



HAL
open science

ORBIFOLD CHERN CLASSES INEQUALITIES AND APPLICATIONS

Erwan Rousseau, Behrouz Taji

► **To cite this version:**

Erwan Rousseau, Behrouz Taji. ORBIFOLD CHERN CLASSES INEQUALITIES AND APPLICATIONS. 2018. hal-01887365v1

HAL Id: hal-01887365

<https://hal.science/hal-01887365v1>

Preprint submitted on 4 Oct 2018 (v1), last revised 11 Sep 2023 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ORBIFOLD CHERN CLASSES INEQUALITIES AND APPLICATIONS

ERWAN ROUSSEAU AND BEHROUZ TAJI

ABSTRACT. In this paper we prove that given a pair (X, D) of a threefold X and a boundary divisor D with mild singularities, if $(K_X + D)$ is movable, then the orbifold second Chern class c_2 of (X, D) is pseudoeffective. This generalizes the classical result of Miyaoka on the pseudoeffectivity of c_2 for minimal models. As an application, we give a simple solution to Kawamata’s effective non-vanishing conjecture in dimension 3, where we prove that $H^0(X, K_X + H) \neq 0$, whenever $K_X + H$ is nef and H is an ample, effective, reduced Cartier divisor. Furthermore, we study Lang-Vojta’s conjecture for codimension one subvarieties and prove that minimal threefolds of general type have only finitely many Fano, Calabi-Yau or Abelian subvarieties of codimension one that are mildly singular and whose numerical classes belong to the movable cone.

CONTENTS

1. Introduction	1
2. Basic definitions and background	3
3. Restriction results for semistable sheaves	6
4. Semipositivity of adapted sheaf of forms	11
5. Pseudoeffectivity of the orbifold c_2	14
6. An effective non-vanishing result for threefolds	16
7. A Miyaoka-Yau inequality in higher dimensions	18
8. Remarks on Lang-Vojta’s conjecture in codimension one	21
References	22

1. INTRODUCTION

It is well known that the Chern classes of nef vector bundles over smooth projective varieties satisfy certain inequalities [DPS94]. More generally, a theorem of Miyaoka [Miy87] states that over a normal, projective variety (that is smooth in codimension two) any torsion free, coherent sheaf \mathcal{E} that is *semipositive* with respect to the tuple of ample divisors (H_1, \dots, H_{n-1}) and whose determinant $\det(\mathcal{E})$ is nef, verifies the inequality

$$c_2(\mathcal{E}) \cdot H_1 \dots H_{n-2} \geq 0.$$

On the other hand, thanks to Miyaoka’s celebrated *generic semipositivity* result, cf. [Miy87], and the result of Boucksom, Demailly, Păun and Peternell ([BDPP13]),

2010 *Mathematics Subject Classification.* 14E30, 14J70, 14B05.

Key words and phrases. Classification theory, Miyaoka-Yau inequality, Movable cone of divisors, Minimal Models, Effective non-vanishing, Lang-Vojta’s conjecture.

Behrouz Taji was partially supported by the DFG-Graduiertenkolleg GK1821 “Cohomological Methods in Geometry” at the University of Freiburg. Erwan Rousseau was partially supported by the ANR project “FOLIAGE”, ANR-16-CE40-0008.

when K_X is pseudoeffective, the cotangent bundle Ω_X^1 of a smooth projective variety is generically semipositive. As a result, for a smooth projective variety X with K_X nef, the inequality

$$(1.0.1) \quad c_2(X) \cdot H_1 \dots H_{n-2} \geq 0$$

holds, for any tuple of ample divisors (H_1, \dots, H_{n-2}) .

Recent works of Campana and Păun ([CP15], [CP16]) have generalized some parts of Miyaoka's results, showing in particular that if X is a smooth projective variety with K_X pseudoeffective, then Ω_X^1 is semipositive with respect to any *movable* class $\alpha \in \text{Mov}_1(X)$.

Our first result is a natural generalization of the inequality (1.0.1) to the setting of pairs with movable log-canonical divisors.

Theorem 1.1. *Let X be a normal projective threefold that is smooth in codimension two and D a reduced effective divisor such that (X, D) has only isolated lc singularities. If $(K_X + D) \in \text{Mov}^1(X)_{\mathbb{Q}}$, then for any ample divisor A , the inequality*

$$c_2((\Omega_X^1 \log(D))^{**}) \cdot A \geq 0$$

holds.

The second result is another generalization of an inequality established by Miyaoka [Miy87], which is sometimes referred to as the Miyaoka-Yau inequality.

Theorem 1.2. *Let X be a normal projective threefold that is smooth in codimension two and D a reduced effective divisor such that (X, D) has only isolated lc singularities. If $(K_X + D) \in \text{Mov}^1(X)_{\mathbb{Q}}$, then*

$$c_1^2((\Omega_X^1 \log(D))^{**}) \cdot A \leq 3c_2((\Omega_X^1 \log(D))^{**}) \cdot A,$$

for any ample divisor A .

There are two main ingredients in the proof of the above inequalities. The first one is a restriction result for semistable sheaves with respect to some *strongly movable* curves. This is described in section 3. The second component involves the semipositivity of the *orbifold cotangent sheaves* and is treated in section 4.

The rest of the paper is devoted to two applications of Theorems 1.1 and 1.2. The first one concerns the so-called effective non-vanishing conjecture.

Conjecture 1.3 (Effective non-vanishing conjecture of Kawamata). *Let Y be a normal projective variety and D_Y an effective \mathbb{R} -divisor such that (Y, D_Y) is klt. Let H be an ample, or more generally big and nef, divisor such that $(K_Y + D_Y + H)$ is Cartier and nef. Then $H^0(X, K_Y + D_Y + H) \neq 0$.*

Using Theorem 1.1, in Section 6, we obtain a simple proof of the following weak version of Conjecture 1.3 in dimension three.

Theorem 1.4 (Non-vanishing for canonical threefolds). *Let Y be a normal projective threefold with only canonical singularities. Let H be a very ample divisor. If $(K_Y + H)$ is a nef and Cartier divisor, then $H^0(Y, K_Y + H) \neq 0$.*

We note that Theorem 1.4 is stated in [Hör12] under the weaker assumption that H is a nef and big Cartier divisor. The proof relies on an inequality similar to that of Theorem 1.1 but under the weaker assumption that the first Chern class is *nef in codimension one*. It seems that there is a gap in the proof of that inequality, but according to the author one can get rid of this assumption and use only the classical result of Miyaoka where c_1 is assumed to be nef (cf. the inequality 1.0.1).

A second application is given in section 8 vis-à-vis Lang-Vojta's conjectures on subvarieties of varieties of general type:

Geometric Lang-Vojta conjecture: *In a projective variety of general type X , subvarieties that are not of general type are contained in a proper algebraic subvariety of X .*

In particular, a variety of general type should have only finitely many codimension one subvarieties that are not of general type. We partially establish this conjecture in the setting of the following theorem.

Theorem 1.5. *Let X be a normal projective \mathbb{Q} -factorial threefold such that $K_X \in \text{Mov}^1(X)_{\mathbb{Q}}$. If X is of general type then X has only a finite number of movable codimension one, normal subvarieties D verifying the following conditions.*

(1.5.1) *The subvariety D has only canonical singularities.*

(1.5.2) *The anticanonical divisor $-K_D$ is pseudoeffective.*

(1.5.3) *The pair (X, D) has only isolated lc singularities.*

In particular, there are only finitely many such Fano, Abelian and Calabi-Yau subvarieties.

Here, by a variety of general type, we mean a normal variety whose resolution has a big canonical bundle.

We remark that—in the smooth setting—a stronger version of Theorems 1.5 and 1.1 has been claimed in [LM97], where the authors establish these results under the weaker assumption that $(K_X + D)$ is pseudoeffective. Unfortunately the arguments in [LM97] are not complete. We refer to Remark 8.2 for a detailed discussion of these problems.

1.1. Acknowledgements. The authors would like to thank Sébastien Boucksom, Junyan Cao, Paolo Cascini, Andreas Höring, Steven Lu and Mihai Păun for fruitful discussions.

2. BASIC DEFINITIONS AND BACKGROUND

2.1. Movable cone. We introduce the *movable* cone of divisors; one of the important cones of divisors that is ubiquitous in birational geometry.

Let X be a normal projective variety and D a \mathbb{Q} -divisor on X . The stable base locus of D is defined by

$$\mathbb{B}(D) := \bigcap_m \text{Bs}(|mD|).$$

The restricted base locus is given by

$$\mathbb{B}_-(D) = \bigcup_{A \text{ ample}} \mathbb{B}(D + A).$$

Definition 2.1. Let X be a normal variety. The movable cone $\text{Mov}^1(X) \subset \mathbb{N}^1(X)$ is the closure of the cone generated by the classes of all effective divisors D such that $\mathbb{B}_-(D)$ has no divisorial components.

The following proposition gives a more geometric picture of this definition.

Proposition 2.2 ([Bou04], Proposition 2.3). *Given any α in the interior of $\text{Mov}^1(X)$, there is a birational map $\phi : Y \rightarrow X$ and an ample divisor A on Y such that $[\phi_* A] = \alpha$.*

2.2. Stability with respect to movable classes. Now we introduce the notion of movable curves which generate the cone dual to the pseudoeffective cone.

Definition 2.3. A class $\gamma \in N_1(X)$ is movable if $\gamma \cdot D \geq 0$ for all effective divisors D . We define $\text{Mov}_1(X)$ to be the closed convex cone of such 1-cycles.

Movable classes form a natural setting for the notion of stability of coherent sheaves (see [CP11] and [GKP15]). We shall now recall the basic definitions and properties.

Definition 2.4. Assume that X is \mathbb{Q} -factorial and let $\gamma \in \text{Mov}_1(X)$. The slope of a coherent sheaf \mathcal{E} with respect to γ is given by

$$\mu_\gamma(\mathcal{E}) := \frac{1}{r} \cdot (\det(\mathcal{E})) \cdot \gamma.$$

Definition 2.5. We say that \mathcal{E} is semistable with respect to γ if $\mu_\gamma(\mathcal{F}) \leq \mu_\gamma(\mathcal{E})$ for any coherent subsheaf $0 \subsetneq \mathcal{F} \subset \mathcal{E}$.

Proposition 2.6 ([GKP15], Corollary 2.27). *Let X be a normal, \mathbb{Q} -factorial, projective variety and $\gamma \in \text{Mov}_1(X)$. There exists a unique Harder-Narasimhan filtration i.e. a filtration $0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \dots \subsetneq \mathcal{E}_r = \mathcal{E}$ where each quotient $\mathcal{Q}_i := \mathcal{E}_i / \mathcal{E}_{i-1}$ is torsion-free, γ -semistable, and where the sequence of slopes $\mu_\gamma(\mathcal{Q}_i)$ is strictly decreasing.*

2.3. \mathbb{Q} -twisted sheaves. It will be quite useful in the sequel to work in the more general setting of \mathbb{Q} -twisted sheaves as introduced in [Miy87].

Definition 2.7 (\mathbb{Q} -twisted sheaves). A \mathbb{Q} -twisted sheaf is a pair $\mathcal{E}\langle B \rangle$, where \mathcal{E} is a coherent sheaf and B is a \mathbb{Q} -Cartier divisor.

Notation 2.8. Let X be a normal projective variety and \mathcal{F} a coherent sheaf on X of rank r . Let D be a Weil divisor in X such that $\det(\mathcal{F}) \cong \mathcal{O}_X(D)$. When D is \mathbb{Q} -Cartier, we define $[\mathcal{F}]$ to denote the numerical class $[D] \in N^1(X)_{\mathbb{Q}}$ of D . For any \mathbb{Q} -Cartier divisor A , we set $[\mathcal{F}\langle A \rangle] = [\mathcal{F}] + r \cdot [A]$. Here by $\det(\mathcal{F})$ we always mean the reflexive hull of the determinant of \mathcal{F} .

Let X be a normal projective variety of dimension n which is smooth in codimension 2 and \mathcal{E} a reflexive sheaf on X . Then, one can define the second Chern class $c_2(\mathcal{E})$, cycle-theoretically, as a multilinear form on

$$\underbrace{N^1(X)_{\mathbb{Q}} \times \dots \times N^1(X)_{\mathbb{Q}}}_{(n-2)\text{-times}}.$$

We now recall the usual formulas for Chern classes of \mathbb{Q} -twisted sheaves.

Definition 2.9 (Chern classes of \mathbb{Q} -twisted sheaves). Let $\mathcal{E}\langle B \rangle$ be a \mathbb{Q} -twisted locally-free sheaf of rank r .

$$\begin{aligned} c_1(\mathcal{E}\langle B \rangle) &= c_1(\mathcal{E}) + rc_1(B), \\ c_2(\mathcal{E}\langle B \rangle) &= c_2(\mathcal{E}) + (r-1)c_1(\mathcal{E}) \cdot c_1(B) + \frac{r(r-1)}{2}c_1(B)^2. \end{aligned}$$

The semipositivity property for sheaves also naturally extends to this setting.

Definition 2.10 (Semipositive \mathbb{Q} -twisted sheaves). Let X be a normal projective variety and $\gamma \in \text{Mov}(X)$. A \mathbb{Q} -twisted, torsion-free sheaf $\mathcal{E}\langle B \rangle$ is said to be semipositive with respect to γ , if for every torsion-free, \mathbb{Q} -twisted, quotient sheaf $\mathcal{E}\langle B \rangle \twoheadrightarrow \mathcal{F}\langle B \rangle$ we have $[\mathcal{F}\langle B \rangle] \cdot \gamma \geq 0$.

We also have the Bogomolov-Gieseker inequality for semistable \mathbb{Q} -twisted sheaves.

Proposition 2.11 (Bogomolov-Gieseker inequality for semistable \mathbb{Q} -twisted sheaves). *Take S to be a smooth projective surface. Let $\mathcal{E}\langle B \rangle$ be a \mathbb{Q} -twisted locally-free sheaf on S of rank r and $A \in \text{Amp}(X)_{\mathbb{Q}}$. If $\mathcal{E}\langle B \rangle$ is semistable with respect to A , then $\mathcal{E}\langle B \rangle$ verifies the Bogomolov-Gieseker inequality:*

$$(2.11.1) \quad 2r \cdot c_2(\mathcal{E}\langle B \rangle) - (r-1) \cdot c_1^2(\mathcal{E}\langle B \rangle) \geq 0.$$

Proof. Let $h : T \rightarrow S$ be the morphism adapted to B so that $\mathcal{E}_T := h^*(\widehat{\mathcal{E}}\langle B \rangle)$ is locally-free. Define $K := \text{Gal}(T/S)$. Notice that as the maximal destabilizing subsheaf of $h^*(\mathcal{E}_T)$ is unique, it is K -invariant. As a result, \mathcal{E}_T is semistable with respect to $A_T := h^*A$. The inequality 2.11.1 now follows from the standard Bogomolov-Gieseker inequality for semistable locally free sheaves. \square

2.4. Orbifold basics. Following the terminology of Campana [Cam04], an *orbifold* is simply a pair (X, D) , consisting of a normal projective variety and a boundary divisor $D = \sum d_i \cdot D_i$, where $d_i = (1 - b_i/a_i) \in [0, 1] \cap \mathbb{Q}$.

Our aim is now to define a notion of cotangent sheaf, adapted to an orbifold. To this end, and since we will not be exclusively working with smooth varieties, we will need a notion of pull-back for Weil divisors (that are not necessarily \mathbb{Q} -Cartier).

Definition 2.12 (Pull-back of Weil divisors). Let $f : Y \rightarrow X$ be a finite morphism between quasi-projective normal varieties X and Y . We define pull-back $f^*(D)$ of a \mathbb{Q} -Weil divisor $D \subset X$ by the Zariski closure of $(f|_{Y_{\text{reg}}})^*(D)$.

To define classical objects for orbifolds, it is quite convenient to use *adapted morphisms*.

Definition 2.13 (Adapted and strongly adapted morphisms). Let (X, D) be an orbifold. A finite, surjective, Galois morphism $f : Y \rightarrow X$ is called *adapted* (to D) if, f^*D is an integral Weil divisor. We say that a given adapted morphism $f : Y \rightarrow X$ is *strictly adapted*, if we have $f^*D_i = a_i \cdot D'_i$, for some Weil divisor $D'_i \subset Y$. Furthermore, we call a strictly adapted morphism f , *strongly adapted*, if the branch locus of f only consists of $\text{supp}(D - \lfloor D \rfloor + A)$, where A is a general member of a linear system of a very ample invertible sheaf on X .

Remark 2.14. For a pair (X, D) , where X is smooth, and D is \mathbb{Q} -effective divisor with simple normal crossing support, the existence of a strongly adapted morphism $f : Y \rightarrow X$ was established by Kawamata, cf. [Laz04, Prop. 4.1]. A similar strategy can be applied to construct strongly adapted morphisms $f : Y \rightarrow X$ when all the irreducible components of D are \mathbb{Q} -Cartier; in particular when X is assumed to be \mathbb{Q} -factorial.

Notation 2.15. Let $f : Y \rightarrow X$ be a morphism adapted to D , where $D = \sum d_i \cdot D_i$, $d_i = 1 - \frac{b_i}{a_i} \in (0, 1] \cap \mathbb{Q}$. For every prime component D_i of $(D - \lfloor D \rfloor)$, let $\{D_{ij}\}_{j(i)}$ be the collection of prime divisors that appear in $f^*(D_i)$. We define new divisors in Y by

$$(2.15.1) \quad D_Y^{ij} := b_i \cdot D_{ij}$$

$$(2.15.2) \quad D_f := f^*(\lfloor D \rfloor).$$

Now, let us explain how to define the cotangent sheaf of an orbifold.

Definition 2.16 (Orbifold cotangent sheaf). In the situation of Notation 2.15, denote Y° to be the snc locus of the pair $(Y, \sum D_{ij} + D_f)$ and define $D_Y^{ij^\circ} := D_Y^{ij}|_{Y^\circ}$. Set $\Omega_{(Y,f,D)}^1$ to be the kernel of the sheaf morphism

$$(f|_{Y^\circ})^*(\Omega_X \log(\Gamma D^\neg)) \longrightarrow \bigoplus_{i,j(i)} \mathcal{O}_{D_Y^{ij^\circ}}$$

induced by the natural residue map. We define the orbifold cotangent sheaf $\Omega_{(Y,f,D)}^{[1]}$ by the coherent extension $(i_{Y^\circ})_*(\Omega_{(Y^\circ,f,D)}^1)$, where i_{Y° is the natural inclusion. We define the orbifold tangent sheaf $\mathcal{T}_{(Y,f,D)}$ by $(\Omega_{(Y,f,D)}^{[1]})^*$.

3. RESTRICTION RESULTS FOR SEMISTABLE SHEAVES

Let $h = (H_1, \dots, H_{n-1})$ be a tuple of ample divisors on a normal projective variety X of dimension n and \mathcal{E} a torsion free sheaf. A theorem of Mehta-Ramanathan [MR82] states that if m is large enough and $Y \in |mH_{n-1}|$ is a generic hypersurface, then the maximal destabilizing subsheaf of $\mathcal{E}|_Y$ is the restriction of the maximal destabilizing subsheaf of \mathcal{E} .

It is natural to try to extend this restriction theorem to movable polarization. Unfortunately, in general, such results are not valid for movable curve. For example, when X is a projective K3 surface then its cotangent bundle Ω_X^1 is not pseudoeffective, which gives rise to the existence of movable curves for which the restriction theorem does not hold (cf. [BDPP13, Sect. 7]).

In this section, we will prove a restriction theorem for some strongly movable curves (see Proposition 3.3 below). The following lemma will serve as the key technical ingredient in the proof of this result.

Lemma 3.1 (Induced destabilizing subsheaves on higher birational models). *Let $\pi : \tilde{S} \rightarrow S$ be a birational morphism between two smooth projective surfaces \tilde{S} and S . Let $\tilde{A}_{\tilde{S}} \subset \tilde{S}$ be an ample divisor and define $P_S := [\pi_*(\tilde{A}_{\tilde{S}})] \in \mathbb{N}^1(S)_{\mathbb{Q}}$. Let $\mathcal{F}_S \subsetneq \mathcal{E}_S$ be a maximal destabilizing subsheaf with respect to P_S of a locally-free sheaf \mathcal{E}_S on S of rank 3. The maximal destabilizing subsheaf $\mathcal{G}_{\tilde{S}}$ of $\pi^*(\mathcal{E}_S)$ with respect to $\tilde{A}_{\tilde{S}}$ is then of the form*

$$\pi^*(\mathcal{F}_S) \otimes \mathcal{O}_{\tilde{S}}(E'),$$

where E' is an exceptional divisor.

Proof. First we notice that by arguing inductively we may assume, without loss of generality, that $\pi : \tilde{S} \rightarrow S$ is a blow up of a single point in S . Let $\tilde{\mathcal{F}}_{\tilde{S}}$ be the pull-back $\pi^*\mathcal{F}_S$. We divide the proof into various cases depending on the ranks of $\tilde{\mathcal{F}}_{\tilde{S}}$ and $\tilde{\mathcal{G}}_{\tilde{S}}$, aiming to show that $\tilde{\mathcal{F}}_{\tilde{S}} \cong \tilde{\mathcal{G}}_{\tilde{S}}$.

Case. 1. ($\text{rank}(\tilde{\mathcal{F}}_{\tilde{S}}) = \text{rank}(\tilde{\mathcal{G}}_{\tilde{S}}) = 2$). As $\tilde{\mathcal{F}}_{\tilde{S}}$ and $\tilde{\mathcal{G}}_{\tilde{S}}$ are both saturated inside $\pi^*\mathcal{E}_S$, thus so are $\Lambda^2 \tilde{\mathcal{F}}_{\tilde{S}}$, $\Lambda^2 \tilde{\mathcal{G}}_{\tilde{S}}$ inside $\Lambda^2(\pi^*\mathcal{E}_S)$. Therefore, if there exists a nontrivial morphism $t : \tilde{\mathcal{G}}_{\tilde{S}} \rightarrow \tilde{\mathcal{F}}_{\tilde{S}}$, then the naturally induced morphism $\Lambda^2 t : \Lambda^2 \tilde{\mathcal{G}}_{\tilde{S}} \rightarrow \Lambda^2 \tilde{\mathcal{F}}_{\tilde{S}}$ is a nontrivial morphism between invertible sheaves. It follows that $\mu_{\tilde{A}_{\tilde{S}}}(\tilde{\mathcal{G}}_{\tilde{S}})$ must be equal to $\mu_{\tilde{A}_{\tilde{S}}}(\tilde{\mathcal{F}}_{\tilde{S}})$ and thus, by the uniqueness of $\tilde{\mathcal{G}}$, we have $\tilde{\mathcal{F}}_{\tilde{S}} \cong \tilde{\mathcal{G}}_{\tilde{S}}$.

So, we may assume that there is no nontrivial morphisms from $\tilde{\mathcal{G}}_{\tilde{S}}$ to $\tilde{\mathcal{F}}_{\tilde{S}}$. Aiming for a contradiction, consider the exact sequence $0 \rightarrow \tilde{\mathcal{F}}_{\tilde{S}} \rightarrow \pi^*(\mathcal{E}_S) \rightarrow \tilde{\mathcal{L}} \rightarrow 0$. By our assumption, there exists a nontrivial morphism $u : \tilde{\mathcal{G}}_{\tilde{S}} \rightarrow \tilde{\mathcal{L}}$ with kernel $\tilde{\mathcal{H}}$:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \widetilde{\mathcal{F}}_{\widetilde{S}} & \longrightarrow & \pi^*(\mathcal{E}_S) & \longrightarrow & \widetilde{\mathcal{L}} \longrightarrow 0, \\
& & \uparrow & & \uparrow & \nearrow u & \\
0 & \longrightarrow & \widetilde{\mathcal{H}} & \longrightarrow & \widetilde{\mathcal{G}}_{\widetilde{S}} & &
\end{array}$$

Now, as $\widetilde{\mathcal{F}}_{\widetilde{S}}$ properly destabilizes $\pi^*(\mathcal{E}_S)$, we have

$$(3.1.1) \quad \mu_{\widetilde{A}_{\widetilde{S}}}(\widetilde{\mathcal{F}}_{\widetilde{S}}) > \mu_{\widetilde{A}_{\widetilde{S}}}(\pi^*\mathcal{E}_S).$$

It thus follows that

$$(3.1.2) \quad \begin{aligned} \mu_{\widetilde{A}_{\widetilde{S}}}(\widetilde{\mathcal{L}}) &< \mu_{\widetilde{A}_{\widetilde{S}}}(\pi^*\mathcal{E}_S) < \mu_{\widetilde{A}_{\widetilde{S}}}(\widetilde{\mathcal{F}}_{\widetilde{S}}), && \text{by Inequality 3.1.1} \\ &< \mu_{\widetilde{A}_{\widetilde{S}}}(\widetilde{\mathcal{G}}_{\widetilde{S}}). \end{aligned}$$

On the other hand, as $\widetilde{\mathcal{G}}_{\widetilde{S}}$ is semistable, from the exact sequence $0 \rightarrow \widetilde{\mathcal{H}} \rightarrow \widetilde{\mathcal{G}}_{\widetilde{S}} \rightarrow \widetilde{\mathcal{L}} \rightarrow 0$, it follows that $\mu_{\widetilde{A}_{\widetilde{S}}}(\widetilde{\mathcal{G}}_{\widetilde{S}}) < \mu_{\widetilde{A}_{\widetilde{S}}}(\widetilde{\mathcal{L}})$, contradicting the inequality (3.1.2).

Case. 2. ($\text{rank}(\widetilde{\mathcal{F}}_{\widetilde{S}}) = 2$ and $\text{rank}(\widetilde{\mathcal{G}}_{\widetilde{S}}) = 1$). We claim that this case does not occur. To this end, consider the exact sequence

$$0 \rightarrow \widetilde{\mathcal{G}}_{\widetilde{S}} \rightarrow \pi^*\mathcal{E}_S \rightarrow \mathcal{Q} \rightarrow 0,$$

where \mathcal{Q} is torsion free, quotient sheaf of rank two. Note that there exists a non-trivial morphism $q : \pi^*\mathcal{F}_S \rightarrow \mathcal{Q}$. Therefore, we have the slope inequality

$$(3.1.3) \quad \mu_{\widetilde{A}_{\widetilde{S}}}(\pi^*\mathcal{F}_S) \leq \mu_{\widetilde{A}_{\widetilde{S}}}(\mathcal{Q}).$$

On the other hand, as $\widetilde{\mathcal{G}}_{\widetilde{S}} \subset \pi^*\mathcal{E}_S$ is the maximal destabilizing subsheaf, we have

$$(3.1.4) \quad \mu_{\widetilde{A}_{\widetilde{S}}}(\mathcal{Q}) < \mu_{\widetilde{A}_{\widetilde{S}}}(\pi^*\mathcal{E}_S).$$

From the inequalities 3.1.3 and 3.1.4 it now follows that

$$\mu_{\widetilde{A}_{\widetilde{S}}}(\pi^*\mathcal{F}_S) < \mu_{\widetilde{A}_{\widetilde{S}}}(\pi^*\mathcal{E}_S),$$

contradicting the fact that \mathcal{F}_S destabilizes \mathcal{E}_S .

Case. 3. ($\text{rank}(\mathcal{F}_S) = 1$). We divide the proof of this case into two subcases based on the existence of a nontrivial morphism from $\widetilde{\mathcal{F}}_{\widetilde{S}}$ to $\widetilde{\mathcal{G}}_{\widetilde{S}}$.

Subcase. 3.1. (No nontrivial morphisms exist). In this case for the projection $p : \widetilde{\mathcal{F}}_{\widetilde{S}} \rightarrow \widetilde{\mathcal{Q}}$, defined by

$$\begin{array}{ccccccc}
0 & \longrightarrow & \widetilde{\mathcal{G}}_{\widetilde{S}} & \longrightarrow & \pi^*(\mathcal{E}_S) & \longrightarrow & \widetilde{\mathcal{Q}} \longrightarrow 0 \\
& & & & \uparrow & \nearrow p & \\
& & & & \widetilde{\mathcal{F}}_{\widetilde{S}} & &
\end{array}$$

we have that $\text{rank}(\text{Image}(p)) \neq 0$. Again, as we are only concerned with the slope of the sheaves $\widetilde{\mathcal{F}}_{\widetilde{S}}$ and $\widetilde{\mathcal{Q}}$, we may assume with no loss of generality that p is an injection and that $\widetilde{\mathcal{F}}_{\widetilde{S}} \subseteq \widetilde{\mathcal{Q}}$. On the other hand, we know, thanks to the assumption that $\widetilde{\mathcal{G}}_{\widetilde{S}}$ properly destabilizes $\pi^*\mathcal{E}_S$ (with respect to $\widetilde{A}_{\widetilde{S}}$), that

$$(3.1.5) \quad \mu_{\tilde{A}_S}(\pi^* \mathcal{E}_S) > \mu_{\tilde{A}_S}(\tilde{\mathcal{Q}}).$$

But $\mu_{\tilde{A}_S}(\tilde{\mathcal{Q}}) \geq \mu_{\tilde{A}_S}(\tilde{\mathcal{F}}_S) \geq \mu_{\tilde{A}_S}(\pi^* \mathcal{F}_S)$, i.e.

$$(3.1.6) \quad \mu_{\tilde{A}_S}(\tilde{\mathcal{Q}}) \geq \mu_{P_S}(\mathcal{F}_S).$$

By combining the two inequalities 3.1.5 and 3.1.6 we find that $\mu_{\tilde{A}_S}(\pi^* \mathcal{E}) > \mu_{P_S}(\mathcal{F}_S)$, that is $\mu_{P_S}(\mathcal{E}_S) > \mu_{P_S}(\mathcal{F}_S)$, contradicting the assumption that \mathcal{F}_S destabilizes \mathcal{E}_S with respect to P_S .

Subcase. 3.2. (A nontrivial morphism exists). Let $j : \tilde{\mathcal{F}}_S \rightarrow \tilde{\mathcal{G}}_S$ be a nontrivial morphism. As $\tilde{\mathcal{G}}_S$ has no torsion, the sheaf morphism j is an injection in codimension one. Moreover since we are only concerned with slope of the image j with respect to \tilde{A}_S , we may assume with no loss of generality, that j is globally an injection. After identifying $\tilde{\mathcal{F}}_S$ with its image (under j), we write $\tilde{\mathcal{F}}_S \subset \tilde{\mathcal{G}}_S$. Our aim is now to show that this set-up for $\tilde{\mathcal{G}}_S$ and $\tilde{\mathcal{F}}_S$ leads to a contradiction and that this subcase does not occur, unless $\tilde{\mathcal{F}}_S \cong \tilde{\mathcal{G}}_S$.

Let \mathcal{G}_S be the locally-free sheaf on S defined by the coherent extension of $(\pi|_{\tilde{S} \setminus \text{Exc}(\pi)})_*(\tilde{\mathcal{G}}_S)$ onto S . Thanks to the reflexivity of both \mathcal{F}_S and \mathcal{G}_S , we have the inclusion of locally free sheaves $\mathcal{F}_S \subset \mathcal{G}_S$. In particular $\text{rank}(\tilde{\mathcal{G}}_S) = 2$. Furthermore, we have $\pi^*(\mathcal{F}_S) \subset \pi^*(\mathcal{G}_S)$ and that, for some $a \in \mathbb{Z}$, the isomorphism $\tilde{\mathcal{G}}_S \cong \pi^*(\mathcal{G}_S) \otimes \mathcal{O}_{\tilde{S}}(a \cdot E)$ holds. Here, the divisor E is the (irreducible) exceptional divisor. Notice that we have

$$(3.1.7) \quad \tilde{A}_S \cdot (a \cdot E) \geq 0,$$

otherwise $\tilde{\mathcal{G}}$ cannot be the maximal destabilizing subsheaf of $\pi^* \mathcal{E}_S$. Therefore $a \geq 0$.

Claim 3.2. Let $\widehat{\mathcal{F}}_S$ be the saturation of $\tilde{\mathcal{F}}_S$ inside $\tilde{\mathcal{G}}_S$. There exists a positive integer $b \geq a$ for which the isomorphism

$$\widehat{\mathcal{F}}_S \cong \pi^*(\mathcal{F}_S) \otimes \mathcal{O}_{\tilde{S}}(b \cdot E)$$

holds.

Proof of Claim 3.2. First notice that we have $\widehat{\mathcal{F}}_S \cong \pi^*(\mathcal{F}_S) \otimes \mathcal{O}_{\tilde{S}}(b \cdot E)$, for some $b \in \mathbb{N}$. As the inclusion $\pi^*(\mathcal{F}_S) \otimes \mathcal{O}_{\tilde{S}}(b \cdot E) \subseteq \pi^*(\mathcal{G}_S) \otimes \mathcal{O}_{\tilde{S}}(a \cdot E)$ is saturated, over a Zariski open subset $E^\circ \subseteq E$ we have $\mathcal{O}_{\tilde{S}}(b \cdot E)|_{E^\circ} \subseteq (\mathcal{O}_{\tilde{S}}(a \cdot E)|_{E^\circ})^{\oplus 2}$. Since $\mathcal{O}_{\tilde{S}}(E)|_{E^\circ} \cong \mathcal{O}_{E^\circ}(-1)$, it follows that $b \geq a$. \square

Using Claim 3.2, and by construction of $\widehat{\mathcal{F}}_S$, we get the following sequence of inequalities.

$$\begin{aligned}
\mu_{\tilde{A}_{\tilde{S}}}(\widehat{\mathcal{F}}_S) &= (\pi^*([\mathcal{F}_S]) + (b \cdot E)) \cdot \tilde{A}_{\tilde{S}} \\
&= [\mathcal{F}_S] \cdot P_S + ((b \cdot E) \cdot \tilde{A}_{\tilde{S}}) \\
&> \mu_{P_S}(\mathcal{G}_S) + ((b \cdot E) \cdot \tilde{A}_{\tilde{S}}), && \text{as } \mathcal{F}_S \text{ is maximal destabilizing} \\
&\geq \frac{1}{2}[\mathcal{G}_S] \cdot P_S + ((a \cdot E) \cdot \tilde{A}_{\tilde{S}}), && \text{as } b \geq a \\
&\geq \frac{1}{2}(\pi^*[\mathcal{G}_S] \cdot \tilde{A}_{\tilde{S}} + ((a \cdot E) \cdot \tilde{A}_{\tilde{S}})),
\end{aligned}$$

i.e.

$$(3.2.1) \quad \mu_{\tilde{A}_{\tilde{S}}}(\widehat{\mathcal{F}}_S) > \mu_{\tilde{A}_{\tilde{S}}}(\tilde{\mathcal{G}}_S)$$

As $\tilde{\mathcal{G}}$ is semistable, the inequality in 3.2.1 yields the desired contradiction. \square

The next proposition is the main result in this section, proving a restriction theorem for semistable sheaves with respect to a particular set of movable classes. As we shall see later in Section 5, these classes naturally arise in the context of positivity problems for second Chern classes.

Proposition 3.3 (A restriction theorem for movable classes). *Let X be a normal projective threefold that is smooth in codimension two. Let $P \in \text{Mov}^1(X)_{\mathbb{Q}}$ and $H_1, H_2 \in \text{Amp}(X)_{\mathbb{Q}}$. Let \mathcal{E} be a torsion free sheaf on X of rank 3. There exists a positive integer M_1 such that for all sufficiently divisible integers $m_1 \geq M_1$, there is a Zariski open subset $V_{m_1} \subset |m_1 \cdot H_1|$ for which the following properties holds.*

(3.3.1) *Every member $S \in V_{m_1}$ is smooth, irreducible and that $S \subset X_{\text{reg}}$.*

(3.3.2) *The restriction $\mathcal{E}|_S$ is torsion free.*

(3.3.3) *The divisor $P|_S$ is nef.*

(3.3.4) *For every such S , there exists $M_2 \in \mathbb{N}^+$ such that every sufficiently divisible integer $m_2 \geq M_2$ gives rise to a Zariski open subset $V_{m_2} \subset |m_2 \cdot (P + H_2)|_S|$, where every $\gamma \in V_{m_2}$ is a smooth, irreducible curve in S verifying the following property:*

(*) *The formation of the HN-filtration of \mathcal{E} with respect to $(H_1, P + H_2)$ commutes with restriction to γ , i.e. $\text{HN}_{\bullet}(\mathcal{E})|_{\gamma} = \text{HN}_{\bullet}(\mathcal{E}|_{\gamma})$.*

Proof. Let $\pi : \tilde{X} \rightarrow X$ be the birational morphism and \tilde{X} the smooth projective variety with ample divisor $\tilde{A} \subset \tilde{X}$ in Proposition 2.2 associated to the Fujita approximation of the big divisor $P + H_2$, i.e.

$$\pi_*[\tilde{A}] = [(P + H_2)].$$

Now, let $N_1 \in \mathbb{N}^+$ be a sufficiently large and divisible integer such that for every $n_1 \geq N_1$, there are open subsets $U_{n_1} \subset |n_1 \cdot \pi^* H_1|$ and $\tilde{U}_{n_1} \subset |n_1 \cdot \tilde{A}|$, where for every subscheme $\tilde{S} := \tilde{D}_{n_1}$ and $\tilde{C} := \tilde{D}_{n_1} \cap D_{n_1}$, with $\tilde{D}_{n_1} \in U_{n_1}$ and $D_{n_1} \in \tilde{U}_{n_1}$, we have:

(3.3.5) Both \tilde{S} and \tilde{C} are smooth and irreducible.

(3.3.6) The restrictions $(\pi^{[*]} \mathcal{E})|_{\tilde{S}}$ is locally free.

(3.3.7) The HN-filtration of $\pi^{[*]} \mathcal{E}$ with respect to $(H_1, P + H_2)$ verifies:
 $\text{HN}_{\bullet}((\pi^{[*]} \mathcal{E})|_{\tilde{S}}) = \text{HN}_{\bullet}(\pi^{[*]} \mathcal{E})|_{\tilde{S}}$.

The positive integer N_1 exists, thanks to Bertini theorem and Langer's restriction theorem for stable sheaves, cf. [Lan04].

Step. 1. (Reflexivity assumption). By the Bertini theorem and [DG65, Thm. 12.2.1], and as $P \in \text{Mov}^1(X)_{\mathbb{Q}}$, there exists a positive integer N_2 such that for every sufficiently divisible $n_2 \geq N_2$ there exists a Zariski open subset $V_{n_2} \subset |n_2 \cdot H_1|$ where every $S \in V_{n_2}$ satisfies the three Properties (3.3.1), (3.3.2) and (3.3.3). We can also ensure that every $S \in V_{n_2}$ is transversal to the exceptional centre of π . Furthermore, as $P|_S$ is nef, we can find $N_3 \in \mathbb{N}^+$ such that for each sufficiently divisible $n_3 \geq N_3$, the general member of $\gamma \in |n_3 \cdot (P + H_2)|_S|$ is smooth and is contained in an open subset of X over which the HN-filtration of \mathcal{E} (with respect to $(H_1, P + H_2)$) is a filtration of \mathcal{E} by locally-free sheaves. Therefore, to prove that Property (*) is verified by γ , we may assume, without loss of generality, that \mathcal{E} is reflexive.

Step. 2. (Construction of S and γ). Let $m \in \mathbb{N}^+$ be a sufficiently divisible integer verifying the inequality $m_1 \geq M_1 := \max\{N_1, N_2\}$. After shrinking V_{m_1} , if necessary, we have, for every $S \in V_{m_1}$ (defined in Step. 1), that $\tilde{S} := \pi^*(S) \in \mathcal{U}_m$.

Let $M_2 \geq N_1$ be a sufficiently large and divisible integer such that for every $m_2 \geq M_2$ there exists a Zariski open subset $V_{m_2} \subset |m_2(P + H_2)|_S|$, where every curve $\gamma \in V_{m_2}$ is smooth and if $\mathcal{E}|_{\gamma}$ is not semistable, then $\mathcal{E}_S := \mathcal{E}|_S$ is not semistable with respect to $(P + H_2)|_S$ and that $\text{HN}_{\bullet}(\mathcal{E}_S)|_{\gamma} = \text{HN}_{\bullet}(\mathcal{E}|_{\gamma})$. The existence of such M_2 is guaranteed by Mehta-Ramanathan restriction Theorem, cf. [MR82].

Now, to prove the proposition, it suffices to show that if \mathcal{E} is semistable with respect to $(H_1, P + H_2)$, then so is $\mathcal{E}|_{\gamma}$. So let us now assume that \mathcal{E} is indeed semistable. The next step is devoted to proving that $\mathcal{E}|_{\gamma}$ is also semistable.

Step. 3. (Extension of maximal destabilizing subsheaves). Aiming for a contradiction, assume that $\mathcal{E}|_{\gamma}$ is not semistable. Then, by our construction in Step. 2, it follows that \mathcal{E}_S is not semistable with respect to $(P + H_2)|_S \equiv (1/m_2) \cdot \gamma$ and that the maximal destabilizing subsheaf $\mathcal{F}_S \subset \mathcal{E}_S$ restricts to the one for $\mathcal{E}|_{\gamma}$. Note that \mathcal{F}_S , being saturated inside $\mathcal{E}|_S$, is locally-free. By applying Lemma 3.1 to $\pi|_{\tilde{S}} : \tilde{S} \rightarrow S$, with $\tilde{A}_{\tilde{S}} := \tilde{A}|_{\tilde{S}}$, we find that the maximal destabilizing subsheaf $\tilde{\mathcal{G}}_{\tilde{S}}$ of $(\pi|_{\tilde{S}})^*(\mathcal{E}_S)$ with respect to $\tilde{A}_{\tilde{S}}$ is of the form

$$(\pi|_{\tilde{S}})^*(\mathcal{F}_S) \otimes \mathcal{O}_{\tilde{S}}(E'),$$

for some exceptional divisor E' . As $m_2 \geq N_1$, by the construction in Step. 1, it follows that $\tilde{\mathcal{G}}_{\tilde{S}} = \tilde{\mathcal{G}}|_{\tilde{S}}$, where $\tilde{\mathcal{G}}$ is the maximal destabilizing subsheaf of $\pi^*(\mathcal{E})$ with respect to (π^*H_1, \tilde{A}) .

Let $\mathcal{G} \subset \mathcal{E}$ be the reflexive sheaf on X defined by the coherent extension of the sheaf $(\pi|_{\tilde{X} \setminus \text{Exc}(\pi)})^*(\tilde{\mathcal{G}})$ onto X . We have, by the construction of the sheaves $\tilde{\mathcal{G}}$, $\tilde{\mathcal{G}}_{\tilde{S}}$, \mathcal{G} , \mathcal{F}_S and the fact that S is transversal to the exceptional centre $Y \subset X$, that

$$(3.3.8) \quad \mathcal{G}|_{(S \setminus Y)} \cong (\mathcal{F}_S)|_{(S \setminus Y)}.$$

As the construction of $\tilde{\mathcal{G}}$, and hence \mathcal{G} , is independent of the choice of S , by shrinking V_{m_1} , if necessary, we can ensure that \mathcal{G} is reflexive. The isomorphism in (3.3.8), together with the fact that $\mathcal{G}|_S$ and \mathcal{F}_S are both reflexive, imply that $\mathcal{G}|_S \cong \mathcal{F}_S$. As \mathcal{F}_S destabilizes $\mathcal{E}|_S$ with respect to $(P + H_2)$, it follows that $\mathcal{G} \subset \mathcal{E}$ is a properly destabilizing subsheaf with respect to $(H_1, P + H_2)$, contradicting the semistability assumption on \mathcal{E} . \square

Remark 3.4 (Restriction of HN-filtration for \mathbb{Q} -twisted sheaves). We note that the consequences of Proposition 3.3 are still valid for \mathbb{Q} -twisted torsion-free sheaves. More precisely, given a \mathbb{Q} -twisted, torsion-free sheaf $\mathcal{E}\langle B \rangle$ and $H_i \in \text{Amp}(X)_{\mathbb{Q}}$, $P \in \text{Mov}^1(X)_{\mathbb{Q}}$, there is a complete intersection surface S and $\gamma \subset S$, as in Proposition 3.3, such that $\text{HN}_{\bullet}(\mathcal{E}\langle B \rangle)|_{\gamma} = \text{HN}_{\bullet}(\mathcal{E}\langle B \rangle|_{\gamma})$. To see this, let $\mathcal{F}\langle B \rangle$ be a \mathbb{Q} -twisted reflexive sheaf, semistable with respect to $(H_1, \dots, P + H_{n-1})$. Let $f : Y \rightarrow X$ be a finite morphism, adapted to B so that the reflexive pull-back $f^{[*]}(\mathcal{F}\langle B \rangle)$ is a coherent reflexive sheaf on Y . Semistability of $f^{[*]}(\mathcal{F}\langle B \rangle)$ is guaranteed by [HL10, Lem. 3.2.2]. According to Proposition 3.3 the reflexive sheaf $f^{[*]}(\mathcal{E}\langle B \rangle)$ verifies the Restriction Theorem, and therefore so does $\mathcal{E}\langle B \rangle$.

Remark 3.5 (Restriction result in higher dimensions). Following the same arguments as those of the proof of Proposition 3.3, we can remove the restriction on the dimension, that is the consequences of Proposition 3.3 are still valid, if X is of dimension $n \geq 3$ and the polarization is $(H_1, H_2, \dots, (P + H_{n-1}))$, for any $H_1, \dots, H_{n-1} \in \text{Amp}(X)_{\mathbb{Q}}$, as long as $\text{rank}(\mathcal{E}) = 3$.

As an immediate consequence we establish a Bogomolov-Gieseker inequality for (\mathbb{Q} -twisted) sheaves that are semistable with respect to movable classes of the form that appear in Proposition 3.3. Although we do not use this inequality in the rest of the paper, we find it to be of independent interest.

Proposition 3.6 (Bogomolov-Gieseker inequality in higher dimensions). *Let X be an n -dimensional, normal projective variety that is smooth in codimension two and $\mathcal{E}\langle B \rangle$ a \mathbb{Q} -twisted, reflexive sheaf of rank at most equal to 3 on X . If $\mathcal{E}\langle B \rangle$ is semistable with respect to $(H_1, P + H_2)$, where $H_1, H_2 \in \text{Amp}(X)_{\mathbb{Q}}$ and $P \in \text{Mov}^1(X)_{\mathbb{Q}}$, then*

$$(2r \cdot c_2(\mathcal{E}\langle B \rangle) - (r-1) \cdot c_1^2(\mathcal{E}\langle B \rangle)) \cdot H_1 \dots \cdot H_{n-2} \geq 0.$$

Proof. This is an immediate consequence of the restriction result in Proposition 3.3 together with Proposition 2.11 (and Remark 3.4). □

4. SEMIPOSITIVITY OF ADAPTED SHEAF OF FORMS

In [CP16] Campana and Păun remarkably prove that the orbifold cotangent sheaf of a log-smooth pair (X, D) is semipositive with respect to movable curve classes on X (see Theorem 4.1 below). Currently it is not clear if this result can be easily extended to the case of singular pairs. In the present section we show that, for a special subset of movable classes, the generalization to singular pairs can be achieved by essentially reducing to the smooth case.

Theorem 4.1 (Orbifold semipositivity with respect to movable classes, cf. [CP16, Thm. 1.2]). *Given an snc pair (X, D) , if $(K_X + D)$ is pseudoeffective, then for any movable class $\gamma \in \text{Mov}_1(X)$ and any adapted morphism $f : Y \rightarrow X$, where Y is smooth, the adapted cotangent sheaf $\Omega_{(Y, f, D)}^1$ is semipositive with respect to $f^*(\gamma)$.*

In the next proposition we slightly refine Theorem 4.1 for a class of movable 1-cycles that we call *complete intersection 1-cycles*. We say that $\gamma \in \text{Mov}_1(X)_{\mathbb{Q}}$ is a complete intersection 1-cycle, if there are classes $B_1, \dots, B_{n-1} \in \text{N}^1(X)_{\mathbb{Q}}$ such that γ is numerically equivalent to the cycle defined by $(B_1 \cdot \dots \cdot B_{n-1}) \in \text{N}_1(X)_{\mathbb{Q}}$. As we will see later in Section 5, such classes appear naturally in our treatment of the pseudoeffectivity of c_2 .

Proposition 4.2 (A refinement of the orbifold semipositivity result). *Let (X, D) be an snc pair and $\gamma \in \text{Mov}_1(X)_{\mathbb{Q}}$ a complete intersection movable cycle. If $(K_X + D)$ is*

pseudoeffective, then for any strictly adapted morphism $g : Z \rightarrow X$, the adapted cotangent sheaf $\Omega_{(Z,g,D)}^{[1]}$ is semipositive with respect to g^γ.*

Proof. Assume that Z is not smooth, otherwise the claim follows from the arguments of Campana and Păun, cf. [CP16]. Let $D = \sum d_i \cdot D_i$, where D_i are Weil divisors and $d_i \in [0, 1] \cap \mathbb{Q}$. For every D_i , let $g^*(D_i) = n_i \cdot D_{Z,i}$, for some $D_{Z,i} \in \text{Div}(Z)$ and $n_i \in \mathbb{N}^+$.

Now, set $f : Y \rightarrow X$ to be a strictly adapted morphism, where, thanks to Kawamata's construction, cf. [Laz04, Prop. 4.1.12], the variety Y is smooth. Let W be the irreducible component of the normalization of fibre product $Y \times_X Z$ with the resulting commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{v} & Z \\ \downarrow u & \searrow h & \downarrow g \\ Y & \xrightarrow{f} & X. \end{array}$$

Aiming for a contradiction, assume that $\Omega_{(Z,g,D)}^{[1]}$ is not semipositive with respect to $g^*\gamma$, that is there exists a reflexive subsheaf $\mathcal{G}_Z \subset \Omega_{(Z,g,D)}^{[1]}$ such that

$$(4.2.1) \quad (\gamma^*(K_X + D) - [\mathcal{G}_Z]) \cdot g^*\gamma < 0.$$

We consider $v^{[*]}(\mathcal{G}_Z) \subset \Omega_{(W,h,D)}^{[1]}$. As γ is, numerically, a complete intersection cycle, we can use the projection formula to conclude that

$$(4.2.2) \quad (h^*(K_X + D) - [v^{[*]}(\mathcal{G}_Z)]) \cdot h^*\gamma < 0,$$

which implies that $\Omega_{(W,h,D)}^{[1]}$ is not semipositive with respect to $h^*\gamma$. Now, let $\Omega_{(W,h,D)}^{[1]} \rightarrow \mathcal{F}_W$ be the torsion free quotient having the minimal slope with the kernel \mathcal{G}_W :

$$(4.2.3) \quad 0 \rightarrow \mathcal{G}_W \rightarrow \Omega_{(W,h,D)}^{[1]} \rightarrow \mathcal{F}_W \rightarrow 0.$$

Let $G := \text{Gal}(W/Y)$. Notice that by the construction of f , we have $\Omega_{(W,h,D)}^{[1]} = u^*(\Omega_{(Y,f,D)}^1)$. Now, as the inclusion $\mathcal{G}_W \subset \Omega_{(W,h,D)}^{[1]}$ is saturated, and since \mathcal{G}_W is a G -subsheaf (thanks to its uniqueness), according to [HL10, Thm. 4.2.15] or [GKPT15, Prop. 2.16], there exists a reflexive subsheaf $\mathcal{G}_Y \subset \Omega_{(Y,f,D)}^1$ such that $u^{[*]}(\mathcal{G}_Y) = \mathcal{G}_W$.

Now by taking the G -invariant sections of Sequence 4.2.3 we find

$$(4.2.4) \quad 0 \rightarrow \mathcal{G}_Y \rightarrow \Omega_{(Y,f,D)}^1 \rightarrow (u_*(\mathcal{F}_W))^G \rightarrow 0.$$

Again, by using the projection formula we find that $\Omega_{(Y,f,D)}^1$ is not semipositive with respect to $f^*\gamma$, contradicting Theorem 4.1. \square

The next proposition is the extension of Theorem 4.1 to a special class of complete intersection, movable 1-cycles on a mildly singular X .

Proposition 4.3 (Semipositivity for mildly singular pairs). *Let X be a normal projective variety. Let $D = \sum d_i \cdot D_i$, $d_i \in [0, 1] \cap \mathbb{Q}$, be an effective \mathbb{Q} -divisor such that the pair (X, D) , in case D is reduced, is at worst lc, and otherwise is assumed to be klt. Let*

$H_1, \dots, H_{n-1} \in \text{Amp}(X)_{\mathbb{Q}}$ and $P \in \text{Mov}^1(X)_{\mathbb{Q}}$. If $(K_X + D)$ is pseudoeffective, then for any strictly adapted morphism $f : Y \rightarrow X$, the adapted cotangent sheaf $\Omega_{(Y,f,D)}^{[1]}$ is semipositive with respect to $f^*(H_1, \dots, H_{n-2}, P + H_{n-1})$.

Proof. Notice that as (X, D) has simple normal crossing in codimension two. According to the construction of adapted covers, cf. [Laz04, Prop. 4.1.12] there exists an adapted morphism $f : Y \rightarrow X$ (which is not unique) such that Y is smooth in codimension two. Now, let $\pi : (\tilde{X}, \tilde{D}) \rightarrow (X, D)$ be a log-resolution and \tilde{Y} the main component of the normalization of the fibre product $Y \times_X \tilde{Y}$ with the commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{X} \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ Y & \xrightarrow{f} & X, \end{array}$$

where $\tilde{\pi} : \tilde{Y} \rightarrow Y$ and $\tilde{f} : \tilde{Y} \rightarrow \tilde{X}$ are the naturally induced projections.

For simplicity, and as the arguments are identical in higher dimensions, we only deal with the case when $\dim X = 3$. Denote $H_{Y,i} = f^*(H_i)$, for $i \in \{1, 2\}$ and $P_Y = f^*(P)$.

Now, aiming for a contradiction, assume that $\Omega_{(Y,f,D)}^{[1]}$ is not semipositive with respect to $(H_{Y,1}, P_Y + H_{Y,2})$. This implies that there exists a saturated subsheaf $\mathcal{G} \subset \mathcal{T}_{(Y,f,D)}$ such that $[\mathcal{G}] \cdot (H_{Y,1}, P_Y + H_{Y,2}) > 0$. Define $\tilde{\mathcal{H}} := (\tilde{\pi}^*[\mathcal{G}]) \cap \mathcal{T}_{(\tilde{Y}, \tilde{f}, \tilde{D})}$. Let m be a sufficiently large positive integer such that the 1-cycle $\gamma \in \text{Mov}^1(Y)_{\mathbb{Q}}$ that is numerically equivalent to the cycle defined by $m^2(H_{Y,1}, P_Y + H_{Y,2})$ is away from the exceptional centre of $\tilde{\pi}$. Existence of such γ in particular guarantees that

$$[\tilde{\mathcal{H}}] \cdot \tilde{\pi}^*(H_{Y,1}, P_Y + H_{Y,2}) > 0.$$

In other words there exists a torsion-free quotient sheaf

$$(4.3.1) \quad \Omega_{(\tilde{Y}, \tilde{f}, \tilde{D})}^{[1]} \twoheadrightarrow \tilde{\mathcal{F}}$$

on \tilde{Y} such that $\deg(\tilde{\mathcal{F}}|_{\tilde{\gamma}}) < 0$, where $\tilde{\gamma} := \tilde{\pi}^{-1}(\gamma)$.

Now, let us consider the logarithmic ramification formula

$$K_{\tilde{X}} + \tilde{D} = \pi^*(K_X + D) + \sum a_i \cdot E_i - \sum b_i \cdot E'_i,$$

where $a_i \in \mathbb{Q}^+$, and, thanks to the assumptions on the singularities, $b_i \in (0, 1] \cap \mathbb{Q}$. Define $\tilde{G} := \sum b_i \cdot E'_i$ and let $\tilde{h} : Z \rightarrow \tilde{X}$ be the morphism adapted to $(\tilde{X}, \tilde{D} + \tilde{G})$, factoring through $\tilde{f} : \tilde{Y} \rightarrow \tilde{X}$:

$$\begin{array}{ccccc} & & \tilde{h} & & \\ & \curvearrowright & & \curvearrowleft & \\ Z & \xrightarrow{r} & \tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{X}. \end{array}$$

Set $B_Z := \tilde{h}^*(\pi^*(H_1, P + H_2))$ and $B_{\tilde{Y}} := \tilde{f}^*(\pi^*(H_1, P + H_2))$. Now, let $\mathcal{G}_{\tilde{Y}}$ be the kernel of the sheaf morphism (4.3.1) so that

$$(4.3.2) \quad (\tilde{f}^*(K_{\tilde{X}} + \tilde{D}) - [\mathcal{G}_{\tilde{Y}}]) \cdot B_{\tilde{Y}} < 0.$$

As γ is away from the exceptional centre of $\tilde{\pi}$ and since \tilde{G} is supported on the exceptional locus of π , we have

$$\begin{aligned}\tilde{h}^*(K_{\tilde{X}} + \tilde{D} + \tilde{G}) \cdot B_Z &= \tilde{h}^*(K_{\tilde{X}} + \tilde{D}) \cdot B_Z \\ &= r^*(\tilde{f}^*(K_{\tilde{X}} + \tilde{D})) \cdot B_Z.\end{aligned}$$

As a result, for the inclusion $r^{[*]}(\mathcal{G}_{\tilde{Y}}) \subset \Omega_{(Z, \tilde{h}, \tilde{D} + \tilde{G})}^{[1]}$, we find that

$$\begin{aligned}\left(\left[\Omega_{(Z, \tilde{h}, \tilde{D} + \tilde{G})}^{[1]} \right] - r^{[*]}(\mathcal{G}_{\tilde{Y}}) \right) \cdot B_Z &= \left(r^*(\tilde{f}^*(K_{\tilde{X}} + \tilde{D})) - r^{[*]}(\mathcal{G}_{\tilde{Y}}) \right) \cdot B_Z \\ &= (\deg r) \left(\tilde{f}^*(K_{\tilde{X}} + \tilde{D}) - [\mathcal{G}_{\tilde{Y}}] \right) \cdot B_{\tilde{Y}} \\ &< 0, \quad \text{by Inequality 4.3.2,}\end{aligned}$$

contradicting Proposition 4.2. □

5. PSEUDOEFFECTIVITY OF THE ORBIFOLD c_2

In [Miy87] Miyaoka famously proved that c_2 of a generically semipositive sheaf with nef determinant is pseudoeffective. Thanks to his result on the semipositivity of cotangent sheaves, Miyaoka then established the pseudoeffectivity of $c_2(X)$ for any minimal model X . Our aim in this section is to generalize this result to the case of pairs (X, D) with movable $(K_X + D)$ (Corollary 5.2) by first extending Miyaoka's result on pseudoeffectivity of c_2 for any semipositive sheaf.

Proposition 5.1 (Pseudoeffectivity of c_2 for semipositive sheaves). *Let X be a normal projective threefold with isolated singularities and $A_1 \in \text{Amp}(X)_{\mathbb{Q}}$. Then, the inequality*

$$c_2(\mathcal{E}) \cdot A_1 \geq 0$$

holds for any reflexive sheaf \mathcal{E} of rank r verifying the following properties.

$$(5.1.1) \quad [\mathcal{E}] \in \text{Mov}^1(X)_{\mathbb{Q}}.$$

$$(5.1.2) \quad \text{For any } A_2 \in \text{Amp}(X)_{\mathbb{Q}}, \text{ the sheaf } \mathcal{E} \text{ is semipositive with respect to } (A_1, [\mathcal{E}] + A_2).$$

Proof. Let c any any positive integer. Consider the \mathbb{Q} -twisted reflexive sheaf $\mathcal{E}\langle \frac{1}{c} \cdot H \rangle$. For the choice of polarization $(A_1, [\mathcal{E}\langle \frac{1}{c} \cdot H \rangle])$, the assumptions of Proposition 3.3 are satisfied, for all c .

Now let S be the complete intersection surface defined in Proposition 3.3 (see also Remark 3.4) so that the restriction $\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle := (\mathcal{E}\langle \frac{1}{c} \cdot H \rangle)|_S$ is semipositive with respect to

$$\beta := c_1(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) = ([\mathcal{E}] + \frac{r}{c} \cdot [H_S])|_S.$$

Following the arguments of Miyaoka, we now consider two cases based on the stability of $\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle$.

First, we consider the case where $\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle$ is semistable with respect to β . Here, the semipositivity of c_2 follows from Bogomolov-Gieseker inequality for \mathbb{Q} -twisted locally-free sheaves (Proposition 2.11).

So we now assume that $\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle$ is not semistable with respect to β . Let

$$(5.1.3) \quad 0 \neq \mathcal{E}_S^1\langle \frac{1}{m} \cdot H_S \rangle \subset \dots \subset \mathcal{E}_S^t\langle \frac{1}{c} \cdot H_S \rangle = \mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle$$

be the the \mathbb{Q} -twisted HN-filtration $\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle$. Denote the semistable, torsion-free, \mathbb{Q} -twisted sheaves

$$\mathcal{E}_S^i\langle \frac{1}{c} \cdot H_S \rangle / \mathcal{E}_S^{i-1}\langle \frac{1}{c} \cdot H_S \rangle$$

of rank r_i by $\mathcal{Q}_S^i\langle\frac{1}{c}\cdot H_S\rangle$ and let $\overline{\mathcal{Q}}_S^i\langle\frac{1}{c}\cdot H_S\rangle$ denote its reflexivization. As the second Chern character $ch_2(\cdot)$ is additive, we have

$$(5.1.4) \quad \begin{aligned} 2 \cdot c_2(\mathcal{E}_S\langle\frac{1}{c}\cdot H_S\rangle) - c_1^2(\mathcal{E}_S\langle\frac{1}{c}\cdot H_S\rangle) &= \sum (2 \cdot c_2(\mathcal{Q}_S^i\langle\frac{1}{c}\cdot H_S\rangle) - c_1^2(\mathcal{Q}_S^i\langle\frac{1}{c}\cdot H_S\rangle)) \\ &\geq \sum (2 \cdot c_2(\overline{\mathcal{Q}}_S^i\langle\frac{1}{c}\cdot H_S\rangle) - c_1^2(\mathcal{Q}_S^i\langle\frac{1}{c}\cdot H_S\rangle)), \end{aligned}$$

where the last inequality follows from the fact that $c_2(\mathcal{Q}_S^i) \geq c_2(\overline{\mathcal{Q}}_S^i)$. Now, by applying the Bogomolov inequality 2.11 to each semistable, \mathbb{Q} -twisted sheaf $\overline{\mathcal{Q}}_S^i\langle\frac{1}{c}\cdot H_S\rangle$ we find that each term in the right-hand side of the inequality (5.1.4) verifies the inequality

$$2 \cdot c_2(\overline{\mathcal{Q}}_S^i\langle\frac{1}{c}\cdot H_S\rangle) - c_1^2(\mathcal{Q}_S^i\langle\frac{1}{c}\cdot H_S\rangle) \geq \frac{-1}{r_i} \cdot c_1^2(\mathcal{Q}_S^i\langle\frac{1}{c}\cdot H_S\rangle).$$

Therefore we have

$$(5.1.5) \quad 2 \cdot c_2(\mathcal{E}_S\langle\frac{1}{c}\cdot H_S\rangle) - c_1^2(\mathcal{E}_S\langle\frac{1}{c}\cdot H_S\rangle) \geq \sum \frac{-1}{r_i} \cdot c_1^2(\mathcal{Q}_S^i\langle\frac{1}{c}\cdot H_S\rangle).$$

Next, we define the rational number $\alpha_i \in \mathbb{Q}$ by the equality

$$(5.1.6) \quad r_i \cdot \alpha_i = \frac{c_1(\mathcal{Q}_S^i\langle\frac{1}{c}\cdot H_S\rangle) \cdot \beta}{c_1^2(\mathcal{E}_S\langle\frac{1}{c}\cdot H_S\rangle)} = \frac{c_1(\mathcal{Q}_S^i\langle\frac{1}{c}\cdot H_S\rangle) \cdot \beta}{\beta^2}.$$

It follows that

$$(5.1.7) \quad \sum r_i \cdot \alpha_i = 1.$$

Furthermore, according to the definition of α_i , and by using the fact that the slopes of the quotients of the HN-filtration (5.1.3) is strictly decreasing, we know that

$$(5.1.8) \quad \alpha_1 > \alpha_2 > \dots > \alpha_t \geq 0,$$

where the last inequality follows from the semipositivity of $\mathcal{E}_S\langle\frac{1}{c}\cdot H_S\rangle$.

Now, as $\alpha_i \geq 0$, for each i , the equality (5.1.7) implies that $\alpha_i \leq 1$. On the other hand, according to the Hodge index theorem we have

$$-c_1^2(\mathcal{Q}_S^i\langle\frac{1}{c}\cdot H_S\rangle) \geq \frac{(c_1(\mathcal{Q}_S^i\langle\frac{1}{c}\cdot H_S\rangle) \cdot \beta)^2}{\beta^2},$$

so that

$$-c_1^2(\mathcal{Q}_S^i\langle\frac{1}{c}\cdot H\rangle) \geq \beta^2(r_i \cdot \alpha_i)^2.$$

After substituting back into the inequality (5.1.5) we now find that

$$\begin{aligned} 2 \cdot c_2(\mathcal{E}_S\langle\frac{1}{c}\cdot H_S\rangle) &\geq \beta^2(1 - \sum r_i \cdot \alpha_i^2) \\ &\geq \beta^2(1 - \alpha_1 \sum r_i \cdot \alpha_i) && \text{by 5.1.8} \\ &= \beta^2(1 - \alpha_1) && \text{by 5.1.7} \\ &\geq 0 && \text{as } \alpha_1 \leq 1. \end{aligned}$$

The inequality $c_2(\mathcal{E}_S) \geq 0$ now follows by taking the limit $c \rightarrow \infty$.

□

As an immediate consequence we can now prove the pseudoeffectivity of c_2 for the orbifold cotangent sheaves of pairs (X, D) in dimension 3, whose $K_X + D$ is movable and has only isolated singularities.

Corollary 5.2 (Positivity of c_2 of orbifold cotangent sheaves). *Let X be a normal projective threefold and D an effective \mathbb{Q} -divisor such that (X, D) has only isolated lc singularities. If $(K_X + D) \in \text{Mov}^1(X)_{\mathbb{Q}}$, then then for any ample divisors $A \subset X$ and strongly adapted morphism $f : Y \rightarrow X$, the inequality*

$$c_2(\Omega_{(Y,f,D)}^{[1]}) \cdot f^*(A) \geq 0$$

holds.

Proof. As $[\Omega_{(Y,f,D)}^{[1]}] = f^*(K_X + D)$, the corollary is a direct consequence of Proposition 5.1 together with Proposition 4.3. \square

5.1. Positivity of orbifold c_2 for log-minimal models. We would like to point out that once we assume that $(K_X + D)$ is nef, then an easy adaptation of the original results of Miyaoka to the case of orbifold Chern classes, together with semipositivity result of [CP14] leads to the following theorem.

Theorem 5.3. *Let X be a projective klt variety of dimension n and $D = \sum(1 - 1/a_i) \cdot D_i$, $a_i \in \mathbb{N}^+ \cup \{\infty\}$, an effective \mathbb{Q} -divisor such that (X, D) is lc. If $(K_X + D)$ is nef, then for any strongly adapted morphism $f : Y \rightarrow X$, we have*

$$c_2(\Omega_{(Y,f,D)}^{[1]}) \cdot f^*(A^{n-2}) \geq 0,$$

where $A \subset X$ is any ample divisor.

6. AN EFFECTIVE NON-VANISHING RESULT FOR THREEFOLDS

The goal of this section is to prove Theorem 1.4. The main point of our strategy is to devise an effective lower bound for $\chi(K_Y + H)$, when Y is terminal (and (Y, H) is lc).

Proposition 6.1 (Lower bounds for the Euler characteristic of adjoint bundles). *Let X be a terminal projective threefold and D an effective divisor. Then, the inequality*

$$(6.1.1) \quad \chi(X, K_X + D + A) \geq \left(\frac{1}{12}\right) \cdot (K_X + D + A) \cdot (D + A) \cdot (D + A + \frac{1}{2}K_X).$$

holds, for any divisor A satisfying the following conditions.

(6.1.2) *The divisor $D + A$ is Cartier and nef and, up to integral linear equivalence, effective and reduced.*

(6.1.3) *The pair $(X, D + A)$ is lc.*

(6.1.4) *The divisors $(D + A)$ and $(K_X + D + A)$ are Cartier and nef.*

Proof. As usual, a key element in the proof is the Hizerbruch-Riemann-Roch for $(K_X + D + A)$:

$$\begin{aligned} \chi(X, K_X + D + A) &= \frac{1}{12} \cdot (K_X + D + A) \cdot (D + A) \cdot (2(K_X + D + A) - K_X) \\ &\quad + \frac{1}{12} \cdot c_2(X) \cdot (K_X + D + A) + \chi(X, \mathcal{O}_X). \end{aligned} \quad (6.1.5)$$

Standard Chern class calculations then show that we have the equality

$$(6.1.6) \quad c_2(X) = c_2(\Omega_X^{[1]}) \log(D + A) - (K_X + D + A) \cdot (D + A),$$

as linear forms on $N^1(X)_\mathbb{Q}$. After substituting back into Equality 6.1.5, we find that the equality

$$\begin{aligned} \chi(X, K_X + D + A) &= (K_X + D + A) \cdot \left\{ (D + A) \cdot (K_X + 2(D + A)) \right. \\ &\quad \left. + c_2(\Omega_X^{[1]} \log(D + A)) - (K_X + D + A) \cdot (D + A) \right\} \\ &\quad + \chi(X, \mathcal{O}_X) \end{aligned}$$

holds, which then simplifies to

$$\begin{aligned} \chi(X, K_X + D + A) &= (K_X + D + A) \cdot \left\{ (D + A)^2 \right. \\ &\quad \left. + c_2(\Omega_X^{[1]} \log(D + A)) \right\} + \chi(X, \mathcal{O}_X). \end{aligned} \quad (6.1.7)$$

On the other hand, as X is terminal, we know, thanks to [Kaw81, Lem. 2.3] (see also [KM98, Cor. 5.39]), that

$$(6.1.8) \quad \chi(X, \mathcal{O}_X) \geq \frac{-1}{24} K_X \cdot c_2(X).$$

After substituting 6.1.8 in 6.1.6 we find:

$$\begin{aligned} \chi(X, \mathcal{O}_X) &= ((K_X + D + A) - (D + A)) \cdot c_2(\Omega_X^{[1]} \log(D + A)) \\ &\quad + (K_X) \cdot (K_X + D + A) \cdot (D + A) \\ &\geq (K_X + D + A) \cdot \left\{ c_2(\Omega_X^{[1]} \log(D + A)) - (K_X) \cdot (D + A) \right\}, \end{aligned}$$

where we have used the assumption that $(D + A)$ is nef and the pseudoeffectivity of c_2 (Theorem 5.3). Now, substituting back into Equation 6.1.7, we get

$$\begin{aligned} \chi(X, K_X + D + A) &= (K_X + D + A) \left\{ (D + A)^2 \right. \\ &\quad \left. + \frac{1}{2} (K_X) \cdot (D + A) + \frac{1}{2} c_2(\Omega_X^{[1]} \log(D + A)) \right\}. \end{aligned} \quad (6.1.9)$$

Again, by using Corollary 5.2 and the nefness assumptions on $(K_X + D + A)$ and $(K_X + A)$, we find that

$$(6.1.10) \quad \chi(X, K_X + D + A) \geq (K_X + D + A) \cdot (D + A) \cdot \left(D + A + \frac{1}{2} K_X \right),$$

as required. □

6.1. Proof of Theorem 1.4. According to Kawamata-Viehweg vanishing, it suffices to prove that $\chi(Y, K_Y + H) \neq 0$. The pair (Y, H) satisfies the assumptions of Proposition 6.1 with $D = 0$, except for the terminal condition.

Now, let $\pi : X \rightarrow Y$ be a terminalization of Y , cf. [KM98, Sect. 6.3]. Set $A := \pi^*(H)$. Since π is small, the adjoint divisor $(K_X + A)$ is nef and big. As a result, the strict positivity of the right-hand side of the inequality (6.1.1) immediately follows: First we rewrite the right-hand side of (6.1.1) as

$$\frac{1}{2} \cdot (K_X + A) \cdot A \cdot ((K_X + A) + A).$$

Now, according to the basepoint freeness theorem for log-canonical threefolds, cf. [Kol92], the divisor $K_X + A$ is semi-ample. Therefore, for sufficiently large integer m , we can find an irreducible surface $S \in |m \cdot (K_X + 2A)|$ such that $(A|_S)$ is big. On the other hand $(K_X + A)|_S$ is nef. It thus follows that $(K_X + A)|_S \cdot A|_S > 0$. \square

7. A MIYAOKA-YAU INEQUALITY IN HIGHER DIMENSIONS

In [Miy87], Miyaoka generalized the famous inequality $c_1^2 \leq 3c_2$ from surfaces with pseudoeffective canonical divisor to higher dimensional varieties with nef canonical divisor. We extend this result to the case of movable canonical divisor.

Theorem 7.1. *Let X be a normal projective threefold and D an effective \mathbb{Q} -divisor such that (X, D) has only isolated lc singularities. If $(K_X + D) \in \text{Mov}^1(X)_{\mathbb{Q}}$, then for any $A \in \text{Amp}(X)_{\mathbb{Q}}$ and for any strongly adapted morphism $f : Y \rightarrow X$,*

$$c_1^2(\Omega_{(Y,f,D)}^{[1]}) \cdot f^*A \leq 3c_2(\Omega_{(Y,f,D)}^{[1]}) \cdot f^*A.$$

Proof. Let $\tilde{H} \in \text{Amp}(X)_{\mathbb{Q}}$, $H := f^*\tilde{H}$ and $\mathcal{E} := \Omega_{(Y,f,D)}^{[1]}$. Let c any any positive integer. Consider the \mathbb{Q} -twisted reflexive sheaf $\mathcal{E}\langle \frac{1}{c} \cdot H \rangle$. For the choice of polarization $(f^*A, [\mathcal{E}\langle \frac{1}{c} \cdot H \rangle])$, the assumptions of Proposition 3.3 are satisfied, for all c .

Now let S be the complete intersection surface defined in Proposition 3.3 (see also Remark 3.4) so that the restriction $\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle := (\mathcal{E}\langle \frac{1}{c} \cdot H \rangle)|_S$ is semipositive with respect to

$$\beta := ([\mathcal{E}] + \frac{r}{c} \cdot H_S)|_S.$$

Let

$$(7.1.1) \quad 0 \neq \mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle \subset \dots \subset \mathcal{E}_S^s\langle \frac{1}{c} \cdot H_S \rangle = \mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle$$

be the the \mathbb{Q} -twisted HN-filtration of $\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle$.

The same arguments as those in the proof of Proposition 5.1 show that

$$(2c_2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle)) \geq \left(\sum \frac{-1}{r_i} c_1^2(\mathcal{Q}_S^i) \right),$$

where $\mathcal{Q}_S^i\langle \frac{1}{c} \cdot H_S \rangle$ is the torsion free, \mathbb{Q} -twisted quotient sheaf of rank r_i of the filtration (7.1.1).

Again, as in the proof of Proposition 5.1, for each i , we define α_i by the equation

$$r_i \cdot \alpha_i = \frac{c_1(\mathcal{Q}_S^i\langle \frac{1}{c} \cdot H_S \rangle) \cdot \beta}{\beta^2}.$$

From the definition of α_i it follows that $\sum r_i \cdot \alpha_i = 1$. Moreover, we have $\alpha_1 > \dots > \alpha_s \geq 0$, where the last inequality is due to the semipositivity of $\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle$.

We now deduce

$$\begin{aligned} & (6c_2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - 2c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle)) \geq \\ & \left(3 \left(\sum_{i>1} \frac{-1}{r_i} c_1^2(\mathcal{G}_i) \right) + 6c_2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle) - 3c_1^2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle) + c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) \right). \end{aligned}$$

And finally,

$$(7.1.2) \quad (6c_2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - 2c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle)) \geq \\ ((1 - 3 \sum_{i>1} r_i \alpha_i^2) \cdot c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) + 6c_2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle) - 3c_1^2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle)).$$

There are three possibilities: $r_1 \geq 3$, $r_1 = 2$ and $r_1 = 1$.

If $r_1 \geq 3$, using Bogomolov-Gieseker inequality and the Hodge index theorem, we obtain

$$(6c_2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - 2c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle)) \geq \\ ((1 - 3 \sum_{i>1} r_i \alpha_i^2) \cdot c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - 3 \frac{1}{r_1} c_1^2(\mathcal{E}_1)) \geq \\ (1 - 3 \sum_i r_i \alpha_i^2) \cdot c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) \geq (1 - 3\alpha_1) \cdot c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) \geq 0.$$

since $3\alpha_1 \leq r_1\alpha_1 \leq \sum_i r_i\alpha_i = 1$.

If $r_1 = 2$, we choose S general enough so that \mathcal{E}_S^1 injects into $\Omega_S(\log f^{-1}[D]|_S)$.

Using the Bogomolov-Miyaoka-Yau inequality, we have either $\kappa(S, c_1(\mathcal{E}_S^1)) \leq 0$ or $c_1^2(\mathcal{E}_S^1) \leq 3c_2(\mathcal{E}_S^1)$.

In the case $\kappa(S, c_1(\mathcal{E}_S^1)) \leq 0$, since $c_1(\mathcal{E}_S^1) \cdot \beta > 0$, we have $c_1^2(\mathcal{E}_S^1) \leq 0$.

Applying Bogomolov-Gieseker inequality to 7.1.2:

$$(6c_2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - 2c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle)) \geq \\ ((1 - 3 \sum_{i>1} r_i \alpha_i^2) \cdot c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - \frac{3}{2}c_1^2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle)) \geq \\ (1 - 3 \sum_{i>1} r_i \alpha_i^2) \cdot c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - \frac{3}{2}c_1^2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle) \geq \\ (1 - 3\alpha_2 \sum_{i>1} r_i \alpha_i) \cdot c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - \frac{3}{2}c_1^2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle) = \\ (1 - 3\alpha_2(1 - 2\alpha_1)) \cdot c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - \frac{3}{2}c_1^2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle) \geq \\ (1 - 3\alpha_1(1 - 2\alpha_1)) \cdot c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - \frac{3}{2}c_1^2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle) = \\ \left(6(\alpha_1 - \frac{1}{4})^2 + \frac{5}{8}\right) \cdot c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - \frac{3}{2}c_1^2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle) \geq -\frac{3}{2}c_1^2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle).$$

Finally, we obtain $(3c_2(\mathcal{E}_S) - c_1^2(\mathcal{E}_S)) \geq 0$.

In the case $c_1^2(\mathcal{E}_S^1) \leq 3c_2(\mathcal{E}_S^1)$ we have from 7.1.2:

$$\begin{aligned}
& (6c_2(\mathcal{E}_S\langle \frac{1}{c}H_S \rangle) - 2c_1^2(\mathcal{E}_S\langle \frac{1}{c}H_S \rangle)) \geq \\
& ((1 - 3 \sum_{i>1} r_i \alpha_i^2) c_1^2(\mathcal{E}_S\langle \frac{1}{c}H_S \rangle) - c_1^2(\mathcal{E}_S^1\langle \frac{1}{c}H_S \rangle) + (6c_2(\mathcal{E}_S\langle \frac{1}{c}H_S \rangle) - 2c_1^2(\mathcal{E}_S^1\langle \frac{1}{c}H_S \rangle)) \geq \\
& ((1 - 4\alpha_1^2 - 3 \sum_{i>1} r_i \alpha_i^2) c_1^2(\mathcal{E}_S\langle \frac{1}{c}H_S \rangle) + (6c_2(\mathcal{E}_S\langle \frac{1}{c}H_S \rangle) - 2c_1^2(\mathcal{E}_S^1\langle \frac{1}{c}H_S \rangle)) \geq \\
& ((1 - 4\alpha_1^2 - 3\alpha_2 \sum_{i>1} r_i \alpha_i) c_1^2(\mathcal{E}_S\langle \frac{1}{c}H_S \rangle) + (6c_2(\mathcal{E}_S\langle \frac{1}{c}H_S \rangle) - 2c_1^2(\mathcal{E}_S^1\langle \frac{1}{c}H_S \rangle)) = \\
& ((1 - 4\alpha_1^2 - 3\alpha_2(1 - 2\alpha_1)) c_1^2(\mathcal{E}_S\langle \frac{1}{c}H_S \rangle) + (6c_2(\mathcal{E}_S\langle \frac{1}{c}H_S \rangle) - 2c_1^2(\mathcal{E}_S^1\langle \frac{1}{c}H_S \rangle)) = \\
& (1 - 2\alpha_1)(1 + 2\alpha_1 - 3\alpha_2) \cdot c_1^2(\mathcal{E}_S\langle \frac{1}{c}H_S \rangle) + (6c_2(\mathcal{E}_S\langle \frac{1}{c}H_S \rangle) - 2c_1^2(\mathcal{E}_S^1\langle \frac{1}{c}H_S \rangle)).
\end{aligned}$$

As $3\alpha_2 < r_1\alpha_1 + r_2\alpha_2 \leq 1$, we have

$$(6c_2(\mathcal{E}_S) - 2c_1^2(\mathcal{E}_S)) \geq 0.$$

Finally, if $r_1 = 1$, a classical result of Bogomolov and Sommes (the Bogomolov-Sommese vanishing) implies that $\mathcal{E}_S^1 \subset \Omega_S(\log f^{-1}[\Delta]_S)$ has Kodaira dimension at most one. Therefore $c_1^2(\mathcal{E}_S^1) \leq 0$. From 7.1.2, one obtains:

$$\begin{aligned}
& (6c_2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - 2c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle)) \geq \\
& ((1 - 3 \sum_{i>1} r_i \alpha_i^2) \cdot c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - 3c_1^2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle)) \geq \\
& ((1 - 3\alpha_1 \sum_{i>1} r_i \alpha_i) \cdot c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - 3c_1^2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle)) = \\
& ((1 - 3\alpha_1(1 - \alpha_1)) \cdot c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - 3c_1^2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle)) \geq \\
& \left(1 - \frac{3}{2}(1 - \frac{1}{2})\right) \cdot c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - 3c_1^2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle) = \\
& \frac{1}{4}c_1^2(\mathcal{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - 3c_1^2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle) \geq \\
& -3c_1^2(\mathcal{E}_S^1\langle \frac{1}{c} \cdot H_S \rangle).
\end{aligned}$$

Therefore, we have

$$(6c_2(\mathcal{E}_S) - 2c_1^2(\mathcal{E}_S)) \geq 0.$$

□

We finish this section by pointing out that when $(K_X + D)$ is nef, the original result of Miyaoka can be adapted to the case of orbifold Chern classes. This can then be combined with the semipositivity result of [CP14] to conclude the following result.

Theorem 7.2. *Let X be a projective klt variety of dimension n and $D = \sum(1 - 1/a_i) \cdot D_i$, $a_i \in \mathbb{N}^+ \cup \{\infty\}$, an effective \mathbb{Q} -divisor such that (X, D) is lc. If $(K_X + D)$ is nef, then for arbitrary ample divisors H_1, \dots, H_{n-2} and any strongly adapted morphism $f : Y \rightarrow X$, we have*

$$(7.2.1) \quad c_1^2(\Omega_{(Y,f,D)}^{[1]}) \cdot f^*(H_1 \dots H_{n-2}) \leq 3c_2(\Omega_{(Y,f,D)}^{[1]}) \cdot f^*(H_1 \dots H_{n-2}).$$

8. REMARKS ON LANG-VOJTA'S CONJECTURE IN CODIMENSION ONE

A classical conjecture of Lang predicts that a variety of general type X , admits a proper algebraic subvariety that contains all subvarieties of X that are *not* of general type. In this section, we will prove a particular case of this conjecture for codimension one subvarieties satisfying certain conditions: The codimension one subvariety will be assumed to be movable and with only canonical singularities.

First, an immediate application of the inequality (7.1) gives the following theorem.

Theorem 8.1. *Let X be a normal projective \mathbb{Q} -factorial threefold such that $K_X \in \text{Mov}^1(X)_{\mathbb{Q}}$. Let H be a nef divisor, D a movable, reduced, irreducible, normal divisor such that (X, D) has only isolated lc singularities. If $-K_D$ is pseudoeffective, then*

$$(8.1.1) \quad K_X \cdot D \cdot H \leq (3c_2 - c_1^2) \cdot H.$$

Proof. From the inequality (7.1), we have $c_1^2(\Omega_X(\log D)) \cdot H \leq 3c_2(\Omega_X(\log D)) \cdot H$. Therefore, $(K_X + D)^2 \cdot H \leq 3(c_2 + (K_X + D) \cdot D) \cdot H$. It follows that

$$2K_X \cdot D \cdot H \leq (3c_2 - c_1^2) \cdot H + 3(K_X + D) \cdot D \cdot H - D^2 \cdot H.$$

Finally, thanks to the adjunction formula, we get $K_X \cdot D \cdot H \leq (3c_2 - c_1^2) \cdot H + 2K_D \cdot H|_D$. The inequality (8.1.1) now follows from the assumption that $-K_D$ is pseudoeffective. \square

Proof of Theorem 1.5. Let H be an ample divisor in X . The divisor K_X is big so we can find a positive integer m such that $(m \cdot K_X - H)$ is linearly equivalent to an effective divisor E .

Let us first prove that the family of polarized varieties $(D, H|_D)$ is bounded. We note that as each D has only rational singularities the theorem of Kollár and Matsusaka [KM83] applies, that is to bound the family $(D, H|_D)$, it suffices to bound the intersection numbers

$$H^2 \cdot D \quad \text{and} \quad H \cdot K_D = H \cdot (K_X + D) \cdot D.$$

For $H^2 \cdot D$, we note that, as long as D is not a component of E we can use the inequality (8.1.1), to get

$$0 \leq H^2 \cdot D \leq mH \cdot (3c_2 - c_1^2).$$

For the second term $K_D \cdot H$, we use Theorem 1.2 to find

$$\begin{aligned} 0 &\leq 3c_2(\Omega_X^1(\log D)) \cdot H - c_1^2(\Omega_X^1(\log D)) \cdot H \\ &= (3c_2 - c_1^2) \cdot H + 2(K_X + D) \cdot D \cdot H - K_X \cdot D \cdot H. \end{aligned}$$

We immediately deduce that

$$-\frac{1}{2}(3c_2 - c_1^2) \cdot H \leq H \cdot (K_X + D) \cdot D = H \cdot K_D \leq 0.$$

Therefore, the family of polarized varieties $(D, H|_D)$ is bounded.

It now remains to show the finiteness of the family $(D, D|_H)$. Aiming for a contradiction assume that the family is not finite. As the family is bounded, after going to a smooth model of X , we are reduced to the case of a fibration. The additivity of the Kodaira dimension ("easy additivity") shows that, as $-K_D$ is pseudoeffective, X cannot be of general type; a contradiction. \square

Remark 8.2. In [LM97, Thm. 4], in the setting where X is non-uniruled and smooth and D is reduced, the Miyaoka-Yau inequality 7.2 is claimed to be valid. As a consequence a stronger version of Theorem 1.5 is obtained. Unfortunately we have been unable to verify the details of the proof of [LM97, Thm. 4]. The main point of difficulty is that within the proof of this theorem, in [LM97, Subsect. 3.1], the authors claim that given a smooth projective, threefold X of general type with an ample divisor H , for sufficiently large m , there is a general member $S \in |m \cdot H|$ for which the following conditions hold.

(8.2.1) The restriction $(\Omega_X \log(D))|_S$ is semipositive with respect to $(P_\sigma(K_X + D))|_S$, where P_σ is the positive part of the divisorial Zariski decomposition of $K_X + D$.

(8.2.2) The restriction $(P_\sigma(K_X + D))|_S$ of the positive part of $K_X + D$ verifies the equality $P_\sigma(K_X + D)|_S \cdot N((K_X + D)|_S) = 0$, where $N(K_X + D|_S)$ is the negative part of the Zariski decomposition of the pseudoeffective divisor $(K_X + D)|_S$.

Although Item (8.2.1) in the conditions above can most likely be recovered by [CP15, Thm. 2.1] and the arguments in Sections 3 and 4 in the current paper, the second condition (8.2.2) is more problematic as the underlying assumption is that Zariski decomposition is functorial; a condition that in general does not hold.

Remark 8.3. Starting with a general type variety X and a divisor D such that (X, D) is lc, thanks to [BCHM10], it is certainly possible to establish a Miyaoka-Yau inequality using a minimal model of (X, D) . More precisely, let $\pi : (X, D) \dashrightarrow (X', D')$ be a LMMP map resulting in the minimal model (X', D') . Let $\tilde{\pi} : \tilde{X} \rightarrow X'$ be a desingularization of π factoring through $\mu : \tilde{X} \rightarrow X$. Now, one can use the original arguments of Miyaoka, together with those of Megyesi, to show that the inequality

$$(3c_2(\Omega_{X'} \log(D') - (K'_X + D')^2)) \cdot H^{n-2} \geq 0$$

holds for any ample divisor $H \subset X'$. Furthermore, we can use known results on the behaviour of Chern classes under birational morphisms to show that

$$(8.3.1) \quad (3c_2(\Omega_{\tilde{X}} \log(\tilde{D})) - (K_{\tilde{X}} + \tilde{D})^2)) \cdot \tilde{\pi}^*(H)^{n-2} \geq 0.$$

But the inequality (8.3.1) is hardly independent of the divisor D . In fact in the inequality (8.3.1) even the polarization (π^*H) depends on D . Therefore, the inequality (8.3.1) is far from being useful in the context of Lang-Vojta's conjecture.

REFERENCES

- [BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *Journal of the AMS*, 23:405–468, 2010. DOI:10.1090/S0894-0347-09-00649-3. ↑ 22
- [BDPP13] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell. The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. *J. Algebraic Geom.*, 22(2):201–248, 2013. arXiv:math/0405285. ↑ 1, 6
- [Bou04] Sébastien Boucksom. Divisorial Zariski decompositions on compact complex manifolds. *Ann. Sci. École Norm. Sup. (4)*, 37(1):45–76, 2004. DOI:10.1016/j.ansens.2003.04.002. ↑ 3
- [Cam04] Frédéric Campana. Orbifolds, special varieties and classification theory. *Ann. Inst. Fourier (Grenoble)*, 54(3):499–630, 2004. ↑ 5
- [CP11] Frédéric Campana and Thomas Peternell. Geometric stability of the cotangent bundle and the universal cover of a projective manifold. *Bull. Soc. Math. France*, 139(1):41–74, 2011. ↑ 4
- [CP14] Frédéric Campana and Mihai Păun. Positivity properties of the bundle of logarithmic tensors on compact Kähler manifolds. Preprint arXiv:1407.3431, July 2014. ↑ 16, 20
- [CP15] Frédéric Campana and Mihai Păun. Orbifold generic semipositivity : an application to families of canonically polarized manifolds. *Ann. Inst. Fourier (Grenoble)*, 65(2):835–861, 2015. Available at http://aif.cedram.org/item?id=AIF_2015__65_2_835_0. Preprint arXiv:1303.3169. ↑ 2, 22

- [CP16] Frédéric Campana and Mihai Păun. Foliations with positive slopes and birational stability of orbifold cotangent bundles. Preprint [arXiv:1508.02456](https://arxiv.org/abs/1508.02456), April 2016. ↑ 2, 11, 12
- [DG65] J. Dieudonné and Alexandre Grothendieck. Éléments de géométrie algébrique iv, Étude locale des schémas et des morphismes de schémas (seconde partie). *Institut des Hautes Études Scientifiques publications mathématiques*, (24), 1965. ↑ 10
- [DPS94] Jean-Pierre Demailly, Thomas Peternell, and Michael Schneider. Compact complex manifolds with numerically effective tangent bundles. *J. Algebraic Geom.*, 3(2):295–345, 1994. ↑ 1
- [GKP15] Daniel Greb, Stefan Kebekus, and Thomas Peternell. Movable curves and semistable sheaves. *Int Math Res Notices*, 2015. Published online May 19, 2015. To appear in print. DOI:10.1093/imrn/rnv126. Preprint [arXiv:1408.4308](https://arxiv.org/abs/1408.4308). ↑ 4
- [GKPT15] Daniel Greb, Stefan Kebekus, Thomas Peternell, and Behrouz Taji. The Miyaoka-Yau inequality and uniformisation of canonical models. Preprint [arXiv:1511.08822](https://arxiv.org/abs/1511.08822), November 2015. ↑ 12
- [HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010. ↑ 11, 12
- [Hör12] Andreas Höring. On a conjecture of Beltrametti and Sommese. *J. Algebraic Geom.*, 21(4):721–751, 2012. ↑ 2
- [Kaw81] Y. Kawamata. On the plurigenera of minimal algebraic 3-folds with $K_X \equiv 0$. *Math. Annalen*, 275:539–546, 1981. ↑ 17
- [KM83] J. Kollár and T. Matsusaka. Riemann-Roch type inequalities. *Amer. J. Math.*, 105(1):229–252, 1983. ↑ 21
- [KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. ↑ 17
- [Kol92] János Kollár. *Flips and abundance for algebraic threefolds*. Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992). ↑ 18
- [Lan04] Adrian Langer. Semistable sheaves in positive characteristic. *Ann. of Math. (2)*, 159(1):251–276, 2004. ↑ 10
- [Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals. ↑ 5, 12, 13
- [LM97] Steven Shin-Yi Lu and Yoichi Miyaoka. Bounding codimension one subvarieties and a general inequality between Chern numbers. *Amer. J. Math.*, 119(3):487–502, 1997. ↑ 3, 22
- [Miy87] Yoichi Miyaoka. The Chern classes and Kodaira dimension of a minimal variety. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 449–476. North-Holland, Amsterdam, 1987. ↑ 1, 2, 4, 14, 18
- [MR82] Vikram B. Mehta and Annamalai Ramanathan. Semistable sheaves on projective varieties and their restriction to curves. *Math. Ann.*, 258(3):213–224, 1981/82. ↑ 6, 10
- [Rei77] Miles Reid. Bogomolov’s theorem $c_1^2 \leq 4c_2$. *Int. Symp. on Algebraic Geometry*, pages 632–642, 1977. ↑

ERWAN ROUSSEAU, INSTITUT UNIVERSITAIRE DE FRANCE & AIX MARSEILLE UNIV, CNRS, CENTRALE MARSEILLE, I2M, MARSEILLE, FRANCE

E-mail address: erwan.rousseau@univ-amu.fr

URL: <https://old.i2m.univ-amu.fr/~erousseau/>

BEHROUZ TAJI, UNIVERSITY OF NOTRE DAME, DEPARTMENT OF MATHEMATICS, 277 HURLEY, NOTRE DAME, USA

E-mail address: btaji@nd.edu

URL: <http://sites.nd.edu/b-taji/>