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Binary morphisms and fractal curves

Alexis Monnerot-Dumaine*

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Abstract

The Fibonacci morphism, the most elementary non-trivial binary morphism, generates the infinite Fibonacci word Fractal through a simple drawing rule, as described in [12]. We study, here, families of fractal curves associated with other simple binary morphisms. Among them the Sierpinsky triangle appears. We establish some properties and calculate their Hausdorff dimension.

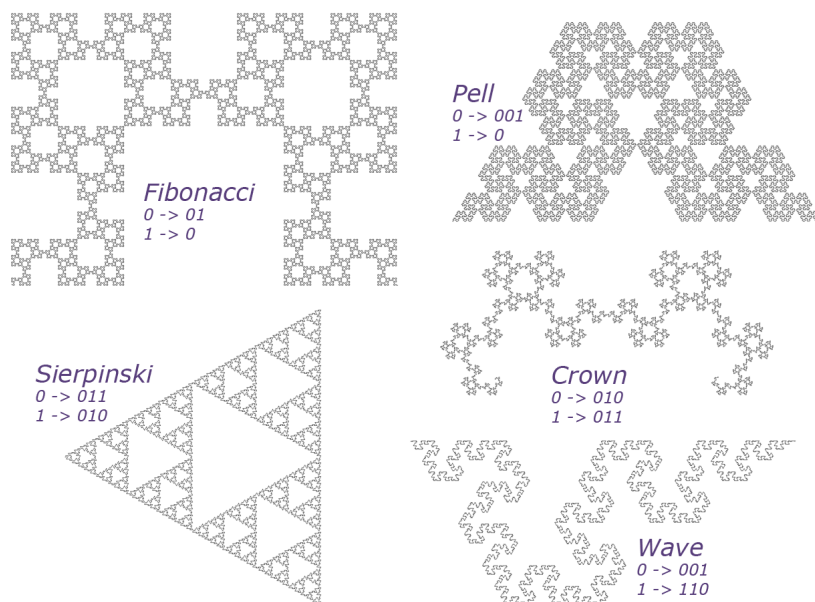


Fig.1:fractal curves generated by simple binary morphisms

1 Definitions

1.1 Morphisms and mirror morphisms

In this paper, we will call "elementary morphisms", binary morphisms σ such that, for any letter a , $|\sigma(a)| < 4$. This includes the well known Fibonacci morphism and the Thue-Morse morphism.

Let a morphism σ defined, for any letter a , by $\sigma(a) = a_1 a_2 a_3 \dots a_n$, then its mirror $\bar{\sigma}$ is such that $\bar{\sigma}(a) = a_n \dots a_3 a_2 a_1$

A morphism σ is said to be "composite" if we can find two other morphisms ρ and τ for which, for any letter a , $\sigma(a) = \rho(\tau(a))$.

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1.2 The odd-even drawing rule (OEDR)

Let w be a binary word defined in a two letters alphabet $0;1$. We define the odd-even drawing rule iteratively as follows : Take the n th letter of w .

- draw a segment forward
 - If the letter is "0" then :
 - . turn left if "n" is even
 - . turn right if "n" is odd
- and iterate

We will define $\mathcal{P}(w)$ be the curve generated from the word w . We define also the instructions F = "draw a segment forward", L = "draw a segment forward and turn left" and R = "draw a segment forward and turn right".

Then, for example:

- $\mathcal{P}(0)$ is generated by the instruction R
- $\mathcal{P}(1)$ is generated by the instruction F
- $\mathcal{P}(10)$ is generated by the instructions FL

1.3 Curves associated to a morphism

The curves studied here are all the result of repeated iterations of an elementary morphism, starting from the value 0, through the OEDR. Let k be an iteration number and σ a morphism, then we will consider the words $\sigma^k(0)$, and the associated curves $\mathcal{P}(\sigma^k(0))$. We will sometimes call these curves \mathcal{P}_k , for short, when the context is sufficiently clear about the morphism concerned.

In this article, we will concentrate also on self-avoiding curves which present interesting properties of density and self-similarity. Every time we choose the most suitable value for the angle so that those properties appear more clearly. We will see $\pi/2$ and $2\pi/3$ are the most interesting angles.

Let's take an example of such a curve, with the Thue-Morse word. The Thue-Morse morphism is defined by $t(0) = 01$ and $t(1) = 10$;

The Thue-Morse word TM is the word generated by the the infinite iteration of the Thue-Morse morphism : TM = 011010010110100110010110... This sequence has been described by Axel Thue in [17] and [17].

Applying the Odd-Even drawing rule to the Thue-Morse word TM generates a path $\mathcal{P}(TM)$ illustrated in the figure below. The path follows the instructions RFFLFLRFRFFLFLRFFLRFRFFL... The pattern unravels in a non-periodic series of turns. It is not self-intersecting. The general aspect of $\mathcal{P}(TM)$ is a straight line, whatever the angle chosen for the turns : $\pi/2$ or $2\pi/3$.

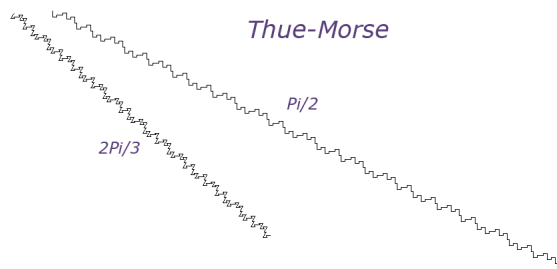


Fig.2: Pattern associated to the Thue-Morse word

1.4 Hausdorff dimension of self-similar curves

In this article, we will use a formula for the Hausdorff dimension described in [8] and [11]. If a given curve displays n distinct self-similarities of ratios r_k , and providing the open condition holds,

then its Hausdorff dimension d is a solution of the equation:

$$\sum_{k=1}^n r_k^d = 1(1)$$

2 The Fibonacci word fractal curve

This curve has been extensively described in [12], the reader should refer to this paper for more details. We will just list here all the elementary morphisms that can generate this pattern and the different variants.

The Fibonacci morphism is defined by $0 \rightarrow 01$ and $1 \rightarrow 0$. Iterating this morphism several times generates the Fibonacci words.

The infinite Fibonacci word starts with : $0100101001001010010100100101001001010010\dots$. It is the fixed point of the Fibonacci morphism.

It is remarkable that no less than 12 different elementary morphisms can generate the Fibonacci word fractal. This reveals the particular role this curve must have among the ones listed here. One can observe, that all the mirror morphisms and the various compositions of those morphisms generate the Fibonacci word fractal. What is more, exchanging the role of 0 and 1 in the rule, also generates this pattern.

morph	0 \rightarrow	1 \rightarrow	Comments
F_1	01	0	Regular Fibonacci morphism
F_2	10	0	mirror of F_1
F_3	001	01	composite : $F_3 = F_1 \circ F_2$
F_4	010	01	composite : $F_4 = F_1^2$
F_5	010	10	composite : $F_5 = F_2^2$ and mirror of F_4
F_6	100	10	composite : $F_6 = F_2 \circ F_1$ and mirror of F_3

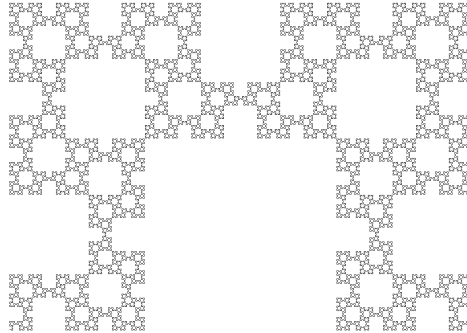


Fig.2:2 The Fibonacci word curve, F23 segments

Other morphisms can generate a 45° extended version of the fractal, where the roles of the "0" and the "1" are exchanged:

morph	0 \rightarrow	1 \rightarrow	Comments
G_1	1	01	Exchanging the role of 0 and 1 in the Fibonacci morphism F_1 .
G_2	1	10	mirror of G_1
G_3	01	011	composite : $G_3 = G_2 \circ G_1$
G_4	01	101	composite : $G_4 = G_2^2$
G_5	10	101	composite : $G_5 = G_1^2$ and mirror of G_4 .
G_6	10	110	composite : $G_6 = G_1 \circ G_2$ and mirror of G_3

It is important to emphasize on the fact that all those curves, although they differ for a limited number of iterations, converge to the same fractal pattern at infinity.

3 The Pell word fractal curve

3.1 The Pell word

We define the Pell morphism P_1 :

$$P_1(0) = 001$$

$$P_1(1) = 0$$

Starting from $w = 0$, we get the sequence of what we will call Pell words :

- 001
- 0010010
- 00100100010010001

Iterating to infinity creates the infinite Pell word.

We can show that, for any iteration k , $P_1^k(0) = 2P_1^{k-1}(0)P_1^{k-2}(0)$. Every term is the concatenation of twice the previous term and the term before. This recalls the Pell numbers P_n , defined by $P_k = 2P_{k-1} + P_{k-2}$ with $P_0 = 0$ and $P_1 = 1$. The Pell words are then defined by analogy with the Pell numbers, as the Fibonacci words were defined from the Fibonacci numbers.

We can show that the length of a Pell word at iteration k is $|s^k(0)| = P_k$

We note this morphism is composed : let F_1 be the Fibonacci morphism $0 \rightarrow 01$ and $1 \rightarrow 0$, and G the morphism defined by $0 \rightarrow 0$ and $1 \rightarrow 01$. Then it is straightforward to see that, for any letter a , $P_1(a) = (GoF_1)(a)$

The Pell word is a Sturmian word. It has $n+1$ distinct factors of length n .

The mirror morphism $P_2 : 0 \rightarrow 100$ and $1 \rightarrow 0$, generates a similar pattern.

3.2 The Pell fractal curve

By the odd-even drawing rule, and choosing an angle of $2\pi/3$, the curve \mathcal{P}_k shows rather dense pattern inscribed in an isosceles trapezoid. The self-similarities appear clearly at any scales. We notice it can be constructed by juxtaposing two copies of \mathcal{P}_{k-1} and one \mathcal{P}_{k-2} in different ways: $\mathcal{P}_{k-1} + \mathcal{P}_{k-1} + \mathcal{P}_{k-2}$ or $\mathcal{P}_{k-1} + \mathcal{P}_{k-2} + \mathcal{P}_{k-1}$, this will be useful to calculate its Hausdorff dimension (see below).

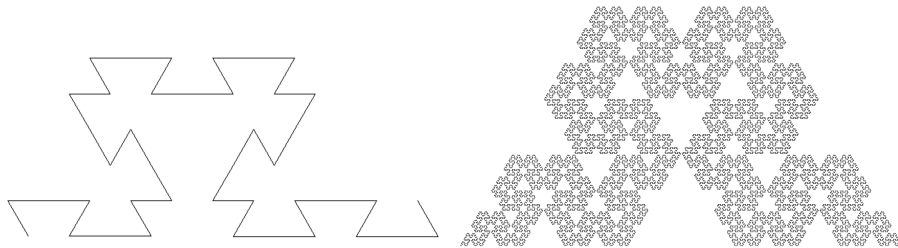


Fig.2:2 iterations of the Pell word fractal curve

We notice also clearly that the path is non intersecting.

3.3 The Hausdorff dimension of the Pell fractal

We notice that the curve can be built using two self-similarities of ratio a and one similarity of ratio a^2 see figure. The figure suggests immediately that $a + a^2 = 1$, so $a = \phi = (1 + \sqrt{5})/2$, the golden ratio.

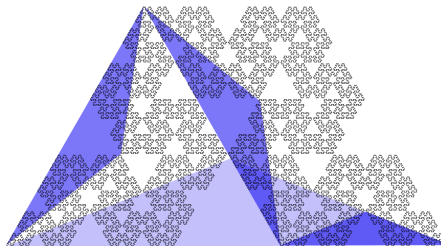


Fig.2:2 Self-similarities

Since the curves can be built with similarities, we can use the following formula for the Hausdorff dimension d : $2\phi^d + \phi^{2d} = 1$, so d equals:

$$d = \frac{\log(1 + \sqrt{2})}{\log \phi} = 1.8315.. \quad (2)$$

We notice here the presence of both the golden and the silver ratios [19]. We can then write the remarkable expression:

$$d = \frac{\log\left(2 + \frac{1}{2 + \frac{1}{2 + \dots}}\right)}{\log\left(1 + \frac{1}{1 + \frac{1}{1 + \dots}}\right)} \quad (3)$$

3.4 Variants

The variants shown here, present a different orientation and can be extended. An extended version is a curve that is larger but has an overall same aspect. We list here, also, a particular case we call "horned version". Although it shares some characteristics with the original curve the original curve, it is clearly different.

morph	0 →	1 →	Comment
P_3	1	011	an extended version tilted $45\hat{A}^\circ$
P_4	1	110	mirror of P_3
P_5	010	0	a "horned" version. Quite different, see figure"
P_6	1	101	mirror of P_5 , a "horned" version, extended

It is straightforward that, appart from the horned version, all those curves converge to the same fractal pattern at infinity.

4 The Sierpinski Triangle, Crown and Wave fractals

Those three patterns are regrouped here, in the same section, because they share notable common properties :

- they all derive from uniform binary morphisms
- each one of them can be created from two different morphisms or their mirrors.
- those morphisms are morphisms of length 3 (for any letter a , $|\sigma(a)| = 3$.)
- none of thoses morphisms is composite
- they all share the same Hausdorff dimension.

morphism	0 →	1 →	Word	Comments
The Sierpinski Triangle				
S_1	011	010	01101001001101001101...	Sierpinski Triangle 1
S_2	101	001	00110100100110100110...	Sierpinski Triangle 2
S_3	110	010	01001011011001011001...	Mirror of S_1
S_4	101	100	10010110110010110010...	Mirror of S_2
The Crown fractal				
C_1	010	011	01001101001001101101...	Crown fractal 1
C_2	001	101	00100110100100110110...	Crown fractal 2
C_3	010	110	01011001011011001001...	mirror of C_1
C_4	100	101	10110010110110010010...	Mirror of C_2
The Wave fractal				
W_1	001	110	00100111000100111011...	Wave fractal 1
W_2	011	100	01110010010001101110...	Wave fractal 2
W_3	100	011	01110010010001101110...	Mirror of W_1
W_4	110	001	00100111000100111011...	Mirror of W_2

4.1 The fractal curves

Applying the odd-even drawing rule, with an angle of $2\pi/3$, the curve \mathcal{P}_k shows 3 kinds of patterns :

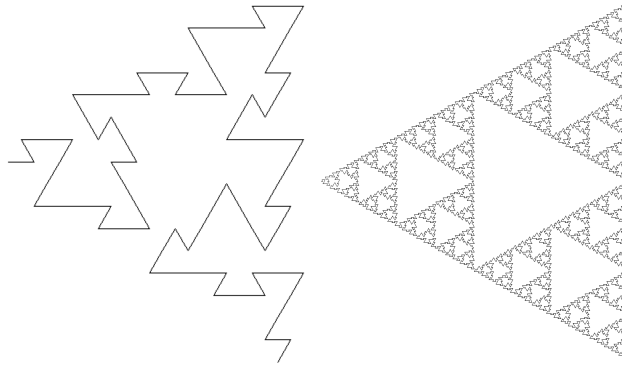


Fig.2:2 iterations of the Sierpinski fractal curve

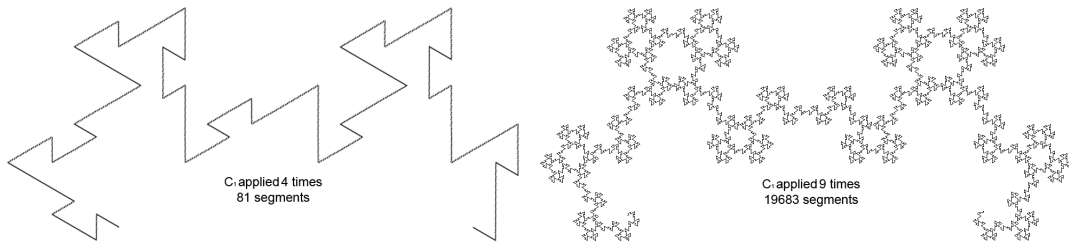


Fig.2:2 iterations of the Crown fractal

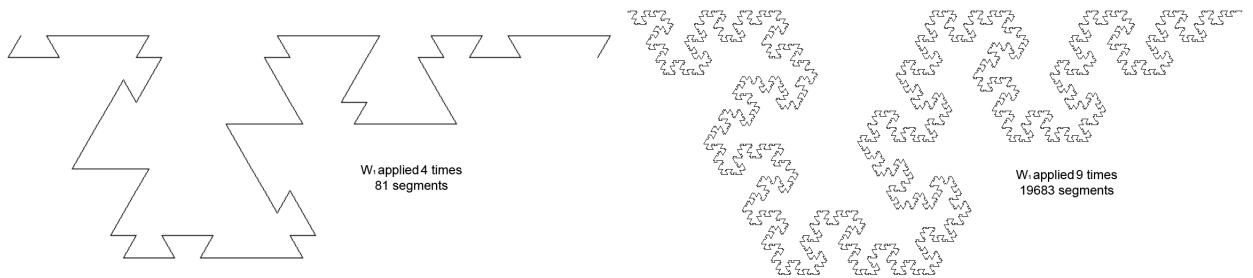


Fig.2:2 iterations of the Wave fractal

Again, the self-similarities appear at any scales. We notice also clearly that the paths are non self-intersecting.

4.2 Their Hausdorff dimension

Each pattern, the Sierpinski triangle, the Crown Fractal or the Wave fractal, can be built by combining 3 copies of itself, twice smaller.

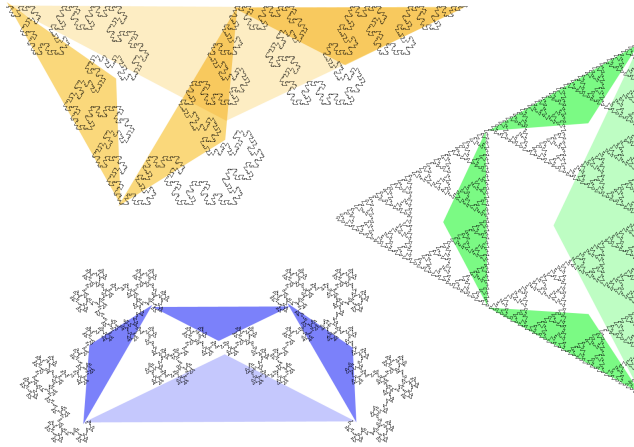


Fig.2:2 Each pattern is made of three copies of itself, twice smaller

Since the curves can be built with similarities, we can use the following formula for the Hausdorff dimension $d : 3(1/2)^d = 1$. The Hausdorff dimension is then :

$$d = \frac{\log 3}{\log 2} = 1.5849.. \quad (4)$$

5 Conclusion

		0→									
		1	01	10	11	001	010	011	100	101	110
12*1→	0		FF	FF		P	PH		P		
	00		T	T							
	01	FFb		TM	T	FF	FF	CO			
	10	FFb	TM		T		FF		FF		
	11										
	001							Sb		S	V
	010							S			S
	011	Pb	FFb			Cb	C		V		
	100			CO				V		S	Sb
	101	PHb	FFb	FFb		C				C	
	110	Pb		FFb		V	C			Cb	
	111						P				

6 Acknowledgements

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- 05B30 Combinatorics - Other designs, configurations
- 11B39 Fibonacci and Lucas numbers and polynomials and generalizations
- 28A80 Fractals
- 68R15 Combinatorics on words