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Key polynomials for simple extensions of valued fields

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Abstract

In this paper we present a refined version of MacLane's theory of key polynomials [16]–[17], similar to those considered by M. Vaquié [24]–[27], and reminiscent of related objects studied by Abhyankar and Moh (approximate roots [1], [2]) and T.C. Kuo [14], [15].

Let (K, ν_0) be a valued field. Given a simple transcendental extension of valued fields $\iota : K \hookrightarrow K(x)$ we associate to ι a countable well ordered set of polynomials of $K[x]$ called key polynomials. We define limit key polynomials and give an explicit description of them. We show that the order type of the set of key polynomials is bounded by $\omega \times \omega$. If $\text{char } k_{\nu_0} = 0$ and $\text{rk } \nu_0 = 1$, the order type is bounded by $\omega + 1$.

1 Introduction

Let $\iota : K \hookrightarrow K(x)$ be a simple transcendental extension of valued fields, where K is equipped with a valuation ν_0 . That is, we assume given a valuation ν_0 of K and its extension ν to $K(x)$. Let $(R_{\nu_0}, M_{\nu_0}, k_{\nu_0})$ denote the valuation ring of ν_0 . The purpose of this paper is to present a refined version of MacLane's theory of key polynomials [16], [17], similar to those considered by M. Vaquié [24]–[27], and reminiscent of related objects studied by Abhyankar and Moh (approximate roots [1], [2]) and T.C. Kuo [14], [15]. Related questions were studied by Ron Brown [5]–[6], Alexandru–Popescu–Zaharescu [3]–[4], S. K. Khanduja [9], [10], F.-V. Kuhlmann [11] and Moyls [19].

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Precisely, we associate to ι a countable well ordered set

$$\mathbf{Q} = \{Q_i\}_{i \in \Lambda} \subset K[x];$$

the Q_i are called **key polynomials**. Key polynomials Q_i that have no immediate predecessor are called **limit key polynomials**. Let $\beta_i = \nu(Q_i)$. In the case when $\text{rk } \nu_0 = 1$ we give an explicit description of the limit key polynomials. The first limit key polynomial can always be chosen to be a p -polynomial in Kaplansky's terminology (p -polynomials may be viewed as a generalization of the Artin–Schreier polynomials). We also give an upper bound on the order type of the set of key polynomials. Namely, we show that the order type of the set of key polynomials is bounded by $\omega \times \omega$, where ω stands for the first infinite ordinal. If $\text{char } k_{\nu_0} = 0$ and $\text{rk } \nu_0 = 1$, the order type is bounded by $\omega + 1$. If $\text{char } k_{\nu_0} = 0$ and $\text{rk } \nu = 1$, the set of key polynomials has order type at most ω .

Let Γ_0 (resp. Γ) denote the value group of ν_0 (resp. ν). Let $\tilde{\Gamma} := \Gamma_0 \otimes_{\mathbb{Z}} \mathbb{Q}$. In the special case when $\text{rk } \nu_0 = 1$ we will fix an embedding $\tilde{\Gamma} \hookrightarrow \mathbb{R}$. In this sense, we will be able to talk about the supremum of a certain subset of $\tilde{\Gamma}$ (the supremum can be either a real number or infinity) or about a certain sequence of elements of $\tilde{\Gamma}$ tending to infinity.

The main application of the theory of key polynomials that we have in mind is proving the local uniformization theorem for quasi-excellent noetherian schemes in positive and mixed characteristic. In Chapter IV of his Ph.D. thesis ([22], Institut de Mathématiques de Toulouse, 2013) J.-C. San Saturnino reduced the local uniformization theorem in the case of positive characteristic to the problem of the monomialization of the first limit key polynomial of a certain explicitly defined simple field extension $K \hookrightarrow K(x)$ assuming local uniformization in lower dimensions (see [23], Theorem 6.5). In Chapter V he proved a similar reduction for local uniformization in the case of mixed characteristic, but under some additional hypotheses.

Chapter 3 of the Ph.D. thesis of W. Mahboub (Institut de Mathématiques de Toulouse, 2013) develops the theory of key polynomials for valuations of arbitrary rank. Here we mostly work with valuations of arbitrary rank but limit ourselves to the case $\text{rk } \nu_0 = 1$ towards the end of the paper for some (but not all) of the finer results on the shape of limit key polynomials.

The particular importance of the case $\text{rk } \nu_0 = 1$ is witnessed by a recent theorem of Novacoski–Spivakovsky that says that local uniformization along rank one valuations implies local uniformization in its full generality [20]–[21].

The well ordered set $\mathbf{Q} = \{Q_i\}_{i \in \Lambda}$ of key polynomials of ν will be defined recursively in i .

Notation. We will use the notation \mathbb{N} for the set of strictly positive integers and \mathbb{N}_0 for the set of non-negative integers.

For an element $\ell \in \Lambda$, we will denote by $\ell + 1$ the immediate successor of ℓ in Λ . The immediate predecessor of ℓ , when it exists, will be denoted by $\ell - 1$. For a positive integer t , $\ell + t$ will denote the immediate successor of $\ell + (t - 1)$. For an element $\ell \in \Lambda$, the initial segment $\{Q_i\}_{i < \ell}$ of the set of key polynomials will be denoted by \mathbf{Q}_ℓ . For the rest of this paper, we let $p = \text{char } k_{\nu_0}$ if $\text{char } k_{\nu_0} > 0$ and $p = 1$ if $\text{char } k_{\nu_0} = 0$. For an element $\beta \in \Gamma \cup \tilde{\Gamma}$, let

$$\begin{aligned} \mathbf{P}_\beta &= \{y \in K(x) \mid \nu(y) \geq \beta\} \cup \{0\} \\ \mathbf{P}_{\beta+} &= \{y \in K(x) \mid \nu(y) > \beta\} \cup \{0\}. \end{aligned}$$

Put $G_\nu = \bigoplus_{\beta \in \Gamma} \frac{\mathbf{P}_\beta}{\mathbf{P}_{\beta+}}$ and $G_{\nu_1} = \bigoplus_{\beta \in \tilde{\Gamma}} \frac{\mathbf{P}_\beta}{\mathbf{P}_{\beta+}}$. We regard G_ν and G_{ν_1} as k_ν -algebras.

Let Δx be an independent variable. For $f \in K[x]$ and $j \in \mathbb{N}$ let $\partial_j f$ denote the j -th **formal derivative** of f with respect to x . The polynomials $\partial_j f$ are, by definition, the

coefficients appearing in the Taylor expansion of $f: f(x + \Delta x) = \sum_j \partial_j f \Delta x^j$. In papers on local uniformization the formal derivatives ∂_j are often denoted by $\frac{1}{j!} \frac{\partial^j}{\partial x^j}$; this notation is regarded as one indivisible symbol; its parts such as $\frac{1}{j!}$ do not make sense on their own.

Details about formal derivatives can be found in [12], Chapter 24.10, starting on p. 701, as well as in [13].

A set of key polynomials is said to be **complete** if for every $\beta \in \Gamma$ the additive group $\mathbf{P}_\beta \cap K[x]$ is generated by the products of the form $a \prod_{j=1}^s Q_{i_j}^{\gamma_j}$, where $a \in K$, $i_j \in \Lambda$ and $\gamma_j \in \mathbb{N}$, contained in $\mathbf{P}_\beta \cap K[x]$. It is said to be $\tilde{\Gamma}$ -**complete** if the above condition holds for all $\beta \in \tilde{\Gamma}$, in other words, if for all $\beta \in \tilde{\Gamma}$ every polynomial $f \in K[x]$ with $\nu(f) = \beta$ belongs to the additive group generated by products of the form $a \prod_{j=1}^s Q_{i_j}^{\gamma_j}$, $a \in K$, such that $\sum_{j=1}^s \gamma_j \nu(Q_{i_j}) + \nu_0(a) \geq \beta$.

Remark 1.1. If $\mathbf{Q} = \{Q_i\}_{i \in \Lambda}$ is a complete set of key polynomials, the data $\{Q_i, \nu(Q_i)\}$ completely determines the ideals \mathbf{P}_β for all $\beta \in \Gamma$, hence also all the ideals $\mathbf{P}_{\beta+}$, since $\mathbf{P}_{\beta+} = \bigcup_{\tilde{\beta} > \beta} \mathbf{P}_{\tilde{\beta}}$.

For an element $y \in K(x)$ we have $\nu(y) = \beta$ if and only if $y \in \mathbf{P}_\beta \setminus \mathbf{P}_{\beta+}$. Thus the valuation ν is completely determined by the data $\{Q_i, \nu(Q_i)\}$.

We will use the following multi-index notation: $\bar{\gamma}_{\ell+1} = \{\gamma_i\}_{i \leq \ell}$, where the γ_i are non-negative integers, all but finitely many of which are equal to 0, and

$$\mathbf{Q}_{\ell+1}^{\bar{\gamma}_{\ell+1}} = \prod_{i \leq \ell} Q_i^{\gamma_i}. \quad (1.1)$$

An ℓ -**standard monomial** in $\mathbf{Q}_{\ell+1}$ is a product of the form

$$c_{\bar{\gamma}_{\ell+1}} \mathbf{Q}_{\ell+1}^{\bar{\gamma}_{\ell+1}}, \quad (1.2)$$

where $c_{\bar{\gamma}_{\ell+1}} \in K$ and the multiindex $\bar{\gamma}_{\ell+1}$ satisfies certain additional conditions to ensure a form of uniqueness (see Definition 3.10). An ℓ -**standard expansion** is a finite sum of ℓ -standard monomials. In §3 we will show that for any $\ell \in \Lambda$ and any $f \in K[x]$ the element f admits an ℓ -standard expansion

$$f = \sum_{j=0}^{s_\ell} c_{j\ell} Q_\ell^j, \quad (1.3)$$

where each $c_{j\ell}$ is a Q_ℓ -free ℓ -standard expansion. We define the ℓ -**truncation** ν_ℓ of ν by putting $\nu_\ell(f) = \min_{0 \leq j \leq s_\ell} \{\nu(c_{j\ell}) + j\beta_\ell\}$ for each $f \in K[x]$. By the axioms for valuations, we have

$$\nu(f) \geq \nu_\ell(f) \quad (1.4)$$

for all $f \in K[x]$. Then the statement that \mathbf{Q} is a complete set of key polynomials can be expressed as follows: for any $f \in K[x]$ there exists $\ell \in \Lambda$ such that equality holds in (1.4); see Remark 3.21 for details.

The paper is organized as follows. §2 is devoted to generalities on algebras, graded by ordered semigroups. There we define the notion of the saturation G^* of a graded algebra G (Definition 2.3). We consider an extension $G \subset G'$ graded algebras and a homogeneous element $x \in G'$. We study the condition that x be algebraic over G . We note that x is algebraic over G if and only if it is integral over G^* . We show that if x is algebraic over G then the algebra

$G^*[x]$ is saturated (Lemma 2.6). Finally, we prove the simple but useful characterization of the strict inequality $\nu\left(\sum_{i=1}^s y_i\right) > \min_{1 \leq i \leq s} \{\nu(y_i)\}$ in terms of the natural images of the elements y_i in the graded algebra G_ν .

In §3 we will construct a well ordered set \mathbf{Q} of key polynomials and prove that this set is complete.

We start our recursive construction of the Q_i by putting $Q_1 := x$. We fix an ordinal $\ell < \omega \times \omega$ and assume, inductively, that the key polynomials $\mathbf{Q}_{\ell+1}$ are already defined. We will then define the next key polynomial $Q_{\ell+1}$. Roughly speaking, $Q_{\ell+1}$ will be defined to be a lifting to $K[x]$ of the monic minimal polynomial, satisfied by $\text{in}_\nu Q_\ell$ over the graded algebra $G_{\nu_0}[\text{in}_\nu \mathbf{Q}_\ell]$, where $\text{in}_\nu Q_\ell$ denotes the natural image of Q_ℓ in G_ν . This gives rise to explicit formulae describing each *non-limit* key polynomial in terms of the preceding key polynomials.

If at some point of the construction we arrive at the situation when $\nu_\ell(f) = \nu(f)$ for all $f \in K[x]$, the algorithm stops: the resulting set $\mathbf{Q}_{\ell+1}$ of key polynomials is already complete by Remark 3.21. In particular, this occurs whenever our algorithm produces a key polynomial whose value does not lie in $\tilde{\Gamma}$. At this stage, starting with \mathbf{Q}_ℓ we will have recursively constructed a set of key polynomials $\{Q_i\}_{i < \ell + \omega}$ of key polynomials of order type at most $\ell + \omega$. This will already complete the construction of \mathbf{Q} in the case when for all $f \in K[x]$ there exists $i \in \mathbb{N}_0$ such that

$$\nu(f) = \nu_{\ell+i}(f). \quad (1.5)$$

However, it may happen that for each $i \in \mathbb{N}$ there exists $f \in K[x]$ satisfying $\nu(f) > \nu_{\ell+i}(f)$ for all $i \in \mathbb{N}_0$. The **limit key polynomial** $Q_{\ell+\omega}$ is defined to be a monic such f of smallest degree.

This completes the recursive definition of the set $\{Q_i\}$ of key polynomials. Then we prove the main property: they form a complete set of key polynomials (Theorem 3.35). As a Corollary we will obtain that the images of the key polynomials in G_ν generate the field of fractions of G_ν over the field of fractions of G_{ν_0} .

The main goal of §§4–8 is to derive explicit formulae for limit key polynomials in terms of the preceding key polynomials. For some (though not all) of the results of §5 and §§7–8 we need to assume that $\text{rk } \nu_0 = 1$.

The breakdown of the contents among these remaining five section goes as follows.

§4 and §6 are auxiliary: there we develop certain technical tools (the integer numerical character $\delta_i(f)$, $i \in \Lambda$, $f \in K[x]$, and differential operators, respectively) to be used in §5 and §§7–8.

In §4 we associate to each $h \in K[x]$ and each ordinal i for which Q_i is defined, a positive integer numerical character $\delta_i(h) \leq \frac{\deg_x h}{\deg_x Q_i}$ of the Newton polygon $\Delta_i(h)$. We prove that $\delta_i(h)$ is non-increasing with i , and hence must stabilize for $i \gg 0$. We also show that the equality

$$\delta_i(h) = \delta_{i+1}(h)$$

imposes strong restrictions on h . The numerical character $\delta_i(h)$ helps analyze infinite ascending sequences $\{Q_i\}_{i \in \Lambda}$ in §5 and §§7–8.

In §6 we study the effect of differential operators ∂_j on key polynomials and standard expansions. Let j denote the element of \mathbb{N} which maximizes the quantity $\frac{\beta_i - \nu(\partial_j Q_i)}{j}$. We show that j is of the form

$$j = p^{e_i} \quad \text{for some } e_i \in \mathbb{N}_0. \quad (1.6)$$

The non-negative integers e_i , $i \in \Lambda$, are important numerical characters of the extension ι of valued fields. Most importantly, given an ℓ -standard monomial $c_{\tilde{\gamma}_{\ell+1}} \mathbf{Q}_{\ell+1}^{\tilde{\gamma}_{\ell+1}}$, we prove the equality

$$\nu\left(\partial_{p^b c_{\tilde{\gamma}_{\ell+1}}} \mathbf{Q}_{\ell+1}^{\tilde{\gamma}_{\ell+1}}\right) = \nu_\ell\left(\partial_{p^b c_{\tilde{\gamma}_{\ell+1}}} \mathbf{Q}_{\ell+1}^{\tilde{\gamma}_{\ell+1}}\right),$$

and we derive an explicit formula for the quantity $\nu\left(\partial_{p^b} c_{\tilde{\gamma}_{\ell+1}} \mathbf{Q}_{\ell+1}^{\tilde{\gamma}_{\ell+1}}\right)$, for integers $b \geq e_i$, and under certain additional conditions. Also, for any ℓ -standard expansion f and an integer $b \geq e_i$, we derive a formula for $\nu_\ell(\partial_{p^b} f)$.

The importance of this type of explicit formulae can be explained as follows. The importance of differential operators for resolution of singularities is well known. One difficulty with dealing with differential operators up to now has been the fact that they obey no simple transformation law under blowing up. Since key polynomials become coordinates after blowing up, the above formulae can be viewed as comparison results for derivatives of the defining equations of a singularity before and after blowing up.

The main subject of study of §5 and §§7–8 are infinite sequences $\{Q_{\ell+t}\}_{t \in \mathbb{N}}$ of key polynomials and the corresponding limit key polynomials $Q_{\ell+\omega}$.

In §5 we assume that the sequence $\{\nu(Q_{\ell+t})\}_{t \in \mathbb{N}}$ is bounded in $\tilde{\Gamma}$ (this situation is referred to as Case 2b in §3). We also assume that $\text{rk } \nu_0 = 1$. We let δ denote the stable value of $\delta_{\ell+t}(Q_{\ell+\omega})$ for a sufficiently large positive integer t . We show that $\deg_x Q_{\ell+\omega} = \delta \deg_x Q_{\ell+t}$.

In §7 we use the results of §6 to show that the stable value $\delta_i(f)$ must be of the form $\delta_i(f) = p^e$ for some $e \in \mathbb{N}_0$.

Next, we assume that $\text{char } k_\nu = \text{char } K$, that the sequence $\{\nu(Q_t)\}_{t \in \mathbb{N}}$ is unbounded in $\tilde{\Gamma}$ (this is Case 2a of §3, where we take $\ell = 1$) and that $\deg_x Q_t = 1$ for all $t \in \mathbb{N}$. We show that $Q_{\ell+\omega} \in K[x^\delta]$.

For the third main results of §7 we assume that $\text{rk } \nu_0 = 1$. Proposition 7.9 asserts that if $\text{char } k_\nu = 0$ then there are no limit key polynomials Q_i such that $\beta_i \in \tilde{\Gamma}$. By definition, we have $\beta_i \in \tilde{\Gamma}$ whenever i is not a maximal element of Λ . The set Λ contains a maximal element λ if and only if it contains an element λ such that $\text{in}_\nu Q_\lambda$ is transcendental over $k_\nu[\text{in}_\nu \mathbf{Q}_\lambda]$, where $\mathbf{Q}_\lambda = \{Q_i\}_{i < \lambda}$. This explains why $\Lambda \leq \omega + 1$ whenever $\text{rk } \nu_0 = 1$ and $\text{char } k_\nu = 0$.

In §8 we assume that $\text{char } k_\nu = p > 0$ and consider an ordinal $\ell + \omega$ that does not have an immediate predecessor. We assume that the key polynomials $\mathbf{Q}_{\ell+t}$ and the limit key polynomial $Q_{\ell+\omega}$ are defined. We assume that $\text{rk } \nu_0 = 1$ and that we are in Case 2b, that is, the sequence $\{\nu(Q_{\ell+t})\}_{t \in \mathbb{N}}$ is bounded in $\tilde{\Gamma}$. We prove that $Q_{\ell+\omega}$ can be chosen in such a way that for some $t \in \mathbb{N}$ the $Q_{\ell+t}$ -standard expansion of $Q_{\ell+\omega}$ is weakly affine. This means, by definition, that, letting $i = \ell + t$, we have

$$Q_{\ell+\omega} = Q_i^{p^{e_{\ell+\omega}}} + \sum_{j=0}^{e_{\ell+\omega}-1} c_{p^j i} Q_i^{p^j} + c_{0i}, \quad (1.7)$$

where c_{0i} and $c_{p^j i}$ are Q_i -free i -standard expansions.

The results of this paper are related to those contained in the paper [7] (see also [27]). However, there are some important differences, which we now explain. We chose to rewrite the whole theory from scratch for several reasons:

1. In [7] we work with an algebraic extension ι while for local uniformization we need to consider purely transcendental extensions. We note that the case of algebraic extensions can easily be reduced to that of transcendental ones using composition of valuations. Indeed, let $\iota_- : (K, \nu_0) \hookrightarrow (K(x), \nu)$ be a simple algebraic extension of valued fields. Write $K(x) = \frac{K[X]}{(f)}$, where f is the minimal polynomial of x over K . Let ν_f denote the (f) -adic valuation of $K[X]$ and put $\nu^* := \nu_f \circ \nu$ (the composition of ν_f with ν). Complete sets $\{Q_i\}_{i \in \Lambda}$ of key polynomials of the transcendental extension $\iota : (K, \nu_0) \rightarrow (K(X), \nu^*)$ constructed in the present paper are very closely related to complete sets $\{Q_i^-\}_{i \in \Lambda_-}$ of key polynomials of the algebraic extension $(K, \nu_0) \rightarrow (K(x), \nu)$, constructed in [7]. Namely,

we have $\Lambda = \Lambda_- \cup \{\Lambda_-\}$ (extension by one element), Q_i^- is the image of Q_i under the natural map $K[X] \rightarrow K(x)$ and $Q_{\Lambda_-} = f$. In other words, a complete set $\{Q_i\}$ of key polynomials for ι can be obtained from that of ι_- by lifting each key polynomial Q_i^- to $K[X]$ and then adding one final key polynomial f . In this sense the theory presented here can be viewed as a generalization of [7].

2. Our main interest in [7] was to classify all the possible extensions ν of a given ν_0 ; in the present paper we content ourselves with a fixed ν .
3. The crucial formulae for $\nu_\ell(\partial_{p^b} f)$ were not made explicit in [7].
4. We take this opportunity to correct numerous mistakes which, unfortunately, made their way into the paper [7]: an inaccuracy in the definition of complete set of key polynomials, the failure to take into account the case of mixed characteristic, a mistake in the definition of the numerical characters e_i and many others which made the paper [7] unreadable.
5. In [7] we make a blanket assumption that $\text{rk } \nu_0 = 1$ whereas in the present paper ν_0 is of arbitrary rank except for some results about the precise form of limite key polynomials at the end of the paper.

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2 Algebras graded by ordered semigroups

Graded algebras associated to valuations will play a crucial role in this paper. In this section, we give some basic definitions and prove several easy results about graded algebras. Throughout this paper, a “graded algebra” will mean “an algebra without zero divisors, graded by an ordered semigroup”. As usual, for a graded algebra G , ord will denote the natural valuation of G , given by the grading.

Let $G = \bigoplus_{\alpha \in \Gamma} G_\alpha$ be a graded algebra where Γ is an ordered abelian group.

Definition 2.1. An element $x \in G$ is said to be **homogeneous** if there exists $\alpha \in \Gamma$ such that $x \in G_\alpha$.

For a homogeneous element $x \in G_\alpha \subset G$ we will write $\text{ord } x = \alpha$.

Now let X be an independent variable and consider the ring $G[X]$. Fix a polynomial $f = \sum_{i=0}^d a_i X^i \in G[x]$ such that a_i is a homogeneous element of G for all $i \in \{0, \dots, d\}$. Fix an element $\beta \in \Gamma$.

Definition 2.2. We say that f is **quasi-homogeneous with** $w(X) = \beta$ if for all $i, j \in \{0, \dots, d\}$ we have $i\beta + \text{ord } a_i = j\beta + \text{ord } a_j$. In this situation we will also say that β is the **weight assigned to** X .

Definition 2.3. Let G be a graded algebra without zero divisors. The **saturation** of G , denoted by G^* , is the graded algebra

$$G^* = \left\{ \frac{g}{h} \mid g, h \in G, h \text{ homogeneous, } h \neq 0 \right\}.$$

G is said to be **saturated** if $G = G^*$.

Of course, we have $G^* = (G^*)^*$ for any graded algebra G , so that G^* is always saturated.

Example 2.4. Let R be a ring and $\nu : R \rightarrow \Gamma \cup \{\infty\}$ a valuation, centered at a prime ideal of R . Let $\Phi = \nu(R \setminus \{0\})$. For each $\alpha \in \Phi$, consider the ideals

$$\begin{aligned} I_\alpha &:= \{x \in R \mid \nu(x) \geq \alpha\} \quad \text{and} \\ I_{\alpha+} &:= \{x \in R \mid \nu(x) > \alpha\}. \end{aligned} \tag{2.1}$$

I_α is called the ν -**ideal** of R of value α .

If $\alpha_1 > \alpha_2 > \dots$ is an infinite descending sequence of elements of Φ then $I_{\alpha_1} \subsetneq I_{\alpha_2} \subsetneq \dots$ is an infinite ascending chain of ideals of R . Thus if R is noetherian then the ordered set $\nu(R)$ contains no infinite descending sequences, that is, $\nu(R)$ is well ordered.

If I is an ideal in a noetherian ring R and ν a valuation of R , $\nu(I)$ will denote

$$\min\{\nu(x) \mid x \in I\}.$$

We now define certain natural graded algebras, associated to valuations. Let R , ν and Φ be as above. For $\alpha \in \Phi$, let I_α and $I_{\alpha+}$ be as in (2.1). We define

$$\text{gr}_\nu R = \bigoplus_{\alpha \in \Phi} \frac{I_\alpha}{I_{\alpha+}}.$$

The algebra $\text{gr}_\nu R$ is an integral domain. For any element $x \in R$ with $\nu(x) = \alpha$, we may consider the natural image of x in $\frac{I_\alpha}{I_{\alpha+}} \subset \text{gr}_\nu R$. This image is a homogeneous element of $\text{gr}_\nu R$ of degree α , which we will denote by $\text{in}_\nu x$. The grading induces an obvious valuation on $\text{gr}_\nu R$ with values in Φ ; this valuation will be denoted by ord .

Next, suppose that (R, M, k) is a local domain and ν is a valuation with value group Γ , centered at R . Let K denote the field of fractions of R . Let (R_ν, M_ν, k_ν) denote the valuation ring of ν . For $\alpha \in \Gamma$, consider the following R_ν -submodules of K :

$$\begin{aligned} \mathbf{I}_\alpha &= \{x \in K \mid \nu(x) \geq \alpha\}, \\ \mathbf{I}_{\alpha+} &= \{x \in K \mid \nu(x) > \alpha\}. \end{aligned} \tag{2.2}$$

We define

$$G_\nu = \bigoplus_{\alpha \in \Gamma} \frac{\mathbf{I}_\alpha}{\mathbf{I}_{\alpha+}}.$$

Again, given $x \in K$, we may speak of the natural image of x in G_ν , also denoted by $\text{in}_\nu x$ (since $\text{gr}_\nu R$ is naturally a graded subalgebra of G_ν , there is no danger of confusion). Then ord is a valuation of the common field of fractions of $\text{gr}_\nu R$ and G_ν , with values in Γ .

We have $G_\nu = (\text{gr}_\nu R)^*$; in particular, G_ν is saturated.

Remark 2.5. Let G, G' be two graded algebras without zero divisors, with $G \subset G'$. Let x be a homogeneous element of G' , satisfying an algebraic dependence relation

$$a_0 x^\alpha + a_1 x^{\alpha-1} + \dots + a_\alpha = 0 \tag{2.3}$$

over G (here $a_j \in G$ for $0 \leq j \leq \alpha$). Without loss of generality, we may assume that the integer α is the smallest possible.

Claim. Without loss of generality, we may further assume that (2.3) is homogeneous (that is, all the a_j are homogeneous and the quantity $j \text{ord } x + \text{ord } a_j$ is constant for $0 \leq j \leq \alpha$ such that $a_j \neq 0$).

Proof of Claim. Let $\mu := \min_{0 \leq j \leq \alpha} \{j \text{ ord } x + \text{ord } a_j\}$. Then each a_j can be written as a finite sum of homogeneous elements of G , all of orders greater than or equal to $\mu - j \text{ ord } x$. For $j \in \{0, \dots, \alpha\}$ write $a_j = a_{0j} + \tilde{a}_j$, where $\text{ord } a_{0j} = \mu - j \text{ ord } x$ and \tilde{a}_j is a sum of homogeneous elements of G of orders strictly greater than $\mu - j \text{ ord } x$ (for some j it may happen that $a_{0j} = 0$, but there exist at least two different values of j for which $a_{0j} \neq 0$). Now, x satisfies the equation

$$a_{00}x^\alpha + a_{10}x^{\alpha-1} + \dots + a_{0\alpha} = 0. \quad (2.4)$$

This proves the Claim. From now on we will always take the coefficients a_j to be homogeneous without mentioning it explicitly.

Dividing (2.3) by a_0 , we see that x satisfies a *monic* homogeneous relation over G^* of degree α and no algebraic relation of degree less than α . In other words, x is *algebraic* over G if and only if it is *integral* over G^* ; the conditions of being “algebraic over G^* ” and “integral over G^* ” are one and the same thing (as usual, “integral” means “satisfying a monic polynomial”).

Let $G \subset G'$, x be as above and let $G[x]$ denote the graded subalgebra of G' , generated by x over G . By Remark 2.5, we may assume that x satisfies a homogeneous integral relation

$$x^\alpha + a_1x^{\alpha-1} + \dots + a_\alpha = 0 \quad (2.5)$$

over G^* and no algebraic relations over G^* of degree less than α .

Lemma 2.6. *Every element of $(G[x])^*$ can be written uniquely as a polynomial in x with coefficients in G^* , of degree strictly less than α .*

Proof. Let y be a homogeneous element of $G[x]$. Since x is integral over G^* , so is y . Let

$$y^\gamma + b_1y^{\gamma-1} + \dots + b_\gamma = 0 \quad (2.6)$$

with $b_j \in G^*$, be an integral dependence relation of y over G^* , with b_j homogeneous elements of G^* , $b_\gamma \neq 0$, such that $j \text{ ord } y + \text{ord } b_j$ is constant for all j such that $b_j \neq 0$. By (2.6),

$$\frac{1}{y} = -\frac{1}{b_\gamma}(y^{\gamma-1} + b_1y^{\gamma-2} + \dots + b_{\gamma-1}).$$

Thus, for any $z \in G[x]$, we have

$$\frac{z}{y} \in G^*[x]. \quad (2.7)$$

Since y was an arbitrary homogeneous element of $G[x]$, we have proved that

$$(G[x])^* = G^*[x].$$

Now, for every element $y \in G^*[x]$ we can add a multiple of (2.5) to y so as to express y as a polynomial in x of degree less than α . Moreover, this expression is unique because x does not satisfy any algebraic relation over G^* of degree less than α . \square

Notation. If $\Delta \subset \Delta'$ are ordered semigroups and β is an element of Δ' , then $\Delta : \beta$ will denote the positive integer defined by

$$\Delta : \beta = \min\{\alpha \in \mathbb{N} \mid \alpha\beta \in \Delta\}.$$

If the set on the right hand side is empty, we take $\Delta : \beta = \infty$. (Note that $\beta \in \Delta$ if and only if $\Delta : \beta = 1$).

Lemma 2.7. *Let G, G' be as in Remark 2.5 and x a homogeneous element of G' . Assume that the degree 0 part of G (that is, the subring of G consisting of all the elements of degree 0) contains a field k and that G is generated as a k -algebra by homogeneous elements w_1, \dots, w_r . Let*

$$\beta_j = \text{ord } w_j, \quad 1 \leq j \leq r,$$

and let Δ denote the group $\Delta = \{\text{ord } y \mid y \in G^*\} = \left\{ \sum_{j=1}^r \alpha_j \beta_j \mid \alpha_j \in \mathbb{Z} \right\}$. Assume that the following two conditions hold:

- (1) $\Delta : (\text{ord } x) < \infty$
- (2) Let $\tilde{\alpha} := \Delta : \text{ord } x$. Let $\alpha_1, \dots, \alpha_r \in \mathbb{Z}$ be such that

$$\tilde{\alpha} \text{ ord } x = \sum_{j=1}^r \alpha_j \beta_j. \quad (2.8)$$

Let $y = \prod_{j=1}^r w_j^{\alpha_j}$. Assume that the element

$$z := \frac{x^{\tilde{\alpha}}}{y} \in (G')^* \quad (2.9)$$

is algebraic over k .

Then x is integral over G^* . An integral dependence relation of x over G^* can be described as follows. Let z be as in (2.9). Let Z be an independent variable and let

$$\lambda(Z) = Z^d + \sum_{i=0}^{d-1} c_i Z^i \quad (2.10)$$

denote the minimal polynomial of z over k . Then x satisfies the integral dependence relation

$$x^{d\tilde{\alpha}} + \sum_{i=0}^{d-1} c_i y^{d-i} x^{i\tilde{\alpha}} = 0. \quad (2.11)$$

Conversely, suppose x is integral over G^* . Then (1) holds. Suppose, furthermore, that β_1, \dots, β_r are \mathbb{Z} -linearly independent. Then (2) also holds. In this case, (2.11) is the minimal polynomial of x over G^* . In particular, the degree α of the minimal polynomial of x over G^* is given by

$$\alpha = d\tilde{\alpha}. \quad (2.12)$$

Proof. If (1) and (2) hold, x is integral over G^* because it satisfies the integral dependence relation (2.11) (this is verified immediately by substituting (2.9) for Z in (2.10) and multiplying through by y^d). In particular, if α denotes the degree of x over G^* , (2.11) shows that

$$\alpha \leq d\tilde{\alpha}. \quad (2.13)$$

Conversely, suppose x is integral over G^* . Then x satisfies a homogeneous integral relation of the form (2.5). Since (2.5) is homogeneous, we have the equality

$$i \text{ ord } x + \text{ord } a_i = j \text{ ord } x + \text{ord } a_j \text{ for some } i, j \text{ such that } 0 \leq i < j \leq \alpha.$$

Hence

$$(j - i)\text{ord } x = \text{ord } a_i - \text{ord } a_j. \quad (2.14)$$

Now, $\frac{a_i}{a_j} \in G^*$ so that

$$\text{ord } a_i - \text{ord } a_j = \text{ord } \frac{a_i}{a_j} \in \Delta. \quad (2.15)$$

Putting together (2.14) and (2.15), we obtain (1) of the Lemma.

Now, assume that β_1, \dots, β_r are \mathbb{Z} -linearly independent. We wish to prove (2). Since β_1, \dots, β_r are \mathbb{Z} -linearly independent, all the monomials $w_1^{\gamma_1} \dots w_r^{\gamma_r}$, $\gamma_j \in \mathbb{Z}$, have different values with respect to ord . Since (2.5) is homogeneous with respect to ord , each a_i must be a *monomial* in the w_j with (not necessarily positive) integer exponents. Also by the \mathbb{Z} -linear independence of β_1, \dots, β_r , the coefficients $\alpha_1, \dots, \alpha_r$ in (2.8) are uniquely determined. Moreover, any relation of the form

$$i \text{ord } x - \sum_{j=1}^r \alpha'_j \beta_j = 0, \quad i \in \mathbb{N}, \alpha'_1, \dots, \alpha'_r \in \mathbb{Z} \quad (2.16)$$

is a positive integer multiple of the relation

$$\tilde{\alpha} \text{ord } x - \sum_{j=1}^r \alpha_j \beta_j = 0. \quad (2.17)$$

This proves that x^i may appear in (2.5) only if $\tilde{\alpha} \mid i$; in particular, $\tilde{\alpha} \mid \alpha$. Let $d' := \frac{\alpha}{\tilde{\alpha}}$. Let $0 \leq i < d'$. To find the coefficient $a_{i\tilde{\alpha}}$ in (2.5), note that

$$\alpha \text{ord } x = d' \tilde{\alpha} \text{ord } x = i \tilde{\alpha} \text{ord } x + \text{ord } a_{i\tilde{\alpha}},$$

so that

$$(d' - i) \tilde{\alpha} \text{ord } x = \text{ord } a_{i\tilde{\alpha}}. \quad (2.18)$$

Since $a_{i\tilde{\alpha}}$ is a monomial in w_1, \dots, w_r , (2.18) gives rise to a \mathbb{Z} -linear dependence relation of the form (2.16), which therefore must be equal to (2.17) multiplied by $d' - i$. This determines the monomial $a_{i\tilde{\alpha}}$ uniquely up to multiplication by an element of k : we must have $a_{i\tilde{\alpha}} = c_i y^{d'-i}$, where $c_i \in k$. Then $z = \frac{x^{\tilde{\alpha}}}{y}$ satisfies the algebraic dependence relation

$$z^{d'} + \sum_{i=0}^{d'-1} c_i z^i = 0. \quad (2.19)$$

This proves (2) of the Lemma. Now, we have shown that, under the hypothesis of linear independence of the β_j , if x has degree α over G^* then $\tilde{\alpha} \mid \alpha$ and z satisfies a polynomial of degree $d' = \frac{\alpha}{\tilde{\alpha}}$. Letting d denote the degree of z over k , as above, we obtain

$$d' = \frac{\alpha}{\tilde{\alpha}} \geq d. \quad (2.20)$$

Combining (2.20) with (2.13), we obtain (2.12); in particular, (2.11) is the minimal polynomial of x over G . This completes the proof of Lemma 2.7. \square

Corollary 2.8. *Let G , w_1, \dots, w_r and β_1, \dots, β_r be as in lemma 2.7. If β_1, \dots, β_r are \mathbb{Z} -linearly independent in Δ then w_1, \dots, w_r are algebraically independent over k .*

Proof. Induction on r . For $r = 1$ there is nothing to prove. For the induction step, assume that the Corollary is true for $r = i$. If w_{i+1} were algebraic over $k[w_1, \dots, w_i]$, we would have

$$(\beta_1, \dots, \beta_i) : \beta_{i+1} < \infty \quad (2.21)$$

by lemma 2.7, applied to the graded algebra $k[w_1, \dots, w_i]$ and the element w_{i+1} . (2.21) contradicts the linear independence of β_1, \dots, β_r , and we are done. Alternatively, the Corollary can be proved by observing that by linear independence of β_1, \dots, β_r , all the monomials in w_1, \dots, w_r have different degrees, thus any polynomial in w_1, \dots, w_r over k contains a unique monomial of smallest degree. Hence it cannot vanish. \square

Definition 2.9. Let G be a graded algebra and $x_\Lambda := \{x_\lambda\}_{\lambda \in \Lambda}$ a collection of elements of G . Let k be a field, contained in the degree 0 part of G . Let $k[x_\Lambda]$ denote the k -subalgebra of G , generated by x_Λ . We say that x_Λ **rationally generate** G over k if $G^* = k[x_\Lambda]^*$.

The following result is an immediate consequence of definitions:

Proposition 2.10. Let G_ν be the graded algebra associated to a valuation $\nu : K \rightarrow \Gamma$, as above.

Consider a sum of the form $y = \sum_{i=1}^s y_i$, with $y_i \in K$. Let $\beta = \min_{1 \leq i \leq s} \nu(y_i)$ and

$$S = \{i \in \{1, \dots, n\} \mid \nu(y_i) = \beta\}.$$

The following two conditions are equivalent:

1. $\nu(y) = \beta$
2. $\sum_{i \in S} \text{in}_\nu y_i \neq 0$.

3 Construction of key polynomials and proof of completeness

Let $K \rightarrow K(x)$ be a simple transcendental field extension, ν a valuation of $K(x)$ and ν_0 the restriction of ν to K . We will assume that

$$\nu(x) > 0. \quad (3.1)$$

Definition 3.1. A **complete set of key polynomials** for ν is a well ordered collection

$$\mathbf{Q} = \{Q_i\}_{i \in \Lambda}$$

of elements of $K[x]$ such that for each $\beta \in \Gamma$ the additive group $\mathbf{P}_\beta \cap K[x]$ is generated by products of the form $a \prod_{j=1}^s Q_{i_j}^{\gamma_j}$, $a \in K$, such that $\sum_{j=1}^s \gamma_j \nu(Q_{i_j}) + \nu_0(a) \geq \beta$. The collection $\mathbf{Q} = \{Q_i\}_{i \in \Lambda}$ is said to be $\tilde{\Gamma}$ -**complete** if for all $\beta \in \tilde{\Gamma}$ any polynomial $f \in K[x]$ with $\nu(f) = \beta$ belongs to the additive group generated by products of the form $a \prod_{j=1}^s Q_{i_j}^{\gamma_j}$, $a \in K$, such that

$$\sum_{j=1}^s \gamma_j \nu(Q_{i_j}) + \nu_0(a) \geq \beta.$$

Note, in particular, that if \mathbf{Q} is a complete set of key polynomials then their images $\text{in}_\nu Q_i \in G_\nu$ rationally generate G_ν over G_{ν_0} ; if \mathbf{Q} is a $\tilde{\Gamma}$ -complete set of key polynomials then their images $\text{in}_\nu Q_i \in G_\nu$ rationally generate G_{ν_1} over G_{ν_0} (see page 2 for the definition of G_{ν_1}). We want to make the set Λ as small as possible, that is, to minimize the order type of Λ (it will turn out that this order type is bounded above by $\omega \times \omega$).

Our algorithm amounts to successively and explicitly constructing key polynomials until the resulting set of key polynomials becomes complete for ν .

Take a polynomial $h = \sum_{i=0}^s d_i x^i \in K[x]$, $d_i \in K$.

Definition 3.2. The first **Newton polygon** of h with respect to ν_0 is the convex hull $\Delta_1(h)$ of the set $\bigcup_{i=0}^s \left((\nu(d_i), i) + \left(\tilde{\Gamma}_+ \times \mathbb{Q}_+ \right) \right)$ in $\tilde{\Gamma} \times \mathbb{Q}$.

To an element $\beta_1 \in \Gamma_+$, we associate the following valuation ν_1 of $K(x)$: for a polynomial $h = \sum_{i=0}^s d_i x^i$, we put

$$\nu_1(h) = \min \{ \nu_0(d_i) + i\beta_1 \mid 0 \leq i \leq s \}.$$

Consider an element $\beta_1 \in \Gamma_+$.

Definition 3.3. We say that β_1 **determines a side** of $\Delta_1(h)$ if the set

$$S_1(h) = \{ i \in \{0, \dots, s\} \mid \nu_0(d_i) + i\beta_1 = \nu_1(h) \}.$$

has at least two elements.

Let $\beta_1 = \nu(x)$. Then for any $h \in K[x]$ we have

$$\nu_1(h) \leq \nu(h) \tag{3.2}$$

by the axioms for valuations. We put $\Lambda = \{1\}$, $Q_1 = x$ and $\alpha_1 = 1$. If equality holds in (3.2) for all $h \in K[x]$, we stop here. The definition of key polynomials is complete. From now on, assume that there exists a polynomial $h \in K[x]$ such that $\nu_1(h) < \nu(h)$.

Proposition 3.4. Take a polynomial $h = \sum_{i=0}^s d_i x^i \in K[x]$ such that

$$\nu_1(h) < \nu(h). \tag{3.3}$$

Then

$$\sum_{i \in S_1(h)} \text{in}_{\nu_0} d_i \text{in}_\nu x^i = 0.$$

Proof. We have

$$\sum_{i \in S_1(h)} d_i x^i = h(x) - \sum_{i \in \{0, \dots, s\} \setminus S_1(h)} d_i x^i,$$

hence

$$\nu \left(\sum_{i \in S_1(h)} d_i x^i \right) > \nu_1(h).$$

Then $\sum_{i \in S_1(h)} \text{in}_{\nu_0} d_i \text{in}_\nu x^i = 0$ in $\frac{\mathbf{P}_{\nu_1(h)}}{\mathbf{P}_{\nu_1(h)}^+} \subset G_\nu$ by Proposition 2.10. \square

Corollary 3.5. *Take a polynomial $h \in K[x]$ such that $\nu_1(h) < \nu(h)$. Then β_1 determines a side of $\Delta_1(h)$.*

Proof. If $S_1(h)$ consisted of a single element i , we would have $\nu(h) = \nu(d_i x^i) = \nu_1(h)$, contradicting the assumption. \square

Notation. Let X be a new variable. Take a polynomial h as above. We denote

$$\text{in}_1 h := \sum_{i \in S_1(h)} \text{in}_{\nu_0} d_i X^i.$$

The polynomial $\text{in}_1 h$ is quasi-homogeneous in $G_{\nu_0}[X]$, where the weight assigned to X is β_1 . Let

$$\text{in}_1 h = v \prod_{j=1}^t g_j^{\gamma_j} \tag{3.4}$$

be the factorization of $\text{in}_1 h$ into irreducible factors in $G_{\nu_0}[X]$. Here $v \in G_{\nu_0}$ and the g_j are monic polynomials in $G_{\nu_0}[X]$ (to be precise, we first factor $\text{in}_1 h$ over the field of fractions of G_{ν_0} and then observe that all the factors are quasi-homogeneous and therefore lie in $G_{\nu_0}[X]$).

Proposition 3.6. (1) *The element $\text{in}_{\nu} x$ is integral over G_{ν_0} .*

(2) *The minimal polynomial of $\text{in}_{\nu} x$ over G_{ν_0} is one of the irreducible factors g_j of (3.4).*

Proof. Both (1) and (2) of the Proposition follow from the fact that $\text{in}_{\nu} x$ is a root of the polynomial $\text{in}_1 h$ (Proposition 3.4). \square

Note that Proposition 3.6 (1) implies that $\nu(x) \in \tilde{\Gamma}$.

Now let g_1 be the minimal polynomial of $\text{in}_{\nu} x$ over G_{ν_0} . Let $\alpha_2 = \deg_X g_1$. Write

$$g_1 = \sum_{i=0}^{\alpha_2} \bar{b}_i X^i,$$

where $\bar{b}_{\alpha_2} = 1$. For each i , $0 \leq i < \alpha_2$, choose a representative b_i of \bar{b}_i in R_{ν_0} (that is, an element of R_{ν_0} such that $\text{in}_{\nu_0} b_i = \bar{b}_i$; in particular, we take $b_{\alpha_2} = 1$). Put $Q_2 = \sum_{i=0}^{\alpha_2} b_i x^i$.

Definition 3.7. The elements $Q_1 := x$ and Q_2 are called, respectively, **the first and second key polynomials** of ν .

Remark 3.8. By convention, Q_0 is not defined; our indexing starts at Q_1 .

Now, every element y of $K[x]$ can be written uniquely as a finite sum of the form

$$y = \sum_{\substack{0 \leq \gamma_1 < \alpha_2 \\ 0 \leq \gamma_2}} b_{\gamma_1 \gamma_2} Q_1^{\gamma_1} Q_2^{\gamma_2} \tag{3.5}$$

where $b_{\gamma_1 \gamma_2} \in K$ (this is proved by Euclidean division by the monic polynomial Q_2). The expression (3.5) is called **the 2-standard expansion of y** .

If $\nu(Q_2) \notin \tilde{\Gamma}$, the algorithm stops here. From now on, assume that $\nu(Q_2) \in \tilde{\Gamma}$.

Take an ordinal number ℓ , $2 \leq \ell < \omega \times \omega$. Assume, inductively, that key polynomials $\mathbf{Q}_{\ell+1}$ and positive integers $\alpha_{\ell+1} = \{\alpha_i\}_{i \leq \ell}$ are already constructed, that $\nu(Q_i) \in \tilde{\Gamma}$ for $i \leq \ell$ and that all but finitely many of the α_i are equal to 1. We want to define the key polynomial $Q_{\ell+1}$.

We will use the following multi-index notation: $\bar{\gamma}_{\ell+1} = \{\gamma_i\}_{i \leq \ell}$, where all but finitely many γ_i are equal to 0, $\mathbf{Q}_{\ell+1}^{\bar{\gamma}_{\ell+1}} = \prod_{i \leq \ell} Q_i^{\gamma_i}$. Let $\beta_i = \nu(Q_i)$.

Definition 3.9. An index $i < \ell$ is said to be ℓ -**essential** if there exists a positive integer t such that either $i + t = \ell$ or $i + t < \ell$ and $\alpha_{i+t} > 1$; otherwise i is called ℓ -**inessential**.

In other words, i is ℓ -inessential if and only if $i + \omega \leq \ell$ and $\alpha_{i+t} = 1$ for all $t \in \mathbb{N}_0$.

Notation. For $i < \ell$, let

$$\begin{aligned} i+ &= i + 1 && \text{if } i \text{ is } \ell\text{-essential} \\ &= i + \omega && \text{otherwise.} \end{aligned}$$

Definition 3.10. A multiindex $\bar{\gamma}_{\ell+1}$ is said to be **standard with respect to $\alpha_{\ell+1}$** if

$$0 \leq \gamma_i < \alpha_{i+} \text{ for } i \leq \ell, \quad (3.6)$$

and if i is ℓ_0 -inessential for some limit ordinal $\ell_0 \leq \ell$ then the set $\{j < i+ \mid j+ = i+ \text{ and } \gamma_j \neq 0\}$ has cardinality at most one. An ℓ -**standard monomial in $\mathbf{Q}_{\ell+1}$** (resp. an ℓ -**standard monomial in $\text{in}_\nu \mathbf{Q}_{\ell+1}$**) is a product of the form $c_{\bar{\gamma}_{\ell+1}} \mathbf{Q}_{\ell+1}^{\bar{\gamma}_{\ell+1}}$, (resp. $c_{\bar{\gamma}_{\ell+1}} \text{in}_\nu \mathbf{Q}_{\ell+1}^{\bar{\gamma}_{\ell+1}}$) where $c_{\bar{\gamma}_{\ell+1}} \in K$ (resp. $c_{\bar{\gamma}_{\ell+1}} \in G_{\nu_0}$) and the multiindex $\bar{\gamma}_{\ell+1}$ is standard with respect to $\alpha_{\ell+1}$.

Keep the notation of Definition 3.10.

Definition 3.11. An ℓ -standard monomial $c_{\bar{\gamma}_{\ell+1}} \mathbf{Q}_{\ell+1}^{\bar{\gamma}_{\ell+1}}$ is said to be Q_ℓ -**free** if it does not involve Q_ℓ , that is, if $\gamma_\ell = 0$.

Definition 3.12. An ℓ -**standard expansion** of an element $g \in K[x]$ is an expression of the form $g = \sum_{j=0}^s c_j Q_\ell^j$, where each $c_j Q_\ell^j$ is an ℓ -standard monomial. A Q_ℓ -**free ℓ -standard expansion** is a finite sum of Q_ℓ -free ℓ -standard monomials.

For an element $G \in G_\nu$, an expression of the form $G = \sum_{\bar{\gamma}} \bar{c}_{\bar{\gamma}} \text{in}_\nu \mathbf{Q}_{\ell+1}^{\bar{\gamma}}$, where each $\bar{c}_{\bar{\gamma}} \in G_{\nu_0}$ and each $\mathbf{Q}_{\ell+1}^{\bar{\gamma}}$ is an ℓ -standard monomial, will be called an ℓ -**standard expansion of G** .

Remark 3.13. We note that a Q_ℓ -free ℓ -standard expansion is not just an i -standard expansion for some $i < \ell$. Namely, if ℓ is not a limit ordinal, it is required, in addition, that the exponent of the last appearing key polynomial be bounded.

Remark 3.14. For an element $g \in K[x]$, let $g = \sum_{j=0}^s c_j Q_\ell^j$ be its ℓ -standard expansion. Then each c_j is a Q_ℓ -free ℓ -standard expansion.

Proposition 3.15. *Let i be an ordinal and t a positive integer. Assume that $i + t + 1 \leq \ell$, so that the key polynomials \mathbf{Q}_{i+t+1} are defined, and that $\alpha_i = \cdots = \alpha_{i+t} = 1$. Then any $(i + t)$ -standard expansion does not involve any Q_q with $i \leq q < i + t$. In particular, a Q_i -free i -standard expansion is the same thing as a Q_{i+t} -free $(i + t)$ -standard expansion.*

Proof. (3.6) implies that for $i \leq q \leq i + t$, Q_q cannot appear in an $(i + t)$ -standard expansion with a positive exponent. \square

We will frequently use this fact in the sequel without mentioning it explicitly.

Definition 3.16. Let $\sum_{\bar{\gamma}} \bar{c}_{\bar{\gamma}} \text{in}_\nu \mathbf{Q}_{\ell+1}^{\bar{\gamma}}$ be an ℓ -standard expansion, where $\bar{c}_{\bar{\gamma}} \in G_{\nu_0}$. A **lifting** of $\sum_{\bar{\gamma}} \bar{c}_{\bar{\gamma}} \text{in}_\nu \mathbf{Q}_{\ell+1}^{\bar{\gamma}}$ to $K[x]$ is an ℓ -standard expansion $\sum_{\bar{\gamma}} c_{\bar{\gamma}} \mathbf{Q}_{\ell+1}^{\bar{\gamma}}$, where $c_{\bar{\gamma}}$ is a representative of $\bar{c}_{\bar{\gamma}}$ in K .

Definition 3.17. Assume that $\text{char } k_\nu = p > 0$. An ℓ -standard expansion $\sum_j c_j Q_\ell^j$ is said to be **weakly affine** if $c_j = 0$ whenever $j > 0$ and j is not of the form p^e for some $e \in \mathbb{N}_0$.

Assume, inductively, that for each ordinal $i \leq \ell$, every element h of $K[x]$ admits an i -standard expansion. Furthermore, assume that for each $i \leq \ell$ and each i_0 with

$$i = i_0 +, \quad (3.7)$$

the i -th key polynomial Q_i admits an i_0 -standard expansion, having the following additional properties.

If i has an immediate predecessor $i - 1$, the $(i - 1)$ -standard expansion of Q_i has the form

$$Q_i = Q_{i-1}^{\alpha_i} + \sum_{j=0}^{\alpha_i-1} \left(\sum_{\tilde{\gamma}_{i-1}} c_{ji\tilde{\gamma}_{i-1}} \mathbf{Q}_{i-1}^{\tilde{\gamma}_{i-1}} \right) Q_{i-1}^j, \quad (3.8)$$

where:

- (1) each $c_{ji\tilde{\gamma}_{i-1}} \mathbf{Q}_{i-1}^{\tilde{\gamma}_{i-1}}$ is an $(i - 1)$ -standard monomial
- (2) the quantity $\nu_0 (c_{ji\tilde{\gamma}_{i-1}}) + j\beta_{i-1} + \sum_{q < i-1} \gamma_q \beta_q$ is constant for all the monomials

$$\left(c_{ji\tilde{\gamma}_{i-1}} \mathbf{Q}_{i-1}^{\tilde{\gamma}_{i-1}} \right) Q_{i-1}^j$$

appearing on the right hand side of (3.8)

(3) the equation

$$\text{in}_\nu Q_{i-1}^{\alpha_i} + \sum_{j=0}^{\alpha_i-1} \left(\sum_{\tilde{\gamma}_{i-1}} \text{in}_{\nu_0} c_{ji\tilde{\gamma}_{i-1}} \text{in}_\nu \mathbf{Q}_{i-1}^{\tilde{\gamma}_{i-1}} \right) \text{in}_\nu Q_{i-1}^j = 0 \quad (3.9)$$

is the minimal algebraic relation satisfied by $\text{in}_\nu Q_{i-1}$ over $G_{\nu_0}[\text{in}_\nu \mathbf{Q}_{i-1}]^*$.

Finally, if i does not have an immediate predecessor then for each i_0 satisfying (3.7), the i_0 -standard expansion

$$Q_i = \sum_{j=0}^{\alpha_i} c_{ji_0} Q_{i_0}^j \quad (3.10)$$

satisfies

$$\nu(Q_i) > \min_{0 \leq j \leq \alpha_i} \left\{ \nu \left(c_{ji_0} Q_{i_0}^j \right) \right\}. \quad (3.11)$$

Moreover, the polynomial Q_i is monic of the smallest degree among those satisfying (3.11) for all i_0 as in (3.7).

Remark 3.18. In §8 we will show, assuming that $\text{rk } \nu_0 = 1$ and the set $\{\nu(Q_{i_0})\}_{i=i_0+}$ is bounded in $\tilde{\Gamma}$, that we can choose i_0 and Q_i so that Q_i is a weakly affine monic i_0 -standard expansion of degree $\alpha_i = p^{e_i}$ for a certain integer e_i . Moreover, there exists a positive element $\bar{\beta}_i \in \mathbb{R}$ such that

$$\bar{\beta}_i > \beta_q \quad \text{for all } q < i, \quad (3.12)$$

$$\beta_i \geq \alpha_i \bar{\beta}_i \quad \text{and} \quad (3.13)$$

$$p^j \bar{\beta}_i + \nu(c_{p^j i_0}) = \alpha_i \bar{\beta}_i \quad \text{for } 0 \leq j \leq \alpha_i. \quad (3.14)$$

Definition 3.19. A well-ordered set of polynomials $\{Q_i\}$ of order type at most $\omega \times \omega$ satisfying (3.8)—(3.11) is said to be a **set of key polynomials**. An element of the set is called a **key polynomial**.

If $i \in \mathbb{N}_0$, we assume, inductively, that the i -standard expansion is unique. For a general i , if $h = \sum_{j=0}^{s_i} d_{ji} Q_i^j$ is an i -standard expansion of h (where $h \in K[x]$), we assume that the elements $d_{ji} \in K[x]$ are uniquely determined by h , but their i -standard expansions need not be. For example, if i is a limit ordinal, d_{ji} admits an i_0 -standard expansion for each $i_0 < i$ such that $i = i_0+$, but there are countably many choices of i_0 for which such an i_0 -standard expansion is a Q_{i_0} -free i_0 -standard expansion in the sense of Definition 3.12.

Proposition 3.20. (1) *The polynomial Q_i is monic in x ; we have*

$$\deg_x Q_i \geq \prod_{j \leq i} \alpha_j. \quad (3.15)$$

(2) *Let z be an Q_i -free i -standard expansion. Then*

$$\deg_x z < \deg_x Q_i. \quad (3.16)$$

Proof. (3.15) and (3.16) are proved simultaneously by transfinite induction on i , using (3.8), (3.10) and (3.6) repeatedly to calculate and bound the degree in x of any standard monomial (recall that by assumption all but finitely many of the α_i are equal to 1). \square

For each ordinal $i \leq \ell$ we define a valuation ν_i of $K(x)$ as follows. Given an i -standard expansion $h = \sum_{j=0}^{s_i} d_{ji} Q_i^j$, put

$$\nu_i(h) = \min_{0 \leq j \leq s_i} \{j\beta_i + \nu(d_{ji})\}. \quad (3.17)$$

The valuation ν_i will be called the **i -truncation** of ν . Note that even though the i -standard expansions elements d_{ji} are not, in general, unique, the elements $d_{ji} \in K[x]$ themselves are unique by Euclidean division, so ν_i is well defined. That ν_i is, in fact, a valuation, rather than a pseudo-valuation, follows from the definition of standard expansion, particularly, from the minimality of $\deg_x Q_i$, stipulated in (3.9) and (3.11). We always have

$$\nu_i(h) \leq \nu(h). \quad (3.18)$$

Remark 3.21. The set \mathbf{Q} is a complete set of key polynomials if and only if for each polynomial $f \in K[x]$ there exists an ordinal ℓ such that $\nu(f) = \nu_\ell(f)$. Indeed, assume that such an ordinal always exists. Take any $\beta \in \Gamma$, and let $f \in \mathbf{P}_\beta \cap K[x]$. Put $\beta' = \nu(f)$ and let ℓ be such that $\beta' = \nu(f) = \nu_\ell(f)$. Write $f = \sum_{j=0}^{s_\ell} c_{j\ell} Q_\ell^j$, where each $c_{j\ell} Q_\ell^j \in \mathbf{P}_{\beta'} \cap K[x] \subset \mathbf{P}_\beta \cap K[x]$.

Conversely, take $f \in K[x]$. Let $\beta = \nu(f)$. Write f as a finite sum of the form $f = \sum_{\gamma} c_\gamma \mathbf{Q}^\gamma$, $c_\gamma \in K$, with $\nu(c_\gamma \mathbf{Q}^\gamma) \geq \beta$ for all γ such that $c_\gamma \neq 0$. Let ℓ denote the greatest ordinal such that Q_ℓ appears in one of the monomials $c_\gamma \mathbf{Q}^\gamma$. Then

$$\beta = \nu(f) \geq \nu_\ell(f) \geq \min_{\gamma} \{\nu_\ell(c_\gamma \mathbf{Q}^\gamma)\} = \min_{\gamma} \{\nu(c_\gamma \mathbf{Q}^\gamma)\} \geq \beta.$$

For $b \in \mathbb{N}_0$, let ∂_b denote the b -th formal derivative with respect to x . For all $b \in \mathbb{N}$, we have

$$\nu_{i_0}(\partial_b Q_i) = \nu(\partial_b Q_i) \quad (3.19)$$

(where the notation is as in (3.7)–(3.11)).

Proposition 3.22. *For a pair of ordinals $i_0 < i \leq \ell$ such that $i = i_0 +$ we have*

$$\nu_{i_0}(Q_i) < \beta_i.$$

Proof. To prove the Proposition, we distinguish the cases when i does or does not admit an immediate predecessor. If i admits an immediate predecessor then $i_0 = i - 1$. In this case the result follows from (3.8) and (3.9). If i does not have an immediate predecessor then the Proposition is nothing but (3.11). \square

The rest of this section is devoted to the definition of $Q_{\ell+1}$. Take an element $h \in K[x]$ and let

$$h = \sum_{j=0}^{s_\ell} d_{j\ell} Q_\ell^j \quad (3.20)$$

be an ℓ -standard expansion of h , where each $d_{j\ell}$ is a Q_ℓ -free standard expansion.

Definition 3.23. The ℓ -th Newton polygon of h with respect to ν is the convex hull $\Delta_\ell(h)$ of the set $\bigcup_{j=0}^{s_\ell} \left((\nu(d_{j\ell}), j) + \left(\tilde{\Gamma}_+ \times \mathbb{Q}_+ \right) \right)$ in $\tilde{\Gamma} \times \mathbb{Q}$.

Consider the valuation ν_ℓ , defined in (3.17). If equality holds in (3.18) for $i = \ell$ and for all $h \in K[x]$, put $\Lambda = \ell + 1$ and stop. In this case, the definition of key polynomials is complete. From now on, assume that strict inequality holds in (3.18) for some $h \in K[x]$.

Let \bar{Q}_ℓ be a new variable and let h be as above. Let β^* be a non-negative element of Γ . We denote

$$S_\ell(\beta^*, h) := \{j \in \{0, \dots, s_\ell\} \mid j\beta^* + \nu(d_{j\ell}) = \nu_\ell(h)\}.$$

Definition 3.24. We say that β^* **determines a side** of $\Delta_\ell(h)$ if the set $\#S_\ell(\beta^*, h)$ has at least two elements.

Notation:

$$S_\ell(h) := S_\ell(\beta_\ell, h). \quad (3.21)$$

$$\text{in}_\ell h := \sum_{j \in S_\ell(h)} \text{in}_\nu d_{j\ell} \bar{Q}_\ell^j; \quad (3.22)$$

The polynomial $\text{in}_\ell h$ is quasi-homogeneous in $G[\text{in}_\nu \mathbf{Q}_\ell, \bar{Q}_\ell]$, where the weight assigned to \bar{Q}_ℓ is β_ℓ .

Take a polynomial h such that

$$\nu_\ell(h) < \nu(h). \quad (3.23)$$

Proposition 3.25. *We have $\sum_{j \in S_\ell(h)} \text{in}_\nu \left(d_{j\ell} Q_\ell^j \right) = 0$ in $\frac{\mathbf{P}_{\nu_\ell(h)}}{\mathbf{P}_{\nu_\ell(h)+}} \subset G_\nu$.*

Proof. This follows immediately from (3.23), the fact that

$$\sum_{j \in S_\ell(h)} d_{j\ell} Q_\ell^j = h - \sum_{j \in \{0, \dots, s\} \setminus S_\ell(h)} d_{j\ell} Q_\ell^j$$

and Proposition 2.10. \square

Corollary 3.26. *The element β_ℓ determines a side of $\Delta_\ell(h)$.*

Proof. Suppose not. Then the ℓ -standard expansion of h contains a unique monomial $d_{j\ell} Q_\ell^j$ of minimal value, so $\nu(h) = \nu(d_{j\ell} Q_\ell^j) = \nu_\ell(h)$, contradicting (3.23). Corollary 3.26 is proved. \square

Let

$$\text{in}_\ell h = v_\ell \prod_{j=1}^t g_{j\ell}^{\gamma_{j\ell}} \quad (3.24)$$

be the factorization of $\text{in}_\ell h$ into (monic) irreducible factors in $G_{\nu_0}[\text{in}_\nu \mathbf{Q}_\ell][\bar{Q}_\ell]$ (to be precise, we first factor $\text{in}_\ell h$ over the field of fractions of $G_{\nu_0}[\text{in}_\nu \mathbf{Q}_\ell]$ and then observe that all the factors are quasi-homogeneous and therefore lie in $G_{\nu_0}[\text{in}_\nu \mathbf{Q}_\ell][\bar{Q}_\ell]$).

Corollary 3.27. *The element $\text{in}_\nu Q_\ell$ is integral over $G_{\nu_0}[\text{in}_\nu \mathbf{Q}_\ell]$. Its minimal polynomial over $G_{\nu_0}[\text{in}_\nu \mathbf{Q}_\ell]$ is one of the irreducible factors $g_{j\ell}$ of (3.24).*

Let $\alpha_{\ell+1}$ denote the degree of $\text{in}_\nu Q_\ell$ over $G_{\nu_0}[\text{in}_\nu \mathbf{Q}_\ell]$. Renumbering the factors in (3.24), if necessary, we may assume that $g_{1\ell}$ is the minimal polynomial of $\text{in}_\nu Q_\ell$ over $G_{\nu_0}[\text{in}_\nu \mathbf{Q}_\ell]$, so that

$$\alpha_{\ell+1} = \deg_{\bar{Q}_\ell} g_{1\ell}. \quad (3.25)$$

Write

$$g_{1\ell} = \bar{Q}_\ell^{\alpha_{\ell+1}} + \sum_{j=0}^{\alpha_{\ell+1}-1} \left(\sum_{\bar{\gamma}_\ell} \bar{c}_{\ell+1, j\bar{\gamma}_\ell} \text{in}_\nu \mathbf{Q}_\ell^{\bar{\gamma}_\ell} \right) \bar{Q}_\ell^j, \quad (3.26)$$

Let X be a new variable, and consider a lifting of $g_{1\ell}$ of (3.26) to $K[X]$, that is, the polynomial

$$X^{\alpha_{\ell+1}} + \sum_{j=0}^{\alpha_{\ell+1}-1} \left(\sum_{\bar{\gamma}_\ell} c_{\ell+1, j\bar{\gamma}_\ell} \mathbf{Q}_\ell^{\bar{\gamma}_\ell} \right) X^j.$$

Define **the $(\ell+1)$ -st key polynomial** of ν to be

$$Q_{\ell+1} = Q_\ell^{\alpha_{\ell+1}} + \sum_{j=0}^{\alpha_{\ell+1}-1} \left(\sum_{\bar{\gamma}_\ell} c_{\ell+1, j\bar{\gamma}_\ell} \mathbf{Q}_\ell^{\bar{\gamma}_\ell} \right) Q_\ell^j. \quad (3.27)$$

In the special case when $\alpha_{\ell+1} = 1$, some additional conditions must be imposed on the polynomial $Q_{\ell+1}$ (3.27). In fact, in this case we will define several consecutive key polynomials at the same time. We will now explain what these additional conditions are, after making some general remarks.

Remark 3.28. We claim that $Q_{\ell+1}$ is an irreducible polynomial in x . Indeed, consider a factorization $Q_{\ell+1} = fg$ in $K[x]$. Passing to the natural images of $Q_{\ell+1}$, f and g in

$$G_{\nu_0}[\text{in}_\nu \mathbf{Q}_\ell][\bar{Q}_\ell] \subset G_{\nu_\ell},$$

we obtain $g_{1\ell} = \text{in}_\ell f \text{in}_\ell g$. Since $g_{1\ell}$ is an irreducible polynomial in \bar{Q}_ℓ by definition, we have, up to interchanging f and g ,

$$\deg_{\bar{Q}_\ell} \text{in}_\ell g = \alpha_{\ell+1}. \quad (3.28)$$

Then

$$\deg_x f \geq \alpha_{\ell+1} \deg_x Q_\ell. \quad (3.29)$$

Since $g_{1\ell}$ has the form (3.26), we must have equality in (3.29) and $\deg_x g = 0$. Thus $g \in K$; this completes the proof of the irreducibility of $Q_{\ell+1}$ in $K[x]$.

Proposition 3.29. *Take an element h of $K[x]$ and an ordinal $i \leq \ell$. Assume that one of the following conditions holds:*

(1)

$$\nu(h) < \beta_i \quad (3.30)$$

and h admits an i -standard expansion

$$h = \sum_{j=0}^s c_j Q_i^j, \quad (3.31)$$

such that

$$\nu(c_j) \geq 0 \quad \text{for all } j. \quad (3.32)$$

(2) $\deg_x h < \deg_x Q_{i+1}$.

Then $\nu(h) = \nu_i(h)$.

Proof. Let (3.31) be an i -standard expansion of h , where in case (1) we assume that (3.32) holds. By definition of standard expansion, each c_j in (3.31) is a Q_i -free standard expansion. Then $\nu_i(c_j) = \nu(c_j)$ for $0 \leq j \leq s$.

(1) By (3.30) and (3.32),

$$\nu(c_j Q_i^j) = \nu(c_j) + j\beta_i > \nu(h) \quad \text{for } j > 0. \quad (3.33)$$

This implies that $\nu(h) = \nu(c_0)$ and $\nu(c_0) < \nu(c_j) + j\beta_i = \nu_i(c_j) + j\beta_i$ for all $j > 0$. Thus in the sum (3.31) the ν_i -value (resp. the ν -value) $\nu_i(c_0) = \nu(c_0)$ of c_0 is strictly smaller than the ν_i -values (resp. the ν -values) $\nu_i(c_j Q_i^j) = \nu(c_j Q_i^j)$ of all the other terms. It is well known and follows easily from the definition of valuation that in this situation we have $\nu_i(h) = \nu_i(c_0) = \nu(c_0) = \nu(h)$, as desired.

(2) Let $S_i(h)$ be as in (3.21). Since the degree of $\text{in}_\nu Q_i$ over $G_{\nu_0}[\text{in}_\nu \mathbf{Q}_i]^*$ is α_{i+1} by (3.9), we see, using the assumption on $\deg_x h$, that $\sum_{j=0}^s \text{in}_\nu c_j \text{in}_\nu Q_i^j \neq 0$ in G_ν . The result now follows from Proposition 2.10. □

We will now describe the additional conditions we impose on the polynomial $Q_{\ell+1}$ (3.27) in the case when $\alpha_{\ell+1} = 1$. Assume that $\alpha_{\ell+1} = 1$. In what follows, we will consider ℓ -standard expansions of the form

$$Q' = Q_\ell + z_\ell + \cdots + z_i, \quad (3.34)$$

where each z_j is a homogeneous Q_ℓ -free standard expansion, such that

$$\beta_\ell = \nu(z_\ell) < \nu(z_{\ell+1}) < \cdots < \nu(z_i). \quad (3.35)$$

Remark 3.30. Note that by (3.16), we have $\deg_x z_q < \deg_x Q_\ell$ for all q .

Let T denote the set of all the ℓ -standard expansions of the form (3.34), where each z_j is a homogeneous Q_ℓ -free ℓ -standard expansion, such that the inequalities (3.35) hold and $\nu(z_i) < \nu(Q')$.

We impose the following partial ordering on T . Given an element

$$Q' = Q_\ell + z_\ell + \cdots + z_i \in T$$

with $i > \ell$, we declare its immediate predecessor in T to be the element $Q_\ell + z_\ell + \cdots + z_{i-1}$. By definition, our partial ordering is the coarsest one among those in which $Q_\ell + z_\ell + \cdots + z_{i-1}$ precedes $Q_\ell + z_\ell + \cdots + z_i$ for all the elements Q' as above.

Remark 3.31. Take an element $Q' := Q_\ell + z_\ell + \cdots + z_i \in T$. For each ordinal j with $\ell \leq j \leq i$, we have

$$\text{in}_\nu(Q_\ell + z_\ell + \cdots + z_{j-1}) = -\text{in}_\nu z_j; \quad (3.36)$$

in particular, $\nu(Q_\ell + z_\ell + \cdots + z_{j-1}) = \nu(z_j)$.

Proposition 3.32. *Consider two elements*

$$Q' = Q_\ell + z'_\ell + \cdots + z'_{i'}, Q'' = Q_\ell + z''_\ell + \cdots + z''_{i''} \in T.$$

Assume that

$$\nu(Q') < \nu(Q'').$$

Then there exists a standard expansion

$$Q''' := Q_\ell + z'_\ell + \cdots + z'_{i'} + z_{i'+1} + \cdots + z_{i'''} \in T$$

such that $Q''' > Q'$ with respect to the partial ordering on standard expansions defined above and $Q''' = Q''$ (here by the equality $Q''' = Q''$ we mean equality as elements of $K[x]$, regardless of the standard expansion (3.34)).

Proof. Let $w = Q'' - Q'$. We have

$$\nu(w) = \nu(Q') < \nu(Q'').$$

Since $\deg w < \deg Q_\ell$, the ℓ -standard expansion of w does not involve Q_ℓ . Let

$$w = z_{i'+1} + \cdots + z_{i'''} \text{ with } \nu(z_{i'+1}) < \cdots < \nu(z_{i'''}) \quad (3.37)$$

be the ℓ -standard expansion of w , where $z_{i'+1}, \dots, z_{i'''}$ are homogeneous ℓ -standard expansions, not involving Q_ℓ . Put

$$Q''' = Q_\ell + z'_\ell + \cdots + z'_{i'} + z_{i'+1} + \cdots + z_{i'''}$$

To prove that $Q''' > Q'$ we still have to prove that $\nu(z'_{i'}) < \nu(z_{i'+1})$. By definition of Q' we have $\nu(z'_{i'}) < \nu(Q')$, by definition of w we have $\nu(Q') = \nu(w)$ and by the properties of a valuation we have $\nu(w) = \nu(z_{i'+1})$.

Clearly

$$Q''' = Q''$$

and the Proposition follows immediately. \square

To define $Q_{\ell+1}$ in the special case when

$$\alpha_{\ell+1} = 1, \quad (3.38)$$

consider two cases:

Case 1. The set T contains a maximal element. Let $Q' = Q_\ell + z_\ell + z_{\ell+1} + \cdots + z_{s-1}$ be such a maximal element, where each z_i is a homogeneous Q_ℓ -free standard expansion, and s is an ordinal of the form $s = \ell + t$, $t \in \mathbb{N}_0$. Define

$$Q_i = Q_\ell + z_\ell + \cdots + z_{i-1} \quad \text{for } \ell + 1 \leq i \leq s.$$

Case 2. The set T does not contain a maximal element. In this case, Proposition 3.32 shows that there exists an infinite sequence $z_\ell, z_{\ell+1}, \dots$ of homogeneous Q_ℓ -free ℓ -standard expansions, such that for each $t \in \mathbb{N}_0$ we have

$$Q_\ell + z_\ell + \cdots + z_{\ell+t} \in T \quad (3.39)$$

and $\lim_{t \rightarrow \infty} \nu(Q_\ell + z_\ell + \cdots + z_{\ell+t}) = \bar{\beta}$; pick and fix one such sequence. Define $\alpha_{\ell+t} = 1$ and

$$Q_{\ell+t} = Q_\ell + z_\ell + z_{\ell+1} + \cdots + z_{\ell+t-1} \quad \text{for } t \in \mathbb{N}_0.$$

Note that (3.35) and Remark 3.31 imply that the sequence $\{\nu(Q_\ell + z_\ell + \cdots + z_{\ell+t})\}_{t \in \mathbb{N}_0}$ is strictly increasing.

For future reference, it will be convenient to distinguish two subcases of Case 2:

Case 2a. The sequence $\{\beta_{\ell+t}\}_{t \in \mathbb{N}_0}$ is unbounded in $\tilde{\Gamma}$.

Case 2b. The set $\{\nu(Q') \mid Q' \in T\}$ has a least upper bound (but no maximum) in $\tilde{\Gamma}$.

Now as we have defined the key polynomial $Q_{\ell+1}$, for every polynomial $h \in K[x]$ we can define the $(\ell + 1)$ -standard expansion of h . Indeed, fix a polynomial $h \in K[x]$. By iterated Euclidean division by $Q_{\ell+1}$ we can write

$$h = \sum_{j=0}^s c_j Q_{\ell+1}^j$$

where each c_j is a polynomial of degree strictly less than degree of $Q_{\ell+1}$. To be precise, we first divide h by $Q_{\ell+1}$ and call the remainder c_0 , then divide the quotient of that division by $Q_{\ell+1}$ and call the remainder c_1 , and so on, until we obtain a quotient of degree strictly less than $\deg_x Q_{\ell+1}$; we call that quotient c_s and stop.

Now, by the induction hypotheses, for each j , we can write the ℓ -standard expansion of c_j . Since $\deg_x c_j < \deg_x Q_{\ell+1}$, by Proposition 3.29 we have $\nu_\ell(c_j) = \nu(c_j)$. Thus each c_j is a $Q_{\ell+1}$ -free standard expansion.

If $\alpha_{\ell+1} > 1$, we construct the polynomial $Q_{\ell+2}$ from $\mathbf{Q}_{\ell+2}$ in the same way as we already constructed $Q_{\ell+1}$ from $\mathbf{Q}_{\ell+1}$. If $\alpha_{\ell+1} = 1$ and we are in case 1, then we construct the polynomial $Q_{\ell+s+1}$ from $\mathbf{Q}_{\ell+s+1}$ in the same way as we already constructed $Q_{\ell+1}$ from $\mathbf{Q}_{\ell+1}$. Finally, if $\alpha_{\ell+1} = 1$ and we are in case 2 then the set $\{Q_{\ell+t}\}_{t \in \mathbb{N}_0}$ is already constructed. Therefore, by recursion on t , we have constructed the set $\{Q_{\ell+t}\}_{t \in \mathbb{N}_0}$.

If for some $t \in \mathbb{N}_0$ we obtain

$$\nu(Q_{\ell+t}) \notin \tilde{\Gamma}, \quad (3.40)$$

put $\Lambda = \ell + t + 1$ and stop (this cannot happen if we fall in case 2). Later in this section we will show that $\mathbf{Q}_{\ell+t+1}$ is a complete set of key polynomials for ν .

Assume that

$$\nu(Q_{\ell+t}) \in \tilde{\Gamma} \quad \text{for all } t \in \mathbb{N}_0. \quad (3.41)$$

We denote the set $\mathbf{Q}_\ell \cup \{Q_{\ell+t}\}_{t \in \mathbb{N}}$ of key polynomials by $\mathbf{Q}_{\ell+\omega}$.

If $\mathbf{Q}_{\ell+\omega}$ is a complete set of key polynomials, stop. The definition of the key polynomials $\{Q_i\}_i$ is complete.

Assume that $\mathbf{Q}_{\ell+\omega}$ is not complete. Then there exists a monic polynomial h such that

$$\nu_{\ell+t}(h) < \nu(h) \quad (3.42)$$

for all $t \in \mathbb{N}$. In this case, define $Q_{\ell+\omega}$ to be a smallest degree monic polynomial h satisfying (3.42).

Remark 3.33. The inequality (3.42) implies that

$$\deg_x Q_{\ell+\omega} \geq \deg_x Q_\ell = \deg_x Q_{\ell+t} \quad \text{for all } t \in \mathbb{N}. \quad (3.43)$$

Moreover,

$$\nu(Q_{\ell+\omega}) > \nu(Q_{\ell+t}) \quad (3.44)$$

by Proposition 3.29 (1) and (3.42).

If the inequality in (3.43) was an equality, we could put $z_{\ell+\omega} = Q_{\ell+\omega} - Q_\ell$. We would then have $Q_{\ell+\omega} = Q_\ell + z_{\ell+\omega} \in T$, which, together with (3.44), contradicts the definition of $\{Q_{\ell+t}\}$. Thus, $\deg_x Q_{\ell+\omega} > \deg_x Q_\ell = \deg_x Q_{\ell+t}$, $t \in \mathbb{N}$.

Remark 3.34. By Remark 3.33 we have $\alpha_{\ell+\omega} > 1$. Hence after constructing at most an infinite sequence of limit key polynomials, we obtain a set of key polynomials with unbounded degrees. Such a set of key polynomials is complete by Proposition 3.29 (2).

We iterate the above recursive procedure for constructing key polynomials until we obtain either a complete set of key polynomials or a key polynomial whose value does not lie in $\tilde{\Gamma}$. We will show in a moment that in the latter case the resulting set of key polynomials is also complete. The construction of key polynomials stops here. By Remark 3.34 our construction stops before reaching the ordinal $\omega \times \omega$. In §7 we will study further properties of $Q_{\ell+\omega}$. Among other things, we will show (Propositions 7.6 and 7.9 and Remark 7.7) that:

(a) if $\text{char } k_\nu = 0$ and $\text{rk } \nu_0 = 1$ then our construction gives a complete set of key polynomials of order type at most $\{Q_i\}_{i \in \mathbb{N}} \cup \{Q_\omega\}$

(b) if, in addition, $\text{rk } \nu = 1$ then the construction produces a *sequence* of key polynomials that may be finite or infinite.

Theorem 3.35. *The well ordered set $\mathbf{Q} := \{Q_i\}_{i \in \Lambda}$ constructed above is a complete set of key polynomials. In other words, for any element $\beta \in \Gamma$ the R_ν -module $\mathbf{P}_\beta \cap K[x]$ is generated as an additive group by all the monomials in the Q_i of value β or higher, multiplied by elements of K . In particular, we have*

$$\bigoplus_{\beta \in \Gamma} \frac{\mathbf{P}_\beta}{\mathbf{P}_{\beta+}} = G_{\nu_0}[\text{in}_\nu \mathbf{Q}]^*.$$

Proof. We argue by contradiction. First, assume that Λ does not contain a maximal element. If $\{Q_i\}_{i \in \Lambda}$ is not complete then $\Lambda < \omega \times \omega$ by Remark 3.34. Then, according to the above

prescription, our construction could not have stopped at \mathbf{Q} and would have produced a further key polynomial Q_Λ . This is a contradiction. Hence \mathbf{Q} is complete.

Next, assume that Λ contains a maximal element λ . If \mathbf{Q} is not complete then our construction must have stopped because

$$\beta_\lambda \notin \tilde{\Gamma}. \quad (3.45)$$

Take an element $h \in K[x]$. Let $h = \sum_{j=0}^s c_{j\lambda} Q_\lambda^j$ be the λ -standard expansion of h . By (3.45) we have

$$\nu_\lambda(h) = \nu_\lambda(c_{\mu\lambda} Q_\lambda^\mu) = \nu(c_{\mu\lambda} Q_\lambda^\mu) = \nu(h).$$

This proves that \mathbf{Q} is a complete set of key polynomials, contradicting the assumption. Theorem 3.35 is proved. \square

Proposition 3.36. *If (3.41) holds and the sequence $\{\beta_{\ell+t}\}_{t \in \mathbb{N}_0}$ is cofinal in $\tilde{\Gamma}$, then the set $\mathbf{Q}_{\ell+\omega}$ of key polynomials defined above is $\tilde{\Gamma}$ -complete. In other words, for any element $\beta \in \tilde{\Gamma}_+$ any polynomial $f \in K[x]$ with $\nu(f) = \beta$ belongs to the additive subgroup of $\mathbf{P}_\beta \cap K[x]$ generated by all the monomials in the Q_i of value β or higher, multiplied by elements of K .*

Proof. To prove Proposition 3.36, it is sufficient to show that for every positive $\beta \in \tilde{\Gamma}_+$ and every $h \in K[x]$ such that $\nu(h) = \beta$, h belongs to the R_ν -submodule of $K[x]$ generated by all the monomials $c\mathbf{Q}^{\tilde{\gamma}}$ such that $\nu(c\mathbf{Q}^{\tilde{\gamma}}) \geq \beta$.

Take an element $h \in K[x]$. Since the sequence $\{\beta_{\ell+t}\}_{t \in \mathbb{N}_0}$ is cofinal in $\tilde{\Gamma}$, there exists i of the form $i = \ell + t$, $t \in \mathbb{N}_0$, such that

$$\beta_i > \nu(h). \quad (3.46)$$

Let

$$h = \sum_{j=0}^{s_i} c_j Q_i^j \quad (3.47)$$

be the i -standard expansion of h . Without loss of generality, we may assume that, writing $h = \sum_{j=0}^s d_j x^j$, we have

$$\nu_0(d_j) \geq 0 \quad \text{for all } j \quad (3.48)$$

(otherwise, multiply h by a suitable element of K). Then the i -standard expansion (3.47) satisfies the hypotheses of Proposition 3.29 (1). Now, Proposition 3.29 says that

$$\nu_i(h) = \nu(h).$$

This means, by definition, that h can be written as a sum of monomials in \mathbf{Q}_{i+1} of value at least $\nu(h)$, hence it belongs to the ideal generated by all such monomials. This completes the proof. \square

Remark 3.37. The proof of Theorem 3.35 (together with Proposition 3.29 (2)) shows that:

(1) if (3.41) holds and

$$\#\{\alpha_{\ell+t} \mid \alpha_{\ell+t} > 1\} = \infty, \quad (3.49)$$

then $\mathbf{Q}_{\ell+\omega}$ is a complete set of key polynomials and

(2) if $\mathbf{Q}_{\ell+\omega}$ is $\tilde{\Gamma}$ -complete then

$$\mathbf{Q}_{\ell+\omega} \cup \{Q_{\ell+\omega}\}$$

is a complete set of key polynomials.

Moreover, we will see below (at the end of §7) that if $\text{rk } \nu_0 = 1$,

$$\text{char } k_\nu = 0. \quad (3.50)$$

and $\mathbf{Q}_{\ell+\omega}$ is not a complete set of key polynomials then the sequence $\{\beta_{\ell+t}\}_{t \in \mathbb{N}_0}$ is cofinal in $\tilde{\Gamma}$.

In the next five sections we introduce the numerical character $\delta_i(h)$ and study the effect of differential operators on the Q_i in order to give a more precise description of the form of limit key polynomials in the case when $\text{rk } \nu_0 = 1$.

4 The numerical character $\delta_i(h)$

Let i be an ordinal such that Q_i is defined, and h an element of $K[x]$. Recall the definition of $\text{in}_i(h)$ ((3.21)–(3.22)). In this section we define the numerical character $\delta_i(h)$, which will play a crucial role in the rest of the paper. We prove that $\delta_i(h)$ does not increase with i . We also show that the equality $\delta_i(h) = \delta_{i+1}(h)$ imposes strong restrictions on $\text{in}_i h$.

Let $h = \sum_{j=0}^{s_i} d_{ji} Q_i^j$ be an i -standard expansion of h , where each d_{ji} is a Q_i -free i -standard expansion. The main definition of this section is: let

$$\delta_i(h) = \deg_{\bar{Q}_i} \text{in}_i h = \max S_i(h) \quad (4.1)$$

in the notation of (3.21)–(3.22).

Definition 4.1. The vertex $(\nu(d_{\delta_i(h),i}), \delta_i(h))$ of the Newton polygon $\Delta_i(h)$ is called the **pivotal vertex** of $\Delta_i(h)$.

Let

$$\nu_i^+(h) = \min \left\{ \nu(d_{ji} Q_i^j) \mid \delta_i(h) < j \leq s_i \right\} \quad (4.2)$$

and

$$S'_i(h) = \left\{ j \in \{\delta_i(h) + 1, \dots, s_i\} \mid \nu(d_{ji} Q_i^j) = \nu_i^+(h) \right\}.$$

If the set on the right hand side of (4.2) is empty, we adopt the convention that $\nu_i^+(h) = \infty$. We have $\delta_i(h) > 0$ whenever $\nu_i(h) < \nu(h)$.

Take an ordinal i such that Q_i and Q_{i+1} are defined. The fact that Q_{i+1} is defined means that there exists a polynomial $h \in K[x]$ such that $\nu_i(h) < \nu(h)$ (if $\nu_i(h) = \nu(h)$ for all $h \in K[x]$, the algorithm stops at Q_i). Take a polynomial h such that $\nu_i(h) < \nu(h)$. Consider the i -th Newton polygon of h . Let $S_i(h)$ be as in (3.21).

The next Proposition shows that $\delta_i(h)$ is non-increasing with i and that the equality $\delta_{i+1}(h) = \delta_i(h)$ imposes strong restrictions on $\text{in}_i h$.

Proposition 4.2. (1) *We have*

$$\alpha_{i+1} \delta_{i+1}(h) \leq \delta_i(h). \quad (4.3)$$

(2) *If $\delta_{i+1}(h) = \delta_i(h)$ then*

$$\text{in}_i h = \text{in}_\nu d_{\delta_i(h),i} (\bar{Q}_i + \text{in}_\nu z_i)^{\delta_i(h)}, \quad (4.4)$$

where z_i is some Q_i -free standard expansion, and $\text{in}_{i+1} h$ contains a monomial of the form $\text{in}_\nu d_{\delta_i(h),i} \bar{Q}_{i+1}^{\delta_i(h)}$; in particular,

$$\text{in}_\nu d_{\delta_i(h),i} = \text{in}_\nu d_{\delta_i(h),i+1}. \quad (4.5)$$

(3) If $\delta_{i+1}(h) = \delta_i(h)$, then for all $j > \delta_i(h)$ we have

$$\nu \left(d_{j,i+1} Q_{i+1}^j \right) - \nu_{i+1}(h) \geq \nu_i^+(h) - \nu_i(h) \quad (4.6)$$

Proof. We start with three Lemmas. First, consider the $(i+1)$ -standard expansion of h :

$$h = \sum_{j=0}^s d_{j,i+1} Q_{i+1}^j, \quad (4.7)$$

where the $d_{j,i+1}$ are Q_{i+1} -free standard expansions.

Lemma 4.3. (1) We have

$$\nu_i(h) = \min_{0 \leq j \leq s} \nu_i \left(d_{j,i+1} Q_{i+1}^j \right) = \min_{0 \leq j \leq s} \{ \nu(d_{j,i+1}) + j\alpha_{i+1}\beta_i \}.$$

(2) Let

$$S_{i,i+1} = \left\{ j \in \{0, \dots, s\} \mid \nu_i \left(d_{j,i+1} Q_{i+1}^j \right) = \nu_i(h) \right\}$$

and $j_0 = \max S_{i,i+1}$. Then $\delta_i(h) = \alpha_{i+1}j_0 + \delta_i(d_{j_0,i+1})$.

Proof. (1) Provisionally, let

$$\mu = \min_{0 \leq j \leq s} \nu_i \left(d_{j,i+1} Q_{i+1}^j \right) = \min_{0 \leq j \leq s} \{ \nu(d_{j,i+1}) + j\alpha_{i+1}\beta_i \},$$

$$S'_{i,i+1} = \left\{ j \in \{0, \dots, s\} \mid \nu_i \left(d_{j,i+1} Q_{i+1}^j \right) = \mu \right\},$$

$j' = \max S'_{i,i+1}$ and $\delta' = \alpha_{i+1}j' + \delta_i(d_{j',i+1})$. We want to show that $\mu = \nu_i(h)$, $S'_{i,i+1} = S_{i,i+1}$, $j' = j_0$ and $\delta_i(h) = \delta'$.

Let $\bar{h} = \sum_{j \in S'_{i,i+1}} d_{j,i+1} Q_{i+1}^j$. Then $\nu_i(h - \bar{h}) > \mu$, so to prove that $\nu_i(h) = \mu$ it is sufficient

to prove that $\nu_i(\bar{h}) = \mu$.

Write $d_{j',i+1} = d'_{t,i} Q_i^t + \dots + d'_{\delta_i(d_{j',i+1})} Q_i^{\delta_i(d_{j',i+1})} + \dots + d'_{0,i}$.

Write $Q_{i+1} = Q_i^{\alpha_{i+1}} + y$.

Now $d_{j',i+1} Q_{i+1}^{j'} = \left(d'_{t,i} Q_i^t + \dots + d'_{\delta_i(d_{j',i+1})} Q_i^{\delta_i(d_{j',i+1})} + \dots + d'_{0,i} \right) (Q_i^{\alpha_{i+1}} + y)^{j'}$. All the terms of the form $d'_{s,i} Q_i^s \binom{j'}{m} Q_i^{\alpha_{i+1}m} y^{j'-m}$ with $s > \delta_i(d_{j',i+1})$ satisfy:

$$\begin{aligned} \nu_i \left(d'_{s,i} Q_i^s \binom{j'}{m} Q_i^{\alpha_{i+1}m} y^{j'-m} \right) &\geq \nu_i \left(d'_{s,i} Q_i^s \right) + \nu_i \left(Q_i^{\alpha_{i+1}m} y^{j'-m} \right) \\ &> \nu_i \left(d'_{\delta_i(d_{j',i+1})} Q_i^{\delta_i(d_{j',i+1})} \right) + \nu_i \left(Q_i^{\alpha_{i+1}m} y^{j'-m} \right) \\ &= \nu_i \left(d'_{\delta_i(d_{j',i+1})} Q_i^{\delta_i(d_{j',i+1})} \right) + \nu_i \left(Q_i^{j'} \right) \\ &= \nu_i \left(d'_{\delta_i(d_{j',i+1})} Q_i^{\delta_i(d_{j',i+1}) + \alpha_{i+1}j'} \right). \end{aligned}$$

Now for $(s = \delta_i(d_{j',i+1})$ and $m \neq j')$ and for $s < \delta_i(d_{j',i+1})$ we have

$$\deg_x d'_{s,i} Q_i^s \binom{j'}{m} Q_i^{\alpha_{i+1}m} y^{j'-m} < \deg_x d'_{\delta_i(d_{j',i+1})} Q_i^{\delta_i(d_{j',i+1}) + \alpha_{i+1}j'}.$$

We have proved that the i -standard expansions of $d_{j',i+1}Q_{i+1}^{j'}$ contains an i -standard monomial of the form $dQ_i^{\delta'}$, such that $\nu_i(dQ_i^{\delta'}) = \nu_i(d_{j',i+1}Q_{i+1}^{j'})$.

All the i -standard monomials appearing in the i -standard expansion of $d_{j,i+1}Q_{i+1}^j$ for $j < j'$, have degree in x strictly less than $\deg_x dQ_i^{\delta'}$. Therefore $dQ_i^{\delta'}$ appears also in the i -standard expansion of \bar{h} . Thus

$$\nu_i(\bar{h}) \leq \nu_i(dQ_i^{\delta'}) = \nu_i(d_{j',i+1}Q_{i+1}^{j'}) = \mu,$$

so $\nu_i(h) \leq \mu$. The opposite inequality is trivial and (1) is proved. (2) follows immediately from this. \square

Lemma 4.4. Consider two terms of the form dQ_{i+1}^j and $d'Q_{i+1}^{j'}$ (where $j, j' \in \mathbb{N}_0$ and d and d' are Q_i -free i -standard expansions. Assume that

$$\nu_i(dQ_{i+1}^j) \leq \nu_i(d'Q_{i+1}^{j'}) \quad (4.8)$$

and

$$\nu(dQ_{i+1}^j) \geq \nu(d'Q_{i+1}^{j'}). \quad (4.9)$$

Then $j \geq j'$. If at least one of the inequalities (4.8), (4.9) is strict then $j > j'$.

Proof. Subtract (4.8) from (4.9) and use the definition of ν_i and the facts that

$$\nu_i(Q_{i+1}) = \alpha_{i+1}\beta_i < \beta_{i+1},$$

$$\nu_i(d) = \nu(d) \text{ and } \nu_i(d') = \nu(d'). \quad \square$$

In the notation of Lemma 4.3, let $\theta_{i+1}(h) = \min S_{i,i+1}$.

Definition 4.5. The vertex $(\nu(d_{\theta_{i+1}(h),i+1}), \theta_{i+1}(h))$ is called the **characteristic vertex** of $\Delta_{i+1}(h)$.

Let

$$\text{in}_i h = \text{in}_\nu d_{\delta_i} \prod_{j=1}^t g_{j_i}^{\gamma_{j_i}} \quad (4.10)$$

be the factorization of $\text{in}_i h$ into (monic) irreducible factors in $G_{\nu_0}[\text{in}_\nu \mathbf{Q}_i][\bar{Q}_i]$, where $\delta = \delta_i(h)$ and g_{1_i} is the minimal polynomial of $\text{in}_\nu Q_i$ over $G_{\nu_0}[\text{in}_\nu \mathbf{Q}_i]$.

Lemma 4.6. We have

$$\gamma_{1_i} = \theta_{i+1}(h) \quad (4.11)$$

(in particular, $d_{\gamma_{1_i},i+1} \neq 0$) and

$$\text{in}_\nu d_{\theta_{i+1}(h),i+1} = \text{in}_\nu d_{\delta_i} \prod_{j=2}^t g_{j_i}^{\gamma_{j_i}}(\text{in}_\nu Q_i). \quad (4.12)$$

Proof. Write

$$h = \sum_{q \in S_{i,i+1}} d_{q,i+1} Q_{i+1}^q + \sum_{q \in \{0, \dots, s\} \setminus S_{i,i+1}} d_{q,i+1} Q_{i+1}^q.$$

By Lemma 4.3,

$$\text{in}_i h = \sum_{q \in S_{i,i+1}} \text{in}_i d_{q,i+1} \text{in}_i Q_{i+1}^q. \quad (4.13)$$

By definition of $\theta_{i+1}(h)$, $\text{in}_i Q_{i+1}^{\theta_{i+1}(h)}$ is the highest power of $\text{in}_i Q_{i+1}$ dividing

$$\sum_{q \in \mathcal{S}_{i,i+1}} \text{in}_i d_{q,i+1} \text{in}_i Q_{i+1}^q.$$

Also by definition, we have

$$\text{in}_i Q_{i+1} = g_{1i}. \quad (4.14)$$

Now (4.11) follows from (4.13). Also from (4.13), we see that $\text{in}_\nu d_{\theta_{i+1}(h),i+1}$ is obtained by substituting $\text{in}_\nu Q_i$ into $\frac{\text{in}_i h}{\text{in}_i Q_{i+1}^{\gamma_{1i}}}$, and (4.12) follows. \square

Now, apply Lemma 4.4 to the monomials $d_{\theta_{i+1}(h),i+1} Q_{i+1}^{\theta_{i+1}(h)}$ and $d_{\delta_{i+1}(h),i+1} Q_{i+1}^{\delta_{i+1}(h)}$. We have

$$\nu \left(d_{\delta_{i+1}(h),i+1} Q_{i+1}^{\delta_{i+1}(h)} \right) \leq \nu \left(d_{\theta_{i+1}(h),i+1} Q_{i+1}^{\theta_{i+1}(h)} \right) \quad (4.15)$$

by definition of δ_{i+1} and

$$\nu_i \left(d_{\theta_{i+1}(h),i+1} Q_{i+1}^{\theta_{i+1}(h)} \right) = \nu_i(h) \leq \nu_i \left(d_{\delta_{i+1}(h),i+1} Q_{i+1}^{\delta_{i+1}(h)} \right) \quad (4.16)$$

by Lemma 4.3, so the hypotheses of Lemma 4.4 are satisfied. By Lemma 4.4

$$\theta_{i+1}(h) \geq \delta_{i+1}(h). \quad (4.17)$$

Since

$$\alpha_{i+1} \theta_{i+1}(h) = \alpha_{i+1} \gamma_{1i} \leq \deg_{\bar{Q}_i} \text{in}_i h = \delta_i(h) \quad (4.18)$$

by Lemma 4.6 and (4.10), (1) of the Proposition follows.

(2). Assume that $\delta_{i+1}(h) = \delta_i(h)$. Then the above two monomials coincide and

$$\alpha_{i+1} = 1. \quad (4.19)$$

Furthermore, we have equality in (4.18), so $\text{in}_i h = \text{in}_\nu d_{\delta_i(h),i} g_{1i}^{\delta_i(h)}$. Combined with (4.19), this proves (4.4).

The equality (4.5) follows from (4.12) and the fact that $\theta_{i+1}(h) = \delta_{i+1}(h)$.

(3). Assume that $\delta_{i+1}(h) = \delta_i(h)$. Fix an integer $j > \delta_i(h)$. For $j' < j$, monomials of the form $d_{j'i} Q_i^{j'}$ contribute nothing to $d_{j,i+1} Q_{i+1}^j$; in other words, the coefficient $d_{j,i+1}$ is completely determined by $\sum_{j'=j}^{s_i} d_{j'i} Q_i^{j'}$.

Fix an integer $j' \in \{j, \dots, s_i\}$. Write

$$d_{j'i} Q_i^{j'} = d_{j'i} (Q_{i+1} - z_i)^{j'} = d_{j'i} \sum_{k=0}^{j'} \binom{j'}{k} (-1)^k Q_{i+1}^k z_i^{j'-k}, \quad (4.20)$$

where z_i is an Q_i -free i -standard expansion. Again, the terms on the right hand side of (4.20) with $k > j$ contribute nothing to $d_{j,i+1} Q_{i+1}^j$. For $k \leq j$, let $d_{j'ki}$ denote the coefficient of Q_{i+1}^k in the $(i+1)$ -standard expansion of $d_{j'i} Q_{i+1}^k z_i^{j'-k}$. To prove (3), it is sufficient to prove that for all $j' \in \{j, \dots, s_i\}$ and all $k \in \{0, \dots, j\}$ we have

$$\nu \left(d_{j'ki} Q_{i+1}^k \right) - \nu_{i+1}(h) \geq \nu_i^+(h) - \nu_i(h). \quad (4.21)$$

To prove (4.21), we start out by noting that $\nu_i \left(d_{j'ki} Q_{i+1}^k z_i^{j'-k} \right) \leq \nu_i \left(d_{j'ki} Q_{i+1}^j \right)$, so

$$\nu \left(d_{j'ki} \right) = \nu_i \left(d_{j'ki} \right) \geq (j' - j)\beta_i + \nu \left(d_{j'i} \right). \quad (4.22)$$

Hence $\nu \left(d_{j'ki} Q_{i+1}^j \right) - \nu_{i+1}(h) = \nu \left(d_{j'ki} \right) - \nu \left(d_{\delta, i+1} \right) + (j - \delta)\beta_{i+1} \geq (j' - j)\beta_i + \nu \left(d_{j'i} \right) - \nu \left(d_{\delta, i+1} \right) + (j - \delta)\beta_i = \nu \left(d_{j'i} Q_i^{j'} \right) - \nu \left(d_{\delta i} Q_i^\delta \right) \geq \nu_i^+(h) - \nu_i(h)$, as desired.

This completes the proof of the Proposition. \square

Remark 4.7. One way of interpreting Lemma 4.4, together with the inequalities (4.15)–(4.17) is that the characteristic vertex $(\nu(d_{\theta_{i+1}(h), i+1}), \theta_{i+1}(h))$ of $\Delta_{i+1}(h)$ always lies above its pivotal vertex $(\nu(d_{\delta_{i+1}(h), i+1}), \delta_{i+1}(h))$.

Now, assume that $\mathbf{Q}_{\ell+1}$ is defined for a certain ordinal number ℓ and that ω iterations of the algorithm of §3 produce an infinite sequence $\{Q_{\ell+t}\}_{t \in \mathbb{N}_0}$.

Corollary 4.8 (of Proposition 4.2). *Assume that the set $\{t \in \mathbb{N} \mid \alpha_{\ell+t} > 1\}$ is infinite. Then the set $\mathbf{Q}_{\ell+\omega}$ of key polynomials constructed in §3 is complete.*

Proof. Take any element $h \in K[x]$. It is sufficient to show that $\nu_i(h) = \nu(h)$ for some i of the form $\ell + t$, $t \in \mathbb{N}$. Proposition 4.2 (1) says that

$$\delta_{i+1}(h) < \delta_i(h) \quad (4.23)$$

whenever $\delta_i(h) > 0$ and $\alpha_{i+1} > 1$. Since the set $\{t \in \mathbb{N} \mid \alpha_{\ell+t} > 1\}$ is infinite, and the inequality cannot occur infinitely many times, we have $\delta_i(h) = 0$ for some $i = \ell + t$, $t \in \mathbb{N}$. Then $\text{in}_i h$ does not involve \bar{Q}_i , hence $\nu_i(h) = \nu(h)$. \square

5 If $\text{rk } \nu_0 = 1$ and Case 2b holds then $\deg_{Q_\ell} Q_{\ell+\omega} = \delta(Q_{\ell+\omega})$

Let ℓ be an ordinal such that $Q_{\ell+t}$, $t \in \mathbb{N}_0$, and $Q_{\ell+\omega}$ are defined. To simplify the notation, in this section we will denote $Q_{\ell+\omega}$ by f . Then

$$\deg_{Q_\ell} f \geq \delta_\ell(f). \quad (5.1)$$

By Proposition 4.2 and Corollary 4.8, there exists $t_0 \in \mathbb{N}_0$ such that

$$\alpha_{\ell+t} = 1 \quad \text{and} \quad \delta_{\ell+t}(f) = \delta_{\ell+t_0}(f) \quad \text{for all } t \geq t_0. \quad (5.2)$$

Let $\delta(f)$ denote the stable value $\delta_{\ell+t}(f)$ for large t . The inequality (5.1) implies that

$$\deg_{Q_\ell} f \geq \delta(f). \quad (5.3)$$

Assume that $\text{rk } \nu_0 = 1$ and Case 2b of §3 holds. The main result of this section says that under these assumptions equality holds in (5.3).

Let

$$\bar{\beta} = \sup \{ \nu(Q_{\ell+t}) \mid t \in \mathbb{N} \}.$$

(here we allow the possibility $\bar{\beta} = \infty$, which means that the set $\{\nu(Q') \mid Q' \in T\}$ is unbounded in $\bar{\Gamma}$). Saying that we are in Case 2b amounts to saying that $\bar{\beta} \in \mathbb{R}$, that is, $\bar{\beta} < \infty$.

Below, in Proposition 7.6, we will show that $\delta(f)$ is of the form $\delta(f) = p^{e_0}$ for some $e_0 \in \mathbb{N}_0$. Together with Remark 3.33 this will prove that, under the assumptions of this section, we have $\text{char } k_\nu > 0$ and $e_0 > 0$.

Replacing ℓ by $\ell + s$ for a suitable positive integer s , we may assume that $\alpha_{\ell+t} = 1$ for all strictly positive t . In what follows, the index i will run over the set $\{\ell + t\}_{t \in \mathbb{N}_0}$. Let $\delta = \delta(f)$.

Lemma 5.1. *Take polynomials $h, g \in K[x]$ such that $\deg_x h, \deg_x g < \deg_x Q_i$. Let*

$$hg = uQ_i + c$$

be the Euclidian division of hg by Q_i . Then $\nu(c) = \nu(hg) < \nu(uQ_i)$.

Proof. We have $\nu_\ell(f) = \nu(h)$ and $\nu_\ell(g) = \nu(g)$, so $\nu_\ell(hg) = \nu(hg) = \nu_i(hg)$. Using Lemma 4.3 we obtain $\min\{\nu(c), \nu_i(uQ_i)\} = \nu_i(hg) = \nu_\ell(hg) = \min\{\nu(c), \nu_\ell(uQ_i)\}$. Since

$$\nu_i(uQ_i) > \nu_\ell(uQ_i),$$

we see that $\nu(c) = \nu_i(hg) = \nu_\ell(hg) = \nu(hg) < \nu_i(uQ_i) = \nu(uQ_i)$, as desired. \square

Proposition 5.2. *For each*

$$i \in \{\ell + t\}_{t \in \mathbb{N}_0}$$

we have

$$\deg_x f = \delta \deg_x Q_i \tag{5.4}$$

and

$$\bar{\beta} \leq \frac{1}{\delta} \nu(f). \tag{5.5}$$

Proof. The inequality (5.5) is equivalent to saying that

$$\nu(f) > \delta \nu(Q_\ell + z_\ell + \cdots + z_{\ell+t}) \tag{5.6}$$

for all $t \in \mathbb{N}_0$. This follows from Remark 3.33 (cf. (3.44)). The main point is to prove (5.4).

For i of the form $\ell + t$, $t \in \mathbb{N}$, let

$$f = \sum_{j=0}^{n_i} a_{ji} Q_i^j \tag{5.7}$$

be the i -standard expansion of f .

Since $\alpha_i = 1$ for all i , all the i -standard expansions of f have the same degree n_ℓ in Q_i .

Let

$$\theta(i) = \min\{\nu_i^+(f) - \nu_i(f), \beta_i - \beta_\ell\}; \tag{5.8}$$

we have $\theta(i) > 0$.

By (4.6) the quantity $\nu_i^+(f) - \nu_i(f)$ is increasing with i and hence so is $\theta(i)$. Taking into account the fact that $\bar{\beta} = \lim_{i \rightarrow \infty} \beta_i$, we have, for i sufficiently large,

$$\nu(a_{\delta i}) + \delta \bar{\beta} - \nu_i(f) = \delta(\bar{\beta} - \beta_i) < \theta(i). \tag{5.9}$$

By choosing $\ell_1 > \ell$ sufficiently large, we may assume that (5.9) holds for $i \geq \ell_1$.

Let $a^* \in K[x]$ denote a polynomial such that

$$\text{in}_\nu a^* \text{in}_\nu a_{\delta \ell} = 1 \tag{5.10}$$

in G_ν . According to Lemma 2.6 we may choose a^* to be of degree strictly less than $\deg_x Q_\ell$; this condition determines $\text{in}_\nu a^*$ uniquely. Note that

$$\text{in}_\nu a_{\delta \ell} = \text{in}_\nu a_{\delta i} \quad \text{for all } i \geq \ell \tag{5.11}$$

by Proposition 4.2 (2).

By Proposition 4.2 (2), for all $i \geq \ell$ we have

$$\text{in}_i f = \text{in}_\nu a_{\delta i} (\bar{Q}_i + \text{in}_\nu z_i)^\delta,$$

hence in view of (5.10)–(5.11) we have $\text{in}_i(a^* f) = (\bar{Q}_i + \text{in}_\nu z_i)^\delta$. In particular,

$$\nu(a^* f) > \nu_i(a^* f) \quad \text{for all } i. \quad (5.12)$$

We claim that multiplying by a^* does not affect δ .

Indeed, applying Lemma 5.1 to the pairs of polynomials

$$(h, g) = (a_{\delta i}, a^*)$$

and

$$(h, g) = (d_{\delta-1, i}, a^*),$$

we see that after multiplying f by a^* and applying Euclidean division by Q_i to obtain the i -standard expansion of $a^* f$, only the remainders in the Euclidean division contribute to $\text{in}_i a^* f$. In particular, $\delta_i(a^* f) = \delta$ for all $i \geq \ell$.

Thus, replacing f by $a^* f$, we may assume that $\text{in}_\nu a_{\delta i} = 1$ for all i .

Next, write $a_{\delta i} = 1 + a_i^\dagger$ with

$$\nu(a_i^\dagger) \geq \beta_i - \beta_\ell > 0 \quad (5.13)$$

Write

$$f = \bar{f} + \tilde{f},$$

where

$$\bar{f} = Q_{\ell_1}^\delta + \sum_{j=0}^{\delta-1} a_{j\ell_1} Q_{\ell_1}^j$$

and

$$\tilde{f} = a_{\ell_1}^\dagger Q_{\ell_1}^\delta + \sum_{j=\delta+1}^{n_\ell} a_{j\ell_1} Q_{\ell_1}^j.$$

(5.9) implies that for all j with $\delta < j \leq n_\ell$ and all $i \geq \ell_1$, we have

$$\nu_i(a_{j\ell_1} Q_{\ell_1}^j) \geq \nu_{\ell_1}(a_{j\ell_1} Q_{\ell_1}^j) \geq \nu_{\ell_1}^+(f) > \delta \bar{\beta} > \delta \beta_i = \nu_i(f).$$

From (5.9) and (5.13), we see that

$$\nu_i(a_{\ell_1}^\dagger Q_{\ell_1}^\delta) \geq \nu_{\ell_1}(a_{\ell_1}^\dagger Q_{\ell_1}^\delta) = \nu_{\ell_1}(a_{\ell_1}^\dagger) + \delta \beta_{\ell_1} \geq \beta_{\ell_1} - \beta_\ell + \delta \beta_{\ell_1} \geq \theta(\ell_1) + \delta \beta_{\ell_1} > \delta \bar{\beta} > \nu_i(f).$$

Hence, for all i , $\nu(\tilde{f}) \geq \nu_i(\tilde{f}) > \nu_i(\bar{f})$ which implies that $\text{in}_i f = \text{in}_i \bar{f}$ and so $\nu_i(f) = \nu_i(\bar{f}) < \nu(\bar{f})$. Since f was chosen of minimal degree with respect to the latter property, we must have $\deg_x f = \deg_x \bar{f} = \delta \deg_x Q_{\ell_1}$. \square

6 Key polynomials and differential operators

This section is devoted to proving some basic results about the effect of differential operators on key polynomials. Here and below, for a non-negative integer b , ∂_b will denote the differential operator $\frac{1}{b!} \frac{\partial^b}{\partial x^b}$. Given an ℓ -standard expansion h , we are interested in proving lower bounds on (and, in some cases, exact formulae for) the quantities $\nu(\partial_b h)$ and $\nu_\ell(\partial_b h)$ and the elements $\text{in}_\nu \partial_b h$ and $\text{in}_\ell \partial_b h$. In particular, we will give sufficient conditions for the element $\partial_b h$ to be non-zero.

Take an ordinal $i \leq \ell$ and assume that the key polynomials \mathbf{Q}_{i+1} are defined. Let b_i denote the smallest positive integer which maximizes the quantity $\frac{\beta_i - \nu(\partial_{b_i} \mathbf{Q}_i)}{b_i}$ (later in this section, we will show that b_i is necessarily of the form p^{e_i} , for some $e_i \in \mathbb{N}_0$ and, in particular, that $b_i = 1$ if $\text{char } k_\nu = 0$).

Let h be any element of $K[x]$. One of our main tasks in this section is studying the quantities $\nu(\partial_b h)$ and $\nu_i(\partial_b h)$. We use the following convention for binomial coefficients: if $s < t$, the binomial coefficient $\binom{s}{t}$ is considered to be 0. We may view the binomial coefficients as elements of K via the natural map $\mathbb{Z} \rightarrow K$.

Notation:

Let p be as defined in the Introduction. If $p > 1$, for an integer a we shall denote by $\nu^{(p)}(a)$ the p -adic value of a , that is, the greatest power of p which divides a . If $p = 1$, we adopt the convention $\nu^{(p)}(a) = 1$ for all non-zero a and $\nu^{(p)}(0) = \infty$.

Proposition 6.1. *Take an element $h \in K[x]$.*

(1) *For all $b \in \mathbb{N}_0$ we have*

$$\nu_i(h) - \nu_i(\partial_b h) \leq \frac{b}{b_i} (\beta_i - \nu(\partial_{b_i} \mathbf{Q}_i)). \quad (6.1)$$

(2) *Let $h = \sum_{j=0}^s d_{ji} \mathbf{Q}_i^j$ be the i -standard expansion of h . Assume that*

$$\left\{ j \in \{0, \dots, s\} \mid \nu(d_{ji} \mathbf{Q}_i^j) = \nu_i(h) \right\} \neq \{0\}$$

(in particular, we have $s > 0$). Let $d_{ji} \mathbf{Q}_i^j$ denote the term in the i -standard expansion of h which minimizes the triple $(\nu_i(d_{ji} \mathbf{Q}_i^j), \nu^{(p)}(j), j)$ in the lexicographical ordering. Let $e = \nu^{(p)}(j)$ and $b(i, h) = b_i p^e$. Then equality holds in (6.1) for $b = b(i, h)$.

Remark 6.2. Let $i_0 = i - 1$ if i admits an immediate predecessor and let i_0 be as in (3.10) otherwise. For all $b \in \mathbb{N}$ we have

$$\nu(\partial_b \mathbf{Q}_i) = \nu_{i_0}(\partial_b \mathbf{Q}_i); \quad (6.2)$$

this holds by Proposition 3.29 (2). In particular $\nu(\partial_{b_i} \mathbf{Q}_i) = \nu_{i_0}(\partial_{b_i} \mathbf{Q}_i)$. Thus replacing $\nu(\partial_{b_i} \mathbf{Q}_i)$ by $\nu_{i_0}(\partial_{b_i} \mathbf{Q}_i)$ in (6.1) gives rise to an equivalent inequality. Also, $\nu_i(\partial_b h) \leq \nu(\partial_b h)$, so replacing $\nu_i(\partial_b h)$ by $\nu(\partial_b h)$ in (6.1) gives rise to a true, but an a priori weaker inequality.

Proof of Proposition 6.1. We prove Proposition 6.1 by transfinite induction. For $i = 1$ we have $b_i = 1$ and the result is obvious. Assume that $i > 1$ and that the result is known for all the ordinals strictly smaller than i .

Lemma 6.3. *Consider a pair of ordinals i', i'' such that $i' < i'' \leq i$. Then*

$$\frac{\beta_{i'} - \nu(\partial_{b_{i'}} Q_{i'})}{b_{i'}} < \frac{\beta_{i''} - \nu(\partial_{b_{i''}} Q_{i''})}{b_{i''}}. \quad (6.3)$$

Proof. By transfinite induction on the ordinal $i'' - i'$, we may assume that $i'' = i' +$, and that $Q_{i''}$ admits an i' -standard expansion of the form (3.8) or (3.10), depending on whether or not i'' is a limit ordinal. Moreover, we may assume that for every positive integer \tilde{b} we have $\nu(\partial_{\tilde{b}} Q_{i''}) = \nu_{i'}(\partial_{\tilde{b}} Q_{i''})$.

By definition of $b_{i''}$, it is sufficient to prove that there exists a strictly positive integer \tilde{b} such that (6.3) holds with $b_{i''}$ replaced by \tilde{b} .

We take $\tilde{b} := b(i', Q_{i''})$. We have:

$$\begin{aligned} \beta_{i''} - \nu(\partial_{\tilde{b}} Q_{i''}) &> \nu_{i'}(Q_{i''}) - \nu(\partial_{\tilde{b}} Q_{i''}) = \nu_{i'}(Q_{i''}) - \nu_{i'}(\partial_{\tilde{b}} Q_{i''}) = \\ &= \frac{\tilde{b}}{b_{i'}} (\beta_{i'} - \nu(\partial_{b_{i'}} Q_{i'})). \end{aligned}$$

Here the first inequality is given by Proposition 3.22 (1), the first equality by Remark 6.2 and the second equality by Proposition 6.1 (2) applied to $i' < i$, which we are allowed to use by the induction assumption. This completes the proof of the Lemma. \square

To prove Proposition 6.1 (1), it is sufficient to prove it for each i -standard monomial appearing in the i -standard expansion of h . Indeed, assume that the result is true for each i -standard monomial $d_{ji} Q_i^j$ appearing in the i -standard expansion of h . This means that for each j we have

$$\nu(\partial_b d_{ji} Q_i^j) \geq \nu(d_{ji} Q_i^j) - \frac{b}{b_i} (\beta_i - \nu(\partial_{b_i} Q_i)).$$

Thus $\nu_i(\partial_b h) \geq \min_j \nu(\partial_b d_{ji} Q_i^j) \geq \min_j \nu(d_{ji} Q_i^j) - \frac{b}{b_i} (\beta_i - \nu(\partial_{b_i} Q_i)) = \nu_i(h) - \frac{b}{b_i} (\beta_i - \nu(\partial_{b_i} Q_i))$.

Let $\mathbf{Q}_{i+1}^{\bar{\gamma}_{i+1}}$ be such an i -standard monomial. Let $\bar{\gamma}_{i+1} = (\gamma_j \mid j \leq i)$ and write

$$\mathbf{Q}_{i+1}^{\bar{\gamma}_{i+1}} = \mathbf{Q}_i^{\bar{\gamma}_i} Q_i^{\gamma_i}.$$

We want to expand $\partial_b \mathbf{Q}_{i+1}^{\bar{\gamma}_{i+1}}$ in terms of products of the form $Q_i^{\gamma_i - q} (\partial_{j_0} \mathbf{Q}_i^{\bar{\gamma}_i}) \prod_{t=1}^q (\partial_{j_t} Q_i)$, where $q \leq \gamma_i$ and $j_0 + j_1 + \dots + j_q = b$. Each such product appears in the sum with a certain positive integer coefficient that we will now compute explicitly.

To do that, we first prove some general formulae about formal derivatives of products and powers of polynomials.

Lemma 6.4. *For any two polynomials A and B and any positive integer b , we have*

$$\partial_b(AB) = \sum_{j=0}^b (\partial_j A)(\partial_{b-j} B). \quad (6.4)$$

Proof. Let $m = \deg A$ and $n = \deg B$. By definition, formal derivatives are the coefficients in Taylor expansions:

$$A(X+Y) = \sum_{i=0}^m \partial_i A(X) Y^i \quad \text{and} \quad B(X+Y) = \sum_{i=0}^n \partial_i B(X) Y^i.$$

We obtain

$$AB(X + Y) = A(X + Y)B(X + Y) = \sum_{b=0}^{m+n} \left(\sum_{j=0}^b \partial_j A(X) \partial_{b-j} B(X) \right) Y^b.$$

Since the coefficients in the Taylor expansion are uniquely determined, this proves (6.4). \square

For positive integers a_1, \dots, a_q and an integer $n \geq q$, we define the multiplicities n_1, \dots, n_k of a_1, \dots, a_q as follows. Let n_1 be the number of appearances of the smallest element of $\{a_1, \dots, a_q\}$ in the sequence a_1, \dots, a_q . Let n_2 be the number of appearances of the second smallest element, and so on until n_k , which is, by definition, the number of appearances of the largest element.

Notation. Let $C_n(a_1, \dots, a_q) = \frac{n!}{(n-q)! \cdot n_1! \cdots n_k!}$.

Lemma 6.5. *For any polynomial B and any positive integers b and n we have*

$$\partial_b(B^n) = \sum_{\substack{j_1 + \dots + j_q = b \\ 1 \leq q \leq n}} C_n(j_1, \dots, j_q) B^{n-q} \prod_{t=1}^q (\partial_{j_t} B), \quad (6.5)$$

Proof. Let $m = \deg B$. We have

$$B(X + Y)^n = \left(\sum_{i=0}^m \partial_i B(X) Y^i \right)^n = \sum_{b=0}^{mn} \left(\sum_{j_1 + \dots + j_n = b} \prod_{t=1}^n \partial_{j_t} B(X) \right) Y^b, \quad (6.6)$$

where the j_t run over non-negative integers.

For each product appearing in parentheses on the right hand side of (6.6), let

$$q = \# \{t \in \{1, \dots, n\} \mid j_t \neq 0\}.$$

Then we can rewrite (6.6) as

$$B(X + Y)^n = \sum_{b=0}^{mn} \left(\sum_{j_1 + \dots + j_q = b} B^{n-q} \prod_{t=1}^q \partial_{j_t} B(X) \right) Y^b, \quad (6.7)$$

where the j_t run over strictly positive integers. Now we have to count how often the same product $B^{n-q} \prod_{t=1}^q \partial_{j_t} B(X)$ appears in the second sum on the right hand side. How many distinct n -tuples can we obtain from the numbers j_1, \dots, j_n ? If all the j_t are distinct and $q \leq 1$, then the number is $n!$. But if some of the j_t 's are equal, then permuting them only among themselves does not produce new tuples. Similarly, if $q \geq 2$, permuting the $(n - q)$ factors in B^{n-q} among themselves does produce new tuples. Let the numbers n_1, \dots, n_k be the multiplicities of the numbers j_1, \dots, j_n , defined above. Then the number of appearances of $B^{n-q} \prod_{t=1}^q \partial_{j_t} B(X)$ is

$$\frac{n!}{(n-q)! \cdot n_1! \cdots n_k!} = C_n(j_1, \dots, j_q). \quad (6.8)$$

This completes the proof. \square

Lemma 6.6. *For any two polynomials A and B and any positive integers b and n , we have*

$$\partial_b(AB^n) = \sum_{\substack{j_0+j_1+\dots+j_q=b \\ q \leq n}} C_n(j_1, \dots, j_q) B^{n-q} (\partial_{j_0} A) \prod_{t=1}^q (\partial_{j_t} B), \quad (6.9)$$

Proof. By Lemma 6.4, we have $\partial_b(AB^n) = \sum_{j_0=0}^b (\partial_{j_0} A) (\partial_{b-j_0} B^n)$. Now Lemma 6.6 follows from Lemma 6.5. \square

Coming back to the proof of Proposition 6.1 (1), we have

$$\partial_b \mathbf{Q}_{i+1}^{\tilde{\gamma}^{i+1}} = \sum_{\substack{j_0+j_1+\dots+j_q=b \\ q \leq \gamma_i}} C_{\gamma_i}(j_1, \dots, j_q) Q_i^{\gamma_i-q} (\partial_{j_0} \mathbf{Q}_i^{\tilde{\gamma}^i}) \prod_{t=1}^q (\partial_{j_t} Q_i), \quad (6.10)$$

where $0 < j_1 \leq \dots \leq j_q$, by Lemma 6.6.

By Proposition 3.29 (2), we have

$$\nu_i(\partial_{j_t} Q_i) = \nu(\partial_{j_t} Q_i).$$

We have

$$\beta_i - \nu_i(\partial_{j_t} Q_i) = \beta_i - \nu(\partial_{j_t} Q_i) \leq \frac{j_t}{b_i} (\beta_i - \nu(\partial_{b_i} Q_i)) \quad (6.11)$$

by Remark 6.2 and definition of b_i . Further,

$$\nu_i(\mathbf{Q}_i^{\tilde{\gamma}^i}) - \nu_i(\partial_{j_0} \mathbf{Q}_i^{\tilde{\gamma}^i}) = \nu_{i_0}(\mathbf{Q}_i^{\tilde{\gamma}^i}) - \nu_{i_0}(\partial_{j_0} \mathbf{Q}_i^{\tilde{\gamma}^i}) \leq \frac{j_0}{b_i} (\beta_i - \nu(\partial_{b_i} Q_i)), \quad (6.12)$$

where i_0 is sufficiently large with $i_0 + = i$, the equality holds because Q_i does not appear in $\mathbf{Q}_i^{\tilde{\gamma}^i}$ and by Remark 6.2, and the inequality by the induction assumption and in view of Lemma 6.3. Note that the last inequality in (6.12) is strict whenever $j_0 > 0$. Adding the inequalities (6.11) for $1 \leq t \leq q$ and (6.12), we obtain:

$$\begin{aligned} & \nu_i(\mathbf{Q}_{i+1}^{\tilde{\gamma}^{i+1}}) - \nu_i \left(Q_i^{\gamma_i-q} (\partial_{j_0} \mathbf{Q}_i^{\tilde{\gamma}^i}) \prod_{t=1}^q (\partial_{j_t} Q_i) \right) \leq \\ & \leq \frac{j_0 + j_1 + \dots + j_q}{b_i} (\beta_i - \nu(\partial_{b_i} Q_i)) = \frac{b}{b_i} (\beta_i - \nu(\partial_{b_i} Q_i)). \end{aligned} \quad (6.13)$$

By (6.10), (6.13) and since ν is non-negative on \mathbb{N} (in particular, $\nu(C_{\gamma_i}(j_1, \dots, j_q)) \geq 0$), we have

$$\begin{aligned} & \nu_i(\mathbf{Q}_{i+1}^{\tilde{\gamma}^{i+1}}) - \nu_i(\partial_b \mathbf{Q}_{i+1}^{\tilde{\gamma}^{i+1}}) \leq \\ & \leq \nu_i(\mathbf{Q}_{i+1}^{\tilde{\gamma}^{i+1}}) - \min_{(j_0, \dots, j_q)} \left\{ \nu_i \left(Q_i^{\gamma_i-q} (\partial_{j_0} \mathbf{Q}_i^{\tilde{\gamma}^i}) \prod_{t=1}^q (\partial_{j_t} Q_i) \right) \right\} \leq \frac{b}{b_i} (\beta_i - \nu(\partial_{b_i} Q_i)), \end{aligned} \quad (6.14)$$

as desired. Proposition 6.1 (1) is proved.

Now let the notation be as in Proposition 6.1 (2). To prove this part, we first show that replacing $\nu_i(\partial_b h)$ by $\nu \left(\left(\partial_{b_i} Q_i^{p^e} \right) d_{j_i} Q_i^{j_i - p^e} \right)$ gives equality in (6.1).

Indeed, we have $\nu_i(h) - \nu \left(\left(\partial_{b_i} Q_i^{p^e} \right) d_{j_i} Q_i^{j-p^e} \right) = \nu_i(h) - (p^e \nu_i(\partial_{b_i} Q_i) + \nu(d_{j_i}) + (j - p^e) \nu(Q_i)) = p^e \nu(Q_i) - p^e \nu(\partial_{b_i} Q_i) = \frac{b_i p^e}{b_i} (\beta_i - \nu_i(\partial_{b_i} Q_i)) = \frac{b_i}{b_i} (\beta_i - \nu_i(\partial_{b_i} Q_i))$.

Therefore by part (1) of the Proposition, $\nu_i(\partial_b h)$ must be at least equal to

$$\nu \left(\left(\partial_{b_i} Q_i^{p^e} \right) d_{j_i} Q_i^{j-p^e} \right)$$

and it is sufficient to prove that the i -standard expansion of $\partial_b h$ contains a term of the form $dQ_i^{j-p^e}$ with $\nu(d) = \nu \left(\left(\partial_{b_i} Q_i^{p^e} \right) d_{j_i} \right)$ and all the other terms $d'Q_i^{j'}$ satisfy either $j' \neq j - p^e$ or $\nu_i(d'Q_i^{j'}) > \nu_i(dQ_i^{j-p^e})$.

We proceed by considering all the terms of the form $\mathbf{Q}_{i+1}^{\tilde{\gamma}_{i+1}}$ that appear in the i -standard expansion of h .

First, consider such a monomial appearing in $d_{j_i} Q_i^j$. Write $b = \sum_{t=1}^q j_t$, where $q = p^e$ and $j_t = b_i$ for all t . For each Q_i -free standard monomial $\mathbf{Q}_i^{\tilde{\gamma}_i}$, appearing in d_{j_i} , the corresponding term in (6.10) is $\binom{j}{p^e} Q_i^{j-p^e} \mathbf{Q}_i^{\tilde{\gamma}_i} (\partial_{b_i} Q_i)^{p^e}$, by (6.8). Put $d = \binom{j}{p^e} \mathbf{Q}_i^{\tilde{\gamma}_i} (\partial_{b_i} Q_i)^{p^e}$. Hence

$$\nu(d) = \nu \left(\left(\partial_{b_i} Q_i^{p^e} \right) d_{j_i} \right).$$

Now for any other choice of j_0, j_1, \dots, j_t such that $q = p^e$ we would have either $j_0 \neq 0$ or at least one t such that $j_t < b_i$, therefore such terms satisfy strict inequality in (1) since they satisfy strict equality in (6.12) or in (6.11) and hence their valuation is strictly greater than $\nu_i(dQ_i^{j-p^e})$.

Therefore we have the equality

$$\nu_i(d_{j_i} Q_i^j) - \nu_i(\partial_b d_{j_i} Q_i^j) = \frac{b}{b_i} (\beta_i - \nu(\partial_{b_i} Q_i)). \quad (6.15)$$

Now assume $\mathbf{Q}_{i+1}^{\tilde{\gamma}_{i+1}}$ is such that $\nu \left(\mathbf{Q}_{i+1}^{\tilde{\gamma}_{i+1}} \right) > \nu_i \left(d_{j_i} Q_i^j \right)$. By (6.1) we have

$$\nu \left(\mathbf{Q}_{i+1}^{\tilde{\gamma}_{i+1}} \right) - \nu \left(\partial_b \mathbf{Q}_{i+1}^{\tilde{\gamma}_{i+1}} \right) \leq \frac{b}{b_i} (\beta_i - \nu(\partial_{b_i} Q_i))$$

and using (6.15) we find that $\nu \left(\partial_b \mathbf{Q}_{i+1}^{\tilde{\gamma}_{i+1}} \right) > \nu_i \left(dQ_i^{j-p^e} \right)$.

Now consider terms $\mathbf{Q}_{i+1}^{\tilde{\gamma}_{i+1}}$ that appear in an expression $d_{m_i} Q_i^m$ such that

$$\nu \left(\mathbf{Q}_{i+1}^{\tilde{\gamma}_{i+1}} \right) = \nu_i \left(d_{j_i} Q_i^j \right).$$

It is sufficient to show that for $j' = m - q = j - p^e$ such terms satisfy the strict inequality in (6.1), so in view of (6.15) we deduce that their valuation is strictly greater than $\nu_i \left(dQ_i^{j-p^e} \right)$.

Take one such term. We have two cases. If $m > j$ then for $j' = m - q = j - p^e$, we must have $q > p^e$, so for any choice of j_0, j_1, \dots, j_t we must have at $j_0 \neq 0$ or at least one t such that $j_t < b_i$. Therefore such terms satisfy the strict inequality in (6.1) since they satisfy the strict inequality in (6.12) or (6.11).

If $m < j$ then $m = up^s$ with $s > e$ and hence if $m - q = j - p^e$ then $q = n.p^e$ so n must be positive so $q > p^e$, so for any choice of j_0, j_1, \dots, j_t we must have at $j_0 \neq 0$ or at least one t such that $j_t < b_i$, therefore such terms satisfy strict inequality in (1) by (6.12) and (6.11). \square

Remark 6.7. Let

$$I_{i,max} = \left\{ \tilde{b}_i \in \mathbb{N}_0 \mid \frac{\beta_i - \nu(\partial_{b_i} Q_i)}{b_i} = \frac{\beta_i - \nu(\partial_{\tilde{b}_i} Q_i)}{\tilde{b}_i} \right\}. \quad (6.16)$$

By definitions, we have $b_i = \min I_{i,max}$. Of course, Proposition 6.1 holds equally well with b_i replaced by \tilde{b}_i . Similarly, Lemma 6.3 holds if the pair $(b_{i'}, b_{i''})$ is replaced by $(\tilde{b}_{i'}, \tilde{b}_{i''})$ with $\tilde{b}_{i'} \in \mathbb{N}$, $\tilde{b}_{i''} \in I_{i'',max}$.

Corollary 6.8. *For each ordinal $i \leq \ell$, each $\tilde{b}_i \in I_{i,max}$ is of the form $\tilde{b}_i = p^{\tilde{e}_i}$ for some $\tilde{e}_i \in \mathbb{N}_0$. In particular, $b_i = p^{e_i}$ for some $e_i \in \mathbb{N}_0$. In the special case when $\text{char } k_\nu = 0$ we have $p = 1$ and so $I_{i,max} = \{b_i\} = \{1\}$.*

Proof. Write $\tilde{b}_i = p^{\tilde{e}_i} u$, where $p \nmid u$ if $\text{char } k_\nu = p > 0$, and $p^{\tilde{e}_i} = 1$ if $\text{char } k_\nu = 0$. We want to prove that $u = 1$. We argue by contradiction. Assume that $u > 1$. We claim that we can write

$$\binom{\tilde{b}_i}{b'} \partial_{\tilde{b}_i} = \partial_{b'} \circ \partial_{b''}, \quad (6.17)$$

where b', b'' are strictly positive integers such that

$$b' + b'' = \tilde{b}_i \quad (6.18)$$

and

$$\nu \left(\binom{\tilde{b}_i}{b'} \right) = 0 \quad (6.19)$$

Indeed, we can take $b' = p^{\tilde{e}_i}$ and $b'' = p^{\tilde{e}_i}(u - 1)$. Now, by Remark 6.11 below, p does not divide $\binom{\tilde{b}_i}{b'}$ and therefore its natural image in K is non-zero and its value is 0 (as usual, we view $\binom{\tilde{b}_i}{b'}$ as an element of K via the natural map $\mathbb{N} \rightarrow K$).

Let $i_0 = i - 1$ if i admits an immediate predecessor and let i_0 be as in (3.10) otherwise. We have

$$\beta_i - \nu(\partial_{\tilde{b}_i} Q_i) = (\beta_i - \nu(\partial_{b''} Q_i)) + (\nu_{i_0}(\partial_{b''} Q_i) - \nu_{i_0}(\partial_{\tilde{b}_i} Q_i)) \quad (6.20)$$

by (6.2). By (6.17), we have $\partial_{\tilde{b}_i} Q_i = \frac{b'!b''!}{\tilde{b}_i!} \partial_{b'}(\partial_{b''} Q_i)$. Hence

$$\nu_{i_0}(\partial_{b''} Q_i) - \nu_{i_0}(\partial_{\tilde{b}_i} Q_i) \leq \frac{b'}{b_{i_0}} \left(\beta_{i_0} - \nu(\partial_{\tilde{b}_{i_0}} Q_{i_0}) \right) < \frac{b'}{b_i} \left(\beta_i - \nu(\partial_{\tilde{b}_i} Q_i) \right) \quad (6.21)$$

by (6.19), Proposition 6.1 (1) and Lemma 6.3. From (6.20)–(6.21) we obtain

$$\beta_i - \nu(\partial_{b''} Q_i) > \left(1 - \frac{b'}{\tilde{b}_i} \right) \left(\beta_i - \nu(\partial_{\tilde{b}_i} Q_i) \right) = \frac{b''}{\tilde{b}_i} \left(\beta_i - \nu(\partial_{\tilde{b}_i} Q_i) \right)$$

which contradicts the fact that $\tilde{b}_i \in I_{i,max}$. Corollary 6.8 is proved. \square

Next, we investigate further the case of equality in (6.1). We give a necessary condition on h and b for the equality to hold in (6.1) and prove that this condition is sufficient under some additional assumptions. Finally, we derive a formula for $\text{in}_i h$ in the case when this criterion for equality in (6.1) holds. We start with the case when h is a single i -standard monomial.

Proposition 6.9. Consider an i -standard monomial $h = \mathbf{Q}_{i+1}^{\tilde{\gamma}_{i+1}}$. Write

$$b_i = p^{e_i} \quad \text{and} \quad (6.22)$$

$$\gamma_i = p^e u, \quad \text{where } p \nmid u \text{ if } p > 1. \quad (6.23)$$

(1) If equality holds in (6.1) then

$$p^{e+e_i} \mid b. \quad (6.24)$$

(2) We have the following partial converse to (1). Assume that (6.24) holds and that

$$\text{either } b = p^{e+e_i} \quad \text{or} \quad I_{i,max} = \{b_i\}. \quad (6.25)$$

Then equality holds in (6.1) if and only if

$$\nu \left(\binom{u}{b/p^{e+e_i}} \right) = 0. \quad (6.26)$$

(3) Assume that (6.24)–(6.26) hold. Then

$$\text{in}_i \partial_b \mathbf{Q}_{i+1}^{\tilde{\gamma}_{i+1}} = \binom{u}{b/p^{e+e_i}} \text{in}_i \left(\mathbf{Q}_i^{\tilde{\gamma}_i} Q_i^{\gamma_i - \frac{b}{b_i}} (\partial_{b_i} Q_i)^{\frac{b}{b_i}} \right); \quad (6.27)$$

in particular, $\partial_b \mathbf{Q}_{i+1}^{\tilde{\gamma}_{i+1}} \neq 0$.

Remark 6.10. If $b = p^{e+e_i}$ holds in Proposition 6.9 (2) then $\frac{b}{p^{e+e_i}} = 1$ and $\binom{u}{b/p^{e+e_i}} = u$, so (6.26) holds automatically in this case.

Proof of Proposition 6.9. We go through the proof of Proposition 6.1 and analyze the case of equality at each step. We start with a general remark about binomial coefficients in positive and mixed characteristic.

Remark 6.11. If $\text{char } k_\nu = 0$, we have

$$\nu \left(\binom{\gamma}{j} \right) = 0 \quad (6.28)$$

for any non-negative integers $j \leq \gamma$; this implies that $\nu(C(j_1, \dots, j_q)) = 0$ for any q -tuple (j_1, \dots, j_q) as in (6.10). If $\text{char } k_\nu = p > 0$, the following is a well known characterization of the equality (6.28). Let $\gamma = k_0 + pk_1 + \dots + p^s k_s$ and $j = t_0 + pt_1 + \dots + p^s t_s$, with $k_0, \dots, k_s, t_0, \dots, t_s \in \{0, 1, \dots, p-1\}$, denote the respective p -adic expansions of γ and j (where we allow one of the $(s+1)$ -tuples (k_0, \dots, k_s) and (t_0, \dots, t_s) to end in zeroes). Then (6.28) holds if and only if

$$k_j \geq t_j \quad \text{for all } j \in \{0, \dots, s\}. \quad (6.29)$$

We recall the proof for the reader's convenience. For a positive integer n , let $\nu^{(p)}(n!)$ denotes the p -adic value of $n!$. If $n = n_0 + pn_1 + \dots + p^s n_s$ is the p -adic expansion of n , we have

$$\nu^{(p)}(n!) = n_1 + \frac{p^2 - 1}{p - 1} n_2 + \dots + \frac{p^s - 1}{p - 1} n_s.$$

Let $\gamma - j = l_0 + pl_1 + \dots + p^s l_s$ be the p -adic expansion of $\gamma - j$.

First, suppose that (6.29) holds. Then $k_j = t_j + l_j$ for all j . We have

$$\nu^{(p)}(\gamma!) = k_1 + \frac{p^2-1}{p-1}k_2 + \cdots + \frac{p^s-1}{p-1}k_s, \quad (6.30)$$

$$\nu^{(p)}(j!) = t_1 + \frac{p^2-1}{p-1}t_2 + \cdots + \frac{p^s-1}{p-1}t_s, \quad (6.31)$$

$$\nu^{(p)}((\gamma - j)!) = l_1 + \frac{p^2-1}{p-1}l_2 + \cdots + \frac{p^s-1}{p-1}l_s \quad (6.32)$$

Thus $\nu^{(p)}(\gamma!) = \nu^{(p)}(j!) + \nu^{(p)}((\gamma - j)!)$ and (6.28) holds.

Conversely, assume that (6.29) is not true. Let

$$(j_0, j_0 + 1, \dots, j_1 - 1, j_1) \quad (6.33)$$

be a maximal subsequence of $(1, \dots, s)$ consisting of consecutive integers such that $k_j \neq t_j + l_j$ for $j_0 \leq j \leq j_1$. Then $k_{j_0} = t_{j_0} + l_{j_0} - p$, $k_j = t_j + l_j - p + 1$ for $j_0 < j < j_1$ and $k_{j_1} = t_{j_1} + l_{j_1} + 1$. Thus the total contribution of (6.33) to $\nu^{(p)}(\gamma!) - \nu^{(p)}(j!) - \nu^{(p)}((\gamma - j)!)$ is

$$\frac{p^{j_1} - 1}{p - 1} - \sum_{j=j_0+1}^{j_1-1} (p^j - 1) - p \frac{p^{j_0} - 1}{p - 1} = j_1 - j_0 \geq 1.$$

The quantity $\nu^{(p)}(\gamma!) - \nu^{(p)}(j!) - \nu^{(p)}((\gamma - j)!)$ is obtained by summing the contributions of all the subsequences of the form (6.33), hence it is strictly positive, as desired.

Below, we will be particularly interested in the following special cases of (6.29):

(1) If

$$\gamma = p^e u \quad \text{with } p \nmid u \quad (6.34)$$

then (6.28) implies that $p^e \mid j$.

(2) We have the following partial converse to (1): if γ is as in (6.34) and $j = p^e$ then (6.29) holds. Indeed, we have $t_e = 1$, $t_j = 0$ for $j \neq e$ and $k_e \geq 1$. In this case

$$\binom{\gamma}{j} = \binom{p^e u}{p^e} = \frac{p^e u (p^e u - 1) \cdots (p^e u - p^e + 1)}{p^e!} = u \pmod{m_\nu}$$

since $p^e u - j = p^e - j \pmod{m_\nu}$ for all j .

This is the main situation in which Proposition 6.9 will be applied in this paper.

Lemma 6.12. (1) *The inequality in (6.11) is strict unless $j_t \in I_{i, \max}$.*

(2) *Let γ_i and b_i be as in (6.22)–(6.23). Assume that $j_0 = 0$, and*

$$j_t \in I_{i, \max} \quad \text{for } 1 \leq t \leq q. \quad (6.35)$$

If

$$\nu(C(j_1, \dots, j_q)) = 0 \quad (6.36)$$

then

$$p^{e+e_i} \mid b. \quad (6.37)$$

(3) *Let the assumptions be as in (2) and assume, in addition, that $b = p^{e+e_i}$. Then (6.36) holds if and only if $q = p^e$ and $j_1 = \cdots = j_q = b_i$.*

Proof. (1) is immediate from definitions.

(2) Let $\{p^{c_1}, \dots, p^{c_\ell}\} \subset I_{i, \max}$ with

$$e_i \leq c_1 < c_2 < \dots < c_\ell \quad (6.38)$$

denote the set of *distinct* natural numbers appearing among $\{j_1, \dots, j_q\}$ (cf. (6.35) and Corollary 6.8). For $1 \leq j \leq \ell$, let $a_j = \#\{t \in \{1, \dots, q\} \mid j_t \leq p^{c_j}\}$; let $a_0 = 0$. Then

$$b = \sum_{j=1}^{\ell} (a_j - a_{j-1}) p^{c_j}. \quad (6.39)$$

Assume that (6.36) holds. By (6.8) and Remark 6.11 (1), we have

$$p^e \mid a_j \quad \text{for } 1 \leq j \leq \ell. \quad (6.40)$$

(6.38)–(6.40) imply (6.37), as desired.

(3) Assume, in addition, that $b = p^{e+e_i}$.

“Only if”. From (6.38)–(6.40), we see that $\ell = 1$ and $a_1 = p^e$; the result follows immediately.

“If”. By assumptions, we have $\ell = 1 = q$ and $a_1 = p^e$. By (6.8) and Remark 6.11 (2), we have

$$C(j_1, \dots, j_q) = C(\underbrace{b_i, \dots, b_i}_{p^e}) = \binom{\gamma_i}{p^e} = u \pmod{m_\nu}$$

and the result follows. □

We can now finish the proof of Proposition 6.9.

By (6.12) and Lemma 6.12 (1), the inequality in (6.13) is strict unless $j_0 = 0$, and

$$j_t \in I_{i, \max}. \quad (6.41)$$

Hence, by Lemma 6.12 (2), the first inequality in (6.14) is strict unless $j_0 = 0$ and $p^{e+e_i} \mid b$. This proves (1) of the Proposition.

(2) Assume that (6.24) holds. If $b = p^{e+e_i}$, by Lemma 6.12 (3) there is exactly one term on the right hand side of (6.10) for which equality holds in (6.13), namely, the term with $q = p^e$ and $j_1 = \dots = j_q = b_i$. If $I_{i, \max} = \{b_i\}$, then by Lemma 6.12 (1) there is at most one term on the right hand side of (6.10) for which equality holds in (6.13); if such a term exists, it is the term with $q = \frac{b}{b_i}$ and $j_1 = \dots = j_q = b_i$. Moreover, this term satisfies equality in (6.13) if and only if $\nu(C(\underbrace{b_i, \dots, b_i}_{b/b_i})) = \nu\left(\binom{\gamma_i}{b/b_i}\right) = \nu\left(\binom{u}{b/p^{e+e_i}}\right) = 0$. In either case, there is at most one

term on the right hand side of (10.4) for which equality holds in (6.13), and there is exactly one such term if and only if $\nu\left(\binom{u}{b/p^{e+e_i}}\right) = 0$. This proves (2).

(3) of the Proposition follows from (2) and (6.10). □

In the notation of Proposition 6.9, assume that (6.24) holds. Let

$$\gamma_i = k_0 + pk_1 + \dots + p^s k_s,$$

with $k_0, \dots, k_s \in \{0, 1, \dots, p-1\}$, denote the p -adic expansion of γ_i . Take integer $s' \in \{0, \dots, s\}$, $k'_{s'} \in \{0, \dots, k_{s'}\}$. Let $b = (k_0 + pk_1 + \dots + p^{s'-1} k'_{s'-1} + p^{s'} k'_{s'}) b_i$.

Corollary 6.13. *Equality holds in (6.1) for the generalized monomial $h = \mathbf{Q}_{i+1}^{\bar{\gamma}_{i+1}}$. The element $\text{in}_i \partial_b \mathbf{Q}_{i+1}^{\bar{\gamma}_{i+1}}$ is given by the formula (6.27).*

Proof. Repeated application of Proposition 6.9 (2) and (3), first k_0 times with b replaced by 1, then k_1 times with b replaced by p , and so on. \square

Let $h = \sum_{j=0}^s d_{ji} Q_i^j$ be an i -standard expansion. Let $S_i = S_i(h)$, where the notation is as in (3.21). Write $\text{in}_i h = \sum_{j \in S_i} \text{in}_i (d_{ji} Q_i^j)$. Write $b_i = p^{e_i}$, as above. Let e be the greatest non-negative integer such that for all $j \in S_i$ we have $p^e \mid j$.

Proposition 6.14. (1) *If equality holds in (6.1) then*

$$p^{e+e_i} \mid b. \quad (6.42)$$

(2) *Assume that*

$$b = p^{e+e_i}. \quad (6.43)$$

Then equality holds in (6.1). In particular, we have $\partial_b h \neq 0$.

(3) *Assume that (6.43) holds. Let $S_{bi} = \{j \in S_i \mid p^{e+1} \text{ does not divide } j\}$. Then*

$$\text{in}_i \partial_b h = \sum_{j \in S_{bi}} \text{in}_i \left(\binom{j}{p^e} d_{ji} Q_i^{j-p^e} (\partial_{b_i} Q_i)^{p^e} \right).$$

Proof. (1), (2) and (3) of Proposition 6.14 follow, respectively, from (1), (2) and (3) of Proposition 6.9. \square

Corollary 6.15. *In the notation of Proposition 6.14, we have*

$$h \notin K \left[x^{p^{e+e_i+1}} \right]. \quad (6.44)$$

Proof. Take b as in (6.43). Now the result follows from Proposition 6.14 (2). \square

Let the notation be as in Proposition 6.14.

Proposition 6.16. *Take an element $j \in S_i$. Write $j = p^e u$, where*

$$\text{if } \text{char } k_\nu = p > 0 \text{ then } p \nmid u.$$

Assume that

$$p^{e+1} \mid j' \text{ for all } j' \in S_i, j' < j. \quad (6.45)$$

Let $u = t_0 + t_1 p + \cdots + t_s p^s$ be the p -adic expansion of u . Then

$$\nu_i(\partial_{j b_i} h) = \nu_i(h) - j(\beta_i - \nu(\partial_{b_i} Q_i)), \quad (6.46)$$

$$\text{in}_i \partial_{j b_i} h = \left(\prod_{q=1}^s t_q! \right) d_{ji} (\text{in}_i \partial_{b_i} Q_i)^j + \text{terms involving } \text{in}_i Q_i. \quad (6.47)$$

For any $j' \neq j$ we have

$$\frac{\nu_i(h) - \nu_i(\partial_{j' b_i} h)}{j'} \leq \frac{\nu_i(h) - \nu_i(\partial_{j b_i} h)}{j}, \quad (6.48)$$

and the inequality is strict whenever $j' \notin S_i$ or $j' < j$.

Proof. By (6.45) and Proposition 6.9 (1), terms of the form $d_{j'i}Q_i^{j'}$ with

$$j' \in S_i, j' < j$$

satisfy strict inequality in (6.1) with $b = jb_i$. Thus replacing h by $\sum_{j'=j}^s d_{j'i}Q_i^{j'}$ does not change the problem. Now apply Proposition 6.14 repeatedly $t_0 + t_1 + \dots + t_s$ times. By (2) of Proposition 6.14,

$$\nu_i \left(\partial_{jb_i} \left(d_{ji}Q_i^j \right) \right) = \nu_i \left(d_{ji}Q_i^j \right) - j(\beta_i - \nu(\partial_{b_i}Q_i)) \quad (6.49)$$

and

$$\nu_i(\partial_{jb_i}h) = \nu_i(h) - j(\beta_i - \nu(\partial_{b_i}Q_i)), \quad (6.50)$$

this proves (6.46). (6.47) follows from Proposition 6.14 (3), by induction on u . Finally, the last statement of the Proposition follows from Proposition 6.14 (1) and (3), by induction on u . \square

Remark 6.17. Here is an alternative, more explicit explanation of (6.47). Take $j' \in \{j, \dots, s\}$ and apply (6.10) to one of the generalized monomials appearing in $d_{j'i}Q_i^{j'}$ (we take $\gamma_i = j'$ and $b = jb_i$ in (6.10)), in order to decide which values of j' and which decompositions $j_0 + \dots + j_q = b$ contribute to $\text{in}_i \partial_b h$.

If either $j' > j$, $q \neq j$, $j_0 \neq 0$ or $j_t \neq b_i$ for some $t \in \{1, \dots, j\}$ then, by definition of b_i , the corresponding term in (6.10) is either divisible by Q_i or has ν_i -value strictly greater than $\nu_i \left(d_{ji}Q_i^j \right) - j(\beta_i - \nu(\partial_{b_i}Q_i))$. This proves (6.47).

Let $\sum_q c_{qi}Q_i^q$ denote the i -standard expansion of $\partial_{jb_i}h$. The above considerations prove that c_{0i} coincides with the coefficient of Q_i^0 in the i -standard expansion of $d_{ji}(\partial_{b_i}Q_i)^j$ modulo an element of higher ν_i -value. In particular, $c_{0i} \neq 0$ and

$$\nu(c_{0i}) = \nu_i(c_{0i}). \quad (6.51)$$

We have

$$\nu(c_{0i}) = \nu_i(\partial_{jb_i}h) = \nu_i \left(\partial_{jb_i} \left(d_{ji}Q_i^j \right) \right) = \nu_i \left(d_{ji}Q_i^j \right) - j(\beta_i - \nu(\partial_{b_i}Q_i)). \quad (6.52)$$

Corollary 6.18. *We have*

$$\nu_i(h) = \min_{0 \leq j \leq s} \{ \nu_i(\partial_{jb_i}h) + j(\beta_i - \nu(\partial_{b_i}Q_i)) \} \quad (6.53)$$

and the minimum in (6.53) is attained for all $j \in S_i$, satisfying (6.45).

7 Infinite sequences of key polynomials

In this section, we assume that $\mathbf{Q}_{\ell+1}$ is defined for a certain ordinal number ℓ and that ω iterations of the algorithm of §3 produce an infinite sequence $\{Q_{\ell+t}\}_{t \in \mathbb{N}_0}$; in particular, we have $\alpha_{\ell+t} = 1$ for $t \gg 0$. Take an element $h \in K[x]$. Proposition 4.2 (1) implies that $\delta_{\ell+t}(h)$ stabilizes for t sufficiently large. Let $\delta(h)$ denote this stable value of $\delta_{\ell+t}(h)$. For a positive integer t , we have

$$\delta_{\ell+t}(h) = 0 \implies \nu(h) = \nu_{\ell+t}(h) \implies \delta_{\ell+t+1}(h) = 0.$$

Thus saying that $\nu(h) = \nu_{\ell+t}(h)$ for all t sufficiently large is equivalent to saying that $\delta_{\ell+t}(h) = 0$ for all t sufficiently large.

Assume that there exists $h \in K[x]$ such that

$$\nu(h) > \nu_{\ell+t}(h) \quad \text{for all } t \in \mathbb{N} \quad (7.1)$$

(in other words, $\delta_{\ell+t}(h) > 0$ for all $t \in \mathbb{N}$ and the key polynomial $Q_{\ell+\omega}$ is defined); put $h = Q_{\ell+\omega}$. One of the three main results of this section says that $\delta(h)$ has the form p^e for some $e \in \mathbb{N}_0$ (in particular, $\delta(h) = 1$ if $\text{char } k_\nu = 0$). To prove this, we use differential operators and their properties derived in §6 on differential operators.

The second main result of this section is the statement that if

$$\text{rk } \nu_0 = 1. \quad (7.2)$$

and if either $\text{char } k_\nu = 0$ or $p \nmid \delta(h)$ then the sequences $\{\beta_{\ell+t}\}_{t \in \mathbb{N}}$ and $\nu_{\ell+t}(h)$ are unbounded in $\tilde{\Gamma}$ (this is precisely Case 2a of Section 3); in particular, the set $\mathbf{Q}_{\ell+\omega}$ of key polynomials is $\tilde{\Gamma}$ -complete by Proposition 3.36. Finally, in Remark 7.7 (without the hypothesis (7.2)) we take $\ell = 1$ and assume that $\alpha_t = 1$ for all $t \in \mathbb{N}$ and that the sequence $\{\beta_t\}_{t \in \mathbb{N}}$ is unbounded in $\tilde{\Gamma}$, that is, we are in Case 2a. We show that $h \in K[x^\delta]$.

Replacing ℓ by $\ell+t$ for a sufficiently large t , we may assume that $\alpha_{\ell+t} = 1$ for all (strictly) positive integers t . Below the ordinal i will run over the set $\{\ell+t \mid t \in \mathbb{N}_0\}$. By definition, for all such i we have

$$Q_{i+1} = Q_i + z_i, \quad (7.3)$$

where z_i is a homogeneous Q_ℓ -free standard expansion of value β_i (cf. Proposition 3.15). By Proposition 3.20 (2), we have

$$\deg_x z_i < \deg_x Q_i. \quad (7.4)$$

Finally,

$$\text{in}_\nu Q_i = -\text{in}_\nu z_i \quad (7.5)$$

by (3.36).

As before, let

$$h = \sum_{j=0}^{s_i} d_{ji} Q_i^j$$

be an i -standard expansion of h for $i \geq \ell$, where each d_{ji} is a Q_ℓ -free standard expansion. Note that since $\alpha_{\ell+t} = 1$ for $t \in \mathbb{N}_0$, we have

$$\deg_x Q_i = \prod_{j=2}^{\alpha_i} \alpha_j = \prod_{j=2}^{\alpha_\ell} \alpha_j = \deg_x Q_\ell$$

and so

$$s_i = \left\lceil \frac{\deg_x h}{\deg_x Q_i} \right\rceil = \left\lceil \frac{\deg_x h}{\deg_x Q_\ell} \right\rceil = s_\ell. \quad (7.6)$$

Proposition 7.1. *For each i of the form $i = \ell + t$, $t \in \mathbb{N}$, we have $b_{i+1} \leq b_i$.*

Proof. Write $Q_{i+1} = Q_i + z_i$, as above.

Lemma 7.2. *For any $b \in \mathbb{N}$ we have*

$$\frac{\beta_i - \nu(\partial_b z_i)}{b} < \frac{\beta_i - \nu(\partial_{b_i} Q_i)}{b_i}. \quad (7.7)$$

If, in addition, $b \geq b_i$, then

$$\frac{\beta_{i+1} - \nu(\partial_b z_i)}{b} < \frac{\beta_{i+1} - \nu(\partial_{b_i} Q_i)}{b_i}. \quad (7.8)$$

Proof. Let i' denote the smallest ordinal such that

$$\nu_{i'}(z_i) = \nu(z_i); \quad (7.9)$$

by Proposition 3.29 (2) and (7.4), $i' < i$. Let $z_i = \sum_{j=0}^{s_{i'}} c_{j i'} Q_{i'}^j$ be the i' -standard expansion of z_i .

By Lemma 6.3 we have

$$\frac{\beta_i - \nu(\partial_{b_i} Q_i)}{b_i} > \frac{\beta_{i'} - \nu(\partial_{b_{i'}} Q_{i'})}{b_{i'}}. \quad (7.10)$$

Combining (7.10) with (6.1), we obtain

$$\frac{\nu_{i'}(z_i) - \nu_{i'}(\partial_b z_i)}{b} \leq \frac{\beta_{i'} - \nu(\partial_{b_{i'}} Q_{i'})}{b_{i'}} < \frac{\beta_i - \nu(\partial_{b_i} Q_i)}{b_i}, \quad (7.11)$$

which gives the inequality (7.7). If $b \geq b_i$, (7.8) follows immediately by adding the inequality $\frac{\beta_{i+1} - \beta_i}{b} \leq \frac{\beta_{i+1} - \beta_i}{b_i}$ to (7.7). \square

Corollary 7.3. *We have*

$$\nu(\partial_{b_i} z_i) > \nu(\partial_{b_i} Q_i) = \nu(\partial_{b_i} Q_{i+1}). \quad (7.12)$$

Proof. The inequality in (7.12) is a special case of (7.7) when $b = b_i$. The equality in (7.12) follows immediately from the inequality. \square

To prove Proposition 7.1, we argue by contradiction. Suppose that

$$b_{i+1} > b_i. \quad (7.13)$$

Letting $b = b_{i+1}$ in Lemma 7.2, we obtain

$$\frac{\beta_{i+1} - \nu(\partial_{b_{i+1}} z_i)}{b_{i+1}} < \frac{\beta_{i+1} - \nu(\partial_{b_i} Q_i)}{b_i}. \quad (7.14)$$

We have

$$\frac{\beta_i - \nu(\partial_{b_{i+1}} Q_i)}{b_{i+1}} \leq \frac{\beta_i - \nu(\partial_{b_i} Q_i)}{b_i} \quad (7.15)$$

by definition of b_i . Combining (7.15) with (7.13), we obtain

$$\frac{\beta_{i+1} - \nu(\partial_{b_{i+1}} Q_i)}{b_{i+1}} < \frac{\beta_{i+1} - \nu(\partial_{b_i} Q_i)}{b_i}. \quad (7.16)$$

We can rewrite (7.14) and (7.16) as

$$\min\{\nu(\partial_{b_{i+1}} Q_i), \nu(\partial_{b_{i+1}} z_i)\} > \beta_{i+1} - \frac{b_{i+1}}{b_i} (\beta_{i+1} - \nu(\partial_{b_i} Q_i)). \quad (7.17)$$

Since $\partial_{b_{i+1}} Q_{i+1} = \partial_{b_{i+1}} Q_i + \partial_{b_{i+1}} z_i$, (7.17) shows that

$$\nu(\partial_{b_{i+1}} Q_{i+1}) > \beta_{i+1} - \frac{b_{i+1}}{b_i} (\beta_{i+1} - \nu(\partial_{b_i} Q_i)),$$

which contradicts the definition of b_{i+1} . This completes the proof of Proposition 7.1. \square

Corollary 7.4. *Keep the above notation. Assume that $b_{i+1} = b_i$. Then $I_{i+1, \max} = \{b_{i+1}\}$.*

Proof. Take an integer

$$b > b_{i+1} = b_i. \quad (7.18)$$

Then

$$\frac{\beta_{i+1} - \beta_i}{b} < \frac{\beta_{i+1} - \beta_i}{b_i}. \quad (7.19)$$

By definition of b_i , we have

$$\frac{\beta_i - \nu(\partial_b Q_i)}{b} \leq \frac{\beta_i - \nu(\partial_{b_i} Q_i)}{b_i}. \quad (7.20)$$

Adding up (7.19) and (7.20) and using Corollary 7.12, we obtain

$$\frac{\beta_{i+1} - \nu(\partial_b Q_{i+1})}{b} < \frac{\beta_{i+1} - \nu(\partial_{b_{i+1}} Q_{i+1})}{b_{i+1}},$$

so $b \notin I_{i+1, \max}$. This proves the Corollary. \square

Recall that $\delta(h)$ denotes the stable value of $\delta_{\ell+t}(h)$ for all sufficiently large integers t . Set $\delta := \delta(h)$. Write $\delta = p^e u$, where if $p > 1$ then $p \nmid u$.

If $\text{char } k_\nu = 0$, equations (3.24) and (4.4) imply that $d_{\delta-1, \ell} \neq 0$ and

$$g_{1\ell} = \bar{Q}_\ell + \text{in}_\nu \frac{d_{\delta-1, \ell}}{\delta d_{\delta\ell}}. \quad (7.21)$$

If $\text{char } k_\nu = p > 0$ then, according to Proposition 4.2 (2) and using the notation of (3.21), we see that for $i = \ell + t$, $t \in \mathbb{N}_0$,

$$\delta - p^e \in S_i(h) \quad (7.22)$$

(in particular, $d_{\delta-p^e, i} \neq 0$) and that

$$\text{in}_i z_i = \left(\frac{\text{in}_i d_{\delta-p^e, i}}{u \text{in}_i d_{\delta i}} \right)^{\frac{1}{p^e}}. \quad (7.23)$$

We have $v_\ell = \text{in}_\nu d_{\delta\ell}$ and (3.24) rewrites as

$$\text{in}_\ell h = \text{in}_\nu d_{\delta\ell} g_{1\ell}^\delta. \quad (7.24)$$

Next, we prove a comparison result which expresses the coefficients d_{ji} in terms of $d_{j\ell}$ for $\delta - p^e \leq j \leq \delta$, modulo terms of sufficiently high value.

Proposition 7.5. *Take ordinals i and ℓ_1 such that $\ell < \ell_1 < i < \ell + \omega$. Assume that*

$$\delta_{i+1}(h) = \delta_\ell(h) = \delta. \quad (7.25)$$

We have

$$d_{\delta i} \equiv d_{\delta \ell_1} \pmod{\mathbf{P}_{\nu(d_{\delta \ell_1}) + \min\{\nu_{\ell_1}^+(h) - \nu_{\ell_1}(h), \beta_{\ell_1} - \beta_\ell\}}}, \quad (7.26)$$

Proof. By definitions, we have $Q_i = Q_{\ell_1} + z_{\ell_1} + \cdots + z_{i-1}$. For simplicity, write $z := z_{\ell_1} + \cdots + z_{i-1}$. We will compare the ℓ_1 -standard expansion of h with the i -standard one. To this end, we substitute $Q_i = Q_{\ell_1} + z$ into the i -standard expansion of h . We obtain

$$h = \sum_{j=0}^{s_i} d_{ji} Q_i^j = \sum_{j=0}^{s_i} d_{ji} (Q_{\ell_1} + z)^j. \quad (7.27)$$

First note that $\deg_x \sum_{j=0}^{\delta-1} d_{ji} (Q_{\ell_1} + z)^j < \delta \deg_x Q_i$. Hence d_{δ, ℓ_1} is completely determined by $d_{\delta, i}, d_{\delta+1, i}, \dots, d_{s_i, i}$. Next, by (4.22) in the proof of Proposition 4.2 (3) note that for

$$0 < j \leq s_i - \delta$$

the coefficient $d_{\delta+j, i}$ is a sum of terms of the form $d_{j'k\ell_1}$ with $j' \geq j + \delta$ satisfying

$$\nu(d_{j'k\ell_1}) = \nu_{\ell_1}(d_{j'k\ell_1}) \geq (j' - j - \delta)\beta_{\ell_1} + \nu(d_{j'\ell_1}).$$

Hence

$$\nu_{\ell_1}(d_{\delta+j, i} Q_i^{\delta+j}) \geq \nu_{\ell_1}^+(h), \quad (7.28)$$

so for $0 < j \leq s_i - \delta$ the terms $d_{\delta+j, i} Q_i^{\delta+j}$ in (7.27) contribute nothing to

$$d_{\delta\ell_1} \pmod{\mathbf{P}_{(\nu_{\ell_1}(h) - \delta\beta_{\ell_1}) + \min\{\nu_{\ell_1}^+(h) - \nu_{\ell_1}(h), \beta_i - \beta_{\ell_1}\}}}. \quad (7.29)$$

Therefore, the only term on the left hand side of (7.27) that affect the element (7.29) is $d_{\delta i} (Q_{\ell_1} + z)^\delta$.

We have

$$d_{\delta i} Q_i^\delta = d_{\delta i} \sum_{j=0}^{\delta} \binom{\delta}{j} Q_{\ell_1}^{\delta-j} z^j.$$

For $j < \delta$, the coefficient of $Q_{\ell_1}^j$ in the ℓ_1 -standard expansion of $d_{\delta i} z^j$ contributes to $d_{\delta\ell_1}$. Let us denote this coefficient by d'_j . We have

$$\nu_{\ell}(z) = \nu(z)$$

and

$$\nu_{\ell}(d_{\delta i}) = \nu(d_{\delta i}).$$

By Lemma 4.3 (1), the quantity $\nu_{\ell}(d_{\delta i} z^j)$ is the minimum of the ν_{ℓ} -values of the summands appearing in the ℓ_1 -standard expansion of $d_{\delta i} z^j$. Thus

$$\nu_{\ell}(d_{\delta i} z^j) = \nu(d_{\delta i} z^j) \leq \nu_{\ell}(d'_j Q_{\ell_1}^j).$$

Hence

$$\nu(d'_j Q_{\ell_1}^j) = \nu_{\ell}(d'_j Q_{\ell_1}^j) + j(\beta_{\ell_1} - \beta_{\ell}) \geq \nu(d_{\delta i} z^j) + j(\beta_{\ell_1} - \beta_{\ell}) = \nu(d_{\delta i}) + 2j\beta_{\ell_1} - j\beta_{\ell}.$$

This shows that $\nu(d'_j) \geq \nu(d_{\delta i}) + j(\beta_{\ell_1} - \beta_{\ell}) \geq \nu(d_{\delta i}) + (\beta_{\ell_1} - \beta_{\ell})$, so for $j > 0$ the term $d'_j Q_{\ell_1}^j$ does not affect the element (7.29). This completes the proof. \square

Proposition 7.6. *The integer δ is of the form $\delta = p^e$ for some $e \in \mathbb{N}_0$ (in particular, $\delta = 1$ whenever $\text{char } k_{\nu} = 0$).*

Proof. We give a proof by contradiction. Write

$$\delta = p^e v, \quad \text{where if } \text{char } k_\nu = p > 0 \text{ then } p \nmid v. \quad (7.30)$$

Suppose that $v > 1$. By Proposition 7.1, the sequence $\{b_i\}$ is non-increasing with t and hence stabilizes for t sufficiently large. Let b_∞ denote the stable value of b_i . Write $b_\infty = p^{e_\infty}$. Let $b = p^{e+e_\infty}$ and let $g = \partial_b h$. By Proposition 4.2 (2), $\text{in}_i h$ has the form (4.4) for $i = \ell + t$, as t runs over \mathbb{N}_0 , in particular, p^e is the same as in Proposition 6.14. Hence h and b satisfy the hypotheses of Proposition 6.14. By Proposition 6.14 (3) and (4.4), $g \neq 0$ and, for t sufficiently large we have $\text{in}_i g = v \text{ in}_\nu (d_{\delta_i} \partial_{b_i} Q_i)^{p^e} (\bar{Q}_i + \text{in}_\nu z_i)^{\delta - p^e}$. In particular,

$$\nu(g) > \nu \left((d_{\delta_i} \partial_{b_i} Q_i)^{p^e} \right) + \delta \beta_{\ell+t} - p^e \beta_{\ell+t} = \nu \left((d_{\delta_i} \partial_{b_i} Q_i)^{p^e} \right) + p^e (v-1) \beta_{\ell+t} = \nu_i(g)$$

(here is where we are using $v > 1$), which contradicts the fact that h has minimal degree among all the polynomials satisfying $\nu(h) > \nu_i(h)$. \square

Remark 7.7. Let the notation be as in Proposition 7.6. Assume, in addition, that the sequence $\{\beta_{\ell+t}\}$ is unbounded in $\tilde{\Gamma}$. Then $\nu(h) \notin \tilde{\Gamma}$.

Assume that $\text{char } k_\nu = \text{char } K$,

$$\ell = 1 \text{ and } \alpha_t = 1 \text{ for all } t \in \mathbb{N}. \quad (7.31)$$

In particular, $b_t = 1$ for all $t \in \mathbb{N}$ and $b_\infty = 1$.

We have $\alpha_\omega = \deg_x h$. Let e_ω be the integer e of Proposition 7.6 and put $\tilde{\alpha}_\omega = \frac{\alpha_\omega}{p^{e_\omega}}$.

We have $b = p^{e_\omega} = \delta$. We claim that

$$h \in K \left[x^\delta \right]; \quad (7.32)$$

in particular, for all $b' < \delta$ we have

$$\partial_{b'} h = 0. \quad (7.33)$$

Clearly (7.32) implies (7.33). We prove (7.32) by contradiction. Assume the contrary. Let e' denote the greatest non-negative integer such that $h \in K \left[x^{p^{e'}} \right]$; by assumption, $e' < e_\omega$. Then $\deg_x \partial_{p^{e'}} h < \deg_x h$, so there exists $t_0 \in \mathbb{N}$ such that

$$\nu_{t_0} \left(\partial_{p^{e'}} h \right) = \nu \left(\partial_{p^{e'}} h \right). \quad (7.34)$$

Take an integer $t > t_0$. Let $\sum_{p^{e'+1} \nmid j} c_{jt} Q_t^j$ denote the sum of all those monomials appearing in the t -standard expansion of h whose exponent j is not divisible by $p^{e'+1}$. The operator $\partial_{p^{e'}}$ annihilates all the monomials whose exponents are divisible by $p^{e'+1}$. Thus

$$\partial_{p^{e'}} h = \partial_{p^{e'}} \left(\sum_{p^{e'+1} \nmid j} c_{jt} Q_t^j \right) = \sum_{p^{e'+1} \nmid j} c_{jt} \binom{j}{p^{e'}} Q_t^{j-p^{e'}}. \quad (7.35)$$

Formulas (7.34) and (7.35) imply that the t -standard expansion of h contains a monomial of the form $c_{p^{e'},t} Q_t^{p^{e'}}$ and that for each j with $p^{e'+1} \nmid j$ we have

$$\nu_{t_0} \left(c_{jt} Q_t^j \right) \geq \nu \left(c_{p^{e'},t} Q_t^{p^{e'}} \right).$$

Then for each j with $p^{e'+1} \nmid j$ and $j > p^{e'}$ we have $\nu_t(c_{jt}Q_t^j) > \nu_t(c_{p^{e'},t}Q_t^{p^{e'}})$. Hence

$$\nu(c_{jt}Q_t^j) > \nu(c_{p^{e'},t}Q_t^{p^{e'}}) \text{ for all } j \text{ with } p^{e'+1} \nmid j \text{ and } j > p^{e'}. \quad (7.36)$$

We obtain that for all t sufficiently large the t -standard expansion of h contains a monomial of the form $c_{p^{e'},t}Q_t^{p^{e'}}$ and all the other monomials not divisible by $Q_t^{p^{e'+1}}$ have values strictly greater than $\nu(c_{p^{e'},t}Q_t^{p^{e'}})$.

Then for all $t' > t$ we have $\nu(c_{p^{e'},t}) = \nu(c_{p^{e'},t'})$. Choosing t' sufficiently large, we obtain $\nu(c_{p^{e'},t'}Q_{t'}^{p^{e'}}) < \nu(c_{\delta,t'}Q_{t'}^\delta)$, which contradicts the definition of δ . This completes the proof of (7.32) and (7.33).

In fact, by a similar argument this statement can be proved not only for h , but for any polynomial satisfying the strict inequalities (7.1).

Remark 7.8. Keep the assumption that $\{\beta_{\ell+t}\}$ is unbounded in $\tilde{\Gamma}$, as well as (7.31), but now assume that $\text{char } K = 0$ and $\text{char } k_\nu = p > 0$. By studying the coefficient of $Q_t^{\delta-1}$ in the t -standard expansion of h for different t , one can prove that $\delta = p^e = 1$. A detailed proof of this will appear in a subsequent paper.

From now till the end of the paper, assume that $\text{rk } \nu_0 = 1$.

Proposition 7.9. *Keep the notation and assumptions stated in the beginning of this section. Assume that $\delta = p^e = 1$ in the notation of (7.30) (this assumption holds automatically if $\text{char } k_\nu = 0$). Then the sequences*

$$\{\nu_i(h)\} \quad (7.37)$$

and

$$\{\beta_i\}_i, \quad (7.38)$$

where i runs over the set $\{\ell + t \mid t \in \mathbb{N}\}$, are unbounded in $\tilde{\Gamma}$.

Proof. Proposition 4.2 (2) implies that $\nu_i(h) = \beta_i + \nu(d_{1i})$ and that $\nu(d_{1i})$ is independent of i . Thus to show that the sequence (7.37) is unbounded in $\tilde{\Gamma}$ it is sufficient to show that (7.38) is unbounded in $\tilde{\Gamma}$.

Moreover, to prove that (7.38) is unbounded, it is sufficient to show that the set $\nu(T)$ itself is unbounded in $\tilde{\Gamma}$.

To prove the unboundedness of $\nu(T)$, let $\ell_1 = \ell + 1$. Let $d_{1\ell_1}^* \in K[x]$ denote a polynomial such that $\text{in}_\nu d_{1\ell_1}^* \text{in}_\nu d_{1\ell_1} = 1$ in G_ν . According to Lemma 2.6 we may choose $d_{1\ell_1}^*$ to be of degree strictly less than $\deg_x Q_{\ell_1} = \deg_x Q_\ell$; this condition determines $d_{1\ell_1}^*$ uniquely. We have that $d_{1\ell_1}^* d_{1\ell_1} - 1$ is divisible by Q_{ℓ_1} in $K[x]$. We have $\nu_\ell(d_{1\ell_1}^*) = \nu(d_{1\ell_1}^*)$ by Proposition 3.29 (2), hence $\nu_i(d_{1\ell_1}^*) = \nu(d_{1\ell_1}^*)$ for all $i \geq \ell$. We claim that after multiplying h by $d_{1\ell_1}^*$ we still have $\delta_i(d_{1\ell_1}^* h) = 1$ for $i = \ell + t$ with $t \in \mathbb{N}$ sufficiently large.

Indeed, applying Lemma 5.1 to the pairs $(h, g) = (d_{1\ell_1}, d_{1\ell_1}^*)$ and $(h, g) = (d_{0\ell_1}, d_{1\ell_1}^*)$, we see that after multiplying h by $d_{1\ell_1}^*$ and applying Euclidean division by Q_{ℓ_1} to obtain the ℓ_1 -standard expansion of $d_{1\ell_1}^* h$, only the remainders in the Euclidean division contribute to $\text{in}_{\ell_1} d_{1\ell_1}^* h$. In particular, $\delta_i(d_{1\ell_1}^* h) = 1$ for all $i \geq \ell$.

Thus multiplying h by $d_{1\ell_1}^*$ does not change the problem. Therefore we may assume that

$$\text{in}_\nu d_{1i} = \text{in}_i d_{1i} = 1 \text{ for all } i \text{ of the form } \ell + t, t \in \mathbb{N}. \quad (7.39)$$

Clearly, the result of the Proposition does not depend on the choice of $z_{\ell+t}$ and $Q_{\ell+t}$ which was made in §3. We will now modify our choice of $z_{\ell+t}$ and $Q_{\ell+t}$ in such a way as to make the unboundedness $\nu(T)$ obvious.

In view of (7.39) and Proposition 4.2 (2), we have

$$\text{in}_{\ell_1} h = \text{in}_{\ell_1} (Q_{\ell_1} + z_{\ell_1}) = \text{in}_{\ell_1} (Q_{\ell_1} + d_{0\ell_1}). \quad (7.40)$$

Comparing the right and the left hand side of (7.40), we obtain

$$\text{in}_{\ell_1} z_{\ell_1} = \text{in}_{\nu_{\ell_1}} d_{0\ell_1}. \quad (7.41)$$

In particular, (7.41) shows that $d_{0\ell_1} \neq 0$ and

$$\nu(Q_{\ell_1}) < \nu(Q_{\ell_1} + d_{0\ell_1}). \quad (7.42)$$

By (7.42), $Q_{\ell_1} + d_{0\ell_1} \in T$. Replace z_{ℓ_1} by $d_{0\ell_1}$, in other words, put

$$z_{\ell_1} := d_{0\ell_1}.$$

and $Q_{\ell+2} = Q_{\ell_1} + d_{0\ell_1}$. We now iterate the procedure. Precisely, assume that $z_{\ell+1}, \dots, z_{\ell+t}$ and Q_i for $i \leq \ell + t + 1$ are already constructed. By Proposition 4.2 (2) and (7.39) we have

$$\text{in}_{\ell+t+1} h = \text{in}_{\ell+t+1} Q_{\ell+t+1} + \text{in}_{\ell+t+1} z_{\ell+t+1}. \quad (7.43)$$

Note that (7.43) implies that $d_{0,\ell+t} \neq 0$. We now redefine

$$z_{\ell+t+1} := d_{0,\ell+t}$$

and $Q_{\ell+t+2} = Q_{\ell+t+1} + z_{\ell+t+1}$.

This completes the recursive construction. Notice that all the elements $z_{\ell+t}$ and $Q_{\ell+t}$ lie in a fixed noetherian ring A , namely, the \mathbb{Z} -subalgebra of $K[x]$, generated by x and the finitely many coefficients of the polynomial Q_ℓ .

Lemma 7.10. *Let μ be a rank one valuation with value group contained in $\tilde{\Gamma}$, centered in a local noetherian domain (R, M, k) (that is, non-negative on R and strictly positive on M). Let*

$$\Phi = \mu(R \setminus \{0\}) \subset \tilde{\Gamma}.$$

Then Φ contains no infinite bounded sequences.

Proof. An infinite ascending sequence $\alpha_1 < \alpha_2 < \dots$ in Φ , bounded above by an element $\beta \in \Phi$, would give rise to an infinite descending chain of ideals in $\frac{R}{I_\beta}$, where I_β denotes the μ -ideal of R of value β . Thus it is sufficient to prove that $\frac{R}{I_\beta}$ has finite length.

Let $\delta := \mu(M) \equiv \min(\Phi \setminus \{0\})$. Since μ is of rank one, there exists $n \in \mathbb{N}$ such that $\beta \leq n\delta$. Then $M^n \subset I_\beta$, so that there is a surjective map $\frac{R}{M^n} \twoheadrightarrow \frac{R}{I_\beta}$. Thus $\frac{R}{I_\beta}$ has finite length, as desired. \square

Coming back to the proof of the Proposition, let $H = \{a \in A \mid \nu(a) \notin \tilde{\Gamma}\}$ and

$$M = \{a \in A \mid \nu(a) > 0\}.$$

Applying Lemma 7.10 to the local noetherian ring $\frac{A_M}{H A_M}$ and using the fact that the sequence β_i is strictly increasing with i , we obtain that $\{\beta_i\}$ is unbounded in $\tilde{\Gamma}$, as desired. \square

Remark 7.11. Take a polynomial $g \in K[x]$ such that $\nu_{\ell+t}(g) < \nu(g)$ for all $t \in \mathbb{N}$, not necessarily of the smallest degree. Let $\delta := \delta(g)$ denote the stable value of $\delta_{\ell+t}(h)$ for $t \in \mathbb{N}$ sufficiently large. Assume that $p^e = 1$ in the notation of (7.30) (in other words, either $\text{char } k_\nu = 0$ or $\text{char } k_\nu = p > 0$ and $p \nmid \delta$). For $i = \ell + t$ with t sufficiently large we have $\nu_i(g) = \nu(d_{\delta i}) + \delta \beta_i$ with $\nu(d_{\delta i})$ independent of i . Thus $\nu_i(g)$ is unbounded in $\tilde{\Gamma}$.

8 Key polynomials indexed by limit ordinals

In this section, we assume that $\text{char } k_\nu = \text{char } K = p > 0$. We assume that we have a set of key polynomials $\{Q_i\}_{i \in \Lambda}$ such that Λ contains at least one limit ordinal. Let $\ell + \omega \in \Lambda$ be a limit ordinal. Assume that the sequence $\{\nu(Q_{\ell+t})\}_{t \in \mathbb{N}}$ is bounded in $\tilde{\Gamma}$ (in other words, we are in Case 2b of §3). The main result of this section, Proposition 8.2, says that the polynomial $Q_{\ell+\omega}$ can be chosen in such a way that there exist $i_0 = \ell + t_0 \in \Lambda$, $t_0 \in \mathbb{N}$ (so that $i_0 + = \ell + \omega$), such that the i_0 -standard expansion of $Q_{\ell+\omega}$ is weakly affine.

Remark 8.1. If $\deg_x Q_{i_0+t} = 1$ for all $t \in \mathbb{N}$, this result was proved by I. Kaplansky. In Kaplansky's terminology x is a limit of a pseudo-convergent sequence $\{\rho\}_{j < \lambda}$ of algebraic type in K , and $Q_{\ell+\omega}$ is a monic polynomial of minimal degree, not fixing the values of $\{\rho\}_{j < \lambda}$. See [8], Lemma 10, page 311.

Let the notation be as in §5.

Proposition 8.2. *The polynomial $Q_{\ell+\omega}$ can be chosen in such a way that there exist $i \in \{\ell + t\}_{t \in \mathbb{N}_0}$ such that the i -standard expansion of $Q_{\ell+\omega}$ is weakly affine and monic of degree p^{e_0} in Q_i , with*

$$\bar{\beta} \leq \frac{1}{p^{e_0}} \nu(Q_{\ell+\omega}). \quad (8.1)$$

Proof. Let f a limit key polynomial with index $\ell + \omega$, that is, a monic polynomial of smallest degree such that

$$\nu(f) > \nu_{\ell+t}(f) \quad \text{for all } t \in \mathbb{N}. \quad (8.2)$$

The idea is to gradually modify the polynomial f until we arrive at $g = Q_{\ell+\omega}$ satisfying the conclusion of the Proposition.

For $i = \ell + t$, $t \in \mathbb{N}$, let $f = \sum_{j=0}^{\delta} a_{ji} Q_i^j$ denote the i -standard expansion of f . By Proposition 5.2, the polynomial f is of degree $\delta \deg_x Q_\ell$. In other words, we have $a_{\delta i} = 1$.

Write $\delta = p^{e_0}$ with $e_0 \in \mathbb{N}$. Choose $i_0 \geq \ell_1$ sufficiently large so that

$$\beta_{i_0} - \alpha_\ell \beta_{\ell-1} > 2p^{e_0}(\bar{\beta} - \beta_{i_0}). \quad (8.3)$$

Remark 8.3. Assume that there exist $i \geq i_0$ and j , $1 \leq j < p^{e_0}$, such that

$$\nu(a_{ji}) + j\bar{\beta} \geq 2p^{e_0}\bar{\beta} - p^{e_0}\beta_i.$$

Then for each $i' \geq i$ we have

$$\nu_{i'} \left(a_{ji} Q_i^j \right) = \nu \left(a_{ji} Q_i^j \right) > 2p^{e_0} \bar{\beta} - p^{e_0} \beta_i + j\beta_i - j\bar{\beta} > p^{e_0} \bar{\beta} > p^{e_0} \beta_{i'}.$$

Hence $\text{in}_{i'} \left(f - a_{ji} Q_i^j \right) = \text{in}_{i'} f$ and

$$\nu_{i'} \left(f - a_{ji} Q_i^j \right) < \nu \left(f - a_{ji} Q_i^j \right).$$

Thus replacing f by $f - a_{ji} Q_i^j$ does not affect the condition (8.2); $f - a_{ji} Q_i^j$ is still a limit key polynomial with index $\ell + \omega$.

Definition 8.4. Take an $i \geq i_0$. Consider an i -standard expansion (5.7) of f and let $a_{ji} Q_j^i$ be a monomial appearing in this expansion. We say that $a_{ji} Q_j^i$ is **bad** if

$$\nu(a_{ji}) + j\bar{\beta} < 2p^{e_0} \bar{\beta} - p^{e_0} \beta_i. \quad (8.4)$$

and at least one of the following three conditions holds:

$$(1) \quad \nu(a_{ji}) < (p^{e_0} - j)\bar{\beta} \quad (8.5)$$

(2) j is not a power of p

$$(3) \quad \nu(a_{ji}) > (p^{e_0} - j)\bar{\beta}. \quad (8.6)$$

In view of Remark 8.3, to say that the i -standard expansion (5.7) satisfies the conclusion of Proposition 8.2 it is sufficient to show that it contains no bad monomials, in which case there is nothing more to do. Assume that there exists at least one bad monomial. Let $j(i)$ denote the greatest $j \in \{1, \dots, p^{e_0} - 1\}$ such that the monomial $a_{ji} Q_j^i$ is bad. Let $j^\bullet(i)$ denote the element $j \in \{1, \dots, p^{e_0} - 1\}$ which minimizes the pair $(\nu(a_{ji}) + j\beta_i, -j)$ in the lexicographical ordering among all the elements of $\{1, \dots, p^{e_0} - 1\}$ such that the monomial $a_{ji} Q_j^i$ is bad.

Take $i \geq i_0$. To finish the proof of Proposition 8.2, we will first prove the following three Lemmas:

Lemma 8.5. *We have*

$$j(i+1) \leq j(i) \quad (8.7)$$

and

$$j^\bullet(i+1) \leq j^\bullet(i). \quad (8.8)$$

If $j \in \{j(i), j^\bullet(i)\}$ then

$$\text{in}_\nu a_{j,i+1} = \text{in}_\nu a_{ji}. \quad (8.9)$$

Lemma 8.6. *If $j \in \{j(i), j^\bullet(i)\}$ then (8.5) does not hold.*

Lemma 8.7. *If $j = j(i)$ then (8.6) holds.*

According to Lemmas 8.6 and 8.7, if $j = j(i)$ then (8.6) must hold. In that case, we will prove that increasing i , if necessary, and replacing f by $f - a_{ji} Q_j^i$ preserves the strict inequalities (8.2) for all t and either eliminates the last bad monomial or strictly decreases $j(i)$.

At that point the proof of Proposition 8.2 will be finished by induction on $j(i)$.

Proof of Lemma 8.5. First, suppose $j = j(i)$. Since j is the greatest element of $\{1, \dots, p^{e_0} - 1\}$ satisfying (8.4), and one of the conditions (1)–(3) of Definition 8.4, any $j' \in \{j + 1, \dots, p^{e_0} - 1\}$ satisfying (8.4) is a power of p and

$$\nu(a_{j'i}) = (p^{e_0} - j')\bar{\beta}. \quad (8.10)$$

We want to analyze the monomial $a_{j,i+1}Q_{i+1}^j$ in the $(i+1)$ -standard expansion of f . To do that, write

$$f = \sum_{j'=0}^{\delta} a_{j'i}(Q_{i+1} - z_i)^{j'}. \quad (8.11)$$

Now, terms in (8.11) with $j' < j$ do not affect the statement of the Lemma. We claim that the same is true of the terms with $j' > j$. Indeed, take a

$$j' \in \{j + 1, \dots, \delta\}.$$

Write $j' = p^{e'}$. Then

$$a_{j'i}(Q_{i+1} - z_i)^{j'} = a_{j'i}Q_{i+1}^{p^{e'}} - a_{j'i}z_i^{p^{e'}}. \quad (8.12)$$

In general, the right hand side of (8.12) need not be an $(i+1)$ -standard expansion, since $\deg_x a_{j'i}z_i^{p^{e'}}$ may be quite large, even as large or larger than $\deg_x Q_{i+1}^{p^{e'}}$. However, the $(i+1)$ -standard expansion of $a_{j'i}(Q_{i+1} - z_i)^{p^{e'}}$ is obtained from it by iterating Euclidean division by Q_{i+1} . The first Euclidean division we perform consists of writing $a_{j'i}z_i^{p^{e'}} = Q_{i+1}g + h$ with $\deg_x h < \deg_x Q_{i+1}$. We have

$$\begin{aligned} \nu_{\ell-1} \left(a_{j'i}z_i^{p^{e'}} \right) &= \nu \left(a_{j'i}z_i^{p^{e'}} \right) = \nu_{i+1} \left(a_{j'i}z_i^{p^{e'}} \right), \\ \nu_{\ell-1}(h) &= \nu(h) = \nu_{i+1}(h) \end{aligned}$$

and

$$\nu_{i+1}(Q_{i+1}g) - \nu_{\ell-1}(Q_{i+1}g) \geq \nu_{i+1}(Q_{i+1}) - \nu_{\ell-1}(Q_{i+1}) = \beta_{i+1} - \alpha_{\ell}\beta_{\ell-1}.$$

Hence $\nu_{i+1}(Q_{i+1}g) - (\beta_{i+1} - \alpha_{\ell}\beta_{\ell-1}) \geq \nu_{i+1} \left(a_{j'i}z_i^{p^{e'}} \right) = \nu_{i+1}(h) \geq p^{e_0}\beta_i$. This implies that all the new monomials appearing after all the subsequent Euclidean divisions have ν_{i+1} -value greater than or equal to $p^{e_0}\beta_i + (\beta_{i+1} - \alpha_{\ell}\beta_{\ell-1})$, and hence, in view of (8.3)–(8.4), strictly greater than $\nu(a_{ji}) + j\beta_{i+1}$. In particular, if such a new monomial is of the form dQ_{i+1}^j , with d and Q_{i+1} -free standard expansion, we have $\nu(d) > \nu(a_{ji})$. This proves that the passage to an $(i+1)$ -standard expansion does not affect $\text{in}_{\nu} a_{ji}$, and (8.9) holds for $j = j(i)$.

The fact that all the new monomials arising from iterated Euclidean divisions of

$$a_{j'i}(Q_{i+1} - z_i)^{p^{e'}}, \quad j' > j,$$

have ν_{i+1} -value greater than or equal to

$$p^{e_0}\beta_i + (\beta_{i+1} - \alpha_{\ell}\beta_{\ell-1}),$$

together with (8.3)–(8.4) also shows that after the passage to the $(i+1)$ -standard expansion, no new bad monomials $a_{j',i+1}Q_{i+1}^{j'}$ are produced with $j' > j$ (the monomial $a_{j,i+1}Q_{i+1}^j$ may or may not be bad). This proves (8.7).

The proof of the Lemma in the case $j = j^\bullet(i)$ is very similar to that of $j = j(i)$, except for the following minor change. We can no longer assert that j' is a power of p . On the other hand, j' satisfies $\nu(a_{j'i}) + j'\beta_i > \nu_i(f)$, which allows us to use similar arguments as in the $j = j(i)$ case. This completes the proof of Lemma 8.5. \square

Proof of Lemma 8.6. We give a proof by contradiction. Suppose that $j = j(i)$ and that (8.5) holds for this j . (8.5) can be rewritten as $\nu(a_{ji}) + j\bar{\beta} < p^{e_0}\bar{\beta}$. Combining this with (8.9) we obtain that (8.5) holds with i replaced by $i + 1$,

$$\nu(a_{j,i+1}) + j\bar{\beta} < 2p^{e_0}\bar{\beta} - p^{e_0}\beta_{i+1}. \quad (8.13)$$

so the monomial $a_{j,i+1}Q_{i+1}^j$ is also bad and $j(i+1) = j(i)$. By induction on $i' \geq i$ we see that $j(i')$ is independent of i' , so the i' -standard expansion of f contains a monomial $a_{j'i'}Q_{i'}^j$ with

$$\nu\left(a_{j'i'}Q_{i'}^j\right) = \nu(a_{j'i'}) + j\beta_{i'} = \nu(a_{ji}) + j\beta_{i'} < \nu(a_{ji}) + j\bar{\beta}.$$

Then $p^{e_0}\beta_{i'} = \nu_{i'}(f) < \nu(a_{ji}) + j\bar{\beta} < p^{e_0}\bar{\beta}$ for all i' , hence the least upper bound of $\beta_{i'}$ is bounded above by $\frac{1}{p^{e_0}}\nu(a_{ji}) + j\bar{\beta}$ and hence is strictly less than $\bar{\beta}$. This contradicts the definition of $\bar{\beta}$.

The proof in the case $j = j^\bullet(i)$ is similar to that with $j = j(i)$ and we omit it. \square

Proof of Lemma 8.7. We argue by contradiction. Assume that $j = j(i)$ and that (8.6) does not hold. In view of Lemma 8.6 this implies that

$$\nu(a_{ji}) + j\bar{\beta} = p^{e_0}\bar{\beta}. \quad (8.14)$$

Then, by definition of $j(i)$, j is not a power of p . Write $j = p^e u$, $u \geq 2$ and $p \nmid u$, and

$$Q_{i+1} = Q_i + z_i.$$

Lemma 8.6, applied to $j^\bullet(i)$, implies that

$$\nu(a_{j'i}) + j'\bar{\beta} \geq p^{e_0}\bar{\beta} \quad \text{for all } j' \in \{1, \dots, p^{e_0}\}. \quad (8.15)$$

Let $b = p^e b_\infty$. Arguing as in the proof of Lemma 8.5, we can show that $j(i)$ remains unchanged as i increases. Take i sufficiently large so that

$$\nu_i(\partial_b f) = \nu(\partial_b f). \quad (8.16)$$

Moreover, after suitably increasing ℓ and i , we may assume that $b_\ell = b_i = b_\infty$, $\deg_x Q_\ell = \deg_x Q_i$ and $\ell < i$. By Corollary 7.4 there is at most one value of $i = \ell + t$, $t \in \mathbb{N}$, for which $\#I_{i,max} > 1$. Hence we may assume, in addition, that $I_{i,max} = \{b_i\}$. Finally, we will assume that

$$\beta_i - \beta_\ell \geq 2b_i p^{e_0}(\bar{\beta} - \beta_i). \quad (8.17)$$

Let $f = \sum_s a_{s,i+1}Q_{i+1}^s$ be the $(i+1)$ -standard expansion of f and let $\tilde{f} = \sum_{s=0}^j a_{s,i+1}Q_{i+1}^s$.

Claim. We have $S_{i,i+1}(\tilde{f}) \cap \{1, \dots, j-1\} \neq \emptyset$, that is, $S_{i,i+1}(\tilde{f})$ contains an element j' such that $1 \leq j' < j$.

Proof of Claim. We argue by contradiction. Assume that there is no such j' , that is,

$$\{j\} = S_{i,i+1}(\tilde{f}) \setminus \{0\}. \quad (8.18)$$

We will now show that

$$0 \notin S_{i,i+1}(\partial_b f) \quad (8.19)$$

which will contradict (8.16).

First of all, for each polynomial a with

$$\deg_x a < \deg_x Q_i \quad (8.20)$$

and each $b' \in \mathbb{N}$ we have

$$\nu_i(\partial_{b'} a) = \nu_\ell(\partial_{b'} a) \geq \nu_\ell(a) - \frac{b'}{b_\ell}(\beta_\ell - \nu(\partial_{b_\ell} Q_i)) > \nu_i(a) - \frac{b'}{b_i}(\beta_i - \nu(\partial_{b_i} Q_i)) \quad (8.21)$$

by Proposition 6.1. Next, for any positive integers s and b' , consider $\partial_{b'}(Q_i^s)$. We have

$$\nu_i(\partial_{b'} Q_i^s) \geq \nu_i(Q_i^s) - \frac{b'}{b_i}(\beta_i - \nu(\partial_{b_i} Q_i)). \quad (8.22)$$

by Proposition 6.1. Moreover, consider the case $b' = b$ and $s = j$ in (8.22). Then, by Proposition 6.9, equality holds in (8.22) since $\binom{j}{p^e} = u \neq 0$. Proposition 6.9 (3) says that

$$\text{in}_i \partial_b Q_i^j = u \text{in}_i \left(Q_i^{j-p^e} (\partial_{b_i} Q_i)^{p^e} \right). \quad (8.23)$$

Next, for any positive integers s and b' and a polynomial a satisfying (8.20), consider $\partial_{b'}(aQ_{i+1}^s)$. Again, we have

$$\nu_i(\partial_{b'} aQ_{i+1}^s) \geq \nu_i(aQ_{i+1}^s) - \frac{b'}{b_i}(\beta_i - \nu(\partial_{b_i} Q_i)) \quad (8.24)$$

by Proposition 6.1. Note that, together with (8.17) and (8.18) this implies that

$$\nu_i(\partial_b(a_{s,i+1}Q_{i+1}^s)) > \nu_i(a_{j,i+1}Q_{i+1}^j) - p^e(\beta_i - \nu(\partial_{b_i} Q_i)) \quad \text{for } s \neq j. \quad (8.25)$$

By the generalized Leibnitz rule, we have $\partial_{b'}(a_{s,i+1}Q_{i+1}^s) = \sum_{l=0}^{b'} (\partial_l a_{s,i+1} \partial_{b'-l} Q_{i+1}^s)$.

Letting $b' = b$, $s = j$, $a = a_{j,i+1}$, using the equality in (8.22) and the strict inequality (8.21) for both $a_{j,i+1}$ and z_i , we see that all the terms coming from the differentiation of $a_{j,i+1}$ and z_i are negligible, so that $\nu_i(\partial_b(a_{j,i+1}Q_{i+1}^j)) = \nu_i(a_{j,i+1}Q_{i+1}^j) - p^e(\beta_i - \nu(\partial_{b_i} Q_i))$ and (comparing with (8.23)),

$$\text{in}_i \left(\partial_b \left(a_{j,i+1} Q_{i+1}^j \right) \right) = u \text{in}_i \left(Q_{i+1}^{j-p^e} (\partial_{b_i} Q_i)^{p^e} \right). \quad (8.26)$$

Combining (8.25) with (8.26) we see that $S_{i,i+1}(\partial_b f) = \{j - p^e\}$. This proves (8.19), which gives the desired contradiction. This completes the proof of the Claim.

Now, let j' be as in the Claim. Then $\nu(a_{j',i'+1}) + j'\beta_{i'+1} \leq \nu(a_{j,i'+1}) + j\beta_{i'+1}$ and

$$\nu(a_{j',i'+1}) + j'\bar{\beta} < \nu(a_{j,i'+1}) + j\bar{\beta},$$

which contradicts (8.14) and (8.15).

This completes the proof of Lemma 8.7. \square

Let $j = j(i)$. By Lemmas 8.6 and 8.7 the inequality (8.6) holds for j .

By Lemma 8.5 (8.9), $\text{in}_\nu a_{ji}$ is independent of i .

Since $\bar{\beta} - \beta_{i'}$ can be made arbitrarily small as $t \rightarrow \infty$, by (8.6), taking i_1 sufficiently large, we can ensure that

$$\nu \left(a_{ji} Q_{i_1}^j \right) > 2p^{e_0} \bar{\beta} - p^{e_0} \beta_{i_1}. \quad (8.27)$$

Take the smallest i_1 satisfying (8.27). By the minimality of i_1 , Lemma 8.5 (8.9) and induction on i' , $i \leq i' \leq i_1$, we see that the monomial $a_{ji'} Q_{i'}^j$ remains bad for $i \leq i' < i_1$ and that

$$\nu(a_{ji}) = \nu(a_{ji_1}). \quad (8.28)$$

From (8.27)–(8.28) we obtain

$$\nu \left(a_{ji_1} Q_{i_1}^j \right) > 2p^{e_0} \bar{\beta} - p^{e_0} \beta_{i_1}. \quad (8.29)$$

Replace f by $f - a_{ji_1} Q_{i_1}^j$; Remark 8.3 says that strict inequality (8.2) is satisfied with f replaced by $f - a_{i_1 j} Q_{i_1}^j$. This procedure strictly decreases the integer $j(i)$. Hence after finitely many repetitions of this procedure we obtain a polynomial f containing no bad monomials. The non-existence of bad monomials is preserved as we pass from i to $i+1$. Thus, by choosing i sufficiently large, we may assume that f contains no bad monomials; we will make this assumption from now on.

If $Q_{\ell+\omega} = f$ satisfies the conclusion of Proposition 8.2 there is nothing more to prove.

Otherwise, there exists $j \in \{1, \dots, p^e - 1\}$ such that

$$\nu(a_{ji}) + j\bar{\beta} > 2p^{e_0} \bar{\beta} - p^{e_0} \beta_i. \quad (8.30)$$

Let \mathcal{A} denote the set of all such j . Replace f by $f - \sum_{j \in \mathcal{A}} a_{ji} Q_i^j$. Remark 8.3 says that strict inequality (8.2) is satisfied for this new f . In this way, we obtain a polynomial f such that $Q_{\ell+\omega} = f$ satisfies the conclusion of Proposition 8.2. This completes the proof of Proposition 8.2. \square

Remark 8.8. For $i \geq i_0$ let $\tilde{\mathcal{S}}_i = \{j \in \{1, \dots, p^{e_0}\} \mid a_{ji} \neq 0 \text{ and } \nu(a_{ji}) = (p^{e_0} - j)\bar{\beta}\}$. Write $f = \sum_{j=0}^{\delta} a_{ji} Q_i^j = \sum_{j=0}^{\delta} a_{ji} (Q_{i+1} - z_i)^j$. Opening the parentheses on the right hand side of this formula, we can derive information about the coefficients $a_{j,i+1}$ in the $(i+1)$ -standard expansion of f . Namely, the absence of bad monomials in the i -standard expansion and the inequality $\nu(a_{ji}) + j\bar{\beta} \geq 2p^{e_0} \bar{\beta} - p^{e_0} \beta_i$ ensures that the terms $a_{ji} (Q_{i+1} - z_i)^j$ with $j \notin \tilde{\mathcal{S}}_i$ contribute nothing to $\tilde{\mathcal{S}}_{i+1}$. For $j \in \tilde{\mathcal{S}}_i$ the terms $a_{ji} (Q_{i+1} - z_i)^j$ may contribute to $a_{j',i+1} Q_{i+1}^{j'}$ for $j' > j$. However, the inequality (7.12) implies that this contribution has value strictly greater than $(p^{e_0} - j')\bar{\beta}$. Hence the sets

$$\mathcal{S} := \{j \in \{1, \dots, p^{e_0}\} \mid a_{ji} \neq 0 \text{ and } \nu(a_{ji}) = (p^{e_0} - j)\bar{\beta}\}$$

and $\{\text{in}_\nu a_{ji} \mid j \in \mathcal{S}\}$ are independent of i for $i \geq i_0$.

Remark 8.9. We do not claim that the property that f is a weakly affine expansion in Q_i is preserved when we pass from i to some other ordinal $ii + t$, $t \in \mathbb{N}$. However, the above results show that for any $i' \geq i$ of the form $i' = \ell + t$, $t \in \mathbb{N}_0$, f is a sum of a weakly affine expansion in $Q_{i'}$ all of whose monomials $a_{ji'} Q_{i'}^j$ for $j > 0$ lie on the critical line $\nu(a_{ji}) = (p^{e_0} - j)\bar{\beta}$ and another standard expansion of degree strictly less than p^{e_0} in Q_i , all of whose monomials have value greater than or equal to $2p^{e_0} \bar{\beta} - p^{e_0} \beta_{i'}$.

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