Binary morphisms and fractal curves

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Abstract

The Fibonacci morphism, the most elementary non-trivial binary morphism, generates the infinite Fibonacci word Fractal through a simple drawing rule, as described in [12]. We study, here, families of fractal curves associated with other simple binary morphisms. Among them the Sierpinsky triangle appears. We establish some properties and calculate their Hausdorff dimension.

Fig.1: fractal curves generated by simple binary morphisms

1 Definitions

1.1 Morphisms and mirror morphisms

In this paper, we will call "elementary morphisms", binary morphisms σ such that, for any letter a, |σ(a)| < 4. This includes the well known Fibonacci morphism and the Thue-Morse morphism.

Let a morphism σ defined, for any letter a, by σ(a) = α_1α_2α_3...α_n, then its mirror σ is such that σ(a) = a_n...α_3α_2α_1

A morphism σ is said to be "composite" if we can find two other morphisms ρ and τ for which, for any letter a, σ(a) = ρ(τ(a)).

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1.2 The odd-even drawing rule (OEDR)

Let \( w \) be a binary word defined in a two letters alphabet 0;1. We define the odd-even drawing rule iteratively as follows: Take the \( n \)th letter of \( w \).
- draw a segment forward
- If the letter is "0" then :
  . turn left if "n" is even
  . turn right if "n" is odd
and iterate

We will define \( \mathcal{P}(w) \) be the curve generated from the word \( w \). We define also the instructions 
\( F = \) "draw a segment forward", \( L = \) "draw a segment forward and turn left" and \( R = \) "draw a segment forward and turn right". 

Then, for example:
- \( \mathcal{P}(0) \) is generated by the instruction \( R \)
- \( \mathcal{P}(1) \) is generated by the instruction \( F \)
- \( \mathcal{P}(10) \) is generated by the instructions \( FL \)

1.3 Curves associated to a morphism

The curves studied here are all the result of repeated iterations of an elementary morphism, starting from the value 0, through the OEDR. Let \( k \) be an iteration number and \( \sigma \) a morphism, then we will consider the words \( \sigma^k(0) \), and the associated curves \( \mathcal{P}(\sigma^k(0)) \). We will sometimes call these curves \( \mathcal{P}_k \), for short, when the context is sufficiently clear about the morphism concerned.

In this article, we will concentrate also on self-avoiding curves which present interesting properties of density and self-similarity. Every time we choose the most suitable value for the angle so that those properties appear more clearly. We will see \( \pi/2 \) and \( 2\pi/3 \) are the most interesting angles.

Let’s take an example of such a curve, with the Thue-Morse word. The Thue-Morse morphism is defined by \( t(0) = 01 \) and \( t(1) = 10 \);

The Thue-Morse word TM is the word generated by the the infinite iteration of the Thue-Morse morphism : \( TM = 0110100110100110010110... \) This sequence has been described by Axel Thue in [17] and [17].

Applying the Odd-Even drawing rule to the Thue-Morse word TM generates a path \( \mathcal{P}(TM) \) illustrated in the figure below. The path follows the instructions RFFLFLRFRFFLFLRFFLR- FRFFL... The pattern unravels in a non-periodic series of turns. It is not self-intersecting. The general aspect of \( \mathcal{P}(TM) \) is a straight line, whatever the angle chosen for the turns : \( \pi/2 \) or \( 2\pi/3 \).

Fig.2: Pattern associated to the Thue-Morse word

1.4 Hausdorff dimension of self-similar curves

In this article, we will use a formula for the Hausdorff dimension described in [8] and [11]. If a given curve displays \( n \) distinct self-similarities of ratios \( r_k \), and providing the open condition holds,
then its Hausdorff dimension \( d \) is a solution of the equation:

\[
\sum_{k=1}^{n} r_k^d = 1(1)
\]

2 The Fibonacci word fractal curve

This curve has been extensively described in [12], the reader should refer to this paper for more details. We will just list here all the elementary morphisms that can generate this pattern and the different variants.

The Fibonacci morphism is defined by \( 0 \rightarrow 01 \) and \( 1 \rightarrow 0 \). Iterating this morphism several times generates the Fibonacci words.

The infinite Fibonacci word starts with: 01001001001010010010100100101001001010010...

It is remarkable that no less than 12 different elementary morphisms can generate the Fibonacci word fractal. This reveals the particular role this curve must have among the ones listed here.

One can observe, that all the mirror morphisms and the various compositions of those morphisms generate the Fibonacci word fractal. What is more, exchanging the role of 0 and 1 in the rule, also generates this pattern.

<table>
<thead>
<tr>
<th>morph</th>
<th>0 ( \rightarrow )</th>
<th>1 ( \rightarrow )</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1 )</td>
<td>01</td>
<td>0</td>
<td>Regular Fibonacci morphism</td>
</tr>
<tr>
<td>( F_2 )</td>
<td>10</td>
<td>0</td>
<td>mirror of ( F_1 )</td>
</tr>
<tr>
<td>( F_3 )</td>
<td>001</td>
<td>01</td>
<td>composite: ( F_3 = F_1 \circ F_2 )</td>
</tr>
<tr>
<td>( F_4 )</td>
<td>010</td>
<td>01</td>
<td>composite: ( F_4 = F_2^2 )</td>
</tr>
<tr>
<td>( F_5 )</td>
<td>010</td>
<td>10</td>
<td>composite: ( F_5 = F_2^2 ) and mirror of ( F_4 )</td>
</tr>
<tr>
<td>( F_6 )</td>
<td>100</td>
<td>10</td>
<td>composite: ( F_6 = F_2 \circ F_1 ) and mirror of ( F_3 )</td>
</tr>
</tbody>
</table>

Fig.2:2 The Fibonacci word curve, F23 segments

Other morphisms can generate a 45\(^\circ\) extended version of the fractal, where the roles of the "0" and the "1" are exchanged:

<table>
<thead>
<tr>
<th>morph</th>
<th>0 ( \rightarrow )</th>
<th>1 ( \rightarrow )</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 )</td>
<td>1</td>
<td>01</td>
<td>Exchanging the role of 0 and 1 in the Fibonacci morphism ( F_1 ).</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>1</td>
<td>10</td>
<td>mirror of ( G_1 )</td>
</tr>
<tr>
<td>( G_3 )</td>
<td>01</td>
<td>011</td>
<td>composite: ( G_3 = G_2 \circ G_1 )</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>01</td>
<td>101</td>
<td>composite: ( G_4 = G_2^2 )</td>
</tr>
<tr>
<td>( G_5 )</td>
<td>10</td>
<td>101</td>
<td>composite: ( G_5 = G_2^2 ) and mirror of ( G_4 ).</td>
</tr>
<tr>
<td>( G_6 )</td>
<td>10</td>
<td>110</td>
<td>composite: ( G_6 = G_1 \circ G_2 ) and mirror of ( G_3 ).</td>
</tr>
</tbody>
</table>

It is important to emphasize on the fact that all those curves, although they differ for a limited number of iterations, converge to the same fractal pattern at infinity.

3 The Pell word fractal curve

3.1 The Pell word

We define the Pell morphism \( P_1 \):

\[
P_1(0) = 001 \\
P_1(1) = 0
\]
Starting from $w = 0$, we get the sequence of what we will call Pell words:
- 001
- 0010010
- 0010010001001001

Iterating to infinity creates the infinite Pell word. We can show that, for any iteration $k$, $P_k(0) = 2P_{k-1}(0)P_{k-2}(0)$. Every term is the concatenation of twice the previous term and the term before. This recalls the Pell numbers $P_n$, defined by $P_k = 2P_{k-1} + P_{k-2}$ with $P_0 = 0$ and $P_1 = 1$. The Pell words are then defined by analogy with the Pell numbers, as the Fibonacci words were defined from the Fibonacci numbers.

We can show that the length of a Pell word at iteration $k$ is $|s_k(0)| = P_k$. The Pell word is a sturmian word. It has $n+1$ distinct factors of length $n$.

The mirror morphism $P_2 : 0 \rightarrow 100$ and $1 \rightarrow 0$, generates a similar pattern.

### 3.2 The Pell fractal curve

By the odd-even drawing rule, and choosing an angle of $2\pi/3$, the curve $P_k$ shows rather dense pattern inscribed in an isosceles trapezoid. The self-similarities appear clearly at any scales. We notice it can be constructed by juxtaposing two copies of $P_{k-1}$ and one $P_{k-2}$ in different ways: $P_{k-1} + P_{k-1} + P_{k-2}$ or $P_{k-1} + P_{k-2} + P_{k-1}$, this will be useful to calculate its Hausdorff dimension (see below).

![Fig.2:2 iterations of the Pell word fractal curve](image)

We notice also clearly that the path is non intersecting.

### 3.3 The Hausdorff dimension of the Pell fractal

We notice that the curve can be built using two self-similarities of ratio $a$ and one similarity of ratio $a^2$ see figure. The figure suggests immediately that $a + a^2 = 1$, so $a = \phi = (1 + \sqrt{5})/2$, the golden ratio.

![Fig.2:2 Self-similarities](image)
Since the curves can be built with similarities, we can use the following formula for the Hausdorff dimension $d$:  
\[ 2\phi^d + \phi^{-d} = 1, \]

so $d$ equals:  
\[ d = \log \left( \frac{1 + \sqrt{2}}{\log \phi} \right) = 1.8315. \] (2)

We notice here the presence of both the golden and the silver ratios [19]. We can then write the remarkable expression:

\[ d = \log \left( \frac{2 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ldots}}}}}{} \right) \] (3)

### 3.4 Variants

The variants shown here, present a different orientation and can be extended. An extended version is a curve that is larger but has an overall same aspect. We list here, also, a particular case we call "horned version". Although it shares some characteristics with the original curve, it is clearly different.

<table>
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<th>$0 \rightarrow$</th>
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<th>Comment</th>
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</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>1</td>
<td>011</td>
<td>an extended version tilted 45°</td>
</tr>
<tr>
<td>$P_2$</td>
<td>1</td>
<td>110</td>
<td>mirror of $P_1$</td>
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<tr>
<td>$P_3$</td>
<td>010</td>
<td>0</td>
<td>a &quot;horned&quot; version. Quite different, see figure</td>
</tr>
<tr>
<td>$P_4$</td>
<td>1</td>
<td>101</td>
<td>mirror of $P_3$, a &quot;horned&quot; version, extended</td>
</tr>
</tbody>
</table>

It is straightforward that, apart from the horned version, all those curves converge to the same fractal pattern at infinity.

### 4 The Sierpinski Triangle, Crown and Wave fractals

Those three patterns are regrouped here, in the same section, because they share notable common properties:

- they all derive from uniform binary morphisms
- each one of them can be created from two different morphisms or their mirrors.
- those morphisms are morphisms of length 3 (for any letter $a$, $|\sigma(a)| = 3$.)
- none of these morphisms is composite
- they all share the same Hausdorff dimension.

#### 4.1 The fractal curves

Applying the odd-even drawing rule, with an angle of $2\pi/3$, the curve $P_k$ shows 3 kinds of patterns:
Fig. 2: 2 iterations of the Sierpinski fractal curve

Fig. 2: 2 iterations of the Crown fractal

Fig. 2: 2 iterations of the Wave fractal

Again, the self-similarities appear at any scales. We notice also clearly that the paths are non self-intersecting.

4.2 Their Hausdorff dimension

Each pattern, the Sierpinski triangle, the Crown Fractal or the Wave fractal, can be built by combining 3 copies of itself, twice smaller.
Each pattern is made of three copies of itself, twice smaller.

Since the curves can be built with similarities, we can use the following formula for the Hausdorff dimension $d$:

$$d = \frac{\log 3}{\log 2} = 1.5849.$$  \hspace{1cm} (4)

## 5 Conclusion

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## 6 Acknowledgements

References


A. Monnerot-Dumaine, The Fibonacci word fractal, published electronically at http://hal.archives-ouvertes.fr/hal-00367972/fr/ (2009)


AMS classification:
- 05B30 Combinatorics - Other designs, configurations
- 11B39 Fibonacci and Lucas numbers and polynomials and generalizations
- 28A80 Fractals
- 68R15 Combinatorics on words