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Asymptotic stability of the solitary waves for the generalized Kawahara equation

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Abstract

In Kabakouala and Molinet [7] we construct two families of solitary waves of the gKW equation and we prove that they are orbitally stable. In this paper by applying the method of Martel and Merle [12] we prove that these families of solitary waves are asymptotically stable in the energy space.

1 Introduction

The generalized Kawahara equation is given by:

$$u_t + u^p \partial_x u + \partial_x^3 u - \mu \partial_x^5 u = 0, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{R}, \quad (1.1)$$

where $p \in \mathbb{N}^*$ denotes the power of nonlinearity, and $\mu > 0$ the parameter which controls the fifth-order dispersion term. For $p = 1$ and 2 , the gKW equation has applications for instance in fluid mechanics and plasma physics. For $p \geq 3$, what interests us is the interaction between the nonlinear forces produced by $\frac{1}{p+1} \partial_x u^{p+1}$ and the linear forces produced by $\partial_x(\partial_x^2 u - \mu \partial_x^4 u)$. The most interesting property generated by this interaction is the appearance of solitary waves.

The Cauchy problem associated to (1.1) is locally well-posed in $H^2(\mathbb{R})$ (see for instance [1]). The H^2 -solutions of (1.1) satisfy the following two conservation laws in time:

$$E_{p,\mu}(u(t)) = \int_{\mathbb{R}} \left[\frac{\mu}{2} (\partial_x^2 u)^2(t) + \frac{1}{2} (\partial_x u)^2(t) - \frac{1}{(p+1)(p+2)} u^{p+2}(t) \right] = E_{\mu}(u_0) \quad (\text{energy}) \quad (1.2)$$

and

$$V(u(t)) = \frac{1}{2} \int_{\mathbb{R}} u^2(t) = V(u_0) \quad (\text{mass}). \quad (1.3)$$

These conserved quantities enable us to extend the solutions for all positive times so that (1.1) is actually globally well-posed in $H^2(\mathbb{R})$. Moreover, thanks to the conservation laws one can rewrite equation (1.1) in the Hamiltonian form:

$$\partial_t V'(u) = \partial_x E'(u), \quad (1.4)$$

where $V'(u)$ and $E'(u)$ denote respectively the Fréchet derivative of $V(u)$ and $E(u)$.

The solitons of gKW are solution of (1.1) of the form $u(t, x) = \varphi_{c,p,\mu}(x - ct)$, which travels to the right direction with the constant speed $c > 0$. Substituting u by $\varphi_{c,p,\mu}$ in (1.1), integrating on \mathbb{R} with the assumption $\partial_x^k \varphi_{c,p,\mu}(\pm\infty) = 0$ for $k = 0, \dots, 4$, we obtain the equation of gKW-solitons:

$$\mu \partial_x^4 \varphi_{c,p,\mu}(x) - \partial_x^2 \varphi_{c,p,\mu}(x) + c \varphi_{c,p,\mu}(x) = \frac{1}{p+1} \varphi_{c,p,\mu}^{p+1}(x), \quad \forall x \in \mathbb{R}. \quad (1.5)$$

In 1996 Dey , Khare and Kumar [5] compute the explicit solitons of gKW, they found:

$$\varphi_{c,p,\mu}(x) = \left[\frac{(p+1)(p+4)(3p+4)c}{8(p+2)} \right]^{1/p} \operatorname{sech}^{4/p} \left[\frac{p\sqrt{(p^2+4p+8)c}}{4(p+2)} x \right], \quad \forall x \in \mathbb{R}, \quad (1.6)$$

with a speed completely determined when we fix the value of μ :

$$c = \frac{2^2(p+2)^2}{(p^2+4p+8)^2 \mu}. \quad (1.7)$$

Due to the necessary condition of stability of the soliton $\varphi_{c,p,\mu}$ introduced by Karpman [8]:

$$\frac{\mu \int_{\mathbb{R}} (\partial_x^2 \varphi_{c,p,\mu})^2(x)}{c \int_{\mathbb{R}} \varphi_{c,p,\mu}^2(x)} > \frac{p(p-4)}{(p^2+4p+32)}, \quad (1.8)$$

Dey et al.[5] obtain that the single soliton $\varphi_{c,p,\mu}$ is unstable with respect to the small perturbation in the case $p \geq 5$.

Now we fix $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$ then the speed of the explicit soliton (1.6)-(1.7) is $c_0 = 1$. In [7], we construct a family of even solitons $\{\varphi_{c_0,p,\mu_p}, c_0 \in]1 - \delta_p, 1 + \delta_p[\}$, with $0 < \delta_p \ll 1$, by applying the Implicit Function Theorem in the neighborhood of φ_{1,p,μ_p} (see Lemma 2.2 for the sketch of the proof). Moreover, combining the spectral method introduced by Benjamin [3] with the continuity arguments, we prove that this family of solitons is orbitally stable in the energy space H^2 for the nonlinearity $p \in \{1, 2, 3, 4\}$ (see Theorem 2.1 for the sketch of the proof).

Second, for any $c > 0$ and any $\mu > 0$ (independent of each other) we construct a family of even solitons $\{\psi_{c,p,\mu} : \mu > 0\}$ by solving the global minimization problem $\inf \{I_{c,\mu}(g) : g \in H_c^2(\mathbb{R}) \text{ and } K_p(g) = K_p(\psi_{c,p})\}$, where $H_c^2(\mathbb{R}) = \{g \in H^2(\mathbb{R}) : g(-x) = g(x)\}$, $I_{c,\mu}(g) = \int [\mu(\partial_x^2 g)^2 + (\partial_x g)^2 + cg^2]$, $K_p(g) = \frac{1}{p+1} \int g^{p+1}$, and $\psi_{c,p}(x) = \left[\frac{(p+1)(p+2)c}{2} \right]^{1/p} \operatorname{sech}^{2/p} \left[\frac{p\sqrt{c}}{2} x \right]$ the explicit soliton of the generalized Korteweg-de Vries equation (gKdV). Moreover, we prove that the family $\{\psi_{c,p,\mu} : 0 < \mu \ll 1\}$ converge to $\psi_{c,p}$ strongly in H^1 as μ tends to 0. We also obtain the orbital stability for the nonlinearity $p \in \{1, 2, 3\}$. In this paper we transform this global minimization problem by a local minimization problem and we give a more simple proof. Moreover, we obtain the convergence in all Sobolev space (see Proposition 2.1).

In Kabakouala [6], following the method of Martel and Merle [12] on the gKdV equation, we prove the nonlinear Liouville property for the gKW equation: if $u(t, x)$ the H^2 -global solution of gKW equation is close to the soliton $\varphi_{c_0,p,\mu}$ (up to a translation) in the energy space: $\|u(t, \cdot + \xi(t)) - \varphi_{c_0,p,\mu_p}\|_{H^2(\mathbb{R})} \leq \gamma$, with $\gamma \ll 1$, moreover if $u(t, x + \xi(t))$ is uniformly localized: $|u(t, x + \xi(t))| \leq e^{-\sigma|x|}$, then $u(t, x)$ is identically equal to a soliton $\varphi_{c_1,p,\mu_p}(x - \xi_1 - c_1 t)$, where $|c_1 - c_0| \leq \gamma$ and $|\xi_1| \leq \gamma$ (See Theorem 6.2 and Corollary 6.1). In this paper, following the method of Kenig-Martel-Robbiano [9] we have established the same result with the family $\{\psi_{c,p,\mu}, 0 < \mu < \mu_0\}$ with $0 < \mu_0 \ll 1$ (see Theorem 5.1).

The main result of this paper is the asymptotic stability of the two families of solitons: the solution of gKW which is close to a soliton at initial time converge in some sense to a soliton.

Theorem 1.1 (Asymptotic stability). *(a) Let $p \in \{1, 2\}$, $c_0 \in]1 - \delta_p, 1 + \delta_p[$, with $\delta_p \ll 1$, and $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$. Let $u(t, x)$ be the global H^2 -solution of the gKW equation which is close at the initial time ($t = 0$):*

$$\|u_0 - \varphi_{c_0,p,\mu_p}\|_{H^2} \leq \gamma, \quad \text{with } \gamma \ll 1, \quad \text{and } u_0(x) = u(0, x). \quad (1.9)$$

Then there exists $c(t) \in C^1(\mathbb{R})$ and $\xi(t) \in C^1(\mathbb{R})$ such that:

$$u(t, x) - \varphi_{c(t)}(x - \xi(t)) \rightarrow 0 \text{ in } H^2(x > \theta t) \text{ as } t \rightarrow +\infty, \text{ where } \theta > 0. \quad (1.10)$$

Moreover there exists \tilde{c}_0 such that:

$$\xi'(t) \rightarrow \tilde{c}_0 \text{ as } t \rightarrow +\infty \text{ and } |\tilde{c}_0 - c_0| \leq \gamma. \quad (1.11)$$

(b) The part (a) holds with the family of solitons $\{\psi_{c,p,\mu} : 0 < \mu \ll 1\}$ with $0 < \mu < \mu_0 \ll 1$, for any speed $c > 0$, and for all nonlinearity power $p > 1$.

2 Preliminaries

Lemma 2.1 (Spectral properties of \mathcal{L}_{1,p,μ_p}). *Let $p \in \mathbb{N}^*$ and $\mu_p = \frac{2^2(p+2)}{(p^2+4p+8)^2}$. We consider the unbounded operator $\mathcal{L}_{1,p,\mu_p} : L^2 \rightarrow L^2$, defined by: $u \mapsto \mu_p \partial_x^4 u - \partial_x^2 u + u - \varphi_{1,p,\mu_p}^p u$. We claim that \mathcal{L}_{1,p,μ_p} possesses, among others, the following three crucial properties:*

(P1) *The essential spectrum of \mathcal{L}_{1,p,μ_p} is $[1, +\infty[$;*

(P2) *\mathcal{L}_{1,p,μ_p} has only one negative eigenvalue λ_{1,p,μ_p} which is simple;*

(P3) *The kernel of \mathcal{L}_{1,p,μ_p} is spanned by φ'_{1,p,μ_p} .*

Proof. First, the property (P1) is due to the fact that \mathcal{L}_{1,p,μ_p} is a compact perturbation of $\mu_p \partial_x^4 - \partial_x^2 + 1$, since φ_{1,p,μ_p}^p is smooth and decay exponentially to 0. Second, Albert [2] has given a sufficient condition based on the qualitative properties of the solitons φ_{1,p,μ_p} which confers the operator \mathcal{L}_{1,p,μ_p} the spectral properties (P2) and (P3):

$$\mathcal{F}(\varphi_{1,p,\mu_p})(\omega) > 0, \quad \forall \omega \in \mathbb{R}, \quad \text{and} \quad \frac{d^2}{d\omega^2} \log \mathcal{F}(\varphi_{1,p,\mu_p}^p)(\omega) < 0, \quad \forall \omega \in \mathbb{R}^*, \quad (2.1)$$

where $\mathcal{F}(g(x))(\omega) = \int e^{-ix\omega} g(x)$ denotes the fourier transformation. The sufficient conditions (2.1) are clearly satisfied for the solitons of gKW (see [7] Lemma 2.1 for detailed calculations).

Remark 2.1 (Spectral properties of $\mathcal{L}_{1,p,0}$). We claim that Lemma 2.1 holds for $\mathcal{L}_{1,p,0}$ the linearized operator of gKdV around the soliton $\psi_{1,p}$ of gKdV. It suffices to prove (2.1):

$$\mathcal{F}(\psi_{1,p}(x))(\omega) = \frac{[2(p+1)(p+2)]^{1/p}}{p\Gamma(2/p)} \left| \Gamma\left(\frac{1}{p} + i\frac{\omega}{p}\right) \right|^2 \geq 0, \quad \forall \omega \in \mathbb{R}, \quad (2.2)$$

where $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} \forall z \in \mathbb{C}$ denotes the gamma function. We also compute that:

$$\mathcal{F}(\psi_{1,p}^p(x))(\omega) = \frac{(p+1)(p+2)}{p} \frac{\pi\omega}{\sinh(\pi\omega/p)}, \quad (2.3)$$

and then:

$$\frac{d^2}{d\omega^2} \log \mathcal{F}(\psi_{1,p}^p(x))(\omega) = \frac{\pi^2}{p^2 \sinh^2(\pi\omega/p)} - \frac{1}{\omega^2} < 0, \quad \forall \omega \in \mathbb{R}^*. \quad (2.4)$$

Lemma 2.2 (Existence of Solitons φ_{c_0,p,μ_p} for c_0 close to 1). *Let $p \in \mathbb{N}^*$ and $\mu_p = \frac{2^2(p+2)}{(p^2+4p+8)^2}$. There exist $\delta_p > 0$ and $\tilde{\delta}_p > 0$ such that for any $c_0 > 0$ with $|c_0 - 1| < \delta_p$, there exists a unique H^4 even soliton φ_{c_0,p,μ_p} of gKW in the ball of H^4 centered at φ_{1,p,μ_p} with radius $\tilde{\delta}_p > 0$. Moreover, the function $c_0 \mapsto \varphi_{c_0,p,\mu_p}$ is of class C^1 from $]1 - \delta_p, 1 + \delta_p[$ into $B_{H^4}(\varphi_{1,p,\mu_p})$.*

Proof. The proof is based on the application of the Implicit Function Theorem (See K-M. Lemma 2.2). We consider the functional $T :]1 - \delta_p, 1 + \delta_p[\times H_e^4 \mapsto L_e^2$ defined by: $T(c, g) = \mu_p \partial_x^4 \psi - \partial_x^2 g + cg - \frac{1}{p+1} g^{p+1}$, where $H_e^4 = \{u \in H^4 : u(-x) = u(x)\}$ and $L_e^2 = H_e^0$. One can easily compute that $T(1, \varphi_{1,p,\mu_p}) = 0$ and

$\frac{\partial}{\partial \psi} T(1, \varphi_{1,p,\mu_p}) = \mathcal{L}_{1,p,\mu_p} |_{H_e^4}$. Let us prove that $\mathcal{L}_{1,p,\mu_p} |_{H_e^4}$ is an isomorphism. We have clearly the linearity and the continuity: $\|\mathcal{L}_{1,p,\mu_p}\|_{L^2} \leq C\|u\|_{H^4}$. Now, since the kernel of \mathcal{L}_{1,p,μ_p} is spanned by φ'_{1,p,μ_p} (see Lemma 2.1 (P3)) and φ'_{1,p,μ_p} is an odd function, then $\mathcal{L}_{1,p,\mu_p} |_{H_e^4}$ is injective. Let $g \in L_e^2$, then there exists $f \in H^4$ such that $\mathcal{L}_{1,p,\mu_p} f = g$. We deduce also that $\mathcal{L}_{1,p,\mu_p} f(-\cdot) = g$. By setting $u(x) = \frac{f(x)+f(-x)}{2} \in H_e^4$ we get $\mathcal{L}_{1,p,\mu_p} u(-\cdot) = g$, and this prove the surjectivity of $\mathcal{L}_{1,p,\mu_p} |_{H_e^4}$. Finally, by applying the Implicit Function Theorem we obtain the desired result. \square

Remark 2.2 (Important properties on φ_{c_0,p,μ_p} and $\mathcal{L}_{c_0,p,\mu_p}$ for c_0 close to 1). For c_0 close enough to 1, the even soliton $\varphi_{c_0,p,\mu_p} \in H^4$ satisfies the following nonlinear fourth-order ODE:

$$\mu_p \partial_x^4 \varphi_{c_0,p,\mu_p} - \partial_x^2 \varphi_{c_0,p,\mu_p} + c_0 \varphi_{c_0,p,\mu_p} - \frac{1}{p+1} \varphi_{c_0,p,\mu_p}^{p+1} = 0. \quad (2.5)$$

The classical bootstrap argument implies that $\varphi_{c_0,p,\mu_p} \in H^k \forall k \in \mathbb{N}$. Observe that by differentiating equation 2.5 with respect to the speed c we obtain:

$$\mathcal{L}_{c,p,\mu_p} \left(-\frac{d}{dc} \varphi_{c,p,\mu_p} \right) = \varphi_{c,p,\mu_p} \quad \text{and we set} \quad \rho_{c_0,p,\mu_p} = -\frac{d}{dc} \varphi_{c,p,\mu_p} |_{c=c_0}. \quad (2.6)$$

We claim that:

$$\lim_{c_0 \rightarrow 1} \|\varphi_{c_0,p,\mu_p} - \varphi_{1,p,\mu_p}\|_{H^k} = 0 \quad \text{and} \quad \lim_{c_0 \rightarrow 1} \|\rho_{c_0,p,\mu_p} - \rho_{1,p,\mu_p}\|_{H^k} = 0, \quad \forall k \in \mathbb{N}. \quad (2.7)$$

Note that by Lemma 2.2 the claim (2.7) holds for $k \in \{0, 1, 2, 3, 4\}$. Now, We make the difference between the equation of φ_{c_0,p,μ_p} and φ_{1,p,μ_p} :

$$\begin{aligned} & \mu_p \partial_x^4 (\varphi_{c_0,p,\mu_p} - \varphi_{1,p,\mu_p}) - \partial_x^2 (\varphi_{c_0,p,\mu_p} - \varphi_{1,p,\mu_p}) + c_0 (\varphi_{c_0,p,\mu_p} - \varphi_{1,p,\mu_p}) + (c_0 - 1) \varphi_{1,p,\mu_p} \\ & - \frac{1}{p+1} (\varphi_{c_0,p,\mu_p}^{p+1} - \varphi_{1,p,\mu_p}^{p+1}) = 0. \end{aligned} \quad (2.8)$$

By differentiating (2.8) with respect to the space variable x , multiplying by $\partial_x^5 (\varphi_{c_0,p,\mu_p} - \varphi_{1,p,\mu_p})$, and integrating on \mathbb{R} , we get:

$$\begin{aligned} \mu_p \int [\partial_x^5 (\varphi_{c_0,p,\mu_p} - \varphi_{1,p,\mu_p})]^2 &= - \int [\partial_x^4 (\varphi_{c_0,p,\mu_p} - \varphi_{1,p,\mu_p})]^2 - c_0 \int [\partial_x^3 (\varphi_{c_0,p,\mu_p} - \varphi_{1,p,\mu_p})]^2 \\ & - (c_0 - 1) \int (\partial_x^3 \varphi_{1,p,\mu_p}) \partial_x^3 (\varphi_{c_0,p,\mu_p} - \varphi_{1,p,\mu_p}) + \frac{1}{p+1} \int \partial_x^3 (\varphi_{c_0,p,\mu_p}^{p+1} - \varphi_{1,p,\mu_p}^{p+1}) \partial_x^3 (\varphi_{c_0,p,\mu_p} - \varphi_{1,p,\mu_p}) \\ & \lesssim |c_0 - 1| \|\partial_x^3 (\varphi_{c_0,p,\mu_p} - \varphi_{1,p,\mu_p})\|_{L^2}. \end{aligned} \quad (2.9)$$

Proceeding similarly with the equation of $\rho_{c_0,p,\mu_p} - \rho_{1,p,\mu_p}$, we get:

$$\begin{aligned} \mu_p \int [\partial_x^5 (\rho_{c_0,p,\mu_p} - \rho_{1,p,\mu_p})]^2 &= - \int [\partial_x^4 (\rho_{c_0,p,\mu_p} - \rho_{1,p,\mu_p})]^2 - c_0 \int [\partial_x^3 (\rho_{c_0,p,\mu_p} - \rho_{1,p,\mu_p})]^2 \\ & - (c_0 - 1) \int (\partial_x^3 \rho_{1,p,\mu_p}) \partial_x^3 (\rho_{c_0,p,\mu_p} - \rho_{1,p,\mu_p}) + \frac{1}{p+1} \int \partial_x^3 (\varphi_{c_0,p,\mu_p} - \varphi_{1,p,\mu_p}) \partial_x^3 (\rho_{c_0,p,\mu_p} - \rho_{1,p,\mu_p}) \\ & \lesssim (|c_0 - 1| + \|\partial_x^3 (\varphi_{c_0,p,\mu_p} - \varphi_{1,p,\mu_p})\|_{L^2}) \|\partial_x^3 (\rho_{c_0,p,\mu_p} - \rho_{1,p,\mu_p})\|_{L^2}. \end{aligned} \quad (2.10)$$

Then we deduce that (2.7) holds for $k = 5$. By iterating the method, we obtain the desired result for all $k \in \mathbb{N}$.

From (2.7), we infer that for c_0 close to 1, the soliton φ_{c_0,p,μ_p} and all these derivatives decay exponentially:

$$|\partial_x^k \varphi_{c_0,p,\mu_p}(x)| \lesssim e^{-\sqrt{c_0}|x|}, \quad \forall x \in \mathbb{R}, \quad \forall k \in \mathbb{N}. \quad (2.11)$$

Indeed, let $x \in \mathbb{R}$, and choose $R_0 > 0$ such that $|x| \leq R_0$. From (2.7) and the Sobolev embedding of H^1 into C^0 there exists $\varepsilon(R_0) > 0$ such that for all $|c_0 - 1| \leq \varepsilon$ we have:

$$|\varphi_{c_0,p,\mu_p}(x)| \leq |\varphi_{1,p,\mu_p}(x)| + |\varphi_{c_0,p,\mu_p}(x) - \varphi_{1,p,\mu_p}(x)| \leq C(p)e^{-\frac{\sqrt{p^2+4p+8}}{p+2}|x|} + e^{-R_0} \lesssim e^{-|x|}. \quad (2.12)$$

We argue similarly for the derivatives of φ_{c_0,p,μ_p} .

Next, we claim that for $p \in \{1, 2, 3, 4\}$ and c_0 close enough to 1, the kernel of $\mathcal{L}_{c_0,p,\mu_p}$ is spanned by φ'_{c_0,p,μ_p} . Proof: by differentiating equation 2.5 with respect to x , we have $\mathcal{L}_{c_0,p,\mu_p}\varphi'_{c_0,p,\mu_p} = 0$. Recall that for $p \in \{1, 2, 3, 4\}$, we prove in [7]. (see Subsection 2.3) that: $\langle \rho_{1,p,\mu_p}, \varphi_{1,p,\mu_p} \rangle_{L^2} < 0$. Then, for all $v \in H^2$ satisfying the orthogonality conditions: $\langle v, \varphi'_{c_0,p,\mu_p} \rangle_{L^2} = \langle v, \varphi_{c_0,p,\mu_p} \rangle_{L^2} = 0$ for c_0 close 1, the operator $\mathcal{L}_{c_0,p,\mu_p}$ is strictly positive: $\langle \mathcal{L}_{c_0,p,\mu_p} v, v \rangle_{L^2} > 0$. Assume that the dimension of the kernel of $\mathcal{L}_{c_0,p,\mu_p}$ is larger than or equal to 2, and choose $v_0 = v_1 - \frac{\langle v_1, \varphi'_{c_0,p,\mu_p} \rangle_{L^2}}{\|\varphi'_{c_0,p,\mu_p}\|_{L^2}^2} \varphi'_{c_0,p,\mu_p}$ with $v_1 \in \text{Ker} \mathcal{L}_{c_0,p,\mu_p} \setminus \{\varphi'_{c_0,p,\mu_p}\}$. Then v_0 satisfies the orthogonalities: $\langle v_0, \varphi'_{c_0,p,\mu_p} \rangle_{L^2} = 0$ and $\langle v_0, \varphi_{c_0,p,\mu_p} \rangle_{L^2} = \langle \mathcal{L}_{c_0,p,\mu_p} v_0, \rho_{c_0,p,\mu_p} \rangle_{L^2} = 0$. But we have $\langle \mathcal{L}_{c_0,p,\mu_p} v_0, v_0 \rangle_{L^2} = 0$ which is a contradiction.

One can deduce that for $p \in \{1, 2, 3, 4\}$ and c_0 close to 1 the function ρ_{c_0,p,μ_p} is even. Indeed, since φ_{c_0,p,μ_p} is odd we have $\mathcal{L}_{c_0,p,\mu_p}(\rho_{c_0,p,\mu_p}(x) - \rho_{c_0,p,\mu_p}(-x)) = 0$. Then $\rho_{c_0,p,\mu_p}(x) - \rho_{c_0,p,\mu_p}(-x) \in \{0, \varphi'_{c_0,p,\mu_p}(x)\}$. Since φ'_{c_0,p,μ_p} is odd and $\langle \rho_{c_0,p,\mu_p}, \varphi_{c_0,p,\mu_p} \rangle_{L^2} \sim \langle \rho_{1,p,\mu_p}, \varphi_{1,p,\mu_p} \rangle_{L^2} < 0$ pour $p \in \{1, 2, 3, 4\}$, we have $\rho_{c_0,p,\mu_p}(x) = \rho_{c_0,p,\mu_p}(-x)$.

Proposition 2.1 (Existence of the solitons $\psi_{c,p,\mu}$ for all $c > 0$ and $\mu > 0$). *Let $c > 0$, $\mu > 0$, $R > \frac{1}{2}$ and $p > 1$. We define the functionals $I_{\mu,c,R}(g) = \int_{-R}^R (\frac{\mu}{2}(\partial_x^2 g)^2 + \frac{1}{2}(\partial_x g)^2 + \frac{c}{2}g^2)$ and $K_{p,R}(g) = \frac{1}{(p+1)(p+2)} \int_{-R}^R g^{p+2}$.*

(a) *There exists $\psi_{c,p,\mu,R}$ solution of the following minimization problem:*

$$S_{c,p,\mu,R} = \inf\{I_{\mu,c,R}(g) : g \in H_e^2(-R, R) \text{ and } K_{p,R}(g) = K_{p,R}(\psi_{c,p})\}, \quad (2.13)$$

where $\psi_{c,p}$ is a soliton of $gKdV$ travelling with the speed c . Let $P_R : H^2(-R, R) \mapsto H^2(\mathbb{R})$ the linear bounded extension operator defined by: $P_R g = \tilde{g}$. Then the family $\{\tilde{\psi}_{c,p,\mu,R} : R > 1/2\}$ is uniformly bounded in H^2 , and there exists $\tilde{\psi}_{c,p,\mu,\infty} \in H_e^2$ satisfying:

$$\tilde{\psi}_{c,p,\mu,R} \rightharpoonup \tilde{\psi}_{c,p,\mu,\infty} \text{ in } H^2, \text{ as } R \rightarrow +\infty, \quad (2.14)$$

$$\tilde{\psi}_{c,p,\mu,R} \rightarrow \tilde{\psi}_{c,p,\mu,\infty} \text{ in } H_{loc}^1, \text{ as } R \rightarrow +\infty, \quad (2.15)$$

and

$$\tilde{\psi}_{c,p,\mu,R} \rightarrow \tilde{\psi}_{c,p,\mu,\infty} \text{ a.e in } \mathbb{R}, \text{ as } R \rightarrow +\infty. \quad (2.16)$$

Moreover $\tilde{\psi}_{c,p,\mu,\infty}$ is a even soliton of gKW travelling with the speed c .

(b) *The family $\{\tilde{\psi}_{c,p,\mu,\infty} : 0 < \mu \ll 1\}$ is uniformly bounded in H^2 , and there exists $\tilde{\psi}_{c,p,0,\infty} \in H^2$ satisfying:*

$$\tilde{\psi}_{c,p,\mu,\infty} \rightarrow \tilde{\psi}_{c,p,0,\infty} \text{ in } H^1, \text{ as } \mu \rightarrow 0^+. \quad (2.17)$$

Moreover $\tilde{\psi}_{c,p,0,\infty} = \psi_{c,p}$ is a soliton of $gKdV$.

(c) *For $p \in \{1, 2, 3\}$ there exists $\mu_0 > 0$ such that for all $0 < \mu < \mu_0 \ll 1$ $\tilde{\psi}_{c,p,\mu,\infty}$ is the unique soliton of gKW .*

Proof. We will present the proof in four steps.

Step 1: Resolution of the minimization problem.

We choose $c = 1$. Let (g_k) be a minimizing sequence of the problem (2.13), that is to say, for all $k \in \mathbb{N}$, $g_k \in H_e^2(-R, R)$, and satisfying:

$$K_{p,R}(g_k) = K_{p,R}(\psi_{1,p}) \text{ and } S_{1,p,\mu,R} \leq I_{\mu,R}(g_k) \leq S_{1,p,\mu,R} - \frac{1}{k}. \quad (2.18)$$

We infer that (g_k) is bounded $H_e^2([-R, R])$. Since it is a reflexive space, then there exists $\psi_{1,p,\mu,R} \in H_e^2([-R, R])$ such that up to a subsequence $g_k \rightharpoonup \psi_{1,p,\mu,R}$ converge weakly in $H^2([-R, R])$. And since we have the compact injection of $H^2([-R, R])$ into $C^1([-R, R])$, we infer that $g_k \rightarrow \psi_{1,p,\mu,R}$ strongly in $C^1([-R, R])$. In particular, we also have $g_k \rightarrow \psi_{1,p,\mu,R}$ strongly in $L^{p+2}([-R, R])$:

$$\int_{-R}^R |g_k - \psi_{1,p,\mu,R}|^{p+2} \leq \|g_k - \psi_{1,p,\mu,R}\|_{L^\infty([-R, R])}^p \|g_k - \psi_{1,p,\mu,R}\|_{L^2([-R, R])}^2 \rightarrow 0 \text{ as } k \rightarrow +\infty. \quad (2.19)$$

Passing to the limit in (2.18), using (2.19) and the lower semi-continuity: $I_{\mu,R}(\psi_{1,p,\mu,R}) \leq \liminf_{k \rightarrow +\infty} I_{\mu,R}(g_k)$, we obtain:

$$K_{p,R}(\psi_{1,p,\mu,R}) = \lim_{k \rightarrow +\infty} K_{p,R}(g_k) = K_{p,R}(\psi_{1,p}) \text{ and } S_{1,p,\mu,R} = \liminf_{k \rightarrow +\infty} I_{\mu,R}(g_k) \geq I_{\mu,R}(\psi_{1,p,\mu,R}). \quad (2.20)$$

This prove that $\psi_{1,p,\mu,R}$ is the solution of the minimization problem (2.18).

We can now rewrite the Euler-Lagrange equation related to our minimization problem. There exists $\alpha_{1,p,\mu,R} \in \mathbb{R}^*$ the multiplier of lagrange such that for all $g \in H_e^2([-R, R])$:

$$\mu \int_{-R}^R (\partial_x^2 \psi_{1,p,\mu,R})(\partial_x^2 g) + \int_{-R}^R (\partial_x \psi_{1,p,\mu,R})(\partial_x g) + \int_{-R}^R \psi_{1,p,\mu,R} g - \frac{\alpha_{1,p,\mu,R}}{p+1} \int_{-R}^R \psi_{1,p,\mu,R}^{p+1} g = 0. \quad (2.21)$$

It is clear that (2.21) is satisfied, since the scalar product of an even and odd function is zero. This implies that (2.21) is satisfied $g \in H^2([-R, R])$.

Step 2: Existence of solitons of gKW.

Let $P_R : H^2([-R, R]) \mapsto H^2(\mathbb{R})$ the linear bounded extension operator defined by: $P_R g = \tilde{g}$. According to the Theorem 8.6 of Brezis [4]:

$$\|\tilde{\psi}_{1,p,\mu,R}\|_{H^2(\mathbb{R})} \leq 6(1 + \frac{1}{2R}) \|\psi_{1,p,\mu,R}\|_{H^2([-R, R])}. \quad (2.22)$$

Since $\psi_{1,p}$ the soliton of gKdV is a potential solution of the minimization problem (2.18) then for $R > 1/2$:

$$I_{\mu,\infty}(\tilde{\psi}_{1,p,\mu,R}) \leq 12I_{\mu,R}(\psi_{1,p,\mu,R}) \leq 12I_{\mu,\infty}(\psi_{1,p}). \quad (2.23)$$

This implies that $\{\tilde{\psi}_{1,p,\mu,R} : R > 1/2\}$ is uniformly bounded $H^2(\mathbb{R})$. Let $R_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Since $H^2(\mathbb{R})$ is reflexive, there exists $\psi_{1,p,\mu,\infty} \in H^2(\mathbb{R})$ such that up to a subsequence:

$$\tilde{\psi}_{c,p,\mu,R_n} \rightharpoonup \tilde{\psi}_{c,p,\mu,\infty} \text{ in } H^2, \text{ as } n \rightarrow +\infty, \quad (2.24)$$

$$\tilde{\psi}_{c,p,\mu,R_n} \rightarrow \tilde{\psi}_{c,p,\mu,\infty} \text{ in } H_{loc}^1, \text{ as } n \rightarrow +\infty, \quad (2.25)$$

and

$$\tilde{\psi}_{c,p,\mu,R_n} \rightarrow \tilde{\psi}_{c,p,\mu,\infty} \text{ a.e in } \mathbb{R}, \text{ as } n \rightarrow +\infty. \quad (2.26)$$

directly, form the pointwise convergence (2.26), we deduce that $\tilde{\psi}_{1,p,\mu,\infty}$ is even. Indeed, let $x_0 \in \mathbb{R}$, there exists $N(x_0)$ such that for $n > N$ $|x| \leq R_n$, then: $\tilde{\psi}_{1,p,\mu,\infty}(-x_0) = \lim_{n \rightarrow +\infty} \tilde{\psi}_{1,p,\mu,R_n}(-x_0) = \lim_{n \rightarrow +\infty} \tilde{\psi}_{1,p,\mu,R_n}(x_0) = \tilde{\psi}_{1,p,\mu,\infty}(x_0)$.

On the other hand if we replace g by $\psi_{1,p,\mu,R}$ in the equation (2.21), we obtain:

$$\alpha_{1,p,\mu,R} = \frac{2I_{\mu,R}(\psi_{1,p,\mu,R})}{(p+2)K_{p,R}(\psi_{1,p,\mu,R})} \leq \frac{2I_{\mu,R}(\psi_{1,p})}{(p+2)K_{p,R}(\psi_{1,p})}. \quad (2.27)$$

When $R \rightarrow +\infty$ we deduce that:

$$0 < \lim_{R \rightarrow +\infty} \alpha_{1,p,\mu,R} := \alpha_{1,p,\mu,\infty} \leq \frac{2I_{\mu,\infty}(\psi_{1,p})}{(p+2)K_{p,\infty}(\psi_{1,p})}. \quad (2.28)$$

We now have all the elements to pass to the limit in (2.21). Using that $\lim_{n \rightarrow +\infty} \mathbb{1}_{]-R_n, R_n[} = 1$ and (2.24), we deduce that: $\forall k \in \{0, 1, 2\}$, $(\partial_x^k \tilde{\psi}_{1,p,\mu,R}) \mathbb{1}_{]-R_n, R_n[} \rightharpoonup \partial_x^k \tilde{\psi}_{1,p,\mu,\infty}$ weakly in $L^2(\mathbb{R})$. This allows to pass to the limit in the linear part of the equation.

For the nonlinear part of the equation we use that: $\lim_{n \rightarrow +\infty} \tilde{\psi}_{1,p,\mu,R_n}^{p+1} \mathbb{1}_{]-R_n, R_n[} = \tilde{\psi}_{1,p,\mu,\infty}^{p+1}$ and we apply Lebesgue's dominated convergence theorem. Therefore $\tilde{\psi}_{1,p,\mu,\infty}$ satisfied for all $g \in H^2(\mathbb{R})$:

$$\mu \int (\partial_x^2 \tilde{\psi}_{1,p,\mu,\infty})(\partial_x^2 g) + \int (\partial_x \tilde{\psi}_{1,p,\mu,\infty})(\partial_x g) + \int \tilde{\psi}_{1,p,\mu,\infty} g - \frac{\alpha_{1,p,\mu,\infty}}{p+1} \int \tilde{\psi}_{1,p,\mu,\infty}^{p+1} g = 0. \quad (2.29)$$

By applying elliptic regularization arguments we deduce that $\tilde{\psi}_{1,p,\mu,\infty} \in H^k(\mathbb{R}) \forall k \in \mathbb{N}$ and satisfied:

$$\mu \partial_x^4 \tilde{\psi}_{1,p,\mu,\infty} - \partial_x^2 \tilde{\psi}_{1,p,\mu,\infty} + \tilde{\psi}_{1,p,\mu,\infty} - \frac{\alpha_{1,p,\mu,\infty}}{p+1} \tilde{\psi}_{1,p,\mu,\infty}^{p+1} = 0. \quad (2.30)$$

By setting $\bar{\psi}_{1,p,\mu,p,\infty} = \alpha_{1,p,\mu,\infty}^{1/p} \tilde{\psi}_{1,p,\mu,\infty}$ we obtain a soliton of gKW.

Knowing that for all $k \in \mathbb{N}$ $\partial_x^k \tilde{\psi}_{1,p,\mu,\infty}(\pm\infty) = 0$, then for $|x|$ large enough and $p \geq 1$, $\tilde{\psi}_{1,p,\mu,\infty}$ behaves like the solution of a linear fourth order ODE: $\mu \partial_x^4 \tilde{\psi}_{1,p,\mu,\infty} - \partial_x^2 \tilde{\psi}_{1,p,\mu,\infty} + \tilde{\psi}_{1,p,\mu,\infty} = 0$. We deduce that:

$$|\partial_x^k \tilde{\psi}_{1,p,\mu,\infty}(x)| \lesssim e^{-\frac{\sqrt{1-\sqrt{1-4\mu}}}{\sqrt{2\mu}}|x|} \lesssim e^{-|x|}, \quad \text{for } |x| \gg 1, k \in \mathbb{N}, \text{ and } 0 < \mu \ll 1. \quad (2.31)$$

For more rigorous proof of (2.31) see Li-Zhao.

Step 3: Relation between the soliton of gKW and the soliton of gKdV.

Directly we remarque that by passing to the limit in (2.28):

$$0 < \lim_{\mu \rightarrow 0^+} \alpha_{1,p,\mu,\infty} := \alpha_{1,p,0,\infty} \leq \frac{2I_{0,\infty}(\psi_{1,p})}{(p+2)K_{p,\infty}(\psi_{1,p})} = 1. \quad (2.32)$$

On the other hand, using the lower semi-continuity and (2.23), we obtain:

$$I_{\mu,\infty}(\tilde{\psi}_{1,p,\mu,\infty}) \leq \limsup_{n \rightarrow +\infty} I_{\mu,\infty}(\tilde{\psi}_{1,p,\mu,R_n}) \leq 12I_{\mu,\infty}(\psi_{1,p}). \quad (2.33)$$

It means that this family $\{\tilde{\psi}_{1,p,\mu,\infty} : 0 < \mu \ll 1\}$ is uniformly bounded in $H^1(\mathbb{R})$.

We claim that the family $\{\tilde{\psi}_{1,p,\mu,\infty} : 0 < \mu \ll 1\}$ is uniformly bounded in $H^2(\mathbb{R})$. Indeed, by applying the Fourier transformation (2.30), we obtain:

$$(\mu|\omega|^4 + |\omega|^2 + 1)\mathcal{F}(\tilde{\psi}_{1,p,\mu,\infty}(x))(\omega) = \frac{\alpha_{1,p,\mu,\infty}}{p+1} \mathcal{F}(\tilde{\psi}_{1,p,\mu,\infty}^{p+1}(x))(\omega). \quad (2.34)$$

Since $\{\alpha_{1,p,\mu,\infty} : 0 < \mu \ll 1\}$ is bounded, then it suffice to prove that: $\{\mathcal{F}(\tilde{\psi}_{1,p,\mu,\infty}^{p+1}) : 0 < \mu \ll 1\}$ is bounded in $L^2(\mathbb{R})$. Thanks to the Plancherel Parseval identity this is equivalent to prove that $\{\tilde{\psi}_{1,p,\mu,\infty} : 0 < \mu \ll 1\}$ is bounded in $L^{2p+2}(\mathbb{R})$. We know that:

$$\|\tilde{\psi}_{1,p,\mu,\infty}\|_{L^{2p+2}(\mathbb{R})}^{2p+2} \leq \|\tilde{\psi}_{1,p,\mu,\infty}\|_{L^\infty(\mathbb{R})}^{2p} \|\tilde{\psi}_{1,p,\mu,\infty}\|_{L^2(\mathbb{R})}^2 \leq 2^{-p} \|\tilde{\psi}_{1,p,\mu,\infty}\|_{H^1(\mathbb{R})}^{2p+2}. \quad (2.35)$$

Let $\mu_n \rightarrow 0$ as $n \rightarrow +\infty$. Then there exists $\tilde{\psi}_{1,p,0,\infty} \in H^2(\mathbb{R})$ such that up to subsequence:

$$\tilde{\psi}_{1,p,\mu_n,\infty} \rightharpoonup \tilde{\psi}_{1,p,0,\infty} \text{ in } H^2 \text{ as } n \rightarrow +\infty, \quad (2.36)$$

$$\tilde{\psi}_{1,p,\mu_n,\infty} \rightarrow \tilde{\psi}_{1,p,0,\infty} \text{ in } H_{loc}^1 \text{ as } n \rightarrow +\infty, \quad (2.37)$$

and

$$\tilde{\psi}_{1,p,\mu_n,\infty} \rightarrow \tilde{\psi}_{1,p,0,\infty} \text{ a.e in } \mathbb{R}, \text{ as } n \rightarrow +\infty. \quad (2.38)$$

By (2.31) we know that $(\tilde{\psi}_{1,p,\mu_n,\infty})$ is H^k -compact, that is to say, for $R_0 > 0$ large enough:

$$\int_{|x|>R_0} |\partial_x^k \tilde{\psi}_{1,p,\mu,\infty}(x)|^2 \lesssim e^{-2R_0}. \quad (2.39)$$

Then (2.37) and (2.39) give us:

$$\tilde{\psi}_{1,p,\mu_n,\infty} \rightarrow \tilde{\psi}_{1,p,0,\infty} \text{ in } H^1 \text{ as } n \rightarrow +\infty. \quad (2.40)$$

Finally, by passing to limit in (2.29), using (2.32), (2.36) and (2.38), we infer that for all $g \in H^2(\mathbb{R})$:

$$\int (\partial_x \tilde{\psi}_{1,p,0,\infty})(\partial_x g) + \int \tilde{\psi}_{1,p,0,\infty} g - \frac{\alpha_{1,p,0,\infty}}{p+1} \int \tilde{\psi}_{1,p,0,\infty}^{p+1} g = 0. \quad (2.41)$$

Thus by setting $\bar{\psi}_{1,p,0,\infty} = \alpha^{1/p} \tilde{\psi}_{1,p,0,\infty}$ we obtain a soliton of gKdV.

Step 4: Uniqueness of the solitons of gKW.

Now we will prove (c) arguing by contradiction. Assume that we have two families $\{\tilde{\psi}_{1,p,\mu_n,\infty}^I : 0 \leq \mu \ll 1\}$ et $\{\tilde{\psi}_{1,p,\mu_n,\infty}^{II} : 0 < \mu \ll 1\}$ satisfying (a) and (b) and such that: $\tilde{\psi}_{1,p,\mu_n,\infty}^I \neq \tilde{\psi}_{1,p,\mu_n,\infty}^{II}$. We set $\tilde{\eta}_\mu = \tilde{\psi}_{1,p,\mu_n,\infty}^I - \tilde{\psi}_{1,p,\mu_n,\infty}^{II}$. From (a) and (b) we know that $\tilde{\eta}_\mu \rightarrow 0$ in H^1 as $\mu \rightarrow 0^+$ and such $\{\tilde{\eta}_\mu : 0 < \mu \ll 1\}$ is bounded in H^2 . We will normalize this family $\eta_\mu = \frac{\tilde{\eta}_\mu}{\|\tilde{\eta}_\mu\|_{H^1}}$. The equation of $\eta_\mu(t, x)$ is given by:

$$\mu \partial_x^4 \eta_\mu - \partial_x^2 \eta_\mu + \eta_\mu - \frac{1}{p+1} \eta_\mu \sum_{k=0}^p (\tilde{\psi}_{1,p,\mu,\infty}^I)^k (\tilde{\psi}_{1,p,\mu,\infty}^{II})^{p-k} = 0. \quad (2.42)$$

One can easily compute:

$$\mathcal{L}_{1,p,\mu} \eta_\mu = \eta_\mu \tilde{\eta}_\mu \left[\frac{1}{p+1} \sum_{k=0}^{p-k} (\tilde{\psi}_{1,p,\mu,\infty}^I)^k \sum_{j=0}^{p-k-1} (\tilde{\psi}_{1,p,\mu,\infty}^{II})^j (\tilde{\psi}_{1,p,\mu,\infty}^I)^{p-k-j-1} \right]. \quad (2.43)$$

This implies:

$$\|\mathcal{L}_{1,p,\mu} \eta_\mu\|_{L^2} \lesssim \|\eta_\mu\|_{L^2} \|\tilde{\eta}_\mu\|_{L^2} \quad \text{and} \quad \|\mathcal{L}_{1,p,\mu} \eta_\mu\|_{H^1} \lesssim \|\tilde{\eta}_\mu\|_{H^1}. \quad (2.44)$$

On the other hand one can compute that:

$$\begin{aligned} \langle \mathcal{L}_{1,p,0} \eta_\mu, \eta_\mu \rangle_{L^2} &= \langle \mathcal{L}_{1,p,\mu} \eta_\mu, \eta_\mu \rangle_{L^2} - \mu \int (\partial_x^2 \eta_\mu)^2 + \int \eta_\mu^2 [(\tilde{\psi}_{1,p,\mu,\infty}^I)^p - \psi_{1,p}^p] \\ &\lesssim \|\tilde{\eta}_\mu\|_{L^2} + \mu + \|(\tilde{\psi}_{1,p,\mu,\infty}^I)^p - \psi_{1,p}^p\|_{H^1}. \end{aligned} \quad (2.45)$$

It is clear $\tilde{\eta}_\mu$ which is an even function is orthogonal to $\psi_{1,p}^I$ which is an odd function. We will prove that $\tilde{\eta}_\mu$ is almost orthogonal to $\psi_{1,p}$. We have:

$$\begin{aligned} |\langle \eta_\mu, \psi_{1,p} \rangle_{L^2}| &= |\langle \eta_\mu, \mathcal{L}_{1,p,0} \rho_{1,p} \rangle_{L^2}| = |\langle \mathcal{L}_{1,p,\mu} \eta_\mu, \rho_{1,p} \rangle_{L^2} - \mu \int \eta_\mu \partial_x^4 \rho_{1,p} + \int \eta_\mu \rho_{1,p} [(\tilde{\psi}_{1,p,\mu,\infty}^I)^p - \psi_{1,p}^p]| \\ &\lesssim (\|\tilde{\eta}_\mu\|_{L^2} + \mu + \|(\tilde{\psi}_{1,p,\mu,\infty}^I)^p - \psi_{1,p}^p\|_{L^2}) \|\eta_\mu\|_{L^2}. \end{aligned} \quad (2.46)$$

Since $\langle \mathcal{L}_{1,p,0}^{-1} \psi_{1,p}, \psi_{1,p} \rangle_{L^2} = -\frac{1}{2} \frac{d}{dc} \|\psi_{c,p}\|_{L^2}|_{c=1} < 0$ for $p \in \{1, 2, 3\}$, then there exists $C > 0$ such that:

$$\langle \mathcal{L}_{1,p,0} \eta_\mu, \eta_\mu \rangle_{L^2} \geq C. \quad (2.47)$$

For μ small (2.45) contradicts (2.47). Hence the uniqueness.

Remark 2.3 (Approximation dans H^k des solitons de gKdV par des solitons de gKW). *We claim that for all $k \in \mathbb{N}$ the family $\{\tilde{\psi}_{1,p,\mu,\infty} : 0 < \mu \ll 1\}$ is bounded in H^k . We will just prove the control in H^4 . By differentiating twice equation (2.30) and applying the Fourier transformation we get:*

$$(\mu|\omega|^6 + |\omega|^4 + |\omega|^2)\mathcal{F}(\tilde{\psi}_{1,p,\mu,\infty}(x))(\omega) = \frac{\alpha_{1,p,\mu,\infty}}{p+1}|\omega|^2\mathcal{F}(\tilde{\psi}_{1,p,\mu,\infty}^{p+1}(x))(\omega). \quad (2.48)$$

Then it suffice to prove that $\tilde{\psi}_{1,p,\mu,\infty}^{p+1} \in H^2$. One can easily compute that:

$$\begin{aligned} \|\partial_x^2 \tilde{\psi}_{1,p,\mu,\infty}^{p+1}\|_{L^2}^2 &\leq 2(p+1)^2 \|\tilde{\psi}_{1,p,\mu,\infty}\|_{L^\infty}^{2p} \|\partial_x^2 \tilde{\psi}_{1,p,\mu,\infty}\|_{L^2}^2 \\ 2p^2(p+1)^2 \|\tilde{\psi}_{1,p,\mu,\infty}\|_{L^\infty}^{2p-2} \|\partial_x \tilde{\psi}_{1,p,\mu,\infty}\|_{L^\infty}^2 &\|\partial_x \tilde{\psi}_{1,p,\mu,\infty}^{p+1}\|_{L^2}^2 \lesssim \|\tilde{\psi}_{1,p,\mu,\infty}\|_{H^2}^{2p+2}. \end{aligned} \quad (2.49)$$

By iterating this method we obtain the control in H^k .

Theorem 2.1 (Orbital stability). (a) *Let $p \in \{1, 2, 3, 4\}$ and $\mu_p = \frac{2^2(p+2)}{(p^2+4p+8)^2}$. There exists $\delta_p > 0$ such that for all $|c_0 - 1| < \delta_p$ the soliton φ_{c_0,p,μ_p} of gKW given by Lemma 2.2 is orbitally stable.*

(b) *Let $p \in \{1, 2, 3\}$ and $c > 0$. There exists $\mu_0 > 0$ such that $0 < \mu < \mu_0$ the soliton $\psi_{c,\mu,p}$ of gKW given by Proposition 2.1 is orbitally stable.*

Sketch of the proof. We explain the prove of (a). Assume that the global solution $u(t, x)$ of gKW is close φ_{c_0,p,μ_p} at the initial time in the energy space: $\|u_0 - \varphi_{c_0,p,\mu_p}\|_{H^2} \leq \varepsilon^2$, where $u_0(x) = u(0, x)$ and $\varepsilon \ll 1$.

Let $\xi(t) \in \mathbb{R}$ a translation that we will choose appropriately. We set: $\eta(t, x) = u(t, x + \xi(t)) - \varphi_{c_0,p,\mu_p}(x)$. We introduce the following action functional: $S_{c_0,p,\mu_p}(u) = E_{\mu_p,p}(u) + c_0V(u)$ which is preserved over time. By applying the Taylor expansion of $S_{c_0,p,\mu_p}(u)$ in the neighbourhood φ_{c_0,p,μ_p} , we obtain $\forall t \in \mathbb{R}$:

$$S_{c_0,p,\mu_p}(u_0) - S_{c_0,p,\mu_p}(\varphi_{c_0,p,\mu_p}) = \frac{1}{2}\langle \mathcal{L}_{c_0,p,\mu_p}\eta(t), \eta(t) \rangle_{L^2} + R(\varphi_{c_0,p,\mu_p}, \eta(t)), \quad (2.50)$$

where

$$R(f, g) = -\frac{1}{(p+1)(p+2)} \sum_{k=3}^{p+2} C_k^{p+2} \int f^k g^{p-k+2}. \quad (2.51)$$

Using the initial assumption: $|S_{c_0,p,\mu_p}(u_0) - S_{c_0,p,\mu_p}(\varphi_{c_0,p,\mu_p})| \leq \varepsilon^2$. The soliton φ_{c_0,p,μ_p} is orbitally stable if and only if $\mathcal{L}_{c_0,p,\mu_p}$ is coercive, i.e, there exists $C > 0$ such that:

$$\langle \mathcal{L}_{c_0,p,\mu_p}\eta(t), \eta(t) \rangle_{L^2} \geq C\|\eta(t)\|_{H^2}^2. \quad (2.52)$$

Thanks to the initial assumption of and the continuity of the map: $t \mapsto \|\eta(t)\|_{H^2}$, we can suppose that: $\|\eta(t)\|_{H^2} \sim \varepsilon$ for $t \in [0, t_0]$ with $|t_0| \ll 1$. For to obtain (2.52). We will choose $\xi(t)$ such that $\langle \eta(t), \varphi'_{c_0,p,\mu_p} \rangle_{L^2} = 0$. And using the conservation of the L^2 -norm, one can see that $\eta(t, x)$ is almost orthogonal to $\varphi_{c_0,p,\mu_p}(x)$:

$$\langle \eta(t), \varphi_{c_0,p,\mu_p} \rangle_{L^2} = (V(u_0) - V(\varphi_{c_0,p,\mu_p})) - V(\eta(t)) \leq \varepsilon\|\eta(t)\|_{L^2}. \quad (2.53)$$

One can easy check that φ_{1,p,μ_p} is a negative vector: $\langle \mathcal{L}_{1,p,\mu_p}\varphi_{1,p,\mu_p}, \varphi_{1,p,\mu_p} \rangle_{L^2} = -\frac{p}{p+1}\|\varphi_{1,p,\mu_p}\|_{L^{p+2}}^{p+2} < 0$. And since $\langle \mathcal{L}_{1,p,\mu_p}^{-1}\varphi_{1,p,\mu_p}, \varphi_{1,p,\mu_p} \rangle_{L^2} < 0$ for $p \in \{1, 2, 3, 4\}$, and moreover $|\langle \eta(t), \varphi'_{1,p,\mu_p} \rangle_{L^2}| + |\langle \eta(t), \varphi_{1,p,\mu_p} \rangle_{L^2}| \leq \varepsilon\|\eta(t)\|_{L^2}$, for c_0 close to 1, so according to Weinstein [13], there exists $C > 0$ such that:

$$\langle \mathcal{L}_{1,p,\mu_p}\eta(t), \eta(t) \rangle_{L^2} \geq C\|\eta(t)\|_{H^2}^2. \quad (2.54)$$

This implies (2.52) for c_0 close 1. Hence we deduce that: $\|u(t, \cdot + \xi(t)) - \varphi_{c_0,p,\mu}\|_{H^2} \leq \varepsilon$.

For the proof of (b) we argue similarly using that $\langle \mathcal{L}_{c,p,0}^{-1}\psi_{c,p}, \psi_{c,p} \rangle_{L^2} < 0$ for $p \in \{1, 2, 3\}$, and that $\psi_{c,p,\mu} \sim \psi_{c,p}$ in H^1 and $\mathcal{L}_{c,p,\mu_p} \sim \mathcal{L}_{1,p,0}$ when μ is small enough.

3 Linear Liouville property with the family of solitons of gKW which is close to the explicit soliton of gKW

Theorem 3.1 (Linear Liouville property with φ_{c_0,p,μ_p}). *Assume that $p \in \{1, 2\}$, $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$ and $c_0 \in]1 - \delta_p, 1 + \delta_p[$, with $0 < \delta \ll 1$. Let $u \in C(\mathbb{R}, H^2(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^2(\mathbb{R}))$ be the solution of*

$$\partial_t u = \partial_x(\mathcal{L}_{c_0,p,\mu_p} u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (3.1)$$

Moreover, assume that $u(t, x)$ is L^2 -compact:

$$\forall \varepsilon > 0, \exists R_\varepsilon > 0, \text{ such that } \forall t \in \mathbb{R} \int_{|x| > R_\varepsilon} |u(t, x)|^2 \leq \varepsilon. \quad (3.2)$$

Then there exists $a_1 \in \mathbb{R}$ such that

$$u(t, x) \equiv a_1 \varphi'_{c_0,p,\mu_p}(x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (3.3)$$

Remark 3.1. It is clear that φ'_{c_0,p,μ_p} satisfies (3.1) and (3.2), since $\mathcal{L}_{c_0,p,\mu_p} \varphi'_{c_0,p,\mu_p} = 0$. By making the change of variable $w = \partial_x(\mathcal{L}_{c_0,p,\mu_p} u)$, we obtain that $w(t, x)$ satisfies equation (3.1) and is orthogonal in L^2 to φ_{c_0,p,μ_p} . Therefore the solution of (3.1) must be orthogonal to φ_{c_0,p,μ_p} .

Corollary 3.1 (Linear Liouville property with $\psi_{c,p,\mu}$ for $p = 1$ and $c > 0$). *The Theorem 3.1 holds with the family of solitons $\{\psi_{c,p,\mu} : 0 < \mu \ll 1\}$ with $0 < \mu < \mu_0 \ll 1$, for any speed $c > 0$, and for the nonlinearity power $p = 1$.*

3.1 Monotonicity formula related to the linearized gKW equation

The aim of this section is to prove the smoothness and exponential decay properties of the L^2 -compact solutions $u(t, x)$ of the linearized gKW equation around φ_{c_0,p,μ_p} .

Proposition 3.1 (Smoothness and exponential decay). *Let $p \in \mathbb{N}^*$ and $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$. Let $x_0 > 0$, $K > 0$ chosen correctly, $t_0 > t$, and $\phi_K(x) = \frac{1}{2} + \frac{1}{2} \tanh(x/2K)$. Let $u \in C(\mathbb{R}, H^2(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^2(\mathbb{R}))$ satisfying (3.1). We set $\tilde{x} = x - \frac{c_0}{2}(t_0 - t) - x_0$, and we define $\forall k \in \mathbb{N}$:*

$$I_{x_0,t_0}^{(k)}(t) = \int (\partial_x^k u(t, x))^2 \phi_K(\tilde{x}). \quad (3.4)$$

Then $\forall k \in \mathbb{N}$:

$$I_{x_0,t_0}^{(k)}(t_0) - I_{x_0,t_0}^{(k)}(t) \lesssim \sup_{t \in \mathbb{R}} \|u(t)\|_{H^k}^2 e^{-x_0/K}. \quad (3.5)$$

Moreover, if $u(t, x)$ is L^2 -compact, then $\forall k \in \mathbb{N}$ $u(t) \in H^k(\mathbb{R})$, and

$$|\partial_x^k u(t, x)| \lesssim e^{-|x|/2K}, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (3.6)$$

Proof. We follow the arguments of Laurent and Martel [10].

For more clarity we will present the proof in four steps.

Step 1: Monotonicity formula at order 0.

For $x_0 > 0$ and $t_0 \geq t$, we claim that:

$$I_{x_0,t_0}^{(0)}(t_0) - I_{x_0,t_0}^{(0)}(t) \lesssim \sup_{t \in \mathbb{R}} \|u(t)\|_{L^2}^2 e^{-x_0/K}. \quad (3.7)$$

Proof of (3.7). We compute the time variation of $I_{x_0,t_0}^{(0)}(t)$:

$$\frac{d}{dt} I_{x_0,t_0}^{(0)}(t) = 2 \int (\partial_t u) u \phi_K(\tilde{x}) + \frac{c_0}{2} \int u^2 \phi'_K(\tilde{x}). \quad (3.8)$$

Using that $u(t, x)$ satisfies equation (3.1), the first term in (3.8) gives us:

$$\begin{aligned} 2 \int (\partial_t u) u \phi_K(\tilde{x}) &= 2\mu_p \int (\partial_x^5 u) u \phi_K(\tilde{x}) - 2 \int (\partial_x^3 u) u \phi_K(\tilde{x}) + 2c_0 \int (\partial_x u) u \phi_K(\tilde{x}) - 2 \int \partial_x (\varphi_{c_0, p, \mu_p}^p u) u \phi_K(\tilde{x}) \\ &= I + J + K + L. \end{aligned} \quad (3.9)$$

We compute I by applying several integration by parts:

$$I = -2\mu_p \int (\partial_x^4 u) (\partial_x u) \phi_K(\tilde{x}) - 2\mu_p \int (\partial_x^4 u) u \phi'_K(\tilde{x}) = I_1 + I_2. \quad (3.10)$$

I_1 give us

$$I_1 = 2\mu_p \int (\partial_x^3 u) (\partial_x^2 u) \phi_K(\tilde{x}) + 2\mu_p \int (\partial_x^3 u) (\partial_x u) \phi'_K(\tilde{x}) = -3\mu_p \int (\partial_x^2 u)^2 \phi'_K(\tilde{x}) + \mu_p \int (\partial_x u)^2 \phi'''_K(\tilde{x}). \quad (3.11)$$

I_2 give us

$$I_2 = 2\mu_p \int (\partial_x^3 u) (\partial_x u) \phi'_K(\tilde{x}) + 2\mu_p \int (\partial_x^3 u) u \phi''_K(\tilde{x}) = I_{21} + I_{22}, \quad (3.12)$$

with

$$I_{21} = -2\mu_p \int (\partial_x^2 u)^2 \phi'_K(\tilde{x}) + \mu_p \int (\partial_x u)^2 \phi'''_K(\tilde{x}), \quad (3.13)$$

and

$$I_{22} = -2\mu_p \int (\partial_x^2 u) (\partial_x u) \phi''_K(\tilde{x}) - 2\mu_p \int (\partial_x^2 u) u \phi'''_K(\tilde{x}) = 3\mu_p \int (\partial_x u)^2 \phi'''_K(\tilde{x}) - \mu_p \int u^2 \phi_K^{(5)}(\tilde{x}). \quad (3.14)$$

Combining (3.10)-(3.14) we get I :

$$I = -5\mu_p \int (\partial_x^2 u)^2 \phi'_K(\tilde{x}) + 5\mu_p \int (\partial_x u)^2 \phi'''_K(\tilde{x}) - \mu_p \int u^2 \phi_K^{(5)}(\tilde{x}), \quad (3.15)$$

We compute J :

$$J = 2 \int (\partial_x^2 u) (\partial_x u) \phi_K(\tilde{x}) + 2 \int (\partial_x^2 u) u \phi'_K(\tilde{x}) = -3 \int (\partial_x u)^2 \phi'_K(\tilde{x}) + \int u^2 \phi'''_K(\tilde{x}). \quad (3.16)$$

K and L give us

$$K = -c_0 \int u^2 \phi'_K(\tilde{x}), \quad L = \int u^2 \left[\varphi_{c_0, p, \mu_p}^p \phi'_K(\tilde{x}) - (\varphi_{c_0, p, \mu_p}^p)' \phi_K(\tilde{x}) \right]. \quad (3.17)$$

Therefore, combining (3.8),(3.15)-(3.17), we obtain

$$\begin{aligned} \frac{d}{dt} I_{x_0, t_0}(t) &= -5\mu_p \int (\partial_x^2 u)^2 \phi'_K(\tilde{x}) - 3 \int (\partial_x u)^2 \phi'_K(\tilde{x}) - \frac{c_0}{2} \int u^2 \phi'_K(\tilde{x}) \\ &\quad + 5\mu_p \int (\partial_x u)^2 \phi'''_K(\tilde{x}) - \mu_p \int u^2 \phi_K^{(5)}(\tilde{x}) + \int u^2 \phi'''_K(\tilde{x}) \\ &\quad + \int u^2 \left[\varphi_{c_0, p, \mu_p}^p \phi'_K(\tilde{x}) - (\varphi_{c_0, p, \mu_p}^p)' \phi_K(\tilde{x}) \right]. \end{aligned} \quad (3.18)$$

Note that the term $5\mu_p \int (\partial_x^2 u)^2 \phi'_K(\tilde{x}) + 3 \int (\partial_x u)^2 \phi'_K(\tilde{x})$ will permit us to gain regularity. One can notice that: $\phi'_K(x) = \frac{1}{4K} \operatorname{sech}^2(x/2K)$ and $|\phi^{(j+1)}(x)| \leq \frac{1}{K^j} \phi'_K(x)$, $\forall x \in \mathbb{R}$, $\forall j \in \mathbb{N}$. Let us estimate the terms on the second line of (3.18):

$$\begin{aligned} &\left| 5\mu_p \int (\partial_x u)^2 \phi'''_K(\tilde{x}) - \mu_p \int u^2 \phi_K^{(5)}(\tilde{x}) + \int u^2 \phi'''_K(\tilde{x}) \right| \\ &\leq \max \left(\frac{5\mu_p}{K^2}, \frac{\mu_p}{K^4} + \frac{1}{K^2} \right) \int ((\partial_x u)^2 + u^2) \phi'_K(\tilde{x}). \end{aligned} \quad (3.19)$$

Now, let us estimate the last term in (3.18). We first remark that, using the properties of the soliton φ_{c_0, p, μ_p} and the test function ϕ_K , we have for all $x \in \mathbb{R}$:

$$\left| \varphi_{c_0, p, \mu_p}^p(x) \right| \phi_K'(\tilde{x}) + \left| (\varphi_{c_0, p, \mu_p}^p)'(x) \right| \phi_K(\tilde{x}) \lesssim e^{-p\sqrt{c_0}|x|} e^{\tilde{x}/K}. \quad (3.20)$$

Then we deduce that for $K \geq 1/p\sqrt{c_0}$:

$$\left| \int u^2 \left[\varphi_{c_0, p, \mu_p}^p \phi_K'(\tilde{x}) - (\varphi_{c_0, p, \mu_p}^p)' \phi_K(\tilde{x}) \right] \right| \leq \|u\|_{L^2}^2 \sup_{x \in \mathbb{R}} (e^{-p\sqrt{c_0}|x|} e^{\tilde{x}/K}) = \|u\|_{L^2}^2 e^{(-\frac{c_0}{2}(t_0-t) - x_0)/K}. \quad (3.21)$$

Finally, by adding (3.18), (3.19) and (3.21), integrating between t and t_0 , we obtain for K chosen correctly:

$$\begin{aligned} \int u^2(x, t_0) \phi_K(x - x_0) dx + \int_t^{t_0} \int_{\mathbb{R}} [\mu_p (\partial_x^2 u)^2 + (\partial_x u)^2 + u^2](x, s) \phi_K'(\tilde{x}) dx ds \\ \lesssim \sup_{t \in \mathbb{R}} \|u(t)\|_{L^2}^2 e^{-x_0/K} + \int_{\mathbb{R}} u^2(x, t) \phi_K(x - \frac{c_0}{2}(t_0 - t) - x_0) dx. \end{aligned} \quad (3.22)$$

This prove in particular (3.7).

Step 2: Asymptotic limit of the L^2 -localized energy.

We claim that:

$$\lim_{t \rightarrow -\infty} I_{x_0, t_0}^{(0)}(t) = 0. \quad (3.23)$$

Proof of (3.23). Let $\varepsilon > 0$, since $u(t, x)$ is L^2 -compact, there exists $R_\varepsilon > 0$ such that:

$$\int_{|x| > R_\varepsilon} u^2(t, x) \leq \varepsilon. \quad (3.24)$$

Then we deduce that:

$$\int_{\mathbb{R}} u^2(t, x) \phi_K(\tilde{x}) = \int_{x > R_\varepsilon} u^2(t, x) \phi_K(\tilde{x}) + \int_{x < R_\varepsilon} u^2(t, x) \phi_K(\tilde{x}) \leq \varepsilon + A. \quad (3.25)$$

For the convergence of A we use that: $\|u(t)\|_{L^2} \leq C$, $\phi_K(x)$ is increasing, and $\phi_K(-\infty) = 0$:

$$A \leq \|u\|^2 \phi_K(R_\varepsilon - \frac{c_0}{2}(t_0 - t) - x_0) \rightarrow 0 \text{ as } t \rightarrow -\infty. \quad (3.26)$$

Moreover, passing to the limit in (3.22), we deduce that:

$$I_{x_0, t_0}^{(0)}(t_0) + \int_{-\infty}^{t_0} \int [\mu_p (\partial_x^2 u)^2 + (\partial_x u)^2 + u^2](t, x) \phi_K'(\tilde{x}) dx dt \lesssim \sup_{t \in \mathbb{R}} \|u(t)\|_{L^2}^2 e^{-x_0/K}. \quad (3.27)$$

Step 3: Monotonicity formula at any order.

For $x_0 > 0$, $t_0 \geq t$ and $k \in \mathbb{N}$, we claim that:

$$\int (\partial_x^k u)^2(t_0, x) \phi_K(x - x_0) dx + \int_{-\infty}^{t_0} \int [\mu_p (\partial_x^{k+2} u)^2 + (\partial_x^{k+1} u)^2 + (\partial_x^k u)^2](t, x) \phi_K'(\tilde{x}) dx dt \lesssim \sup_{t \in \mathbb{R}} \|u(t)\|_{H^k}^2 e^{-x_0/K}, \quad (3.28)$$

Proof of (3.28). We proceed by induction on k . From Step 1 and Step 2, the estimate (3.28) holds for $k = 0$. Now, we assume that (3.28) is true for $1 \leq j \leq k - 1$. By repeating the computations done on Step 1, we get:

$$\begin{aligned} \frac{d}{dt} I_{x_0, t_0}^{(k)}(t) &= -5\mu_p \int (\partial_x^{k+2} u)^2 \phi_K'(\tilde{x}) - 3 \int (\partial_x^{k+1} u)^2 \phi_K'(\tilde{x}) - \frac{c_0}{2} \int (\partial_x^k u)^2 \phi_K'(\tilde{x}) \\ &\quad + 5\mu_p \int (\partial_x^{k+1} u)^2 \phi_K'''(\tilde{x}) - \mu_p \int (\partial_x^k u)^2 \phi_K^{(5)}(\tilde{x}) + \int (\partial_x^k u)^2 \phi_K''''(\tilde{x}) \\ &\quad - 2 \int \partial_x^{k+1} (\varphi_{c_0, p, \mu_p}^p u) (\partial_x^k u) \phi_K(\tilde{x}). \end{aligned} \quad (3.29)$$

By applying the similar arguments as for the estimate (3.19), we have:

$$\begin{aligned} & \left| 5\mu_p \int (\partial_x^{k+1} u)^2 \phi_K'''(\tilde{x}) - \mu_p \int (\partial_x^k u)^2 \phi_K^{(5)}(\tilde{x}) + \int (\partial_x^k u)^2 \phi_K'''(\tilde{x}) \right| \\ & \leq \max\left(\frac{5\mu_p}{K^2}, \frac{\mu_p}{K^4} + \frac{1}{K^2}\right) \int ((\partial_x^{k+1} u)^2 + (\partial_x^k u)^2) \phi_K'(\tilde{x}). \end{aligned} \quad (3.30)$$

Let us estimate the last term in (3.29). Applying the Leibniz formula on $\partial_x^{k+1}(\varphi_{c_0, p, \mu_p}^p u)$, it holds:

$$\begin{aligned} -2 \int \partial_x^{k+1}(\varphi_{c_0, p, \mu_p}^p u)(\partial_x^k u) \phi_K(\tilde{x}) &= \int (\partial_x^k u)^2 [-(2k+1)(\varphi_{c_0, p, \mu_p}^p)' \phi_K(\tilde{x}) + \varphi_{c_0, p, \mu_p}^p \phi_K'(\tilde{x})] \\ &\quad - 2 \sum_{j=0}^{k-1} C_{k+1}^j \int (\partial_x^k u)(\partial_x^j u)(\varphi_{c_0, p, \mu_p}^p)^{(k-j+1)} \phi_K(\tilde{x}). \end{aligned} \quad (3.31)$$

Arguing as for the estimate (3.21), we get

$$\left| \int (\partial_x^k u)^2 [-(2k+1)(\varphi_{c_0, p, \mu_p}^p)' \phi_K(\tilde{x}) + \varphi_{c_0, p, \mu_p}^p \phi_K'(\tilde{x})] \right| \lesssim \|\partial_x^k u\|_{L^2}^2 e^{(-\frac{c_0}{2}(t_0-t)-x_0)/K}. \quad (3.32)$$

Similarly, with the last term we apply the Young's inequality and we argue as estimate (3.21):

$$\left| 2 \sum_{j=2}^{k+1} C_{k+1}^j \int (\partial_x^k u)(\partial_x^{k-j+1} u)(\varphi_{c_0, p, \mu_p}^p)^{(j)} \phi_K(\tilde{x}) \right| \lesssim \|u\|_{H^k}^2 e^{(-\frac{c_0}{2}(t_0-t)-x_0)/K}. \quad (3.33)$$

Therefore, by integrating (3.29) between on t and t_0 , and combining the estimates (3.30), (3.32) and (3.33), we obtain

$$I_{x_0, t_0}^{(k)}(t_0) + \int_t^{t_0} \int [\mu_p (\partial_x^{k+2} u)^2 + (\partial_x^{k+1} u)^2 + (\partial_x^k u)^2] \phi_K'(\tilde{x}) \leq \sup_{t \in \mathbb{R}} \|u\|_{H^k}^2 e^{-x_0/K} + I_{x_0, t_0}^{(k)}(t). \quad (3.34)$$

Letting t tends to $-\infty$ we get:

$$I_{x_0, t_0}^{(k)}(t_0) + \int_{-\infty}^{t_0} \int [\mu_p (\partial_x^{k+2} u)^2 + (\partial_x^{k+1} u)^2 + (\partial_x^k u)^2] \phi_K'(\tilde{x}) \leq \sup_{t \in \mathbb{R}} \|u\|_{H^k}^2 e^{-x_0/K} + \liminf_{t \rightarrow -\infty} I_{x_0, t_0}^{(k)}(t). \quad (3.35)$$

Let us proof that $I_{x_0, t_0}^{(k)}(t) \rightarrow 0$ as $t \rightarrow -\infty$. From the induction hypothesis with $j = k - 1$, we know that

$$\int_{-\infty}^{t_0} \int (\partial_x^k u)^2 \phi_K'(\tilde{x}) \lesssim e^{-x_0/K}. \quad (3.36)$$

Then

$$\int_{-\infty}^{t_0} \int_{x < \frac{c_0}{2}(t_0-t)+x_0} (\partial_x^k u)^2 e^{(x - \frac{c_0}{2}(t_0-t))/K} \lesssim 1, \quad (3.37)$$

since $e^{\tilde{x}/K} \lesssim \phi_K'(\tilde{x})$ for $\tilde{x} < 0$. Thus, passing to the limit as x_0 tends to $+\infty$ in (3.37) and multiplying by $e^{-x_0/K}$, we get:

$$\int_{-\infty}^{t_0} \int (\partial_x^k u)^2 e^{\tilde{x}/K} \lesssim e^{-x_0/K}, \quad (3.38)$$

and then, since $\phi(\tilde{x}) \lesssim e^{\tilde{x}/K}$ for all $x \in \mathbb{R}$,

$$\int_{-\infty}^{t_0} \int (\partial_x^k u)^2 \phi_K(\tilde{x}) \lesssim e^{-x_0/K}. \quad (3.39)$$

Thus, we infer that

$$\liminf_{t \rightarrow -\infty} \int (\partial_x^k u)^2 \phi_K(\tilde{x}) = 0. \quad (3.40)$$

Combining (3.35) and (3.40), we get the claim (3.28).

Step 4: Smoothness and exponential decay.

For all $k \in \mathbb{N}$, we claim that:

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^{k+2}} \leq C \quad (3.41)$$

and

$$\int (\partial_x^k u)^2(t_0, x) e^{|x|/K} \leq C. \quad (3.42)$$

Proof of (3.42) assuming (3.41). From Step 3, we know that:

$$\int (\partial_x^k u)^2(t_0, x) \phi_K(x - x_0) \lesssim e^{-x_0/K}, \quad (3.43)$$

and then

$$\int_{x < x_0} (\partial_x^k u)^2(t_0, x) e^{x/K} \lesssim 1, \quad (3.44)$$

since $e^{(x-x_0)/K} \lesssim \phi_K(x - x_0)$ for $x < x_0$. Now, passing to the limit as x_0 tends to $+\infty$ in (3.44), we get the exponential decay property by the right:

$$\int (\partial_x^k u)^2(t_0, x) e^{x/K} \lesssim 1. \quad (3.45)$$

For the left exponential decay property, one can remark that $u(-t, -x)$ satisfies (3.1) and (3.2). Then, repeating the same analysis with $u(-t, -x)$, we get the claim (3.42). In particular, applying the Gagliardo-Nirenberg inequality, we infer that for all $k \in \mathbb{N}$ and $t_0 \in \mathbb{R}$:

$$\|(\partial_x^k u)(t_0) e^{|\cdot|/2K}\|_{L^\infty} \leq C_G \|(\partial_x^k u)(t_0) e^{|\cdot|/2K}\|_{L^2}^{1/2} \left(\|(\partial_x^k u)(t_0) e^{|\cdot|/2K}\|_{L^2} + \|(\partial_x^{k+1} u)(t_0) e^{|\cdot|/2K}\|_{L^2} \right)^{1/2} \lesssim 1, \quad (3.46)$$

and this prove (3.6).

Let us prove (3.41). Assume that $k = 1$. From (3.28) we know that for all $t_0 \in \mathbb{R}$:

$$\int_{t_0-1}^{t_0} \int (\partial_x^3 u)^2(t, x) \phi'_K(\tilde{x}) \lesssim \sup_{t \in \mathbb{R}} \|u(t)\|_{H^1}^2 e^{-x_0/K}. \quad (3.47)$$

Now arguing as from the estimate (3.46), and using the invariance of the problem by the transformation: $x \mapsto -x$ and $t \mapsto -t$, we get:

$$\int_{t_0-1}^{t_0} \int (\partial_x^3 u)^2(t, x) e^{|x|/K} \leq \sup_{t \in \mathbb{R}} \|u(t)\|_{H^1}^2 \leq C. \quad (3.48)$$

This implies that there exists $t_1 \in [t_0 - 1, t_0]$ such that:

$$\int (\partial_x^3 u)^2(t_1, x) \leq C. \quad (3.49)$$

On the other hand, using the equation of $u(t, x)$, one can easily check that:

$$\frac{1}{2} \frac{d}{dt} \int (\partial_x^3 u)^2 = \int \partial_x^3 (\varphi_{c_0, p, \mu_p}^p u) (\partial_x^4 u). \quad (3.50)$$

We claim that:

$$\frac{1}{2} \frac{d}{dt} \int (\partial_x^3 u)^2 \lesssim \int (\partial_x^3 u)^2 + \sup_{t \in \mathbb{R}} \|u(t)\|_{H^2}^2. \quad (3.51)$$

Indeed, applying the Leibniz formula, we have:

$$\partial_x^3(\varphi_{c_0,p,\mu_p}^p u) = \sum_{k=0}^3 C_3^k(\varphi_{c_0,p,\mu_p}^p)^{(k)}(\partial_x^{3-k}u). \quad (3.52)$$

Let us explain the estimate of the first term:

$$\int \varphi_{c_0,p,\mu_p}^p(\partial_x^3u)(\partial_x^4u) = -\frac{1}{2} \int (\varphi_{c_0,p,\mu_p}^p)'(\partial_x^3u)^2 \lesssim \int (\partial_x^3u)^2. \quad (3.53)$$

By Young's inequality the estimate of the last term is given by:

$$\int (\varphi_{c_0,p,\mu_p})^{(3)}u(\partial_x^4u) = - \int (\varphi_{c_0,p,\mu_p})^{(4)}u(\partial_x^3u) - \int (\varphi_{c_0,p,\mu_p})^{(3)}(\partial_xu)(\partial_x^3u) \lesssim \int u^2 + \int (\partial_xu)^2 + \int (\partial_x^3u)^2. \quad (3.54)$$

Therefore, by integrating (3.51) between $[t_1, t_0]$ and using (3.49), we obtain that:

$$\int (\partial_x^3u)^2(t_0, x) \leq C. \quad (3.55)$$

The iteration of this method implies (3.41).

3.2 Proof of Theorem 3.1.

We follow the ideas of Martel [11] and we split the proof in four steps.

Step 1. Numerical computing of $\mathcal{K}_p = \langle \mathcal{L}_{1,p,\mu_p}^{-2}\varphi_{1,p,\mu_p}, \mathcal{L}_{1,p,\mu_p}^{-1}\varphi_{1,p,\mu_p} \rangle_{L^2([0,10])}$ for $p \in \{1, 2, 3\}$

We work with $c_0 = 1$ and $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$. We know that $\rho_{1,p,\mu_p} \in \mathcal{L}_{1,p,\mu_p}^{-1}\varphi_{1,p,\mu_p}$, and since \mathcal{L}_{1,p,μ_p} is self-ajoint on L^2 the value of \mathcal{K}_p does not depend on the choice of ρ_{1,p,μ_p} . We recall that ρ_{1,p,μ_p} is an even function since φ_{1,p,μ_p} is even, then $\rho_{1,p,\mu_p} \in (\text{Ker } \mathcal{L}_{1,p,\mu_p})^\perp$. Moreover $\langle \mathcal{L}_{1,p,\mu_p}^{-1}\rho_{1,p,\mu_p}, \chi_{1,p,\mu_p} \rangle_{L^2} = \lambda_{1,p,\mu_p}^{-2} \langle \chi_{1,p,\mu_p}, \varphi_{1,p,\mu_p} \rangle_{L^2} \neq 0$, where the pair $(\lambda_{1,p,\mu_p}, \chi_{1,p,\mu_p})$ is such that $\mathcal{L}_{1,p,\mu_p}\chi_{1,p,\mu_p} = \lambda_{1,p,\mu_p}\chi_{1,p,\mu_p}$ with $\lambda_{1,p,\mu_p} < 0$ (see Lemma 2.1). Recall also that ρ_{1,p,μ_p} is a negative direction: $\langle \mathcal{L}_{1,p,\mu_p}\rho_{1,p,\mu_p}, \rho_{1,p,\mu_p} \rangle_{L^2} = \langle \rho_{1,p,\mu_p}, \varphi_{1,p,\mu_p} \rangle_{L^2} < 0$ for $p \in \{1, 2, 3, 4\}$ (see [7] Subsection 2.4).

It is well known that if $K_p < 0$, then for all $v \in H^2$ satisfying the orthogonalities: $\langle v, \varphi'_{1,p,\mu_p} \rangle_{L^2} = \langle v, \rho_{1,p,\mu_p} \rangle_{L^2} = 0$, there exists $C > 0$ such that: $\langle \mathcal{L}_{1,p,\mu_p}v, v \rangle_{L^2} \geq C\|v\|_{H^2}^2$. In the proof of the Theorem 3.1 we will use this coercivity.

Following exactly the same numerical scheme described in [7] (see Subsection 2.4), we compute that: $\mathcal{K}_1 \approx -6.4912 < 0$ (see Fig.1a-1b), $\mathcal{K}_2 \approx -0.0713 < 0$ (see Fig.1c-1d) and $\mathcal{K}_3 \approx 0.1090 > 0$ (see Fig.1e-1f).

Step 2. Linear dual problem related to (3.1)

We set $\tilde{v} = \mathcal{L}_{c_0,p,\mu_p}u$. Since $\varphi'_{c_0,p,\mu_p} \in \text{Ker}\mathcal{L}_{c_0,p,\mu_p}$, we get the first orthogonality condition: $\langle \tilde{v}, \varphi'_{c_0,p,\mu_p} \rangle_{L^2} = 0$, and from (3.1) we infer that \tilde{v} satisfies: $\partial_t\tilde{v} = \mathcal{L}_{c_0,p,\mu_p}(\partial_x\tilde{v})$ on $\mathbb{R} \times \mathbb{R}$. Next, we modify \tilde{v} to obtain the second orthogonality condition with the function ρ_{c,p,μ_p} :

$$v(t, x) = \tilde{v}(t, x) - \left(\frac{\int \tilde{v}(t, x)\rho_{c_0,p,\mu_p}(x)}{\int \varphi_{c_0,p,\mu_p}(x)\rho_{c_0,p,\mu_p}(x)} \right) \varphi_{c_0,p,\mu_p}(x). \quad (3.56)$$

One can see that $\langle v, \varphi'_{c_0,p,\mu_p} \rangle_{L^2} = \langle v, \rho_{c_0,p,\mu_p} \rangle_{L^2} = 0$ and satisfies for $(t, x) \in \mathbb{R} \times \mathbb{R}$:

$$\partial_tv = \mathcal{L}_{c_0,p,\mu_p}(\partial_xv) + \delta(t)\varphi_{c_0,p,\mu_p}, \quad (3.57)$$

with

$$\delta(t) = -\frac{1}{\int \varphi_{c_0,p,\mu_p}(x)\rho_{c_0,p,\mu_p}(x)} \frac{d}{dt} \int \tilde{v}(t, x)\rho_{c_0,p,\mu_p}(x), \quad (3.58)$$

since $\mathcal{L}_{c_0,p,\mu_p}(\partial_x v) = \mathcal{L}_{c_0,p,\mu_p}(\partial_x \tilde{v})$. Now, substituting $\partial_t \tilde{v}$ by $\mathcal{L}_{c_0,p,\mu_p}(\partial_x \tilde{v})$ in (3.58), using that $\mathcal{L}_{c_0,p,\mu_p} \rho_{c_0,p,\mu_p} = \varphi_{c_0,p,\mu_p}$, doing intergration by parts, and from the fact \tilde{v} is orthogonal to φ'_{c_0,p,μ_p} in L^2 , we obtain that $\delta(t) = 0$. This justifies our choice of orthogonality condition between v and ρ_{c_0,p,μ_p} (as Côte and al.). Therefore, v is a solution of the following linear dual problem:

$$\partial_t v = \mathcal{L}_{c_0,p,\mu_p}(\partial_x v), \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (3.59)$$

Step 3. Virial Type Identity related to equation (3.59)

We compute a Virial Type Identity for the linear dual problem (3.59). By multiplying (3.59) with the function $v(t, x)x$ and integrating on \mathbb{R} , we obtain

$$\frac{1}{2} \frac{d}{dt} \int v^2 x = \int (\partial_t v) v x = \int \mathcal{L}_{c_0,p,\mu_p}(\partial_x v) v x. \quad (3.60)$$

Using the definition of $\mathcal{L}_{c_0,p,\mu_p}$, it holds

$$\begin{aligned} \int \mathcal{L}_{c_0,p,\mu_p}(\partial_x v) v x &= \mu_p \int (\partial_x^5 v) v x - \int (\partial_x^3 v) v x + c_0 \int (\partial_x v) v x - \int \varphi_{c_0,p,\mu_p}^p(\partial_x v) v x \\ &= I + J + K + L. \end{aligned} \quad (3.61)$$

Now, with the aid of integration by parts, I give us

$$I = -\mu_p \int (\partial_x^4 v) \partial_x (v x) = -\mu_p \int (\partial_x^4 v) v - \mu_p \int (\partial_x^4 v) (\partial_x v) x = I_1 + I_2, \quad (3.62)$$

with

$$I_1 = \mu_p \int (\partial_x^3 v) (\partial_x v) = -\mu_p \int (\partial_x^2 v)^2 \quad (3.63)$$

and

$$\begin{aligned} I_2 &= \mu_p \int (\partial_x^3 v) \partial_x [(\partial_x v) x] = \mu_p \int (\partial_x^3 v) (\partial_x^2 v) x + \mu_p \int (\partial_x^3 v) (\partial_x v) \\ &= \frac{\mu_p}{2} \int \partial_x [(\partial_x^2 v)^2] x - \mu_p \int (\partial_x^2 v)^2 = -\frac{3\mu_p}{2} \int (\partial_x^2 v)^2. \end{aligned} \quad (3.64)$$

Summing (3.63) and (3.64), we deduce that

$$I = -\frac{5\mu_p}{2} \int (\partial_x^2 v)^2. \quad (3.65)$$

We compute J :

$$\begin{aligned} J &= \int (\partial_x^2 v) \partial_x (v x) = \int (\partial_x^2 v) v + \int (\partial_x^2 v) (\partial_x v) x \\ &= -\int (\partial_x v)^2 + \frac{1}{2} \int \partial_x [(\partial_x v)^2] x = -\frac{3}{2} \int (\partial_x v)^2. \end{aligned} \quad (3.66)$$

Finally, we compute K and L :

$$K = \frac{c_0}{2} \int \partial_x (v^2) x = -\frac{1}{2} \int v^2 \quad (3.67)$$

and

$$L = -\frac{1}{2} \int \partial_x (v^2) \varphi_{c_0,p,\mu_p}^p x = \frac{1}{2} \int v^2 \varphi_{c_0,p,\mu_p}^p + \frac{p}{2} \int v^2 \varphi_{c_0,p,\mu_p}^{p-1} \varphi'_{c_0,p,\mu_p} x. \quad (3.68)$$

Thus, summing (3.65)-(3.68), we obtain the following identity:

$$-\frac{d}{dt} \int v^2 x = 5\mu_p \int (\partial_x^2 v)^2 + 3 \int (\partial_x v)^2 + c_0 \int v^2 - \int v^2 \varphi_{c_0,p,\mu_p}^p - p \int v^2 \varphi_{c_0,p,\mu_p}^{p-1} \varphi'_{c_0,p,\mu_p} x. \quad (3.69)$$

Step 4. End of the proof of Theorem 3.1.

The goal is to prove that $v(t, x) = \mathcal{L}_{c_0, p, \mu_p} u(t, x) = 0$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, and then we will deduce that $u(t, x) = a_1 \varphi'_{c_0, p, \mu_p}(x)$, with $a_1 \in \mathbb{R}$, since $\text{Ker } \mathcal{L}_{c_0, p, \mu_p} = \{0, \varphi'_{c_0, p, \mu_p}\}$. The definition of $\mathcal{L}_{c_0, p, \mu_p}$ permit us to rewrite (3.69) as:

$$-\frac{d}{dt} \int v^2 x = \langle \mathcal{L}_{c_0, p, \mu_p} v, v \rangle_{L^2} + 4\mu_p \int (\partial_x^2 v)^2 + 2 \int (\partial_x v)^2 - p \int v^2 \varphi_{c_0, p, \mu_p}^{p-1} \varphi'_{c_0, p, \mu_p} x. \quad (3.70)$$

With the last term we deal as follow, let us set $\varphi_{c_0, p, \mu_p}(x) = \varphi_{1, p, \mu_p}(x) + (\varphi_{c_0, p, \mu_p}(x) - \varphi_{1, p, \mu_p}(x)) = \varphi_{c_0, p, \mu_p}(x) + \gamma(x)$. For c_0 close to 1, we have $\sup_{x \in \mathbb{R}} (|\gamma(x)| + |\gamma'(x)|) \leq \gamma$, with $0 < \gamma \ll 1$, and $|\gamma(x)| + |\gamma'(x)| \lesssim e^{-|x|}$ for all $x \in \mathbb{R}$. Then we deduce the following estimate:

$$-p \int v^2 \varphi_{c_0, p, \mu_p}^{p-1} \varphi'_{c_0, p, \mu_p} x \geq -p \int v^2 \varphi_{1, p, \mu_p}^{p-1} \varphi'_{1, p, \mu_p} x - \gamma \int v^2. \quad (3.71)$$

Since $\varphi_{c_0, p, \mu_p}(x) > 0$ and $-\varphi'_{c_0, p, \mu_p}(x)x > 0$ for all $x \in \mathbb{R}$, we get

$$-p \int v^2 \varphi_{c_0, p, \mu_p}^{p-1} \varphi'_{c_0, p, \mu_p} x \geq -\gamma \int v^2. \quad (3.72)$$

The identity (3.70) becomes

$$-\frac{d}{dt} \int v^2 x \geq \langle \mathcal{L}_{c_0, p, \mu_p} v, v \rangle_{L^2} + 4\mu_p \|\partial_x^2 v\|_{L^2}^2 + 2\|\partial_x v\|_{L^2}^2 - \gamma \|v\|_{L^2}^2. \quad (3.73)$$

Recall that $\langle v, \varphi'_{c_0, p, \mu_p} \rangle_{L^2} = \langle v, \rho_{c_0, p, \mu_p} \rangle_{L^2} = 0$ (by Step 2). Now, we claim that v still almost ortogonal to φ_{1, p, μ_p} and ρ_{1, p, μ_p} . Indeed, applying the Cauchy-Schwarz inequality, it holds

$$|\langle v, \varphi_{1, p, \mu_p} \rangle| \leq \gamma \|v\|_{L^2} \quad \text{and} \quad |\langle v, \rho_{1, p, \mu_p} \rangle| = \left| \langle v, \frac{d}{dc}|_{c=c_0} \varphi_{c, p, \mu_p} - \frac{d}{dc}|_{c=1} \varphi_{c, p, \mu_p} \rangle_{L^2} \right| \leq \gamma \|v\|_{L^2}. \quad (3.74)$$

Since v is almost orthogonal to φ'_{1, p, μ_p} and ρ_{1, p, μ_p} , and since $\mathcal{K}_p < 0$ for $p = 1, 2$ (by Step 1), and $\mathcal{L}_{1, p, \mu_p}$ possesses the properties (P1)-(P2) (see Lemma Spectral), arguing as [7] (see Subsection 2.3), there exists $C_0 > 0$ such that

$$\langle \mathcal{L}_{1, p, \mu_p} v, v \rangle_{L^2} \geq C_0 \|v\|_{H^2}^2. \quad (3.75)$$

It follows immediatly that for c_0 close to 1,

$$\begin{aligned} \langle \mathcal{L}_{c_0, p, \mu_p} v, v \rangle_{L^2} &= \langle \mathcal{L}_{1, p, \mu_p} v, v \rangle_{L^2} + (c_0 - 1) \|v\|_{L^2}^2 - \left\langle v^2, \varphi_{c_0, p, \mu_p}^p - \varphi_{1, p, \mu_p}^p \right\rangle_{L^2} \\ &\geq C_0 \|v\|_{H^2}^2 + (c_0 - 1 - \gamma) \|v\|_{L^2}^2 \\ &\geq \frac{C_0}{2} \|v\|_{H^2}^2. \end{aligned} \quad (3.76)$$

Then combining (3.73) and (3.76), we get the following inequality

$$-\frac{d}{dt} \int v^2 x \geq C_1 \|v\|_{H^2}^2. \quad (3.77)$$

Lemma 3.1, the definition of $\mathcal{L}_{c_0, p, \mu_p}$, the smoothness and exponential decay properties of φ_{c_0, p, μ_p} , imply that $v \in C^\infty(\mathbb{R} \times \mathbb{R})$, and there exists $\sigma_0 > 0$ such that

$$|v(t, x)| \lesssim e^{-\sigma_0 |x|}, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (3.78)$$

For all $t \in \mathbb{R}$, we set

$$\mathcal{J}(t) = \int v^2(t, x) x. \quad (3.79)$$

Thanks to (3.78), one can check that for all $t \in \mathbb{R}$:

$$|\mathcal{J}(t)| \leq \int v^2(t, x) |x| \lesssim \int_{\mathbb{R}} e^{-2\sigma|x|} |x| = \frac{1}{2\sigma_1^2}. \quad (3.80)$$

Now, integrating (3.77) with respect to time and using (3.80), we obtain

$$\int_{-\infty}^{+\infty} \|v(t)\|_{H^2}^2 dt \leq C_1^{-1} (\mathcal{J}(-\infty) - \mathcal{J}(+\infty)) < +\infty. \quad (3.81)$$

It follows that for a time sequence $t_n \rightarrow +\infty$, we have $v(t_n) \rightarrow 0$ in $H^2(\mathbb{R})$. In particular, we have

$$|\mathcal{J}(t_n)| \leq \|v(t_n)x\|_{L^2} \|v(t_n)\|_{L^2} \lesssim \frac{1}{\sqrt{2\sigma_1^3}} \|v(t_n)\|_{L^2} \rightarrow 0 \text{ as } t_n \rightarrow +\infty. \quad (3.82)$$

We infer that $\mathcal{J}(+\infty) = 0$. Similarly, $\mathcal{J}(-\infty) = 0$. Thus (3.81) becomes

$$\int_{-\infty}^{+\infty} \|v(t)\|_{H^2}^2 dt = 0,$$

which implies that $v(t, x) \equiv 0$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$. Then $\mathcal{L}_{c_0, p, \mu_p} u(t, x) = \tilde{v}(t, x) = \beta(t) \varphi_{c_0, p, \mu_p}(x)$, and using that $\mathcal{L}_{c, p, \mu_p} \rho_{c, p, \mu_p}(x) = \varphi_{c, p, \mu_p}(x)$ and $\text{Ker} \mathcal{L}_{c_0, p, \mu_p} = \langle \varphi'_{c_0, p, \mu_p} \rangle$, we get $u(t, x) = a(t) \rho_{c_0, p, \mu_p}(x) + b(t) \varphi'_{c_0, p, \mu_p}(x)$. One can easily compute that $\partial_x \mathcal{L}_{c_0, p, \mu_p} u = a(t) \varphi'_{c_0, p, \mu_p}(x)$ and $\partial_t u = a'(t) \rho_{c_0, p, \mu_p}(x) + b'(t) \varphi'_{c_0, p, \mu_p}(x)$. Now, using that $\partial_t u = \partial_x (\mathcal{L}_{c_0, p, \mu_p} u)$, we obtain $a'(t) = 0$ and $b'(t) = a(t)$, and this implies that $a(t) = a_0$ and $b(t) = a_0 t + a_1$. Finally, since $\sup_{t \in \mathbb{R}} \|u(t)\|_{H^2} \leq C$, then $a_0 = 0$ and we deduce that $u(t, x) \equiv a_1 \varphi'_{c_0, p, \mu_p}(x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$.

Proof of Corollary 3.1. The corollary 3.1 follows since we have: $\langle \mathcal{L}_{1, 1, 0}^{-1} \psi_{1, 1}, \psi_{1, 1} \rangle_{L^2([0, 10])} = -7.5784 < 0$ (see Fig. 2a-2b). Note that $\langle \mathcal{L}_{1, 2, 0}^{-1} \psi_{1, 2}, \psi_{1, 2} \rangle_{L^2([0, 10])} = 0.521 > 0$ (see Fig. 2c-2d).

4 Linear Liouville property with the family of solitons of gKW which converge to the soliton of gKdV

Theorem 4.1 (Linear Liouville property with $\psi_{c, p, \mu}$ for $p > 1$ and $c > 0$). *The Theorem 3.1 holds with the family of solitons $\{\psi_{c, p, \mu} : 0 < \mu \ll 1\}$ with $0 < \mu < \mu_0 \ll 1$, for any speed $c > 0$, and for all nonlinearity power $p > 1$.*

Proof. The proof of the Theorem 4.1 is done by contradiction following the ideas of Kenig-Martel-Robbiano [9] and assuming that the Linear Liouville property holds for the solitons of gKdV. We choose $c = 1$, and we assume that $\mu_n \rightarrow 0^+$ as $n \rightarrow +\infty$, and the functions $w_n(t, x)$ satisfy:

$$\partial_t w_n = \partial_x (\mathcal{L}_{1, p, \mu_n} w_n), \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (4.1)$$

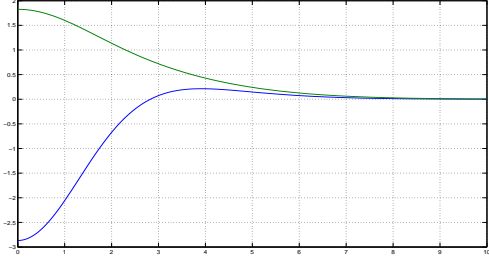
$$w_n(t, x) \neq a_n(t) \psi'_{1, p, \mu_n}(x), \quad \sup_{t \in \mathbb{R}} \|w_n(t)\|_{H^2} \leq C_n, \quad (4.2)$$

and

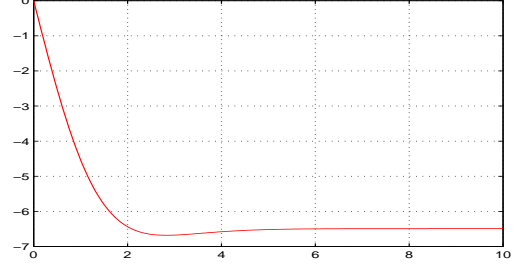
$$\forall \varepsilon > 0, \exists R_n(\varepsilon) > 0, \forall t \in \mathbb{R}, \int_{|x| > R_n(\varepsilon)} |w_n(t, x)|^2 dx \leq \varepsilon. \quad (4.3)$$

We start by constructing the functions $\tilde{w}_n(t, x)$ orthogonal to $\psi'_{1, p, \mu_n}(x)$ in L^2 . We set:

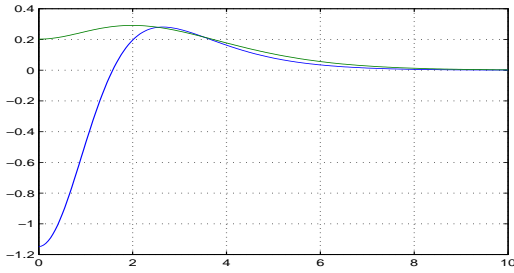
$$\tilde{w}_n(t, x) = w_n(t, x) - \frac{\int w_n(t, x) \psi'_{1, p, \mu_n}(x)}{\int (\psi'_{1, p, \mu_n}(x))^2} \psi'_{1, p, \mu_n}(x). \quad (4.4)$$



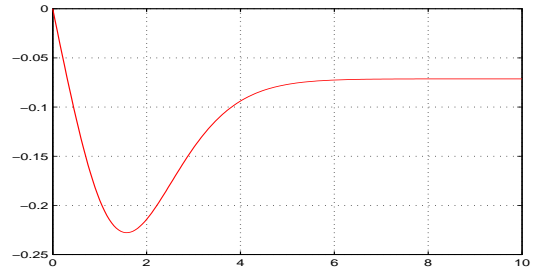
(a) $\rho_{1,1,\mu_1}(x)$ and $\mathcal{L}_{1,1,\mu_1}^{-1}\rho_{1,1,\mu_1}(x)$ profiles.



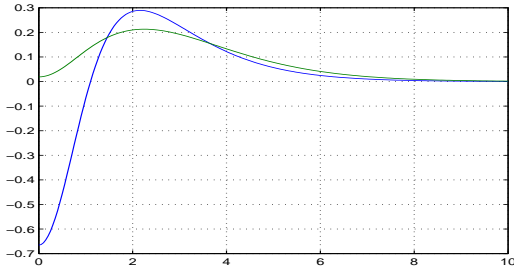
(b) $\int_0^x \mathcal{L}_{1,1,\mu_1}^{-1}\rho_{1,1,\mu_1}(x)(y)\rho_{1,1,\mu_1}(y)$ profile.



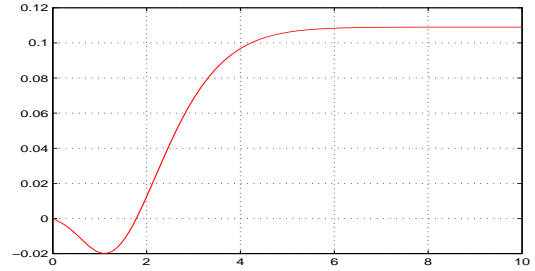
(c) $\rho_{1,2,\mu_2}(x)$ and $\mathcal{L}_{1,2,\mu_2}^{-1}\rho_{1,2,\mu_2}(x)$ profiles.



(d) $\int_0^x \mathcal{L}_{1,2,\mu_2}^{-1}\rho_{1,2,\mu_2}(y)\rho_{1,2,\mu_2}(y)$ profile.



(e) $\rho_{1,3,\mu_3}(x)$ and $\mathcal{L}_{1,3,\mu_3}^{-1}\rho_{1,3,\mu_3}(x)$ profiles.



(f) $\int_0^x \mathcal{L}_{1,3,\mu_3}^{-1}\rho_{1,3,\mu_3}(y)\rho_{1,3,\mu_3}(y)$ profile.

Figure 1: Variation of \mathcal{K}_p for $p \in \{1, 2, 3\}$ (with $c_0 = 1$ and $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$).

Using (4.1)-(4.3), and the fact that $(\varphi_{\mu_n})_{n \in \mathbb{N}}$ is bounded in $H^s(\mathbb{R})$ and decays exponentially, we deduce that:

$$\partial_t \tilde{w}_n = \partial_x (\mathcal{L}_{1,p,\mu_n} \tilde{w}_n) + \delta_n(t) \psi'_{1,p,\mu_n}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (4.5)$$

$$\tilde{w}_n(t, x) \neq 0, \quad \sup_{t \in \mathbb{R}} \|\tilde{w}_n(t)\|_{H^2} \leq C_n, \quad \int \tilde{w}_n(t) \psi'_{1,p,\mu_n} = 0, \quad (4.6)$$

$$\delta_n(t) = \frac{1}{\int (\psi'_{1,p,\mu_n})^2} \int \tilde{w}_n(t) \mathcal{L}_{1,p,\mu_n} (\psi''_{1,p,\mu_n}), \quad (4.7)$$

and

$$\forall \varepsilon > 0, \exists R_n(\varepsilon) > 0, \forall t \in \mathbb{R}, \int_{|x| > R_n(\varepsilon)} |\tilde{w}_n(t, x)|^2 dx \leq \varepsilon. \quad (4.8)$$

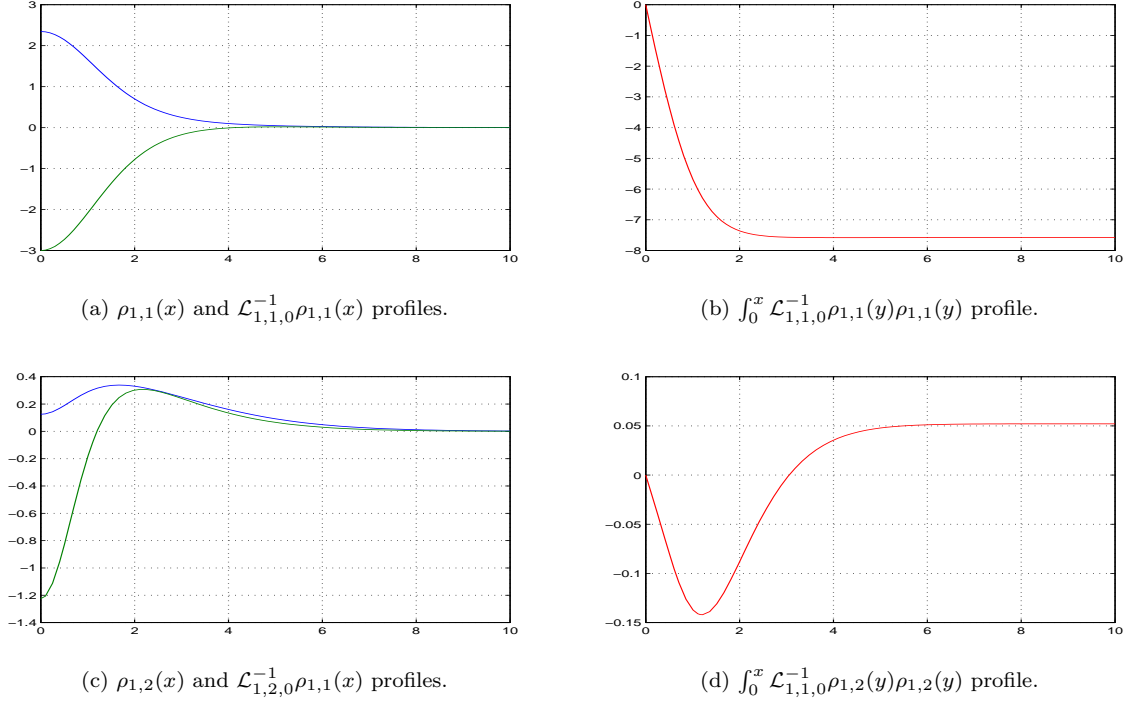


Figure 2: Sufficient condition of the Liouville linear property with $\psi_{1,p}$ for $p \in \{1, 2\}$.

Now, we claim that for all $t \in \mathbb{R}$, and $K > 0$ chosen correctly:

$$\sum_{k=0}^5 \int (\partial_x^k \tilde{w}_n)^2(t, x) e^{|x|/K} \lesssim \sup_{t \in \mathbb{R}} \|\tilde{w}_n(t)\|_{H^5}^2. \quad (4.9)$$

By the monotonicity property related to the Linear Liouville property, we know that (4.9) holds for $w_n(t, x)$. Then, It suffices to explain how we can estimate the term which contain $\delta_n(t)$. We have:

$$\begin{aligned} \delta_n(t) \int \varphi_{1,p,\mu_n}^{(6)}(x) (\partial_x^5 \omega_n)(t, x) \phi_K(\tilde{x}) &= -\delta_n(t) \int \partial_x^5 [\varphi_{1,p,\mu_n}^{(6)}(x) \phi_K(\tilde{x})] \tilde{w}_n(t, x) \\ &\lesssim |\delta_n(t)| \|\tilde{w}_n(t)\|_{L^2} \left(\int e^{-2|x|} e^{2\tilde{x}/K} \right)^{1/2} \lesssim \|\tilde{w}_n(t)\|_{L^2}^2 e^{(-\frac{c_0}{2}(t_0-t)-x_0)/K}. \end{aligned} \quad (4.10)$$

Then integrating between $[t, t_0]$ we obtain:

$$\int_t^{t_0} \delta_n(t) \int \varphi_{1,p,\mu_n}^{(6)}(x) (\partial_x^5 \omega_n)(t, x) \phi_K(\tilde{x}) \lesssim \sup_{t \in \mathbb{R}} \|\tilde{w}_n(t)\|_{L^2}^2 e^{-x_0/K}. \quad (4.11)$$

We set: $b_n = \sup_{t \in \mathbb{R}} \|\tilde{w}_n(t)\|_{H^5}$. Let t_n be such that $\|\tilde{w}_n(t_n)\|_{H^5} \geq \frac{1}{2} b_n$, and we define:

$$\bar{w}_n(t, x) = \frac{\tilde{w}_n(t_n + t, x)}{b_n}. \quad (4.12)$$

So that it holds:

$$\partial_t \bar{w}_n = \partial_x (\mathcal{L}_{1,p,\mu_n} \bar{w}_n) + \bar{\delta}_n(t) \psi'_{1,p,\mu_n}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (4.13)$$

$$\|\bar{w}_n(0)\|_{H^5} \geq \frac{1}{2}, \quad \sup_{t \in \mathbb{R}} \|\bar{w}_n(t)\|_{H^5} \leq 1, \quad \int \bar{w}_n(t) \psi'_{1,p,\mu_n} = 0, \quad (4.14)$$

$$\bar{\delta}_n(t) = \frac{1}{\int (\psi'_{1,p,\mu_n})^2} \int \bar{w}_n(t) \mathcal{L}_{1,p,\mu_n}(\psi''_{1,p,\mu_n}), \quad (4.15)$$

and

$$\forall x_0 > 0, \quad \forall t \in \mathbb{R}, \quad \sum_{k=0}^5 \int_{|x| > x_0} (\partial_x^k \bar{w}_n(t, x))^2 dx \lesssim e^{-x_0/K}. \quad (4.16)$$

Finally, we set:

$$\hat{w}_n(t, x) = \bar{w}_n(t, x) - \psi'_{1,p,\mu_n} \int_0^t \bar{\delta}_n(s) ds, \quad (4.17)$$

so that

$$\partial_t \hat{w}_n = \partial_x(\mathcal{L}_{1,p,\mu_n} \hat{w}_n) + \bar{\delta}_n(t) \psi'_{1,p,\mu_n}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (4.18)$$

$$\|\hat{w}_n(0)\|_{H^5} \geq \frac{1}{2}, \quad \|\hat{w}_n(0)\|_{H^5} \leq 1, \quad \int \hat{w}_n(0) \psi'_{1,p,\mu_n} = 0, \quad (4.19)$$

and

$$\forall x_0 > 0, \quad \sum_{k=0}^5 \int_{|x| > x_0} (\partial_x^k \hat{w}_n(0, x))^2 dx \lesssim e^{-x_0/K}. \quad (4.20)$$

Since $(\hat{w}_n(0))$ is bounded in H^5 , there exists $\hat{w}_0 \in H^5$, such that up to a subsequence:

$$\hat{w}_n(0) \rightharpoonup \hat{w}_0 \text{ in } H^5 \text{ as } n \rightarrow +\infty, \quad (4.21)$$

and

$$\hat{w}_n(0) \rightarrow \hat{w}_0 \text{ in } H_{loc}^4 \text{ as } n \rightarrow +\infty. \quad (4.22)$$

Then, combining (4.20) and (4.22), we infer that:

$$\hat{w}_n(0) \rightarrow \hat{w}_0 \text{ in } H^4 \text{ as } n \rightarrow +\infty. \quad (4.23)$$

Let $\hat{w} \in C([0, T], H^5(\mathbb{R}))$ the unique solution of the linearized gKdV equation around $\psi_{1,p}$:

$$\begin{cases} \partial_t \hat{w} = \partial_x(\mathcal{L}_{1,p,0} \hat{w}), & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ \hat{w}(0, x) = \hat{w}_0 \in H^5(\mathbb{R}). \end{cases} \quad (4.24)$$

We claim that $\forall t \in \mathbb{R}$:

$$\hat{w}_n(t) \rightarrow \hat{w}(t) \text{ in } H^2 \text{ as } n \rightarrow +\infty. \quad (4.25)$$

Since (4.18) is invariant by the transformation: $x \rightarrow -x$ and $t \rightarrow -t$, it suffice to prove (4.25) for $t > 0$. We define $\forall t \in \mathbb{R}$:

$$v_n(t, x) = \hat{w}_n(t, x) - \hat{w}(t, x). \quad (4.26)$$

Then $v(t, x)$ satisfies:

$$\partial_t v_n = \partial_x(\mathcal{L}_{1,p,\mu_n} v_n) + \mu_n \partial_x^5 \hat{w} - \partial_x[(\psi_{1,p,\mu_n}^p - \psi_{1,p}^p) \hat{w}], \quad (4.27)$$

and we have:

$$\hat{v}_n(0) \rightarrow 0 \text{ in } H^4 \text{ as } n \rightarrow +\infty \quad \text{and} \quad \sup_{t \in [0, T]} \|v_n(t)\|_{H^5} \leq C_T. \quad (4.28)$$

Let us compute $\frac{d}{dt} \|\hat{w}(t)\|_{H^2}^2$:

$$\frac{1}{2} \frac{d}{dt} \int (\partial_x^2 v)^2 = \int \partial_x^3(\mathcal{L}_{1,p,\mu_n} v) + \mu_n \int (\partial_x^7 \hat{w})(\partial_x^2 v_n) - \int \partial_x^3[(\psi_{1,p,\mu_n}^p - \psi_{1,p}^p) \hat{w}](\partial_x^2 v_n) = A + B + C. \quad (4.29)$$

Let us estimate C :

$$C = \int (\psi_{1,p,\mu_n}^p - \psi_{1,p}^p) \widehat{w} (\partial_x^5 v_n) \leq \|\widehat{w}\|_{L^\infty} \|\psi_{1,p,\mu_n}^p - \psi_{1,p}^p\|_{L^2} \|\partial_x^5 v_n\|_{L^2} \leq C_T \|\psi_{1,p,\mu_n}^p - \psi_{1,p}^p\|_{L^2}. \quad (4.30)$$

Let us estimate B :

$$B = -\mu_n \int (\partial_x^4 \widehat{w}) (\partial_x^5 v_n) \leq \mu_n \|\partial_x^4 \widehat{w}\|_{L^2} \|\partial_x^5 v_n\|_{L^2} \leq C_T \mu_n. \quad (4.31)$$

Let us estimate A :

$$A = - \int \partial_x^3 (\psi_{1,p,\mu_n}^p v_n) (\partial_x^2 v_n), \quad (4.32)$$

and recall that:

$$\partial_x^3 (\psi_{1,p,\mu_n}^p v_n) = \sum_{k=0}^3 C_3^k (\psi_{1,p,\mu_n}^p)^{(k)} (\partial_x^{3-k} v_n). \quad (4.33)$$

We have:

$$- \int \psi_{1,p,\mu_n} (\partial_x^3 v_n) (\partial_x^2 v_n) = \frac{1}{2} \int \psi'_{1,p,\mu_n} (\partial_x^2 v_n)^2 \lesssim \int (\partial_x^2 v_n)^2, \quad (4.34)$$

and applying Young's inequality:

$$\int \psi_{1,p,\mu_n}^{(3)} v_n (\partial_x^2 v_n) \lesssim \int v_n^2 + \int (\partial_x^2 v_n)^2. \quad (4.35)$$

Now combining (4.27)-(4.35) we get:

$$\frac{1}{2} \frac{d}{dt} \int (\partial_x^2 v_n)^2(t, x) \lesssim \int (\partial_x^2 v_n)^2(t, x) + C_T (\mu_n + \|\psi_{1,p,\mu_n}^p - \psi_{1,p}^p\|_{L^2}). \quad (4.36)$$

One can easily deduce that:

$$\frac{d}{dt} \|v_n(t)\|_{H^2}^2 \lesssim \|v_n(t)\|_{H^2}^2 + C_T (\mu_n + \|\psi_{1,p,\mu_n}^p - \psi_{1,p}^p\|_{L^2}). \quad (4.37)$$

Then integrating between $0 < t < T$ we obtain:

$$\|v_n(t)\|_{H^2}^2 \lesssim C_T (\|v_n(0)\|_{H^2}^2 + \mu_n + \|\psi_{1,p,\mu_n}^p - \psi_{1,p}^p\|_{L^2}) \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (4.38)$$

Moreover, using that: $\forall k \in \mathbb{N} \lim_{n \rightarrow +\infty} \|\psi_{1,p,\mu_n} - \psi_{1,p}\|_{H^k} = 0$, we get:

$$\bar{\delta}_n(t) \rightarrow \bar{\delta}(t) = \frac{1}{\int (\psi'_{1,p})^2} \int \widehat{w}(t) \mathcal{L}_{1,p,0} (\psi''_{1,p}). \quad (4.39)$$

Let us define: $\bar{w}(t, x) = \widehat{w}(t, x) + \psi'_{1,p}(x) \int_0^t \bar{\delta}(s) ds$. Then

$$\forall t \in \mathbb{R}, \quad \bar{w}_n(t) \rightarrow \bar{w}(t) \text{ in } H^2, \quad (4.40)$$

$$\partial_t \bar{w} = \partial_x (\mathcal{L}_{1,p,0} \bar{w}) + \bar{\delta} \psi'_{1,p}, \quad (4.41)$$

$$\bar{w}(0) \neq 0, \quad \int \bar{w}(t, x) \psi'_{1,p} = 0, \quad (4.42)$$

and

$$\forall x_0 > 0, \quad \forall t \in \mathbb{R}, \quad \sum_{k=0}^5 \int_{|x| > x_0} (\partial_x^k \bar{w}(t, x))^2 dx \lesssim e^{-x_0/K}. \quad (4.43)$$

But the existence of such a $\bar{w}(t, x)$ is a contradiction with the linear Liouville Property for the gKdV equation (See Theorem 1 in Martel [11]).

5 Nonlinear Liouville property with the family of solitons of gKW which converge to the soliton of gKdV

Theorem 5.1 (Nonlinear Liouville property with $\psi_{c,p,\mu}$ for $p > 1$ and $c > 0$). *Let $p > 1$, $c > 0$, and $0 < \mu < \mu_0 \ll 1$. Assume that $u(t, x)$ the H^2 global solution of the gKW equation which is close to the soliton $\psi_{c,p,\mu}$ at the initial time ($t = 0$):*

$$\|u_0 - \psi_{c,p,\mu}\|_{H^2} \leq \gamma, \text{ with } \gamma \ll 1, \text{ and } u_0(x) = u(0, x), \quad (5.1)$$

and assume that $u(t, x + \xi(t))$ is L^2 -compact:

$$\forall \varepsilon > 0 \exists R_\varepsilon > 0 \text{ such that } \forall t \in \mathbb{R} \int_{|x| > R_\varepsilon} |u(t, x + \xi(t))|^2 \leq \varepsilon, \quad (5.2)$$

then

$$u(t, x) = \psi_{c,p,\mu}(x - \xi(0) - c(0)t), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (5.3)$$

where the translation function $\xi(t)$ and the speed function $c(t)$ are given by Lemma 5.1 below.

5.1 Construction of modulation parameters I

In this section we recall the lemma that give us the existence of the modulation parameters.

Lemma 5.1 (Construction of modulation parameters). *Let $p > 1$, $c > 0$, and $0 < \mu < \mu_0 \ll 1$. Assume that $u(t, x)$ the H^2 global solution of the gKW equation which is close to the soliton $\psi_{c,p,\mu}$ at the initial time ($t = 0$):*

$$\|u_0 - \psi_{c,p,\mu}\|_{H^2} \leq \gamma, \text{ with } \gamma \ll 1, \text{ and } u_0(x) = u(0, x), \quad (5.4)$$

Then there exists $c(t) \in C^1(\mathbb{R})$ and $\xi(t) \in C^1(\mathbb{R})$ such:

$$\eta(t, x) = u(t, x + \xi(t)) - \psi_{c(t),p,\mu}(x), \quad (5.5)$$

satisfying for all $t \in \mathbb{R}$,

$$|\xi(0)| + |c(t) - c_0| + \|\eta(t)\|_{H^2} \leq K_0 \gamma, \quad (5.6)$$

$$\int \eta(t, x) \psi'_{c(t),p,\mu}(x) = \int \eta(t, x) \psi_{c(t),p,\mu}(x) = 0, \quad (5.7)$$

and

$$|c'(t)|^{1/2} + |\xi'(t) - c(t)| \leq K_0 \left(\int \eta^2(t, x) \psi'_{c,p,\mu}(x) \right)^{1/2} \leq K_0 \|\eta(t)\|_{L^2}, \quad (5.8)$$

where K_0 is a positive constant.

5.2 Monotonicity formula related to the nonlinear gKW equation I

Proposition 5.1 (Monotonicity formula I). *Let $p > 1$, $c > 0$, and $0 < \mu < \mu_0 \ll 1$. Assume that $u(t, x)$ is the H^2 global solution of the gKW equation which is close to the soliton $\psi_{c,p,\mu}$ at the initial time ($t = 0$):*

$$\|u_0 - \psi_{c,p,\mu}\|_{H^2} \leq \gamma, \text{ with } \gamma \ll 1, \text{ and } u_0(x) = u(0, x). \quad (5.9)$$

Let $x_0 > 0$, $K > 0$ chosen correctly, $t_0 > t$, and $\phi_K(x) = \frac{1}{2} + \frac{1}{2} \tanh(x/2K)$. We set: $\tilde{x} = x - \frac{c}{2}(t_0 - t) - x_0$, and we redefine:

$$I_{x_0, t_0}(t) = \int \eta^2(t, x) \phi_K(\tilde{x}) \text{ and } J_{x_0, t_0}(t) = \int \left(\frac{\mu}{2} (\partial_x^2 \eta)^2 + \frac{1}{2} (\partial_x \eta)^2 - R(\eta, \psi_{c,p,\mu}) \right)(t, x) \phi_K(\tilde{x}), \quad (5.10)$$

where

$$R(\eta, \psi_{c,p,\mu}) = \frac{1}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k \eta^k \psi_{c,p,\mu}^{p-k+2}. \quad (5.11)$$

Assume that $u(t, x)$ is the H^2 global solution of the gKW equation which is close to the soliton $\psi_{c,p,\mu}$ at the initial time ($t = 0$):

$$\|u_0 - \psi_{c,p,\mu}\|_{H^2} \leq \gamma, \text{ with } \gamma \ll 1, \text{ and } u_0(x) = u(0, x). \quad (5.12)$$

Then we have:

$$I_{x_0,t_0}(t_0) - I_{x_0,t_0}(t) \lesssim e^{-x_0/K} \text{ et } J_{x_0,t_0}(t_0) - J_{x_0,t_0}(t) \lesssim e^{-x_0/K}. \quad (5.13)$$

Moreover if we assume that $u(t, x + \xi(t))$ is L^2 -compact:

$$\forall \varepsilon > 0 \exists R_\varepsilon > 0 \text{ such that } \forall t \in \mathbb{R} \int_{|x| > R_\varepsilon} |u(t, x + \xi(t))|^2 \leq \varepsilon, \quad (5.14)$$

then we have:

$$\sum_{k=0}^2 \int (\partial_x^k \eta)^2(t_0, x) e^{|x|/K} + \int_{t_0-1}^{t_0} \int [\mu(\partial_x^4 \eta)^2 + \sum_{k=0}^3 (\partial_x^k \eta)^2](t, x) e^{|x|/K} dx dt \lesssim \sup_{t \in \mathbb{R}} \|\eta(t)\|_{H^2}^2, \quad (5.15)$$

where $\xi(t)$ is given by Lemma 5.1.

Proof. We present the proof in four steps.

Step 1: Monotonicity formula for the L^2 localized mass.

For all $x_0 > 0$, and $t_0 > t$, we claim that:

$$I_{x_0,t_0}(t_0) - I_{x_0,t_0}(t) \lesssim \sup_{t \in \mathbb{R}} \|\eta(t)\|_{L^2}^2 e^{-x_0/K}. \quad (5.16)$$

Proof. Recall that the equation of $\eta(t, x)$ is given by:

$$\partial_t \eta = \partial_x (\mathcal{L}_{c,p,\mu} \eta) + \partial_x R_1(\eta, \psi_{c,p,\mu}) + (\xi' - c) \partial_x (\eta + \psi_{c,p,\mu}) + c' \rho_{c,p,\mu}, \quad (5.17)$$

where

$$R_1(\eta, \psi_{c,p,\mu}) = -\frac{1}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k \eta^k \psi_{c,p,\mu}^{p-k+1}. \quad (5.18)$$

Now by differentiating I_{x_0,t_0} with respect to time:

$$\begin{aligned} \frac{d}{dt} I_{x_0,t_0}(t) &= 2 \int (\partial_t \eta) \eta \phi_K(\tilde{x}) + \frac{c}{2} \int \eta^2 \phi'_K(\tilde{x}) = 2 \int \partial_x (\mathcal{L}_{c,p,\mu} \eta) \eta \phi_K(\tilde{x}) + 2 \int \partial_x (R_1(\eta, \psi_{c,p,\mu})) \eta \phi_K(\tilde{x}) \\ &+ 2(\xi' - c) \int \partial_x (\eta + \psi_{c,p,\mu}) \eta \phi_K(\tilde{x}) + 2c' \int \rho_{c,p,\mu} \eta \phi_K(\tilde{x}) = A + B + C + D. \end{aligned} \quad (5.19)$$

The estimate of A is given by the monotonicity formula related to the linearized gKW equation:

$$A \lesssim -\mu \int (\partial_x^2 \eta)^2 \phi'_K(\tilde{x}) - \int (\partial_x \eta)^2 \phi'_K(\tilde{x}) - \int \eta^2 \phi'_K(\tilde{x}) + \|\eta\|_{L^2}^2 e^{(-\frac{\varepsilon}{2}(t_0-t)-x_0)/K}. \quad (5.20)$$

The estimate of B is the following:

$$\begin{aligned} B &= \frac{2}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k \int \eta^{k+1} \left[-\frac{1}{k+1} \partial_x (\psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x})) + \psi_{c,p,\mu}^{p-k+1} \phi'_K(\tilde{x}) \right] \\ &\lesssim \int \eta^3 \psi'_{c,p,\mu} \phi_K(\tilde{x}) + \int \eta^3 \phi'_K(\tilde{x}) \lesssim \sup_{x \in \mathbb{R}} (e^{-c|x|} e^{-\tilde{x}/K}) \|\eta\|_{H^1} \|\eta\|_{L^2}^2 + \|\eta\|_{H^1} \int \eta^2 \phi'_K(\tilde{x}) \\ &\lesssim \|\eta\|_{H^1} \left(\|\eta\|_{L^2}^2 e^{(-\frac{\varepsilon}{2}(t_0-t)-x_0)/K} + \int \eta^2 \phi'_K(\tilde{x}) \right). \end{aligned} \quad (5.21)$$

The estimate of C is:

$$\begin{aligned}
C &= -2(\xi' - c) \int \eta^2 \phi'_K(\tilde{x}) + 2(\xi' - c) \int \eta \psi'_{c,p,\mu} \phi_K(\tilde{x}) \\
&\lesssim \|\eta\|_{L^2} \int \eta^2 \phi'_K(\tilde{x}) + \|\eta\|_{L^2}^2 \left(\int e^{-2c|x|} e^{-2\tilde{x}/K} \right)^{1/2} \\
&\lesssim \|\eta\|_{L^2} \int \eta^2 \phi'_K(\tilde{x}) + \|\eta\|_{L^2}^2 e^{(-\frac{c}{2}(t_0-t)-x_0)/K}.
\end{aligned} \tag{5.22}$$

Since $\rho_{c,p,\mu}$ decays exponentially, the estimate of D is similar to C :

$$D \lesssim \|\eta\|_{L^2}^3 e^{(-\frac{c}{2}(t_0-t)-x_0)/K}. \tag{5.23}$$

Finally, by combining (5.19)-(5.22), and integrating between $[t, t_0]$ we obtain for γ small enough and K chosen correctly:

$$I_{x_0, t_0}(t_0) - I_{x_0, t_0}(t) + \int_t^{t_0} \int (\mu(\partial_x^2 u)^2 + (\partial_x u)^2 + u^2)(s, x) dx ds \lesssim \sup_{t \in \mathbb{R}} \|\eta(t)\|_{L^2}^2 e^{-x_0/K}, \tag{5.24}$$

and this prove (5.16).

Step 2: The asymptotic limit of the L^2 localized energy.

We claim that $\forall t_0 \in \mathbb{R}$:

$$I_{x_0, t_0}(t_0) + \int_{-\infty}^{t_0} \int (\mu(\partial_x^2 u)^2 + (\partial_x u)^2 + u^2)(s, x) dx ds \lesssim \sup_{t \in \mathbb{R}} \|\eta(t)\|_{L^2}^2 e^{-x_0/K}. \tag{5.25}$$

In view of (5.24) to obtain (5.25) we just need to prove that: $\lim_{t \rightarrow -\infty} I_{x_0, t_0}(t) = 0$. Let $\varepsilon > 0$, since by assumption $u(t, x + \xi(t))$ is L^2 -compact and $\psi_{c,p,\mu}(x)$ decays exponentially, then $\eta(t, x)$ is also L^2 -compact. So, there exists $R_\varepsilon > 0$ such that $\forall t \in \mathbb{R}$ we have: $\int_{|x| > R_\varepsilon} \eta^2(t, x) \leq \varepsilon$. Now we deduce that:

$$I_{x_0, t_0}(t) = \int_{|x| > R_\varepsilon} \eta^2(t, x) \phi_K(\tilde{x}) + \int_{|x| < R_\varepsilon} \eta^2(t, x) \phi_K(\tilde{x}) \leq \varepsilon + Z. \tag{5.26}$$

For the estimate of Z we use that $\phi_K(\tilde{x})$ is increasing and $\phi_K(\pm\infty) = 0$:

$$A \leq \phi_K(R_\varepsilon - \frac{c}{2}(t_0 - t) - x_0) \|\eta(t)\|_{L^2}^2 \rightarrow 0 \text{ as } t \rightarrow -\infty. \tag{5.27}$$

Step 3: Monotonicity formula for the H^2 - localized energy.

We claim that $\forall x_0 > 0$ and $t_0 > t$, we have:

$$J_{x_0, t_0}(t_0) - J_{x_0, t_0}(t) + \int_t^{t_0} \int [\mu(\partial_x^4 \eta)^2 + \sum_{k=1}^3 (\partial_x^k \eta)^2](t, x) \phi'_K(\tilde{x}) \lesssim \sup_{t \in \mathbb{R}} \|\eta(t)\|_{H^2}^2 e^{-x_0/K}. \tag{5.28}$$

Proof. We compute the time variation of $\frac{1}{2} \frac{d}{dt} \int (\partial_x \eta)^2 \phi_K(\tilde{x})$:

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int (\partial_x \eta)^2 \phi_K(\tilde{x}) &= \int \partial_x^2 (L_{c,p,\mu} \eta) (\partial_x \eta) \phi_K(\tilde{x}) + \int \partial_x^2 (R_1(\eta, \psi_{c,p,\mu})) (\partial_x \eta) \phi_K(\tilde{x}) \\
&+ (\xi' - c) \int \partial_x^2 (\eta + \psi_{c,p,\mu}) (\partial_x \eta) \phi_K(\tilde{x}) + c' \int \rho'_{c,p,\mu} (\partial_x \eta) \phi_K(\tilde{x}) + \frac{c_0}{2} \int (\partial_x^2 \eta) \phi'_K(\tilde{x}) = A + B + C + D.
\end{aligned} \tag{5.29}$$

The estimate of A is given by the monotonicity formula related to the linearized gKW equation:

$$A \lesssim -\mu \int (\partial_x^3 \eta)^2 \phi'_K(\tilde{x}) - \int (\partial_x^2 \eta)^2 \phi'_K(\tilde{x}) - \int (\partial_x \eta)^2 \phi'_K(\tilde{x}) + \|\eta\|_{H^1}^2 e^{(-\frac{c}{2}(t_0-t)-x_0)/K}. \tag{5.30}$$

Arguing as in Step 1 the estimate of C and D is given by:

$$C + D \lesssim \|\eta\|_{L^2} \int (\partial_x \eta)^2 \phi'_K(\tilde{x}) + \|\eta\|_{H^1}^2 e^{(-\frac{\varepsilon}{2}(t_0-t)-x_0)/K}. \quad (5.31)$$

Let us explain the estimate of the nonlinear term B :

$$B = - \int \partial_x (R_1(\eta, \psi_{c,p,\mu})) (\partial_x^2 \eta) \phi_K(\tilde{x}) - \int \partial_x (R_1(\eta, \psi_{c,p,\mu})) (\partial_x \eta) \phi'_K(\tilde{x}) = B_1 + B_2. \quad (5.32)$$

We have:

$$- \partial_x (R_1(\eta, \psi_{c,p,\mu})) = \frac{1}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k [k(\partial_x \eta) \eta^{k-1} \psi_{c,p,\mu}^{p-k+1} + \eta^k (\psi_{c,p,\mu}^{p-k+1})'], \quad (5.33)$$

then

$$B_1 = \frac{1}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k k \int (\partial_x^2 \eta) (\partial_x \eta) \eta^{k-1} \psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x}) + \frac{1}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k \int (\partial_x^2 \eta) \eta^k (\psi_{c,p,\mu}^{p-k+1})' \phi_K(\tilde{x}) = B_{11} + B_{12}. \quad (5.34)$$

We compute:

$$\begin{aligned} B_{11} &= -\frac{p}{2} \int (\partial_x \eta)^3 \eta^{p-1} \phi_K(\tilde{x}) - \frac{1}{p+1} \sum_{k=2}^p C_{p+1}^k \frac{k}{2} (k-1) \int (\partial_x \eta)^3 \eta^{k-2} \psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x}) \\ &\quad - \frac{1}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k \frac{k}{2} \int (\partial_x \eta)^2 \eta^{k-1} (\psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x}))', \end{aligned} \quad (5.35)$$

and

$$\begin{aligned} & -\frac{1}{p+1} \sum_{k=2}^p C_{p+1}^k \frac{k}{2} (k-1) \int (\partial_x \eta)^3 \eta^{k-2} \psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x}) - \frac{1}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k \frac{k}{2} \int (\partial_x \eta)^2 \eta^{k-1} (\psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x}))' \\ & \lesssim \int (\partial_x \eta)^3 \psi_{c,p,\mu} \phi_K(\tilde{x}) + \int (\partial_x \eta)^2 \eta \psi'_{c,p,\mu} \phi_K(\tilde{x}) \lesssim \|\eta\|_{H^1}^3 e^{(-\frac{\varepsilon}{2}(t_0-t)-x_0)/K}. \end{aligned} \quad (5.36)$$

We have:

$$\begin{aligned} B_{12} &= -\frac{1}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k k \int (\partial_x^2 \eta)^2 \eta^{k-1} (\psi_{c,p,\mu}^{p-k+1})' \phi_K(\tilde{x}) + \frac{1}{p+1} \sum_{k=2}^{p+1} \frac{C_{p+1}^k}{k+1} \int \eta^{k+1} [(\psi_{c,p,\mu}^{p-k+1})' \phi_K(\tilde{x})]' \\ & \lesssim \|\eta\|_{H^1}^3 e^{(-\frac{\varepsilon}{2}(t_0-t)-x_0)/K}. \end{aligned} \quad (5.37)$$

The estimate of B_2 is similar more simple since it contains $\phi'_K(\tilde{x})$ and the nonlinearity term: $\eta[\eta^2 + (\partial_x \eta)^2]$.

Let us compute the variation of $\frac{\mu}{2} \frac{d}{dt} \int (\partial_x^2 \eta)^2 \phi_K(\tilde{x})$:

$$\begin{aligned} \frac{\mu}{2} \int (\partial_x^2 \eta)^2 \phi_K(\tilde{x}) &= \mu \int \partial_x^3 (\mathcal{L}_{c,p,\mu} \eta) (\partial_x^2 \eta) \phi_K(\tilde{x}) + \mu \int \partial_x (R_1(\eta, \psi_{c,p,\mu})) (\partial_x^2 \eta) \phi_K(\tilde{x}) \\ &+ \mu(\xi' - c) \int \partial_x^3 (\eta + \psi_{c,p,\mu}) (\partial_x^2 \eta) \phi_K(\tilde{x}) + \mu c' \int (\partial_x^2 \rho_{c,p,\mu}) (\partial_x^2 \eta) \phi_K(\tilde{x}) + \frac{\mu c}{4} \int (\partial_x^2 \eta)^2 \phi'_K(\tilde{x}) = A + B + C + D. \end{aligned} \quad (5.38)$$

The estimate of A is given by the monotonicity formula related to the linearized gKW equation:

$$A \lesssim -\mu \int (\partial_x^4 \eta)^2 \phi'_K(\tilde{x}) - \int (\partial_x^3 \eta)^2 \phi'_K(\tilde{x}) - \int (\partial_x^2 \eta)^2 \phi'_K(\tilde{x}) + \|\eta\|_{H^2}^2 e^{(-\frac{\varepsilon}{2}(t_0-t)-x_0)/K}. \quad (5.39)$$

Arguing as in Step 1 the estimate of C and D is given by:

$$C + D \lesssim \|\eta\|_{L^2} \int (\partial_x^2 \eta)^2 \phi'_K(\tilde{x}) + \|\eta\|_{H^2}^2 e^{(-\frac{\varepsilon}{2}(t_0-t)-x_0)/K}. \quad (5.40)$$

Let us show how estimate the nonlinear term B :

$$B = -\mu \int \partial_x^2 (R_1(\eta, \psi_{c,p,\mu})) (\partial_x^3 \eta) \phi_K(\tilde{x}) + \int \partial_x^2 (R_1(\eta, \psi_{c,p,\mu})) (\partial_x^2 \eta) \phi'_K(\tilde{x}) = B_1 + B_2. \quad (5.41)$$

We have:

$$\begin{aligned} -\partial_x^2 (R_1(\eta, \psi_{c,p,\mu})) &= \frac{1}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k [(\psi_{c,p,\mu}^{p-k+1})'' \eta^k + 2k(\psi_{c,p,\mu}^{p-k+1})' (\partial_x \eta) \eta^{k-1} \\ &\quad + k(\psi_{c,p,\mu}^{p-k+1}) (\partial_x^2 \eta) \eta^{k-1} + k(k-1)(\psi_{c,p,\mu}^{p-k+1}) (\partial_x \eta)^2 \eta^{k-2}]. \end{aligned} \quad (5.42)$$

Let us estimate B_{13} :

$$\begin{aligned} B_{13} &= \frac{\mu}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k k \int (\partial_x^3 \eta) (\partial_x^2 \eta) \eta^{k-1} (\psi_{c,p,\mu}^{p-k+1}) \phi_K(\tilde{x}) \\ &= -\frac{\mu}{2} \int (\partial_x^2 \eta)^2 (\partial_x \eta) \eta^{p-1} \phi_K(\tilde{x}) - \frac{\mu}{2(p+1)} \sum_{k=2}^p C_{p+1}^k k(k-1) \int (\partial_x^2 \eta)^2 (\partial_x \eta) \eta^{k-2} \psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x}) \\ &\quad - \frac{\mu}{2(p+1)} \sum_{k=2}^{p+1} C_{p+1}^k k \int (\partial_x^2 \eta)^2 \eta^{k-1} (\psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x}))', \end{aligned} \quad (5.43)$$

and we have:

$$\begin{aligned} & -\frac{\mu}{2(p+1)} \sum_{k=2}^p C_{p+1}^k k(k-1) \int (\partial_x^2 \eta)^2 (\partial_x \eta) \eta^{k-2} \psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x}) - \frac{\mu}{2(p+1)} \sum_{k=2}^{p+1} C_{p+1}^k k \int (\partial_x^2 \eta)^2 \eta^{k-1} (\psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x}))' \\ & \lesssim \|\eta\|_{H^2}^3 e^{(-\frac{\varepsilon}{2}(t_0-t)-x_0)/K}. \end{aligned} \quad (5.44)$$

Let us estimate B_{14} :

$$\begin{aligned} B_{14} &= \frac{\mu}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k k(k-1) \int (\partial_x^3 \eta) (\partial_x^2 \eta)^2 \eta^{k-2} \psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x}) = -2\mu \int (\partial_x^2 \eta)^2 (\partial_x \eta) \eta^{p-1} \phi_K(\tilde{x}) \\ &\quad - \frac{2\mu}{p+1} \sum_{k=2}^p C_{p+1}^k k(k-1) \int (\partial_x^2 \eta)^2 (\partial_x \eta) \eta^{k-2} \psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x}) \\ &\quad - \frac{\mu}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k k(k-1)(k-2) \int (\partial_x^2 \eta) (\partial_x \eta)^3 \eta^{k-3} \psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x}) \\ &\quad - \frac{\mu}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k k(k-1) \int (\partial_x^2 \eta) (\partial_x \eta)^2 \eta^{k-2} (\psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x}))' \\ &= -2\mu \int (\partial_x^2 \eta)^2 (\partial_x \eta) \eta^{p-1} \phi_K(\tilde{x}) + B_{141} + B_{142} + B_{143}, \end{aligned} \quad (5.45)$$

It is clear that the term B_{141} and B_{143} contain just the exponential decay.

$$\begin{aligned}
B_{142} &= \frac{\mu}{4(p+1)} \sum_{k=2}^{p+1} C_{p+1}^k k(k-1)(k-2)(k-3) \int (\partial_x \eta)^5 \eta^{k-4} \psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x}) \\
&\quad + \frac{\mu}{4(p+1)} \sum_{k=2}^{p+1} C_{p+1}^k k(k-1)(k-2) \int (\partial_x \eta)^4 \eta^{k-3} (\psi_{c,p,\mu}^{p-k+1} \phi_K(\tilde{x}))' \\
&\leq \frac{\mu p(p-1)(p-2)}{4} \int (\partial_x \eta)^5 \eta^{p-3} \phi_K(\tilde{x}) + C \|\eta\|_{H^2}^4 e^{(-\frac{\varepsilon}{2}(t_0-t)-x_0)/K}.
\end{aligned} \tag{5.46}$$

The estimate of B_2 is similar and more simple since it contains $\phi'_K(\tilde{x})$.

Let us compute the time variation of $-\frac{d}{dt} \int R(\eta, \psi_{c,p,\mu})$:

$$\begin{aligned}
-\frac{d}{dt} \int R(\eta, \psi_{c,p,\mu}) &= -\frac{1}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k \int \psi_{c,p,\mu}^{p-k+2} \eta^{k-1} (\partial_t \eta) \phi_K(\tilde{x}) \\
&\quad - \frac{c}{2(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k \int \psi_{c,p,\mu}^{p-k+2} \eta^k \phi'_K(\tilde{x}) = -\frac{1}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k \int \partial_x (L_{c,p,\mu} \eta) \psi_{c,p,\mu}^{p-k+2} \eta^{k-1} \phi_K(\tilde{x}) \\
&\quad + \frac{1}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k \int (\partial_x R_1(\eta, \psi_{c,p,\mu})) \psi_{c,p,\mu}^{p-k+2} \eta^{k-1} \phi_K(\tilde{x}) \\
&\quad - \frac{(\xi' - c)}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k \int \partial_x (\eta + \psi_{c,p,\mu}) \psi_{c,p,\mu}^{p-k+2} \eta^{k-1} \phi_K(\tilde{x}) \\
&\quad - \frac{c'}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k \int \rho_{c,p,\mu} \psi_{c,p,\mu}^{p-k+2} \eta^{k-1} \phi_K(\tilde{x}) = A + B + C + D.
\end{aligned} \tag{5.47}$$

The estimate of C and D is simple:

$$C + D \lesssim \|\eta\|_{L^2} \int \eta^2 \phi'_K(\tilde{x}) + \|\eta\|_{H^1}^2 e^{(-\frac{\varepsilon}{2}(t_0-t)-x_0)/K}. \tag{5.48}$$

Let us compute A :

$$A = -\frac{1}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k \int \psi_{c,p,\mu}^{p-k+2} \eta^{k-1} \phi_K(\tilde{x}) \partial_x (\mu \partial_x^4 \eta - \partial_x^2 \eta + c\eta - \partial_x (\psi_{c,p,\mu}^p \eta)), \tag{5.49}$$

$$\begin{aligned}
A_1 &= -\frac{\mu}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k \int (\partial_x^5 \eta) \eta^{k-1} \psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}) \\
&= \frac{\mu}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k \int (\partial_x^4 \eta) [(k-1)(\partial_x \eta) \eta^{k-2} \psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}) + \eta^{k-1} (\psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}))'] = A_{11} + A_{12}.
\end{aligned} \tag{5.50}$$

$$\begin{aligned}
A_{11} &= -\frac{\mu}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k(k-1) \int (\partial_x^3 \eta) [(\partial_x^2 \eta) \eta^{k-2} \psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}) + (k-2)(\partial_x \eta)^2 \eta^{k-3} \psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}) \\
&\quad + (\partial_x \eta) \eta^{k-2} (\psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}))'] = A_{111} + A_{112} + A_{113}.
\end{aligned} \tag{5.51}$$

$$\begin{aligned}
A_{111} &= \frac{\mu}{2} \int (\partial_x^2 \eta)^2 (\partial_x \eta) \eta^{p-1} \phi_K(\tilde{x}) + \frac{\mu}{2(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k(k-1)(k-2) \int (\partial_x^2 \eta)^2 (\partial_x \eta) \eta^{k-3} \psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}) \\
&+ \frac{\mu}{2(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k(k-1) \int (\partial_x^2 \eta)^2 \eta^{k-2} (\psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}))' \\
&\leq \frac{\mu}{2} \int (\partial_x^2 \eta)^2 (\partial_x \eta) \eta^{p-1} \phi_K(\tilde{x}) + C \|\eta\|_{H^2}^2 e^{(-\frac{\varepsilon}{2}(t_0-t)-x_0)/K}.
\end{aligned} \tag{5.52}$$

$$\begin{aligned}
A_{112} &= \frac{\mu}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k(k-1)(k-2) \int (\partial_x^2 \eta) [2(\partial_x^2 \eta) (\partial_x \eta) \eta^{k-3} \psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}) \\
&(k-3)(\partial_x \eta)^3 \eta^{k-4} \psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}) + (\partial_x \eta)^2 \eta^{k-3} (\psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}))'] = A_{1121} + A_{1122} + A_{1123}.
\end{aligned} \tag{5.53}$$

$$\begin{aligned}
A_{1121} &= 2\mu p \int (\partial_x^2 \eta)^2 (\partial_x \eta)^{p-1} \phi_K(\tilde{x}) + \frac{2\mu}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k(k-1)(k-2) \int (\partial_x^2 \eta)^2 (\partial_x \eta) \eta^{k-3} \psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}) \\
&\leq 2\mu p \int (\partial_x^2 \eta)^2 (\partial_x \eta)^{p-1} \phi_K(\tilde{x}) + C \|\eta\|_{H^2}^3 e^{(-\frac{\varepsilon}{2}(t_0-t)-x_0)/K}.
\end{aligned} \tag{5.54}$$

$$\begin{aligned}
A_{1122} &= -\frac{\mu p(p-1)(p-2)}{4} \int (\partial_x \eta)^5 \eta^{p-3} \phi_K(\tilde{x}) \\
&- \frac{\mu}{4(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k(k-1)(k-2)(k-3)(k-4) \int (\partial_x \eta)^5 \eta^{k-5} \psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}) \\
&- \frac{\mu}{4(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k(k-1)(k-2)(k-3) \int (\partial_x \eta)^4 \eta^{k-4} (\psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}))' \\
&\leq -\frac{\mu p(p-1)(p-2)}{4} \int (\partial_x \eta)^5 \eta^{p-3} \phi_K(\tilde{x}) + C \|\eta\|_{H^2}^4 e^{(-\frac{\varepsilon}{2}(t_0-t)-x_0)/K}.
\end{aligned} \tag{5.55}$$

$$\begin{aligned}
A_{1123} &= -\frac{\mu}{3(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k(k-1)(k-2)(k-3) \int (\partial_x \eta)^4 \eta^{k-4} (\psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}))' \\
&- \frac{\mu}{3(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k(k-1)(k-2) \int (\partial_x \eta)^3 \eta^{k-3} (\psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}))'' \\
&\lesssim \|\eta\|_{H^2}^3 e^{(-\frac{\varepsilon}{2}(t_0-t)-x_0)/K}.
\end{aligned} \tag{5.56}$$

One can easily check that:

$$A_{113} \lesssim \|\eta\|_{H^2}^2 e^{(-\frac{\varepsilon}{2}(t_0-t)-x_0)/K}. \tag{5.57}$$

The estimate of A_{12} is simple since it contains the localization function $\psi'_{c,p,\mu} \phi_K(\tilde{x})$.

Let us compute A_2 :

$$\begin{aligned}
A_2 &= -\frac{1}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k \int \partial_x^2 \eta [(k-1)(\partial_x \eta) \eta^{k-2} \psi_{c,p,\mu}^{p-k-2} \phi_K(\tilde{x}) + \eta^{k-1} (\psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}))'] = A_{21} + A_{22},
\end{aligned} \tag{5.58}$$

$$\begin{aligned}
A_{21} &= \frac{1}{2(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k(k-1) \int (\partial_x \eta)^2 [(k-2)(\partial_x \eta) \eta^{k-3} \psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}) + \eta^{k-2} (\psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}))'] \\
&\leq \frac{p}{2} \int (\partial_x \eta)^3 \eta^{p-1} \phi_K(\tilde{x}) + \|\eta\|_{H^1}^2 e^{(-\frac{c}{2}(t_0-t)-x_0)/K},
\end{aligned} \tag{5.59}$$

$$A_{22} \lesssim \|\eta\|_{H^1}^2 e^{(-\frac{c}{2}(t_0-t)-x_0)/K}. \tag{5.60}$$

Let us compute A_3 :

$$A_3 = -\frac{c}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k \int \eta^k (\psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}))' \lesssim \|\eta\|_{L^2}^2 e^{(-\frac{c}{2}(t_0-t)-x_0)/K}. \tag{5.61}$$

Let us compute A_4 :

$$\begin{aligned}
A_4 &= \frac{1}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k \int \eta^k [k \psi_{c,p,\mu}^{p-k+2} (\psi_{c,p,\mu}^p)' \phi_K(\tilde{x}) - (\psi_{c,p,\mu}^{2p-k+2} \phi_K(\tilde{x}))'] \\
&\lesssim \|\eta\|_{L^2}^2 e^{(-\frac{c}{2}(t_0-t)-x_0)/K}.
\end{aligned} \tag{5.62}$$

Let us compute B :

$$B = -\frac{1}{(p+1)(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k \int R_1(\psi_{c,p,\mu}, \eta) [(k-1)(\partial_x \eta) \eta^{k-2} \psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}) + \eta^{k-1} (\psi_{c,p,\mu}^{p-k+2} \phi_K(\tilde{x}))'] = B_1 + B_2, \tag{5.63}$$

It is clear that the estimate of B_2 is:

$$B_2 \lesssim \|\eta\|_{H^1}^3 e^{(-\frac{c}{2}(t_0-t)-x_0)/K}. \tag{5.64}$$

One can compute that:

$$\begin{aligned}
B_1 &= -\frac{1}{(p+1)^2(p+2)} \sum_{k=2}^{p+2} C_{p+2}^k k(k-1) \sum_{j=2}^{p+1} C_{p+1}^k \frac{1}{j+k-1} \int \eta^{j+k-1} (\psi_{c,p,\mu}^{2p-k-j+3} \phi_K(\tilde{x}))' \\
&\lesssim \|\eta\|_{H^1}^3 e^{(-\frac{c}{2}(t_0-t)-x_0)/K}.
\end{aligned} \tag{5.65}$$

The term: $-\frac{p}{2} \int (\partial_x \eta)^3 \eta^{p-1} \phi_K(\tilde{x})$ content in (5.35) will cancel with this opposite content in (5.59). Similarly, the term: $-\frac{5\mu p}{2} \int (\partial_x^2 \eta)^2 (\partial_x \eta)^{p-1} \phi_K(\tilde{x})$ content in (5.43)+(5.45) will cancel with this opposite content in (5.52)+(5.54). Also, the term: $\frac{\mu p(p-1)(p-2)}{4(p+1)} \int (\partial_x \eta)^5 \eta^{p-3} \phi_K(\tilde{x})$ content in (5.46) will cancel with this opposite in (5.55).

Finally, by integrating $J_{x_0, t_0}(t)$ between $[t, t_0]$ and taking γ small enough we obtain (5.28).

Step 4: Asymptotic limit of the H^2 localized energy.

We claim that for all $x_0 > 0$ and $t_0 \in \mathbb{R}$:

$$J_{x_0, t_0}(t_0) + \int_{-\infty}^{t_0} \int [\mu (\partial_x^4 \eta)^2 + \sum_{k=1}^3 (\partial_x^k \eta)^2](t, x) \phi_K'(\tilde{x}) \lesssim \sup_{t \in \mathbb{R}} \|\eta(t)\|_{H^2}^2 e^{-x_0/K}. \tag{5.66}$$

In view of (5.28) it suffice to prove that: $\lim_{t \rightarrow -\infty} J_{x_0, t_0}(t) = 0$. It is easy to see that: $J_{x_0, t_0}(t) \leq \sum_{k=0}^2 \int (\partial_x^k \eta)^2 \phi_K'(\tilde{x})$ and for Step 2 we have:

$$\sum_{k=0}^2 \int_{-\infty}^{t_0} \int (\partial_x^k \eta)^2 \phi_K'(\tilde{x}) \lesssim e^{-x_0/K}. \tag{5.67}$$

Since $\phi'_K(\tilde{x}) \sim e^{-|\tilde{x}|/K} \sim \phi_K(\tilde{x})$ for $\tilde{x} < 0$:

$$\sum_{k=0}^2 \int_{-\infty}^{t_0} \int_{x < \frac{\varepsilon}{2}(t_0-t) + x_0} (\partial_x^k \eta)^2 e^{(x - \frac{\varepsilon}{2}(t_0-t))/K} \lesssim 1, \quad (5.68)$$

letting $x_0 \rightarrow +\infty$ and mutiplying by $e^{-x_0/K}$ we get:

$$\sum_{k=0}^2 \int_{-\infty}^{t_0} (\partial_x^k \eta)^2 \phi'_K(\tilde{x}) \lesssim \sum_{k=0}^2 \int_{-\infty}^{t_0} (\partial_x^k \eta)^2 e^{-\tilde{x}/K} \lesssim e^{-x_0/K}, \quad (5.69)$$

and this implies that:

$$\lim_{t \rightarrow -\infty} \sum_{k=0}^2 \int (\partial_x^k \eta)^2(t, x) \phi'_K(\tilde{x}) = 0. \quad (5.70)$$

Step 4: Exponential decay of $\partial_x \eta(t, x)$.

We claim that:

$$\sum_{k=0}^2 \int (\partial_x^k \eta)^2(t_0, x) e^{|x|/K} + \int_{t_0-1}^{t_0} \int [\mu(\partial_x^4 \eta)^2 + \sum_{k=0}^3 (\partial_x^k \eta)^2](t, x) e^{|x|/K} dx dt \lesssim \sup_{t \in \mathbb{R}} \|\eta(t)\|_{H^2}^2. \quad (5.71)$$

Proof. From Step 2 and Step 3 we have:

$$\sum_{k=0}^2 \int (\partial_x^k \eta)^2(t_0, x) \phi_K(x - x_0) \lesssim \sup_{t \in \mathbb{R}} \|\eta(t)\|_{H^2}^2 e^{-x_0/K}. \quad (5.72)$$

Using that $\phi_K(x - x_0) \sim e^{(x-x_0)/K}$ for $x < x_0$ we have:

$$\sum_{k=0}^2 \int_{x < x_0} (\partial_x^k \eta)^2(t_0, x) e^{x/K} \lesssim 1. \quad (5.73)$$

Letting $x_0 \rightarrow +\infty$ and multiplying by $e^{-x_0/K}$ we obtain the exponential decay by the right:

$$\sum_{k=0}^2 \int_{\mathbb{R}} (\partial_x^k \eta)^2(t_0, x) e^{x/K} \lesssim 1. \quad (5.74)$$

Now using that $\eta(-t_0, -x)$ is L^2 -compact and that $\eta(-t_0, -x)$ satisfies the same equation as $\eta(t_0, x)$ with $\xi(t_0)$ substituting by $-\xi(-t_0)$ and $c(t_0)$ by $c(-t_0)$ we obtain the left exponential decay:

$$\sum_{k=0}^2 \int_{\mathbb{R}} (\partial_x^k \eta)^2(-t_0, x) e^{-x/K} \lesssim 1, \quad (5.75)$$

and this prove (5.71).

5.3 Proof of Theorem 5.1

The proof of Theorem 5.1 is done by contradiction following the idea of Kening-Martel-Robbiano [9]. Let $(u_n(0, x))$ be the sequence of initial data satisfying (5.1), i.e:

$$\lim_{n \rightarrow +\infty} \|u_n(0) - \psi_{1,p,\mu}\|_{H^2} = 0. \quad (5.76)$$

Then thanks to the orbital stability we have:

$$\lim_{n \rightarrow +\infty} \sup_{t \in \mathbb{R}} \inf_{r \in \mathbb{R}} \|u_n(t) - \psi_{1,p,\mu}(\cdot - r)\|_{H^2} = 0. \quad (5.77)$$

By applying the Lemma of construction of the modulation parameters, there exists $\xi_n(t)$ and $c_n(t)$ satisfying:

$$\lim_{n \rightarrow +\infty} \sup_{t \in \mathbb{R}} (|\xi_n(0)| + |c_n(t) - 1| + \|\eta_n(t)\|_{H^2}) = 0, \quad (5.78)$$

where $\eta_n(t, x) = u_n(t, x + \xi_n(t)) - \psi_{c_n, p, \mu}(x)$. Moreover $\forall t \in \mathbb{R}$:

$$\int \eta_n(t, x) \psi_{c_n(t), p, \mu}(x) = \int \eta_n(t, x) \psi'_{c_n(t), p, \mu}(x) = 0, \quad (5.79)$$

and

$$\lim_{n \rightarrow +\infty} \sup_{t \in \mathbb{R}} (|\xi'_n(t) - c_n(t)| + |c'_n(t)|) = 0. \quad (5.80)$$

And we assume that $\forall n \in \mathbb{N}$ $\eta_n(t, x) \not\equiv 0$, and

$$\forall n \in \mathbb{N}, \forall \varepsilon > 0, \exists R_{n, \varepsilon} > 0, \forall t \in \mathbb{R} \int_{|x| > R_{n, \varepsilon}} (u_n(t, x + \xi_n(t)))^2 dx \leq \varepsilon. \quad (5.81)$$

We set $0 \neq b_n = \sup_{t \in \mathbb{R}} \|\eta_n(t)\|_{H^2}$. By (5.78) $b_n \rightarrow 0$ as $n \rightarrow +\infty$. Then, there exists t_n such that $\|\eta_n(t_n)\|_{H^2} \geq \frac{1}{2}$. We define:

$$w_n(t, x) = \frac{\eta_n(t_n + t, x)}{b_n}. \quad (5.82)$$

One can see that $\|w_n(0)\| \geq \frac{1}{2}$ and $\|w_n(t)\|_{H^2} \leq 1$.

Firts, we claim that $(w_n(t, x))$ is H^2 -compact. Ideed, applying the monotonicity formula with $\eta_n(t, x)$, we have $\forall n \in \mathbb{N}$, $K > 0$ large enough, $\forall t \in \mathbb{R}$:

$$\sum_{k=0}^2 \int (\partial_x^k \eta_n(t, x))^2 e^{-|x|/K} + \int_{t-1}^t \int [\mu(\partial_x^4 \eta_n)^2 + \sum_{k=0}^3 (\partial_x^k \eta_n)^2](t, x) e^{-|x|/K} \lesssim \sup_{t \in \mathbb{R}} \|\eta_n(t)\|_{H^2}^2. \quad (5.83)$$

Then we infer that:

$$\sum_{k=0}^2 \int (\partial_x^k w_n)^2(t, x) e^{-|x|/K} + \int_{t-1}^t \int [\mu(\partial_x^4 w_n)^2 + \sum_{k=0}^3 (\partial_x^k w_n)^2](t, x) e^{-|x|/K} \lesssim 1. \quad (5.84)$$

In particular this implies that $w_n(t, x)$ is H^2 -compact:

$$\forall x_0 > 0, \sum_{k=0}^2 \int_{|x| > x_0} (\partial_x^k w_n)^2(t, x) \lesssim e^{-x_0/K}. \quad (5.85)$$

Second, we claim that:

$$\sup_{t \in \mathbb{R}} \|w_n(t)\|_{H^3} \lesssim 1. \quad (5.86)$$

Proof. The equation of $w_n(t, x)$ is given by:

$$\begin{aligned} \partial_t w_n &= \partial_x (L_{c_n(t+t_n), p, \mu} w_n) - \frac{1}{b_n} \partial_x (R_1(b_n w_n, \psi_{c_n(t_n+t), p, \mu})) \\ &+ \frac{1}{b_n} (\xi'_n(t_n + t) - c_n(t + t_n)) \psi'_{c_n(t_n+t), p, \mu} + (\xi'_n(t_n + t) - c(t_n + t)) \partial_x w_n + \frac{c'_n(t_n + t)}{b_n} \rho_{c_n(t_n+t), p, \mu}. \end{aligned} \quad (5.87)$$

Then we obtain

$$\begin{aligned}
\frac{1}{2} \int (\partial_x^3 w_n)^2 &= \int \partial_x^3 (L_{c_n(t_n+t)} w_n) (\partial_x^3 w) - \frac{1}{b_n} \int \partial_x^3 (R_1(b_n w_n, \psi_{c_n(t_n+t)})) (\partial_x^3 w_n) \\
&+ \frac{1}{b_n} (\xi'_n(t_n+t) - c_n(t_n+t)) \int (\partial_x^4 \psi_{c_n(t_n+t)}) (\partial_x^3 w_n) + (\xi'_n(t_n+t) - c_n(t_n+t)) \int (\partial_x^4 w_n) (\partial_x^3 w_n) \\
&+ \frac{c'_n(t_n+t)}{b_n} \int (\partial_x^3 \rho_{c_n(t_n+t)}) (\partial_x^3 w_n) = A + B + C + D + E.
\end{aligned} \tag{5.88}$$

Let us estimate E :

$$E = -\frac{c'_n(t_n+t)}{b_n} \int (\partial_x^6 \rho_{c_n(t_n+t)}) w_n \lesssim b_n \|w_n\|_{L^2}^3. \tag{5.89}$$

It is clear that $D = 0$. Let us estimate C :

$$C = -\frac{1}{b_n} (\xi'_n(t_n+t) - c_n(t_n+t)) \int (\partial_x^7 \psi_{c_n(t_n+t), p, \mu}) w_n \lesssim \|w_n\|_{L^2}^2. \tag{5.90}$$

Let us estimate B :

$$B = \frac{1}{b_n} \int \partial_x^3 (R_1(b_n w_n, \psi_{c_n(t_n+t), p, \mu})) (\partial_x^4 w_n), \tag{5.91}$$

$$\begin{aligned}
\frac{1}{b_n} \partial_x^3 (R_1(b_n w_n, \psi_{c_n(t_n+t), p, \mu})) &= \frac{1}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k b_n^{k-1} [k(\partial_x^3 w_n) w_n^{k-1} \psi_{c_n, p, \mu}^{p-k+1} + 3k(k-1)(\partial_x^2 w_n)(\partial_x w_n) w_n^{k-2} \psi_{c_n, p, \mu}^{p-k+1} \\
&+ 3k(\partial_x^2 w_n) w_n^{k-1} (\psi_{c_n, p, \mu}^{p-k+1})' + k(k-1)(k-2)(\partial_x w_n)^3 w_n^{p-3} \psi_{c_n, p, \mu}^{p-k+1} + 3k(k-1)(\partial_x w_n)^2 w_n^{k-2} (\psi_{c_n, p, \mu}^{p-k+1})' \\
&+ k(k-1)(k-2)(\partial_x w_n)^3 w_n^{p+3} \psi_{c_n, p, \mu}^{p-k+1} + 3k(k-1)(\partial_x w_n^2) w_n^{k-2} (\psi_{c_n, p, \mu}^{p-k+1})' + 2k(\partial_x w_n) w_n^{k-1} (\psi_{c_n, p, \mu}^{p-k+1})''].
\end{aligned} \tag{5.92}$$

We have:

$$\begin{aligned}
B_1 &= \frac{1}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k b_n^{k-1} k \int (\partial_x^3 w_n) (\partial_x^4 w_n) w_n^{k-1} \psi_{c_n, p, \mu}^{p-k+1} = -\frac{1}{2(p+1)} \sum_{k=2}^{p+1} C_{p+1}^k b_n^{k-1} k \int (\partial_x^3 w_n)^2 (w_n^{k-1} \psi_{c_n, p, \mu}^{p-k+1})' \\
&\lesssim b_n \int (\partial_x^3 w_n)^2,
\end{aligned} \tag{5.93}$$

$$\begin{aligned}
B_2 &= -\frac{1}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k 3k(k-1) b_n^{k-1} \int (\partial_x^3 w_n) [(\partial_x^3 w_n) (\partial_x w_n) w_n^{k-2} \psi_{c_n, p, \mu}^{p-k+1} + (\partial_x^2 w_n)^2 w_n^{k-2} \psi_{c_n, p, \mu}^{p-k+1} \\
&+ (\partial_x^2 w_n) (\partial_x w_n) (w_n^{k-2} \psi_{c_n, p, \mu}^{p-k+1})'],
\end{aligned} \tag{5.94}$$

$$B_{21} = -\frac{1}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k 3k(k-1) b_n^{k-1} \int (\partial_x^3 w_n)^2 (\partial_x w_n) w_n^{k-2} \psi_{c_n, p, \mu}^{p-k+1} \lesssim b_n \int (\partial_x^3 w_n)^2, \tag{5.95}$$

$$\begin{aligned}
B_{22} &= \frac{1}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k b_n^{k-1} k(k-1) \int (\partial_x^2 w_n)^3 (w_n^{k-2} \psi_{c_n, p, \mu}^{p-k+1})' \lesssim b_n \int (\partial_x^2 w_n)^3 \lesssim b_n \|\partial_x^2 w_n\|_{L^2}^2 \|\partial_x w_n\|_{L^\infty} \\
&\leq C_G b_n \|\partial_x^2 w_n\|_{L^2}^{5/2} \|\partial_x^3 w_n\|_{L^2}^{1/2} \leq C_G b_n \left(\frac{1}{4} \|\partial_x^3 w_n\|_{L^2}^2 + \frac{3}{4} \|\partial_x^2 w_n\|_{L^2}^{10/3} \right),
\end{aligned} \tag{5.96}$$

$$B_{23} = \frac{1}{2(p+1)} \sum_{k=2}^{p+1} C_{p+1}^k 3k(k-1) b_n^{k-1} \int (\partial_x^2 w_n)^2 [(\partial_x^2 w_n) (w_n^{k-2} \psi_{c_n, p, \mu}^{p-k+1})' + (\partial_x w_n) (w_n^{k-2} \psi_{c_n, p, \mu}^{p-k+1})''], \tag{5.97}$$

$$\begin{aligned}
B_{231} &= \frac{1}{2(p+1)} \sum_{k=2}^{p+1} C_{p+1}^k 3k(k-1)b_n^{k-1} \int (\partial_x^2 w_n)^3 (w_n^{k-2} \psi_{c_n, p, \mu}^{p-k+1})' \lesssim b_n \int (\partial_x^2 w_n)^3 \\
&\lesssim b_n \left(\frac{3}{4} \|\partial_x^2 w_n\|_{L^2}^{10/3} + \frac{1}{4} \|\partial_x^3 w_n\|_{L^2}^2 \right), \tag{5.98}
\end{aligned}$$

$$B_{232} = \frac{1}{2(p+1)} \sum_{k=2}^{p+1} C_{p+1}^k 3k(k-1)b_n^{k-1} \int (\partial_x^2 w_n)^2 (\partial_x w_n) (w_n^{k-2} \psi_{c_n, p, \mu}^{p-k+1})'', \tag{5.99}$$

$$\begin{aligned}
\partial_x^2 (w_n^{k-2} \psi_{c_n, p, \mu}^{p-k+1}) &= (k-2)(\partial_x^2 w_n) w_n^{k-3} \psi_{c_n, p, \mu}^{p-k+1} + (k-2)(k-3)(\partial_x w_n)^2 w_n^{k-4} \psi_{c_n, p, \mu}^{p-k+1} \\
&+ 2(k-2)(\partial_x w_n) w_n^{k-3} (\psi_{c_n, p, \mu}^{p-k+1})' + w_n^{k-2} (\psi_{c_n, p, \mu}^{p-k+1})'', \tag{5.100}
\end{aligned}$$

and then

$$B_{232} \lesssim b_n \left(\int (\partial_x^2 w_n)^3 + \int (\partial_x^2 w_n)^2 \right). \tag{5.101}$$

The estimate of B_3 is similar to B_2 since it contains the product $(\partial_x^4 w_n)(\partial_x^2 w_n)$, and the estimate of B_4 - B_8 is more simple since it contains the product: $(\partial_x^4 w_n)(\partial_x w_n)$.

Let us now estimate A :

$$A = - \int \partial_x^4 (\psi_{c_n, p, \mu}^p w_n) (\partial_x^3 w_n) = \int \partial_x^3 (\psi_{c_n, p, \mu}^p w_n) (\partial_x^4 w_n), \tag{5.102}$$

$$\partial_x (\psi_{c_n, p, \mu}^p w_n) = \partial_x^3 (\psi_{c_n, p, \mu}^p) w_n + 3\partial_x^2 (\psi_{c_n, p, \mu}^p) (\partial_x w_n) + 3\partial_x (\psi_{c_n, p, \mu}^p) (\partial_x^2 w_n) + \psi_{c_n, p, \mu}^p (\partial_x^3 w_n), \tag{5.103}$$

$$A_1 = \int (\partial_x^4 w_n) w_n (\partial_x^3 (\psi_{c_n, p, \mu}^p) = - \int (\partial_x^3 w_n) [(\partial_x w_n) \partial_x^3 (\psi_{c_n, p, \mu}^p) + w_n (\partial_x^4 \psi_{c_n, p, \mu}^p)], \tag{5.104}$$

$$\begin{aligned}
A_{11} &= \int (\partial_x^2 w_n) [(\partial_x^2 w_n) \partial_x^3 (\psi_{c_n, p, \mu}^p) + (\partial_x w_n) \partial_x^4 (\psi_{c_n, p, \mu}^p)] = \int (\partial_x^2 w_n) \partial_x^3 (\psi_{c_n, p, \mu}^p) - \frac{1}{2} \int (\partial_x w_n)^2 \partial_x (\psi_{c_n, p, \mu}^p) \\
&\lesssim \int (\partial_x^2 w_n)^2 + \int (\partial_x w_n)^2, \tag{5.105}
\end{aligned}$$

$$\begin{aligned}
A_{12} &= \int (\partial_x^2 w_n) [(\partial_x w_n) \partial_x^4 (\psi_{c_n, p, \mu}^p) + w_n \partial_x^5 (\psi_{c_n, p, \mu}^p)] = -\frac{3}{2} \int (\partial_x w_n)^2 \partial_x^5 (\psi_{c_n, p, \mu}^p) + \frac{1}{2} \int w_n^2 \partial_x^7 (\psi_{c_n, p, \mu}^p) \\
&\lesssim \int (\partial_x w_n)^2 + \int w_n^2, \tag{5.106}
\end{aligned}$$

$$A_2 = -3 \int (\partial_x^3 w_n) [(\partial_x^2 w_n) \partial_x^2 (\psi_{c_n, p, \mu}^p) + (\partial_x w_n) \partial_x^3 (\psi_{c_n, p, \mu}^p)], \tag{5.107}$$

$$A_{21} = \frac{3}{2} \int (\partial_x^2 w_n)^2 (\partial_x^3 \psi_{c_n, p, \mu}^p) \lesssim \int (\partial_x^2 w_n)^2, \tag{5.108}$$

$$A_{22} = 3 \int (\partial_x^2 w_n)^2 \partial_x^3 (\psi_{c_n, p, \mu}^p) - \frac{3}{2} \int (\partial_x w_n)^2 \partial_x^5 (\psi_{c_n, p, \mu}^p) \lesssim \int (\partial_x^2 w_n)^2 + \int (\partial_x w_n)^2. \tag{5.109}$$

$$\begin{aligned}
A_3 &= 3 \int (\partial_x^4 w_n) (\partial_x^2 w_n) \partial_x (\psi_{c_n, p, \mu}^p) = -3 \int (\partial_x^3 w_n) [(\partial_x^3 w_n) \partial_x (\psi_{c_n, p, \mu}^p)' + (\partial_x^2 w_n) \partial_x^2 (\psi_{c_n, p, \mu}^p)] \\
&= -3 \int (\partial_x^3 w_n)^2 (\partial_x \psi_{c_n, p, \mu}^p) + \frac{3}{2} \int (\partial_x^2 w_n)^2 \partial_x^3 (\psi_{c_n, p, \mu}^p) \\
&\lesssim \int (\partial_x^3 w_n)^2 + \int (\partial_x^2 w_n)^2, \tag{5.110}
\end{aligned}$$

$$A_4 = \int (\partial_x^4 w_n)(\partial_x^3 w_n) \psi_{c_n, p, \mu}^p = -\frac{1}{2} \int (\partial_x^3 w_n)^2 \partial_x (\psi_{c_n, p, \mu}^p) \lesssim \int (\partial_x^3 w_n)^2. \quad (5.111)$$

Then we infer that:

$$\frac{d}{dt} \int (\partial_x^3 w_n)^2 \lesssim \int (\partial_x^3 w_n)^2 + 1. \quad (5.112)$$

From (5.84) we know that $\forall t \in \mathbb{R}$:

$$\int_{t-1}^t \int (\partial_x^3 w_n)^2(s, x) \lesssim 1, \quad (5.113)$$

then there exists $t_1 \in [t-1, t]$ such that:

$$\int (\partial_x^3 w_n)^2(t_1, x) \lesssim 1. \quad (5.114)$$

Integrating (5.112) between $[t_1, t]$ and using (5.114) we obtain (5.86).

Now we will transform the equation of $w_n(t, x)$. We set:

$$\tilde{\beta}_n(t) = \frac{1}{b_n} (\xi'_n(t_n + t) - c_n(t_n + t)) \text{ and } \tilde{\beta}_n(t) = \frac{1}{\int (\psi'_{c_n(t_n+t)})^2} \int w_n(t, x) L_{c_n(t_n+t), p, \mu} \psi''_{c_n(t_n+t), p, \mu}(x), \quad (5.115)$$

and we rewrite the equation of $w_n(t, x)$ as:

$$\begin{aligned} \partial_t w_n &= \partial_x (L_{c_n(t_n+t), p, \mu} w_n) - \frac{1}{b_n} \partial_x (R(b_n w_n, \psi_{c_n(t_n+t), p, \mu})) + \beta_n(t) \psi'_{c_n(t_n+t), p, \mu} + (\tilde{\beta}_n(t) - \beta_n(t)) \psi'_{c_n(t_n+t), p, \mu} \\ &+ b_n \tilde{\beta}_n(t) (\partial_x w_n) + \frac{c'_n(t_n+t)}{b_n} \rho'_{c_n(t_n+t), p, \mu}. \end{aligned} \quad (5.116)$$

Now by developing $\frac{d}{dt} \int w_n(t, x) \psi'_{c_n, p, \mu} = 0$ we obtain:

$$\begin{aligned} &\frac{1}{b_n} (\xi'_n(t_n + t) - c_n(t_n + t)) - \frac{1}{\int (\psi'_{c_n(t_n+t), p, \mu})^2} \int w_n(t, x) L_{c_n(t_n+t), p, \mu} \psi''_{c_n(t_n+t), p, \mu} \\ &= -\frac{1}{b_n} \int R(b_n w_n, \psi_{c_n(t_n+t), p, \mu}) \psi''_{c_n(t_n+t), p, \mu} + (\xi'_n(t_n + t) - c_n(t_n + t)) \int w_n \psi''_{c_n(t_n+t), p, \mu} \\ &- \frac{c'_n(t_n+t)}{b_n} \int \rho_{c_n(t_n+t), p, \mu} \psi'_{c_n, p, \mu}(t_n + t) + c'_n(t_n + t) \int w_n \rho'_{c_n(t_n+t), p, \mu}, \end{aligned} \quad (5.117)$$

Using that $(\psi_{c_n, p, \mu})$ and $(\rho_{c_n, p, \mu})$ is bounded in H^s for all $s \in \mathbb{R}$, we obtain:

$$\left| \frac{1}{b_n} (\xi'_n(t_n + t) - c_n(t_n + t)) - \frac{1}{\int (\psi'_{c_n(t_n+t), p, \mu})^2} \int w_n(t, x) L_{c_n(t_n+t), p, \mu} \psi''_{c_n(t_n+t), p, \mu} \right| \lesssim b_n \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (5.118)$$

Next, we define:

$$\tilde{w}_n(t, x) = w_n(t, x) - \psi'_{c_n(t_n+t), p, \mu} \int_0^t \beta_n(s) ds, \quad (5.119)$$

Then the equation of $\tilde{w}_n(t, x)$ is given by:

$$\begin{aligned} \partial_t \tilde{w}_n &= \partial_x (L_{c_n(t_n+t), p, \mu} \tilde{w}_n) - \frac{1}{b_n} \partial_x (R_1(b_n w_n, \psi_{c_n(t_n+t), p, \mu})) + (\tilde{\beta}_n(t) - \beta_n(t)) \psi'_{c_n(t_n+t), p, \mu} \\ &+ b_n \tilde{\beta}_n(t) \partial_x \tilde{w}_n + b_n \tilde{\beta}_n(t) \psi''_{c_n(t_n+t), p, \mu} \int_0^t \beta_n(s) ds + c'_n(t_n + t) \rho'_{c_n(t_n+t), p, \mu} \int_0^t \beta_n(s) ds + \frac{c'_n(t_n + t)}{b_n} \rho_{c_n(t_n+t), p, \mu}. \end{aligned} \quad (5.120)$$

From (5.86) we know that $(w_n(0, x))$ is bounded in H^3 , then there exists $w_0 \in H^3$ such that up to a subsequence we have:

$$w_n(0) \rightharpoonup w_0 \text{ in } H^3 \text{ as } n \rightarrow +\infty, \quad (5.121)$$

$$w_n(0) \rightarrow w_0 \text{ in } H_{loc}^2 \text{ as } n \rightarrow +\infty. \quad (5.122)$$

By combining the convergence in H_{loc}^2 (5.122) and the H^2 -compactness (5.85) we obtain:

$$w_n(0) = \tilde{w}_n(0) \rightarrow w_0 = \tilde{w}_0 \text{ in } H^2 \text{ as } n \rightarrow +\infty. \quad (5.123)$$

Let $\tilde{w} \in C([0, T], H^2(\mathbb{R}))$ the unique global solution of the linear Liouville problem related to the gKW equation:

$$\begin{cases} \partial_t \tilde{w} = \partial_x(\mathcal{L}_{1,p,\mu} \tilde{w}), & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ \tilde{w}(0, x) = \tilde{w}_0 \in H^2(\mathbb{R}). \end{cases} \quad (5.124)$$

We claim that $\forall t \in \mathbb{R}$:

$$\tilde{w}_n(t) \rightarrow \tilde{w}(t) \text{ in } H^2 \text{ as } n \rightarrow +\infty. \quad (5.125)$$

Since (5.120) is invariant by the transformation: $x \rightarrow -x$ and $t \rightarrow -t$, it suffice to prove (5.126) for $t > 0$. We define $\forall t \in \mathbb{R}$:

$$v_n(t, x) = \tilde{w}_n(t, x) - \tilde{w}(t, x). \quad (5.126)$$

Then $v(t, x)$ satisfies:

$$\begin{aligned} \partial_t v_n &= \partial_x(\mathcal{L}_{c_n,p,\mu_n} v_n) + (c_n - 1)\partial_x \tilde{w} - \partial_x[(\psi_{c_n,p,\mu}^p - \psi_{1,p,\mu}^p)\tilde{w}] - \frac{1}{b_n}\partial_x(R_1(b_n w_n, \psi_{c_n,p,\mu})) \\ &+ (\tilde{\beta}_n(t) - \beta_n(t))\psi'_{c_n,p,\mu} + b_n \tilde{\beta}_n \partial_x \tilde{w}_n + b_n \tilde{\beta}_n \psi''_{c_n,p,\mu} \int_0^t \beta_n(s) ds + c'_n \rho'_{c_n,p,\mu} \int_0^t \beta_n(s) ds. \end{aligned} \quad (5.127)$$

and we have:

$$\tilde{v}_n(0) \rightarrow 0 \text{ in } H^2 \text{ as } n \rightarrow +\infty \text{ and } \sup_{t \in [0, T]} \|v_n(t)\|_{H^2} \leq C_T. \quad (5.128)$$

Let us compute $\frac{d}{dt} \|v(t)\|_{H^2}^2$:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int (\partial_x^2 v_n)^2 &= \int \partial_x^3(L_{c_n,p,\mu} v_n)(\partial_x^2 v_n) + (c_n - 1) \int (\partial_x^3 \tilde{w})(\partial_x^2 v_n) - \int \partial_x^3[(\psi_{c_n,p,\mu}^p - \psi_{1,p,\mu}^p)\tilde{w}](\partial_x^2 v_n) \\ &- \frac{1}{b_n} \int \partial_x^3(R_1(b_n w_n, \psi_{c_n,p,\mu}))(\partial_x^2 v_n) + (\tilde{\beta}_n - \beta_n) \int \partial_x^3(\psi_{c_n,p,\mu})(\partial_x^2 v_n) + b_n \tilde{\beta}_n \int (\partial_x^3 \tilde{w}_n)(\partial_x^2 v_n) \\ &+ b_n \tilde{\beta}_n \int_0^t \beta_n(s) ds \int (\partial_x^4 \psi_{c_n,p,\mu})(\partial_x^2 v_n) + c'_n \int_0^t \beta_n(s) ds \int (\partial_x^3 \rho_{c_n,p,\mu})(\partial_x^2 v_n) + \frac{c'_n}{b_n} \int (\partial_x^2 \rho_{c_n,p,\mu})(\partial_x^2 v_n) \\ &= \sum_{k=0}^9 I_k. \end{aligned} \quad (5.129)$$

Let us estimate I_9 :

$$I_9 = c'_n \int_0^t \beta_n(s) ds \int (\partial_x^5 \rho_{c_n,p,\mu}) v_n \leq |c'_n| \left| \int_0^t \beta_n(s) ds \right| \|\partial_x^5 \rho_{c_n,p,\mu}\|_{L^2} \|v_n\|_{L^2} \leq C_T b_n^2, \quad (5.130)$$

Let us estimate I_8 :

$$I_8 = \frac{c'_n}{b_n} \int (\partial_x^4 \rho_{c_n,p,\mu}) v_n \leq C_T b_n. \quad (5.131)$$

The estimate of I_7 is similar to I_8 . Let us estimate I_6 :

$$I_6 \leq b_n |\tilde{\beta}_n| \|\partial_x^3 \tilde{w}_n\|_{L^2} \|\partial_x^2 v_n\|_{L^2} \leq C_T b_n. \quad (5.132)$$

Let us estimate I_5 :

$$I_5 = (\tilde{\beta}_n - \beta_n) \int (\partial_x^5 \psi_{c_n, p, \mu}) v_n \leq |\tilde{\beta}_n - \beta_n| \|\partial_x^5 \psi_{c_n, p, \mu}\|_{L^2} \|v_n\|_{L^2} \leq C_T b_n. \quad (5.133)$$

Let us estimate I_4 . Applying the Young's inequality:

$$\begin{aligned} I_4 &= -\frac{1}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k b_n^{k-1} \int \partial_x^3 (w_n^k \psi_{c_n, p, \mu}^{p-k+1}) (\partial_x^2 v_n) \leq \frac{1}{p+1} \sum_{k=2}^{p+1} C_{p+1}^k b_n^{k-1} \int \left(\frac{1}{2} (\partial_x^3 (w_n^k \psi_{c_n, p, \mu}^{p-k+1}))^2 + \frac{1}{2} (\partial_x^2 v_n)^2 \right) \\ &\leq C_T b_n. \end{aligned} \quad (5.134)$$

Let us estimate I_3 and I_2 . One can easy check that:

$$I_3 \leq C_T \|\psi_{c_n, p, \mu}^p - \psi_{1, p, \mu}^p\|_{H^4} \quad \text{and} \quad I_2 \leq C_T |c_n - 1|. \quad (5.135)$$

Let us estimate I_1 :

$$I_1 = - \int \partial_x^3 (\psi_{c_n, p, \mu}^p v_n) (\partial_x^2 v_n) = - \sum_{k=0}^3 C_3^k \int (\partial_x^k v_n) (\partial_x^2 v_n) (\partial_x^{3-k} \psi_{c_n, p, \mu}^p), \quad (5.136)$$

we have:

$$I_{11} = - \int (\partial_x^2 v_n) v_n (\partial_x^3 \psi_{c_n, p, \mu}^p) = \int (\partial_x v_n)^2 (\partial_x^3 \psi_{c_n, p, \mu}^p) - \frac{1}{2} \int v_n^2 (\partial_x^5 \psi_{c_n, p, \mu}^p) \lesssim \int (\partial_x v_n)^2 + \int v_n^2, \quad (5.137)$$

and

$$I_{13} = - \int (\partial_x^3 v_n) (\partial_x^2 v_n) \psi_{c_n, p, \mu}^p = \frac{1}{2} \int (\partial_x^2 v_n)^2 \partial_x \psi_{c_n, p, \mu}^p \lesssim \int (\partial_x^2 v_n)^2, \quad (5.138)$$

then

$$I_1 \lesssim \int (\partial_x^2 v_n)^2 + \int (\partial_x v_n)^2 + \int v_n^2. \quad (5.139)$$

All of this implies that:

$$\frac{1}{2} \int (\partial_x^2 v_n)^2 \leq \sum_{k=0}^2 \int (\partial_x^k v_n)^2 + C_T (|c_n - 1| + \|\psi_{c_n, p, \mu}^p - \psi_{1, p, \mu}^p\|_{H^4} + b_n). \quad (5.140)$$

Now one can easy deduce that:

$$\frac{d}{dt} \|v_n(t)\|_{H^2}^2 \leq \|v_n(t)\|_{H^2}^2 + C_T (|c_n - 1| + \|\psi_{c_n, p, \mu}^p - \psi_{1, p, \mu}^p\|_{H^4} + b_n). \quad (5.141)$$

Finally by integrating (5.142) between $[0, t]$ we obtain:

$$\|v_n(t)\|_{H^2}^2 \leq C_T (\|v_n(0)\|_{H^2}^2 + |c_n - 1| + \|\psi_{c_n, p, \mu}^p - \psi_{1, p, \mu}^p\|_{H^4} + b_n) \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (5.142)$$

Using (5.143) and that: $\lim_{n \rightarrow +\infty} \|\psi_{c_n, p, \mu} - \psi_{1, p, \mu}\|_{H^s(\mathbb{R})} = 0$ we also obtain that:

$$\beta_n(t) = \frac{1}{\int (\psi'_{c_n, p, \mu})^2} \int \tilde{w}_n L_{c_n, p, \mu} \psi_{c_n, p, \mu}'' \rightarrow \tilde{\beta}(t) = \frac{1}{\int (\psi'_{1, p, \mu})^2} \int \tilde{w} L_{1, p, \mu} \psi_{1, p, \mu}''. \quad (5.143)$$

Now we set: $w(t, x) = \tilde{w}(t, x) + \psi_{1, p, \mu}'' \int_0^t \tilde{\beta}(s) ds$. Thus

$$w_n(t) \rightarrow w(t) \text{ in } H^2 \text{ as } n \rightarrow +\infty, \quad (5.144)$$

and we deduce that $w(t, x)$ satisfies the following equation:

$$\partial_t w = \partial_x(L_{1,p,\mu} w) + \tilde{\beta}(t)\psi'_{1,p,\mu}. \quad (5.145)$$

We also have:

$$\lim_{n \rightarrow +\infty} \|w_n(0)\|_{H^2} = \|w(0)\|_{H^2} \geq \frac{1}{2}, \quad (5.146)$$

$$\forall t \in \mathbb{R}, \quad \int w(t, x)\psi_{1,p,\mu} = \int w(t, x)\psi'_{1,p,\mu} = 0, \quad (5.147)$$

and

$$\forall x_0 > 0, \quad \forall t \in \mathbb{R}, \quad \int_{|x| > x_0} [\mu(\partial_x^2 w)^2 + (\partial_x w)^2 + w^2](t, x) \lesssim e^{-x_0/K}. \quad (5.148)$$

We define:

$$\bar{w}(t, x) = w(t, x) + \frac{1}{\int \rho_{1,p,\mu}\psi_{1,p,\mu}} J(t)\psi'_{1,p,\mu} \quad \text{where} \quad J(t) = \int_0^y w(t, x) \rho_{1,p,\mu}(x) dx dy. \quad (5.149)$$

One can easily compute that:

$$J'(t) = -\beta_n(t) \int \rho_{1,p,\mu}\psi_{1,p,\mu}, \quad (5.150)$$

then $\bar{w}(t, x)$ satisfies:

$$\partial_t \bar{w} = \partial_x(L_{1,p,\mu}\bar{w}), \quad (5.151)$$

and it is simple to see that $\bar{w}(t, x)$ is H^2 -compact:

$$\forall x_0 > 0, \quad \forall t \in \mathbb{R}, \quad \sum_{k=0}^2 \int_{|x| > x_0} (\partial_x^k \bar{w})^2(t, x) \leq e^{-x_0/K}. \quad (5.152)$$

By the linear Liouville property related to the gKW equation, we infer that:

$$\bar{w}(t, x) = a(t)\psi'_{1,p,\mu}(x), \quad (5.153)$$

and then

$$\tilde{w}(t, x) = \left(a(t) - \frac{1}{\int \rho_{1,p,\mu}\psi_{1,p,\mu}} J(t) \right) \psi'_{1,p,\mu}(x). \quad (5.154)$$

Since $\langle w, \psi'_{1,p,\mu} \rangle_{L^2} = 0$ we deduce that:

$$\forall t \in \mathbb{R}, \quad x \in \mathbb{R}, \quad w(t, x) \equiv 0, \quad (5.155)$$

this contradicts (5.146) and concludes the proof of the theorem.

6 Asymptotic stability of the solitons of gKW equation

In this Section, by applying the method of Martel and Merle [12], we will prove the asymptotic stability of the solitary waves for the generalized Kawahara equation (gKW).

Theorem 6.1 (Asymptotic stability of φ_{c_0,p,μ_p}). *Let $p \in \{1, 2\}$, $c_0 \in]1 - \delta_p, 1 + \delta_p[$, with $\delta_p \ll 1$, and $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$. Let $u(t, x)$ be the global H^2 -solution of the gKW equation which is close at the initial time ($t = 0$):*

$$\|u_0 - \varphi_{c_0,p,\mu_p}\|_{H^2} \leq \gamma, \quad \text{with } \gamma \ll 1, \quad \text{and } u_0(x) = u(0, x). \quad (6.1)$$

Then there exists $c(t) \in C^1(\mathbb{R})$ and $\xi(t) \in C^1(\mathbb{R})$ such that:

$$u(t, x) - \varphi_{c(t)}(x - \xi(t)) \rightarrow 0 \quad \text{in } H^2(x > \theta t) \quad \text{as } t \rightarrow +\infty, \quad \text{where } \theta > 0. \quad (6.2)$$

Moreover there exists \tilde{c}_0 such that:

$$\xi'(t) \rightarrow \tilde{c}_0 \quad \text{as } t \rightarrow +\infty \quad \text{and } |\tilde{c}_0 - c_0| \leq \gamma. \quad (6.3)$$

6.1 Construction of modulation parameters II

In this Subsection we recall one more time the Lemma which give us the existence and the estimate of the modulation parameters.

Lemma 6.1 (Construction of modulation parameters). *Let $p = \{1, 2, 3, 4, 5, \dots\}$, $c_0 \in]1 - \delta_p, 1 + \delta_p[$ with $\delta_p \ll 1$, and $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$. Assume that $u(t, x)$ is the global H^2 solution of the gKW equation which is close at the initial time ($t = 0$):*

$$\|u_0 - \varphi_{c_0, p, \mu_p}\|_{H^2} \leq \gamma, \text{ avec } \gamma \ll 1, \text{ et } u_0(x) = u(0, x). \quad (6.4)$$

Then there exist $c(t) \in C^1(\mathbb{R})$ and $\xi(t) \in C^1(\mathbb{R})$ such that:

$$\eta(t, x) = u(t, x + \xi(t)) - \varphi_{c(t), p, \mu_p}(x), \quad (6.5)$$

satisfying for all $t \in \mathbb{R}$,

$$|\xi(0)| + |c(t) - c_0| + \|\eta(t)\|_{H^2} \leq K_0 \gamma, \quad (6.6)$$

$$\int \eta(t, x), \varphi'_{c(t)}(x) = \int \eta(t, x), \varphi_{c(t)}(x) = 0, \quad (6.7)$$

and

$$|c'(t)|^{1/2} + |\xi'(t) - c(t)| \leq K_0 \left(\int \eta^2(t, x) \varphi'_{c, p, \mu_p}(x) \right)^{1/2} \leq K_0 \|\eta(t)\|_{L^2}, \quad (6.8)$$

where K_0 is a positive constant.

6.2 Monotonicity formula retated to the gKW equation II

In this Subsection, we establish the monotonicity property with the mass L^2 and energy H^2 of the global solution of the gKW equation localized by the right. Thanks to the reversibility fo the gKW equation, we deduce the monotonicity formula with the mass and the energy localized by the left.

Proposition 6.1 (Monotonicity formula by the right). *Let $p \in \mathbb{N}^*$, $c_0 \in]1 - \delta_p, 1 + \delta_p[$ with $\delta_p \ll 1$, and $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$. Let $x_0 > 0$, $K > 0$ chosen correctly, $t_0 > t$, and $\phi_K(x) = \frac{1}{2} + \frac{1}{2} \tanh(x/2K)$. We set: $\bar{x} = x - \xi(t_0) + \frac{c_0}{2}(t_0 - t) - x_0$, and we define:*

$$I_{x_0, t_0}(t) = \int u^2(t, x) \phi_K(\bar{x}) \text{ and } J_{x_0, t_0}(t) = \int \left(\frac{\mu_p}{2} (\partial_x^2 u)^2 + \frac{1}{2} (\partial_x u)^2 - \frac{1}{(p+1)(p+2)} u^{p+2} \right) (t, x) \phi_K(\bar{x}). \quad (6.9)$$

Assume that $u(t, x)$ is the global H^2 -solution of the gKW equation which is close at the initial time ($t = 0$):

$$\|u_0 - \varphi_{c_0, p, \mu_p}\|_{H^2} \leq \gamma, \text{ with } \gamma \ll 1, \text{ and } u_0(x) = u(0, x). \quad (6.10)$$

Then we have:

$$I_{x_0, t_0}(t_0) - I_{x_0, t_0}(t) \lesssim e^{-x_0/K} \text{ and } J_{x_0, t_0}(t_0) - J_{x_0, t_0}(t) \lesssim e^{-x_0/K}. \quad (6.11)$$

Moreover if we assume that $u(t, x + \xi(t))$ is L^2 -compact:

$$\forall \varepsilon > 0 \exists R_\varepsilon > 0 \text{ such that } \forall t \in \mathbb{R} \int_{|x| > R_\varepsilon} |u(t, x + \xi(t))|^2 \leq \varepsilon, \quad (6.12)$$

Then $u(t) \in H^k$ for all $k \in \mathbb{N}$, and we have:

$$|\partial_x^k u(t, x + \xi(t))| \lesssim e^{-|x|/2K}, \forall (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (6.13)$$

Where the translation function $\xi(t)$ is given by Lemma 6.1.

Remark 6.1 (Monotonicity formula by the left). Assume that $u(t, x)$ satisfied (6.4) and we set: $v(t, x) = u(-t, -x)$. It is simple to check that $v(t, x)$ is the solution of gKW, and that: $\|v_0 - \varphi_{c_0, p, \mu_p}\|_{H^2} \leq \gamma$ where $v_0(x) = v(0, x)$ (since the soliton $\varphi_{c_0, p, \mu_p}(x)$ is even). Thanks to the stability result established by K-M, we know that: $\sup_t \inf_r \|u(t) - \varphi_{c_0, p, \mu_p}(\cdot - r)\|_{H^2} \leq \gamma^{1/2}$ for the nonlinearity $p \in \{1, 2, 3, 4\}$. Now, by Lemma 6.1, there exists $\xi_1(t)$ and $c_1(t)$ such that $v(t, x)$ satisfied at the two orthogonality conditions (6.7):

$$\int (v(t, x + \xi_1(t)) - \varphi_{c_1(t), p, \mu_p}(x)) \varphi'_{c_1(t), p, \mu_p}(x) = - \int (u(-t, x + (-\xi_1(t))) - \varphi_{c_1(t), p, \mu_p}(x)) \varphi'_{c_1(t), p, \mu_p}(x) = 0 \quad (6.14)$$

and

$$\int (v(t, x + \xi_1(t)) - \varphi_{c_1(t), p, \mu_p}(x)) \varphi_{c_1(t), p, \mu_p}(x) = \int (u(-t, x + (-\xi_1(t))) - \varphi_{c_1(t), p, \mu_p}(x)) \varphi_{c_1(t), p, \mu_p}(x) = 0 \quad (6.15)$$

By uniqueness of the modulation parameters (given by the Implicite Function Theorem), we infer that $\xi_1(t) = -\xi(-t)$ and $c_1(t) = c(-t)$.

Applying the monotonicity formula by the right (6.11) to $v(t, x)$, we obtain:

$$\int u^2(-t_0, -x) \phi_K(x + \xi(-t_0) - x_0) \lesssim \int u^2(-t, -x) \phi_K(x + \xi(-t_0 + \frac{c_0}{2}(t_0 - t) - x_0) + e^{-x_0/K}. \quad (6.16)$$

Using the transformation $-x \rightarrow x$ and that $\int u^2(-t_0) = \int u^2(-t)$, we get:

$$\int u^2(-t, x) (1 - \phi_K(-x + \xi(-t_0) + \frac{c_0}{2}(t_0 - t) - x_0)) \lesssim \int u^2(-t_0, x) (1 - \phi_K(-x + \xi(-t_0) - x_0)) + e^{-x_0/K}. \quad (6.17)$$

Finally, by setting $t_1 = -t$, $t_2 = -t_0$, and using that $\phi_K(x) = 1 - \phi_K(-x)$, we obtain for all $t_1 > t_2$ (since $t_0 > t$) the monotonicity formula by the right:

$$\int u^2(t_1, x) \phi_K(x - \xi(t_2) - \frac{c_0}{2}(t_1 - t_2) + x_0) \lesssim \int u^2(t_2, x) \phi_K(x - \xi(t_2) + x_0) + e^{-x_0/K}. \quad (6.18)$$

Arguing similar with the energy localized by the right (6.11), we also obtain:

$$\begin{aligned} & \int \left(\frac{\mu_p}{2} (\partial_x^2 u)^2 + \frac{1}{2} (\partial_x u)^2 - \frac{1}{(p+1)(p+2)} u^{p+2} \right) (t_1, x) \phi_K(x - \xi(t_2) - \frac{c_0}{2}(t_1 - t_2) + x_0) \\ & \lesssim \int \left(\frac{\mu_p}{2} (\partial_x^2 u)^2 + \frac{1}{2} (\partial_x u)^2 - \frac{1}{(p+1)(p+2)} u^{p+2} \right) (t_2, x) \phi_K(x - \xi(t_2) + x_0) + e^{-x_0/K}, \end{aligned} \quad (6.19)$$

since the energy is also the conservation law for the gKW.

Proof of the Proposition (5.1). All the elements of the proof of this Proposition are content in [6], we will recall them for a better understanding.

One can easy compute that the test function satisfied: $\phi'_K(x) = \frac{1}{4K} \operatorname{sech}^2\left(\frac{x}{2K}\right) \lesssim e^{-|x|/K}$ and that $|\phi_K^{(j+1)}(x)| \leq \frac{1}{K^j} \phi'_K(x) \forall j \in \mathbb{N}$.

Step 1: Monotonicity formula with the L^2 -mass localized by the right.

Using only that: $\|u(t) - \varphi_{c_0, p, \mu_p}(\cdot - \xi(t))\|_{H^1} \leq \gamma$ (which is given by (6.6)), we will prove that for $x_0 > 0$, $K > 0$ chosen correctly, and $t_0 > t$:

$$I_{x_0, t_0}(t_0) - I_{x_0, t_0}(t) \lesssim e^{-x_0/K}. \quad (6.20)$$

By differentiating $I_{x_0, t_0}(t)$ with respect to time (see Virial Identity I in K. (2019)):

$$\begin{aligned} \frac{d}{dt} I_{x_0, t_0}(t) &= -5\mu_p \int (\partial_x^2 u)^2 \phi'_K(\bar{x}) - 3 \int (\partial_x u)^2 \phi'_K(\bar{x}) - \frac{c_0}{2} \int u^2 \phi'_K(\bar{x}) \\ & \quad + 5\mu_p \int (\partial_x u)^2 \phi_K^{(3)}(\bar{x}) - \mu_p \int u^2 \phi_K^{(5)}(\bar{x}) + \int u^2 \phi_K^{(3)}(\bar{x}) \\ & \quad + \frac{2}{p+2} \int u^{p+2} \phi'_K(\bar{x}). \end{aligned} \quad (6.21)$$

Let us make the estimate of the terms in the second line of identity (6.21). Using that $\left| \phi_K^{(j+1)}(x) \right| \lesssim \frac{1}{K^j} \phi_K'(x)$, we obtain:

$$\begin{aligned} & \left| 5\mu_p \int (\partial_x u)^2 \phi_K^{(3)}(\bar{x}) - \mu_p \int u^2 \phi_K^{(5)}(\bar{x}) + \int u^2 \phi_K^{(3)}(\bar{x}) \right| \\ & \leq \frac{5\mu_p}{K^2} \int (\partial_x u)^2 \phi_K'(\bar{x}) + \left(\frac{\mu_p}{K^4} + \frac{1}{K^2} \right) \int u^2 \phi_K'(\bar{x}). \end{aligned} \quad (6.22)$$

Let us make the estimate of the term which contains the nonlinearity: $\frac{2}{p+2} \int u^{p+2} \phi_K'(\bar{x})$. Let $R_0 > 0$, we divide \mathbb{R} in two parts:

$$\begin{aligned} \frac{2}{p+2} \int_{\mathbb{R}} u^{p+2} \phi_K'(\bar{x}) &= \frac{2}{p+2} \int_{|x-\xi(t)| > R_0} u^{p+2} \phi_K'(\bar{x}) + \frac{2}{p+2} \int_{|x-\xi(t)| < R_0} u^{p+2} \phi_K'(\bar{x}) \\ &= A + B. \end{aligned} \quad (6.23)$$

On the one hand, using the Sobolev embedding of H^1 into C^0 , we deduce that: $\sup_{|x-\xi(t)| > R_0} |u(t, x)| \leq \sup_{|x-\xi(t)| > R_0} |\varphi_{c_0, p, \mu_p}(x - \xi(t))| + \gamma$. Then we have:

$$|A| \lesssim (e^{-\sqrt{c_0}R_0} + \gamma)^p \int u^2 \phi_K'(\bar{x}). \quad (6.24)$$

On the other hand, for $\gamma \ll 1$, (6.8) give us: $\dot{\xi}(t) \geq 3c_0/4$, and we deduce that: $|\bar{x}| \geq -|x - \xi(t)| + |(\xi(t_0) - \xi(t)) - \frac{c_0}{2}(t_0 - t) + x_0| \geq -R_0 + \frac{c_0}{4}(t_0 - t) + x_0$. Then, using that $\phi_K'(x) \lesssim e^{-|x|/K}$, and that $u(t)$ is uniformly bounded H^1 , we obtain:

$$|B| \lesssim e^{-x_0/K} e^{-c_0(t_0-t)/4K}. \quad (6.25)$$

Finally, adding (6.21)-(6.25), choosing R_0, K and γ correctly, and by integrating $\frac{d}{dt} I_{x_0, t_0}(t)$ between $[t, t_0]$, we get:

$$I_{x_0, t_0}(t_0) + \sum_{k=0}^2 \int_t^{t_0} \int (\partial_x^k u)^2(s, x) \phi_K'(\bar{x}) dx ds \lesssim e^{-x_0/K} + I_{x_0, t_0}(t). \quad (6.26)$$

This implies in particular (6.20).

Step 2: Asymptotic limit of the L^2 -mass localized by right.

Using that (6.26) and that $u(t, x + \xi(t))$ is L^2 -compact (assumption (6.12)), we will prove that:

$$\int u^2(t_0, x) \phi_K(x - \xi(t_0) - x_0) dx + \int_{-\infty}^{t_0} \int ((\partial_x^2 u)^2 + (\partial_x u)^2 + u^2)(s, x) \phi_K'(\bar{x}) dx ds \lesssim e^{-x_0/K}. \quad (6.27)$$

Form (6.26), it suffice to prove that: $\lim_{t \rightarrow -\infty} I_{x_0, t_0}(t) = 0$. Let $\varepsilon > 0$, form (6.12), there exists $R_\varepsilon > 0$ such that $\forall t \in \mathbb{R}$:

$$\int_{x > R_\varepsilon} u^2(t, x + \xi(t)) < \varepsilon. \quad (6.28)$$

we divide \mathbb{R} in two parts: $\{x < R_\varepsilon\}$ and $\{x > R_\varepsilon\}$, and we compute:

$$I_{x_0, t_0}(t) = \int_{x > R_\varepsilon} u^2(t, x + \xi(t)) \phi_K(\bar{x} + \xi(t)) + \int_{x < R_\varepsilon} u^2(t, x + \xi(t)) \phi_K(\bar{x} + \xi(t)) = A_1 + B_1. \quad (6.29)$$

First, using that (6.28) and that $\|\phi_K\|_{L^\infty} \leq 1$, we obtain:

$$A_1 \lesssim \varepsilon. \quad (6.30)$$

Second, using that: $\dot{\xi}(t) \geq 3c_0/4$, one can estimate that: $\bar{x} + \xi(t) \leq x - \frac{c_0}{4}(t_0 - t) - x_0$. Since ϕ_K is increasing on \mathbb{R} , and that L^2 -norm of $u(t, x)$ is conserved, we have:

$$B_1 \lesssim \phi_K(R_\varepsilon - \frac{c_0}{4}(t_0 - t) - x_0) \rightarrow \phi_K(-\infty) = 0 \text{ as } t \rightarrow -\infty. \quad (6.31)$$

Thus (6.30) and (6.31) give us $\lim_{t \rightarrow -\infty} I_{x_0, t_0}(t) = 0$, and we infer (6.27).

Step 3: Monotonicity formula with the H^2 -energy localized by the right.

Using only that: $\|u(t) - \varphi_{c_0, p, \mu_p}(\cdot - \xi(t))\|_{H^2} \leq \gamma$, we will prove that for $x_0 > 0$, $K > 0$ chosen correctly, and $t_0 > t$:

$$J_{x_0, t_0}(t_0) - J_{x_0, t_0}(t) \lesssim e^{-x_0/K}. \quad (6.32)$$

By differentiating $J_{x_0, t_0}(t)$ with respect to time (see Virial identity II in K. (2019)), we obtain:

$$\begin{aligned} \frac{d}{dt} J_{x_0, t_0}(t) = & -\frac{5\mu_p^2}{2} \int (\partial_x^4 u)^2 \phi'_K(\bar{x}) - 4\mu_p \int (\partial_x^3 u)^2 \phi'_K(\bar{x}) - \left(\frac{3}{2} + \frac{c_0 \mu_p}{4}\right) \int (\partial_x^2 u)^2 \phi'_K(\bar{x}) - \frac{c_0}{4} \int (\partial_x u)^2 \phi'_K(\bar{x}) \\ & - \frac{1}{2(p+1)^2} \int u^{2p+2} \phi'_K(\bar{x}) + \frac{5\mu_p^2}{2} \int (\partial_x^3 u)^2 \phi_K^{(3)}(\bar{x}) - \frac{\mu_p^2}{2} \int (\partial_x^2 u)^2 \phi_K^{(5)}(\bar{x}) + 3\mu_p \int (\partial_x^2 u)^2 \phi_K^{(3)}(\bar{x}) \\ & - \frac{\mu_p}{2} \int (\partial_x u)^2 \phi_K^{(5)}(\bar{x}) + \frac{1}{2} \int (\partial_x u)^2 \phi_K^{(3)}(\bar{x}) + 3\mu_p \int (\partial_x^2 u)^2 u^p \phi'_K(\bar{x}) - \mu_p p(p-1) \int (\partial_x u)^4 u^{p-2} \phi'_K(\bar{x}) \\ & - \frac{15\mu_p}{6} \int (\partial_x u)^3 u^{p-1} \phi_K''(\bar{x}) + 2 \int (\partial_x u)^2 u^p \phi'_K(\bar{x}) - \frac{5\mu_p}{2} \int (\partial_x u)^2 u^p \phi_K^{(3)}(\bar{x}) \\ & + \frac{\mu_p}{(p+1)(p+2)} \int u^{p+2} \phi_K^{(5)}(\bar{x}) - \frac{1}{(p+1)(p+2)} \int u^{p+2} \phi_K^{(3)}(\bar{x}) + \frac{c_0}{2(p+1)(p+2)} \int u^{p+2} \phi'_K(\bar{x}). \end{aligned} \quad (6.33)$$

The estimate of all terms in (6.33) are made following the same logic Step 1. For clarity we will estimate two of them. For instance the estimate of: $\frac{5\mu_p^2}{2} \int (\partial_x^3 u)^2 \phi_K^{(3)}(\bar{x})$. We have:

$$\left| \frac{5\mu_p^2}{2} \int (\partial_x^3 u)^2 \phi_K^{(3)}(\bar{x}) \right| \leq \frac{5\mu_p^2}{2K^2} \int (\partial_x^3 u)^2 \phi'_K(\bar{x}). \quad (6.34)$$

Let's now estimate a term which contains the nonlinearity for to apply the assumption made on $u(t, x)$. For instance: $-\frac{15\mu_p}{6} \int (\partial_x u)^3 u^{p-1} \phi_K''(\bar{x})$. First, we know that $u(t)$ is bounded in H^2 . Second, by the Sobolev embedding of H^2 in C^1 , we know that: $\|u(t)\|_{C^1} \leq \|\varphi_{c_0, p, \mu_p}\|_{C^1} + \gamma$. Thus, arguing as for the estimate of $\frac{2}{p+2} \int u^{p+2} \phi'_K(\bar{x})$, we obtain:

$$\left| \frac{15\mu_p}{6} \int (\partial_x u)^3 u^{p-1} \phi_K''(\bar{x}) \right| \lesssim (e^{-\sqrt{c_0}R_0} + \gamma)^p \int (\partial_x u)^2 \phi'_K(\bar{x}) + e^{-x_0/K} e^{-c_0(t_0-t)/4K}. \quad (6.35)$$

Thus, by combining (6.33)-(6.35), choosing K , R_0 and γ correctly, and integrating $\frac{d}{dt} J_{x_0, t_0}(t)$ between $[t, t_0]$, we obtain:

$$J_{x_0, t_0}(t_0) + \int_t^{t_0} \int \left(\sum_{k=1}^4 (\partial_x^k u)^2 + u^{2p+2} \right)(s, x) \phi'_K(\bar{x}) dx ds \lesssim e^{-x_0/K} + J_{x_0, t_0}(t), \quad (6.36)$$

this implies that (6.32).

Step 4: The asymptotic limit of the H^2 -energy localized by the right.

Using that (6.27) and (6.36), we will prove that:

$$J_{x_0, t_0}(t_0) + \int_{-\infty}^{t_0} \int \left(\sum_{k=1}^4 (\partial_x^k u)^2 + u^{2p+2} \right)(s, x) \phi'_K(\bar{x}) dx ds \lesssim e^{-x_0/K}. \quad (6.37)$$

According to (6.36) it suffice to prove that: $\lim_{t \rightarrow -\infty} J_{x_0, t_0}(t) = 0$. First, one can see that: $J_{x_0, t_0}(t) \lesssim \sum_{k=0}^2 \int (\partial_x^k u)^2 \phi_K(\bar{x})$. Now, using that (6.27) and that $\phi'_K(\bar{x}) \sim \phi_K(\bar{x}) \sim e^{\bar{x}/K}$ for $\bar{x} < 0$, we have:

$$\sum_{k=0}^2 \int_{-\infty}^{t_0} \int_{x < \xi(t_0) + \frac{c_0}{2}(t_0-t) + x_0} (\partial_x^k u)^2 e^{(x - \xi(t_0) + \frac{c_0}{2}(t_0-t))/K} \lesssim 1. \quad (6.38)$$

Letting $x_0 \rightarrow +\infty$ and multiplying $e^{-x_0/K}$, we have:

$$\sum_{k=0}^2 \int_{-\infty}^{t_0} \int (\partial_x^k u)^2 \phi_K(\bar{x}) \lesssim e^{-x_0/K}, \quad (6.39)$$

and we deduce that:

$$\lim_{t \rightarrow -\infty} \sum_{k=0}^2 \int (\partial_x^k u)^2 \phi_K(\bar{x}) = 0. \quad (6.40)$$

Which leads to (6.37).

Step 5: Exponential decay u et u_x .

Using (6.27) and (6.37), we will prove that $k \in \{1, 2\}$:

$$|\partial_x^k u(t_0, x + \xi(t_0))| \lesssim e^{-|x|/2K}. \quad (6.41)$$

Indeed, thanks to (6.27) and (6.37), and that $\phi_K(x - x_0) \gtrsim e^{(x-x_0)/K}$ if $x < x_0$, we have:

$$\sum_{k=0}^2 \int_{x < x_0} (\partial_x^k u)^2(t_0, x + \xi(t_0)) e^{x/K} \lesssim 1, \quad (6.42)$$

So, letting $x_0 \rightarrow +\infty$, we obtain the right exponential decay:

$$\sum_{k=0}^2 \int (\partial_x^k u)^2(t_0, x + \xi(t_0)) e^{x/K} \lesssim 1. \quad (6.43)$$

The remarque 6.1 give us the left exponential decay:

$$\sum_{k=0}^2 \int (\partial_x^k u)^2(-t_0, x + \xi(-t_0)) e^{-x/K} \lesssim 1. \quad (6.44)$$

We deduce (6.41) by applying the Gagliardo-Nirenberg inequality:

$$\begin{aligned} \|(\partial_x^k u(t_0, x + \xi(t_0))) e^{|x|/2K}\|_{C^0} &\leq C_G \|(\partial_x^k u(t_0, x + \xi(t_0))) e^{|x|/2K}\|_{L^2}^{1/2} \times \\ &\left(\|(\partial_x^k u(t_0, x + \xi(t_0))) e^{|x|/2K}\|_{L^2}^2 + \|(\partial_x^{k+1} u(t_0, x + \xi(t_0))) e^{|x|/2K}\|_{L^2}^2 \right)^{1/4} \lesssim 1. \end{aligned} \quad (6.45)$$

Step 6: Exponential decay of all derivatives of u .

We claim that $k \in \mathbb{N}$:

$$\int_{\mathbb{R}} (\partial_x^k u)^2(t, x) \phi_K(x - \xi(t_0) - x_0) dx + \sum_{j=0}^2 \int_{-\infty}^{t_0} \int_{\mathbb{R}} (\partial_x^{k+j} u)^2(s, x) \phi'_K(\bar{x}) dx ds \lesssim e^{-x_0/K}. \quad (6.46)$$

The proof of (6.46) is made by induction on k applying the similar arguments in Step 1-4 (see details in [6]).

6.3 Nonlinear Liouville property with the family of solitons of gKW which is close to the explicit soliton of gKW

In this Subsection we recall the nonlinear Liouville property established in [6]. This property is the main tools for the prove of Theorem 6.1.

Theorem 6.2 (Nonlinear Liouville property with φ_{c_0,p,μ_p}). *Let $p = \{1, 2\}$, $c_0 \in]1 - \delta_p, 1 + \delta_p[$ with $\delta_p \ll 1$, and $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$. Assume that $u(t, x)$ is the global H^2 -solution of the gKW equation which is close at the initial time ($t = 0$):*

$$\|u_0 - \varphi_{c_0,p,\mu_p}\|_{H^2} \leq \gamma, \text{ with } \gamma \ll 1, \text{ and } u_0(x) = u(0, x), \quad (6.47)$$

and assume moreover that $u(t, x + \xi(t))$ is L^2 -compact:

$$\forall \varepsilon > 0 \exists R_\varepsilon > 0 \text{ such that } \forall t \in \mathbb{R} \int_{|x| > R_\varepsilon} u^2(t, x + \xi(t)) \leq \varepsilon, \quad (6.48)$$

then

$$u(t, x) = \varphi_{c(0),p,\mu_p}(x - \xi(0) - c(0)t), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (6.49)$$

where the translation function $\xi(t)$ and the speed function $c(t)$ are given by Lemma 6.1.

By applying the method presented in [6]. we directly obtain the following corollary:

Corollary 6.1 (Nonlinear Liouville with $\psi_{c,p,\mu}$). *The Theorem 6.2 holds with the family of solitons $\{\psi_{c,p,\mu} : 0 < \mu \ll 1\}$ with $0 < \mu < \mu_0 \ll 1$, for any speed $c > 0$, and for the nonlinearity power $p = 1$.*

6.4 Convergence to a compact limit

Assume that (6.4), then using (6.6), we infer that the sequence $\{u(t, x + \xi(t))\}$ is bounded in H^2 . Thus there exists a asymptotic limit $\tilde{u}_0(x) \in H^2$ such that: $u(t, x + \xi(t)) \rightharpoonup \tilde{u}_0(x)$ weakly in H^2 as $t \rightarrow +\infty$. Martel and Merle understood that the asymptotic object $\tilde{u}_0(x)$ will be naturally close to φ_{c_0,p,μ_p} in H^2 , and the most remarkable $\tilde{u}_0(x)$ will be L^2 -compact (6.48). Thanks to the nonlinear Liouville Theorem 6.2, we obtain: $\tilde{u}_0(x) = \varphi_{\tilde{c}(0),p,\mu_p}(x)$, where $\tilde{\xi}(0) = 0$ and $\tilde{c}(0)$ are given by Lemma 6.1.

Proposition 6.2 (Convergence to a compact solution). *Let $p = \{1, 2, 3, 4\}$, $c_0 \in]1 - \delta_p, 1 + \delta_p[$ with $\delta_p \ll 1$, and $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$. Assume that $u(t, x)$ is the global H^2 -solution of the gKW equation which is close at the initial time ($t = 0$):*

$$\|u_0 - \varphi_{c_0,p,\mu_p}\|_{H^2} \leq \gamma, \text{ with } \gamma \ll 1, \text{ and } u_0(x) = u(0, x). \quad (6.50)$$

Let $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Then there exists $(t_{\tau(n)})$, $\tilde{c}_0 > 0$, and $\tilde{u}_0 \in H^2$ such that:

$$u(t_{\tau(n)}, \cdot + \xi(t_{\tau(n)})) \rightharpoonup \tilde{u}_0 \text{ weakly in } H^2 \text{ as } n \rightarrow +\infty, \quad (6.51)$$

for all $A > 0$,

$$u(t_{\tau(n)}, \cdot + \xi(t_{\tau(n)})) \rightarrow \tilde{u}_0 \text{ strongly in } H^2(x > -A) \text{ as } n \rightarrow +\infty \quad (6.52)$$

and

$$c(t_{\tau(n)}) \rightarrow \tilde{c}_0 \text{ as } n \rightarrow +\infty, \quad (6.53)$$

where $\xi(t)$ and $c(t)$ are given by Lemma 6.1.

Moreover, $\tilde{u}(t, x)$ the global H^2 -solution of gKW emanating by the initial data $\tilde{u}(0, x) = \tilde{u}_0(x)$ satisfied:

$$\|\tilde{u}_0 - \varphi_{c_0,p,\mu_p}\|_{H^2} \leq \gamma \quad (6.54)$$

and $\tilde{u}(t, x + \tilde{\xi}(t))$ is L^2 -compact:

$$\forall \varepsilon > 0 \exists R_\varepsilon > 0 \text{ such that } \forall t \in \mathbb{R} \int_{|x| > R_\varepsilon} \tilde{u}^2(t, x + \tilde{\xi}(t)) \leq \varepsilon, \quad (6.55)$$

where $\tilde{\xi}(t)$ and $\tilde{c}(t)$ are given by Lemma 6.8 and $\tilde{\xi}(0) = 0$.

Proof. We will follow the ideas of Martel and Merle and we will present the proof in five steps.

Assume that $u(t, x)$ satisfied (6.1), then thanks the stability result established by K-M, we know that: $\sup_t \inf_r \|u(t) - \varphi_{c_0, p, \mu_p}(\cdot - r)\|_{H^2} \leq \gamma$ for the nonlinearity $p \in \{1, 2, 3, 4\}$. Moreover, by applying the Lemma 6.1, we know that there exists $\xi(t)$ and $c(t)$ such that: $\|u(t, \cdot + \xi(t)) - \varphi_{c(t), p, \mu_p}\|_{H^2} \leq \gamma$ and $|c(t) - c_0| \leq \gamma \forall t \in \mathbb{R}$. Using that: $\lim_{c \rightarrow c_0} \|\varphi_{c, p, \mu_p} - \varphi_{c_0, p, \mu_p}\|_{H^2} = 0$, We infer that:

$$\|u(t, \cdot + \xi(t))\|_{H^2} \leq \|u(t, \cdot + \xi(t)) - \varphi_{c(t), p, \mu_p}\|_{H^2} + \|\varphi_{c(t), p, \mu_p} - \varphi_{c_0, p, \mu_p}\|_{H^2} + \|\varphi_{c_0, p, \mu_p}\|_{H^2} \leq \|\varphi_{c_0, p, \mu_p}\|_{H^2} + \gamma. \quad (6.56)$$

Let $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$, by (6.56) the sequence $\{u(t_n, x + \xi(t_n))\}$ is bounded in H^2 , then up to a subsequence, there exists $\tilde{u}_0 \in H^2$ such that:

$$u(t_n, \cdot + \xi(t_n)) \rightharpoonup \tilde{u}_0 \text{ weakly in } H^2 \text{ as } n \rightarrow +\infty, \quad (6.57)$$

and also:

$$u(t_n, \cdot + \xi(t_n)) \rightarrow \tilde{u}_0 \text{ strongly in } H^1_{loc} \text{ as } n \rightarrow +\infty. \quad (6.58)$$

Moreover, using the lower semi-continuity, we infer that:

$$\|\tilde{u}_0 - \varphi_{c_0, p, \mu_p}\|_{H^2} \leq \liminf_{n \rightarrow +\infty} \|u(t_n, \cdot + \xi(t_n)) - \varphi_{c_0, p, \mu_p}\|_{H^2} \leq \gamma. \quad (6.59)$$

Then, thanks to the stability result: $\sup_t \inf_r \|\tilde{u}(t) - \varphi_{c_0, p, \mu_p}(\cdot - r)\|_{H^2} \leq \gamma$, where $\tilde{u}(t, x)$ is the global H^2 -solution of gKW emanating by the initial datum $\tilde{u}(0, x) = \tilde{u}_0(x)$. Moreover, by Lemma 6.1, there exist $\tilde{\xi}(t)$ and $\tilde{c}(t)$ such that (6.5)-(6.8) are satisfied.

Step 1: Strong convergence in H^1 of $u(t_n, x + \xi(t_n))$ to $\tilde{u}_0(x)$ by the right and exponential decay of $\tilde{u}_0(x)$ and $\partial_x \tilde{u}_0(x)$ by the right.

We will prove that for all $A > 0$:

$$u(t_n, \cdot + \xi(t_n)) \rightarrow \tilde{u}_0 \text{ in } H^1(x > -A) \text{ as } n \rightarrow +\infty, \quad (6.60)$$

and for all $x_0 > 0$ and $K > 0$ chosen correctly:

$$\sum_{k=0}^2 \int (\partial_x^k \tilde{u}_0)^2(x) \phi_K(x - x_0) \lesssim e^{-x_0/K}. \quad (6.61)$$

Using the monotonicity property by the right (6.11), one can see that:

$$\limsup_{t \rightarrow +\infty} \sum_{k=0}^2 \int (\partial_x^k u)^2(t, x + \xi(t)) \phi_K(x - x_0) \lesssim e^{-x_0/K}. \quad (6.62)$$

Indeed, according to (6.11), we have:

$$\int u^2(t, x) \phi_K(x - \xi(t) - x_0) \lesssim \int u_0^2(x + \xi(0)) \phi_K(x - (\xi(t) - \xi(0)) + \frac{c_0}{2}t - x_0) + e^{-x_0/K}. \quad (6.63)$$

Recall that for γ small enough, (6.6) and (6.8) imply that: $\xi(t) - \xi(0) \sim c_0 t$. Thus, using that: $\lim_{t \rightarrow +\infty} \phi_K(x - \frac{c_0}{2}t + x_0) = 0$, and applying Lebesgue's Dominated Convergence Theorem, we obtain:

$$\limsup_{t \rightarrow +\infty} \int u^2(t, x) \phi_K(x - \xi(t) - x_0) \lesssim e^{-x_0/K}. \quad (6.64)$$

Arguing similarly with the H^2 -energy localized by the right $J_{x_0, t_0}(t)$, we obtain (6.62).

It is clear that the strong convergence in H^1_{loc} (6.58) combined with (6.62), give us (6.60).

Thanks to (6.57), we remark that: $\forall k \in \{0, 1, 2\}$ $(\partial_x^k u)(t_n, \cdot + \xi(t_n))\sqrt{\phi_K(\cdot - x_0)} \rightarrow (\partial_x^k \tilde{u}_0)\sqrt{\phi_K(\cdot - x_0)}$ in L^2 (since $\forall g \in L^2$, we have $\sqrt{\phi_K(\cdot - x_0)}g \in L^2$). Thus, using the lower semi-continuity and (6.62):

$$\sum_{k=0}^2 \int (\partial_x^k \tilde{u}_0)^2(x) \phi_K(x - x_0) \leq \liminf_{n \rightarrow +\infty} \sum_{k=0}^2 \int (\partial_x^k u)^2(t_n, x + \xi(t_n)) \phi_K(x - x_0) \lesssim e^{-x_0/K}. \quad (6.65)$$

Moreover, proceeding as in Step 5 (Monotonicity formula), we deduce that:

$$\|\tilde{u}_0(x)e^{x/2K}\|_{C^1} \lesssim 1. \quad (6.66)$$

Step 2: Strong convergence in L^2 of $u(t + t_n, x + \xi(t_n))$ to $\tilde{u}(t, x)$ by the right and exponential decay of $\tilde{u}(t, x)$ and $\partial_x \tilde{u}(t, x)$ by the right.

We will prove that for all $A > 0$ and $\forall t \in \mathbb{R}$:

$$u(t + t_n, \cdot + \xi(t_n)) \rightarrow \tilde{u}(t, \cdot) \text{ in } L^2(x > -A) \text{ as } n \rightarrow +\infty, \quad (6.67)$$

$$u(t + t_n, \cdot + \xi(t_n)) \rightarrow \tilde{u}(t, \cdot) \text{ in } H^2 \text{ as } n \rightarrow +\infty. \quad (6.68)$$

Moreover, for all $x_0 > 0$ and $K > 0$ chosen correctly, $t \in \mathbb{R}$:

$$\sum_{k=0}^2 \int (\partial_x^k \tilde{u})^2(t, x) \phi_K(x - x_0) \lesssim e^{-x_0/K}, \quad (6.69)$$

and

$$\|\tilde{u}(t, x)e^{x/2K}\|_{C^1} \lesssim 1. \quad (6.70)$$

Assume (6.68), it is clear that using the lower semi-continuity and (6.62), we have (6.69). (6.70) follows by applying the same arguments as in Step 5 (Monotonicity formula).

Assume now (6.67). Let $t_1 \in \mathbb{R}$, we have in particular:

$$u(t_1 + t_n, \cdot + \xi(t_n)) \rightarrow \tilde{u}(t_1, \cdot) \text{ in } L_{loc}^2 \text{ as } n \rightarrow +\infty. \quad (6.71)$$

Since by hypothesis: $\|u(t_n, \cdot + \xi(t_n)) - \varphi_{c_0, p, \mu_p}\|_{H^2} \leq \gamma$, then thanks to the orbital stability: $\sup_t \inf_r \|u(t + t_n, \cdot + \xi(t_n)) - \varphi_{c_0, p, \mu_p}(\cdot - r)\|_{H^2} \leq \gamma$, and by Lemma 6.1 we know that there exist $\xi_2(t)$ et $c_2(t)$ such that: $\|u(t_1 + t_n, \cdot + \xi(t_n)) - \varphi_{c_2(t), p, \mu_p}(x - \xi_2(t))\|_{H^2} \leq \gamma$ with $|c_2(t) - c_0| \leq \gamma$. Thus the sequence $\{u(t_1 + t_n, x + \xi(t_n))\}$ is bounded in H^2 . Then up to a subsequence, there exists $\tilde{u}_1 \in H^2$ such that:

$$u(t_1 + t_n, \cdot + \xi(t_n)) \rightarrow \tilde{u}_1 \text{ in } H^2 \text{ as } n \rightarrow +\infty, \quad (6.72)$$

and

$$u(t_1 + t_n, \cdot + \xi(t_n)) \rightarrow \tilde{u}_1 \text{ in } H_{loc}^1 \text{ as } n \rightarrow +\infty. \quad (6.73)$$

We deduce that $\forall g \in C_c^\infty$:

$$\begin{aligned} \int |\tilde{u}(t_1, x) - \tilde{u}_1(x)|^2 g &\leq 2 \int |u(t_1 + t_n, x + \xi(t_n)) - \tilde{u}(t_1, x)|^2 g \\ &\quad + 2 \int |u(t_1 + t_n, x + \xi(t_n)) - \tilde{u}_1(x)|^2 g \rightarrow 0 \text{ as } n \rightarrow +\infty. \end{aligned} \quad (6.74)$$

Hence $\tilde{u}_1(x) = \tilde{u}(t_1, x) \forall x \in \mathbb{R}$, this implies (6.68).

It remains to prove (6.67). For all $t \in \mathbb{R}$ we define:

$$v_n(t, x) = u(t + t_n, x + \xi(t_n)) - \tilde{u}(t, x). \quad (6.75)$$

Using (6.60) one can see that:

$$\int v_n^2(0, x) \phi_K(x) \rightarrow 0 \text{ as } n \rightarrow +\infty, \quad (6.76)$$

and using the equation of $u(t, x)$ and of $\tilde{u}(t, x)$, we obtain the equation of $v_n(t, x)$:

$$\partial_t v_n = \mu_p \partial_x^5 v_n - \partial_x^3 v_n - \partial_x (f(\tilde{u} + v_n) - f(\tilde{u})), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (6.77)$$

where $f(x) = \frac{1}{p+1} x^{p+1}$. Let us prove (6.67) firstly for $t \geq 0$. By differentiating $\int v_n^2(t, x) \phi_K(x)$ with respect to time, we get:

$$\begin{aligned} \frac{d}{dt} \int v_n^2(t, x) \phi_K(x) &= -5\mu_p \int (\partial_x^2 v_n)^2 \phi'_K(x) - 3 \int (\partial_x v_n)^2 \phi'_K(x) + 5\mu_p \int (\partial_x v_n)^2 \phi_K^{(3)}(x) - \mu_p \int v_n^2 \phi_K^{(5)}(x) \\ &\quad + \int v_n^2 \phi_K^{(3)}(x) + \frac{2}{p+1} \sum_{k=1}^{p+1} C_{p+1}^k \int v_n^{k+1} \left[\tilde{u}^{p-k+1} \phi'_K(x) - \frac{1}{k+1} \partial_x (\tilde{u}^{p-k+1} \phi_K(x)) \right]. \end{aligned} \quad (6.78)$$

Using that: $\|u(t)\|_{H^1} \lesssim 1$, $\phi_K(x) \gtrsim \phi'_K(x) > 0$, and $\phi_K^{(j+1)}(x) \lesssim \frac{1}{K^j} \phi'_K(x) \forall j \in \mathbb{N}$, we obtain for K chosen correctly:

$$\frac{d}{dt} \int v_n^2(t, x) \phi_K(x) + C_0 \sum_{k=1}^2 \int (\partial_x^k v_n)^2(t, x) \phi'_K(x) \lesssim \int v_n^2(t, x) \phi_K(x), \quad (6.79)$$

where C_0 is a positive constant. So, by applying the Gronwall Lemma between $0 \leq t \leq t_0$, and using (6.76):

$$\sup_{t \in [0, t_0]} \int v_n^2(t, x) \phi_K(x) \lesssim e^{t_0} \int v_n^2(0, x) \phi_K(x) \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (6.80)$$

Now we will prove that (6.67) for $t \leq 0$. Let $t_1 < 0$, we know that (6.72) is true. Let $\tilde{u}_1(t, x)$ be the global H^2 -solution of gKW emanating from $\tilde{u}_1(0, x) = \tilde{u}_1(x)$. By repeating the previous analysis with \tilde{u}_1 , we have in particular:

$$u(t_n, \cdot + \xi(t_n)) \rightarrow \tilde{u}_1(-t_1) \text{ as } n \rightarrow +\infty, \quad (6.81)$$

Then combining (6.57) and (6.81), we deduce that $\forall g \in C_c^\infty$:

$$\int (\tilde{u}_0(x) - \tilde{u}_1(-t_1, x)) g = \int (\tilde{u}_0(x) - u(t_n, x + \xi(t_n))) g + \int (u(t_n, x + \xi(t_n)) - \tilde{u}_1(-t_1, x)) g \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (6.82)$$

This implies that: $\tilde{u}_0(x) = \tilde{u}_1(-t_1, x) \forall x \in \mathbb{R}$. By uniqueness of the global H^2 -solution of the gKW, we obtain $\tilde{u}_1(x) = \tilde{u}_1(0, x) = \tilde{u}(t_1, x)$. Thus for all $t_1 < 0$, (6.72) give us:

$$u(t_1 + t_n, \cdot + \xi(t_n)) \rightarrow \tilde{u}(t_1, \cdot) \text{ in } H^2 \text{ as } n \rightarrow +\infty. \quad (6.83)$$

Step 3: Exponential decay of $\partial_x^2 \tilde{u}(t, x)$ by the right.

We will prove that for all $x_0 > 0$, $K > 0$ chosen correctly, and $t \in \mathbb{R}$:

$$\int (\partial_x^3 \tilde{u})^2(t, x) \phi_K(x - x_0) \lesssim e^{-x_0/3K} \quad (6.84)$$

and

$$\|\tilde{u}(t, x) e^{x/2K}\|_{C^2} \lesssim 1. \quad (6.85)$$

We observe that (6.85) is a direct consequence of (6.69) and (6.84).

To simplify writing we define: $h(\tilde{u}) = \frac{\mu_p}{2}(\partial_x^2 \tilde{u})^2 + \frac{1}{2}(\partial_x \tilde{u})^2 - \frac{1}{(p+1)(p+2)}\tilde{u}^{p+2}$. We will differentiating $\int h(\tilde{u}(t, x))\phi_K(x - x_0)$ with respect to time:

$$\begin{aligned}
\frac{d}{dt} \int h(\tilde{u}(t, x))\phi_K(x - x_0) &= -\frac{5\mu_p^2}{2} \int (\partial_x^4 \tilde{u})^2 \phi'_K(x - x_0) - 4\mu_p \int (\partial_x^3 \tilde{u})^2 \phi'_K(x - x_0) - \frac{3}{2} \int (\partial_x^2 \tilde{u})^2 \phi'_K(x - x_0) \\
&\quad - \frac{1}{2(p+1)^2} \int \tilde{u}^{2p+2} \phi'_K(x - x_0) + \frac{5\mu_p^2}{2} \int (\partial_x^3 \tilde{u})^2 \phi_K^{(3)}(x - x_0) - \frac{\mu_p^2}{2} \int (\partial_x^2 \tilde{u})^2 \phi_K^{(5)}(x - x_0) \\
&\quad + 3\mu_p \int (\partial_x^2 \tilde{u})^2 \phi_K^{(3)}(x - x_0) - \frac{\mu_p}{2} \int (\partial_x \tilde{u})^2 \phi_K^{(5)}(x - x_0) + \frac{1}{2} \int (\partial_x \tilde{u})^2 \phi_K^{(3)}(x - x_0) + 3\mu_p \int (\partial_x^2 \tilde{u})^2 \tilde{u}^p \phi'_K(x - x_0) \\
&\quad - \mu_p p(p-1) \int (\partial_x \tilde{u})^4 \tilde{u}^{p-2} \phi'_K(x - x_0) - \frac{15\mu_p p}{6} \int (\partial_x \tilde{u})^3 \tilde{u}^{p-1} \phi_K''(x - x_0) + 2 \int (\partial_x \tilde{u})^2 \tilde{u}^p \phi'_K(x - x_0) \\
&\quad - \frac{5\mu_p}{2} \int (\partial_x \tilde{u})^2 \tilde{u}^p \phi_K^{(3)}(x - x_0) - \frac{1}{(p+1)(p+2)} \int \tilde{u}^{p+2} \phi_K^{(3)}(x - x_0) + \frac{\mu_p}{(p+1)(p+2)} \int \tilde{u}^{p+2} \phi_K^{(5)}(x - x_0).
\end{aligned} \tag{6.86}$$

Using that: $\|u(t)\|_{H^2} \lesssim 1$, $\phi(x) \gtrsim \phi'(x) > 0$, and $\phi_K^{(j+1)}(x) \lesssim \frac{1}{K^j} \phi'_K(x) \forall j \in \mathbb{N}$, we obtain for K chosen corretly:

$$\frac{d}{dt} \int h(\tilde{u}(t, x))\phi_K(x - x_0) + C_1 \sum_{k=3}^4 \int (\partial_x^k \tilde{u})^2(t, x) \phi'_K(x - x_0) \lesssim \int h(\tilde{u}(t, x))\phi_K(x - x_0) + \int \tilde{u}^2(t, x) \phi_K(x - x_0), \tag{6.87}$$

where C_1 is a positive constant. Let $t_0 \in \mathbb{R}$. By integrating between $t_0 - 1 \leq t \leq t_0$, and using (6.69), we get:

$$\sum_{k=3}^4 \int_{t_0-1}^{t_0} \int (\partial_x^k \tilde{u})^2(t, x) \phi_K(x - x_0) \leq C_2 e^{-x_0/K}, \tag{6.88}$$

where C_2 is a positive constant. This implies that there exists $t_1 \in [t_0 - 1, t_0]$ such that:

$$\int ((\partial_x^4 \tilde{u})^2 + (\partial_x^3 \tilde{u})^2)(t_1, x) \phi'_K(x - x_0) \leq C_2 e^{-x_0/K}, \tag{6.89}$$

and since $\phi'_K(x - x_0) \sim \phi_K(x - x_0) \sim e^{(x-x_0)/K}$ for $x < x_0$, we deduce that:

$$\int ((\partial_x^4 \tilde{u})^2 + (\partial_x^3 \tilde{u})^2)(t_1, x) \phi_K(x - x_0) \leq C_2 e^{-x_0/K}. \tag{6.90}$$

We claim that:

$$\begin{aligned}
\frac{d}{dt} \int (\partial_x^3 \tilde{u})^2(t, x) \phi_K(x - x_0) - 5\mu_p \int (\partial_x^5 \tilde{u})^2 \phi'_K(x - x_0) &\lesssim \int (\partial_x^3 \tilde{u})^2(t, x) \phi_K(x - x_0) \\
&\quad + \left(\int (\partial_x^2 \tilde{u})^2(t, x) \phi_K(x - x_0) \right)^{1/3} + \sum_{k=1}^2 \int (\partial_x^k \tilde{u})^2(t, x) \phi_K(x - x_0).
\end{aligned} \tag{6.91}$$

Then by applying the Gronwall between $t_0 - 1 \leq t_1 \leq t_0$, and using (6.69) and (6.90), we have:

$$\begin{aligned}
\int (\partial_x^3 \tilde{u})^2(t_0, x) \phi_K(x - x_0) &\lesssim \exp\left(\int_{t_1}^{t_0} ds\right) \int (\partial_x^3 \tilde{u})^2(t_1, x) \phi_K(x - x_0) \\
&\quad + (e^{-x_0/3K} + e^{-x_0/K}) \int_{t_1}^{t_0} \exp\left(\int_{\tau}^{t_0} ds\right) d\tau \lesssim e e^{-x_0/3K}.
\end{aligned} \tag{6.92}$$

It remains to prove (6.91). We will differentiate: $\int (\partial_x^3 \tilde{u})^2(t, x) \phi_K(x - x_0)$ with respect to time and gradually estimate each terms. To simplify writing we set: $g(x) = \phi_K(x - x_0)$.

$$\begin{aligned} \frac{d}{dt} \int (\partial_x^3 \tilde{u})^2(t, x) g(x) &= 2 \int (\partial_x^3 \tilde{u}) \partial_t (\partial_x^3 \tilde{u}) g = -2 \int (\partial_x^6 \tilde{u}) (\partial_x^3 \tilde{u}) g + 2\mu_p \int (\partial_x^8 \tilde{u}) (\partial_x^3 \tilde{u}) g - 2 \int (\partial_x^4 f(\tilde{u})) (\partial_x^3 \tilde{u}) g \\ &= A + B + C. \end{aligned} \quad (6.93)$$

Let us compute A:

$$A = 2 \int (\partial_x^5 \tilde{u}) (\partial_x^4 \tilde{u}) g + 2 \int (\partial_x^5 \tilde{u}) (\partial_x^3 \tilde{u}) g' = A_1 + A_2, \quad (6.94)$$

$$A_1 = \int \partial_x [(\partial_x^4 \tilde{u})^2] g = - \int (\partial_x^4 \tilde{u})^2 g', \quad (6.95)$$

$$A_2 = -2 \int (\partial_x^4 \tilde{u})^2 g' - \int \partial_x [(\partial_x^3 \tilde{u})^2] g'' = -2 \int (\partial_x^4 \tilde{u})^2 g' + \int (\partial_x^3 \tilde{u})^2 g^{(3)}, \quad (6.96)$$

adding (6.131)-(6.96):

$$A = -3 \int (\partial_x^4 \tilde{u})^2 g' + \int (\partial_x^3 \tilde{u})^2 g^{(3)}. \quad (6.97)$$

Let us compute B:

$$B = -2\mu_p \int (\partial_x^7 \tilde{u}) (\partial_x^4 \tilde{u}) g - 2\mu_p \int (\partial_x^7 \tilde{u}) (\partial_x^3 \tilde{u}) g' = B_1 + B_2, \quad (6.98)$$

$$B_1 = 2\mu_p \int (\partial_x^6 \tilde{u}) (\partial_x^5 \tilde{u}) g + 2\mu_p \int (\partial_x^6 \tilde{u}) (\partial_x^4 \tilde{u}) g' = B_{11} + B_{12}, \quad (6.99)$$

$$B_{11} = \mu_p \int \partial_x [(\partial_x^5 \tilde{u})^2] g = -\mu_p \int (\partial_x^5 \tilde{u})^2 g', \quad (6.100)$$

$$\begin{aligned} B_{12} &= -2\mu_p \int (\partial_x^5 \tilde{u})^2 g' - 2\mu_p \int (\partial_x^5 \tilde{u}) (\partial_x^4 \tilde{u}) g'' = -2\mu_p \int (\partial_x^5 \tilde{u})^2 g' - \mu_p \int \partial_x [(\partial_x^4 \tilde{u})^2] g'' \\ &= -2\mu_p \int (\partial_x^5 \tilde{u})^2 g' + \mu_p \int (\partial_x^4 \tilde{u})^2 g^{(3)}, \end{aligned} \quad (6.101)$$

$$B_2 = 2\mu_p \int (\partial_x^6 \tilde{u}) (\partial_x^4 \tilde{u}) g' + 2\mu_p \int (\partial_x^6 \tilde{u}) (\partial_x^3 \tilde{u}) g'' = -2\mu_p \int (\partial_x^5 \tilde{u})^2 g' + \mu_p \int (\partial_x^4 \tilde{u})^2 g^{(3)} + B_{21}, \quad (6.102)$$

$$\begin{aligned} B_{21} &= -2\mu_p \int (\partial_x^5 \tilde{u}) (\partial_x^4 \tilde{u}) g'' - 2\mu_p \int (\partial_x^5 \tilde{u}) (\partial_x^3 \tilde{u}) g^{(3)} \\ &= -\mu_p \int \partial_x [(\partial_x^4 \tilde{u})^2] g'' + 2\mu_p \int (\partial_x^4 \tilde{u})^2 g^{(3)} + \mu_p \int \partial_x [(\partial_x^3 \tilde{u})^2] g^{(4)} \\ &= 3\mu_p \int (\partial_x^4 \tilde{u})^2 g^{(3)} - \mu_p \int (\partial_x^3 \tilde{u})^2 g^{(5)}, \end{aligned} \quad (6.103)$$

adding (6.98)-(6.103):

$$B = -5\mu_p \int (\partial_x^5 \tilde{u})^2 g' + 5\mu_p \int (\partial_x^4 \tilde{u})^2 g^{(3)} - \mu_p \int (\partial_x^3 \tilde{u})^2 g^{(5)}. \quad (6.104)$$

We deduce that:

$$A + B \lesssim -5\mu_p \int (\partial_x^5 \tilde{u})^2 \phi'_K(x - x_0) - \left(3 - \frac{5\mu_p}{K^2}\right) \int (\partial_x^4 \tilde{u})^2 \phi'_K(x - x_0) + \int (\partial_x^3 \tilde{u})^2 \phi_K(x - x_0). \quad (6.105)$$

Let us compute C :

$$C = \frac{2}{p+1} \int \partial_x^3(\tilde{u}^{p+1})(\partial_x^4 \tilde{u})g + \frac{2}{p+1} \int \partial_x^3(\tilde{u}^{p+1})(\partial_x^3 \tilde{u})g' = C_1 + C_2. \quad (6.106)$$

We have:

$$\frac{1}{p+1} \partial_x^3(\tilde{u}^{p+1}) = (\partial_x^3 \tilde{u})\tilde{u}^p + 3p(\partial_x^2 \tilde{u})(\partial_x \tilde{u})\tilde{u}^{p-1} + p(p-1)(\partial_x \tilde{u})^3 \tilde{u}^{p-2}, \quad (6.107)$$

then

$$\begin{aligned} C_1 &= 2 \int (\partial_x^4 \tilde{u})(\partial_x^3 \tilde{u})\tilde{u}^p g + 6p \int (\partial_x^4 \tilde{u})(\partial_x^2 \tilde{u})(\partial_x \tilde{u})\tilde{u}^{p-1} g + 2p(p-1) \int (\partial_x^4 \tilde{u})(\partial_x \tilde{u})^3 \tilde{u}^{p-2} g \\ &= C_{11} + C_{12} + C_{13}, \end{aligned} \quad (6.108)$$

$$C_{11} = \int \partial_x [(\partial_x^3 \tilde{u})^2] \tilde{u}^p g = - \int (\partial_x^3 \tilde{u})^2 \partial_x (\tilde{u}^p g), \quad (6.109)$$

$$\begin{aligned} C_{12} &= -6p \int (\partial_x^3 \tilde{u})^2 (\partial_x \tilde{u}) \tilde{u}^{p+1} g - 6p \int (\partial_x^3 \tilde{u})(\partial_x^2 \tilde{u})^2 \tilde{u}^{p-1} g - 6p(p-1) \int (\partial_x^3 \tilde{u})(\partial_x^2 \tilde{u})(\partial_x \tilde{u})^2 \tilde{u}^{p-2} g \\ &\quad - 6p \int (\partial_x^3 \tilde{u})(\partial_x^2 \tilde{u})(\partial_x \tilde{u}) \tilde{u}^{p-1} g' = -6p \int (\partial_x^3 \tilde{u})^2 (\partial_x \tilde{u}) \tilde{u}^{p+1} g + C_{121} + C_{122} + C_{123}, \end{aligned} \quad (6.110)$$

$$C_{121} = -2p \int \partial_x [(\partial_x^2 \tilde{u})^3] \tilde{u}^{p-1} g = 2p \int (\partial_x^2 \tilde{u})^3 \partial_x (\tilde{u}^{p-1} g), \quad (6.111)$$

$$\begin{aligned} C_{122} &= -3p(p-1) \int \partial_x [(\partial_x^2 \tilde{u})^2] (\partial_x \tilde{u})^2 \tilde{u}^{p-2} g \\ &= 6p(p-1) \int (\partial_x^2 \tilde{u})^3 (\partial_x \tilde{u}) \tilde{u}^{p-2} g + 3p(p-1) \int (\partial_x^2 \tilde{u})^2 (\partial_x \tilde{u})^2 \partial_x (\tilde{u}^{p-2} g), \end{aligned} \quad (6.112)$$

$$C_{123} = -3p \int \partial_x [(\partial_x^2 \tilde{u})^2] (\partial_x \tilde{u}) \tilde{u}^{p-1} g' = 3p \int (\partial_x^2 \tilde{u})^3 \tilde{u}^{p-1} g' + 3p \int (\partial_x^2 \tilde{u})^2 (\partial_x \tilde{u}) \partial_x (\tilde{u}^{p-1} g'), \quad (6.113)$$

$$\begin{aligned} C_{13} &= -6p(p-1) \int (\partial_x^3 \tilde{u})(\partial_x^2 \tilde{u})(\partial_x \tilde{u})^2 \tilde{u}^{p-2} g - 2p(p-1)(p-2) \int (\partial_x^3 \tilde{u})(\partial_x \tilde{u})^4 \tilde{u}^{p-3} g \\ &\quad - 2p(p-1) \int (\partial_x^3 \tilde{u})(\partial_x \tilde{u})^3 \tilde{u}^{p-2} g' = C_{131} + C_{132} + C_{133}, \end{aligned} \quad (6.114)$$

$$\begin{aligned} C_{131} &= -3p(p-1) \int \partial_x [(\partial_x^2 \tilde{u})^2] (\partial_x \tilde{u})^2 \tilde{u}^{p-2} g = 6p(p-1) \int (\partial_x^2 \tilde{u})^3 (\partial_x \tilde{u}) \tilde{u}^{p-2} g \\ &\quad + 3p(p-1)(p-2) \int (\partial_x^2 \tilde{u})^2 (\partial_x \tilde{u})^3 \tilde{u}^{p-3} g + 3p(p-1) \int (\partial_x^2 \tilde{u})^2 (\partial_x \tilde{u})^2 \tilde{u}^{p-2} g', \end{aligned} \quad (6.115)$$

$$\begin{aligned} C_{132} &= 8p(p-1)(p-2) \int (\partial_x^2 \tilde{u})^2 (\partial_x \tilde{u})^3 \tilde{u}^{p-3} g + 2p(p-1)(p-2)(p-3) \int (\partial_x^2 \tilde{u})(\partial_x \tilde{u})^5 \tilde{u}^{p-4} g \\ &\quad + 2p(p-1)(p-2) \int (\partial_x^2 \tilde{u})(\partial_x \tilde{u})^4 \tilde{u}^{p-3} g' = 8p(p-1)(p-2) \int (\partial_x^2 \tilde{u})^2 (\partial_x \tilde{u})^3 \tilde{u}^{p-3} g + C_{1321} + C_{1322}, \end{aligned} \quad (6.116)$$

$$C_{1321} = \frac{p}{3}(p-1)(p-2)(p-3) \int \partial_x [(\partial_x \tilde{u})^6] \tilde{u}^{p-4} g = -\frac{p}{3}(p-1)(p-2)(p-3) \int (\partial_x \tilde{u})^6 \partial_x (\tilde{u}^{p-4} g), \quad (6.117)$$

$$C_{1322} = \frac{2}{5}p(p-1)(p-2) \int \partial_x [(\partial_x \tilde{u})^5] \tilde{u}^{p-3} g' = -\frac{2}{5}p(p-1)(p-2) \int (\partial_x \tilde{u})^5 \partial_x (\tilde{u}^{p-3} g'), \quad (6.118)$$

$$\begin{aligned} C_{133} &= 6p(p-1) \int (\partial_x^2 \tilde{u})^2 (\partial_x \tilde{u})^2 \tilde{u}^{p-2} g' + 2p(p-1)(p-2) \int (\partial_x^2 \tilde{u}) (\partial_x \tilde{u})^4 \tilde{u}^{p-3} g' \\ &\quad + 2p(p-1) \int (\partial_x^2 \tilde{u}) (\partial_x \tilde{u})^3 \tilde{u}^{p-2} g'' = 6p(p-1) \int (\partial_x^2 \tilde{u})^2 (\partial_x \tilde{u})^2 \tilde{u}^{p-2} g' + C_{1331} + C_{1332}, \end{aligned} \quad (6.119)$$

$$C_{1331} = \frac{2}{5}p(p-1)(p-2) \int \partial_x [(\partial_x \tilde{u})^5] \tilde{u}^{p-3} g' = -\frac{2}{5}p(p-1)(p-2) \int (\partial_x \tilde{u})^5 \partial_x (\tilde{u}^{p-3} g'), \quad (6.120)$$

$$C_{1332} = \frac{1}{2}p(p-1) \int \partial_x [(\partial_x \tilde{u})^4] \tilde{u}^{p-2} g'' = -\frac{1}{2}p(p-1) \int (\partial_x \tilde{u})^4 \partial_x (\tilde{u}^{p-2} g''), \quad (6.121)$$

$$\begin{aligned} C_2 &= 2 \int (\partial_x^3 \tilde{u})^2 \tilde{u}^p g' + 6p \int (\partial_x^3 \tilde{u}) (\partial_x^2 \tilde{u}) (\partial_x \tilde{u}) \tilde{u}^{p-1} g' + 2p(p-1) \int (\partial_x^3 \tilde{u}) (\partial_x \tilde{u})^3 \tilde{u}^{p-2} g' \\ &= 2 \int (\partial_x^3 \tilde{u})^2 \tilde{u}^p g' + C_{21} + C_{22}, \end{aligned} \quad (6.122)$$

$$C_{21} = 3p \int (\partial_x [(\partial_x^2 \tilde{u})^2] (\partial_x \tilde{u}) \tilde{u}^{p-1} g' = -3p \int (\partial_x^2 \tilde{u})^3 \tilde{u}^{p-1} g' - 3p \int (\partial_x^2 \tilde{u})^2 (\partial_x \tilde{u}) \partial_x (\tilde{u}^{p-1} g'), \quad (6.123)$$

$$\begin{aligned} C_{22} &= -6p(p-1) \int (\partial_x^2 \tilde{u})^2 (\partial_x \tilde{u})^2 \tilde{u}^{p-2} g' - 2p(p-1)(p-2) \int (\partial_x^2 \tilde{u}) (\partial_x \tilde{u})^4 \tilde{u}^{p-3} g' \\ &\quad - 2p(p-1) \int (\partial_x^2 \tilde{u}) (\partial_x \tilde{u})^3 \tilde{u}^{p-2} g'' = -6p(p-1) \int (\partial_x^2 \tilde{u})^2 (\partial_x \tilde{u})^2 \tilde{u}^{p-2} g' + C_{221} + C_{222}, \end{aligned} \quad (6.124)$$

$$C_{221} = -\frac{2}{5}p(p-1)(p-2) \int \partial_x [(\partial_x \tilde{u})^5] \tilde{u}^{p-3} g' = \frac{2}{5}p(p-1)(p-2) \int (\partial_x \tilde{u})^5 \partial_x (\tilde{u}^{p-3} g'), \quad (6.125)$$

$$C_{222} = -\frac{1}{2}p(p-1) \int \partial_x [(\partial_x \tilde{u})^4] \tilde{u}^{p-2} g'' = \frac{1}{2}p(p-1) \int (\partial_x \tilde{u})^4 \partial_x (\tilde{u}^{p-2} g''). \quad (6.126)$$

Thus, using that $\tilde{u}(t)$ is uniformly bounded in H^2 , and the property of ϕ_K , we get:

$$C \lesssim \int (\partial_x^3 \tilde{u})^2 \phi_K(x-x_0) + \int (\partial_x^2 \tilde{u})^3 \phi_K(x-x_0) + \int (\partial_x^2 \tilde{u})^2 \phi_K(x-x_0) + \int (\partial_x \tilde{u})^2 \phi_K(x-x_0). \quad (6.127)$$

The only term that we need to transform is: $\int (\partial_x^2 \tilde{u})^3 \phi_K(x-x_0)$. By applying the Gagliardo-Nirenberg inequality:

$$\begin{aligned} \int (\partial_x^2 \tilde{u})^3 \phi_K(x-x_0) &\leq \|\partial_x^2 \tilde{u}\|_{L^2}^2 \|(\partial_x^2 \tilde{u}) \phi_K(x-x_0)\|_{L^\infty} \leq C_G \|\partial_x^2 \tilde{u}\|_{L^2}^2 \|(\partial_x^2 \tilde{u}) \phi_K(x-x_0)\|_{L^2}^{1/2} \times \\ &\quad (\|(\partial_x^2 \tilde{u}) \phi'_K(x-x_0)\|_{L^2} + \|(\partial_x^3 \tilde{u}) \phi_K(x-x_0)\|_{L^2})^{1/2}, \end{aligned} \quad (6.128)$$

Now, applying the Young inequality we deduce that:

$$\int (\partial_x^2 \tilde{u})^3 \phi_K(x-x_0) \leq C_G \|\partial_x^2 \tilde{u}\|_{L^2}^2 \left(\frac{3}{4} \|(\partial_x^2 \tilde{u}) \phi_K(x-x_0)\|_{L^2}^{2/3} + \frac{1}{2} \|(\partial_x^2 \tilde{u}) \phi'_K(x-x_0)\|_{L^2}^2 + \frac{1}{2} \|(\partial_x^3 \tilde{u}) \phi_K(x-x_0)\|_{L^2}^2 \right). \quad (6.129)$$

Finally, by combining (6.105), (6.127) and (6.129), we get (6.91).

Step 4: Strong convergence in H^2 of $u(t+t_n, x+\xi(t_n))$ to $\tilde{u}(t, x)$ by the right.

We will prove that for all $A > 0$ and $\forall t \in \mathbb{R}$:

$$u(t+t_n, \cdot + \xi(t_n)) \rightarrow \tilde{u}(t, \cdot) \text{ in } H^2(x > -A) \text{ as } n \rightarrow +\infty. \quad (6.130)$$

Let $t_0 \in \mathbb{R}$. First, by integrating (6.79) between $[t_0 - 1, t_0]$ and using (6.80), we obtain:

$$\sum_{k=1}^2 \int_{t_0-1}^{t_0} \int (\partial_x^k v_n)^2(t, x) \phi'_K(x) dx dt \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (6.131)$$

Second, combining (6.62) and (6.69), we deduce that for all $x_0 > 0$ and $K > 0$ chosen correctly:

$$\limsup_{n \rightarrow +\infty} \sum_{k=0}^2 \int (\partial_x^k v_n)^2(t, x) \phi_K(x - x_0) \lesssim e^{-x_0/K}. \quad (6.132)$$

(6.131) and (6.132) implies that:

$$\sum_{k=1}^2 \int_{t_0-1}^{t_0} \int (\partial_x^k v_n)^2(t, x) \phi_K(x) dx dt \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (6.133)$$

We claim that:

$$\frac{d}{dt} \int h(v_n(t, x)) \phi_K(x) + C_2 \sum_{k=3}^4 \int (\partial_x^k v_n)^2 \phi'_K(x) \lesssim \int h(v_n(t, x)) \phi_K(x) + \int v_n^2(t, x) \phi_K(x) + \left(\int v_n^2(t, x) \phi_K(x) \right)^{1/4}, \quad (6.134)$$

where C_2 is positive constant. Then integrating between $t_0 - 1 \leq t \leq t_0$:

$$\begin{aligned} & \int h(v_n(t_0, x)) \phi_K(x) + C_2 \sum_{k=3}^4 \int_{t_0-1}^{t_0} \int (\partial_x^k v_n)^2 \phi'_K(x) \lesssim \int h(v_n(t, x)) \phi_K(x) + \int_{t_0-1}^{t_0} \int h(v_n(s, x)) \phi_K(x) \\ & + \sup_{s \in [t_0-1, t_0]} \int v_n^2(s, x) \phi_K(x) + \sup_{s \in [t_0-1, t_0]} \left(\int v_n^2(s, x) \phi_K(x) \right)^{1/4}. \end{aligned} \quad (6.135)$$

By integrating one more time between $t_0 - 1 \leq t \leq t_0$, and using (6.80) and (6.133):

$$\begin{aligned} & \int h(v_n(t_0, x)) \phi_K(x) + C_2 \sum_{k=3}^4 \int_{t_0-1}^{t_0} \int (\partial_x^k v_n)^2 \phi'_K(x) \lesssim \int_{t_0-1}^{t_0} \int h(v_n(s, x)) \phi_K(x) + \sup_{s \in [t_0-1, t_0]} \int v_n^2(s, x) \phi_K(x) \\ & + \sup_{s \in [t_0-1, t_0]} \left(\int v_n^2(s, x) \phi_K(x) \right)^{1/4} \rightarrow 0 \text{ as } n \rightarrow +\infty. \end{aligned} \quad (6.136)$$

This prove (6.130) for $A = 0$.

Now let us prove (6.134). For simplicity we set $g(x) = \phi_K(x)$. Let us differentiating $\int h(v_n(t, x))g(x)$ with respect to time:

$$\frac{d}{dt} \int h(v_n(t, x))g(x) = \frac{\mu_p}{2} \frac{d}{dt} \int (\partial_x^2 v_n)^2 g + \frac{1}{2} \frac{d}{dt} \int (\partial_x v_n)^2 - \frac{1}{(p+1)(p+2)} \frac{d}{dt} \int v^{p+2} g = A + B + C. \quad (6.137)$$

Let us compute C :

$$\begin{aligned} C &= -\frac{1}{p+1} \int (\partial_t v_n) v_n^{p+1} g = -\frac{\mu_p}{p+1} \int (\partial_x^5 v_n) v^{p+1} g + \frac{1}{p+1} \int (\partial_x^3 v_n) v_n^{p+1} g \\ &+ \frac{1}{p+1} \int \partial_x (f(\tilde{u} + v_n) - f(\tilde{u})) v_n^{p+1} g = C_1 + C_2 + C_3, \end{aligned} \quad (6.138)$$

$$\begin{aligned}
C_1 &= \frac{5\mu_p p}{2} \int (\partial_x^2 v_n)^2 (\partial_x v_n) v_n^{p-1} g - \frac{\mu_p p(p-1)(p-2)}{4} \int (\partial_x v_n)^5 v_n^{p-3} g + \frac{5\mu_p}{2} \int (\partial_x^2 v_n)^2 v_n^p g', \\
&\quad - \frac{15\mu_p p(p-1)}{12} \int (\partial_x v_n)^4 v_n^{p-2} g' - \frac{15\mu_p p}{6} \int (\partial_x v_n)^3 v_n^{p-1} g'' - \frac{5\mu_p}{2} \int (\partial_x v_n)^2 v_n^p g^{(3)} \\
&\quad + \frac{\mu_p}{(p+1)(p+2)} \int v_n^{p+2} g^{(5)}, \tag{6.139}
\end{aligned}$$

$$C_2 = \frac{p}{2} \int (\partial_x v_n)^3 v_n^{p+1} g + \frac{3}{2} \int (\partial_x v_n)^2 v_n^p g' - \frac{1}{(p+1)(p+2)} \int v_n^{p+2} g^{(3)}, \tag{6.140}$$

$$C_3 = -\frac{1}{p+1} \int (f(\tilde{u} + v_n) - f(\tilde{u})) [\partial_x(v_n^{p+1})g + v_n^{p+1}g']. \tag{6.141}$$

Thus using that: $\|v_n(t)\|_{H^2} \lesssim 1$, we claim that:

$$C \lesssim \sum_{k=0}^2 \int (\partial_x^k v_n)^2 \phi_K(x). \tag{6.142}$$

We will just explain the estimation of C_3 . Recall that: $f(\tilde{u} + v_n) - f(\tilde{u}) = \frac{1}{p+1} \sum_{k=1}^{p+1} C_{p+1}^k v_n^k \tilde{u}^{p-k+1}$. By applying the Young inequality:

$$\begin{aligned}
\left| \frac{1}{p+1} \int (f(\tilde{u} + v_n) - f(\tilde{u})) \partial_x(v_n^{p+1}) \phi_K(x) \right| &\leq \frac{1}{2(p+1)} \int (f(\tilde{u} + v_n) - f(\tilde{u}))^2 \phi_K(x) \\
&\quad + \frac{1}{2(p+1)} \int [\partial_x(v_n^{p+1})]^2 \phi_K(x) \lesssim \int v_n^2 \phi_K(x). \tag{6.143}
\end{aligned}$$

Let us compute B :

$$\begin{aligned}
B &= \int \partial_t[(\partial_x v_n)](\partial_x v_n) g = \mu_p \int (\partial_x^6 v_n)(\partial_x v_n) g - \int (\partial_x^4 v_n)(\partial_x v_n) g - \int \partial_x^2 (f(\tilde{u} + v_n) - f(\tilde{u}))(\partial_x v_n) g \\
&= B_1 + B_2 + B_3, \tag{6.144}
\end{aligned}$$

$$B_1 = -\frac{5\mu_p}{2} \int (\partial_x^3 v_n)^2 g' + \frac{5\mu_p}{2} \int (\partial_x^2 v_n)^2 g^{(3)} - \frac{\mu_p}{2} \int (\partial_x v_n)^2 g^{(5)}, \tag{6.145}$$

$$B_2 = -\frac{3}{2} \int (\partial_x^2 v_n)^2 g' + \frac{1}{2} \int (\partial_x v_n)^2 g^{(3)}, \tag{6.146}$$

$$B_3 = \int \partial_x (f(\tilde{u} + v_n) - f(\tilde{u})) [(\partial_x^2 v_n)g + (\partial_x v_n)g']. \tag{6.147}$$

One can easily check that:

$$B \lesssim -\frac{5\mu_p}{2} \int (\partial_x^3 v_n)^2 \phi'_K(x) + \sum_{k=1}^2 \int (\partial_x^2 v_n)^2 \phi_K(x). \tag{6.148}$$

We will just explain the estimate of B_3 . By applying the Young inequality:

$$\begin{aligned}
\int \partial_x (f(\tilde{u} + v_n) - f(\tilde{u})) (\partial_x^2 v_n) \phi_K(x) &\leq \frac{1}{2} \int [\partial_x (f(\tilde{u} + v_n) - f(\tilde{u}))]^2 \phi_K(x) + \frac{1}{2} \int (\partial_x^2 v_n)^2 \phi_K(x) \\
&\lesssim \sum_{k=1}^2 \int (\partial_x^2 v_n)^2 \phi_K(x). \tag{6.149}
\end{aligned}$$

Let us compute A:

$$\begin{aligned} A &= \mu_p \int \partial_t [(\partial_x^2 v_n)] (\partial_x^2 v_n) g = \mu_p^2 \int (\partial_x^7 v_n) (\partial_x^2 v_n) g - \mu_p \int (\partial_x^5 v_n) (\partial_x^2 v_n) g - \mu_p \int \partial_x^3 (f(\tilde{u} + v_n) - f(\tilde{u})) (\partial_x^2 v_n) g \\ &= A_1 + A_2 + A_3, \end{aligned} \quad (6.150)$$

$$A_1 = -\frac{5\mu_p^2}{2} \int (\partial_x^4 v_n)^2 g' + \frac{5\mu_p^2}{2} \int (\partial_x^3 v_n)^2 g^{(3)} - \frac{\mu_p^2}{2} \int (\partial_x^2 v_n)^2 g^{(5)}, \quad (6.151)$$

$$A_2 = -\frac{3\mu_p}{2} \int (\partial_x^3 v_n)^2 g' + \frac{\mu_p}{2} \int (\partial_x^2 v_n)^2 g^{(3)}, \quad (6.152)$$

and we directly infer that

$$A_1 + A_2 \lesssim -\frac{5\mu_p^2}{2} \int (\partial_x^4 v_n)^2 \phi'_K(x) - \left(\frac{3\mu_p}{2} - \frac{5\mu_p^2}{2K^2}\right) \int (\partial_x^3 v_n)^2 \phi'_K(x) + \int (\partial_x^2 v_n)^2 \phi_K(x). \quad (6.153)$$

Next:

$$A_3 = \mu_p \int \partial_x^2 (f(\tilde{u} + v_n) - f(\tilde{u})) (\partial_x^3 v_n) g + \mu_p \int \partial_x^2 (f(\tilde{u} + v_n) - f(\tilde{u})) (\partial_x^2 v_n) g' = A_{31} + A_{32}. \quad (6.154)$$

One can compute that:

$$\begin{aligned} \partial_x^2 (f(\tilde{u} + v_n) - f(\tilde{u})) &= \frac{1}{p+1} \sum_{k=1}^{p+1} C_{p+1}^k [k(\partial_x^2 v_n) v_n^{k-1} \tilde{u}^{p-k+1} + k(k-1)(\partial_x v_n)^2 v_n^{k-2} \tilde{u}^{p-k+1} \\ &\quad + 2k(p-k+1)(\partial_x v_n) v_n^{k-1} (\partial_x \tilde{u}) \tilde{u}^{p-k} + (p-k+1) v_n^k (\partial_x^2 \tilde{u}) \tilde{u}^{p-k} + (p-k+1)(p-k) v_n^k (\partial_x \tilde{u})^2 \tilde{u}^{p-k-1}]. \end{aligned} \quad (6.155)$$

We claim that:

$$A_{32} \lesssim \sum_{k=0}^2 \int (\partial_x^k v_n)^2 \phi_K(x). \quad (6.156)$$

We will explain the estimate of the more complicated terms:

$$\begin{aligned} &\frac{1}{p+1} \sum_{k=1}^{p+1} C_{p+1}^k k(k-1) \int (\partial_x^2 v_n) (\partial_x v_n)^2 v_n^{k-2} \tilde{u}^{p-k+1} \phi'_K(x) \\ &= -\frac{1}{p+1} \sum_{k=1}^{p+1} C_{p+1}^k \frac{k(k-1)}{3} \int (\partial_x v_n)^3 \partial_x [v_n^{k-2} \tilde{u}^{p-k+1} \phi'_K(x)] \lesssim \int (\partial_x v_n)^2 \phi_K(x), \end{aligned} \quad (6.157)$$

$$\begin{aligned} &\frac{1}{p+1} \sum_{k=1}^{p+1} C_{p+1}^k (p-k+1)(p-k) \int (\partial_x^2 v_n) v_n^k (\partial_x \tilde{u})^2 \tilde{u}^{p-k-1} \phi'_K(x) \lesssim \int (\partial_x^2 v_n) v_n \phi_K(x) \\ &\leq \frac{1}{2} \int (\partial_x^2 v_n)^2 \phi_K(x) + \frac{1}{2} \int (\partial_x v_n)^2 \phi_K(x). \end{aligned} \quad (6.158)$$

Here is the most interesting term:

$$\begin{aligned} A_{321} &= \frac{1}{p+1} \sum_{k=1}^{p+1} C_{p+1}^k (p-k+1) \int (\partial_x^2 v_n) v_n^k (\partial_x^2 \tilde{u}) \tilde{u}^{p-k} \phi'_K(x) \lesssim \int (\partial_x^2 v_n) v_n (\partial_x^2 \tilde{u}) \phi_K(x) \\ &\lesssim \|v_n \sqrt{\phi_K}\|_{L^\infty} \left(\int (\partial_x^2 v_n)^2 \phi_K(x) \right)^{1/2} \left(\int (\partial_x^2 \tilde{u})^2 \right)^{1/2}, \end{aligned} \quad (6.159)$$

Using the Sobolev embedding of H^1 into C^0 and that $\frac{\phi'_K(x)}{2\sqrt{\phi_K(x)}} \leq \frac{\sqrt{\phi_K(x)}}{2}$:

$$\|v_n \sqrt{\phi_K}\|_{L^\infty} \leq C_S \left(\|v_n \sqrt{\phi_K}\|_{L^2}^2 + \|(\partial_x v_n) \sqrt{\phi_K}\|_{L^2}^2 \right)^{1/2}, \quad (6.160)$$

thus by applying the young inequality:

$$A_{321} \lesssim \sum_{k=0}^2 \int (\partial_x^k v_n)^2 \phi_K(x). \quad (6.161)$$

We also claim that:

$$A_{31} \lesssim \sum_{k=1}^2 (\partial_x^k v_n)^2 \phi_K(x) + \left(\int v_n^2 \phi_K(x) \right)^{1/4}. \quad (6.162)$$

Indeed:

$$\begin{aligned} \frac{1}{p+1} \sum_{k=1}^{p+1} C_{p+1}^k k \int (\partial_x^3 v_n) (\partial_x^2 v_n) v_n^{k-1} \tilde{u}^{p-k+1} \phi_K(x) &= -\frac{1}{p+1} \int (\partial_x^2 v_n)^2 \partial_x [v_n^{k-1} \tilde{u}^{p-k+1} \phi_K(x)] \\ &\lesssim \int (\partial_x^2 v_n)^2 \phi_K(x), \end{aligned} \quad (6.163)$$

$$\begin{aligned} \frac{1}{p+1} \sum_{k=1}^{p+1} C_{p+1}^k k(k-1) \int (\partial_x^3 v_n) (\partial_x v_n)^2 v_n^{k-2} \tilde{u}^{p-k+1} \phi_K(x) \\ = -\frac{1}{p+1} \sum_{k=1}^{p+1} C_{p+1}^k k(k-1) \int (\partial_x^2 v_n) \{ \partial_x [(\partial_x v_n)^2] v_n^{k-2} \tilde{u}^{p-k+1} \phi_K(x) + (\partial_x v_n)^2 \partial_x [v_n^{k-2} \tilde{u}^{p-k+1} \phi_K(x)] \} \\ \lesssim \int (\partial_x^2 v_n)^2 \phi_K(x) + \int (\partial_x^2 v_n) (\partial_x v_n) \phi_K(x) \lesssim \frac{3}{2} \int (\partial_x^2 v_n)^2 \phi_K(x) + \frac{1}{2} \int (\partial_x v_n)^2 \phi_K(x), \end{aligned} \quad (6.164)$$

$$\begin{aligned} \frac{2}{p+1} \sum_{k=1}^{p+1} C_{p+1}^k k(p-k+1) \int (\partial_x^3 v_n) (\partial_x v_n) v_n^{k-1} (\partial_x \tilde{u}) \tilde{u}^{p-k} \phi_K(x) \\ = -\frac{2}{p+1} \sum_{k=1}^{p+1} C_{p+1}^k k(p-k+1) \int (\partial_x^2 v_n) \{ (\partial_x^2 v_n) v_n^{k-1} (\partial_x \tilde{u}) \tilde{u}^{p-k} \phi_K(x) + (\partial_x v_n) (\partial_x^2 \tilde{u}) v_n^{k-1} \tilde{u}^{p-k} \phi_K(x) \\ + (\partial_x v_n) (\partial_x \tilde{u}) \partial_x [v_n^{k-1} \tilde{u}^{p-k} \phi_K(x)] \} \lesssim \int (\partial_x^2 v_n)^2 \phi_K(x) + \int (\partial_x^2 v_n) (\partial_x v_n) (\partial_x^2 \tilde{u}) \phi_K(x) \\ + \int (\partial_x^2 v_n) (\partial_x v_n) \phi_K(x) \leq \frac{3}{2} \int (\partial_x^2 v_n)^2 \phi_K(x) + \frac{1}{2} \int (\partial_x v_n)^2 \phi_K(x) + \int (\partial_x^2 v_n) (\partial_x v_n) (\partial_x^2 \tilde{u}) \phi_K(x). \end{aligned} \quad (6.165)$$

We estimate the term: $\int (\partial_x^2 v_n) (\partial_x v_n) (\partial_x^2 \tilde{u}) \phi_K(x)$ as A_{321} :

$$\begin{aligned} \int (\partial_x^2 v_n) (\partial_x v_n) (\partial_x^2 \tilde{u}) \phi_K(x) &\leq \|(\partial_x v_n) \sqrt{\phi_K}\|_{L^\infty} \left(\int (\partial_x^2 v_n)^2 \phi_K(x) \right)^{1/2} \left(\int (\partial_x^2 \tilde{u})^2 \right)^{1/2} \\ &\lesssim \left(\|(\partial_x v_n) \sqrt{\phi_K}\|_{L^2}^2 + \|(\partial_x^2 v_n) \sqrt{\phi_K}\|_{L^2}^2 \right)^{1/2} \left(\int (\partial_x^2 v_n)^2 \phi_K(x) \right)^{1/2} \lesssim \sum_{k=1}^2 \int (\partial_x^k v_n)^2 \phi_K(x). \end{aligned} \quad (6.166)$$

Here is the estimate of the most important term:

$$\begin{aligned}
A_{131} &= \frac{1}{p+1} \sum_{k=1}^{p+1} C_{p+1}^k (p-k+1) \int (\partial_x^3 v_n) v_n^k (\partial_x^2 \tilde{u}) \tilde{u}^{p-k} \phi_K(x) \\
&= -\frac{1}{p+1} \sum_{k=1}^{p+1} C_{p+1}^k (p-k+1) \int (\partial_x^2 v_n) \{(\partial_x^3 \tilde{u}) v_n^k \tilde{u}^{p-k} \phi_K(x) + (\partial_x^2 \tilde{u}) \partial_x [v_n^k \tilde{u}^{p-k} \phi_K(x)]\} \\
&\lesssim \int (\partial_x^2 v_n) v_n (\partial_x^3 \tilde{u}) \phi_K(x) + \int (\partial_x^2 v_n) (\partial_x v_n) (\partial_x^2 \tilde{u}) \phi_K(x). \tag{6.167}
\end{aligned}$$

Thus it remains to estimate: $\int (\partial_x^2 v_n) v_n (\partial_x^3 \tilde{u}) \phi_K(x)$. Thanks to (6.84) we know that: $\int (\partial_x^3 \tilde{u})^2(t, x) \phi_K(x) \lesssim 1$. Then:

$$\begin{aligned}
\int (\partial_x^2 v_n) v_n (\partial_x^3 \tilde{u}) \phi_K(x) &\leq \|v_n \sqrt{\phi_K}\|_{L^\infty} \left(\int (\partial_x^3 \tilde{u})^2 \phi_K(x) \right)^{1/2} \left(\int (\partial_x^2 v_n)^2 \right)^{1/2} \\
&\leq C_G \|v_n \sqrt{\phi_K}\|_{L^2}^{1/2} \|\partial_x (v_n \sqrt{\phi_K})\|_{L^2}^{1/2} \left(\int (\partial_x^3 \tilde{u})^2 \phi_K(x) \right)^{1/2} \left(\int (\partial_x^2 v_n)^2 \right)^{1/2} \lesssim \|v_n \sqrt{\phi_K}\|_{L^2}^{1/2}. \tag{6.168}
\end{aligned}$$

So we obtain (6.162). By combining (6.142), (6.148), (6.153), (6.156) and (6.162), we get (6.134).

Step 5: Convergence of the modulation parameters and exponential decay of $\tilde{u}(t, x)$ by the left.

We will prove that $\forall t \in \mathbb{R}$:

$$\tilde{\xi}(0) = 0, \quad \xi(t + t_n) - \xi(t_n) \rightarrow \tilde{\xi}(t) \text{ as } n \rightarrow +\infty, \tag{6.169}$$

$$\tilde{c}(0) = \tilde{c}_0, \quad c(t + t_n) \rightarrow \tilde{c}(t) \text{ as } n \rightarrow +\infty, \tag{6.170}$$

where $\tilde{\xi}(t)$ and $\tilde{c}(t)$ are given by Lemma 6.1 (see the begining of the proof of this Proposition). Moreover, for all $x_0 > 0$, $K > 0$ chosen correctly, and $t \in \mathbb{R}$:

$$\int \tilde{u}^2(t, x + \tilde{\xi}(t)) (1 - \phi_K(x + x_0)) \lesssim e^{-x_0/K}, \tag{6.171}$$

and

$$|\tilde{u}(t, x + \tilde{\xi}(t))| \lesssim e^{x/4K}, \text{ for all } x \leq 0. \tag{6.172}$$

We know by hypothesis that (6.6) and (6.8) are satisfied, then the sequences $\{\xi(t+t_n) - \xi(t_n)\}$ and $\{c(t+t_n)\}$ are uniformly equi-continuous. Indeed, for all $t_0 \geq t_1$ we have: $|\xi(t_0 + t_n) - \xi(t_1 + t_n)| \lesssim c_0(t_0 - t_1)$ and $|c(t_0 + t_n) - c(t_1 + t_n)| \lesssim \gamma^2(t_0 - t_1)$. Thus by the Theorem of Arzelà-Ascoli, there exists a subsequence still denoted by (t_n) , and $\beta \in C^0(\mathbb{R})$, $\alpha \in C^0(\mathbb{R})$ such that for all $T > 0$:

$$\tilde{\beta}(0) = 0, \quad \xi(\cdot + t_n) - \xi(t_n) \rightarrow \beta \text{ in } C([-T, T]) \text{ as } n \rightarrow +\infty, \tag{6.173}$$

and

$$c(\cdot + t_n) \rightarrow \alpha \text{ in } C([-T, T]) \text{ as } n \rightarrow +\infty. \tag{6.174}$$

We set now: $\eta_n(t, x) = u(t + t_n, x + \xi(t + t_n)) - \varphi_{c(t+t_n)}(x)$ and $\tilde{\eta}(t, x) = \tilde{u}(t, x + \beta(t)) - \varphi_{\alpha(t)}(x)$. We know that (6.7) are satisfied:

$$\int \eta_n(t, x) \varphi'_{c(t+t_n)}(x) = \int \eta_n(t, x) \varphi_{c(t+t_n)}(x) = 0. \tag{6.175}$$

Using that: $u(t + t_n, x + \xi(t + t_n)) \rightharpoonup \tilde{u}(t, x + \beta(t))$ weakly in H^2 thanks to (6.68) and (6.173), and that: $\lim_{n \rightarrow +\infty} \|\varphi_{c(t+t_n)} - \varphi_{\alpha(t)}\|_{H^1} = 0$ thanks to (6.174), we obtain:

$$\int \tilde{\eta}(t, x) \varphi'_{\alpha(t)}(x) = \int \tilde{\eta}(t, x) \varphi_{\alpha(t)}(x) = 0. \tag{6.176}$$

Then by uniqueness of the modulation parameters (which is given by the Implicit Function Theorem) we obtain: $\beta(t) = \tilde{\xi}(t)$ and $\alpha(t) = \tilde{c}(t)$.

We set $V_0 = \int \tilde{u}_0^2(x)$ (we recall that the norm L^2 is a conservation law for the gKW). Let $x_0 > 0$ and $t_0 \in \mathbb{R}$, we have:

$$V_0 - \int \tilde{u}^2(t_0, x)(1 - \phi_K(x - \tilde{\xi}(t_0) + x_0)) = \int \tilde{u}^2(t_0, x)\phi_K(x - \tilde{\xi}(t_0) + x_0), \quad (6.177)$$

using (6.67) and (6.169) for $n(x_0) > 0$ large enough:

$$V_0 - \int \tilde{u}^2(t_0, x)(1 - \phi_K(x - \tilde{\xi}(t_0) + x_0)) \geq \int u^2(t_0 + t_n, x)\phi_K(x - \xi(t_0 + t_n) + x_0) - \frac{1}{3}e^{-x_0/K}. \quad (6.178)$$

So, applying the monotonicity formula by the left (6.18), for $m \geq n(x_0)$ such that $t_m \geq t_n + t_0$ we obtain:

$$V_0 - \int \tilde{u}^2(t_0, x)(1 - \phi_K(x - \tilde{\xi}(t_0) + x_0)) \geq \int u^2(t_m, x)\phi_K(x - \xi(t_0 + t_m) - \frac{c_0}{2}[t_m - (t_0 + t_n)] + x_0) - \frac{2}{3}e^{-x_0/K}. \quad (6.179)$$

Recall that γ small enough (6.6) and (6.8) imply: $\xi(t_0 + t_m) + \frac{c_0}{2}[t_m - (t_0 + t_n)] \sim \xi(t_m) - \frac{c_0}{2}[t_m - (t_0 + t_n)]$. Moreover for n fixed, one can remark that:

$$u(t_m, \cdot + \xi(t_m))\sqrt{\phi_K(\cdot + \frac{c_0}{2}[t_m - (t_0 + t_n)] + x_0)} \rightarrow \tilde{u}_0 \text{ in } L^2 \text{ as } m \rightarrow +\infty \quad (6.180)$$

and

$$V_0 \leq \liminf_{m \rightarrow +\infty} \int u^2(t_m, x + \xi(t_m))\phi_K(x + \frac{c_0}{2}[t_m - (t_0 + t_n)] + x_0). \quad (6.181)$$

Indeed, for all $g \in L^2$:

$$\begin{aligned} & \left| \int u(t_m, x + \xi(t_m))\sqrt{\phi_K(x + \frac{c_0}{2}[t_m - (t_0 + t_n)] + x_0)}g - \int \tilde{u}_0(x)g \right| \\ & \leq \left| \int u(t_m, x + \xi(t_m))g \left(\sqrt{\phi_K(x + \frac{c_0}{2}[t_m - (t_0 + t_n)] + x_0)} - 1 \right) \right| + \left| \int u(t_m, x + \xi(t_m))g - \int \tilde{u}_0(x)g \right| \\ & = A + B. \end{aligned} \quad (6.182)$$

Using (6.57) $B \rightarrow 0$, and with A we use that $\phi_K(+\infty) = 1$ and we apply the Lebesgue's Dominated Convergence Theorem:

$$A \leq \left(\int u^2(t_m, x + \xi(t_m)) \right)^{1/2} \left(\int g^2 \left(\sqrt{\phi_K(x + \frac{c_0}{2}[t_m - (t_0 + t_n)] + x_0)} - 1 \right)^2 \right)^{1/2} \rightarrow 0. \quad (6.183)$$

Thus by combining (6.179) and (6.181) we obtain for m large enough:

$$V_0 - \int \tilde{u}^2(t_0, x)(1 - \phi_K(x - \tilde{\xi}(t_0) + x_0)) \geq V_0 - e^{-x_0/K}, \quad (6.184)$$

this gives (6.171).

(6.172) follows from (6.171). It suffice to use that: $1 - \phi_K(x + x_0) \geq \frac{1}{2}$ for $x \leq -x_0$, the Gagliardo-Nirenberg inequality, and that $\tilde{u}(t)$ is bounded in H^1 :

$$\sup_{x \leq -x_0} |\tilde{u}(t, x + \tilde{\xi}(t))| \leq C_G \|\tilde{u}(t, \cdot + \tilde{\xi}(t))\|_{L^2(x \leq -x_0)}^{1/2} \|\partial_x \tilde{u}(t, \cdot + \tilde{\xi}(t))\|_{L^2(x \leq -x_0)}^{1/2} \lesssim e^{-x_0/4K}. \quad (6.185)$$

This completes the proof of the Proposition 6.2.

6.5 End of the proof of the Theorem 6.1

We have now all the elements to prove the asymptotic stability of the solitons of gKW.

Let $u(t, x)$ be the global H^2 -solution of gKW which satisfied (6.1) with γ small enough. Then the Propostion 6.2 holds. Let $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$, we know that there exists a subsequence $(t_{\tau(n)})$, \tilde{c}_0 and $\tilde{u}_0 \in H^2$ such that: $c(t_{\tau(n)}) \rightarrow \tilde{c}_0$, $|\tilde{c}_0 - c_0| \leq \gamma$, and $u(t_{\tau(n)}, \cdot + \xi(t_{\tau(n)})) - \tilde{u}_0 \rightarrow 0$ strongly in $H^2(x > -A)$ for all $A > 0$. Moreover, $\tilde{u}(t, x)$ the global H^2 -solution of gKW emanating for the inital datum: $\tilde{u}(0, x) = \tilde{u}_0(x)$ satisfied (6.54) and (6.55), and thanks to (6.169) and (6.170): $\tilde{\xi}(0) = 0$ and $\tilde{c}(0) = \tilde{c}_0$. Thus the nonlinear Liouville Theorem 6.2 implies that: $\tilde{u}(t, x) = \varphi_{\tilde{c}(0), p, \mu_p}(x - \tilde{\xi}(0) - \tilde{c}(0)t)$. At the initial time we have: $\tilde{u}_0(x) = \varphi_{\tilde{c}_0, p, \mu_p}(x)$. We deduce that: $u(t_{\tau(n)}, \cdot + \xi(t_{\tau(n)})) - \varphi_{c(t_{\tau(n)}), p, \mu_p} \rightarrow 0$ in $H^2(x > -A)$ for all $A > 0$. Since this is true for all sequence of time $t_n \rightarrow +\infty$, we conclude that for all $A > 0$:

$$u(t, \cdot + \xi(t)) - \varphi_{c(t), p, \mu_p} \rightarrow 0 \text{ in } H^2(x > -A) \text{ as } t \rightarrow +\infty, \quad (6.186)$$

and

$$c(t) \rightarrow \tilde{c}_0 \text{ as } t \rightarrow +\infty \text{ et } |\tilde{c}_0 - c_0| \leq \gamma. \quad (6.187)$$

By combining (6.8) and (6.187) we directly infer that:

$$c'(t) \rightarrow 0 \text{ and } \xi'(t) \rightarrow \tilde{c}_0 \text{ as } t \rightarrow +\infty. \quad (6.188)$$

Now we redefine: $\eta(t, x) = u(t, x) - \varphi_{c(t), p, \mu_p}(x - \xi(t))$. Thanks to (6.186) we know that for $x_0 > 0$:

$$\lim_{t \rightarrow +\infty} \sum_{k=0}^2 \int (\partial_x^k \eta)^2(t, x) \phi_K(x - \xi(t) + x_0) = 0. \quad (6.189)$$

Let $t \geq t_0$, we compute:

$$\begin{aligned} \sum_{k=0}^2 \int (\partial_x^k \eta)^2(t, x) \phi_K(x - \xi(t_0) - \frac{c_0}{2}(t - t_0) + x_0) &= \sum_{k=0}^2 \int (\partial_x^k u)^2(t, x) \phi_K(x - \xi(t_0) - \frac{c_0}{2}(t - t_0) + x_0) \\ &- 2 \sum_{k=0}^2 \int (\partial_x^k u)(t, x) (\partial_x^k \varphi_{c(t), p, \mu_p})(x - \xi(t)) \phi_K(x - \xi(t_0) - \frac{c_0}{2}(t - t_0) + x_0) \\ &+ \sum_{k=0}^2 \int (\partial_x^k \varphi_{c(t), p, \mu_p})^2 \phi_K(x - \xi(t_0) - \frac{c_0}{2}(t - t_0) + x_0). \end{aligned} \quad (6.190)$$

From the monotonicity formula by the left (6.18) and (6.19) we know that:

$$\sum_{k=0}^2 \int (\partial_x^k u)^2(t, x) \phi_K(x - \xi(t_0) - \frac{c_0}{2}(t - t_0) + x_0) \lesssim \sum_{k=0}^2 \int (\partial_x^k u)^2(t_0, x) \phi_K(x - \xi(t_0) + x_0) + e^{-x_0/K}. \quad (6.191)$$

We claim that $k \in \{0, 1, 2\}$:

$$\lim_{t \rightarrow +\infty} \left| \int (\partial_x^k u)(t, x) (\partial_x^k \varphi_{c(t), p, \mu_p})(x - \xi(t)) \phi_K(x - \xi(t_0) - \frac{c_0}{2}(t - t_0) + x_0) - \int (\partial_x^k \varphi_{\tilde{c}_0, p, \mu_p})^2(x) \right| = 0, \quad (6.192)$$

$$\lim_{t \rightarrow +\infty} \left| \int (\partial_x^k \varphi_{c(t), p, \mu_p})^2(x + \xi(t)) \phi_K(x - \xi(t_0) - \frac{c_0}{2}(t - t_0) + x_0) - \int (\partial_x^k \varphi_{\tilde{c}_0, p, \mu_p})^2(x) \right| = 0, \quad (6.193)$$

and

$$\lim_{t_0 \rightarrow +\infty} \left| \int (\partial_x^k u)(t_0, x + \xi(t_0)) (\partial_x^k \varphi_{c(t_0), p, \mu_p})(x) \phi_K(x + x_0) - \int (\partial_x^k \varphi_{\tilde{c}_0, p, \mu_p})^2(x) \right| \lesssim e^{-x_0/K}, \quad (6.194)$$

$$\lim_{t_0 \rightarrow +\infty} \left| \int (\partial_x^k \varphi_{c(t_0), p, \mu_p})^2(x) \phi_K(x + x_0) - \int (\partial_x^k \varphi_{\bar{c}_0, p, \mu_p})^2(x) \right| \lesssim e^{-x_0/K}. \quad (6.195)$$

Proof of (6.192). We recall that for γ small enough (6.6) and (6.8) gives us: $\xi(t_0) + \frac{c_0}{2}(t-t_0) \sim \xi(t) - \frac{c_0}{2}(t-t_0)$. We have:

$$\begin{aligned} & \left| \int (\partial_x^k u)(t, x) (\partial_x^k \varphi_{c(t), p, \mu_p})(x - \xi(t)) \phi_K(x - \xi(t_0) - \frac{c_0}{2}(t-t_0) + x_0) - \int (\partial_x^k \varphi_{\bar{c}_0, p, \mu_p})^2(x) \right| \\ & \leq \left| \int (\partial_x^k u)(t, x + \xi(t)) \phi_K(x + \frac{c_0}{2}(t-t_0) + x_0) [(\partial_x^k \varphi_{c(t), p, \mu_p})(x) - (\partial_x^k \varphi_{\bar{c}_0, p, \mu_p})(x)] \right| \\ & + \left| \int (\partial_x^k u)(t, x + \xi(t)) (\partial_x^k \varphi_{\bar{c}_0, p, \mu_p})(x) \left[\phi_K(x + \frac{c_0}{2}(t-t_0) + x_0) - 1 \right] \right| \\ & + \left| \int [(\partial_x^k u)(t, x + \xi(t)) - (\partial_x^k \varphi_{\bar{c}_0, p, \mu_p})(x)] (\partial_x^k \varphi_{\bar{c}_0, p, \mu_p})(x) \right| = A + B + C. \end{aligned} \quad (6.196)$$

By applying the Holder inequality, and using: $\|u(t)\|_{H^2} \lesssim 1$, $\|\phi_K\|_{L^\infty} \leq 1$ and that $\lim_{c(t) \rightarrow \bar{c}_0} \|\varphi_{c(t), p, \mu_p} - \varphi_{\bar{c}_0, p, \mu_p}\|_{H^2} = 0$ on obtient:

$$A \leq \|\phi_K\|_{L^\infty} \left(\int (\partial_x^k u)^2(t, x + \xi(t)) \right)^{1/2} \left(\int [(\partial_x^k \varphi_{c(t), p, \mu_p})(x) - (\partial_x^k \varphi_{\bar{c}_0, p, \mu_p})(x)]^2 \right)^{1/2} \rightarrow 0. \quad (6.197)$$

Again by applying the Holder inequality, and using that $\phi_K(+\infty) = 1$ and the Lebesgue's Dominated Convergence Theorem we get:

$$B \leq \left(\int (\partial_x^k u)^2(t, x + \xi(t)) \right)^{1/2} \left(\int (\partial_x^k \varphi_{\bar{c}_0, p, \mu_p})^2(x) \left[\phi_K(x + \frac{c_0}{2}(t-t_0) + x_0) - 1 \right]^2 \right)^{1/2} \rightarrow 0. \quad (6.198)$$

Finally $C \rightarrow 0$ since: $u(t, x + \xi(t)) \rightharpoonup \varphi_{\bar{c}_0, p, \mu_p}$ weakly in H^2 . (6.193) follows arguing similarly.

Now let us prove (6.194). We have:

$$\begin{aligned} & \left| \int (\partial_x^k u)(t_0, x + \xi(t_0)) (\partial_x^k \varphi_{c(t_0), p, \mu_p})(x) \phi_K(x + x_0) - \int (\partial_x^k \varphi_{\bar{c}_0, p, \mu_p})^2(x) \right| \\ & \leq \left| \int (\partial_x^k u)(t_0, x + \xi(t_0)) \phi_K(x + x_0) [(\partial_x^k \varphi_{c(t_0), p, \mu_p})(x) - (\partial_x^k \varphi_{\bar{c}_0, p, \mu_p})(x)] \right| \\ & + \left| \int [(\partial_x^k u)(t_0, x + \xi(t_0)) - (\partial_x^k \varphi_{\bar{c}_0, p, \mu_p})(x)] (\partial_x^k \varphi_{\bar{c}_0, p, \mu_p})(x) \phi_K(x + x_0) \right| \\ & + \left| \int (\partial_x^k \varphi_{\bar{c}_0, p, \mu_p})^2(x) [1 - \phi_K(x + x_0)] \right| = A + B + C. \end{aligned} \quad (6.199)$$

Then $A \rightarrow 0$ since: $\lim_{c(t_0) \rightarrow \bar{c}_0} \|\varphi_{c(t_0), p, \mu_p} - \varphi_{\bar{c}_0, p, \mu_p}\|_{H^2} = 0$, and $B \rightarrow 0$ since: $u(t_0, x + \xi(t_0)) \rightharpoonup \varphi_{\bar{c}_0, p, \mu_p}$ in H^2 and $(\partial_x^k \varphi_{\bar{c}_0, p, \mu_p}) \phi_K(\cdot + x_0) \in L^2$. Let us explain the exponential decay of C :

$$C = \int_{x < -x_0} (\partial_x^k \varphi_{\bar{c}_0, p, \mu_p})^2(x) [1 - \phi_K(x + x_0)] + \int_{x \geq -x_0} (\partial_x^k \varphi_{\bar{c}_0, p, \mu_p})^2(x) [1 - \phi_K(x + x_0)] = C_1 + C_2. \quad (6.200)$$

Using that: $\partial_x^k \varphi_{\bar{c}_0, p, \mu_p}(x) \lesssim e^{-\sqrt{\bar{c}_0}|x|}$ and $1 - \phi_K(x + x_0) \leq 1 \forall x \in \mathbb{R}$, we obtain $K \geq \frac{1}{2\sqrt{\bar{c}_0}}$:

$$C_1 \lesssim \int_{x < -x_0} e^{-2\sqrt{\bar{c}_0}|x|} = \frac{1}{2\sqrt{\bar{c}_0}} e^{-2\sqrt{\bar{c}_0}x_0} \lesssim e^{-x_0/K}, \quad (6.201)$$

and using that: $1 - \phi_K(x + x_0) \lesssim e^{-(x+x_0)/K} \forall x \in \mathbb{R}$, we obtain:

$$C_2 \lesssim \|\partial_x^k \varphi_{\bar{c}_0, p, \mu_p}\|_{L^\infty}^2 e^{-x_0/K} \int_{x \geq -x_0} e^{-x/K} \lesssim e^{-x_0/K}. \quad (6.202)$$

This completes the proof of (6.194). The proof of (6.195) is similar.

So, by combining (6.190), (6.191) and (6.192)-(6.195), we obtain $t \geq t_0 \geq T$:

$$\sum_{k=0}^2 \int (\partial_x^k \eta)^2(t, x) \phi_K(x - \xi(t_0) - \frac{c_0}{2}(t - t_0) + x_0) \leq \sum_{k=0}^2 \int (\partial_x^k \eta)^2(t_0, x) \phi_K(x - \xi(t_0) + x_0) + C_3 e^{-x_0/K}, \quad (6.203)$$

where C_3 is a positive constant. Finally by setting: $\theta t = \xi(t_0) + \frac{c_0}{2}(t - t_0) - x_0$ and using (6.189):

$$\sum_{k=0}^2 \int (\partial_x^k \eta)^2(t, x) \phi_K(x - \theta t) \leq C_3 e^{-x_0/K}, \quad (6.204)$$

and this prove (6.2).

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