha \nabla_X \cdot \mathbf{M}_\alpha -$$

$$1162 \quad \nabla_X \cdot \mathbf{Q} + \sum_\alpha ((\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{F}) \cdot \mathbf{M}_\alpha + R_Q - \frac{1}{2} \bar{\omega}_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] \quad .(\text{IV-3})$$

1163 which can be written as follows,

$$1164 \quad \frac{dw}{dt} + T \frac{d\theta}{dt} + \theta \frac{dT}{dt} = \boldsymbol{\pi}: \frac{d\Delta}{dt} - \sum_\alpha \mathbf{M}_\alpha \cdot \nabla_X g_\alpha - \sum_\alpha g_\alpha \nabla_X \cdot \mathbf{M}_\alpha - \sum_\alpha \mathbf{M}_\alpha \cdot T \nabla_X (\theta_\alpha) - \sum_\alpha \theta_\alpha \mathbf{M}_\alpha \cdot \nabla_X T -$$

$$1165 \quad T \sum_\alpha \theta_\alpha \nabla_X \cdot \mathbf{M}_\alpha - \nabla_X \cdot \mathbf{Q} + \sum_\alpha ((\mathbf{g} - \boldsymbol{\gamma}^\alpha) \cdot \mathbf{F}) \cdot \mathbf{M}_\alpha + \mathbf{R} - \frac{1}{2} \bar{\omega}_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] \quad .(\text{IV-4})$$

1166 Considering that

$$1167 \quad T \frac{d\theta}{dt} + \sum_\alpha \mathbf{M}_\alpha \cdot T \nabla_X (\theta_\alpha) + T \sum_\alpha \theta_\alpha \nabla_X \cdot \mathbf{M}_\alpha = T \left[ \frac{d\theta}{dt} + \sum_\alpha (\mathbf{M}_\alpha \cdot \nabla_X \theta_\alpha + \theta_\alpha \nabla_X \cdot \mathbf{M}_\alpha) \right]$$



$$1168 \quad = T \left[ \frac{d\theta}{dt} + \sum_{\alpha} \nabla_x \cdot (\theta_{\alpha} \mathbf{M}_{\alpha}) \right] \quad ,(\text{IV-5})$$

$$1169 \quad \text{and } \sum_{\alpha} g_{\alpha} \nabla_x \cdot \mathbf{M}_{\alpha} = g_c \nabla_x \cdot \mathbf{M}_c + g_w \nabla_x \cdot \mathbf{M}_w = g_c \left( -\overline{\omega}_{c \rightarrow w} - \frac{dm_c}{dt} \right) + g_w \left( \overline{\omega}_{c \rightarrow w} - \frac{dm_w}{dt} \right)$$

$$1170 \quad = (-g_c + g_w) \overline{\omega}_{c \rightarrow w} - \sum_{\alpha} g_{\alpha} \frac{dm_{\alpha}}{dt} \quad ,(\text{IV-6})$$

1171 Equation (IV-4) can be rewritten as follows,

$$1172 \quad T \left( \frac{d\theta}{dt} + \sum_{\alpha=c,w} \nabla_x \cdot (\theta_{\alpha} \mathbf{M}_{\alpha}) \right) = \mathbf{R}_Q - \nabla_x \cdot \mathbf{Q} + \Phi_1 + \Phi_3 + \Phi_{\rightarrow} = \mathbf{R}_Q - \nabla_x \cdot \mathbf{Q} + \Phi_M + \Phi_{\rightarrow}. \quad (\text{IV-7}) \text{ or } (80)$$

1173 Combining the energy equation (40) and Equation (89), we can have

$$1174 \quad \frac{de}{dt} + e \nabla_x \cdot \mathbf{V}^s = \frac{d\psi}{dt} + T \frac{d\theta}{dt} + \theta \frac{dT}{dt} + e \nabla_x \cdot \mathbf{V}^s = \frac{d\psi}{dt} + T \frac{d\theta}{dt} + \theta \frac{dT}{dt} + (\psi + T\theta) \nabla_x \cdot \mathbf{V}^s = \boldsymbol{\sigma} : \mathbf{d}^s - \sum_{\alpha} \nabla_x \cdot$$

$$1175 \quad (h_{\alpha} \mathbf{w}^{\alpha}) - \nabla_x \cdot \mathbf{q} + \sum_{\alpha} (\mathbf{g} - \boldsymbol{\gamma}^{\alpha}) \cdot (\mathbf{w}^{\alpha}) + \mathbf{r}_Q - \frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] \quad . \quad (\text{IV-8})$$

1176 Therefore, the above equation can be rewritten as follows,

$$1177 \quad T \frac{d\theta}{dt} + T\theta \nabla_x \cdot \mathbf{V}^s = -\frac{d\psi}{dt} - \theta \frac{dT}{dt} - \psi \nabla_x \cdot \mathbf{V}^s + \boldsymbol{\sigma} : \mathbf{d}^s - \sum_{\alpha} \nabla_x \cdot (h_{\alpha} \mathbf{w}^{\alpha}) - \nabla_x \cdot \mathbf{q} + \sum_{\alpha} (\mathbf{g} - \boldsymbol{\gamma}^{\alpha}) \cdot$$

$$1178 \quad (\mathbf{w}^{\alpha}) + \mathbf{r}_Q - \frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2]. \quad (\text{IV-9})$$

1179 Considering that  $h_{\alpha} = g_{\alpha} + T\theta_{\alpha}$ , we can have

$$1180 \quad T \frac{d\theta}{dt} + T\theta \nabla_x \cdot \mathbf{V}^s = -\frac{d\psi}{dt} - \theta \frac{dT}{dt} - \psi \nabla_x \cdot \mathbf{V}^s + \boldsymbol{\sigma} : \mathbf{d}^s - \sum_{\alpha} \nabla_x \cdot (g_{\alpha} \mathbf{w}^{\alpha}) - \sum_{\alpha} \nabla_x \cdot (T\theta_{\alpha} \mathbf{w}^{\alpha}) - \nabla_x \cdot \mathbf{q} +$$

$$1181 \quad \sum_{\alpha} (\mathbf{g} - \boldsymbol{\gamma}^{\alpha}) \cdot (\mathbf{w}^{\alpha}) + \mathbf{r}_Q - \frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2] \quad . \quad (\text{IV-10})$$

1182 Use of  $\nabla \cdot (\zeta \mathbf{V}) = \zeta \nabla \cdot \mathbf{V} + \nabla \zeta \cdot \mathbf{V}$ , we can have

$$1183 \quad \nabla_x \cdot (g_{\alpha} \mathbf{w}^{\alpha}) = g_{\alpha} \nabla_x \cdot \mathbf{w}^{\alpha} + \mathbf{w}^{\alpha} \cdot \nabla_x g_{\alpha}, \text{ and } \nabla_x \cdot (T\theta_{\alpha} \mathbf{w}^{\alpha}) = T \nabla_x \cdot (\theta_{\alpha} \mathbf{w}^{\alpha}) + (\theta_{\alpha} \mathbf{w}^{\alpha}) \cdot \nabla_x T. \quad (\text{IV-11})$$

1184 Substituting Equation (IV-11) into Equation (IV-10), we can obtain

$$1185 \quad T \frac{d\theta}{dt} + T\theta \nabla_x \cdot \mathbf{V}^s + T \sum_{\alpha} \nabla_x \cdot (\theta_{\alpha} \mathbf{w}^{\alpha}) = \mathbf{r}_Q - \nabla_x \cdot \mathbf{q} + \boldsymbol{\sigma} : \mathbf{d}^s - \frac{d\psi}{dt} - \theta \frac{dT}{dt} - \psi \nabla_x \cdot \mathbf{V}^s - \sum_{\alpha} g_{\alpha} \nabla_x \cdot \mathbf{w}^{\alpha}$$

$$1186 \quad - \sum_{\alpha} \mathbf{w}^{\alpha} \cdot \nabla_x g_{\alpha} - \sum_{\alpha} (\theta_{\alpha} \mathbf{w}^{\alpha}) \cdot \nabla_x T + \sum_{\alpha} (\mathbf{g} - \boldsymbol{\gamma}^{\alpha}) \cdot (\mathbf{w}^{\alpha}) - \frac{1}{2} \Lambda_{c \rightarrow w} [(\mathbf{V}^w - \mathbf{V}^s)^2 - (\mathbf{V}^c - \mathbf{V}^s)^2], \quad (\text{IV-12})$$

1187 which can be rewritten as follows,

$$1188 \quad T \frac{d\theta}{dt} + T\theta \nabla_x \cdot \mathbf{V}^s + T \sum_{\alpha} \nabla_x \cdot (\theta_{\alpha} \mathbf{w}^{\alpha}) = \mathbf{r}_Q - \nabla_x \cdot \mathbf{q} + \varphi_1 + \varphi_3 + \varphi_{\rightarrow}. \quad (\text{IV-13}) \text{ or } (90)$$

## 1189 Appendix V: Deriving Clausius-Duhem inequality related to soil skeleton.

1190 From Equations (110) and (111), we have

$$1191 \quad \theta = \theta_s + \sum_{\alpha} m_{\alpha} \theta_{\alpha}, \text{ and } \psi = \psi_s + \sum_{\alpha} m_{\alpha} \psi_{\alpha} = \psi_s + \sum_{\alpha} m_{\alpha} \left( \mu_{\alpha} - \frac{p_{\alpha}}{\rho_{\alpha}} \right). \quad (\text{V-1})$$

1192 Substituting Equations (V-1) into Equation (108), we can have

$$1193 \quad \sigma: \frac{d\varepsilon}{dt} + \sum \mu_\alpha \frac{dm_\alpha}{dt} - (\theta_s + \sum_\alpha m_\alpha \theta_\alpha) \frac{dT}{dt} - \frac{d(\psi_s + \sum_\alpha m_\alpha (\mu_\alpha - \frac{p_\alpha}{\rho_\alpha}))}{dt} \geq 0, \quad (V-2)$$

1194 which can be rewritten as follows,

$$1195 \quad \sigma: \frac{d\varepsilon}{dt} - \theta_s \frac{dT}{dt} - \frac{d\psi_s}{dt} + \sum \mu_\alpha \frac{dm_\alpha}{dt} - \sum_\alpha m_\alpha \theta_\alpha \frac{dT}{dt} - \sum_\alpha \left( \mu_\alpha - \frac{p_\alpha}{\rho_\alpha} \right) \frac{dm_\alpha}{dt} - \sum_\alpha m_\alpha \frac{d(\mu_\alpha - \frac{p_\alpha}{\rho_\alpha})}{dt} \geq 0. \quad (V-3)$$

1196 The last four items in (V-3) can be written as follows,

$$1197 \quad \sum \mu_\alpha \frac{dm_\alpha}{dt} - \sum_\alpha m_\alpha \theta_\alpha \frac{dT}{dt} - \sum_\alpha \left( \mu_\alpha - \frac{p_\alpha}{\rho_\alpha} \right) \frac{dm_\alpha}{dt} - \sum_\alpha m_\alpha \frac{d(\mu_\alpha - \frac{p_\alpha}{\rho_\alpha})}{dt} = \sum_{\alpha=c,w} \left\{ \mu_\alpha \frac{dm_\alpha}{dt} - m_\alpha \frac{dm_\alpha}{dt} + \right. \\ 1198 \quad \left. \frac{p_\alpha}{\rho_\alpha} \frac{dm_\alpha}{dt} - m_\alpha \theta_\alpha \frac{dT}{dt} - m_\alpha \frac{d\mu_\alpha}{dt} + m_\alpha \frac{d(\frac{p_\alpha}{\rho_\alpha})}{dt} \right\}. \quad (V-4)$$

1199 Considering that  $m_\alpha = \rho_\alpha \phi_\alpha$  and  $\theta_\alpha$  does not change with time, we can obtain

$$1200 \quad \mu_\alpha \frac{dm_\alpha}{dt} - m_\alpha \frac{dm_\alpha}{dt} + \frac{p_\alpha}{\rho_\alpha} \frac{dm_\alpha}{dt} - m_\alpha \theta_\alpha \frac{dT}{dt} - m_\alpha \frac{d\mu_\alpha}{dt} + m_\alpha \frac{d(\frac{p_\alpha}{\rho_\alpha})}{dt} = \frac{p_\alpha}{\rho_\alpha} \frac{dm_\alpha}{dt} - m_\alpha \theta_\alpha \frac{dT}{dt} - m_\alpha \frac{d(\frac{p_\alpha}{\rho_\alpha})}{dt} + \\ 1201 \quad m_\alpha \frac{d(\theta_\alpha dT)}{dt} + m_\alpha \frac{d(\frac{p_\alpha}{\rho_\alpha})}{dt} = \frac{p_\alpha}{\rho_\alpha} \frac{dm_\alpha}{dt} = p_\alpha \frac{d\phi_\alpha}{dt}. \quad (V-5)$$

1202 Substituting Equations (V-4) and (V-5) into Inequality (V-3), we obtain the Clausius-Duhem inequality  
1203 related to the skeleton as follows,

$$1204 \quad \sigma: \frac{d\varepsilon}{dt} - \theta_s \frac{dT}{dt} - \frac{d\psi_s}{dt} + \sum_{\alpha=c,w} p_\alpha \frac{d\phi_\alpha}{dt} \geq 0. \quad (V-6) \text{ or } (112).$$

#### 1205 Appendix VI: Deriving some formula in Section 4.4 for constitutive model.

1206 For plastic strain, we have  $\varepsilon_v^p = \frac{1}{3}(\varepsilon_a^p + 2\varepsilon_c^p)$ ,  $\varepsilon_s^p = \frac{2}{3}(\varepsilon_a^p - \varepsilon_c^p)$ , so  $\sigma_{ij} d\varepsilon_{ij}^p$  in triaxial stress state  
1207 can be expressed as  $\sigma_a d\varepsilon_a^p + 2\sigma_c d\varepsilon_c^p$ . Considering  $\sigma_m, \sigma_s, \varepsilon_v^p$ , and  $\varepsilon_s^p$ , we have

$$1208 \quad \sigma_m d\varepsilon_v^p + \sigma_s d\varepsilon_s^p = \frac{1}{3}(\sigma_a + 2\sigma_c)(d\varepsilon_a^p + 2d\varepsilon_c^p) + \frac{2}{3}(\sigma_a - \sigma_c)(d\varepsilon_a^p - d\varepsilon_c^p) = \sigma_a d\varepsilon_a^p + 2\sigma_c d\varepsilon_c^p. \quad (VI-1)$$

1209 Therefore, the dissipation (148) can be expressed as follows,

$$1210 \quad D = \sigma_m d\varepsilon_v^p + \sigma_s d\varepsilon_s^p + \bar{p}_c d\varphi_c^p + \bar{p}_w d\varphi_w^p + \xi_J d\chi_J. \quad (VI-2)$$

1211 Substituting  $d\varepsilon_v^p = -d\varphi_c^p - d\varphi_w^p$  into (VI-2), we can obtain

$$1212 \quad D = -(\sigma_m - \bar{p}_c)d\varphi_c^p - (\sigma_m - \bar{p}_w)d\varphi_w^p + \sigma_s d\varepsilon_s^p + \xi_J d\chi_J \geq 0. \quad (VI-3) \text{ or } (156)$$

1213 From the yield function (161), we have

$$1214 \quad \frac{\partial f}{\partial \xi_\alpha} = - \left[ 1 - \left( \frac{\eta}{\xi_\alpha} \right)^n \right]^{-2} (-1)\eta^n (-n)(\xi_\alpha)^{-n-1} = - \frac{n}{\xi_\alpha \left[ 1 - \left( \frac{\eta}{\xi_\alpha} \right)^n \right]^2} \left( \frac{\eta}{\xi_\alpha} \right)^n, \quad (VI-4)$$

$$1215 \quad \frac{\partial f}{\partial \xi_\beta} = -1, \quad (VI-5)$$

$$1216 \quad \frac{\partial f}{\partial \sigma_m^E} = \frac{1}{1 - (\eta/\xi_\alpha)^n} + \frac{n}{\eta^2 [1 - (\eta/\xi_\alpha)^n]^2} \left( \frac{\eta}{\xi_\alpha} \right)^n, \quad (VI-6)$$

$$1217 \quad \frac{\partial f}{\partial \sigma_s} = - \frac{n}{\eta [1 - (\eta/\xi_\alpha)^n]^2} \left( \frac{\eta}{\xi_\alpha} \right)^n. \quad (VI-7)$$

1218 From (158), (162) and (163), we know that  $\chi_\alpha = \varepsilon_s^p$  and  $\chi_\beta = \varepsilon_v^p$ , so we have

$$1219 \quad \frac{\partial \xi_\alpha}{\partial \chi_\alpha} = \frac{\partial \xi_\alpha}{\partial \varepsilon_s^p} = \frac{b}{c} \alpha_m \exp\left(-\frac{\varepsilon_s^p}{c}\right), \quad (\text{VI-8})$$

$$1220 \quad \frac{\partial \xi_\beta}{\partial \chi_\beta} = \frac{\partial \xi_\beta}{\partial \varepsilon_v^p} = \beta \sigma_{mr}^E \exp(\beta \varepsilon_v^p). \quad (\text{VI-9})$$

1221 From the plastic potential function  $h$  (164), we have

$$1222 \quad \frac{\partial h}{\partial \sigma_m^E} = \frac{1}{1-(\eta/\xi_{\alpha 1})^{n_1}} + \frac{n_1}{\eta^2[1-(\eta/\xi_{\alpha 1})^{n_1}]^2} \left(\frac{\eta}{\xi_{\alpha 1}}\right)^{n_1}, \quad (\text{VI-10})$$

$$1223 \quad \frac{\partial h}{\partial \sigma_s} = -\frac{n_1}{\eta[1-(\eta/\xi_{\alpha 1})^{n_1}]^2} \left(\frac{\eta}{\xi_{\alpha 1}}\right)^{n_1}. \quad (\text{VI-11})$$

## 1224 Acknowledgments

1225 This research was supported by the 100-Talent Program of the Chinese Academy of Sciences  
 1226 (Granted to Dr. Enlong Liu), the National Natural Science Foundation of China (Grant Nos. 41230630),  
 1227 Key Research Program of Frontier Sciences of Chinese Academy of Sciences  
 1228 (QYZDY-SSW-DQC015).

## 1229 Nomenclature

1230	<b>X</b>	the position vector relative to a cartesian coordinate of reference configuration
1231	$e_i$	the base vectors
1232	<b>x</b>	the position vector relative to a cartesian coordinate of current configuration
1233	<b>F</b>	the deformation gradient
1234	$\xi$	the displacement vector
1235	$V^s$	the velocities of solid grains
1236	$V^{\alpha(\alpha=c, w)}$	the velocities of in-pore phases (ice crystals and unfrozen water)
1237	$M_\alpha$	the Lagrangian flux attached to the initial configuration of phase $\alpha$
1238	$w_\alpha$	the Eulerian mass flux
1239	$n$	the unit normal of current configuration
1240	$N$	the unit normal of initial configuration
1241	$\vartheta$	the filtration vector
1242	$\gamma^s$	the accelerations of soil skeleton
1243	$\gamma^\alpha(\alpha = c, w)$	the acceleration of phase $\alpha$
1244	$g$	the gravitational acceleration
1245	$\pi$	the Piola-Kirchhoff stress tensor

1246	$\sigma$	the Cauchy stress
1247	$\sigma^s$	the separate existence of a partial volumetric stress related to the soil skeleton
1248	$\sigma^\alpha(\alpha = c, w)$	the partial volumetric stress tensor related to the fluids
1249	$T^s$	the traction vector of soil solid grain
1250	$T^\alpha$	the traction vector of phase $\alpha$
1251	$T$	the total traction vector
1252	$\Delta$	the Green-Lagrange strain tensor
1253	$d^\pi(\pi = s, c, w)$	the Eulerian strain rate tensor of phase $\pi$
1254	$J_Q$	a surface rate of heat supply by conduction
1255	$r_Q$	a volume density of the heat provided to REV by an external heat sources
1256	$q$	an outgoing heat flow vector in current configuration
1257	$Q$	the Lagrangian heat flow vector
1258	$R_Q$	the Lagrangian volume density of the heat provided to REV
1259	$k_\alpha$	the permeability tensor relative to the current configuration
1260	$K_\alpha$	the permeability tensor relative to the initial configuration
1261	$\kappa$	the thermal conductivity tensor relative to the current configuration
1262	$K$	the thermal conductivity tensor relative to the initial configuration
1263	$\varepsilon$	the strain tensor in infinitesimal transformation
1264	$X_i$	the components of a position vector of reference configuration
1265	$x_i$	the components of a position vector of current configuration
1266	$F_{ij}$	the components of the deformation gradient
1267	$d\Omega_0$	the volume of the undeformed element at the reference configuration
1268	$d\Omega$	the volume of the deformed element in the current configuration
1269	$J$	the Jacobian
1270	$n$	the Eulerian porosity
1271	$\phi$	the Lagrangian porosity
1272	$\phi_\alpha(\alpha=c, w)$	the Lagrangian porosity of ice crystals or unfrozen water
1273	$S_\alpha(\alpha = c, w)$	the Lagrangian saturation degree of phase $\alpha$
1274	$\varphi_c$ et $\varphi_w$	the volume change due to deformation of the porous space
1275	$n_\alpha(\alpha = c, w)$	the Eulerian partial porosities of phase $\alpha$

1276	$s_\alpha(\alpha = c, w)$	the Eulerian saturation degree of phase $\alpha$
1277	$\rho_s$	the intrinsic mass densities of the solid matrix
1278	$\rho_\alpha(\alpha=c, w)$	the intrinsic mass densities of the in-pore component of phase $\alpha$
1279	$M_\alpha$	the Lagrangian flux attached to the initial configuration of fluid $\alpha$
1280	$m_\alpha$	the Lagrangian fluid mass content related to fluid $\alpha$
1281	$m_s$	the mass of soil skeleton at current configuration
1282	$w_\alpha$	the Eulerian mass flux of fluid $\alpha$
1283	$m_s^0$	the mass of soil skeleton at initial configuration
1284	$da$	the current surface
1285	$dA$	the initial surface
1286	$\rho_s^0$	the initial matrix mass density
1287	$\Lambda_{\alpha\rightarrow\beta}$	the mass of phase $\alpha$ transforming into phase $\beta$ per unit overall current volume and
1288		per unit time
1289	$\varpi_{\alpha\rightarrow\beta}$	the mass of phase $\alpha$ transforming into phase $\beta$ per unit overall initial volume and
1290		per unit time
1291	$\rho$	the total apparent mass density
1292	$p_\alpha$	the fluid pressure of phase $\alpha$
1293	$K_s$	the kinetic energy associated with soil matrix in current configuration
1294	$K_\alpha$	the kinetic energy associated with phase $\alpha$ in current configuration
1295	$P_{\text{def}}$	the strain work rate of the RVE in Eulerian formulation
1296	$P_{f,T}$	the work rate of the RVE in Lagrangian formulation
1297	$P_{f,T}$	the work rate of the RVE in Eulerian formulation
1298	$\mathcal{K}_s$	the kinetic energy associated with soil matrix in Lagrangian formulation
1299	$\mathcal{K}_\alpha$	the kinetic energy associated with phase $\alpha$ in Lagrangian formulation
1300	$\mathcal{P}_{\text{def}}$	the strain work rate of the RVE
1301	$Q^0$	the external heat supply
1302	$e_s$	the specific (i.e. per unit mass) internal energy of the soil matrix
1303	$e_\alpha(\alpha = c, w)$	the specific (i.e. per unit mass) internal energy of the phase $\alpha$
1304	$e$	the total internal energy per unit overall current volume
1305	$h_\alpha(\alpha = c, w)$	the fluid-specific enthalpy of phase $\alpha$

1306	$E$	the overall Lagrangian densities of internal energy per unit of initial volume
1307	$\theta_\pi(\pi = s, c, w)$	the specific entropy in current configuration of phase $\pi$
1308	$\theta$	the total entropy per unit of overall current volume
1309	$\psi$	the Helmholtz free energy in Eulerian formulation
1310	$\Theta$	the Lagrangian entropy density
1311	$\Psi$	the Lagrangian free energy density
1312	$g_\alpha(\alpha = c, w)$	the fluid-specific free enthalpy of phase $\alpha$
1313	$T$	temperature
1314	$\Theta_\alpha(\alpha = c, w)$	the Lagrangian entropy density of phase $\alpha$
1315	$\Phi$	the total dissipation per unit of initial volume in Lagrangian formulation
1316	$\Phi_i$	the dissipation per unit of initial volume with $i = 1,2,3$ in Lagrangian form
1317	$\Phi_\rightarrow$	the dissipation with phase change in Lagrangian formulation
1318	$\varphi$	the dissipations in Eulerian formulation
1319	$\varphi_i$	the Eulerian dissipation volume densities with $i = 1,2,3$ in Eulerian form
1320	$\varphi_\rightarrow$	the dissipation with phase change in Eulerian formulation
1321	$m_{c \rightarrow w}$	the mass of phase ice crystal transforming into unfrozen water per unit overall
1322		current volume and per unit time in infinitesimal transformation
1323	$\mu_\alpha(\alpha = c, w)$	the specific chemical potential of the saturating solution $\alpha$
1324	$\psi_s$	the skeleton free energy
1325	$\theta_s$	the skeleton entropy density
1326	$T_m$	the melting temperature
1327	$\Sigma_m$	the melting entropy
1328	$\wp$	the function expression
1329	$\Gamma$	the $\Gamma$ function
1330	$U$	the interfacial energy
1331	$f$	the yield function
1332	$h$	the plastic potential function

### 1333 **References**

1334 Andersland, O.B. and Ladanyi, B., 2004. Frozen ground engineering. John Wiley & Sons.

- 1335 Al-Rub, R.K.A., Darabi, M., 2012. A thermodynamic framework for constitutive modelling of time-  
1336 and rate-dependent materials. Part I: Theory. *International Journal of Plasticity* 34, 61-92.
- 1337 Biot, M. A., 1941. General theory of three dimensional consolidation. *Journal for Applied Physics* 12,  
1338 155-164.
- 1339 Biot, M. A., 1956. The theory of propagation of elastic waves in a fluid-saturated porous solid. I.  
1340 low-frequency range. *Journal of the Acoustic Society of America* 28, 168-178.
- 1341 Boukpeti, N., 2008. One-dimensional analysis of a poroelastic medium during freezing. *International*  
1342 *Journal for Numerical and Analytical Methods in Geomechanics* 32, 1661-1691.
- 1343 Collard, C., Favier, V., Berbenni, S., Berveiller, M., 2010. Role of discrete intra-granular slip bands on  
1344 the strain-hardening of polycrystals. *International Journal of Plasticity* 26, 310-328.
- 1345 Coussy, O., 1989. A general theory of thermoporoelasticity for saturated porous materials. *Transport in*  
1346 *Media* 4, 281-293.
- 1347 Coussy, O., 1995. *Mechanics of porous continua*. John Wiley & Sons Ltd.
- 1348 Coussy, O., 2004. *Poromechanics*. John Wiley & Sons Ltd.
- 1349 Coussy, O., 2005. Poromechanics of freezing materials. *Journal of the Mechanics and Physics of Solids*  
1350 53, 1689-1718.
- 1351 Coussy, O., Monteiro P. J.M., 2008. Poroelastic model for concrete exposed to freezing temperature.  
1352 *Cement and Concrete* 38, 40-48.
- 1353 Darabi, M., Al-Rub, R.K.A., Masad, E. A., Huang, C.-W., 2012. A modified viscoplastic model to  
1354 predict the permanent deformation of asphaltic materials under cyclic-compression loading at high  
1355 temperatures. *International Journal of Plasticity* 35, 100-134.
- 1356 De Sciarra, F. M., 2012. Hardening plasticity with nonlocal strain damage. *International Journal of*  
1357 *Plasticity* 34, 114-138.
- 1358 Halphen, B., Nguyen, Q. S., 1974. Plastic and visco-plastic materials with generalized potential. *Mech.*  
1359 *Res. Comm.* 1, 43-47.
- 1360 Henann, D.L., Kamrin, K., 2014. Continuum thermomechanics of the nonlocal granular rheology.  
1361 *International Journal of Plasticity* 60, 145-162.
- 1362 Hill, R., 1950. *The mathematical theory of plasticity*. Clarendon Press.
- 1363 Kamrin, K., 2010. Nonlinear elasto-plastic model for dense granular flow. *International Journal of*  
1364 *Plasticity* 26, 167-188.

- 1365 Khan, A.S., Xiang, Y., Huang, S., 1991. Behaviour of Berea sandstone under confining pressure part I:  
1366 Yield and failure surfaces, and nonlinear elastic response. *International Journal of Plasticity* 7(6):  
1367 607-624.
- 1368 Krairi, A., Doghri, I., 2014. A thermodynamically-based constitutive model for thermoplastic polymers  
1369 coupling viscoelasticity, viscoplasticity and ductile damage. *International Journal of Plasticity* 60,  
1370 163-181.
- 1371 Lai, W. M., Rubin, D., Krempl, E., 2010. *Introduction to continuum mechanics*. Elsevier Inc..
- 1372 Lai Y., Jin, L., Chang, X., 2009. Yield criterion and elasto-plastic damage constitutive model for frozen  
1373 sandy soil. *International Journal of Plasticity* 25, 1177-1205.
- 1374 Lai, Y., Yang, Y., Chang, X., Li, S., 2010. Strength criterion and elastoplastic constitutive model of  
1375 frozen silt in generalized plastic mechanics. *International Journal of Plasticity* 26, 1461-1484.
- 1376 Lai, Y., Liao, M., Hu, K., 2016. A constitutive model of frozen saline sandy soil based on energy  
1377 dissipation theory. *International Journal of Plasticity* 78, 84-113.
- 1378 Lai, Y., Xu, X., Dong, Y., Li, S., 2013. Present situation and prospect of mechanical research on frozen  
1379 soils in China. *Cold Regions Science and Technology* 87, 6-18.
- 1380 Lai, Y. M., Wu, Z. W., Zhu, Y. L., Zhu, L. N., 1998. Nonlinear analysis for the coupled problem of  
1381 temperature, seepage and stress fields in cold-region tunnels. *Tunneling and Underground Space*  
1382 *Technology* 13(4), 434-440.
- 1383 Lee, M.Y., Fossum, A, Costin, L.S, Bronowski, D., 2002. Frozen soil material testing and constitutive  
1384 modelling. Sandia report, Sand2002-0524, Sandia National Laboratories.
- 1385 Lei, X., Wong, H., Fabbri, A., Liman, A., Cheng, Y. M., 2014. A thermo-chemo-electro-mechanical  
1386 framework of unsaturated expansive clays. *Computers and Geotechnics* 62, 175-192.
- 1387 Lemaitre, J., Chaboche, J. L., 1994. *Mechanics of solid materials*. Cambridge University Press.
- 1388 Li, N., Chen, B., Chen, F., Xu, X., 2000. The coupled heat-moisture-mechanic model of the frozen soil.  
1389 *Cold Regions Science and Technology* 31, 199-205.
- 1390 Liu, E., Xing, H., 2009. A double hardening thermos-mechanical constitutive model for  
1391 overconsolidated clays. *Acta Geotechnica* 4, 1-6.
- 1392 Liu, E., Yu, H.S., Zhou, C., Nie, Q., Luo, K., 2017. A binary-medium constitutive model for artificially  
1393 structured soils based on the disturbed state concept and homogenization theory. *International Journal*  
1394 *of Geomechanics*, 04016154.



- 1395 Loria, A. F. Loria, Frigo, B., Chiaia, B., 2017. A non-linear constitutive model for describing the  
1396 mechanical behaviour of frozen ground and permafrost. *Cold Regions Science and Technology* 133,  
1397 63-69.
- 1398 Lu, J. F., Tan, Y. P., Wang, J. H., 2011. A phase field model for the freezing saturated porous medium.  
1399 *International Journal of Engineering Science* 49, 768-780.
- 1400 Muraleetharan, K.K., Liu, C., Wei, C., Kibbey, T.C.G., Chen, L., 2009. An elastoplastic framework for  
1401 coupling hydraulic and mechanical behaviour of unsaturated soils. *International Journal of Plasticity* 25,  
1402 473-490.
- 1403 Na, S. H., Sun, W. C., 2017. Computational thermo-hydro-mechanics for multiphase freezing and  
1404 thawing porous media in the finite deformation range. *Comput. Methods Appl. Meth. Engrg.* 318,  
1405 667-700.
- 1406 Neaupane, K.M., Yamabe, T., Yoshinaka, R., 1999. Simulation of a fully coupled  
1407 thermo-hydro-mechanical system in freezing and thawing rock. *International Journal of Rock*  
1408 *Mechanics and Mining Science* 36, 563-580.
- 1409 Neaupane, K.M., Yamabe, T., 2001. A fully coupled thermo-hydro-mechanical nonlinear model for a  
1410 frozen medium. *Computers and Geotechnics* 28, 613-637.
- 1411 Nicot, F., Sibille, L., Darve, F., 2012. Failure in rate-independent granular materials as a bifurcation  
1412 toward a dynamic regime. *International Journal of Plasticity* 2012, 136-154.
- 1413 Nishimura, S., Gens, A., Olivella, S., Jardine, R.J., 2009. THM-coupled finite element analysis of  
1414 frozen soil: formulation and application. *Geotechnique* 59(3), 159-171.
- 1415 Sciarra, G., 2016. Phase field modeling of partially saturated deformable porous media. *Journal of the*  
1416 *Mechanics and Physics of Solids* 94, 230-256.
- 1417 Shen, W.Q., Shao, J.F., 2016. An incremental micro-macro model for porous geomaterials with double  
1418 porosity and inclusion. *International Journal of Plasticity* 83, 37-54.
- 1419 Sheng, D., Zhang, S., Niu, F., Cheng, G., 2014. A potential new frost heave mechanism in high-speed  
1420 railway embankments. *Geotechnique* 64(2), 144-154.
- 1421 Steinhäuser, M.O., Grass, K., Strassburger, E., Blumen, A., 2009. Impact failure of granular  
1422 materials-Non-equilibrium multiscale simulations and high-speed experiments. *International Journal of*  
1423 *Plasticity* 2009, 161-182.
- 1424 Thomas, H. R., Cleall, P., Li, Y.-C., Harris, C., Kern-Luetsch, M., 2009. Modelling of cryogenic

- 1425 processes in permafrost and seasonally frozen soils. *Geotechnique* 59, 173-184.
- 1426 Tsytoich, N. A., 1985. The mechanics of frozen ground. [Translated by Zhang, C. Q. and Zhu, Y.L.]  
1427 Science Press.
- 1428 Xu, X., Wang, J., Zhang, L., 2010. Physics of frozen soils. Science Press.
- 1429 Xu, X., Wang, Y., Yin, Z., Zhang, H., 2017. Effect of temperature and strain rate on mechanical  
1430 characteristics and constitutive model of frozen Helin loess. *Cold Regions Science and Technology* 136,  
1431 44-51.
- 1432 Yang, P., Ke, J., Wang, J.G., Chow, Y.K., Zhu, F., 2006. Numerical simulation of frost heave with  
1433 coupled water freezing, temperature and stress fields in tunnel excavation. *Computers and Geotechnics*  
1434 33, 330-340.
- 1435 Yang, R., Lemarchand, E., Fen.-C., T., Azouni, A., 2015. A micromechanics model for partial freezing  
1436 in porous media. *International Journal of Solids and Structures* 75-76, 109-121.
- 1437 Yao, Y. P., Hou, W., and Zhou, A. N., 2009. UH model: Three dimensional unified hardening model for  
1438 overconsolidated clays. *Geotechnique* 59(5), 451–469.
- 1439 Yao, Y.P., Zhou, A. N., 2013. Non-isothermal unified hardening model: a thermo-elasto-plastic model  
1440 for clays. *Geotechnique* 63(15), 1328-1345.
- 1441 Yao, Y. P., Kong, L. M., Zhou, A. N., and Yin, J. H., 2015. Timedependent unified hardening model:  
1442 Three-dimensional elasto-viscoplastic constitutive model for clays. *J. Eng. Mech.*,  
1443 10.1061/(ASCE)EM.1943-7889.0000885, 04014162.
- 1444 Zhang, S., Leng, W., Zhang, F., Xiong, Y., 2012. A simple thermo-elastoplastic model for geomaterials.  
1445 *International Journal of Plasticity* 34, 93-113.
- 1446 Zhang, Y., 2014. Thermal-hydro-mechanical model for freezing and thawing of soils (Ph. D thesis), the  
1447 University of Michigan.
- 1448 Zhang, Z., 2017. A thermodynamics-based theory for the thermos-poro-mechanical modeling of  
1449 saturated clay. *International Journal of Plasticity* 92, 164-185.
- 1450 Zhou, J., Li, D., 2012. Numerical analysis of coupled water, heat and stress in saturated freezing soil.  
1451 *Cold Regions Science and Technology* 72, 43-49.
- 1452 Zhou, M. M., Meschke, G., 2013. A three-phase thermo-hydro-mechanical finite element model for  
1453 freezing soils. *International Journal for Numerical and Analytical Methods in Geomechanics* 37,  
1454 3173-3193.

- 1455 Zhou, Y. W., Guo, D. X., Cheng, G. D., 2000. Frozen soil of China. Science Press.
- 1456 Zhu, Q.Z., Shao, J.F., Mainguy, M., 2010. A micromechanics-based elastoplastic damage model for  
1457 granular materials at low confining pressure. International Journal of Plasticity 26(4), 586-602.

## 1458 Lists of Figures

- 1459 Fig. 1 Macroscopic model of the REV of saturated frozen soils
- 1460 Fig. 2 Comparisons of tested and simulated results of frozen sand soil at  $-1^{\circ}\text{C}$ ,  $-5^{\circ}\text{C}$  and  $-10^{\circ}\text{C}$  with  
1461 1MPa confining pressure: (a) Deviatoric stress-shear strain curves; (b) Volumetric strain-shear strain  
1462 curves; (c) Predicted volumetric strain- pore pressure curves
- 1463 Fig. 3 Comparisons of tested and simulated results of frozen saline sand soil at  $-6^{\circ}\text{C}$  with 1MPa-10MPa  
1464 confining pressures: (a) Deviatoric stress-shear strain curves; (b) Volumetric strain-shear strain curves;  
1465 (c) Predicted volumetric strain- pore pressure curves
- 1466 Fig. 4 Simulated results with varying  $\alpha_m = \alpha_{m1}$  at 2.0MPa confining pressu(a) Deviatoric  
1467 stress-shear strain curves; (b) Volumetric strain-shear strain curves; (c) Stress path of  $\sigma_m^E - \sigma_s$
- 1468 Fig. 5 Simulated results with varying  $\kappa$  at 2.0MPa confining pressures: (a) Deviatoric stress-shear  
1469 strain curves; (b) Volumetric strain-shear strain curves; (c) Stress path of  $\sigma_m^E - \sigma_s$
- 1470 Fig. 6 Simulated results with varying  $\gamma_1$  at 2.0MPa confining pressures: (a) Deviatoric stress-shear  
1471 strain curves; (b) Volumetric strain-shear strain curves; (c) Stress path of  $\sigma_m^E - \sigma_s$
- 1472 Fig. 7 Simulated results with varying  $\sigma_{mr}^E$  at 2.0MPa confining pressures: (a) Deviatoric stress-shear  
1473 strain curves; (b) Volumetric strain-shear strain curves; (c) Stress path of  $\sigma_m^E - \sigma_s$
- 1474 Fig. 8 Simulated results with varying  $\beta = \beta_1$  at 2.0MPa confining pressures: (a) Deviatoric  
1475 stress-shear strain curves; (b) Volumetric strain-shear strain curves; (c) Stress path of  $\sigma_m^E - \sigma_s$
- 1476 Fig. 9 Simulated results with varying  $z = z_1$  at 2.0MPa confining pressures: (a) Deviatoric  
1477 stress-shear strain curves; (b) Volumetric strain-shear strain curves; (c) Stress path of  $\sigma_m^E - \sigma_s$
- 1478 Fig. 10 Simulated results with varying  $\kappa_1$  at 2.0MPa confining pressures: (a) Deviatoric stress-shear  
1479 strain curves; (b) Volumetric strain-shear strain curves; (c) Stress path of  $\sigma_m^E - \sigma_s$
- 1480 Fig. 11 Simulated results with varying  $\gamma_2$  at 2.0MPa confining pressures: (a) Deviatoric stress-shear  
1481 strain curves; (b) Volumetric strain-shear strain curves; (c) Stress path of  $\sigma_m^E - \sigma_s$

1482

1483