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Averaged controllability in a long time horizon

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Abstract

In this article we study a linear control system with unknown parameter. However, we assume that the possible realisations of this parameter are finite and each realisation can appear with a known probability. We aim to design a control independent of the parameter such that the expectation of the system's realisation reach a given target at a final time $T > 0$. This problem is now well-known as averaged controllability.

We extend this notion to the one of *long time averaged controllability*. More precisely, we consider the question: is it possible to steer the averaged of the system to some prescribed value in time $T > 0$ and then keep the averaged of the system at this prescribed value for all times $t > T$? We provide a necessary and sufficient condition for this property to hold. Once the condition is satisfied, one can apply a feedback control that keeps the average fixed during a given time period. We also address the L^2 -norm optimality of such controls.

Relations between the introduced and previously existing different control notions of parameter dependent systems are discussed, accompanied by numerical examples.

keywords: Parameter dependent ODEs, Averaged control, Controlled invariant subspace, Linear optimal control, Open-loop control systems.

1 Introduction

In this paper, we consider the averaged controllability property of finite-dimensional, parameter dependent systems. The notion of averaged controllability has been introduced by E. Zuazua in [11], and afterwards generalised to PDE setting in [3, 6, 5]. Its goal is to control the average (or more generally a suitable linear combination) of system components by a single control. The problem is relevant in practice when the system depends on a number of variable parameters, and the control has to be chosen independently of their values.

More precisely, let us consider d realisation of control systems,

$$\dot{x}_i = A_i x_i + B_i u, \quad x_i(0) = x_i^0 \quad (i \in \{1, \dots, d\}), \quad (1.1)$$

with $A_i \in M_n(\mathbb{R})$, $x_i(t) \in \mathbb{R}^n$, $B_i \in M_{n,m}(\mathbb{R})$ and $u(t) \in \mathbb{R}^m$, and d parameters $\sigma_1, \dots, \sigma_d \in [0, 1]$ such that $\sum \sigma_i = 1$. Note that in this paper, there is no controllability assumptions on the pairs (A_i, B_i) . The only assumption we make is the *averaged controllability* of this system, see Definition 1.2 below.

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Definition 1.1. We say that the system (1.1) is *controllable in average in some time* $T > 0$ for the parameters $\sigma_1, \dots, \sigma_d$ if for every $x_1^0, \dots, x_d^0 \in \mathbb{R}^n$ and every $x^1 \in \mathbb{R}^n$, there exist a control $u \in L^2(0, T)^m$ such that the solution of (1.1) satisfies

$$\sum_{i=1}^d \sigma_i x_i(T) = x^1. \quad (1.2)$$

This notion is closely related to the simultaneous control problem, in which all system components need to be controlled by a single control.

Definition 1.2. We say that the system (1.1) is *simultaneously controllable in some time* $T > 0$ if for every $x_1^0, \dots, x_d^0 \in \mathbb{R}^n$ and every $x_1^1, \dots, x_d^1 \in \mathbb{R}^n$, there exist a control $u \in L^2(0, T)^m$ such that the solution of (1.1) satisfies

$$x_i(T) = x_i^1 \quad (i \in \{1, \dots, d\}).$$

Note that since we are dealing with finite dimensional control systems, these two notions are independent of the time $T > 0$. In addition, due to the time reversibility and linearity, both of the above defined controllability properties is enough to check for the null target only.

It is obvious that if the system is simultaneously controllable, then it is controllable in average. More detailed relations between these notions have been studied in [4], in which the authors analyse deviations of each system component from the averaged value. To this effect they identify the optimal averaged control as the one minimising a quadratic functional which, together with the control norm, contains a penalisation term measuring deviation of each system component from the average. Under suitable assumptions they show that by increasing the penalisation constant, the averaged control converges to the simultaneous one.

Let us note that all the above cited papers analyse average controllability properties in a single time point, i.e. they investigate the conditions under which there exist a control such that relation (1.2) holds for $t = T$. However, here we must emphasise an essential difference compared to classical control problems. Namely, application of a control steering the averaged of a system to zero at time T does not imply the system average will remain zero for the subsequent time period. Contrarily to classical control problems, the null control applied for $t > T$ does not lead to a constant null average. This is because the different system components are in general governed by different dynamics, which, starting from a same state, will result in trajectories that do not coincide.

In this paper our main concern is whether it is possible or not to find a control such that the average of the solution of (1.1), once brought to zero, remains zero for the subsequent time period.

More precisely, we assume that it is possible to steer the system from any initial condition x_i^0 to the zero averaged level in some time $T > 0$, that is to say that $\sum_{i=1}^d \sigma_i x_i(T) = 0$. Then we aim to construct a control u for time $t > T$ such that $\sum_{i=1}^d \sigma_i x_i(t) = 0$ for every $t > T$. If this is possible, we will say that the system (1.1) satisfies the *long time averaged control property*.

This paper is organised as follows. In Section 2, we extend the notion of averaged controllability to *L-controllability* and prove general result about averaged and long time averaged controllability. The main result of this section and of this paper is Theorem 2.8. It provides condition ensuring *long time L-controllability*. In Section 3, we give the formal expression of L^2 -norm optimal control. More precisely, for a system having the long time averaged controllability property, we fix T_0 and T_1 , two positive times, and we aim to find the minimal L^2 -norm optimal control such that the state averaged is constant equal to 0 for every $t \in [T_0, T_0 + T_1]$. The Section 4 is an application of the results of Section 2 to averaged and long time average controllability. In particular, we focus to the case where $d = 2$ and $\sigma_1 = \sigma_2 = 1/2$. We provide many illustrative examples. In particular, we note that *simultaneous controllability* implies *long time averaged controllability*, and that *long time averaged controllability* implies *averaged controllability*. The examples of Section 4.2

show that none of these notions are equivalent, i.e. *long time averaged controllability* does not imply *simultaneous controllability* and *averaged controllability* does not imply *long time averaged controllability*. Finally, we conclude this article in Section 5, with some concluding remarks and open problems.

2 General result

2.1 Generalisation of the problem

Using the notations introduced in the above system, we define $\tilde{A} \in M_{nd}(\mathbb{R})$, $\tilde{B} \in \mathbb{R}^{nd}$, $\tilde{x}(t) \in \mathbb{R}^{nd}$ and $L \in M_{n,nd}(\mathbb{R})$ by:

$$\tilde{A} = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_d \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B_1 \\ \vdots \\ \vdots \\ B_d \end{pmatrix}, \quad \tilde{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ \vdots \\ x_d(t) \end{pmatrix} \quad \text{and} \quad L = (\sigma_1 I_n \quad \cdots \quad \sigma_d I_n). \quad (2.1)$$

With these new notations, the system (1.1) writes,

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \quad \tilde{x}(0) = \tilde{x}^0.$$

with \tilde{x}^0 , the column vector formed by x_1^0, \dots, x_d^0 . Then the average controllability problem is, given $\bar{x}^1 \in \mathbb{R}^n$, to find a control $u \in L^2(0, T)^m$ such that the solution of the above system satisfies

$$L\tilde{x}(T) = \bar{x}^1,$$

while the long time averaged control problem is to find a control u such that the solution of the above system satisfies

$$L\tilde{x}(t) = \bar{x}^1 \quad (t > T).$$

As shown in [11], the system (1.1) is controllable in average if and only if the following rank condition is satisfied:

$$\text{rank} \left[\sum_{i=1}^d \sigma_i A_i^k B_i, \quad k \in \mathbb{N} \right] = n.$$

Thanks to Hamilton Cayley Theorem, the above rank condition reduces to

$$\text{rank} \left[\sum_{i=1}^d \sigma_i A_i^k B_i, \quad k \in \{0, \dots, nd-1\} \right] = n. \quad (2.2)$$

Similarly, the system (1.1) is simultaneously controllable if and only if the Kalman rank condition holds:

$$\text{rank} [\tilde{A}^k \tilde{B}, \quad k \in \{0, \dots, n-1\}] = nd. \quad (2.3)$$

In this paper we shall see that the introduced, long time averaged control property can be also expressed in terms of a suitable rank condition. In addition, we allow for a general linear operator L that does not have to be of a special form given in (2.1).

Consequently, in full generality, we will consider $A \in M_n(\mathbb{R})$, $B \in M_{n,m}(\mathbb{R})$ and the system

$$\dot{x} = Ax + Bu, \quad x(0) = x^0 \in \mathbb{R}^n. \quad (2.4)$$

Given $L \in M_{q,n}(\mathbb{R})$, the generalisation of the averaged controllability problem is: given $T > 0$ and $\bar{x}^1 \in \text{Ran } L$, find a control $u \in L^2(0, T)^m$ such that the solution x of (2.4) satisfies

$$Lx(T) = \bar{x}^1. \quad (2.5)$$

This question will be treated in Section 2.2.

In order to generalise the long time averaged controllability problem, tackled in Section 2.3, we discuss the conditions under which one can find a control u such that the solution x of (2.4) satisfies

$$Lx(t) = Lx(0) \quad (t > 0). \quad (2.6)$$

2.2 Generalisation of averaged controllability results

Here we present a general result by which a system can be steered to the image of a suitable linear operator L .

We consider the system (2.4) and we extend the notion of averaged controllability introduced in Definition 1.1 to the notion of L -output controllability.

Definition 2.1. We say that the system (2.4) is *L -output controllable in some time $T > 0$* if for every $x^0 \in \mathbb{R}^n$ and every $\bar{x}^1 \in \text{Ran } L$, there exist a control $u \in L^2(0, T)^m$ such that the solution of (2.4) satisfies (2.5).

The characterisation of the introduced notion is provided by the following proposition. This result is a slight extension of the Kalman rank condition for the output controllability (see for instance [2]). In particular, when L is a full rank matrix, these two results coincide.

Proposition 2.2. *For every $x^0 \in \mathbb{R}^n$ and $\bar{x}^1 \in \text{Ran } L$ there exists a control $u \in L^2(0, T)^m$ steering the solution x of (2.4) from x^0 to some $x(T) \in \mathbb{R}^n$ such that $Lx(T) = \bar{x}^1$, if and only if A , B and L satisfy:*

$$\text{rank } L(A^0 B \cdots A^{n-1} B) = \text{rank } L. \quad (2.7)$$

Proof. First of all, since $\bar{x}^1 \in \text{Ran } L$, there exist $x^1 \in \mathbb{R}^n$ such that $Lx^1 = \bar{x}^1$ and the condition $Lx(T) = \bar{x}^1$ now reads $x(T) - x^1 \in \text{Ker } L$. In addition, let us define a matrix $P \in M_{\text{rank } L, q}(\mathbb{R})$ such that $\text{rank } PL = \text{rank } L$ and $\text{Ker } PL = \text{Ker } L$. Then the condition $Lx(T) = \bar{x}^1$ now reads $x(T) - x^1 \in \text{Ker } PL$. Now since PL is a full rank matrix, we have from [2, Theorem III], that this system is PL -output controllable if and only if

$$\text{rank } PL(A^0 B \cdots A^{n-1} B) = \text{rank } PL.$$

Using the fact that $\text{Ker } PL = \text{Ker } L$ and $\text{rank } PL = \text{rank } L$, we obtain that

$$\text{rank } PL(A^0 B \cdots A^{n-1} B) = \text{rank } L(A^0 B \cdots A^{n-1} B).$$

This ends the proof. □

Remark 2.3. Actually, the statement of the above theorem can be slightly extended to

- Given $\bar{x}^1 \in \text{Ran } L$ and given any $T > 0$, then every $x^0 \in \mathbb{R}^n$ can be steered to some $x(T)$ such that $Lx(T) = \bar{x}^1$ if and only if (2.7) is satisfied;
- Given $\bar{x}^0 \in \mathbb{R}^n$ and given any $T > 0$, then for every $\bar{x}^1 \in \text{Ran } L$, $x^0 \in \mathbb{R}^n$ can be steered to some $x(T)$ such that $Lx(T) = \bar{x}^1$ if and only if (2.7) is satisfied.

2.3 General long time control result

The aim of this paragraph is to determine whether it is possible or not to find a control u such that the solution x of the general control problem (2.4) satisfies (2.6), i.e. that it remains constant under action of an operator L for the subsequent time period.

Remark 2.4. Of course, if $\text{rank } L \geq n$, the constraint (2.6) immediately gives $x(t) = x(0)$ for every $t \geq 0$. Thus, in particular, we have $\dot{x}(t) = 0$, implying $Ax(0) = -Bu$. Thus, there is a (constant) control u ensuring condition (2.6) if and only if $Ax(0) \in \text{Ran } B$.

In order to treat a more general case we develop the following procedure. By time derivation of (2.6), we have:

$$L Ax(t) + L B u(t) = 0 \quad (t > 0).$$

That is to say that the control u shall satisfy:

$$L B u(t) = -L A x(t) \quad (t > 0). \quad (2.8)$$

In order to satisfy the last relation, one need to have $L A x(t) \in \text{Ran}(L B)$ for every $t \geq 0$. More precisely, let us define $P \in M_q(\mathbb{R})$ the orthogonal projector of \mathbb{R}^q on $\text{Ker}(L B)^\top$, so that we have $\text{rank } P = q - \text{rank}(L B)$. Consequently, (2.8) is

$$-P L A x(t) = P L B u(t) = 0.$$

Let us notice that the projector P has the following form,

$$P = I_q - Q(L B)(L B)^\top Q^\top, \quad (2.9)$$

where Q is the Gram-Schmidt matrix ensuring that $\text{Ran } Q(L B) = \text{Ran}(L B)$ and the columns of $Q(L B)$ are orthonormal.

Thus, in order to satisfy (2.6), one also has to satisfy:

$$\begin{pmatrix} L \\ P L A \end{pmatrix} x(t) = \begin{pmatrix} L x(0) \\ 0 \end{pmatrix} \quad (t \geq 0). \quad (2.10)$$

This leads to a condition similar to (2.6) with L_1 in place of L , where $L_1 = \begin{pmatrix} L \\ P L A \end{pmatrix}$. Iterating this process, we can define a sequence (L_k) by the following iterative algorithm.

Algorithm 1 General long time control

STEP 0: Set $L_0 = L$.

STEP 1: Set $\Lambda_0 = P_0 L_0 A$, where P_0 is the orthogonal projector of \mathbb{R}^q on $\text{Ker}(L_0 B)^\top$.

Define $L_1 := \begin{pmatrix} L_0 \\ \Lambda_0 \end{pmatrix}$.

STEP k+1: Set $\Lambda_k = P_k L_k A$, where P_k is the orthogonal projector of $\mathbb{R}^{(k+1)q}$ on $\text{Ker}(L_k B)^\top$.

Define $L_{k+1} := \begin{pmatrix} L_0 \\ \Lambda_k \end{pmatrix}$.

The algorithm stops when $\text{Ker } L_{K+1} = \text{Ker } L_K$.

Let us notice here that $\text{Ker } L_{k+1} \subset \text{Ker } L_k \subset \mathbb{R}^n$ which ensures the stopping criteria of the Algorithm 1 will be satisfied after a finite number of steps. Indeed, from the definition of the operator L_1 it follows $\text{Ker } L_1 = \text{Ker } L_0 \cap \text{Ker } \Lambda_0$, trivially implying $\text{Ker } L_1 \subseteq \text{Ker } L_0$. Now suppose that $\text{Ker } L_k \subseteq \text{Ker } L_{k-1}$ for some $k \in \mathbb{N}$. We want to show that it implies $\text{Ker } L_{k+1} \subseteq \text{Ker } L_k$. From the definition of operators L_k , the last inclusion follows if we show that $\text{Ker } \Lambda_k \subseteq \text{Ker } \Lambda_{k-1}$. To this effect, notice that by the induction hypothesis it follows

$$\begin{aligned} \Lambda_k x = 0 &\Leftrightarrow L_k(Ax - Bu) = 0 \quad \text{for some } u \\ &\Rightarrow L_{k-1}(Ax - Bu) = 0 \Leftrightarrow \Lambda_{k-1}x = 0 \end{aligned}$$

thus obtaining $\text{Ker } L_k \subseteq \text{Ker } L_{k-1} \subseteq \mathbb{R}^q$ for every $k \in \mathbb{N}$. Consequently, there exist $K \in \mathbb{N}$ such that $\text{Ker } L_{K+1} = \text{Ker } L_K$, and in particular, from the structure of operators L_k , $\text{Ker } \Lambda_{K+1} = \text{Ker } \Lambda_K$.

In addition, let us notice that due to the structure of the projectors P_k (2.9), we have $\text{Ran } \Lambda_k \subset \text{Ran } L_k$.

Using Algorithm 1 we can now state the conditions by which we are able to stay in $\text{Ker } L$ if we start from a point in $\text{Ker } L$.

Proposition 2.5. *Let $K \in \mathbb{N}$ be the integer such that $\text{Ker } L_{K+1} = \text{Ker } L_K$, where L_k are operators defined by the iterative process described above. Then there exists a control $u \in L^\infty(\mathbb{R}_+)^m$ such that $Lx(t) = Lx(0)$ for every $t \geq 0$, where x is solution of (2.4), if and only if $\Lambda_K x(0) = 0$.*

Proof. First of all let us notice that by using the relation (2.10), and repeating the arguments behind it for different values of k , without loosing generality we may replace the condition $Lx(t) = Lx(0)$ for every $t \geq 0$ with $L_K x(t) = \begin{pmatrix} Lx(0) \\ 0 \end{pmatrix}$, for every $t \geq 0$.

Of course, the last condition implies $\Lambda_K x(0) = 0$.

Assume now that $\Lambda_K x(0) = 0$. Since x is solution of (2.4), for every control u we have

$$L_K \dot{x}(t) = L_K Ax + L_K Bu. \quad (2.11)$$

The claim follows if we show that $L_K \dot{x}(t) = 0$.

As in the iteration procedure, we set P_K the projection on $\text{Ker}(L_K B)^\top$. Then we have

$$L_K \dot{x}(t) = P_K L_K Ax + (I_q - P) L_K Ax + L_K Bu.$$

Since by definition $(I_q - P)L_K Ax$ belongs to $\text{Ran } L_K B$, whatever x is, one can find $u = u(x)$ such that $(I_q - P)L_K Ax + L_K Bu = 0$. For such control u , relation (2.11) reduces to

$$L_K \dot{x}(t) = P_K L_K Ax = \Lambda_K x, \quad (2.12)$$

where $\Lambda_K = P_K L_K A$ is defined as in the recursive process. By using the decomposition

$$x - x(0) = x_0 + x_1, \quad (2.13)$$

where $x_0 \in \text{Ker } L_K$ and $x_1 \in \text{Ran } L_K^\top$ we directly obtain

$$L_K \dot{x}(t) = L_K \dot{x}_1(t) = \Lambda_K (x(0) + x_0 + x_1). \quad (2.14)$$

By the assumption, $x(0)$ belongs to $\text{Ker } \Lambda_K$. On the other hand, since $\text{Ker } L_K = \text{Ker } L_{K+1} = \text{Ker} \begin{pmatrix} L_0 \\ \Lambda_K \end{pmatrix} = \text{Ker } L_0 \cap \text{Ker } \Lambda_K$, we also have $\text{Ker } L_K \subset \text{Ker } \Lambda_K$, implying

$$L_K \dot{x}_1(t) = \Lambda_K x_1. \quad (2.15)$$

Set $z = L_K x_1$, and note that there exist an invertible and linear operator $T = \left(L_K|_{\text{Ran } L_K^\top} \right)^{-1} : \text{Ran } L_K \rightarrow \text{Ran } L_K^\top$ such that $Tz = x_1$.

Thus, (2.15) can be rewritten as

$$\dot{z}(t) = \Lambda_K Tz.$$

From the decomposition (2.13) $x_1(0) = z(0) = 0$ implying $z(t) = 0, t \geq 0$. The conclusion follows. \square

Remark 2.6. The Algorithm 1 developed above resembles the one presented in [10, Theorem 4.3] for construction of a supremal controlled invariant subspace of an arbitrary space K . To be recalled, a subspace $V \subset K$ is controlled invariant if it has the property that if $x(0) \in V$ then there exists a control u such that the solution to system (2.4) satisfies $x(t) \in V$ for all $t \geq 0$.

In the case when $K = \text{Ker } L$ the supremal invariant subspace is exactly the kernel of the operator L_K constructed above. This confirms the optimality of the result obtained in Proposition 2.5 in the sense that $\text{Ker } L_K$ is the largest space we can start our system from, still being sure that we shall be able to remain within $\text{Ker } L_K$, and consequently within $\text{Ker } L$, for all subsequent time periods.

Remark 2.7. From the proof of the last proposition it follows that the operators L_k constructed by the Algorithm 1 can be replaced by \tilde{L}_k as long as $\text{Ker } L_k = \text{Ker } \tilde{L}_k$.

In addition, it is easy to check that the Algorithm stopping condition $\text{Ker } L_{K+1} = \text{Ker } L_K$ implies that $\text{Ker } L_{K+i} = \text{Ker } L_K$ for every $i \in \mathbb{N}$.

Let us now consider the complete problem, i.e. we consider a time $T > 0$ and any starting point $x_0 \in \mathbb{R}^n$, and we aim to steer the solution of (2.4) to a point $x(T) \in \text{Ker } L$ and then for $t \geq T$ keep the trajectory fixed with respect to L .

Merging the results of Propositions 2.2 and 2.5 we can formulate the following.

Theorem 2.8. *For every $x^0 \in \mathbb{R}^n$ and $\bar{x}^1 \in \text{Ran } L$ there exists a control $u \in L^\infty(\mathbb{R}_+)^m$ such that the solution to the system (2.4) satisfies $Lx(t) = \bar{x}^1, t \geq T$ if and only if*

$$\begin{pmatrix} \bar{x}^1 \\ 0 \end{pmatrix} \in \text{Ran } L_{K+1} \quad \text{and} \quad \text{rank } L_K \begin{pmatrix} A^0 B & \cdots & A^{n-1} B \end{pmatrix} = \text{rank } L_K, \quad (2.16)$$

where L_K is the operator constructed by the Algorithm 1.

Proof. According to Proposition 2.5 the required control exists if and only if the system is steered to the state $x(T)$ such that $L_{K+1}x(T) = \begin{pmatrix} \bar{x}^1 \\ 0 \end{pmatrix}$. By Proposition 2.2 this is possible if and only if

$$\text{rank } L_{K+1} \begin{pmatrix} A^0 B & \cdots & A^{n-1} B \end{pmatrix} = \text{rank } L_{K+1}. \quad (2.17)$$

However, as by the construction $\text{Ker } L_K = \text{Ker } L_{K+1}$, the operator L_{K+1} in (2.17) can be replaced by L_K , thus yielding the result. \square

3 Norm optimal controls

In this section, we analyse L^2 -norm optimal controls. More precisely, given two positive times T_0 and T_1 , and given some initial condition $x^0 \in \mathbb{R}^n$, an operator $L \in M_{q,n}(\mathbb{R})$ and $\bar{x}^1 \in \text{Ran } L$, we aim to find the control of minimal L^2 norm such that the solution x of (2.4) satisfies

$$Lx(T_0) = \bar{x}^1 \quad (3.1a)$$

and

$$Lx(t) = Lx(T_0) \quad (t \in [T_0, T_0 + T_1]). \quad (3.1b)$$

According to Theorem 2.8 this problem admits a solution for every starting point x^0 if and only if $\begin{pmatrix} \bar{x}^1 \\ 0 \end{pmatrix} \in \text{Ran } L_{K+1}$ and the rank condition (2.16) is satisfied, with L_K obtained from L by the Algorithm 1.

Given some $x^1 \in R_{T_0}(x^0)$, where $R_{T_0}(x^0)$ is the set of reachable points from x^0 in time $T_0 > 0$, the control of minimal L^2 -norm steering x^0 to x^1 in time T_0 is uniquely defined and its L^2 norm is a function J_0 depending only on x^1 and T_0 (and x^0).

Similarly, we shall show that the norm optimal control such that the solution of (2.4) satisfies (3.1b) is uniquely determined and the corresponding norm is a function J_1 depending only on $x^1 = x(T_0)$ and T_1 .

Consequently, in order to find the global optimal control ensuring the long time controllability result we have to minimise $J_0 + J_1$ over $x^1 \in \text{Ker } \Lambda_K \cap L^{\leftarrow}(\bar{x}^1) \cap R_{T_0}(x^0)$, where $L^{\leftarrow}(\bar{x}^1)$ denotes the preimage of \bar{x}^1 under L .

We will divide the study into several parts: first we find the L^2 -norm optimal controls on $[0, T_0]$ such that $x(T_0) = x^1$ (see Section 3.1), then we look for the optimal control such that the condition (3.1b) is fulfilled (see Section 3.2) and finally, we will combine this two results to obtain the global optimal control such that the state solution of (2.4) satisfies the constraints (3.1) (see Section 3.3).

3.1 Norm optimal controls steering the system to a prescribed state

Set $T > 0$, $x^0 \in \mathbb{R}^n$ and $x^1 \in R_T(x^0)$, we aim to find the optimal control u of minimal L^2 -norm such that the solution x of (2.4) satisfies $x(T) = x^1$.

Following the lines of the proof of Proposition 2.2, this control u is given by

$$u(t) = B^\top p(t),$$

where $p \in \mathbb{R}^n$ is the solution of the adjoint problem

$$-\dot{p} = A^\top p, \quad p(T) = p^1$$

whose terminal datum $p^1 \in \mathbb{R}^n$ is given as solution to

$$\Gamma_T p^1 = x^1 - e^{TA} x^0,$$

with $\Gamma_T = \int_0^T e^{(T-s)A} B B^\top e^{(T-s)A^\top} ds$ denoting the control Gramian. Consequently, the norm of the optimal control is given by $\|u\|_{L^2(0,T)^m}^2 = (p^1)^\top \Gamma_T p^1$.

Note that here, since the controllability of the pair (A, B) is not assumed, the matrix Γ_T is not necessarily regular. However, the fact that $x^1 \in R_T(x^0)$, ensures that $x^1 - e^{TA} x^0 \in \text{Ran } \Gamma_T$ and consequently, the existence of a particular $p_p^1 \in \mathbb{R}^n$ such that $\Gamma_T p_p^1 = x^1 - e^{TA} x^0$. In addition, we have

$$\{p^1 \in \mathbb{R}^n \mid \Gamma_T p^1 = x^1 - e^{TA} x^0\} = \{p_p^1\} + \text{Ker } \Gamma_T$$

and it is easy to check that the L^2 -norm of the control u given by $u(t) = B^\top e^{(T-t)A^\top} p^1$, for $p^1 \in \{p_p^1\} + \text{Ker } \Gamma_T$ is independent of p^1 . Furthermore, there exist a non-negative matrix $Q_T \in M_n(\mathbb{R})$ such that $Q_T \Gamma_T p^1 = p^1$ for $p^1 \in \text{Ran } \Gamma_T$, and

$$\|u\|_{L^2(0,T)^m}^2 = (x^1 - e^{TA} x^0)^\top Q_T (x^1 - e^{TA} x^0) =: J_0(x^1; T).$$

In particular, we have the optimal control u of minimal L^2 -norm such that the solution x of (2.4) satisfies $x(T) = x^1$ is given by

$$u(t) = B^\top e^{(T-t)A^\top} Q_T (x^1 - e^{TA} x^0). \quad (3.2)$$

3.2 Norm optimal controls keeping the state fixed with respect to L

Set $x^0 \in \mathbb{R}^n$ and $T > 0$. We aim to find the control u of minimal L^2 norm such that the solution x of (2.4) satisfies $Lx(t) = Lx^0$ for every $t \in [0, T]$.

Let us decompose \mathbb{R}^m in $\text{Ker } LB \oplus (\text{Ker } LB)^\perp = \text{Ker } LB \oplus \text{Ran}(LB)^\top$ and set $u(t) = u_0(t) + (LB)^\top v_0(t)$, with $u_0(t) \in \text{Ker } LB$ and $v_0(t) \in \mathbb{R}^q$. Consequently, we aim to minimise

$$\frac{1}{2} \int_0^T |u(t)|^2 dt = \frac{1}{2} \int_0^T (|u_0(t)|^2 + |(LB)^\top v_0(t)|^2) dt$$

under the constraint $Lx(t) = Lx(0)$ for every $t \in [0, T]$. Notice that without modifications, we can assume that $v_0(t) \in (\text{Ker}(LB)^\top)^\perp = \text{Ran } LB$. Furthermore, the constraint $Lx = \text{const}$ is satisfied if and only if $-LAx = (LB)(LB)^\top v_0$. For every $x \in \mathbb{R}^n$ the last equation admits one and only one solution $v_0 = v_0(x) \in \text{Ran } LB$. We define $M \in M_{q,q}(\mathbb{R})$ such that $M^\top = M$ and for every $w \in \text{Ran } LB$, $M(LB)(LB)^\top w = w$, then we have, $v_0(x) = -MLAx$.

Let us also define $K \in M_{m,d(LB)}(\mathbb{R})$, where $d(LB) = \dim(\text{Ker } LB)$ such that $\text{Ran } K = \text{Ker } LB$. In addition, K can be chosen to be a matrix of isometry, i.e. setting $u_0 = Kw_0$, we have $|u_0| = |w_0|$.

Consequently, the problem is to find $w_0 \in L^2(0, T)^{d(LB)}$ such that it minimises

$$\frac{1}{2} \int_0^T (|w_0(t)|^2 + \langle MLa x(t), La x(t) \rangle) dt,$$

where x is the solution of

$$\dot{x} = (A - B(LB)^\top MLa) x + BKw_0, \quad x(0) = x^0.$$

This is an unconstrained linear quadratic problem and according to [1], see also [9, Theorem 4.11], the corresponding minimiser is given by

$$w_0(t) = (BK)^\top E(t)x(t) \quad (t \in [0, T]),$$

where $E \in M_n(\mathbb{R})$ is solution of the backward Riccati equation:

$$\dot{E} = (LA)^\top MLa - (A - B(LB)^\top MLa)^\top E - E(A - B(LB)^\top MLa) - EBK(BK)^\top E, \quad (3.3a)$$

$$E(T) = 0. \quad (3.3b)$$

Furthermore, we have,

$$\min_{w_0 \in L^2(0, T)^{d(LB)}} \frac{1}{2} \int_0^T (|w_0(t)|^2 + \langle MLa x(t), La x(t) \rangle) dt = -\frac{1}{2} (x^0)^\top E(0)x^0 =: J_1(x^0; T), \quad (3.4)$$

and $E(0)$ is a non-positive matrix.

Thus, we have obtained that the optimal long time control keeping the solution fixed with respect to the operator L during then time period $t \in [0, T]$ is a feedback control of the form

$$u(t) = \left(K(BK)^\top E(t) - (LB)^\top MLa \right) x(t),$$

whose norm is given by (3.4).

Remark 3.1. In order to solve the Riccati equation (3.3), the classical approach (see for instance [8, 7]) is to perform the change of variables $E(t) = P(t)Q(t)^{-1}$, with $P(t) \in M_n(\mathbb{R})$ and $Q(t) \in GL_n(\mathbb{R})$, and where P and Q are solutions of

$$\begin{aligned} \dot{P}(t) &= (LA)^\top MLa Q(t) - (A - B(LB)^\top MLa)^\top P(t), \\ \dot{Q}(t) &= BB^\top P(t) + (A - B(LB)^\top MLa) Q(t), \end{aligned}$$

with the terminal conditions $P(T) = 0$ and $Q(T) = I_n$.

3.3 Global minimisation problem

We now aim to couple the two optimisation problem given in Sections 3.1 and 3.2 in order to find the L^2 -norm optimal control such that $Lx(t) = \bar{x}^1 \in \text{Ran } L$ for every $t \in [T_0, T_0 + T_1]$.

According to Bellman principle, we have

$$\min_{u \in \mathcal{U}(x^0, T_0, T_1)} \frac{1}{2} \int_0^{T_0+T_1} |u(t)|^2 dt = \min_{\substack{x^1 \in \text{Ker } \Lambda_K \\ x^1 \in L^\leftarrow(\bar{x}_1) \cap R_{T_0}(x^0)}} (J_0(x^1; T_0) + J_1(x^1, T_1)), \quad (3.5)$$

where we have set $\mathcal{U}(x^0, T_0, T_1)$ as the set of admissible L^2 controls, that is to say,

$$\mathcal{U}(x^0, T_0, T_1) = \left\{ u \in L^2(0, T_0 + T_1)^m \mid Lx(t) = \bar{x}^1, \forall t \in [T_0, T_0 + T_1] \right\},$$

while x is the solution of (2.4) with initial condition x^0 and control u .

Several remarks are in order.

- The problem (3.5) is a quadratic minimisation problem determined by positive semi-definite matrices, thus allowing for a solution which in general does not have to be unique. However, the norm of the corresponding optimal control is independent of a choice of a solution to (3.5).
- Note that minimiser x^1 of the problem (3.5) does not have to produce the control $u \in L^2(0, T_0)^m$ minimising the energy required to steer the system to the pre-image of \bar{x}_1 under L . This is because here we consider the global minimisation problem, in which we also take into account the cost of keeping the solution fixed with respect to L for $t > T_0$. Putting a larger effort in the initial time period might be compensated with a lower cost in the subsequent period.
- If $\bar{x}^1 = 0$ then the set $\text{Ker } \Lambda_K \cap L^\leftarrow(\bar{x}_1)$ coincides with the kernel of the operator L_{K+1} constructed by the Algorithm 1.
- Note that the problem (3.5) can also be reset as the minimisation problem:

$$\begin{array}{l} \min \quad \frac{1}{2} \left(\int_0^{T_0} |u(t)|^2 dt - x(T_0)^\top E(0)x(T_0) \right) \\ \left| \begin{array}{l} u \in L^2(0, T_0)^m, \\ L_{K+1}x(T_0) = (\bar{x}_1 \quad 0)^\top, \end{array} \right. \end{array}$$

where x is the solution of (2.4) with initial condition x^0 and control u and E is the solution of the Riccati equation (3.3), with terminal time $T = T_1 - T_0$.

4 Application to long time averaged control problem

We aim to apply in this section the obtained result, (2.16), to our initial, the long time averaged control problem, which has motivated the analysis performed in Sections 2 and 3. For the sake of simplicity, we restrict the analysis to the case of the null control, i.e. the goal is to steer and keep the average equal to zero.

We consider the system (2.1) with $d = 2$ components, $\sigma_1 = \sigma_2 = 1/2$, and $B_1 = B_2 = B$, for which the long time averaged controllability can be formulated as

$$L \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (t) = 0, \quad t \geq T \quad (4.1)$$

with the operator $L = \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix} \in M_{n,2n}(\mathbb{R})$.

In Section 4.1, we first give long time averaged controllability conditions for this type of systems and finally in Section 4.2 we provide different examples of this system.

4.1 Long time averaged controllability result

In the sequel P_B denotes the projector on $\text{Ker}(LB)^\top = \text{Ker } B^\top$, and we use $D = (A_1 - A_2)/2$ and $S = (A_1 + A_2)/2$, as well as \tilde{A}, \tilde{B} defined in (2.1). We will use the abbreviation

$$K = (\tilde{B}, \tilde{A}\tilde{B}, \dots, \tilde{A}^{2n-1}\tilde{B}) \in M_{2n,2nm}(\mathbb{R}) \quad (4.2)$$

for the Kalman controllability matrix.

We will also use the notation $\tilde{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \mathbb{R}^{2n}$, so that \tilde{x} is solution of

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \quad \tilde{x}(0) = \tilde{x}^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}. \quad (4.3)$$

The application of the iteratively algorithm introduced in Section 2.3 to this specific problem is presented by Algorithm 2. Its aim is to find the form of the operator L_K obtained such that $\text{Ker } L_K$ is the supremal invariant subspace of $\text{Ker } L$.

Remark 4.1. Note that in the iterating process described by Algorithm 2, it might happen that $\text{Ker } L_k = \text{Ker } L_{k-1}$ for some $k < n$. However, as commented in Remark 2.7, it does not affect the result as the last equality implies $\text{Ker } L_{k+i} = \text{Ker } L_{k-1}$ for every $i \in \mathbb{N}$.

From Theorem 2.8 we obtain the following result.

Corollary 4.2. *The long time averaged control condition (4.1) is feasible if and only if the rank condition*

$$\text{rank } L_n K = \text{rank } L_n,$$

holds, with K defined by (4.2), and the operator L_n given by (4.4), for $k = n$.

Remark 4.3. By Proposition 2.2 the above corollary ensures that the system can be steered to $(x_1(T), x_2(T)) \in \text{Ker } L_{n-1}$. This condition can be equivalently rewritten as

$$\begin{aligned} (x_1 + x_2)(T) &= 0 \\ (x_1 - x_2)(T) &\in V = \{d \in \mathbb{R}^n, DS^k d \in \text{Ran } B, k \in \{0, \dots, n-1\}\}. \end{aligned}$$

The last inclusion implies the existence of control u such that

$$Bu(t) = -De^{tS}(x_1 - x_2)(T) \quad t \geq T \quad (4.5)$$

and it is easy to check that such control ensures long time averaged controllability. Indeed, denoting $s = (x_1 + x_2)/2$ and $d = (x_1 - x_2)/2$, the system (1.1) (with $d = 2$) can be rewritten as

$$\begin{aligned} \dot{s} &= Ss + Dd + Bu \\ \dot{d} &= Ds + Sd. \end{aligned}$$

Now it becomes obvious that $s(t) = 0$ for $t \geq T$ if and only if there exists a control of the form (4.5).

Finally, let us recall and put together characterisations of different control notions discussed in this paper.

Algorithm 2 Long time averaged control

STEP 0 : Set $L_0 = L$.

STEP 1 : Denote by P_0 the projector on $\text{Ker}(L_0\tilde{B})^\top$.

As $L_0\tilde{B} = 2B$, we note that P_0 actually coincides with P_B , the orthogonal projection to the $\text{Ker} B^\top$.

Then we set $\tilde{L}_1 = \begin{pmatrix} L_0 \\ \Lambda_0 \end{pmatrix}$ with $\Lambda_0 = P_B L_0 \tilde{A} = P_B(A_1 \ A_2)$. Let us notice that $\text{Ker} \tilde{L}_1 = \text{Ker} \begin{pmatrix} I_n & I_n \\ P_B D & -P_B D \end{pmatrix}$.

Consequently, we set $L_1 = \begin{pmatrix} I_n & I_n \\ P_B D & -P_B D \end{pmatrix}$.

STEP 2 : Set P_1 a projector on $\text{Ker}(L_1 B)^\top = \text{Ker}(2B^\top \ 0) = \text{Ker} B^\top \times \mathbb{R}^n$. Consequently, we

choose $P_1 = \begin{pmatrix} P_B & 0 \\ 0 & I_n \end{pmatrix}$, we set $\Lambda_1 = P_1 L_1 \tilde{A} = \begin{pmatrix} P_B A_1 & P_B A_2 \\ P_B D A_1 & -P_B D A_2 \end{pmatrix}$ and $\tilde{L}_2 = \begin{pmatrix} L_0 \\ \Lambda_1 \end{pmatrix}$. Similarly to

the first step, we have $\text{Ker} \tilde{L}_2 = \text{Ker} \begin{pmatrix} I_n & I_n \\ P_B D & -P_B D \\ P_B D S & -P_B D S \end{pmatrix}$.

Consequently, we chose $L_2 = \begin{pmatrix} I_n & I_n \\ P_B D & -P_B D \\ P_B D S & -P_B D S \end{pmatrix}$.

STEP k: By using induction one can easily prove that in the k^{th} step of the algorithm the resulting matrix can be chosen of the form

$$L_k = \begin{pmatrix} I_n & I_n \\ P_B D & -P_B D \\ P_B D S & -P_B D S \\ \vdots & \vdots \\ P_B D S^{k-1} & -P_B D S^{k-1} \end{pmatrix}. \quad (4.4)$$

STEP n: Iterating the process, and due to Cayley-Hamilton theorem it follows the stopping criteria of the algorithm is satisfied after the most n iterations, i.e. it holds $\text{Ker} L_{n+1} = \text{Ker} L_n$.

- The system (4.3) is simultaneously controllable if and only if the rank of the matrix K is equal to $2n$;
- The system (4.3) is controllable in average if and only if the rank of the matrix LK is equal to n ;
- The system (4.3) has the long time averaged property if and only if the rank of the matrix $L_n K$ is equal to the rank of L_n defined by (4.4), with $D = (A_1 - A_2)/2$, $S = (A_1 + A_2)/2$ and P_B denoting the orthonormal projector on $\text{Ker} B^\top$. In the sequel, we shall also use the notation $\mathcal{L} = L_n$ for the operator L_n constructed by the Algorithm 2.

Obviously, from the above characterisations it follows that simultaneous controllability implies long time averaged controllability and that long time averaged controllability implies averaged controllability. However, both implications are strictly one-directional, as it will be demonstrated by the following examples of the following section.

4.2 Examples

In the following examples, we will consider the particular situation, introduced in the beginning of Section 4, where $d = 2$, $\sigma_1 = \sigma_2 = 1/2$ and $B_1 = B_2 = B$. Furthermore, for all these examples, we consider the case $n = 1$, and the matrices

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $A_2 \in M_2(\mathbb{R})$ will be defined in the examples. Note that the matrix K defined by (4.2) is

$$K = \begin{pmatrix} B & A_1 B & A_1^2 B & A_1^3 B \\ B & A_2 B & A_2^2 B & A_2^3 B \end{pmatrix} \in M_4(\mathbb{R}),$$

and the matrix $P_B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is the orthonormal projector of \mathbb{R}^2 on $\text{Ker } B^\top$. Recall also that we set $D = (A_1 - A_2)/2$, $S = (A_1 + A_2)/2$ and $\mathcal{L} = L_2$, the operator L_n ($n = 2$) constructed by the Algorithm 2.

In Example 4.4, we will define A_2 such that the system (1.1) is controllable in average but does not have the long time averaged controllability property. Then, in Example 4.5, we will define A_2 such that the system (1.1) has the long time averaged controllability property, but not the simultaneous one. Finally, in Examples 4.6 and 4.7, we will define A_2 such that the system (1.1) is simultaneously controllable. In Example 4.6, the matrix A_2 is defined so that the only possibility to have the long time averaged controllability property is to steer all the components of the system to zero, while it will not be necessarily the case in the Example 4.7.

Example 4.4. (Averaged controllability without long time averaged controllability)

For this example, we consider in addition the matrix $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We then have

$$K = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad LK = \begin{pmatrix} 2 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

and hence, we compute

$$\text{rank } K = 3 \quad \text{and} \quad \text{rank } LK = 2 = \text{rank } L.$$

This ensures that this system is averaged controllable but not simultaneously controllable. In order to check the long time averaged controllability, we have to compute the matrix \mathcal{L} . To this end, we compute the matrices D and S , and obtain $D = \frac{1}{2} \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, and $S = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, leading to

$$\mathcal{L} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 1/2 \end{pmatrix}.$$

It is easy to see that $\text{Ker } \mathcal{L} = \{0\}$.

This ensures that in order to stay in the kernel of L , one needs to start from 0. But since there is no

simultaneous controllability, this also means that there exist initial conditions x_1^0 and x_2^0 for which there exists no control steering simultaneously the solutions of (4.3) from these initial condition to the null target. Already for this reason, long time averaged controllability does not hold. Alternatively, we can check that

$$\text{rank } \mathcal{L}K = 3 < 4 = \text{rank}(\mathcal{L}),$$

which violates the necessary condition for the long time averaged controllability.

To conclude, the presented system has the averaged controllability property but it is not long time averaged controllable.

Example 4.5. (Long time averaged controllability without simultaneous controllability)

For this example, we consider the matrix $A_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, and we compute

$$K = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad LK = \begin{pmatrix} 2 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix},$$

hence $\text{rank } K = 3$ and $\text{rank } LK = 2 = \text{rank } L$. This ensures that this system is averaged controllable but not simultaneously controllable.

In order to check the long time averaged controllability, we have to compute the matrix \mathcal{L} and we obtain

$$\mathcal{L} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $\text{rank } \mathcal{L} = 3$ and $\text{Ker } \mathcal{L} = (1, 1, -1, -1)^\top \mathbb{R}$.

Similarly, we check that $\text{rank } \mathcal{L}K = 3 = \text{rank } \mathcal{L}$, which ensures this system has the long time averaged controllability property.

Let us now consider the norm optimal control problem. To this end, we set $T_0, T_1 > 0$, and we recall that we aim to minimise the L^2 -norm of the controls u such that the solution of the system (4.3) satisfies $L\tilde{x}(t) = 0$ for every $t \in [T_0, T_0 + T_1]$. By Proposition 2.5 the last condition is equivalent to $\mathcal{L}\tilde{x}(T) = 0$ i.e. we have to steer the system to the $\text{Ker } \mathcal{L}$. Denoting $\tilde{x}^1 = \tilde{x}(T_0) \in \mathbb{R}^4$ the reached point at time T_0 , we have $\tilde{x}^1 \in \text{Ker } \mathcal{L}$, hence

$$\tilde{x}^1 = (\xi, \xi, -\xi, -\xi)^\top,$$

for some $\xi \in \mathbb{R}$.

Minimal L^2 -norm control steering \tilde{x}^0 to \tilde{x}^1 in time T_0 . In order to compute explicitly this L^2 -norm, we follow the lines of Section 3.1. To this end, we compute the Gramian matrix, $\Gamma_{T_0} = \int_0^{T_0} e^{t\tilde{A}} \tilde{B} \tilde{B}^\top e^{t\tilde{A}^\top} dt$. This matrix (not displayed here) takes the form $\begin{pmatrix} G_{T_0} & 0 \\ 0 & 0 \end{pmatrix} \in M_4(\mathbb{R})$, with G_{T_0} a regular 3×3 matrix. In particular, we have $\text{rank } \Gamma_{T_0} = \text{rank } G_{T_0} = 3$ and $\text{Ran } \Gamma_{T_0} = \mathbb{R}^3 \times \{0\}$. Since \tilde{x}^1 should be a reachable point, it must hold that $\tilde{x}^1 - e^{T_0\tilde{A}}\tilde{x}^0 \in \text{Ran } \Gamma_{T_0}$. Denoting by \tilde{x}_i^0 the i^{th} -component of \tilde{x}^0 , we conclude that we shall have,

$$\xi = -e^{T_0}\tilde{x}_4^0. \tag{4.6}$$

Let us now compute a solution $p^1 \in \mathbb{R}^4$ such that $\Gamma_{T_0} p^1 = \tilde{x}^1 - e^{T_0 \tilde{A}} \tilde{x}^0$. It is easy to see that all the solution p^1 of $\Gamma_{T_0} p^1 = \tilde{x}^1 - e^{T_0 \tilde{A}} \tilde{x}^0$ are of the form $p^1 = \begin{pmatrix} \hat{p}^1 \\ p_4^1 \end{pmatrix}$, with $p_4^1 \in \mathbb{R}$ and $\hat{p}^1 \in \mathbb{R}^3$ is solution of

$$G_{T_0} \hat{p}^1 = \hat{x},$$

with \hat{x} the vector of \mathbb{R}^3 formed by the first three components of $\tilde{x}^1 - e^{T_0 \tilde{A}} \tilde{x}^0$. We then deduce that the norm-optimal control u is given by

$$u(t) = \tilde{B}^\top e^{(T_0-t)\tilde{A}} \left(G_{T_0}^{-1} \hat{x} \quad 0 \right)^\top$$

and the corresponding L^2 -norm is,

$$\|u\|_{L^2(0, T_0)}^2 = \hat{x}^\top G_{T_0}^{-1} \hat{x}.$$

Minimal L^2 -norm control in order to stay in the kernel of L . In order to compute this L^2 -norm, we follow the lines of Section 3.2. First of all, we assume we have reached a point $\tilde{x}^1 \in \text{Ker } \mathcal{L}$. According to the above subsection, this point is of the form $\tilde{x}^1 = (\xi, \xi, -\xi, -\xi)^\top$ for ξ given by (4.6). We aim to find the minimal L^2 -norm control such that the solution of (4.3) starting from \tilde{x}^1 in T_0 satisfies $L\tilde{x}(t) = 0$ for every $t \in [T_0, T_0 + T_1]$.

One can easily compute that $\text{Ker } L\tilde{B} = \{0\}$. Consequently, the control keeping the solution in the kernel of L is uniquely determined and takes the form

$$u(t) = (L\tilde{B})^\top v(t),$$

with $v(t) \in \text{Ran } L\tilde{B}$ solution of

$$-L\tilde{A}\tilde{x}(t) = (L\tilde{B})(L\tilde{B})^\top v(t). \quad (4.7)$$

We have, $L\tilde{B} = (2 \quad 0)^\top$ and $\text{Ran } L\tilde{B} = \mathbb{R} \times \{0\}$. Thus, for $M = \begin{pmatrix} 1/4 & 0 \\ 0 & 0 \end{pmatrix}$ it holds $M(L\tilde{B})(L\tilde{B})^\top w = w$ for every $w \in \text{Ran } L\tilde{B}$. Acting by M on (4.7) we deduce that the required optimal control is a feedback control of the form

$$u(t) = (L\tilde{B})^\top M L \tilde{A} \tilde{x} \quad (4.8)$$

and \tilde{x} is solution of

$$\dot{\tilde{x}} = (\tilde{A} - \tilde{B}(L\tilde{B})^\top M L \tilde{A}) \tilde{x}, \quad \tilde{x}(0) = (\xi, \xi, -\xi, -\xi)^\top \in \mathbb{R}^4.$$

After some computation, one can check that $(1, 1, -1, -1)^\top$ is an eigenvector of $(\tilde{A} - \tilde{B}(L\tilde{B})^\top M L \tilde{A})$ for the eigenvalue 1. Consequently, we have

$$\tilde{x}(t) = \xi e^{(t-T_0)} (1, 1, -1, -1)^\top, \quad t \geq T_0$$

and finally, the L^2 -norm of the control is

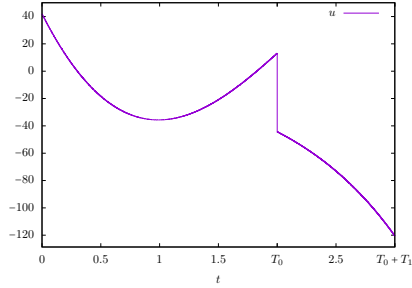
$$\|u\|_{L^2(T_0, T_0+T_1)}^2 = \int_{T_0}^{T_0+T_1} \langle M L \tilde{A} \tilde{x}(t), L \tilde{A} \tilde{x}(t) \rangle dt = 2\xi^2 (e^{2T_1} - 1),$$

with ξ given by (4.6).

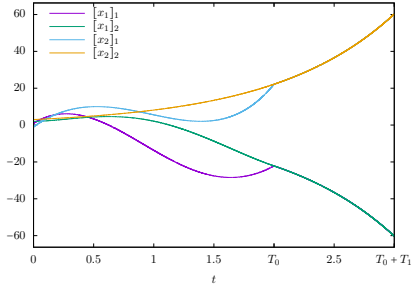
Numerical examples. Let us fix $T_0 = 2$ and $T_1 = 1$.

- Consider for instance $\tilde{x}^0 = (1, 2, -1, 3)^\top$ as initial condition for the system (4.3). Then we compute the control of minimal L^2 -norm, and obtain numerically $\|u\|_{L^2(0, T_0+T_1)} \simeq 86.39$. The corresponding state trajectories and control are displayed on Figure 1.

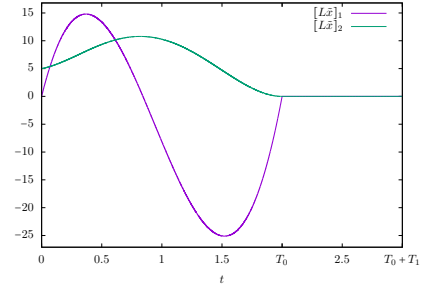
The system is first governed by the control steering it to the unique accessible point of the $\text{Ker } \mathcal{L}$. Reaching it at time T_0 , the feedback control (4.8) is turned on, resulting in a zero average for $t > T_0$ (Figure 1c). Recall that in order to keep the average constant, we have to remain within $\text{Ker } \mathcal{L}$ during the entire second time period. This is clearly visible from (Figure 1b).



(a) Control.



(b) State.



(c) Average of the state.

Figure 1: State trajectories for minimal L^2 -norm control such that $Lx(t) = 0$ for $t \in [T_0, T_0 + T_1]$. We considered $T_0 = 1$, $T_1 = 2$ and the system (4.3) with matrices given in Example 4.5 and initial conditions $x_1^0 = (1, 2)^\top$ and $x_2^0 = (-1, 3)^\top$.

- Consider for instance $\tilde{x}^0 = (1, 1, -1, -1)^\top$ as initial condition. Then we compute the control of minimal L^2 -norm, and obtain numerically $\|u\|_{L^2(0, T)} \simeq 28.36$. The corresponding state trajectories and control are displayed on Figure 2.

Notice that we have $Lx^0 = 0$ and $\mathcal{L}x^0 = 0$, but these properties are not ensured for every $t \in (0, T_0)$ (Figures 2b and 2c). This is natural having in mind that it requires less effort just to steer the system to the $\text{Ker } \mathcal{L}$ at some time instant, than to keep its state fixed within the same kernel all the time.

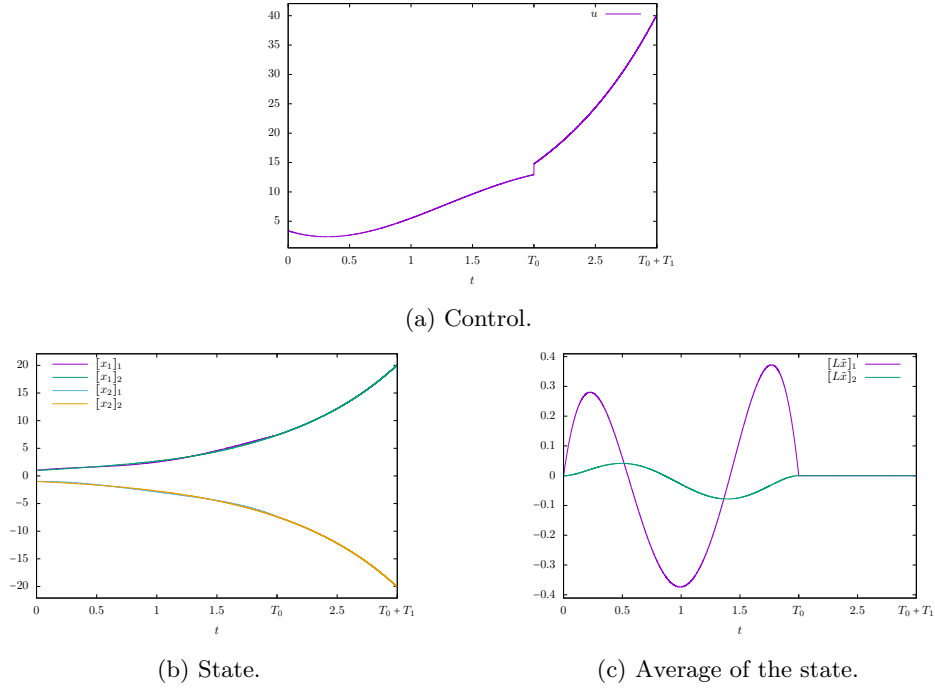


Figure 2: State trajectories for minimal L^2 -norm control such that $Lx(t) = 0$ for $t \in [T_0, T_0 + T_1]$. We considered $T_0 = 1$, $T_1 = 2$ and the system (4.3) with matrices given in Example 4.5 and initial conditions $x_1^0 = (1, 1)^\top$ and $x_2^0 = (-1, -1)^\top$.

Example 4.6. (Simultaneous controllability and $\text{Ker } \mathcal{L}$ is trivial)

For this example, we consider $A_2 = 2A_1$, and we compute

$$K = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -4 & 0 \\ 0 & 2 & 0 & -8 \end{pmatrix} \quad \text{and} \quad \mathcal{L} = \frac{1}{4} \begin{pmatrix} 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -3 \end{pmatrix}.$$

Hence, we have $\text{rank } K = 4$. This ensures that this system is simultaneously controllable, and hence averaged and long time averaged controllable.

It is easy to see that $\text{rank } \mathcal{L} = 4$ and hence, we have $\text{Ker } \mathcal{L} = \{0\}$. Thus, in order to control the average of the system for $t > T_0$, we first have to steer all the system components to zero. This task is feasible as the system satisfies the simultaneous control condition.

Let us now consider the norm optimal control problem. To this end, we set $T_0, T_1 > 0$, and we recall that we aim to minimise the L^2 -norm of the controls u such that the solution of the system (4.3) satisfies $L\tilde{x}(t) = 0$ for every $t \in [T_0, T_0 + T_1]$. We also recall that this last condition is equivalent to $\mathcal{L}\tilde{x}(t) = 0$ for every $t \in [T_0, T_0 + T_1]$. Since we have $\text{Ker } \mathcal{L} = \{0\}$ this ensures that we shall have $\tilde{x}(t) = 0$ for every $t \in [T_0, T_0 + T_1]$. This immediately gives the control $u(t) = 0$ for $t \in (T_0, T_0 + T_1)$. Consequently, in order to obtain the control of minimal L^2 -norm, we only need to compute the control of minimal L^2 -norm steering \tilde{x}^0 to 0 in time T_0 .

Minimal L^2 -norm control steering \tilde{x}^0 to 0 in time T_0 . In order to compute explicitly this L^2 -norm, we follow the lines of Section 3.1. According to (3.2) we deduce that the norm-optimal control u steering the system to zero in time T_0 is given by

$$u(t) = \tilde{B}^\top e^{(T_0-t)\tilde{A}^\top} \Gamma_{T_0}^{-1} \left(-e^{T_0\tilde{A}} \tilde{x}^0 \right)$$

and the corresponding L^2 -norm is,

$$\|u\|_{L^2(0,T_0)}^2 = \left(e^{T_0\tilde{A}} \tilde{x}^0 \right)^\top \Gamma_{T_0}^{-1} \left(e^{T_0\tilde{A}} \tilde{x}^0 \right).$$

Note that for this example, the Gramian matrix $\Gamma_{T_0} = \int_0^{T_0} e^{t\tilde{A}} \tilde{B} \tilde{B}^\top e^{t\tilde{A}^\top} dt$ can be explicitly computed, but its analytic expression is not displayed here.

Minimal L^2 -norm control in order to stay in the kernel of L . This part of the job is completely trivial since we start from 0 initial condition. Then the null control is the norm optimal control.

Numerical example. Let us fix $T_0 = 2$ and $T_1 = 1$ and take $\tilde{x}^0 = (1, 2, -1, 3)^\top$ as initial condition. Then we compute the control of minimal L^2 -norm, and obtain numerically, $\|u\|_{L^2(0,T_0)} \simeq 28.06$. The corresponding state trajectories and control are displayed on Figure 3.

We have an active control during the first time interval, steering the whole system, and not just its average, to zero in T_0 . This is because the supremal invariant subspace of the kernel of the averaged operator L is trivial. Afterwards, no control is required and the system remains at zero.

Example 4.7. (Simultaneous controllability and $\text{Ker } \mathcal{L}$ is not trivial)

For this example, we consider the matrix $A_2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and we have

$$K = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{L} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

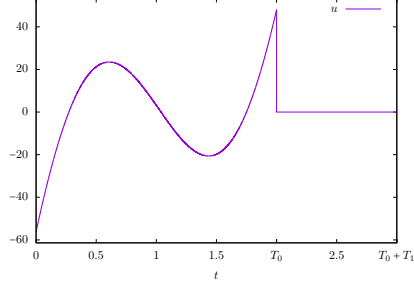
Hence, we have, $\text{rank } K = 4$. This ensures that this system is simultaneously controllable and therefore averaged and long time averaged controllable.

It is easy to see that $\text{rank } \mathcal{L} = 2$ and $\text{Ker } \mathcal{L} = \text{Ker } L = e_1\mathbb{R} \oplus e_2\mathbb{R}$, with $e_1 = (1, 0, -1, 0)^\top$ and $e_2 = (0, 1, 0, -1)^\top$ forming a basis of $\text{Ker } L$.

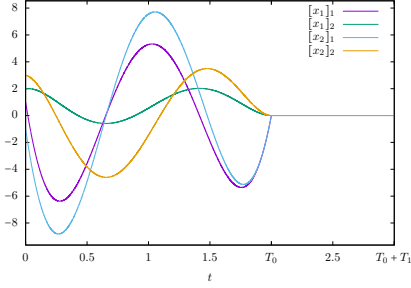
Let us now consider the norm optimal control problem. To this end, we set $T_0, T_1 > 0$, and we recall that we aim to minimise the L^2 -norm of the controls u such that the solution of the system (4.3) satisfies $L\tilde{x}(t) = 0$ for every $t \in [T_0, T_0 + T_1]$. We also recall that this last condition is equivalent to $\mathcal{L}\tilde{x}(t) = 0$ for every $t \in [T_0, T_0 + T_1]$. Let us set $\tilde{x}^1 = \tilde{x}(T_0) \in \mathbb{R}^4$ the reached point at time T_0 . In order to ensure long time average controllability, we shall have $\tilde{x}^1 \in \text{Ker } \mathcal{L}$ and hence,

$$x^1 = (\xi_1, \xi_2, -\xi_1, -\xi_2)^\top,$$

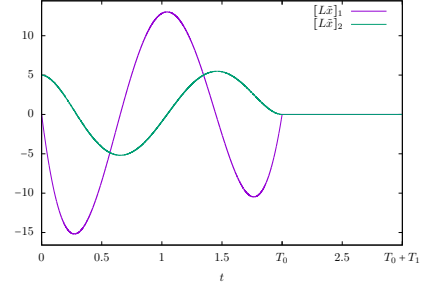
for some $\xi_1, \xi_2 \in \mathbb{R}$.



(a) Control.



(b) State.



(c) Average of the state.

Figure 3: State trajectories for minimal L^2 -norm control such that $Lx(t) = 0$ for $t \in [T_0, T_1]$. We considered $T_0 = 1$, $T_1 = 0.5$ and the system (4.3) with matrices given in Example 4.6 and initial conditions $x_1^0 = (1, 2)^\top$ and $x_2^0 = (-1, 3)^\top$.

Minimal L^2 -norm control steering \tilde{x}^0 to $\tilde{x}^1 \in \text{Ker } \mathcal{L}$ in time T_0 . In order to compute explicitly this L^2 -norm, we follow the lines of Section 3.1. Similarly to the previous examples, the Gramian matrix $\Gamma_{T_0} = \int_0^{T_0} e^{t\tilde{A}} \tilde{B} \tilde{B}^\top e^{t\tilde{A}^\top} dt$ can be explicitly computed (we skip again its analytic expression). Due to the simultaneously controllability of the system, Γ_{T_0} is a positive, invertible matrix

Since \tilde{x}^1 must be in the kernel of \mathcal{L} , we have $\tilde{x}^1 = E\xi^0$, where we have set $E = (e_1 \ e_2) \in M_{4,2}(\mathbb{R})$ and $\xi^0 = (\xi_1^0 \ \xi_2^0)^\top \in \mathbb{R}^2$. We then deduce that the norm-optimal control u steering x^0 to $\xi_1^0 e_1 + \xi_2^0 e_2$ in time T_0 is given by

$$u(t) = \tilde{B}^\top e^{(T_0-t)\tilde{A}^\top} \Gamma_{T_0}^{-1} (E\xi^0 - e^{T_0\tilde{A}} x^0)$$

and the corresponding L^2 -norm is,

$$\begin{aligned} \|u\|_{L^2(0, T_0)}^2 &= E^\top \Gamma_{T_0}^{-1} E \xi^0 \cdot \xi^0 - 2E^\top \Gamma_{T_0}^{-1} e^{T_0\tilde{A}} x^0 \cdot \xi^0 + e^{2T_0\tilde{A}} x^0 \cdot x^0 \\ &= \frac{1}{2} (\xi^0)^\top Q_0(T_0) \xi^0 + R_0(T_0, x_0)^\top \xi^0 + C_0(T_0, x_0), \end{aligned}$$

with $Q_0(T_0)$ a positive 2×2 matrix, $R_0(T_0) \in \mathbb{R}^2$ and $C_0(T_0) \in \mathbb{R}_+$.

Minimal L^2 -norm control in order to stay in the kernel of L . In order to compute this L^2 -norm, we follow the lines of Section 3.2. First of all, we pick $\tilde{x}^1 \in \text{Ker } \mathcal{L}$, and using the notations introduced in the above paragraph we have $\tilde{x}^1 = \xi_1^0 e_1 + \xi_2^0 e_1$, for some $\xi_1^0, \xi_2^0 \in \mathbb{R}$. We then aim to

find the minimal L^2 -norm control so that the solution of (4.3), with initial condition \tilde{x}^0 satisfies $L\tilde{x}(t) = 0$ for every $t \in [T_0, T_1]$.

Same as before, we have that $\text{Ker } L\tilde{B} = \{0\}$. Consequently, the control keeping the solution in the kernel of L is uniquely determined and takes the form

$$u(t) = (L\tilde{B})^\top v(t),$$

with $v(t) \in \text{Ran } L\tilde{B}$ solution of

$$-L\tilde{A}x(t) = (L\tilde{B})(L\tilde{B})^\top v(t).$$

We have, $L\tilde{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\text{Ran } L\tilde{B} = \mathbb{R} \times \{0\}$. Thus, for $M = \begin{pmatrix} 1/4 & 0 \\ 0 & 0 \end{pmatrix}$ it holds $M(L\tilde{B})(L\tilde{B})^\top w = w$ for every $w \in \text{Ran } L\tilde{B}$. We deduce that $v(t) = -ML\tilde{A}x(t)$ for every $t \in [0, T]$, where \tilde{x} is solution of

$$\dot{\tilde{x}} = (\tilde{A} - \tilde{B}(L\tilde{B})^\top ML\tilde{A})\tilde{x}, \quad \tilde{x}(0) = \xi_1^0 e_1 + \xi_2^0 e_2 \in \mathbb{R}^4.$$

Denoting

$$\mathbb{A} = (\tilde{A} - \tilde{B}(L\tilde{B})^\top ML\tilde{A}),$$

one can easily check that

$$\mathbb{A}e_1 = e_1/2 + e_2 \quad \text{and} \quad \mathbb{A}e_2 = -e_1.$$

Consequently, we have $x(t) = E\xi(t)$, where $\xi(t) = (\xi_1(t), \xi_2(t))^\top \in \mathbb{R}^2$ is solution of

$$\dot{\xi} = \mathcal{A}\xi, \quad \xi(0) = (\xi_1^0, \xi_2^0)^\top, \quad \text{with } \mathcal{A} = \begin{pmatrix} 1/2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus we deduce $x(t) = Ee^{t\mathcal{A}}\xi^0$.

The L^2 -norm of the control is given by:

$$\|u\|_{L^2(T_0, T_1)}^2 = \int_{T_0}^{T_1} \langle ML\tilde{A}\tilde{x}(t), L\tilde{A}\tilde{x}(t) \rangle dt.$$

Since $L\tilde{A}E = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, we have

$$\|u\|_{L^2(T_0, T_1)}^2 = e^{t\mathcal{A}^\top} M e^{t\mathcal{A}} \xi^0 \cdot \xi^0 = \frac{1}{2} Q_1(T_1, T_0) \xi^0 \cdot \xi^0,$$

with $Q_1(T_1)$ a non-negative 2×2 matrix.

Global minimisation problem. Given $T_0, T_1 > 0$, in order to find the minimal L^2 -norm control such that $L\tilde{x}(t) = 0$ for every $t \in [T_0, T_0 + T_1]$, we first have to find the point $\tilde{x}(T_0) \in \text{Ker } \mathcal{L}$ in which the kernel of \mathcal{L} is reached. Using the above notations, we have $\tilde{x}(T_0) = \xi_1^0 e_1 + \xi_2^0 e_2$ where $\xi^0 = (\xi_1^0, \xi_2^0) \in \mathbb{R}^2$ minimises

$$\xi^0 \mapsto \frac{1}{2} (Q_0(T_0) + Q_1(T_1, T_0)) \xi^0 \cdot \xi^0 + R(T_0; x^0)^\top \xi^0 + C_0(T_0, x^0). \quad (4.9)$$

That is to say that,

$$\xi^0 = -(Q_0(T_0) + Q_1(T_1, T_0))^{-1} R(T_0; x^0).$$

Note that since Q_0 is positive and Q_1 is non-negative the above expression is well-defined. Then the optimal control, trajectories and cost follows from the expressions obtained above.

Numerical example. Let us fix $T_0 = 2$ and $T_1 = 1$ and take $\tilde{x}^0 = (1, 2, -1, -2)^\top$ as initial condition. Then we compute the control of minimal L^2 -norm, and obtain numerically, $\|u\|_{L^2(0,T)} \simeq 1.94$. The corresponding state trajectories and control are displayed on Figure 4.

Notice that we have $L\tilde{x}^0 = 0$, but similarly as in Example 4.5 this property is not ensured for every $t \in (0, T_0)$ (Figures 4b and 4c). This is natural having in mind that we are using optimal controls of minimal norm, and it requires less effort just to steer the system to the Ker \mathcal{L} at some time instant, than to keep its state fixed within the same kernel all the time.

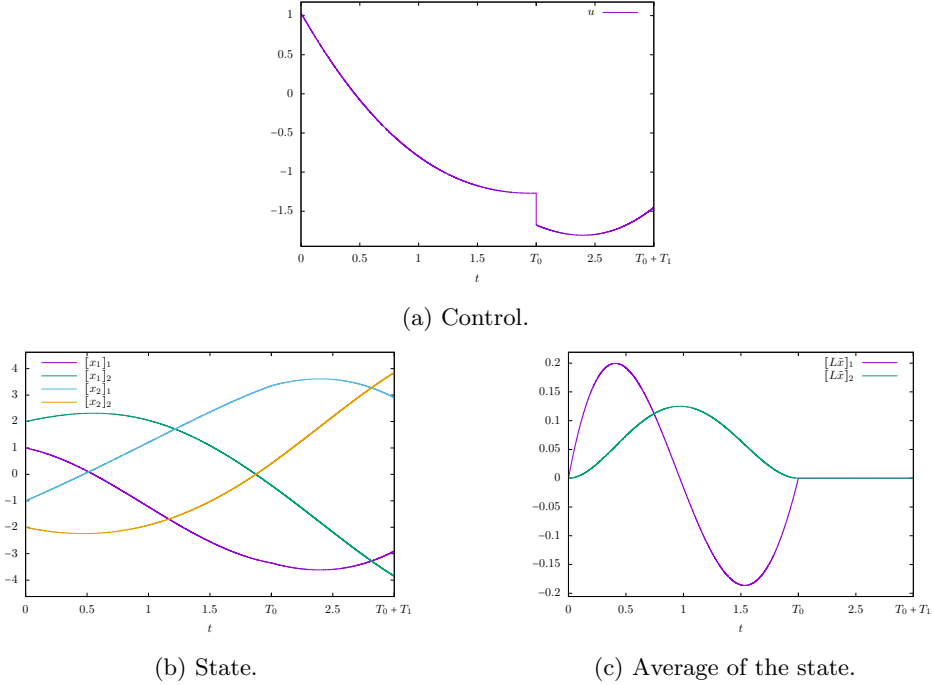


Figure 4: State trajectories for minimal L^2 -norm control such that $L\tilde{x}(t) = 0$ for $t \in [T_0, T_0 + T_1]$. We considered $T_0 = 1$, $T_1 = 2$ and the system (4.3) with matrices given in Example 4.7 and initial conditions $x_1^0 = (1, 2)^\top$ and $x_2^0 = (-1, -2)^\top$.

5 Concluding remarks and open questions

In this paper, we have extended the notion of averaged controllability to the notion of long time averaged controllability. We provide necessary and sufficient condition for this property to hold, as well as the explicit formula for the optimal control keeping the averaged fixed. Recall that simultaneous controllability implies long time averaged controllability, and that long time averaged controllability implies averaged controllability. The notion is also generalised to an arbitrary output operator.

Although we present a rather complete theory for the proposed problem, there are some interesting open questions arising from the obtained results.

Open questions:

- The results provided here deal with a finite number of parameters. The same question has to be addressed to the situation where an infinite number of parameters (either discrete or continuous) are involved. In this case, we consider a system $\dot{x}_\zeta(t) = A_\zeta x_\zeta(t) + B_\zeta u(t)$ with initial condition $x_\zeta(0) = x_\zeta^0$, where $\zeta \in \Omega$ is an unknown parameter and with $(\Omega, \mathcal{F}, \mu)$ a probability space. If we assume that $\int_\Omega x_\zeta^0 d\mu_\zeta = 0$ under which conditions does it exist a control u independent of ζ such that $\int_\Omega x_\zeta(t) d\mu_\zeta = 0$ for every $t > 0$?
- A similar question could have also been addressed for partial differential equations. In this case, the algebraic relation we used surely fails.
- There are many differences between averaged controllability and classical controllability. These differences can be addressed in the general framework of output controllability. More precisely, if we consider the control system $\dot{x} = Ax + Bu$, $y = Cx + Du$, this system is output controllable if and only if $\text{rank}(CB, CAB, \dots, CA^{n-1}B, D) = \text{rank} C$. In this article we point out the fact that for $D = 0$, the notion of output controllability is not strong enough to ensure that if $y(0) = 0$ then there exist a control such that $y(t) = 0$ for every $t > 0$. Some other points can be addressed.
 - If the system is output controllable, for every $x(0)$ and every $T > 0$, one can find a minimal L^2 -norm control u such that $y(T) = 0$. However, the behaviour of this L^2 -norm with respect to time is not clear at all.
 - Does output controllability imply output feed-back stabilisation? More precisely, if the system is output controllable, does it exist a feed-back control $u(t) = Ky(t)$ such that $y(t)$ goes to 0 as t goes to $+\infty$? Note that when the system is controllable, it is automatically stabilisable, with the help of the pole placement Theorem (see for instance [10, Theorem 2.1]).

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References

- [1] R. E. Kalman. Contributions to the theory of optimal control. *Bol. Soc. Mat. Mexicana* (2), 5:102–119, 1960.
- [2] E. Kreindler and P. E. Sarachik. On the concepts of controllability and observability of linear systems. *IEEE Trans. Automatic Control*, AC-9:129–136, 1964.
- [3] Martin Lazar and Enrique Zuazua. Contrôle et observation en moyenne d'équations des ondes dépendant de paramètres. *C. R., Math., Acad. Sci. Paris*, 352(6):497–502, 2014.
- [4] Jérôme Lohéac and Enrique Zuazua. From averaged to simultaneous controllability. *Ann. Fac. Sci. Toulouse, Math.* (6), 25(4):785–828, 2016.
- [5] Jérôme Lohéac and Enrique Zuazua. Averaged controllability of parameter dependent conservative semigroups. *J. Differ. Equations*, 262(3):1540–1574, 2017.
- [6] Qi Lü and Enrique Zuazua. Averaged controllability for random evolution Partial Differential Equations. *J. Math. Pures Appl.* (9), 105(3):367–414, 2016.

- [7] William A. Porter. On the matrix Riccati equation. *IEEE Trans. Automatic Control*, AC-12:746–749, 1967.
- [8] William T. Reid. A matrix differential equation of Riccati type. *Amer. J. Math.*, 68:237–246, 1946.
- [9] Emmanuel Trélat. *Contrôle optimal. Théorie et applications*. Paris: Vuibert, 2005.
- [10] W.Murray Wonham. Linear multivariable control. A geometric approach. 3rd ed. Applications of Mathematics, 10. New York etc.: Springer-Verlag. XVI, 334 p. , 1985.
- [11] Enrique Zuazua. Averaged control. *Automatica*, 50(12):3077–3087, 2014.