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# LARGE DEVIATIONS FOR THE LARGEST EIGENVALUE OF THE SUM OF TWO RANDOM MATRICES

ALICE GUIONNET AND MYLÈNE MAÏDA

ABSTRACT. In this paper, we consider the addition of two matrices in generic position, namely  $A + UBU^*$ , where  $U$  is drawn under the Haar measure on the unitary or the orthogonal group. We show that, under mild conditions on the empirical spectral measures of the deterministic matrices  $A$  and  $B$ , the law of the largest eigenvalue satisfies a large deviation principle, in the scale  $N$ , with an explicit rate function involving the limit of spherical integrals. We cover in particular all the cases when  $A$  and  $B$  have no outliers.

## 1. INTRODUCTION

Understanding the spectrum of the sum  $A + B$  of two Hermitian matrices knowing the spectra of  $A$  and  $B$  respectively is a classical and difficult problem. Since the pioneering works of Voiculescu [1991], we know that free probability provides efficient tools to describe, at least asymptotically, the spectrum of the sum of two large Hermitian matrices in generic position from one another. More precisely, if  $A_N$  and  $B_N$  are two deterministic  $N \times N$  Hermitian matrices and  $U_N$  is a unitary random matrix distributed according to the Haar measure, then, in the large  $N$  limit,  $A_N$  and  $U_N B_N U_N^*$  are asymptotically free and the spectral distribution of  $H_N := A_N + U_N B_N U_N^*$  is given by the free convolution of the spectral distributions of  $A_N$  and  $B_N$ . This global law, that is the convergence of the spectral distribution of  $H_N$  at macroscopic scale, has been studied in details by Speicher [1993], Pastur and Vasilchuk [2000] among others. The local law, that is the comparison of the spectral distribution of  $H_N$  with the free additive convolution of the spectral distributions of  $A_N$  and  $B_N$  below the macroscopic scale was then investigated by Kargin [2012] and Bao et al. [2017]. In this paper, we will be interested in the behavior of the largest eigenvalue of  $H_N$ . As a corollary of the results of Collins and Male [2014] on strong asymptotic freeness, we know that if  $A_N$  and  $B_N$  have no outliers, then the largest eigenvalue of  $H_N$  converges to the right edge of the support of the free convolution of the spectral distributions of  $A_N$  and  $B_N$ . In this work, we investigate the large deviations of this extreme eigenvalue.

In the framework of random matrix theory, there are very few large deviation results known about the spectrum, basically because the eigenvalues are complicated functions of the entries. A notable exception is given by the Gaussian invariant ensembles for which the joint law of the eigenvalues can be explicitly written as a Coulomb gas. Based on this explicit formula, large deviation principles for the

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spectral measure at global scale have been established by [Ben Arous and Guionnet \[1997\]](#) and for the largest eigenvalue by [Ben Arous et al. \[2001\]](#). Another special case is given by the sum of a deterministic matrix and a Gaussian invariant ensemble. Then, the spectrum can be constructed as the realization at time one of a Hermitian (or symmetric) Brownian motion starting from a given deterministic matrix. This point of view was used by [Guionnet and Zeitouni \[2002\]](#) to study the large deviations of the empirical measure, and the large deviations for the process of the largest eigenvalue starting from the origin were derived by [Donati-Martin and Maïda \[2012\]](#). One of the application of this paper is to provide the large deviation for the largest eigenvalue of this sum by using another approach based on spherical integrals. Beyond these cases where specific tools are available, it was observed by [Bordenave and Caputo \[2014\]](#) that deviations of the spectrum of Wigner matrices for which the distribution of the entries has a tail which is heavier than Gaussian are naturally created by big entries. This key remark allowed to obtain the large deviations for the empirical measure in [[Bordenave and Caputo, 2014](#)] (see also [[Groux, 2017](#)] for the counterpart for covariance matrices) and for the largest eigenvalue in [[Augeri, 2016b](#)]. Large deviations for the spectrum of Wigner matrices with sub-gaussian entries is still completely open as far as the empirical measure is concerned. One can mention the deviations results of [Augeri \[2016a\]](#) for the moments of the spectral measure in several models. Concerning the deviations of the largest eigenvalue, beyond the works [[Ben Arous et al., 2001](#), [Donati-Martin and Maïda, 2012](#), [Augeri, 2016b](#)] already cited above, the following models have been so far studied : Gaussian ensembles plus a rank one perturbation by [Maïda \[2007\]](#), very thin covariance matrices by [Fey et al. \[2008\]](#), finite rank perturbations of deterministic matrices or unitarily invariant ensembles by [Benaych-Georges et al. \[2012\]](#). In a companion paper, [Guionnet and Husson \[2018\]](#) have established a large deviation principle for the largest eigenvalue of Wigner matrices with entries having sharp sub-Gaussian tails, such as Rademacher matrices. They show that the speed and the rate function of this large deviation principle are the same as in the Gaussian case.

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## 2. STATEMENT OF THE RESULTS

Let  $(A_N)_{N \geq 1}$  and  $(B_N)_{N \geq 1}$  be two sequences of deterministic real diagonal matrices, with  $A_N$  and  $B_N$  of size  $N \times N$ . We denote by  $\lambda_1^{(A_N)} \geq \dots \geq \lambda_N^{(A_N)}$  and  $\lambda_1^{(B_N)} \geq \dots \geq \lambda_N^{(B_N)}$  their respective eigenvalues in decreasing order, by

$$\|A_N\| := \max(|\lambda_1^{(A_N)}|, |\lambda_N^{(A_N)}|) \text{ and } \|B_N\| := \max(|\lambda_1^{(B_N)}|, |\lambda_N^{(B_N)}|)$$

their respective spectral radius and by

$$\hat{\mu}_{A_N} := \frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j^{(A_N)}} \text{ and } \hat{\mu}_{B_N} := \frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j^{(B_N)}}$$

their respective spectral measures.

For  $\beta = 1$  or  $2$ , we denote by  $m_N^\beta$  the Haar measure on the orthogonal group  $\mathcal{O}_N$  if  $\beta = 1$  and on the unitary group  $\mathcal{U}_N$  if  $\beta = 2$ . For any  $U$  a  $N \times N$  matrix, we

denote by  $H_N(U) := A_N + UB_NU^*$  and by  $\lambda_{\max}^N$  the largest eigenvalue of  $H_N(U)$ . The goal of the present work is to establish a large deviation principle for the law of  $\lambda_{\max}^N$  under the Haar measure  $m_N^\beta$ . This large deviation principle holds under mild assumptions that we now detail.

**Assumption 1.**

- ( $H_{\text{bulk}}$ ) *The sequences of spectral empirical measures  $(\hat{\mu}_{A_N})_{N \geq 1}$  and  $(\hat{\mu}_{B_N})_{N \geq 1}$  converge weakly as  $N$  grows to infinity respectively to  $\mu_a$  and  $\mu_b$ , compactly supported on  $\mathbb{R}$ . Moreover,  $\sup_{N \geq 1} (\|A_N\| + \|B_N\|) < \infty$ .*
- ( $H_{\text{edge}}$ ) *The largest eigenvalues  $\lambda_1^{(A_N)}$  and  $\lambda_1^{(B_N)}$  converge as  $N$  grows to infinity to  $\rho_a$  and  $\rho_b$  respectively.*

A key argument of the proof will be a tilt of the measure by a rank one spherical integral. Similar strategies are used in the companion paper [Guionnet and Husson, 2018] to study some classes of sub-Gaussian Wigner matrices. The rank one spherical integral is defined as follows: for any  $\theta \geq 0$  and  $M_N$  an Hermitian matrix of size  $N$ ,

$$I_N^\beta(\theta, M_N) := \int e^{N\theta(UM_NU^*)_{11}} m_N^\beta(dU) \quad \text{and} \quad J_N^\beta(\theta, M_N) := \frac{1}{N} \log I_N^\beta(\theta, M_N).$$

The rate function of our large deviation principle will crucially involve the limit of  $J_N^\beta(\theta, H_N)$  as  $N$  grows to infinity, which we now describe. For  $\mu$  a compactly supported probability measure on  $\mathbb{R}$ , we denote by  $r(\mu)$  the right edge of the support of  $\mu$  and by  $G_\mu$  the Stieltjes transform of  $\mu$ : for  $\lambda \geq r(\mu)$ ,

$$G_\mu(\lambda) := \int \frac{1}{\lambda - y} \mu(dy).$$

It is decreasing on the interval  $(r(\mu), \infty)$ . By taking the limit as  $\lambda$  decreases to  $r(\mu)$ , one can also define  $G_\mu(r(\mu)) \in \mathbb{R}_+ \cup \infty$ . As  $G_\mu$  is bijective from  $(r(\mu), \infty)$  to  $(0, G_\mu(r(\mu)))$ , one can define its inverse on this latter interval, that we denote by  $K_\mu$ . Then, for any  $z \in (0, G_\mu(r(\mu)))$ , we define

$$R_\mu(z) := K_\mu(z) - \frac{1}{z}.$$

The function  $R_\mu$  is called the  $R$ -transform for  $\mu$ . We can now define, for  $\beta = 1$  or  $2$ ,  $\theta \geq 0$ ,  $\mu$  a compactly supported probability measure and  $\rho \geq r(\mu)$ :

$$J_\mu^\beta(\theta, \rho) := \begin{cases} \frac{\beta}{2} \int_0^{\frac{2\theta}{\beta}} R_\mu(u) du, & \text{if } 0 \leq \frac{2\theta}{\beta} \leq G_\mu(\rho), \\ \theta \rho - \frac{\beta}{2} \log \theta - \frac{\beta}{2} \int \log(\rho - y) \mu(dy) + \frac{\beta}{2} \left( \log \frac{\beta}{2} - 1 \right), & \text{if } \frac{2\theta}{\beta} > G_\mu(\rho). \end{cases}$$

If  $\rho_1$  and  $\rho_2$  are two probability measures compactly supported on  $\mathbb{R}$ , we denote by  $\rho_1 \boxplus \rho_2$  the free convolution of  $\rho_1$  and  $\rho_2$ . It is uniquely determined as the unique probability measure with  $R$ -transform equal to the sum of the  $R$ -transforms of  $\rho_1$  and  $\rho_2$  (see [Voiculescu, 1991]). For any  $\theta \geq 0$  and  $x \geq r(\mu_a \boxplus \mu_b)$ , we denote by

$$I^\beta(\theta, x) := J_{\mu_a \boxplus \mu_b}^\beta(\theta, x) - J_{\mu_a}^\beta(\theta, \rho_a) - J_{\mu_b}^\beta(\theta, \rho_b),$$

and

$$I^\beta(x) := \begin{cases} \sup_{\theta \geq 0} I^\beta(\theta, x), & \text{if } x \geq r(\mu_a \boxplus \mu_b), \\ \infty, & \text{otherwise.} \end{cases} \quad (2.1)$$

It is easy to check the following:

**Lemma 1.** *Let  $\mu_a, \mu_b, \rho_a$  and  $\rho_b$  be given as in Assumption 1. For  $\beta = 1$  or 2, the function  $I^\beta$  is a good rate function. Moreover, for any  $x > \rho_a + \rho_b$ ,  $I^\beta(x) = \infty$ .*

The proof will be given at the beginning of Section 4. We can now state the main results of this paper. The first result is the following large deviation upper bound:

**Proposition 2.** *Under Assumption 1, for  $\beta = 1$  or 2, for any  $x \in \mathbb{R}$ ,*

$$\limsup_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \log m_N^\beta \left( \lambda_{\max}^N \in [x - \delta, x + \delta] \right) \leq -I^\beta(x).$$

We will then derive the following large deviation lower bound:

**Proposition 3.** *Assume that Assumption 1 holds and that  $\mu_a$  is not a Dirac mass at  $\rho_a$  and  $\mu_b$  is not a Dirac mass at  $\rho_b$ . Then, for  $\beta = 1$  or 2, for any  $x \in \mathbb{R}$  such that*

$$G_{\mu_a \boxplus \mu_b}(x) \leq \min(G_{\mu_a}(\rho_a), G_{\mu_b}(\rho_b)), \quad (2.2)$$

we have

$$\liminf_{\delta \downarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N} \log m_N^\beta \left( \lambda_{\max}^N \in [x - \delta, x + \delta] \right) \geq -I^\beta(x).$$

This leads to the following important corollary:

**Theorem 4.** *Under Assumption 1 and if moreover,*

$$G_{\mu_a \boxplus \mu_b}(r(\mu_a \boxplus \mu_b)) \leq \min(G_{\mu_a}(\rho_a), G_{\mu_b}(\rho_b)), \quad (\text{NoOut})$$

then, for  $\beta = 1$  or 2, the law of  $\lambda_{\max}^N$  under  $m_N^\beta$  satisfies a large deviation principle in the scale  $N$  with good rate function  $I^\beta$ .

One can in fact check (see Lemma 11 for more details) that the condition (NoOut) is automatically satisfied if there is no outliers, namely  $\rho_a = r(\mu_a)$  and  $\rho_b = r(\mu_b)$ . This leads to the following corollary

**Corollary 5.** *If  $A_N$  and  $B_N$  have no outliers, then under Assumption 1, for  $\beta = 1$  or 2, the law of  $\lambda_{\max}^N$  under  $m_N^\beta$  satisfies a large deviation principle in the scale  $N$  with good rate function  $I^\beta$ .*

Observe that in the case where one of the measures  $\mu_a$  or  $\mu_b$  is a Dirac mass at  $\rho_a$  or  $\rho_b$  respectively and the other matrix has no outliers,  $r(\mu_a \boxplus \mu_b) = \rho_a + \rho_b$  so that the above result still holds, but with a degenerate rate function which is infinite except at  $\rho_a + \rho_b$ . The rest of the paper will be devoted to the proofs of these results. In the next section, we will first prove a more general result than Proposition 2, that holds not only for  $m_N^\beta$  but also for a whole family of tilted measures. This will be helpful in the proof of Proposition 3, that will be developed in Section 5. Before getting there, we will study in Section 4 some properties of the rate function  $I^\beta$ . The last section will be devoted to the proof of Theorem 4 and Corollary 5, with Lemma 11 as prerequisite.

### 3. LARGE DEVIATION UPPER BOUND FOR TILTED MEASURES

For  $\theta \geq 0$ ,  $\beta = 1$  or 2, we define a tilted measure on  $\mathcal{O}_N$  if  $\beta = 1$  and  $\mathcal{U}_N$  if  $\beta = 2$  as follows

$$m_N^{\beta, \theta}(dU) := \frac{I_N^\beta(\theta, A_N + UB_NU^*)}{I_N^\beta(\theta, A_N)I_N^\beta(\theta, B_N)} m_N^\beta(dU).$$

It is easy to check that  $m_N^{\beta,\theta}$  is a probability measure: indeed, for any  $U$ , we have that  $I_N^\beta(\theta, A_N + UB_NU^*) \geq 0$  and  $\mathbb{E}_{m_N^\beta}(I_N^\beta(\theta, A_N + UB_NU^*)) = I_N^\beta(\theta, A_N)I_N^\beta(\theta, B_N)$ . For these tilted measures, we have the following weak large deviation upper bound :

**Proposition 6.** *Under Assumption 1, for  $\beta = 1$  or 2, for any  $\theta \geq 0$ , for any  $x < r(\mu_a \boxplus \mu_b)$ ,*

$$\limsup_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \log m_N^{\beta,\theta} \left( \lambda_{\max}^N \in [x - \delta, x + \delta] \right) = -\infty, \quad (3.1)$$

and for any  $x \geq r(\mu_a \boxplus \mu_b)$ ,

$$\limsup_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \log m_N^{\beta,\theta} \left( \lambda_{\max}^N \in [x - \delta, x + \delta] \right) \leq - \left[ I^\beta(x) - I^\beta(\theta, x) \right]. \quad (3.2)$$

**Remark 7.** *Applying this proposition for  $\theta = 0$  gives Proposition 2.*

As we will see in Section 5, establishing an upper bound for any  $\theta \geq 0$  will be useful in the proof of Proposition 3. To prove Proposition 6, and in particular its first statement, we will need to check that, under  $m_N^{\beta,\theta}$  the spectral measure

$$\hat{\mu}_N := \frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j^{(H_N(U))}}$$

of  $H_N(U) = A_N + UB_NU^*$  concentrates around a deterministic probability measure  $\nu_N^\beta$  much faster than  $e^{-N}$ . A natural choice for this deterministic equivalent of  $\hat{\mu}_N$  will be its expectation  $\mathbb{E}_{m_N^\beta} \hat{\mu}_N$ . More precisely, we equip the set  $\mathcal{P}(\mathbb{R})$  of probability measures on  $\mathbb{R}$  with the bounded Lipschitz distance  $d$ : for any Lipschitz function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we define  $\|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)|$  and  $\|f\|_{\text{Lip}} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}$ , then for any  $\mu$  and  $\nu$  in  $\mathcal{P}(\mathbb{R})$ ,

$$d(\mu, \nu) := \sup_{\substack{\|f\|_\infty \leq 1 \\ \|f\|_{\text{Lip}} \leq 1}} \int f d\mu - \int f d\nu.$$

We then have the following concentration result:

**Lemma 8.** *Under Assumption ( $H_{\text{bulk}}$ ), for  $\beta = 1$  or 2 and any  $\theta \geq 0$ ,*

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log m_N^{\beta,\theta} \left( d(\hat{\mu}_N, \mathbb{E}_{m_N^\beta} \hat{\mu}_N) > N^{-1/4} \right) = -\infty.$$

*Proof.* Let  $\beta = 1$  or 2 and  $\theta \geq 0$  be fixed. For any Borel subset  $A$  of  $\mathcal{O}_N$  if  $\beta = 1$  and  $\mathcal{U}_N$  if  $\beta = 2$ , we have:

$$\begin{aligned} m_N^{\beta,\theta}(A) &= \frac{1}{I_N^\beta(\theta, A_N) I_N^\beta(\theta, B_N)} \int_A I_N^\beta(\theta, A_N + UB_NU^*) m_N^\beta(dU) \\ &\leq e^{2N\theta K} m_N^\beta(A), \end{aligned}$$

with  $K := \sup_{N \geq 1} (\|A_N\| + \|B_N\|)$ , which is assumed to be finite. Therefore it is enough to prove Lemma 8 for  $\theta = 0$ , that is

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log m_N^\beta \left( d(\hat{\mu}_N, \mathbb{E}_{m_N^\beta} \hat{\mu}_N) > N^{-1/4} \right) = -\infty.$$

For  $\beta = 2$ , Theorem 3.8 in [Meckes and Meckes, 2013] states that there exists  $c, C > 0$  such that

$$m_N^2 \left( d(\hat{\mu}_N, \mathbb{E}_{m_N^2} \hat{\mu}_N) > N^{-1/4} \right) \leq C e^{-cN^{3/2}}, \quad (3.3)$$

from which the lemma follows. A careful reading of [Meckes and Meckes, 2013] shows that the exact same result as (3.3) also holds for  $\beta = 1$ .  $\square$

We can now prove Proposition 6. In the sequel, we will denote by  $\nu_N^\beta := \mathbb{E}_{m_N^\beta} \hat{\mu}_N$ .

*Proof of Proposition 6.* The first claim (3.1) is a direct consequence of the previous lemma. Indeed, let  $x < r(\mu_a \boxplus \mu_b)$  and  $\delta_0 := \frac{r(\mu_a \boxplus \mu_b) - x}{2}$ . Then, for any  $\delta \leq \delta_0$ , there exists  $\varepsilon(\delta) > 0$ ,

$$\{\lambda_{\max}^N \in [x - \delta, x + \delta]\} \subset \{d(\hat{\mu}_N, \mu_a \boxplus \mu_b) > \varepsilon(\delta)\}. \quad (3.4)$$

Using Corollary 5.4.11 for  $\beta = 2$  and Exercise 5.4.18 for  $\beta = 1$  in [Anderson et al., 2010], we know that  $\nu_N^\beta$  converges weakly to  $\mu_a \boxplus \mu_b$  as  $N$  goes to infinity. As the distance  $d$  metrizes the weak convergence, for  $N$  large enough,

$$\{\lambda_{\max}^N \in [x - \delta, x + \delta]\} \subset \{d(\hat{\mu}_N, \nu_N^\beta) > \varepsilon(\delta)/2\}$$

so that, by Lemma 8, for any  $\delta \leq \delta_0$ ,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log m_N^{\beta, \theta} \left( \lambda_{\max}^N \in [x - \delta, x + \delta] \right) = -\infty.$$

We now prove (3.2). Let  $\delta > 0$  and  $x \geq r(\mu_a \boxplus \mu_b)$  be fixed and define the following event:

$$\mathbf{E}_{N, \delta}^x := \left\{ \lambda_{\max}^N \in [x - \delta, x + \delta], d(\hat{\mu}_N, \nu_N^\beta) \leq N^{-1/4} \right\}. \quad (3.5)$$

Then we have,

$$m_N^{\beta, \theta} \left( \lambda_{\max}^N \in [x - \delta, x + \delta] \right) \leq m_N^{\beta, \theta}(\mathbf{E}_{N, \delta}^x) + m_N^{\beta, \theta}(d(\hat{\mu}_N, \nu_N^\beta) > N^{-1/4}).$$

By Lemma 8, it is therefore enough to show that

$$\limsup_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \log m_N^{\beta, \theta} \left( \mathbf{E}_{N, \delta}^x \right) \leq - \left[ I^\beta(x) - I^\beta(\theta, x) \right].$$

To lighten a bit the notations we write  $A, B$  and  $H$  for  $A_N, B_N$  and  $H_N = A_N + UB_NU^*$  respectively. For any  $\theta, \theta' \geq 0$ , we have

$$\begin{aligned} m_N^{\beta, \theta}(\mathbf{E}_{N, \delta}^x) &= \frac{1}{I_N^\beta(\theta, A)I_N^\beta(\theta, B)} \mathbb{E}_{m_N^\beta} \left( \mathbf{1}_{\mathbf{E}_{N, \delta}^x} I_N^\beta(\theta, H) \frac{I_N^\beta(\theta', H)}{I_N^\beta(\theta', H)} \right) \\ &\leq \frac{\mathbb{E}_{m_N^\beta}(I_N^\beta(\theta', H))}{I_N^\beta(\theta, A)I_N^\beta(\theta, B)} \sup_{U \in \mathbf{E}_{N, \delta}^x} \frac{I_N^\beta(\theta, A + UBU^*)}{I_N^\beta(\theta', A + UBU^*)} \\ &= \frac{I_N^\beta(\theta', A)I_N^\beta(\theta', B)}{I_N^\beta(\theta, A)I_N^\beta(\theta, B)} \sup_{U \in \mathbf{E}_{N, \delta}^x} \frac{I_N^\beta(\theta, A + UBU^*)}{I_N^\beta(\theta', A + UBU^*)} \end{aligned}$$

We now have to estimate  $\sup_{U \in \mathbf{E}_{N, \delta}^x} I_N^\beta(\theta, A + UBU^*)$ : we will use the continuity of spherical integrals derived in [Maïda, 2007] that states as follows. Let  $(G_N)_{N \geq 1}$  a sequence of deterministic matrices such that  $\sup_{N \geq 1} \|G_N\| < \infty$  and for any  $N \geq 1$ ,  $\lambda_1^{(G_N)} = x$  and  $d(\hat{\mu}_{G_N}, \nu_N^\beta) \leq N^{-1/4}$ . According to Proposition 2.1 in [Maïda, 2007], for any  $\theta \geq 0$ , there exists a continuous function  $g_\theta$  such that  $g_\theta(0) = 0$  and for any  $U \in \mathbf{E}_{N, \delta}^x$ ,

$$\left| \frac{1}{N} \log I_N^\beta(\theta, A + UBU^*) - \frac{1}{N} \log I_N^\beta(\theta, G_N) \right| \leq g_\theta(\delta).$$

Therefore,

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{1}{N} \log m_N^{\beta, \theta}(\mathbf{E}_{N, \delta}^x) &\leq \lim_{N \rightarrow \infty} (J_N^\beta(\theta', A) + J_N^\beta(\theta', B) - J_N^\beta(\theta, A) - J_N^\beta(\theta, B)) \\ &\quad + \lim_{N \rightarrow \infty} (J_N^\beta(\theta, G_N) - J_N^\beta(\theta', G_N)) + g_\theta(\delta) + g_{\theta'}(\delta), \\ &\leq -(I^\beta(\theta', x) - I^\beta(\theta, x)) + g_\theta(\delta) + g_{\theta'}(\delta), \end{aligned}$$

where at the last line, we have used Theorem 6 in [Guionnet and Maïda, 2005]. Letting  $\delta$  going to zero and then optimizing over  $\theta' \geq 0$ , we get the required upper bound.  $\square$

#### 4. PROPERTIES OF THE RATE FUNCTION $I^\beta$

We now check the properties of the rate function  $I^\beta$  defined in (2.1).

*Proof of Lemma 1.* An ingredient for the proof is the following: for any compactly supported  $\mu$ , for any  $\theta \geq 0$  and  $\rho \geq r(\mu)$  such that  $\theta \leq G_\mu(\rho)$ , we have

$$\rho - \frac{1}{\theta} \leq R_\mu(\theta) \leq \rho - \frac{1}{G_\mu(\rho)}. \quad (4.1)$$

Indeed, as  $K_\mu$  is a decreasing function, we have  $R_\mu(\theta) = K_\mu(\theta) - \frac{1}{\theta} \geq \rho - \frac{1}{\theta}$ . On the other hand, the limit of  $R_\mu(\theta)$  as  $\theta$  grows to  $G_\mu(\rho)$  is  $\rho - \frac{1}{G_\mu(\rho)}$ . As  $R_\mu$  is nondecreasing, we get the upper bound. Moreover, it is easy to check that, for any  $x \geq 0$ , there exists  $C, C' \in \mathbb{R}$  (depending on  $\mu$  and  $x$  but not on  $\theta$ ) such that, for  $\theta$  large enough, we have

$$\theta x - \frac{\beta}{2} \log \theta + C \leq J_\mu^\beta(\theta, x) \leq \theta x + C',$$

so that, for any  $x \geq 0$ , there exists  $c, c' \in \mathbb{R}$  such that, for  $\theta$  large enough,

$$\theta(x - \rho_a - \rho_b) - \frac{\beta}{2} \log \theta + c \leq I^\beta(\theta, x) \leq \theta(x - \rho_a - \rho_b) + \beta \log \theta + c'.$$

If  $x > \rho_a + \rho_b$ , letting  $\theta$  grow to infinity, we obtain that  $I^\beta(x) = \infty$ .

If  $\theta \geq 0$  is small enough,

$$I^\beta(\theta, x) = \frac{\beta}{2} \int_0^{\frac{2\theta}{\beta}} (R_{\mu_a \boxplus \mu_b}(u) - R_{\mu_a}(u) - R_{\mu_b}(u)) du = 0,$$

by the properties of the  $R$ -transform. The function  $I^\beta$  is therefore nonnegative. If we denote by  $g$  the lower semi-continuous function which is equal to  $-\infty$  on  $[r(\mu_a \boxplus \mu_b), \infty)$  and  $\infty$  outside, then  $I^\beta = \sup(g, \sup_\theta I^\beta(\theta, \cdot))$  is lower semi-continuous as a supremum of lower semi-continuous functions. As it is infinite outside the interval  $[r(\mu_a \boxplus \mu_b), \rho_a + \rho_b]$ , it is a good rate function.  $\square$

To perform the tilt leading to the lower bound, we will need to further study the properties of the function  $I^\beta$ .

**Lemma 9.** *Under Assumption 1, for any  $r(\mu_a \boxplus \mu_b) \leq x < \rho_a + \rho_b$  such that*

$$G_{\mu_a \boxplus \mu_b}(x) \leq \min(G_{\mu_a}(\rho_a), G_{\mu_b}(\rho_b)),$$

*then, for  $\beta = 1$  or  $2$ , there exists a unique  $\theta \geq 0$  such that*

$$I^\beta(\theta, x) = \sup_{\theta' \geq 0} I^\beta(\theta', x).$$



We denote by  $\theta_x^\beta := \operatorname{argmax}_{\theta \geq 0} I^\beta(\theta, x)$ . For any  $r(\mu_a \boxplus \mu_b) \leq x < \rho_a + \rho_b$  and  $r(\mu_a \boxplus \mu_b) \leq y \leq \rho_a + \rho_b$  such that  $x \neq y$ ,

$$\sup_{\theta \geq 0} I^\beta(\theta, y) > I^\beta(\theta_x^\beta, y).$$

*Proof of Lemma 9.* Let  $r(\mu_a \boxplus \mu_b) \leq x < \rho_a + \rho_b$  such that

$$G_{\mu_a \boxplus \mu_b}(x) \leq \min(G_{\mu_a}(\rho_a), G_{\mu_b}(\rho_b)).$$

The first remark is that if  $G_{\mu_a}(\rho_a)$  and  $G_{\mu_b}(\rho_b)$  are infinite, then  $r(\mu_a \boxplus \mu_b) \geq \rho_a + \rho_b$  and there is nothing to check. Indeed, if  $G_{\mu_a}(\rho_a) = G_{\mu_b}(\rho_b) = \infty$ , we see by the inequalities (4.1), that

$$\lim_{x \rightarrow \infty} R_{\mu_a}(x) = \rho_a \quad \text{and} \quad \lim_{x \rightarrow \infty} R_{\mu_b}(x) = \rho_b,$$

so that

$$\lim_{x \rightarrow \infty} K_{\mu_a \boxplus \mu_b}(x) = \rho_a + \rho_b \quad \text{and} \quad \lim_{x \rightarrow \rho_a + \rho_b} G_{\mu_a \boxplus \mu_b}(x) = \infty,$$

leading to  $r(\mu_a \boxplus \mu_b) \geq \rho_a + \rho_b$ . By symmetry of the problem, without loss of generality, one can now assume that  $G_{\mu_a}(\rho_a) \leq G_{\mu_b}(\rho_b)$  and  $G_{\mu_a}(\rho_a) < \infty$ .

With the function  $I^\beta$  defined in (2.1), if we denote by  $I_x^\beta$  the function  $\theta \mapsto I^\beta(\theta, x)$ , then there exist some constants  $C_1, C_2$  and  $C_3$  (that may depend on  $\mu_a, \rho_a, \mu_b, \rho_b$  and  $x$  but not on  $\theta$ ) such that

$$I_x^\beta(\theta) = \begin{cases} 0, & \text{if } 0 \leq \frac{2\theta}{\beta} \leq G_{\mu_a \boxplus \mu_b}(x), \\ \theta x - \frac{\beta}{2} \log \theta - \frac{\beta}{2} \int_0^{\frac{2\theta}{\beta}} (R_{\mu_a} + R_{\mu_b})(u) du + C_1, & \text{if } G_{\mu_a \boxplus \mu_b}(x) \leq \frac{2\theta}{\beta} \leq G_{\mu_a}(\rho_a), \\ \theta(x - \rho_a) - \frac{\beta}{2} \int_0^{\frac{2\theta}{\beta}} R_{\mu_b}(u) du + C_2, & \text{if } G_{\mu_a}(\rho_a) \leq \frac{2\theta}{\beta} \leq G_{\mu_b}(\rho_b), \\ \theta(x - \rho_a - \rho_b) + \frac{\beta}{2} \log \theta + C_3, & \text{if } \frac{2\theta}{\beta} \geq G_{\mu_b}(\rho_b), \end{cases}$$

where the last line does not occur if  $G_{\mu_b}(\rho_b) = \infty$ . In the computation, we have used the well known fact that  $R_{\mu_a \boxplus \mu_b} = R_{\mu_a} + R_{\mu_b}$  when the three functions are well defined. Therefore, one can check that the function  $I_x^\beta$  is continuously differentiable and its derivative is given by:

$$(I_x^\beta)'(\theta) = \begin{cases} 0, & \text{if } 0 \leq \frac{2\theta}{\beta} \leq G_{\mu_a \boxplus \mu_b}(x), \\ x - K_{\mu_a \boxplus \mu_b}\left(\frac{2\theta}{\beta}\right), & \text{if } G_{\mu_a \boxplus \mu_b}(x) \leq \frac{2\theta}{\beta} \leq G_{\mu_a}(\rho_a), \\ x - \rho_a - R_{\mu_b}\left(\frac{2\theta}{\beta}\right), & \text{if } G_{\mu_a}(\rho_a) \leq \frac{2\theta}{\beta} \leq G_{\mu_b}(\rho_b), \\ x - \rho_a - \rho_b + \frac{\beta}{2\theta}, & \text{if } \frac{2\theta}{\beta} \geq G_{\mu_b}(\rho_b). \end{cases}$$

We now set  $\alpha_x := \frac{1}{\rho_a + \rho_b - x}$ . We claim that

$$\alpha_x \geq G_{\mu_a}(\rho_a).$$

Indeed,  $K_{\mu_b}$  is well defined on the interval  $(0, G_{\mu_b}(\rho_b))$ , so that  $K_{\mu_b}(G_{\mu_a}(\rho_a))$  and therefore  $K_{\mu_a \boxplus \mu_b}(G_{\mu_a}(\rho_a))$  are well defined. As  $K_{\mu_a \boxplus \mu_b}$  is a decreasing function, we have:

$$G_{\mu_a \boxplus \mu_b}(x) \leq G_{\mu_a}(\rho_a)$$

and this implies:

$$x \leq K_{\mu_a \boxplus \mu_b}(G_{\mu_a}(\rho_a)) = K_{\mu_a}(G_{\mu_a}(\rho_a)) + K_{\mu_b}(G_{\mu_a}(\rho_a)) - \frac{1}{G_{\mu_a}(\rho_a)}$$

As  $K_{\mu_b}$  is also a decreasing function, this yields:

$$x \leq K_{\mu_a}(G_{\mu_a}(\rho_a)) + K_{\mu_b}(G_{\mu_b}(\rho_b)) - \frac{1}{G_{\mu_a}(\rho_a)} = \rho_a + \rho_b - \frac{1}{G_{\mu_a}(\rho_a)},$$

which is equivalent to  $\alpha_x \geq G_{\mu_a}(\rho_a)$ . There are therefore two cases to consider and we claim that:

**Case 1:** If  $G_{\mu_a}(\rho_a) \leq \alpha_x < G_{\mu_b}(\rho_b)$ , then  $I_x^\beta$  reaches its maximum at

$$\theta_x^\beta := \frac{\beta}{2} R_{\mu_b}^{(-1)}(x - \rho_a);$$

**Case 2:** if  $\alpha_x \geq G_{\mu_b}(\rho_b)$ , then  $I_x^\beta$  reaches its maximum at  $\theta_x^\beta := \frac{\beta}{2} \alpha_x$ .

Let us now prove this claim. On the interval  $\left[0, \frac{\beta}{2} G_{\mu_a}(\rho_a)\right]$ , the function  $(I_x^\beta)'$  is nondecreasing and it vanishes at zero, it is therefore nonnegative so that  $I_x^\beta$  is nondecreasing on this interval. We have

$$(I_x^\beta)' \left( \frac{\beta}{2} G_{\mu_a}(\rho_a) \right) \geq 0 \quad \text{and} \quad (I_x^\beta)' \left( \frac{\beta}{2} G_{\mu_b}(\rho_b) \right) = -\frac{1}{\alpha_x} + \frac{1}{G_{\mu_b}(\rho_b)}.$$

Moreover, as  $R_{\mu_b}$  is an increasing function,  $(I_x^\beta)'$  is decreasing on the interval  $\left[\frac{\beta}{2} G_{\mu_a}(\rho_a), \frac{\beta}{2} G_{\mu_b}(\rho_b)\right]$ . We now distinguish the two cases.

In **Case 1**,  $(I_x^\beta)' \left( \frac{\beta}{2} G_{\mu_b}(\rho_b) \right) < 0$ , and therefore there exists

$$\theta_x \in \left[ \frac{\beta}{2} G_{\mu_a}(\rho_a), \frac{\beta}{2} G_{\mu_b}(\rho_b) \right)$$

such that  $I_x^\beta$  is increasing on  $\left[\frac{\beta}{2} G_{\mu_a}(\rho_a), \theta_x\right]$  and then decreasing. One can check that the point where  $(I_x^\beta)'$  cancels is given by  $\frac{\beta}{2} R_{\mu_b}^{(-1)}(x - \rho_a)$ . Moreover,  $(I_x^\beta)'$  is decreasing on  $\left[\frac{\beta}{2} G_{\mu_b}(\rho_b), \infty\right)$  and negative at  $\frac{\beta}{2} G_{\mu_b}(\rho_b)$  so it remains negative and  $I_x^\beta$  is decreasing on this interval. The first claim holds true.

In **Case 2**,  $(I_x^\beta)' \left( \frac{\beta}{2} G_{\mu_b}(\rho_b) \right) \geq 0$ , and therefore  $I_x^\beta$  is increasing on the interval  $\left[\frac{\beta}{2} G_{\mu_a}(\rho_a), \frac{\beta}{2} G_{\mu_b}(\rho_b)\right]$ . But  $(I_x^\beta)'$  is nonnegative at  $\frac{\beta}{2} G_{\mu_b}(\rho_b)$ , decreasing on  $\left[\frac{\beta}{2} G_{\mu_b}(\rho_b), \infty\right)$  and converges to  $x - \rho_a - \rho_b < 0$  as  $\theta$  grows to  $\infty$ . Therefore, there exists  $\theta_x \in \left(\frac{\beta}{2} G_{\mu_b}(\rho_b), \infty\right)$  such that  $I_x^\beta$  is increasing on  $\left(\frac{\beta}{2} G_{\mu_b}(\rho_b), \theta_x\right]$  and then decreasing. One can check that the point where  $(I_x^\beta)'$  cancels is given by  $\frac{\beta}{2} \alpha_x$  and the second claim holds true. This concludes the proof of the uniqueness of  $\theta$ .

Moreover, looking carefully at the definition of  $\theta_x^\beta$  in **Case 1** and **Case 2**, one can see that it is an increasing function of  $x$ . In particular, for  $x \neq y$  such that  $r(\mu_a \boxplus \mu_b) \leq x, y < \rho_a + \rho_b$ ,  $\theta_x^\beta \neq \theta_y^\beta$  and therefore  $\sup_{\theta \geq 0} I^\beta(\theta, y) > I^\beta(\theta_x^\beta, y)$ .

We now have to deal with the case when  $y = \rho_a + \rho_b$ , that is to show that:

$$\sup_{\theta \geq 0} I^\beta(\theta, \rho_a + \rho_b) > I^\beta(\theta_x^\beta, \rho_a + \rho_b). \quad (4.2)$$

If  $G_{\mu_b}(\rho_b)$  is finite, for  $\theta > \frac{\beta}{2} G_{\mu_b}(\rho_b)$ ,

$$I^\beta(\theta, \rho_a + \rho_b) = \frac{\beta}{2} \log \theta + C_3$$

and therefore the supremum is infinite and (4.2) holds. Otherwise let us first consider the case where  $\mu_b = \delta_{\rho_b}$ . We claim that in this case, the condition  $r(\mu_a \boxplus \mu_b) \leq x < \rho_a + \rho_b$  and  $G_{\mu_a \boxplus \mu_b}(x) \leq \min(G_{\mu_a}(\rho_a), G_{\mu_b}(\rho_b))$  are never simultaneously satisfied.

Indeed, in this case,  $\mu_a \boxplus \mu_b$  is just a shift of  $\mu_a$  by  $\rho_b$ , so that, for any  $x < \rho_a + \rho_b$ ,  $G_{\mu_a \boxplus \mu_b}(x) = G_{\mu_a}(x - \rho_b) > G_{\mu_a}(\rho_a)$ , as  $G_{\mu_a}$  is decreasing. If  $\mu_b \neq \delta_{\rho_b}$ , then, there exists  $\alpha \in (0, 1]$  and  $M$  finite such that, for any  $x \geq \rho_b$ ,

$$G_{\mu_b}(x) \leq \frac{1 - \alpha}{x - \rho_b} + M.$$

From there, we get that, for any  $u > G_{\mu_a}(\rho_a) \vee \frac{2M}{\alpha}$ ,

$$u \leq \frac{1 - \alpha}{K_{\mu_b}(u) - \rho_b} + M \quad \text{so that} \quad R_{\mu_b}(u) \leq \rho_b - \frac{\alpha}{2u}.$$

Therefore, there exist  $c, c' \in \mathbb{R}$ , such that for any  $\theta \geq G_{\mu_a}(\rho_a) \vee \frac{2M}{\alpha}$ ,

$$I^\beta(\theta, \rho_a + \rho_b) \geq \theta \rho_b - \frac{\beta}{2} \int_{\frac{2M}{\alpha}}^{\frac{2\theta}{\beta}} \left( \rho_b - \frac{\alpha}{2u} \right) du + c = \frac{\beta \alpha}{4} \log \theta + c'$$

so that, letting  $\theta$  grow to infinity, we get again that  $I^\beta(\rho_a + \rho_b) = \infty$  and (4.2) holds. This concludes the proof of Lemma 9.  $\square$

## 5. LARGE DEVIATION LOWER BOUND

The goal of this section is to show Proposition 3. A classical strategy to get a large deviation lower bound is to tilt the measure in such a way that the rare event  $\{\lambda_{\max}^N \in [x - \delta, x + \delta]\}$  becomes typical under the tilted measure. We now check that it is possible to make such a tilt:

**Lemma 10.** *Under Assumption 1, for any  $x \in [r(\mu_a \boxplus \mu_b), \rho_a + \rho_b)$  such that*

$$G_{\mu_a \boxplus \mu_b}(x) \leq \min(G_{\mu_a}(\rho_a), G_{\mu_b}(\rho_b)),$$

for  $\beta = 1$  or  $2$ , we have

$$\lim_{\delta \downarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N} \log m_N^{\beta, \theta_x^\beta} \left( \mathbb{E}_{N, \delta}^x \right) \geq 0,$$

where  $\mathbb{E}_{N, \delta}^x$  was defined in (3.5) and  $\theta_x^\beta$  in Lemma 9.

*Proof of Lemma 10.* Let  $\beta = 1$  or  $2$  and  $r(\mu_a \boxplus \mu_b) \leq x < \rho_a + \rho_b$  be fixed. Let  $y \neq x$  be such that  $y < r(\mu_a \boxplus \mu_b)$  or  $y > \rho_a + \rho_b$ . By Lemma 1, we know that  $I^\beta(y) = \infty$ , so that, by Proposition 6, we have

$$\lim_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \log m_N^{\beta, \theta_x^\beta} \left( \lambda_{\max}^N \in [y - \delta, y + \delta] \right) = -\infty.$$

Let now  $y \neq x$  be such that  $r(\mu_a \boxplus \mu_b) \leq y \leq \rho_a + \rho_b$ . Then, by Proposition 6, we have

$$\lim_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \log m_N^{\beta, \theta_x^\beta} \left( \lambda_{\max}^N \in [y - \delta, y + \delta] \right) \leq -(\sup_{\theta \geq 0} I^\beta(\theta, y) - I^\beta(\theta_x^\beta, y))$$

As a consequence, if we denote by

$$L_x^\beta(y) := \begin{cases} \sup_{\theta \geq 0} I^\beta(\theta, y) - I^\beta(\theta_x^\beta, y), & \text{if } r(\mu_a \boxplus \mu_b) \leq x \leq \rho_a + \rho_b, \\ \infty, & \text{otherwise,} \end{cases}$$

we know that the law of  $\lambda_{\max}^N$  under  $m_N^{\beta, \theta_x^\beta}$  satisfies a weak large deviation upper bound with good rate function  $L_x^\beta$ . Moreover, for  $N$  large enough,  $\lambda_{\max}^N$  lies with probability one in the compact set  $[r(\mu_a \boxplus \mu_b) - 1, \rho_a + \rho_b + 1]$ , so that it is in fact

a large deviation upper bound. By Lemma 9, we know that  $L_x^\beta$  is nonnegative and vanishes only at  $x$ . Therefore, we deduce that, for any  $\delta > 0$ , for  $N$  large enough,

$$m_N^{\beta, \theta_x^\beta}(\lambda_{\max}^N \in [x - \delta, x + \delta]) \geq \frac{3}{4}.$$

But, in virtue of Lemma 8, for  $N$  large enough, we also have

$$m_N^{\beta, \theta_x^\beta}(d(\hat{\mu}_N, \nu_N^\beta) \leq N^{-1/4}) \geq \frac{3}{4}$$

so that

$$m_N^{\beta, \theta_x^\beta}(\mathbb{E}_{N, \delta}^x) \geq \frac{1}{2},$$

and Lemma 10 follows.  $\square$

From there, one can easily get the large deviation lower bound.

*Proof of Proposition 3.* Let  $\beta = 1$  or  $2$  and  $x \geq r(\mu_a \boxplus \mu_b)$  be fixed. If  $x > \rho_a + \rho_b$  or  $x < r(\mu_a \boxplus \mu_b)$ , Lemma 1 gives that  $I^\beta(x) = \infty$ , so that the lower bound obviously holds. Moreover, as we have seen at the end of the proof of Lemma 9, as  $\mu_b$  is not a Dirac mass at  $\rho_b$ , then  $I^\beta(\rho_a + \rho_b) = \infty$  and the lower bound also holds for  $x = \rho_a + \rho_b$ .

Let us now assume that  $r(\mu_a \boxplus \mu_b) \leq x < \rho_a + \rho_b$  and let  $\theta_x^\beta$  be the corresponding shift defined in Lemma 9. Then, with  $\mathbb{E}_{N, \delta}^x$  defined in (3.5), we have:

$$\begin{aligned} m_N^\beta(\lambda_{\max}^N \in [x - \delta, x + \delta]) &\geq m_N^\beta(\mathbb{E}_{N, \delta}^x) = \mathbb{E}_{m_N^\beta} \left( \mathbf{1}_{\mathbb{E}_{N, \delta}^x} \frac{I_N^\beta(\theta_x^\beta, H)}{I_N^\beta(\theta_x^\beta, H)} \right) \\ &\geq \inf_{U \in \mathbb{E}_{N, \delta}^x} \frac{1}{I_N^\beta(\theta_x^\beta, A + UBU^*)} \\ &\quad \times I_N^\beta(\theta_x^\beta, A) I_N^\beta(\theta_x^\beta, B) m_N^{\beta, \theta_x^\beta}(\mathbb{E}_{N, \delta}^x) \end{aligned}$$

so that, using again Proposition 2.1 in [Maïda, 2007], we get:

$$\begin{aligned} \liminf_{N \rightarrow \infty} \frac{1}{N} \log m_N^\beta(\lambda_{\max}^N \in [x - \delta, x + \delta]) &\geq -I^\beta(\theta_x^\beta, x) - g_{\theta_x^\beta}(\delta) \\ &\quad + \liminf_{N \rightarrow \infty} \frac{1}{N} \log m_N^{\beta, \theta_x^\beta}(\mathbb{E}_{N, \delta}^x). \end{aligned}$$

Letting  $\delta$  going to zero and using Lemma 10, we get that

$$\lim_{\delta \downarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N} \log m_N^\beta(\lambda_{\max}^N \in [x - \delta, x + \delta]) \geq -I^\beta(\theta_x^\beta, x) \geq -I^\beta(x).$$

This concludes the proof.  $\square$

## 6. PROOF OF THE MAIN THEOREM AND ITS COROLLARY

*Proof of Theorem 4.* Assume that Assumption 1 and the condition (NoOut) are satisfied. If we denote by  $K := \sup_{n \geq 1} (\|A_n\| + \|B_n\|)$ , which is assumed to be finite, we have that for any  $N \geq 1$ ,

$$m_N^\beta(\lambda_{\max}^N > 2K) = 0,$$

so that the exponential tightness is obviously satisfied. By [Dembo and Zeitouni, 1998, Lemma 4.1.23], it is therefore enough to show a weak large deviation principle. The upper bound is given by Proposition 2 for  $\theta = 0$ .

As for the lower bound, we distinguish three cases, if  $G_{\mu_a}(\rho_a) = G_{\mu_b}(\rho_b) = \infty$ , as we have seen if the proof of Lemma 9, we have that  $r(\mu_a \boxplus \mu_b) = \rho_a + \rho_b$ . In particular,  $\lambda_{\max}^N$  converges almost surely to  $\rho_a + \rho_b$ , so that the lower bound holds. If  $\mu_b = \delta_{\rho_b}$ , then  $\mu_a \boxplus \mu_b$  is just a shift of  $\mu_a$  by  $\rho_b$ , so that  $r(\mu_a \boxplus \mu_b) = r(\mu_a) + \rho_b$  and  $G_{\mu_a \boxplus \mu_b}(r(\mu_a \boxplus \mu_b)) = G_{\mu_a}(r(\mu_a))$ . Assume that  $G_{\mu_a}(\rho_a) < \infty$ . If  $r(\mu_a) < \rho_a$ , then the condition (NoOut) is not satisfied, because  $G_{\mu_a}$  is a decreasing function. If  $r(\mu_a) = \rho_a$ , then we have a similar situation as in the previous case,  $\lambda_{\max}^N$  converges almost surely to  $\rho_a + \rho_b$ , so that the lower bound holds. By symmetry, the same holds true if  $\mu_a = \delta_{\rho_a}$ . Otherwise and if the condition (NoOut) holds, as  $G_{\mu_a \boxplus \mu_b}$  is decreasing, then for any  $x \geq r(\mu_a \boxplus \mu_b)$ , we have

$$G_{\mu_a \boxplus \mu_b}(x) \leq \min(G_{\mu_a}(\rho_a), G_{\mu_b}(\rho_b)).$$

The lower bound is given by Proposition 3.  $\square$

We now prove Corollary 5. Our goal is to show that if  $A_N$  and  $B_N$  have no outliers, then the condition (NoOut) is automatically satisfied. Indeed, if  $A_N$  and  $B_N$  have no outliers, it means that their respective largest eigenvalues converge to the edge of the support of the limiting measure, that is to say  $\rho_a = r(\mu_a)$  and  $\rho_b = r(\mu_b)$ . Therefore, Corollary 5 is a direct consequence of the following lemma:

**Lemma 11.** *For any probability measures  $\mu$  and  $\nu$  compactly supported on  $\mathbb{R}$ , we have*

$$G_{\mu \boxplus \nu}(r(\mu \boxplus \nu)) \leq \min(G_{\mu}(r(\mu)), G_{\nu}(r(\nu))).$$

*Proof.* If one of the measures  $\mu$  or  $\nu$  is a single point mass, the additive free convolution is just a translation and we have equality. We now assume that none of them is a single point mass. In general, we know (see e.g. [Belinschi, 2008]) that there exists a function  $\omega$ , called the subordination function, which is analytic on  $\mathbb{C}^+ := \{z \in \mathbb{C}, \Im z > 0\}$  such that, for all  $z \in \mathbb{C}^+$ ,

$$G_{\mu \boxplus \nu}(z) = G_{\mu}(\omega(z)) \tag{6.1}$$

By [Belinschi, 2006, Theorem 2.3], as  $\mu$  or  $\nu$  are not a single point mass,  $G_{\mu \boxplus \nu}$  can be continuously extended to  $\mathbb{C}^+ \cup \mathbb{R}$  with values in  $\overline{\mathbb{C}} := \mathbb{C} \cup \infty$ . Moreover, as  $\mu$  and  $\nu$  are compactly supported, by [Belinschi, 2008, Theorem 3.3(3)],  $\omega$  can also be continuously extended to  $\mathbb{C}^+ \cup \mathbb{R}$ . From (6.1), we have that, for any  $z \in \mathbb{C}^+ \cup \mathbb{R}$ ,

$$\Im G_{\mu \boxplus \nu}(z) = -\Im \omega(z) \cdot \int \frac{d\mu(t)}{|t - \omega(z)|^2}.$$

Let  $z$  be a real number in the interval  $(r(\mu \boxplus \nu), \infty)$ . Then  $\int \frac{d\mu(t)}{|t - \omega(z)|^2} > 0$  and  $\Im G_{\mu \boxplus \nu}(z) = 0$ , so that  $\Im \omega(z) = 0$ . Therefore,  $\omega$  restricted to the interval  $(r(\mu \boxplus \nu), \infty)$  takes values in  $\mathbb{R} \cup \infty$ . Moreover  $\omega(z)$  goes to  $\infty$  as  $z$  goes to  $\infty$ , so that  $\omega((r(\mu \boxplus \nu), \infty))$  is an interval  $I_{\omega}$  containing a neighborhood of  $\infty$ .

Let  $a < r(\mu)$  such that  $(a, \infty) \subset I_{\omega}$ . For any  $y > 0$ , we have

$$-\int_a^{r(\mu)} \Im G_{\mu}(x + iy) = \int_a^{r(\mu)} d\mu(t) \left( \arctan\left(\frac{r(\mu) - t}{y}\right) - \arctan\left(\frac{a - t}{y}\right) \right).$$

As  $y$  decreases to zero, the right hand-side converges to  $\pi\mu((a, r(\mu))) > 0$ . On the other hand, for any  $x \in (a, r(\mu)) \subset \omega((r(\mu \boxplus \nu), \infty))$ , there exists  $x' > r(\mu \boxplus \nu)$ , such that  $x = \omega(x')$  and

$$\Im G_{\mu}(x) = \Im G_{\mu}(\omega(x')) = \Im G_{\mu \boxplus \nu}(x') = 0.$$

As  $G_\mu$  is continuous on  $\mathbb{C}^+ \cup \mathbb{R}$ , by dominated convergence, we get that the left hand-side goes to zero, as  $y$  decreases to zero. This leads to a contradiction and we deduce that  $I_\omega \subset [r(\mu), \infty)$ , which means, by continuity of  $\omega$ , that

$$\omega(r(\mu \boxplus \nu)) \geq r(\mu).$$

As  $G_\mu$  is decreasing on  $(r(\mu), \infty)$ , this gives

$$G_{\mu \boxplus \nu}(r(\mu \boxplus \nu)) = G_\mu(\omega(r(\mu \boxplus \nu))) \leq G_\mu(r(\mu)).$$

As  $\mu$  and  $\nu$  play symmetric roles, this concludes the proof of Lemma 11.  $\square$

#### REFERENCES

- G. W. Anderson, A. Guionnet, and O. Zeitouni. *An introduction to random matrices*, volume 118 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010. ISBN 978-0-521-19452-5. 6
- F. Augeri. On the large deviations of traces of random matrices. 2016a. URL <https://arxiv.org/abs/1605.03894v1>. 2
- F. Augeri. Large deviations principle for the largest eigenvalue of Wigner matrices without Gaussian tails. *Electron. J. Probab.*, 21:Paper No. 32, 49, 2016b. ISSN 1083-6489. doi: 10.1214/16-EJP4146. URL <https://doi.org/10.1214/16-EJP4146>. 2
- Z. Bao, L. Erdős, and K. Schnelli. Local law of addition of random matrices on optimal scale. *Comm. Math. Phys.*, 349(3):947–990, 2017. ISSN 0010-3616. doi: 10.1007/s00220-016-2805-6. URL <https://doi.org/10.1007/s00220-016-2805-6>. 1
- S. T. Belinschi. A note on regularity for free convolutions. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(5):635–648, 2006. ISSN 0246-0203. doi: 10.1016/j.anihpb.2005.05.004. URL <https://doi.org/10.1016/j.anihpb.2005.05.004>. 12
- S. T. Belinschi. The Lebesgue decomposition of the free additive convolution of two probability distributions. *Probab. Theory Related Fields*, 142(1-2):125–150, 2008. ISSN 0178-8051. doi: 10.1007/s00440-007-0100-3. URL <https://doi.org/10.1007/s00440-007-0100-3>. 12
- G. Ben Arous and A. Guionnet. Large deviations for Wigner’s law and Voiculescu’s non-commutative entropy. *Probab. Theory Related Fields*, 108(4):517–542, 1997. ISSN 0178-8051. doi: 10.1007/s004400050119. URL <https://doi.org/10.1007/s004400050119>. 2
- G. Ben Arous, A. Dembo, and A. Guionnet. Aging of spherical spin glasses. *Probab. Theory Related Fields*, 120(1):1–67, 2001. ISSN 0178-8051. doi: 10.1007/PL00008774. URL <https://doi.org/10.1007/PL00008774>. 2
- F. Benaych-Georges, A. Guionnet, and M. Maida. Large deviations of the extreme eigenvalues of random deformations of matrices. *Probab. Theory Related Fields*, 154(3-4):703–751, 2012. ISSN 0178-8051. doi: 10.1007/s00440-011-0382-3. URL <https://doi.org/10.1007/s00440-011-0382-3>. 2
- C. Bordenave and P. Caputo. A large deviation principle for Wigner matrices without Gaussian tails. *Ann. Probab.*, 42(6):2454–2496, 2014. ISSN 0091-1798. doi: 10.1214/13-AOP866. URL <https://doi.org/10.1214/13-AOP866>. 2
- B. Collins and C. Male. The strong asymptotic freeness of Haar and deterministic matrices. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(1):147–163, 2014. ISSN 0012-9593. doi: 10.24033/asens.2211. URL <https://doi.org/10.24033/asens.2211>. 1

- A. Dembo and O. Zeitouni. *Large deviations techniques and applications*, volume 38 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 1998. ISBN 0-387-98406-2. doi: 10.1007/978-1-4612-5320-4. URL <https://doi.org/10.1007/978-1-4612-5320-4>. 11
- C. Donati-Martin and M. Maïda. Large deviations for the largest eigenvalue of an Hermitian Brownian motion. *ALEA Lat. Am. J. Probab. Math. Stat.*, 9(2):501–530, 2012. ISSN 1980-0436. 2
- A. Fey, R. van der Hofstad, and M. J. Klok. Large deviations for eigenvalues of sample covariance matrices, with applications to mobile communication systems. *Adv. in Appl. Probab.*, 40(4):1048–1071, 2008. ISSN 0001-8678. URL <http://projecteuclid.org/euclid.aap/1231340164>. 2
- B. Groux. Asymptotic freeness for rectangular random matrices and large deviations for sample covariance matrices with sub-Gaussian tails. *Electron. J. Probab.*, 22:Paper No. 53, 40, 2017. ISSN 1083-6489. doi: 10.1214/17-EJP4326. URL <https://doi.org/10.1214/17-EJP4326>. 2
- A. Guionnet and J. Husson. Large deviations for the largest eigenvalue of Rademacher matrices. working paper or preprint, July 2018. URL <https://hal.archives-ouvertes.fr/hal-01828877>. 2, 3
- A. Guionnet and M. Maïda. A Fourier view on the  $R$ -transform and related asymptotics of spherical integrals. *J. Funct. Anal.*, 222(2):435–490, 2005. ISSN 0022-1236. URL <https://doi.org/10.1016/j.jfa.2004.09.015>. 7
- A. Guionnet and O. Zeitouni. Large deviations asymptotics for spherical integrals. *J. Funct. Anal.*, 188(2):461–515, 2002. ISSN 0022-1236. doi: 10.1006/jfan.2001.3833. URL <https://doi.org/10.1006/jfan.2001.3833>. 2
- V. Kargin. A concentration inequality and a local law for the sum of two random matrices. *Probab. Theory Related Fields*, 154(3-4):677–702, 2012. ISSN 0178-8051. URL <https://doi.org/10.1007/s00440-011-0381-4>. 1
- M. Maïda. Large deviations for the largest eigenvalue of rank one deformations of Gaussian ensembles. *Electron. J. Probab.*, 12:1131–1150, 2007. ISSN 1083-6489. URL <https://doi.org/10.1214/EJP.v12-438>. 2, 6, 11
- E. S. Meckes and M. W. Meckes. Concentration and convergence rates for spectral measures of random matrices. *Probab. Theory Related Fields*, 156(1-2):145–164, 2013. ISSN 0178-8051. doi: 10.1007/s00440-012-0423-6. URL <https://doi.org/10.1007/s00440-012-0423-6>. 5, 6
- L. Pastur and V. Vasilchuk. On the law of addition of random matrices. *Comm. Math. Phys.*, 214(2):249–286, 2000. ISSN 0010-3616. doi: 10.1007/s002200000264. URL <https://doi.org/10.1007/s002200000264>. 1
- R. Speicher. Free convolution and the random sum of matrices. *Publ. Res. Inst. Math. Sci.*, 29(5):731–744, 1993. ISSN 0034-5318. doi: 10.2977/prims/1195166573. URL <https://doi.org/10.2977/prims/1195166573>. 1
- D. Voiculescu. Limit laws for random matrices and free products. *Invent. Math.*, 104(1):201–220, 1991. ISSN 0020-9910. doi: 10.1007/BF01245072. URL <https://doi.org/10.1007/BF01245072>. 1, 3

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