A Linear Liouville Property for the Generalized Kawahara Equation

André Kabakouala

A Linear Liouveille Property for the Generalized Kawahara Equation

André Kabakouala
People's Friendship University of Russia (RUDN University)
Miklukho-Maklaya str. 6, Moscow, 117198 Russia Federation.

kadodo1987@yahoo.fr

Abstract

In Kabakouala and Molinet [6], we construct a family of smooth-even solitons by applying the Implicit Function Theorem in the neighborhood of the explicit soliton of the generalized Kawahara equation (gKW), found by Dey et al [5]. Next, by combining the well-known spectral method introduced by Benjamin [3] with the continuity arguments, we proved the orbital stability of these family for the subcritical and critical generalized Korteweg-de Vries equation (gKdV) nonlinearity power. In this paper, inspired by the Martel's Method [8] on the gKdV equation, we prove that the solution of the linearized gKW equation around these family of solitons which is uniformly localized becomes static (independent of time), and coincides to the first derivative of the soliton.

1 Introduction

The generalized Kawahara equation is given by:

$$u_t + u^p \partial_x u + \partial_x^3 u - \mu \partial_x^5 u = 0, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}, \tag{1.1}$$

where $p \in \mathbb{N}^*$ denotes the power of nonlinearity, and $\mu > 0$ the parameter which control the fifth-order dispersion term. For $p = 1$ and 2, the gKW equation has applications for instance in fuid mechanics and plasma physics. For $p \geq 3$, what interests us is the equilibrium between the nonlinear effect and the scattering effect, this leads to the formation of solitary waves.

The Cauchy problem associated to (1.1) is locally well-posed in $H^2(\mathbb{R})$ (see for instance Abdelouhab et al. [1]). The H^2-solutions of (1.1) satisfy the following two conservation laws in time:

$$E_{p,\mu}(u(t)) = \int_{\mathbb{R}} \left[\frac{\mu}{2} (\partial_x^2 u)^2(t) + \frac{1}{2} (\partial_x u)^2(t) - \frac{1}{(p+1)(p+2)} u^{p+2}(t) \right] = E_\mu(u_0) \quad \text{(energy)} \tag{1.2}$$

and

$$V(u(t)) = \frac{1}{2} \int_{\mathbb{R}} u^2(t) = V(u_0) \quad \text{(mass)}. \tag{1.3}$$

These conserved quantities enable to extend the solutions for all positive times so that (1.1) is actually globally well-posed in $H^2(\mathbb{R})$. Also, note that thanks to the consevartion laws one can rewrite equation (1.1) in the Hamitonian form:

$$\partial_t V'(u) = \partial_x E'(u), \tag{1.4}$$

where $V'(u)$ and $E'(u)$ denote respectively the Fréchet derivative of $V(u)$ and $E(u)$.

The solitons of gKW are solution of (1.1) of the form $u(t,x)=\varphi_{c,p,\mu}(x-ct)$, which travels to the right with the constant speed $c>0$. Substituting u by $\varphi_{c,p,\mu}$ in (1.1), integrating on $\mathbb{R}$ with the assumption $\partial_x^k\varphi_{c,p,\mu}(\pm\infty)=0$ for $k=0,\dots 4$, we obtain the equation of gKW-solitons:

$$\mu\partial_x^4\varphi_{c,p,\mu}(x)-\partial_x^2\varphi_{c,p,\mu}(x)+c\varphi_{c,p,\mu}(x)=\frac{1}{p+1}\varphi_{c,p,\mu}^{p+1}(x),\quad \forall x\in\mathbb{R}. \tag{1.5}$$

In 1996 Dey , Khare and Kumar [5] compute the explicit solitons of gKW, they found:

$$\varphi_{c,p,\mu}(x)=\left[\frac{(p+1)(p+4)(3p+4)c}{8(p+2)}\right]^{1/p}\operatorname{sech}^{4/p}\left[\frac{p\sqrt{(p^2+4p+8)c}}{4(p+2)}x\right],\quad \forall x\in\mathbb{R}, \tag{1.6}$$

with

$$c=\frac{2^2(p+2)^2}{(p^2+4p+8)^2\mu}. \tag{1.7}$$

We first remark that when μ tends to 0 the speed c converge to $+\infty$, then $\varphi_{c,p,\mu}$ do not converge to the soliton of the generalized Korteweg-de Vries (gKdV) equation $\psi_{c,p}$ explicitly defined by:

$$\psi_{c,p}(x)=\left[\frac{(p+1)(p+2)c}{2}\right]^{1/p}\operatorname{sech}^{2/p}\left[\frac{p\sqrt{c}}{2}x\right],\quad \forall x\in\mathbb{R}, \tag{1.8}$$

and satisfying equation (1.5) with $\mu=0$. One can easily observe that $\varphi_{c,p,\mu}$ is lower and narrower than $\psi_{c,p}$. Second, since the speed c is related to the parameter μ, then when we change the value of c, we modify the gKW equation. Due to the necessary condition of stability of the soliton $\varphi_{c,p,\mu}$ introduced by Karpman [7]:

$$\frac{\mu\int_{\mathbb{R}}\left(\partial_x^2\varphi_{c,p,\mu}(x)\right)^2}{c\int_{\mathbb{R}}\varphi_{c,p,\mu}^2(x)}>\frac{p(p-4)}{(p^2+4p+32)}, \tag{1.9}$$

Dey et al.[5] obtain that the single soliton $\varphi_{c,p,\mu}$ is unstable with respect to the small perturbation in the case $p\geq 5$.

Now we fix $c_0=1$ and $\mu_p=\frac{2^2(p+2)^2}{(p^2+4p+8)^2}$. In Kabakouala and Molinet [6], by applying the Implicit Function Theorem in the neighborhood of the explicit soliton of speed 1: φ_{1,p,μ_p}, we construct a continuous in $H^4(\mathbb{R})$ branch $\{\varphi_{c_0,p,\mu_p},c_0\in]1-\delta_p,1+\delta_p[\}$, with $0<\delta_p\ll 1$, of even solutions to equation (1.5). Moreover, for each $c_0\in]1-\delta_p,1+\delta_p[$, φ_{c_0,p,μ_p} is the unique even $H^4(\mathbb{R})$ solution of (1.5) in some H^4-neighborhood of φ_{1,p,μ_p} (see Lemma 2.2). Next, we prove the orbital stability of these family by combining the continuity arguments: for c_0 close to 1 we have φ_{c_0,p,μ_p} is close to φ_{1,p,μ_p}, and the well-known sufficient condition of the stability of the soliton φ_{1,p,μ_p} introduced by Bejamin: $\langle\mathcal{L}_{1,p,\mu_p}^{-1}\varphi_{1,p,\mu_p},\varphi_{1,p,\mu_p}\rangle_{L^2}<0$, where $\mathcal{L}_{1,p,\mu_p}u=\mu_p\partial_x^4u-\partial_x^2u+u-\varphi_{1,p,\mu_p}^pu$. This stability condition has been verified numerically and is satisfied for the subcritical and critical gKdV nonlinearity power $p=1,2,3,4$ (see [6] Subsection 2.3 and Subsection 2.4).

The goal of this paper is the following, by using the method of Martel [8] applied on the gKdV equation, we will prove that if $u(t,x)$ is solution to the linearized gKW equation around φ_{c_0,p,μ_p}: $\partial_tu=\partial_x(\mathcal{L}_{c_0,p,\mu_p}u)$, where $\mathcal{L}_{c_0,p,\mu_p}u=\mu_p\partial_x^4u-\partial_x^2u+c_0u-\varphi_{c_0,p,\mu_p}^pu$, and moreover if $u(t,x)$ is uniformly localized: $|u(t,x)|\lesssim e^{-\sigma|x|}$, then $u(t,x)$ is static and coincides with the unique element of the kernel of $\mathcal{L}_{c_0,p,\mu_p}$ which is φ'_{c_0,p,μ_p}.

Theorem 1.1 (Linear Liouville Property)**.** *Assume that $p\in\{1,2\}$, $\mu_p=\frac{2^2(p+2)^2}{(p^2+4p+8)^2}$ and $c_0\in]1-\delta,1+\delta[$, with $0<\delta\ll 1$. Let $u\in C(\mathbb{R},H^2(\mathbb{R}))\cap L^\infty(\mathbb{R},H^2(\mathbb{R}))$ be the solution of*

$$\partial_tu=\partial_x(\mathcal{L}_{c_0,p,\mu_p}u),\quad (t,x)\in\mathbb{R}\times\mathbb{R}. \tag{1.10}$$

Moreover, assume that there exists $\sigma > 0$ such that

$$|u(t,x)| \lesssim e^{-\sigma|x|}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{1.11}$$

Then there exists $a_1 \in \mathbb{R}$ such that

$$u(t,x) \equiv a_1 \varphi'_{c_0,p,\mu_p}(x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{1.12}$$

It is clear that φ'_{c_0,p,μ_p} satisfies (1.10) and (1.11), since $\mathcal{L}_{c_0,p,\mu_p}\varphi'_{c_0,p,\mu_p} = 0$ (which is equivalent to the space derivative of eqution (1.5)).

2 Linear Liouville property

2.1 Preliminaries

We start by recalling the spectral properties of the operator $\mathcal{L}_{1,p,\mu_p}$, and the result of existence of a familly of solitons (solutions to (1.5)) in a neighborhood of the explicit soliton of velocity 1 (defined in (1.6)-(1.7)). Also, we give the smoothness and exponential decay property of the solution satisfying (1.10) and (1.11).

Lemma 2.1 (Spectral Properties of $\mathcal{L}_{1,p,\mu_p}$)**.** *Let $p \in \mathbb{N}^*$. We consider the unbounded operator $\mathcal{L}_{1,p,\mu_p} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, defined by: $u \mapsto \mu_p \partial_x^4 u - \partial_x^2 u + u - \varphi^p_{1,p,\mu_p} u$. We claim that $\mathcal{L}_{1,p,\mu_p}$ possesses, among others, the following three crucial properties:*
(P1) The essential spectrum of $\mathcal{L}_{1,p,\mu_p}$ is $[1, +\infty[$;
(P2) $\mathcal{L}_{1,p,\mu_p}$ has only one negative eigenvalue λ_{1,p,μ_p} which is simple;
(P3) The kernel of $\mathcal{L}_{1,p,\mu_p}$ is spanned by φ'_{1,p,μ_p}.

Proof. First, the property $(P1)$ is due to the fact that $\mathcal{L}_{1,p,\mu_p}$ is a compact perturbation of $\mu_p\partial_x^4 - \partial_x^2 + 1$, since φ^p_{1,p,μ_p} is smooth and decay exponentially to 0. Second, Albert [2] has given a sufficient condition based on the qualitative properties of the solitons φ_{1,p,μ_p} which confers the operator $\mathcal{L}_{1,p,\mu_p}$ the spectral properties (P2) and (P3):

$$\mathcal{F}(\varphi_{1,p,\mu_p})(\omega) > 0, \quad \forall \omega \in \mathbb{R}, \quad \text{and} \quad \frac{d^2}{dw^2} \log \mathcal{F}(\varphi^p_{1,p,\mu_p})(\omega) < 0, \quad \forall \omega \in \mathbb{R}^*. \tag{2.1}$$

The sufficient conditions (2.1) are clearly satisfied for the gKW-solitons (see [6], Lemma 2.1 for detailed calculations). □

Lemma 2.2 (Existence of Solitons φ_{1,p,μ_p} for c_0 close to 1)**.** *There exist $\delta_p > 0$ and $\tilde{\delta}_p > 0$ such that for any $c_0 > 0$ with $|c_0 - 1| < \delta_p$, there exists a unique $H^4(\mathbb{R})$ even solution φ_{c_0,p,μ_p} of (1.5) in the ball of $H^4(\mathbb{R})$ centered at φ_{1,p,μ_p} with radius $\tilde{\delta}_p > 0$. Moreover, the function $c_0 \mapsto \varphi_{c_0,p,\mu_p}$ is of class C^1 from $]1-\delta_p, 1+\delta_p[$ into $B_{H^4}(\varphi_{1,p,\mu_p})$.*

Proof. The proof is based on the application of the Implicit Function Theorem (See [6] Lemma 2.2) to the fonctional $T :]1-\delta_p, 1+\delta_p[\times H^4_e(\mathbb{R}) \mapsto L^2_e(\mathbb{R})$ defined by: $T(c,\psi) = \mu_p\partial_x^4\psi - \partial_x^2\psi + c\psi - \frac{1}{p+1}\psi^{p+1}$, where $\mu_p = \frac{2^2(p+2)}{(p^2+4p+8)^2}$ and $H^4_e(\mathbb{R}) = \{u \in H^4(\mathbb{R}) : u(-x) = u(x)\}$ and $L^2_e(\mathbb{R}) = H^0_e(\mathbb{R})$. One can easily compute that $T(1, \varphi_{1,p,\mu_p}) = 0$ and $\frac{\partial}{\partial\psi} T(1, \varphi_{1,p,\mu_p}) = \mathcal{L}_{1,p,\mu_p}\Big|_{H^4_e(\mathbb{R})}$. Let us prove that $\mathcal{L}_{1,p,\mu_p}\Big|_{H^4_e(\mathbb{R})}$ is an isomorphism. We have clearly the linearity and the continuity: $\|\mathcal{L}_{1,p,\mu_p}\|_{L^2(\mathbb{R})} \leq C\|u\|_{H^4(\mathbb{R})}$. Now, since the kernel of $\mathcal{L}_{1,p,\mu_p}$ is spanned by φ'_{1,p,μ_p} (see Lemma 2.1 $(P3)$) and φ'_{1,p,μ_p} is an odd function, then $\mathcal{L}_{1,p,\mu_p}\Big|_{H^4_e(\mathbb{R})}$ is injective. Let $\psi \in L^2_e(\mathbb{R})$, then there exists $\phi \in H^4(\mathbb{R})$ such that $\mathcal{L}_{1,p,\mu_p}\psi = \phi$. We deduce also that $\mathcal{L}_{1,p,\mu_p}\psi(-\cdot) = \phi$. By setting $u(x) = \frac{\psi(x)+\psi(-x)}{2} \in H^4_e(\mathbb{R})$ we get $\mathcal{L}_{1,p,\mu_p}u(-\cdot) = \phi$, and

this prove the surjectivity of $\mathcal{L}_{1,p,\mu_p}\Big|_{H^4_e(\mathbb{R})}$. Finally, by applying the Implicit Function Theorem we obtain the desired result. □

Remark 2.1 (Important properties on φ_{c_0,p,μ_p} and $\mathcal{L}_{c_0,p,\mu_p}$)**.** For c_0 close to 1, the even soliton $\varphi_{c_0,p,\mu_p} \in H^4(\mathbb{R})$ satisfies the following nonlinear fourth-order ODE:

$$\mu_p \partial_x^4 \varphi_{c_0,p,\mu_p} - \partial_x^2 \varphi_{c_0,p,\mu_p} + c_0 \varphi_{c_0,p,\mu_p} - \frac{1}{p+1}\varphi_{c_0,p,\mu_p}^{p+1} = 0. \tag{2.2}$$

The classical bootstrap argument implies that $\varphi_{c_0,p,\mu_p} \in H^k(\mathbb{R})$ for all $k \in \mathbb{N}$. In particular, we have $\partial_x^k \varphi_{c_0,p,\mu_p}(\pm\infty) = 0$ for all $k \in \mathbb{N}$. Then the behavior of φ_{c_0,p,μ_p} for $|x| \gg 1$ is given by the linear asymptotic equation: $\mu_p \partial_x^4 \varphi_{c_0,p,\mu_p} - \partial_x^2 \varphi_{c_0,p,\mu_p} + c_0 \varphi_{c_0,p,\mu_p} = 0$. We deduce that φ_{c_0,p,μ_p} and all these derivatives decay exponentially:

$$|\partial_x^k \varphi_{c_0,p,\mu_p}(x)| \lesssim e^{-\sqrt{c_0}|x|}, \ \forall x \in \mathbb{R}, \ \forall k \in \mathbb{N}. \tag{2.3}$$

Observe that by differentiating equation 2.2 with respect to the speed c we obtain

$$\mathcal{L}_{c,p,\mu_p}(-\frac{d}{dc}\varphi_{c,p,\mu_p}) = \varphi_{c,p,\mu_p} \ \text{ and we set } \ \rho_{c_0,p,\mu_p} = -(\frac{d}{dc}\varphi_{c,p,\mu_p})_{c=c_0}. \tag{2.4}$$

We claim that for $p \in \{1,2,3,4\}$ and c_0 close to 1, the kernel of $\mathcal{L}_{c_0,p,\mu_p}$ is spanned by φ'_{c_0,p,μ_p}. Proof: by differentiating equation 2.2 with respect to x, we have $\mathcal{L}_{c_0,p,\mu_p}\varphi'_{c_0,p,\mu_p} = 0$. Now, recall that for $p \in \{1,2,3,4\}$ and c_0 close to 1, we prove in [6] (see Subsectiob 2.3) that for all $v \in H^2(\mathbb{R})$ satisfying the orthogonality conditions: $\langle v, \varphi'_{c_0,p,\mu_p}\rangle_{L^2} = \langle v, \varphi_{c_0,p,\mu_p}\rangle_{L^2} = 0$, the operator $\mathcal{L}_{c_0,p,\mu_p}$ is coercive: $\langle \mathcal{L}_{c_0,p,\mu_p} v, v\rangle_{L^2} > 0$. Assume that the dimension of the kernel of $\mathcal{L}_{c_0,p,\mu_p}$ is larger than or equal to 2, and choose $v_0 = v_1 - \frac{\langle v_1, \varphi'_{c_0,p,\mu_p}\rangle_{L^2}}{\|\varphi'_{c_0,p,\mu_p}\|^2_{L^2(\mathbb{R})}}\varphi'_{c_0,p,\mu_p}$ with $v_1 \in \mathrm{Ker}\mathcal{L}_{c_0,p,\mu_p} \setminus \{\varphi'_{c_0,p,\mu_p}\}$. Then v_0 satisfies the orthogonalities: $\langle v_0, \varphi'_{c_0,p,\mu_p}\rangle_{L^2} = 0$ and $\langle v_0, \varphi_{c_0,p,\mu_p}\rangle_{L^2} = -\langle \mathcal{L}_{c_0,p,\mu_p} v_0, \frac{d}{dc}_{|_{c=c_0}}\varphi_{c,p,\mu_p}\rangle_{L^2} = 0$. But we have $\langle \mathcal{L}_{c_0,p,\mu_p} v_0, v_0\rangle_{L^2} = 0$ which is a contradiction.

Lemma 2.3 (Smoothness and Exponential Decay)**.** *Let $u \in C(\mathbb{R}, H^2(\mathbb{R})) \cap L^\infty(\mathbb{R}, H^2(\mathbb{R}))$ satisfying* (1.10) *and* (1.11)*. Then $u \in C^\infty(\mathbb{R}\times\mathbb{R})$. Moreover, there exists $\sigma > 0$ such that for all $k \in \mathbb{N}$,*

$$\left|\partial_x^k u(t,x)\right| \lesssim e^{-\sigma|x|}, \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{2.5}$$

As for the linearized gKdV equation (see Martel), the proof of Lemma 2.3 is based on the monotonicity property of energies (conservation laws of gKW) localized by the right, and is completely given in the Appendix 2.3

2.2 Proof of Theorem 1.1.

We follows the ideas of Martel [8] and we split the proof in four steps.

Step 1. Numerical computing of $\mathcal{K}_p = \left\langle \mathcal{L}^{-2}_{1,p,\mu_p}\varphi_{1,p,\mu_p}, \mathcal{L}^{-1}_{1,p,\mu_p}\varphi_{1,p,\mu_p}\right\rangle_{L^2}$ for $p = 1,2,3$.

Recall that, we work with $c_0 = 1$ and $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$. Let us choose $\rho_{1,p,\mu_p} \in \mathcal{L}^{-1}_{1,p,\mu_p}\varphi_{1,p,\mu_p}$ and set $\psi_{1,p,\mu_p} = \mathcal{L}^{-2}_{1,p,\mu_p}\varphi_{1,p,\mu_p} = \mathcal{L}^{-1}_{1,p,\mu_p}\rho_{1,p,\mu_p}$. We note that, since $\mathcal{L}_{1,p,\mu_p}$ is self-ajoint on L^2 the value of $\mathcal{K}_p$ does not depend on the choice of ρ_{1,p,μ_p}. We recall that ρ_{1,p,μ_p} is an even function since φ_{1,p,μ_p} is even, then $\rho_{1,p,\mu_p} \in (\mathrm{Ker}\ \mathcal{L}_{1,p,\mu_p})^\perp$. Moreover $\left\langle \psi_1, \chi_{1,p,\mu_p}\right\rangle_{L^2} = \lambda^{-2}_{1,p,\mu_p}\left\langle \chi_{1,p,\mu_p}, \varphi_{1,p,\mu_p}\right\rangle_{L^2} \neq 0$, where the pair $(\lambda_{1,p,\mu_p}, \chi_{1,p,\mu_p})$ is such that $\mathcal{L}_{1,p,\mu_p}\chi_{1,p,\mu_p} = \lambda_{1,p,\mu_p}\chi_{1,p,\mu_p}$ (see Lemma 2.1). Next, by following exactly the same numerical scheme described in [6] (see Subsection 2.4), we compute that: $\mathcal{K}_1 \approx -6.4912 < 0$ (see Fig.1a-1b), $\mathcal{K}_2 \approx -0.0713 < 0$ (see Fig.1c-1d) and $\mathcal{K}_3 \approx 0.1090 > 0$ (see Fig.1e-1f).

Step 2. Linear dual problem related to (1.10).

We set $\tilde{v} = \mathcal{L}_{c_0,p,\mu_p} u$. Since $\varphi'_{c_0,p,\mu_p} \in \mathrm{Ker}\mathcal{L}_{c_0,p,\mu_p}$, we get the first orthogonality condition: $\langle \tilde{v}, \varphi'_{c_0,p,\mu_p} \rangle_{L^2} = 0$, and from (1.10) we infer that $\tilde{v}$ is a solution of: $\partial_t \tilde{v} = \mathcal{L}_{c_0,p,\mu_p}(\partial_x \tilde{v})$ for $(t,x) \in \mathbb{R}\times\mathbb{R}$. Next, we modify $\tilde{v}$ to obtain the second othogonality condition with the function ρ_{c,p,μ_p}:

$$v(t,x) = \tilde{v}(t,x) - \left(\frac{\int_{\mathbb{R}} \tilde{v}(t,x)\rho_{c_0,p,\mu_p}(x)}{\int_{\mathbb{R}} \varphi_{c_0,p,\mu_p}(x)\rho_{c_0,p,\mu_p}(x)} \right) \varphi_{c_0,p,\mu_p}(x), \tag{2.6}$$

where ρ_{c_0,p,μ_p} is defined in 2.4. One can see that $\langle v, \varphi'_{c_0,p,\mu_p} \rangle_{L^2} = \langle v, \rho_{c_0,p,\mu_p} \rangle_{L^2} = 0$ and satisfies for $(t,x) \in \mathbb{R}\times\mathbb{R}$,

$$\partial_t v = \mathcal{L}_{c_0,p,\mu_p}(\partial_x v) + \delta(t)\varphi_{c_0,p,\mu_p}, \tag{2.7}$$

with

$$\delta(t) = -\frac{1}{\int_{\mathbb{R}} \varphi_{c_0,p,\mu_p}(x)\rho_{c_0,p,\mu_p}(x)} \frac{d}{dt} \int_{\mathbb{R}} \tilde{v}(t,x)\rho_{c_0,p,\mu_p}(x), \tag{2.8}$$

since $\mathcal{L}_{c_0,p,\mu_p}(\partial_x v) = \mathcal{L}_{c_0,p,\mu_p}(\partial_x \tilde{v})$. Now, substituting $\partial_t \tilde{v}$ by $\mathcal{L}_{c_0,p,\mu_p}(\partial_x \tilde{v})$ in (2.8), using that $\mathcal{L}_{c_0,p,\mu_p}\rho_{c_0,p,\mu_p} = \varphi_{c_0,p,\mu_p}$, doing intergation by parts, and from the fact $\tilde{v}$ is orthogonal to φ'_{c_0,p,μ_p} in L^2, we obtain that $\delta(t) = 0$. This justifies our choice of orthogonality condition between v and ρ_{c_0,p,μ_p} (as Côte and al. [4]). Therefore, v is a solution of the following linear dual problem:

$$\partial_t v = \mathcal{L}_{c_0,p,\mu_p}(\partial_x v), \quad (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{2.9}$$

Step 3. Virial Type Identity related to equation (2.9).

We compute a Virial Type Identity for the linear dual problem (2.9). By multiplying (2.9) with the function $v(t,x)x$ and integrating on $\mathbb{R}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} v^2 x = \int_{\mathbb{R}} (\partial_t v)vx = \int_{\mathbb{R}} \mathcal{L}_{c_0,p,\mu_p}(\partial_x v)vx. \tag{2.10}$$

Using the definition of $\mathcal{L}_{c_0,p,\mu_p}$, it holds

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{L}_{c_0,p,\mu_p}(\partial_x v)vx &= \mu_p \int_{\mathbb{R}} (\partial_x^5 v)vx - \int_{\mathbb{R}} (\partial_x^3 v)vx + c_0 \int_{\mathbb{R}} (\partial_x v)vx - \int_{\mathbb{R}} \varphi^p_{c_0,p,\mu_p}(\partial_x v)vx \\ &= I + J + K + L.\end{aligned} \tag{2.11}$$

Now, with the aid of integration by parts, I give us

$$I = -\mu_p \int_{\mathbb{R}} (\partial_x^4 v)\partial_x(vx) = -\mu_p \int_{\mathbb{R}} (\partial_x^4 v)v - \mu_p \int_{\mathbb{R}} (\partial_x^4 v)(\partial_x v)x = I_1 + I_2, \tag{2.12}$$

with

$$I_1 = \mu_p \int_{\mathbb{R}} (\partial_x^3 v)(\partial_x v) = -\mu_p \int_{\mathbb{R}} (\partial_x^2 v)^2 \tag{2.13}$$

and

$$\begin{aligned}I_2 &= \mu_p \int_{\mathbb{R}} (\partial_x^3 v)\partial_x[(\partial_x v)x] = \mu_p \int_{\mathbb{R}} (\partial_x^3 v)(\partial_x^2 v)x + \mu_p \int_{\mathbb{R}} (\partial_x^3 v)(\partial_x v) \\ &= \frac{\mu_p}{2} \int_{\mathbb{R}} \partial_x[(\partial_x^2 v)^2]x - \mu_p \int_{\mathbb{R}} (\partial_x^2 v)^2 = -\frac{3\mu_p}{2} \int_{\mathbb{R}} (\partial_x^2 v)^2.\end{aligned} \tag{2.14}$$

Summing (2.13) and (2.14), we deduce that

$$I = -\frac{5\mu_p}{2} \int_{\mathbb{R}} (\partial_x^2 v)^2. \tag{2.15}$$

We compute J:

$$J = \int_{\mathbb{R}} (\partial_x^2 v)\partial_x(vx) = \int_{\mathbb{R}} (\partial_x^2 v)v + \int_{\mathbb{R}} (\partial_x^2 v)(\partial_x v)x$$
$$= -\int_{\mathbb{R}} (\partial_x v)^2 + \frac{1}{2}\int_{\mathbb{R}} \partial_x[(\partial_x v)^2]x = -\frac{3}{2}\int_{\mathbb{R}} (\partial_x v)^2. \tag{2.16}$$

Finally, we compute K and L:

$$K = \frac{c_0}{2}\int_{\mathbb{R}} \partial_x(v^2)x = -\frac{1}{2}\int_{\mathbb{R}} v^2 \tag{2.17}$$

and

$$L = -\frac{1}{2}\int_{\mathbb{R}} \partial_x(v^2)\varphi^p_{c_0,p,\mu_p}x = \frac{1}{2}\int_{\mathbb{R}} v^2\varphi^p_{c_0,p,\mu_p} + \frac{p}{2}\int_{\mathbb{R}} v^2\varphi^{p-1}_{c_0,p,\mu_p}\varphi'_{c_0,p,\mu_p}x. \tag{2.18}$$

Thus, summing (2.15)-(2.18), we obtain the following identity:

$$-\frac{d}{dt}\int_{\mathbb{R}} v^2x = 5\mu_p\int_{\mathbb{R}} (\partial_x^2 v)^2 + 3\int_{\mathbb{R}} (\partial_x v)^2 + c_0\int_{\mathbb{R}} v^2 - \int_{\mathbb{R}} v^2\varphi^p_{c_0,p,\mu_p} - p\int_{\mathbb{R}} v^2\varphi^{p-1}_{c_0,p,\mu_p}\varphi'_{c_0,p,\mu_p}x. \tag{2.19}$$

Step 4. End of the proof of Theorem 1.1.

The goal is to prove that $v(t,x) = 0$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$, and then we will deduce that $u(t,x) = a_1\varphi'_{c_0,p,\mu_p}(x)$, with a_1 a real constant. The definition of $\mathcal{L}_{c_0,p,\mu_p}$ permit us to rewrite (2.19) as

$$-\frac{d}{dt}\int_{\mathbb{R}} v^2x = \langle \mathcal{L}_{c_0,p,\mu_p}v, v\rangle_{L^2} + 4\mu_p\int_{\mathbb{R}} (\partial_x^2 v)^2 + 2\int_{\mathbb{R}} (\partial_x v)^2 - p\int_{\mathbb{R}} v^2\varphi^{p-1}_{c_0,p,\mu_p}\varphi'_{c_0,p,\mu_p}x. \tag{2.20}$$

With the last term we deal as follow, let us set $\varphi_{c_0,p,\mu_p}(x) = \varphi_{1,p,\mu_p}(x) + (\varphi_{c_0,p,\mu_p}(x) - \varphi_{1,p,\mu_p})(x) = \varphi_{c_0,p,\mu_p}(x) + \gamma(x)$. For c_0 close to 1, we have $\sup_{x\in\mathbb{R}}|\gamma(x)| + \sup_{x\in\mathbb{R}}|\gamma'(x)| \le \gamma$, with $0 < \gamma \ll 1$, and $|\gamma(x)| + |\gamma'(x)| \lesssim e^{-\sqrt{\min(c_0,1)}|x|}$ for all $x \in \mathbb{R}$. Then we deduce the following estimate:

$$-p\int_{\mathbb{R}} v^2\varphi^{p-1}_{c_0,p,\mu_p}\varphi'_{c_0,p,\mu_p}x \ge -p\int_{\mathbb{R}} v^2\varphi^{p-1}_{1,p,\mu_p}\varphi'_{1,p,\mu_p}x - \gamma\int_{\mathbb{R}} v^2. \tag{2.21}$$

Since $\varphi_{c_0,p,\mu_p}(x) > 0$ and $-\varphi'_{c_0,p,\mu_p}(x)x > 0$ for all $x \in \mathbb{R}$, we get

$$-p\int_{\mathbb{R}} v^2\varphi^{p-1}_{c_0,p,\mu_p}\varphi'_{c_0,p,\mu_p}x \ge -\gamma\int_{\mathbb{R}} v^2. \tag{2.22}$$

The identity (2.20) becomes

$$-\frac{d}{dt}\int_{\mathbb{R}} v^2x \ge \langle \mathcal{L}_{c_0,p,\mu_p}v, v\rangle_{L^2} + 4\mu_p\|\partial_x^2 v\|^2_{L^2(\mathbb{R})} + 2\|\partial_x v\|^2_{L^2(\mathbb{R})} - \gamma\|v\|^2_{L^2(\mathbb{R})}. \tag{2.23}$$

Recall that $\langle v, \varphi'_{c_0,p,\mu_p}\rangle_{L^2} = \langle v, \rho_{c_0,p,\mu_p}\rangle_{L^2} = 0$ (by Step 2). Now, we claim that v still almost ortogonal to φ_{1,p,μ_p} and ρ_{1,p,μ_p}. Indeed, applying the Cauchy-Schwarz inequality, it holds

$$|\langle v, \varphi_{1,p,\mu_p}\rangle| \le \gamma\|v\|_{L^2(\mathbb{R})} \quad \text{and} \quad |\langle v, \rho_{1,p,\mu_p}\rangle| = \left|\langle v, \frac{d}{dc}_{|_{c=c_0}}\varphi_{c,p,\mu_p} - \frac{d}{dc}_{|_{c=1}}\varphi_{c,p,\mu_p}\rangle_{L^2}\right| \le \gamma\|v\|_{L^2(\mathbb{R})}. \tag{2.24}$$

Since v is almost orthogonal to φ'_{1,p,μ_p} and ρ_{1,p,μ_p}, and since $\mathcal{K}_p < 0$ for $p = 1, 2$ (by Step 1), and $\mathcal{L}_{1,p,\mu_p}$ possesses the properties (P1)-(P2) (see Lemma 2.1), arguing as Kabakouala and Molinet [6](see Subsection 2.3), there exists $C_0 > 0$ such that

$$\langle \mathcal{L}_{1,p,\mu_p}v, v\rangle_{L^2} \ge C_0\|v\|^2_{H^2(\mathbb{R})}. \tag{2.25}$$

It follows immediatly that for c_0 close to 1,

$$\begin{aligned}\left\langle \mathcal{L}_{c_0,p,\mu_p} v, v\right\rangle_{L^2} &= \left\langle \mathcal{L}_{1,p,\mu_p} v, v\right\rangle_{L^2} + (c_0-1)\left\|v\right\|^2_{L^2(\mathbb{R})} - \left\langle v^2, \varphi^p_{c_0,p,\mu_p} - \varphi^p_{1,p,\mu_p}\right\rangle_{L^2} \\ &\geq C_0 \left\|v\right\|^2_{H^2(\mathbb{R})} + (c_0 - 1 - \gamma)\left\|v\right\|^2_{L^2(\mathbb{R})} \\ &\geq \frac{C_0}{2}\left\|v\right\|^2_{H^2(\mathbb{R})}. \end{aligned} \tag{2.26}$$

Then combining (2.23) and (2.26), we get the following inequality

$$-\frac{d}{dt}\int_{\mathbb{R}} v^2 x \geq C_1 \|v\|^2_{H^2(\mathbb{R})}. \tag{2.27}$$

Lemma 2.3, the definition of $\mathcal{L}_{c_0,p,\mu_p}$, the smoothness and exponential decay properties of φ_{c_0,p,μ_p}, imply that $v \in C^\infty(\mathbb{R}\times\mathbb{R})$, and there exists $\sigma_0 > 0$ such that

$$|v(t,x)| \lesssim e^{-\sigma_0|x|}, \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{2.28}$$

For all $t \in \mathbb{R}$, we set

$$\mathcal{J}(t) = \int_{\mathbb{R}} v^2(t,x)x. \tag{2.29}$$

Thanks to (2.28), one can check that for all $t \in \mathbb{R}$,

$$|\mathcal{J}(t)| \leq \int_{\mathbb{R}} v^2(t,x)|x| \lesssim \int_{\mathbb{R}} e^{-2\sigma|x|}|x| = \frac{1}{2\sigma_1^2}. \tag{2.30}$$

Now, integrating (2.27) with respect to time and using (2.30), we obtain

$$\int_{-\infty}^{+\infty} \|v(t)\|^2_{H^2(\mathbb{R})} dt \leq C_1^{-1}(\mathcal{J}(-\infty) - \mathcal{J}(+\infty)) < +\infty. \tag{2.31}$$

It follows that for a time sequence $t_n \to +\infty$, we have $v(t_n) \to 0$ in $H^2(\mathbb{R})$. In particular, we have

$$|\mathcal{J}(t_n)| \leq \|v(t_n)x\|_{L^2(\mathbb{R})}\|v(t_n)\|_{L^2(\mathbb{R})} \lesssim \frac{1}{\sqrt{2\sigma^3}_1}\|v(t_n)\|_{L^2(\mathbb{R})} \to 0 \ \text{ as } \ t_n \to +\infty. \tag{2.32}$$

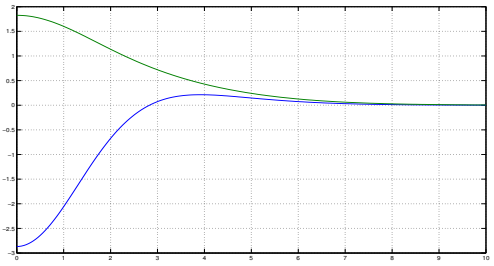
We infer that $\mathcal{J}(+\infty) = 0$. Similarly, $\mathcal{J}(-\infty) = 0$. Thus (2.31) becomes

$$\int_{-\infty}^{+\infty} \|v(t)\|^2_{H^2(\mathbb{R})} dt = 0,$$

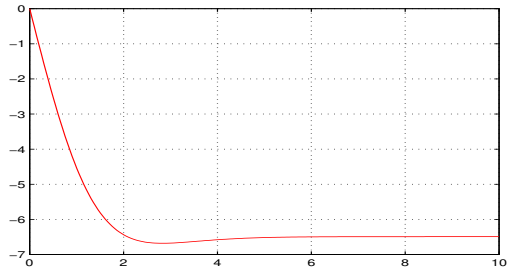
which implies that $v(t,x) \equiv 0$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$. Then $\mathcal{L}_{c_0,p,\mu_p} u(t,x) = \tilde{v}(t,x) = \beta(t)\varphi_{c_0,p,\mu_p}(x)$, and using that $\mathcal{L}_{c,p,\mu_p}\rho_{c,p,\mu_p}(x) = \varphi_{c,p,\mu_p}(x)$ and $\mathrm{Ker}\mathcal{L}_{c_0,p,\mu_p} = \left\langle \varphi'_{c_0,p,\mu_p}\right\rangle$, we get $u(t,x) = a(t)\rho_{c_0,p,\mu_p}(x) + b(t)\varphi'_{c_0,p,\mu_p}(x)$. One can easily compte that $\partial_x\mathcal{L}_{c_0,p,\mu_p}u = a(t)\varphi'_{c_0,p,\mu_p}(x)$ and $\partial_t u = a'(t)\rho_{c_0,p,\mu_p}(x) + b'(t)\varphi'_{c_0,p,\mu_p}(x)$. Now, using that $\partial_t u = \partial_x(\mathcal{L}_{c_0,p,\mu_p}u)$, we obtain $a'(t) = 0$ and $b'(t) = a(t)$, and this implies that $a(t) = a_0$ and $b(t) = a_0 t + a_1$. Finally, since $u(t,x)$ is uniformely bounded, then $a_0 = 0$ and we deduce that $u(t,x) \equiv a_1\varphi'_{c_0,p,\mu_p}(x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$. □

2.3 Appendix: prove of lemma 2.3

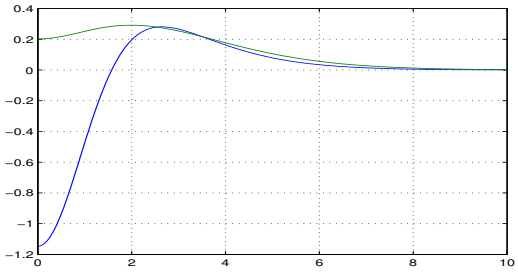
The aim of this section is to prove the smoothness and exponential decay properties of the solutions of the linearized gKW equation around φ_{c_0,p,μ_p} which are uniformly localized on $\mathbb{R}$. We follows the ideas of Martel [8].



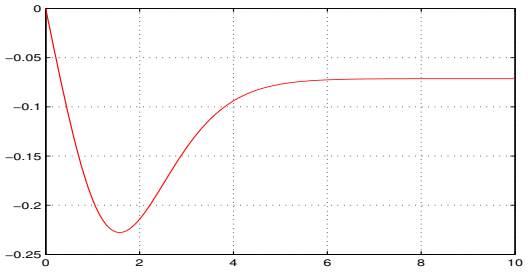
(a) $\rho_1(x)$ and $\psi_1(x)$ profiles.



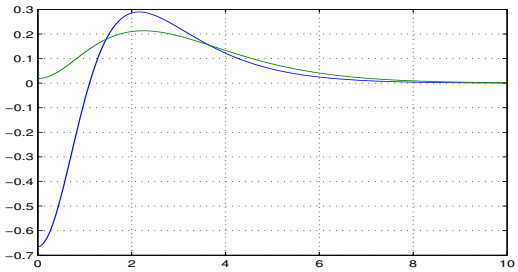
(b) $\int_0^x \psi_1(y)\rho_1(y)$ profile.



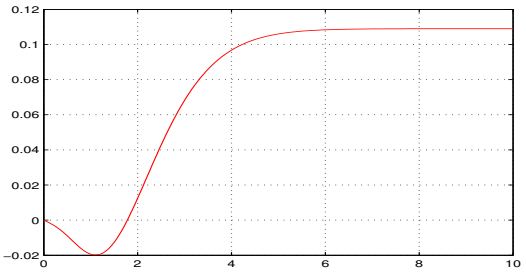
(c) $\rho_1(x)$ and $\psi_1(x)$ profiles.



(d) $\int_0^x \psi_1(y)\rho_1(y)$ profile.



(e) $\rho_1(x)$ and $\psi_1(x)$ profiles.



(f) $\int_0^x \psi_1(y)\rho_1(y)$ profile.

Figure 1: Variation of $\mathcal{K}_p$ for $p = 1, 2, 3$ (with $c_0 = 1$ and $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$).

We define the smooth (C^∞) test-function:

$$\phi(x) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{x}{2}), \quad \forall x \in \mathbb{R}. \tag{2.33}$$

One can easy check the following properties:

$$\phi'(x) = \frac{1}{4}\mathrm{sech}^2\left(\frac{x}{2}\right) \quad \text{and} \quad \left|\phi^{(j+1)}(x)\right| \lesssim \phi'(x), \ \forall j \in \mathbb{N}^*, \ \forall x \in \mathbb{R}. \tag{2.34}$$

Let $K > 0$, we set $\phi_K(x) = \phi(x/K)$. Note that from (2.34) we have $\left|\phi_K^{(j+1)}(x)\right| \leq \frac{1}{K^j}\phi_K'(x)$.

Let $x_0 > 0$, $t_0 \in \mathbb{R}$, for all $t \leq t_0$, we define the L^2-localized energy:

$$I_{x_0,t_0}(t) = \int_{\mathbb{R}} u^2(t,x)\phi_K(x - \frac{c_0}{2}(t_0 - t) - x_0)dx. \tag{2.35}$$

We first prove the monotonicity property of $I_{x_0,t_0}(t)$.

Step 1. For $x_0 > 0$ and $t \le t_0$, we claim that:

$$I_{x_0,t_0}(t_0) - I_{x_0,t_0}(t) \lesssim e^{-x_0/K}. \tag{2.36}$$

Proof of (2.36). We set $\tilde{x}(t) = x - \frac{c_0}{2}(t_0 - t) - x_0 = x - y(t)$. We compute the time variation of $I_{x_0,t_0}(t)$:

$$\frac{d}{dt} I_{x_0,t_0}(t) = 2\int_{\mathbb{R}} (\partial_t u) u \phi_K(\tilde{x}) + \frac{c_0}{2}\int_{\mathbb{R}} u^2 \phi_K'(\tilde{x}). \tag{2.37}$$

Using that $u(t)$ satisfies equation (1.10), the first term in (2.37) gives us:

$$\begin{aligned} 2\int_{\mathbb{R}} (\partial_t u) u \phi_K(\tilde{x}) &= 2\mu_p \int_{\mathbb{R}} (\partial_x^5 u) u \phi_K(\tilde{x}) - 2\int_{\mathbb{R}} (\partial_x^3 u) u \phi_K(\tilde{x}) + 2c_0 \int_{\mathbb{R}} (\partial_x u) u \phi_K(\tilde{x}) - 2\int_{\mathbb{R}} \partial_x(\varphi^p_{c,p,\mu_p} u) u \phi_K(\tilde{x}) \\ &= I + J + K + L. \end{aligned} \tag{2.38}$$

We compute I by applying several integration by parts:

$$I = -2\mu_p \int_{\mathbb{R}} (\partial_x^4 u)(\partial_x u)\phi_K(\tilde{x}) - 2\mu_p \int_{\mathbb{R}} (\partial_x^4 u) u \phi_K'(\tilde{x}) = I_1 + I_2. \tag{2.39}$$

I_1 give us

$$I_1 = 2\mu_p \int_{\mathbb{R}} (\partial_x^3 u)(\partial_x^2 u)\phi_K(\tilde{x}) + 2\mu_p \int_{\mathbb{R}} (\partial_x^3 u)(\partial_x u)\phi_K'(\tilde{x}) = -3\mu_p \int_{\mathbb{R}} (\partial_x^2 u)^2 \phi_K'(\tilde{x}) + \mu_p \int_{\mathbb{R}} (\partial_x u)^2 \phi_K'''(\tilde{x}). \tag{2.40}$$

I_2 give us

$$I_2 = 2\mu_p \int_{\mathbb{R}} (\partial_x^3 u)(\partial_x u)\phi_K'(\tilde{x}) + 2\mu_p \int_{\mathbb{R}} (\partial_x^3 u) u \phi_K''(\tilde{x}) = I_{21} + I_{22}, \tag{2.41}$$

with

$$I_{21} = -2\mu_p \int_{\mathbb{R}} (\partial_x^2 u)^2 \phi_K'(\tilde{x}) + \mu_p \int_{\mathbb{R}} (\partial_x u)^2 \phi_K'''(\tilde{x}), \tag{2.42}$$

and

$$I_{22} = -2\mu_p \int_{\mathbb{R}} (\partial_x^2 u)(\partial_x u)\phi_K''(\tilde{x}) - 2\mu_p \int_{\mathbb{R}} (\partial_x^2 u) u \phi_K'''(\tilde{x}) = 3\mu_p \int_{\mathbb{R}} (\partial_x u)^2 \phi_K'''(\tilde{x}) - \mu_p \int_{\mathbb{R}} u^2 \phi_K^{(5)}(\tilde{x}). \tag{2.43}$$

Combining (2.39)-(2.43) we get I:

$$I = -5\mu_p \int_{\mathbb{R}} (\partial_x^2 u)^2 \phi_K'(\tilde{x}) + 5\mu_p \int_{\mathbb{R}} (\partial_x u)^2 \phi_K'''(\tilde{x}) - \mu_p \int_{\mathbb{R}} u^2 \phi_K^{(5)}(\tilde{x}), \tag{2.44}$$

We compute J:

$$J = 2\int_{\mathbb{R}} (\partial_x^2 u)(\partial_x)\phi_K(\tilde{x}) + 2\int_{\mathbb{R}} (\partial_x^2 u) u \phi_K'(\tilde{x}) = -3\int_{\mathbb{R}} (\partial_x u)^2 \phi_K'(\tilde{x}) + \int_{\mathbb{R}} u^2 \phi_K'''(\tilde{x}). \tag{2.45}$$

K and L give us

$$K = -c_0 \int_{\mathbb{R}} u^2 \phi_K'(\tilde{x}), \quad L = \int_{\mathbb{R}} u^2 \left[\varphi^p_{c_0,p,\mu_p} \phi_K'(\tilde{x}) - (\varphi^p_{c_0,p,\mu_p})' \phi_K(\tilde{x})\right]. \tag{2.46}$$

Therefore, combining (2.37),(2.44)-(2.46), we obtain

$$\begin{aligned} \frac{d}{dt} I_{x_0,t_0}(t) = &-5\mu_p \int_{\mathbb{R}} (\partial_x^2 u)^2 \phi_K'(\tilde{x}) - 3\int_{\mathbb{R}} (\partial_x u)^2 \phi_K'(\tilde{x}) - \frac{c_0}{2}\int_{\mathbb{R}} u^2 \phi_K'(\tilde{x}) \\ &+ 5\mu_p \int_{\mathbb{R}} (\partial_x u)^2 \phi_K'''(\tilde{x}) - \mu_p \int_{\mathbb{R}} u^2 \phi_K^{(5)}(\tilde{x}) + \int_{\mathbb{R}} u^2 \phi_K'''(\tilde{x}) \\ &+ \int_{\mathbb{R}} u^2 \left[\varphi^p_{c_0,p,\mu_p} \phi_K'(\tilde{x}) - (\varphi^p_{c_0,p,\mu_p})' \phi_K(\tilde{x})\right]. \end{aligned} \tag{2.47}$$

Note that the term $5\mu_p \int_{\mathbb{R}} (\partial_x^2 u)^2 \phi_K'(\tilde{x}) + 3\int_{\mathbb{R}} (\partial_x u)^2 \phi_K'(\tilde{x})$ will permit us to gain regularity. Now, using that $\left|\phi_K^{(j)}\right| \leq (1/K^{j-1})\phi_K'$ on $\mathbb{R}$ for $j = 2, 3, 4, 5$, the estimation of the terms on the second line of (2.47) leads to

$$\left|5\mu_p \int_{\mathbb{R}} (\partial_x u)^2 \phi_K'''(\tilde{x}) - \mu_p \int_{\mathbb{R}} u^2 \phi_K^{(5)}(\tilde{x}) + \int_{\mathbb{R}} u^2 \phi_K'''(\tilde{x})\right| \leq \max\left(\frac{5\mu_p}{K^2}, \frac{\mu_p}{K^4} + \frac{1}{K^2}\right) \int_{\mathbb{R}} ((\partial_x u)^2 + u^2)\phi_K'(\tilde{x}). \tag{2.48}$$

Now, let us estimate the last term in (2.47). We first remark that, using the properties of the soliton φ_{c_0,p,μ_p} and the test function ϕ_K, we have for all $x \in \mathbb{R}$,

$$\left|\varphi^p_{c_0,p,\mu_p}(x)\right| \phi_K'(\tilde{x}) + \left|(\varphi^p_{c_0,p,\mu_p})'(x)\right| \phi_K(\tilde{x}) \lesssim e^{-p\sqrt{c_0}|x|}\phi_K(\tilde{x}). \tag{2.49}$$

Let $\alpha > 0$ to be fixed later. We consider the three following cases.

Case: $x < \alpha$. Then $\tilde{x} < \alpha - \frac{c_0}{2}(t_0 - t) - x_0$, so that

$$e^{-p\sqrt{c_0}|x|}\phi_K(\tilde{x}) \lesssim \phi_K(\tilde{x}) \lesssim e^{\tilde{x}/K} \lesssim e^{(\alpha - \frac{c_0}{2}(t_0-t) - x_0)/K}. \tag{2.50}$$

Case: $\alpha < x < \frac{c_0}{2}(t_0 - t) + x_0$. Then, we get that

$$e^{-p\sqrt{c_0}|x|}\phi_K(\tilde{x}) \lesssim e^{-p\sqrt{c_0}x}\phi_K'(\tilde{x}) \lesssim e^{-p\sqrt{c_0}\alpha}\phi_K'(\tilde{x}), \tag{2.51}$$

since $\phi_K(\tilde{x}) \lesssim \phi_K'(\tilde{x})$ for $\tilde{x} < 0$.

Case: $\frac{c_0}{2}(t_0 - t) + x_0 < x$. We have

$$e^{-p\sqrt{c_0}|x|}\phi_K(\tilde{x}) \lesssim e^{-p\sqrt{c_0}(\frac{c_0}{2}(t_0-t)+x_0)} \lesssim e^{-(\frac{c_0}{2}(t_0-t)+x_0)/K}, \tag{2.52}$$

if we choose $K > \frac{1}{\theta p}$.

Combining (2.49)-(2.52), and using that by (1.11) we have $\|u\|_{L^2(\mathbb{R})} \leq 1/\sqrt{\sigma}$, we deduce that

$$\left|\int_{\mathbb{R}} u^2 \left[\varphi^p_{c_0,p,\mu_p}\phi_K'(\tilde{x}) - (\varphi^p_{c_0,p,\mu_p})'\phi_K(\tilde{x})\right]\right| \lesssim e^{(\alpha - \frac{c_0}{2}(t_0-t)-x_0)/K} + e^{-p\sqrt{c_0}\alpha}\int_{\mathbb{R}} u^2\phi_K'(\tilde{x}). \tag{2.53}$$

Finally, integrating (2.47) between t and t_0, using that the first three terms in identity (2.47) are negative, combining the estimates (2.48) and (2.53), and for $K > 0$ and $\alpha > 0$ chosen correctly, we obtain claim (2.36). More general, we get

$$\int_{\mathbb{R}} u^2(x,t_0)\phi_K(x - x_0)dx + \int_t^{t_0}\int_{\mathbb{R}} [(\partial_x^2 u)^2 + (\partial_x u)^2 + u^2](x,s)\phi_K'(\tilde{x})dxds \lesssim e^{-x_0/K} + \int_{\mathbb{R}} u^2(x,t)\phi_K(x - \frac{c_0}{2}(t_0 - t) - x_0)dx. \tag{2.54}$$

Step 2. We claim that:

$$\lim_{t\to-\infty} I_{x_0,t_0}(t) = 0. \tag{2.55}$$

Proof of (2.55). Let $\beta > 0$, we cut the integral into two pieces

$$\int_{\mathbb{R}} u^2(t,x)\phi_K(\tilde{x}) = \int_{x>\beta} u^2(t,x)\phi_K(\tilde{x}) + \int_{x<\beta} u^2(t,x)\phi_K(\tilde{x}) = A + B. \tag{2.56}$$

On the one hand, using the uniform bound of ϕ_K ($\|\phi_K\|_{L^\infty(\mathbb{R})} \leq 1$) and the exponential decay property of $u(t)$, we have

$$A \leq \|\phi_K\|_{L^\infty(\mathbb{R})} \|u\|^2_{L^2(x>\beta)} \leq \frac{1}{2\sigma}e^{-2\sigma\beta}. \tag{2.57}$$

On the other hand, using that $\phi_K(x) \leq e^{x/K}$ for all $x \in \mathbb{R}$, and the L^2 bound of $u(t)$, we have

$$B \leq e^{(\beta - \frac{c_0}{2}(t_0-t)-x_0)/K} \|u\|^2_{L^2(\mathbb{R})} \leq \frac{1}{\sigma} e^{(\beta-\frac{c_0}{2}(t_0-t)-x_0)/K}. \tag{2.58}$$

Since β is an arbitrarily positive constant, then from (2.57) and (2.58), we obtain the claim (2.55).

Step 3. For $x_0 > 0$, $t \leq t_0$ and $k \in \mathbb{N}$, we claim that:

$$\int_{\mathbb{R}} (\partial_x^k u)^2(t_0,x)\phi_K(x-x_0)dx + \int_{-\infty}^{t_0}\int_{\mathbb{R}} \left[(\partial_x^{k+2}u)^2 + (\partial_x^{k+1}u)^2 + (\partial_x^k u)^2(\tilde{x})\right]\phi_K'(s,x)dxds \lesssim e^{-x_0/K}. \tag{2.59}$$

Proof of (2.59). We proceed by induction on k. From Step 1 and Step 2, the estimate (2.59) holds for $k = 0$. Now, we assume that (2.59) is true for $1 \leq j \leq k-1$. By repeating the computations done on Step 1 with the H^k-localized energy: $I^{(k)}_{x_0,t_0}(t) = \int_{\mathbb{R}} (\partial_x^k u)^2(t,x)\phi_K(\tilde{x})dx$, we get

$$\begin{aligned}\frac{d}{dt}I^{(k)}_{x_0,t_0}(t) = &-5\mu_p\int_{\mathbb{R}}(\partial_x^{k+2}u)^2\phi_K'(\tilde{x}) - 3\int_{\mathbb{R}}(\partial_x^{k+1}u)^2\phi_K'(\tilde{x}) - \frac{c_0}{2}\int_{\mathbb{R}}(\partial_x^k u)^2\phi_K'(\tilde{x})\\ &+ 5\mu_p\int_{\mathbb{R}}(\partial_x^{k+1}u)^2\phi_K'''(\tilde{x}) - \mu_p\int_{\mathbb{R}}(\partial_x^k u)^2\phi_K^{(5)}(\tilde{x}) + \int_{\mathbb{R}}(\partial_x^k u)^2\phi_K'''(\tilde{x})\\ &- 2\int_{\mathbb{R}}\partial_x^{k+1}(\varphi^p_{c_0,p,\mu_p}u)(\partial_x^k u)\phi_K(\tilde{x}).\end{aligned} \tag{2.60}$$

By applying the similar arguments as for the estimate (2.48), we have

$$\begin{aligned}&\left|5\mu_p\int_{\mathbb{R}}(\partial_x^{k+1}u)^2\phi_K'''(\tilde{x}) - \mu_p\int_{\mathbb{R}}(\partial_x^k u)^2\phi_K^{(5)}(\tilde{x}) + \int_{\mathbb{R}}(\partial_x^k u)^2\phi_K'''(\tilde{x})\right|\\ &\qquad\leq \max\left(\frac{5\mu_p}{K^2}, \frac{\mu_p}{K^4} + \frac{1}{K^2}\right)\int_{\mathbb{R}}((\partial_x^{k+1}u)^2 + (\partial_x^k u)^2)\phi_K'(\tilde{x}).\end{aligned} \tag{2.61}$$

Let us estimate the last term in (2.60). Applying the Leibniz formula on $\partial_x^{k+1}(\varphi^p_{c_0,p,\mu_p}u)$, it holds

$$\begin{aligned}-2\int_{\mathbb{R}}\partial_x^{k+1}(\varphi^p_{c_0,p,\mu_p}u)(\partial_x^k u)\phi_K(\tilde{x}) = &\int_{\mathbb{R}}(\partial_x^k u)^2\left[-(2k+1)(\varphi^p_{c_0,p,\mu_p})'\phi_K(\tilde{x}) + \varphi^p_{c_0,p,\mu_p}\phi_K'(\tilde{x})\right]\\ &- 2\sum_{j=2}^{k+1}C^j_{k+1}\int_{\mathbb{R}}(\partial_x^k u)(\partial_x^{k-j+1}u)(\varphi^p_{c_0,p,\mu_p})^{(j)}\phi_K(\tilde{x}).\end{aligned} \tag{2.62}$$

Arguing as for the estimate (2.53), we get

$$\left|\int_{\mathbb{R}}(\partial_x^k u)^2\left[-(2k+1)(\varphi^p_{c_0,p,\mu_p})'\phi_K(\tilde{x}) + \varphi^p_{c_0,p,\mu_p}\phi_K'(\tilde{x})\right]\right| \lesssim e^{(\alpha-\frac{c_0}{2}(t_0-t)-x_0)/K} + e^{-p\sqrt{c_0}\alpha}\int_{\mathbb{R}}(\partial_x^k u)^2\phi_K'(\tilde{x}). \tag{2.63}$$

Remark that for all $x \in \mathbb{R}$ and $j \in \mathbb{N}$,

$$|(\varphi^p_{c_0,p,\mu_p})^{(j)}|\phi_K(\tilde{x}) \lesssim e^{-p\sqrt{c_0}|x|}\phi_K(\tilde{x}) \lesssim \phi_K'(\tilde{x}). \tag{2.64}$$

Indeed, for $\tilde{x} \leq 0$, we have $\phi_K(\tilde{x}) \leq \phi_K'(\tilde{x})$. For $\tilde{x} > 0$, we have $x > \tilde{x} > 0$, and then

$$e^{-p\sqrt{c_0}|x|}\phi_K(\tilde{x}) \lesssim e^{-p\sqrt{c_0}x} \lesssim e^{-p\sqrt{c_0}\tilde{x}} \lesssim \phi_K'(\tilde{x}). \tag{2.65}$$

Young's inequality by parts and (2.65) lead to

$$\left|2\sum_{j=2}^{k+1}C^j_{k+1}\int_{\mathbb{R}}(\partial_x^k u)(\partial_x^{k-j+1}u)(\varphi^p_{c_0,p,\mu_p})^{(j)}\phi_K(\tilde{x})\right| \lesssim \sum_{j=0}^{k-1}\int_{\mathbb{R}}(\partial_x^j u)^2\phi_K'(\tilde{x}). \tag{2.66}$$

Therfore, by integrating (2.60) between on t and t_0, and combining the estimates (2.61), (2.63) and (2.66), we obtain

$$I^{(k)}_{x_0,t_0}(t_0) + \int_t^{t_0}\int_{\mathbb{R}} \left[(\partial_x^{k+2}u)^2 + (\partial_x^{k+1}u)^2 + (\partial_x^k u)^2\right]\phi_K'(\tilde{x}) \\ \lesssim e^{-x_0/K} + \sum_{j=0}^{k+1}\int_t^{t_0}\int_{\mathbb{R}} (\partial_x^j u)^2\phi_K'(\tilde{x}) + I^{(k)}_{x_0,t_0}(t). \tag{2.67}$$

Letting t tends to $-\infty$ and using the induction hypothesis, we get

$$I^{(k)}_{x_0,t_0}(t_0) + \int_{-\infty}^{t_0}\int_{\mathbb{R}} \left[(\partial_x^{k+2}u)^2 + (\partial_x^{k+1}u)^2 + (\partial_x^k u)^2\right]\phi_K'(\tilde{x}) \\ \lesssim e^{-x_0/K} + \liminf_{t\to-\infty} I^{(k)}_{x_0,t_0}(t). \tag{2.68}$$

Let us proof that $I^{(k)}_{x_0,t_0}(t) \to 0$ as $t \to -\infty$. From the induction hypothesis with $j = k-1$, we know that

$$\int_{-\infty}^{t_0}\int_{\mathbb{R}} (\partial_x^k u)^2\phi_K'(\tilde{x}) \lesssim e^{-x_0/K}. \tag{2.69}$$

Then

$$\int_{-\infty}^{t_0}\int_{x<\frac{c_0}{2}(t_0-t)+x_0} (\partial_x^k u)^2 e^{(x-\frac{c_0}{2}(t_0-t))/K} \lesssim 1, \tag{2.70}$$

since $e^{\tilde{x}/K} \lesssim \phi_K'(\tilde{x})$ for $\tilde{x} < 0$. Thus, passing to the limit as x_0 tends to $+\infty$ in (2.70) and multiplying by $e^{x_0/K}$, we get

$$\int_{-\infty}^{t_0}\int_{\mathbb{R}} (\partial_x^k u)^2 e^{\tilde{x}} \lesssim e^{-x_0/K}, \tag{2.71}$$

and then, since $\phi(\tilde{x}) \lesssim e^{\tilde{x}/K}$ for all $x \in \mathbb{R}$,

$$\int_{-\infty}^{t_0}\int_{\mathbb{R}} (\partial_x^k u)^2\phi_K(\tilde{x}) \lesssim e^{-x_0/K}. \tag{2.72}$$

Thus, we infer that

$$\liminf_{t\to-\infty}\int_{\mathbb{R}} (\partial_x^k u)^2\phi_K(\tilde{x}) = 0. \tag{2.73}$$

Combining (2.68) and (2.73), we get the claim (2.59).

Step 4. For all $k \in \mathbb{N}$, we claim that:

$$\int_{\mathbb{R}} (\partial_x^k u)^2(t,x)e^{|x|/K} \lesssim 1. \tag{2.74}$$

Proof of (2.74). From Step 3, we know that:

$$\int_{\mathbb{R}} (\partial_x^k u)^2\phi_K(x-x_0) \lesssim e^{-x_0/K}, \tag{2.75}$$

and then

$$\int_{x<x_0} (\partial_x^k u)^2 e^{x/K} \lesssim 1, \tag{2.76}$$

since $e^{(x-x_0)/K} \lesssim \phi_K(x-x_0)$ for $x < x_0$. Now, passing to the limit as x_0 tends to $+\infty$ in (2.76), we get the exponential decay property by the right:

$$\int_{\mathbb{R}} (\partial_x^k u)^2 e^{x/K} \lesssim 1. \tag{2.77}$$

For the left exponential decay proprety, one can remark that $u(-t,-x)$ satisfies (1.10) and (1.11). Then, repeating the same analysis with $u(-t,-x)$, we get the claim (2.74). In particular, applying the Gagliardo-Nirenberg inequality, we infer that for all $k \in \mathbb{N}$,

$$\left\| (\partial_x^k u) e^{|\cdot|/2K} \right\|_{L^\infty(\mathbb{R})} \leq C_G \left\| (\partial_x^k u) e^{|\cdot|/2K} \right\|_{L^2(\mathbb{R})}^{1/2} \left\| (\partial_x^{k+1} u) e^{|\cdot|/2K} \right\|_{L^2(\mathbb{R})}^{1/2} \lesssim 1, \tag{2.78}$$

and this prove (2.5).

Acknowledgements. The author would like to thank Professor Andrey Shishkov for his help and his careful reading of this manuscript. The pulication has been prepared with the support of the RUDN University Program 5-100.

References

[1] L. Abdelouhab, J. L. Bona, M. Felland, and J.-C. Saut. Nonlocal models for nonlinear, dispersive waves. *Phys. D*, 40(3):360–392, 1989.

[2] J. P. Albert. Positivity properties and stability of solitary-wave solutions of model equations for long waves. *Comm. Partial Differential Equations*, 17(1-2):1–22, 1992.

[3] T. B. Benjamin. The stability of solitary waves. *Proc. Roy. Soc. (London) Ser. A*, 328:153–183, 1972.

[4] R. Côte, C. Muñoz, D. Pilod, and G. Simpson. Asymptotic Stability of High-dimensional Zakharov–Kuznetsov Solitons. *Arch. Ration. Mech. Anal.*, 220(2):639–710, 2016.

[5] B. Dey, A. Khare, and C. N. Kumar. Stationary solitons of the fifth order KdV-type. Equations and their stabilization. *Phys. Lett. A*, 223(6):449–452, 1996.

[6] André Kabakouala and Luc Molinet. On the stability of the solitary waves to the (generalized) kawahara equation. working paper or preprint, November 2016.

[7] V. I. Karpman. Stabilization of soliton instabilities by higher order dispersion: KdV-type equations. *Phys. Lett. A*, 210(1-2):77–84, 1996.

[8] Yvan Martel. Linear problems related to asymptotic stability of solitons of the generalized KdV equations. *SIAM J. Math. Anal.*, 38(3):759–781, 2006.