



HAL
open science

On the model-completion of Heyting algebras

Luck Darnière

► **To cite this version:**

| Luck Darnière. On the model-completion of Heyting algebras. 2018. hal-01885531

HAL Id: hal-01885531

<https://hal.science/hal-01885531>

Preprint submitted on 2 Oct 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the model-completion of Heyting algebras

Luck Darnière*

October 2, 2018

Abstract

We axiomatize the model-completion of the theory of Heyting algebras by means of the “Density” and “Splitting” properties in [DJ18], and of a certain “QE Property” that we introduce here. In addition: we prove that this model-completion has a prime model, which is locally finite and which we explicitly construct; we show how the Open Mapping Theorem of [vGR18] can be derived from the QE Property of existentially closed Heyting algebras; and we construct a certain “discriminant” for equations in Heyting algebras, similar to its ring theoretic counterpart.

1 Introduction

It is known since [Pit92] that the second-order intuitionist propositional calculus (IPC²) is interpretable in the first-order one (IPC). This proof-theoretic result, after translation in model-theoretic terms, ensures that the theory of Heyting algebras has a model-completion (see [GZ97]). It has been followed by numerous investigations on uniform interpolation, intermediate logics and model-theory ([Kre97], [Pol98], [Ghi99], [Bil07], [DJ18], [vGR18], among others). In particular, it is proved in [GZ97] that each of the height varieties of Heyting algebras which has the amalgamation property also has a model-completion. In spite of all this work, no intuitively meaningful axiomatization was known by now for the model-completion of the theory of Heyting algebras. So the algebraic nature of its models, the existentially closed Heyting algebras, remains quite mysterious. Our goal in this paper is to fill this lacuna.

Recently, van Gool and Reggio have given in [vGR18] a different proof of Pitts’ result, by showing that it can easily be derived from the following “Open Mapping Theorem”.

Theorem 1.1 (Open Mapping) *Every continuous p -morphism between finitely co-presented Esakia spaces is an open map.*

That both results are tightly connected is further illustrated by the approach in this paper: we prove that, reciprocally, the Open Mapping Theorem can easily be derived from Pitts’ result.

This, and our axiomatization of the model-completion of the theory of Heyting algebras, is based on a new property of existentially closed Heyting algebras

*Département de mathématiques, Faculté des sciences, 2 bd Lavoisier, 49045 Angers, France.

that we introduce here, the “QE property”. Its name comes from the following characterisation: a Heyting algebra has the QE property if and only if its complete theory eliminates the quantifiers (Proposition 4.4). The smart reader may observe that, knowing by Pitts’ result that the theory of existentially closed Heyting algebras eliminates the quantifiers, it is not an achievement to prove that all its models have the QE property!

But things are a bit more complicated: given a model M of a theory T having a model-completion T^* , if $M \models T^*$ then obviously the theory of M eliminates the quantifiers, however the converse is not true in general. For example every dense Boolean algebra has the QE-property (because the theory of dense Boolean algebras eliminates the quantifiers), *but* none of them is existentially closed in the class of all Heyting algebras. Moreover the QE-property itself is probably not first-order in general. Our strategy in the present paper is then: firstly to characterize *among* the Heyting algebras having the QE property, those which are existentially closed; and secondly to prove that the QE-property is first-order axiomatisable in this class. This is done by showing first that a certain Heyting algebra \mathcal{H}_0 , obtained by amalgamating appropriately a set of representatives of all the finite Heyting algebras, is existentially closed. By Pitts’ result \mathcal{H}_0 is then a model of the model-completion of the theory of Heyting algebras, hence it has the QE property. Moreover it is locally finite by construction, and a prime model of this model-completion. Actually it is the only countable and existentially closed Heyting algebra which is locally finite.

This paper is organised as follows. We recall the needed prerequisites in Section 2. In Section 3 we construct the prime model \mathcal{H}_0 . In Section 4, after introducing the QE property, we use that \mathcal{H}_0 is a prime model to prove that a Heyting algebra H is existentially closed if and only if it has the QE property *and* the “Splitting” and “Density” properties that we introduced in [DJ18]. In the next Section 5 we show how the Open Mapping Theorem can be derived from this. Finally we turn to explicit elimination. Indeed the elimination of quantifiers is done by Pitts in a very effective way, but it can be somewhat puzzling from an algebraic point of view. So in Section 6 it seemed to us that it could be worth showing how, given a finite system of equations

$$t(\bar{p}, q) = \mathbf{1} \quad \text{and} \quad s_k(\bar{p}, q) \neq \mathbf{1} \quad (\text{for } 1 \leq k \leq \kappa),$$

we can use the Open Mapping Theorem to explicitly construct a term $\Delta_t(\bar{p})$ (the “discriminant” of t) and terms $\nabla_{t,s_k}(\bar{p})$ for $1 \leq k \leq \kappa$ (the “co-discriminant” of t and s_k), such that for every tuple \bar{a} in a Heyting algebra A , the above system (with parameters \bar{a} replacing \bar{p}), has a solution in an extension of A if and only if $\Delta_t(\bar{a}) = \mathbf{1}$ and $\nabla_{t,s_k}(\bar{a}) \neq \mathbf{1}$ for $1 \leq k \leq \kappa$. The appendix Section 7 is devoted to the Finite Extension Property, a strengthening of the Finite Model Property of IPC which plays a key role in proving that the Heyting algebra constructed Section 3 is indeed existentially closed.

Remark 1.2 It is claimed in [GZ97] that the methods of Pitts, which concern the variety \mathcal{V}_1 of all Heyting algebras, apply *mutatis mutandis* to the variety \mathcal{V}_2 of the logic of the weak excluded middle (axiomatized by $\neg x \vee \neg\neg x = \mathbf{1}$). Consequently all the results presented here for \mathcal{V}_1 remain true for \mathcal{V}_2 . It suffices to restrict the amalgamation in Section 3 to the finite algebras in \mathcal{V}_2 , and to

replace the “Density” and “Splitting” axioms for \mathcal{V}_1 (D1 and S1 in [DJ18]) by the corresponding axioms for \mathcal{V}_2 (D2 and S2 in [DJ18]).

2 Notation and prerequisites

Posets and lattices. For every element a in a poset (E, \leq) we let $a^\uparrow = \{b \in E : a \leq b\}$. For every subset S of E we let $S^\uparrow = \bigcup_{s \in S} s^\uparrow$ be the set of elements of E greater than or equal to some element of S . An **up-set** of E is a set $S \subseteq E$ such that $S = S^\uparrow$. The family of all up-sets of E forms a topology on E that we denote by $\mathcal{O}^\uparrow(E)$.

All our lattices will be assumed distributive and bounded. The language of lattices is then $\mathcal{L}_{\text{lat}} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge\}$, where $\mathbf{0}$ stands for the smallest element, $\mathbf{1}$ for the greatest one, \vee and \wedge for the join and meet operations respectively. Iterated joins and meets will be denoted \mathbb{W} and \mathbb{M} . Of course the order of the lattice is definable by $b \leq a$ if and only if $a \vee b = a$. The language of Heyting algebras is $\mathcal{L}_{\text{HA}} = \mathcal{L}_{\text{lat}} \cup \{\rightarrow\}$ where the new binary operation symbol is interpreted as usually by

$$b \rightarrow a = \max \{c : c \wedge b \leq a\}.$$

Stone spaces and morphisms. The prime filter spectrum, or Stone space, of a distributive bounded lattice A is denoted $\text{Spec}^\uparrow A$. The **Zariski topology** on $\text{Spec}^\uparrow A$ is generated by the sets

$$P^\uparrow(a) = \{x \in \text{Spec}^\uparrow A : a \in x\}$$

where a ranges over A . The **patch topology** is generated by the sets $P^\uparrow(a)$ and their complements in $\text{Spec}^\uparrow A$.

Stone’s duality states that $a \mapsto P^\uparrow(a)$ is an embedding of bounded lattices (an \mathcal{L}_{lat} -embedding for short) of A into $\mathcal{O}^\uparrow(\text{Spec}^\uparrow A)$. If moreover A is a Heyting algebra, it is an embedding of Heyting algebras (an \mathcal{L}_{HA} -embedding for short). In particular, for every $a, b \in A$ and every $x \in \text{Spec}^\uparrow A$,

$$a \rightarrow b \in x \iff \forall y \geq x, [a \in y \Rightarrow b \in y].$$

Facts 2.1 and 2.2 below are folklore. The reader may refer in particular to [BD74], chapters III and IV.

Fact 2.1 *Every filter is the intersection of the prime filters which contain it.*

Given an \mathcal{L}_{lat} -morphism $f : B \rightarrow A$ we let $\text{Ker}^\uparrow f = f^{-1}(\mathbf{1})$ denote its **filter kernel**, and $f^\uparrow : \text{Spec}^\uparrow A \rightarrow \text{Spec}^\uparrow B$ be the dual map defined by

$$f^\uparrow(x) := f^{-1}(x).$$

Fact 2.2 *If f is surjective then f^\uparrow induces an increasing bijection from $\text{Spec}^\uparrow A$ to $\{y \in \text{Spec}^\uparrow B : \text{Ker}^\uparrow f \subseteq y\}$.*

In Heyting algebras, every congruence comes from a filter. So the classical “Isomorphisms Theorems” of universal algebra can be expressed for morphisms of Heyting algebras (\mathcal{L}_{HA} -morphisms for short) in terms of filters, just as analogous statements on morphisms of rings are expressed in terms of ideals. In particular, we will make use of the next property.

Fact 2.3 (Factorisation of Morphisms) *Let $f : B \rightarrow A$ be an \mathcal{L}_{HA} -morphism. Given a filter I of B , let $\pi_I : B \rightarrow B/I$ be the canonical projection. Then f factors through π_I , that is $f = g \circ \pi_I$ for some \mathcal{L}_{HA} -morphism $g : B/I \rightarrow A$, if and only if $\text{Ker}^\uparrow f \subseteq I$.*

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ \pi_I \downarrow & \nearrow g & \\ B/I & & \end{array}$$

In that case, such an \mathcal{L}_{HA} -morphism g is unique. It is injective if and only if $\text{Ker}^\uparrow f = I$.

Esakia spaces and free Heyting algebras. An **Esakia space** (see [Esa74]) is a poset X with a compact topology such that: (i) whenever $x \not\leq y$ in (X, \leq) there is a clopen $U \subseteq X$ that is an up-set for \leq such that $x \in U$ and $y \notin U$; and (ii) $U \downarrow$ is clopen whenever U is a clopen subset of X . Given a Heyting algebra A , its prime filter spectrum ordered by inclusion and endowed with the patch topology is an Esakia space, called the **Esakia dual** of A . We denote it by X_A .

Let $\bar{p} = \{p_1, \dots, p_l\}$ be a finite set of variables, $F(\bar{p})$ be the free Heyting algebra on \bar{p} , and $X_{F(\bar{p})}$ its Esakia space. For every l -tuple \bar{a} in a Heyting algebra A we let

$$\pi_{\bar{a}} : F(\bar{p}) \rightarrow A$$

denote the canonical morphism mapping \bar{p} to \bar{a} .

Every formula $\varphi(\bar{p})$ in the Intuitionistic Propositional Calculus (IPC) can be read as an \mathcal{L}_{HA} -term in the variables \bar{p} , and conversely. The **degree** $\text{deg } \varphi$ is the maximum number of nested occurrences of \rightarrow in φ . Abusing the notation, we will also consider φ as an element of $F(\bar{p})$. We let

$$\llbracket \varphi(\bar{p}) \rrbracket = \{x \in X_{F(\bar{p})} : \varphi(\bar{p}) \in x\}.$$

As a set of prime filters, $\llbracket \varphi(\bar{p}) \rrbracket$ is nothing but $P^\uparrow(\varphi(\bar{p}))$. The different notation simply reminds us to consider it as a subset of $X_{F(\bar{p})}$ with the patch topology, instead of a subset of $\text{Spec}^\uparrow F(\bar{p})$ with the Zariski topology.

Systems of equations and the Finite Extension Property. Every finite system of equations and negated equations in the variables (\bar{p}, \bar{q}) , is equivalent to an \mathcal{L}_{HA} -formula of the form

$$\mathcal{S}_{t, \bar{s}}(\bar{p}, \bar{q}) = \left[t(\bar{p}, \bar{q}) = \mathbf{1} \bigwedge \bigwedge_{k \leq \kappa} s_k(\bar{p}, \bar{q}) \neq \mathbf{1} \right].$$

The **degree** of $\mathcal{S}_{t, \bar{s}}(\bar{p}, \bar{q})$, with $\bar{s} = (s_1, \dots, s_\kappa)$, is the maximum of the degrees of the IPC formulas t and s_k for $1 \leq k \leq \kappa$. We will identify every such formula with the corresponding system of equations and negated equations in (\bar{p}, \bar{q}) , calling $\mathcal{S}_{t, \bar{s}}(\bar{p}, \bar{q})$ itself a **system of degree d** . Given a tuple \bar{a} in a Heyting algebra H , a **solution** of $\mathcal{S}_{t, \bar{s}}(\bar{a}, \bar{q})$ is then a tuple \bar{b} such that $H \models \mathcal{S}_{t, \bar{s}}(\bar{a}, \bar{b})$.

Theorem 2.4 (Finite Extension Property) *Let \mathcal{V} be a variety of Heyting algebras having the Finite Model Property. If an existential \mathcal{L}_{HA} -formula with parameters in a finite \mathcal{V} -algebra A is satisfied in a \mathcal{V} -algebra containing A , then it is satisfied in some finite \mathcal{V} -algebra containing A .*

We will use this improvement of the classical Finite Model Property only for the variety of all Heyting algebras, and for formulas $\exists \bar{q}, \mathcal{S}_{t, \bar{s}}(\bar{a}, \bar{q})$. The proof requires some notions of co-dimension coming from [DJ11]. Since this material is not needed anywhere else in this paper, we delay the proof of the Finite Extension Property to the appendix in Section 7.

Splitting and Density axioms. In [DJ18] we introduced for co-Heyting algebras the notion of “strong order”. It corresponds in Heyting algebras to the following relation.

$$b \gg a \iff b \rightarrow a = a \text{ and } b \geq a$$

This is a strict order on $A \setminus \{\mathbf{1}\}$ (not on A because $\mathbf{1} \gg a$ for every $a \in A$, including $a = \mathbf{1}$). Above all, we introduced the next two axioms (denoted D1 and S1 in [DJ18]), which translate to the following ones in Heyting algebras.

[Density] For every a, c such that $c \gg a \neq \mathbf{1}$ there exists an element $b \neq \mathbf{1}$ such that:

$$c \gg b \gg a$$

[Splitting] For every a, b_1, b_2 such that $b_1 \wedge b_2 \gg a \neq \mathbf{1}$ there exists $a_1 \neq \mathbf{1}$ and $a_2 \neq \mathbf{1}$ such that:

$$\begin{aligned} a_2 \rightarrow a &= a_1 \leq b_1 \\ a_1 \rightarrow a &= a_2 \leq b_2 \\ a_1 \vee a_2 &= b_1 \vee b_2 \end{aligned}$$

The Density axiom clearly says that \gg is a dense order. The Splitting axiom implies the lack of atoms and co-atoms, but it is actually much stronger (and more subtle) than this. For a geometric interpretation of the Splitting Property, we refer the reader to [DJ18] and [Dar]. The two next properties are theorems 1.1 and 1.2 in [DJ18].

Theorem 2.5 *Every existentially closed Heyting algebra has the Density and Splitting Properties.*

Theorem 2.6 *Let H be a Heyting algebra having the Density and Splitting properties. Let A be a finite Heyting subalgebra of H , and B an extension of A . If B is finite then B embeds into H over¹ A .*

In particular, if H is a non-trivial Heyting algebra (that is $\mathbf{0} \neq \mathbf{1}$ in H) having the Density and Splitting properties, then every finite non-trivial Heyting algebra embeds into H : it suffices to apply Theorem 2.6 to H, B and $A = \{\mathbf{0}, \mathbf{1}\}$. Using the model-theoretic Compactness Theorem, the next property follows immediately.

Corollary 2.7 *Let H be a non-trivial Heyting algebra which has the Density and Splitting properties. If H is ω -saturated then every every finitely generated non-trivial Heyting algebra embeds into H .*

¹With other words, there is an \mathcal{L}_{HA} -embedding of B into H which fixes A pointwise.

The Open Mapping Theorem. Following [vGR18], for every integer $d \geq 0$ and every $x, y \in X_{F(\bar{p})}$ we write $x \sim_d y$ if: for every IPC formula $t(\bar{p})$ of degree $\leq d$, $t(\bar{p}) \in x \iff t(\bar{p}) \in y$. The relations \sim_d define on $X_{F(\bar{p})}$ an ultrametric distance

$$d_{\bar{p}}(x, y) = 2^{-\min\{d : x \not\sim_d y\}},$$

with the conventions that $\min \emptyset = +\infty$ and $2^{-\infty} = 0$. The topology induced on $X_{F(\bar{p})}$ by this distance is exactly the patch topology (Lemma 8 in [vGR18]). We let

$$B_{\bar{p}}(x, 2^{-d}) = \{y \in X_{F(\bar{p})} : d_{\bar{p}}(x, y) < 2^{-d}\}$$

denote the **ball of center x and radius 2^{-d}** . Note that it is clopen and that

$$y \in B_{\bar{p}}(x, 2^{-d}) \iff y \sim_d x.$$

So the balls of radius 2^{-d} in $X_{F(\bar{p})}$ are the equivalence classes of \sim_d . Recall that, up to intuitionist equivalence, there are only finitely many IPC formulas in \bar{p} of degree $< d$. In particular, for every ball B of radius 2^{-d} there is a pair (φ_B, ψ_B) of IPC formulas of degree $< d$ such that

$$B = \llbracket \varphi_B \rrbracket \setminus \llbracket \psi_B \rrbracket.$$

We let $\mathcal{B}_d(\bar{p})$ be the set of these balls, and fix for each of them a pair (φ_B, ψ_B) as above.

Claim 2.8 *Let \bar{a} be an l -tuple in some Heyting algebra A . For every ball B in $X_{F(\bar{p})}$, there is $y \in B$ such that $\text{Ker}^\uparrow \pi_{\bar{a}} \subseteq y$ if and only if $\varphi_B(\bar{a}) \rightarrow \psi_B(\bar{a}) \neq \mathbf{1}$.*

Proof: Replacing A if necessary by the Heyting algebra generated in A by \bar{a} , we can assume that $\pi_{\bar{a}}$ is surjective. Note first that for every IPC formula $\theta(\bar{b})$ and every $y \in X_{F(\bar{p})}$ containing $\text{Ker}^\uparrow \pi_{\bar{a}}$,

$$\theta(\bar{p}) \in y \iff \theta(\bar{a}) \in \pi_{\bar{a}}(y).$$

In particular, for every ball $B = \llbracket \varphi_B \rrbracket \setminus \llbracket \psi_B \rrbracket$ in $X_{F(\bar{p})}$, there is $y \in B$ which contains $\text{Ker}^\uparrow \pi_{\bar{a}}$ if and only if there is $y_A \in X_A$ such that $\varphi_B(\bar{a}) \in y_A$ and $\psi_B(\bar{a}) \notin y_A$ (here we use Fact 2.2 and the assumption that $\pi_{\bar{a}}$ is surjective). By Stone's duality, the latter is equivalent to $\varphi_B(\bar{a}) \not\leq \psi_B(\bar{a})$, that is $\varphi_B(\bar{a}) \rightarrow \psi_B(\bar{a}) \neq \mathbf{1}$.

With other words, $\varphi_B(\bar{a}) \rightarrow \psi_B(\bar{a}) = \mathbf{1}$ if and only if there is no $y \in B$ containing $\text{Ker}^\uparrow \pi_{\bar{a}}$, which proves our claim.

■

The main result of [vGR18] is that, for every pair of finitely presented Heyting algebras A, B and every \mathcal{L}_{HA} -morphism $f : A \rightarrow B$, the dual map $f^\uparrow : X_B \rightarrow X_A$ is an open map. We will focus on the special case where f is the natural inclusion from $A = F(\bar{p})$, to $B = F(\bar{p}, q)$, with \bar{p} an l -tuple of variables and q a new variable. Then $f^\uparrow(x) = x \cap F(\bar{p})$, for every $x \in X_{F(\bar{p}, q)}$. In that case, by compactness of $X_{F(\bar{p})}$ the Open Mapping Theorem rephrases as the following property.

Theorem 2.9 (Open Mapping for $X_{F(\bar{p})}$) *For every integer $d \geq 0$, there is an integer $R_{l,d} \geq 0$ such that*

$$\begin{aligned} \forall y \in X_{F(\bar{p})}, \forall x_0 \in X_{F(\bar{p}, q)} \left[d_{\bar{p}}(y, x_0 \cap F(\bar{p})) < 2^{-R_{l,d}} \right. \\ \left. \implies \exists x \in X_{F(\bar{p}, q)} \left(x \cap F(\bar{p}) = y \text{ and } d_{\bar{p}, q}(x, x_0) < 2^{-d} \right) \right] \end{aligned}$$

It is actually from this more precise statement that the Open Mapping Theorem for general finitely presented Heyting algebras is derived in [vGR18]. And it is this one that we will prove in Section 5.

3 The prime existentially closed model

We are going to amalgamate a set of representatives of all the finite Heyting algebras in a countable, locally finite Heyting algebra \mathcal{H}_0 which will prove to be the smallest non-trivial existentially closed Heyting algebra (up to isomorphism), and the only one which is both countable and locally finite. The construction goes by induction.

Construction of \mathcal{H}_0 . Let \tilde{H} be a non-trivial existentially closed Heyting algebra. Let H_0 be the two-element Heyting algebra contained in \tilde{H} . Assume that a finite Heyting algebra $H_n \subseteq \tilde{H}$ has been constructed for some $n \geq 0$. Let A_1, \dots, A_α be the list of all the subalgebras of H_n with $\leq n$ join-irreducible elements. For each $i \leq \alpha$ let $B_{i,1}, \dots, B_{i,\beta_i}$ be a set of representatives of the minimal proper finite extensions of A_i (up to \mathcal{L}_{HA} -isomorphism over A_i). Let $H_{n,1,1} = H_n$ and assume that for some $i_0 \leq \alpha$ and $j_0 \leq \beta_{i_0}$, we have constructed a finite Heyting algebra $H_{n,i_0,j_0} \subseteq \tilde{H}$ containing: firstly H_n ; secondly, for every $i < i_0$ and $j \leq j_i$ a copy of $B_{i,j}$ containing A_i ; and thirdly, for every $j < j_0$ a copy of $B_{i_0,j}$ containing A_{i_0} . This assumption is obviously true if $i_0 = j_0 = 1$. In the induction step we distinguish three cases.

Case 1: $j_0 < \beta_{i_0}$. It is well known that the variety of Heyting algebras has the Finite Amalgamation Property, that is every pair of finite extensions of a given Heyting algebra L amalgamate over L in a finite Heyting algebra. In particular, there is a finite Heyting algebra C containing both H_{n,i_0,j_0} and a copy B of B_{i_0,j_0+1} . By Theorem 2.5, H satisfies the Spitting and Density axioms. So by Theorem 2.6, there is an \mathcal{L}_{HA} -embedding $\varphi : C \rightarrow H$ which fixes H_{n,i_0,j_0} pointwise. We let H_{n,i_0,j_0+1} be the Heyting algebra generated in H by the union of H_{n,i_0,j_0} and $\varphi(B)$. This is a finite Heyting algebra containing H_n and, for every $i < i_0$ and $j \leq j_i$ a copy of $B_{i,j}$ containing A_i , and for every $j < j_0 + 1$ a copy of $B_{i_0,j}$ containing A_{i_0} (namely $\varphi(B_{i_0,j_0+1})$ for $j = j_0 + 1$, the other cases being covered by the induction hypothesis because H_{n,i_0,j_0+1} contains H_{n,i_0,j_0}).

Case 2: $j_0 = \beta_{i_0}$ and $i_0 < \alpha$. By the Finite Amalgamation Property again, there is a finite Heyting algebra C containing both H_{n,i_0,j_0} and a copy B of $B_{i_0+1,1}$. Theorems 2.5 and 2.6 then give an \mathcal{L}_{HA} -embedding $\varphi : C \rightarrow \tilde{H}$ which fixes H_{n,i_0,j_0} pointwise. We let $H_{n,i_0+1,1}$ be the Heyting algebra generated in \tilde{H} by the union of H_{n,i_0,j_0} and $\varphi(B)$. This is a finite Heyting algebra containing H_n and, for every $i < i_0 + 1$ and $j \leq j_i$ a copy of $B_{i,j}$ containing A_i , and a copy of $B_{i_0,1}$ containing A_{i_0} .

Case 3: $j_0 = \beta_{i_0}$ and $i_0 = \alpha$. We let $H_{n+1} = H_{n,\alpha,\beta_\alpha}$.

Repeating this construction, we get an increasing chain of finite Heyting algebras H_n contained in \tilde{H} . We let $\mathcal{H}_0 = \bigcup_{n \in \mathbb{N}} H_n$. Since every H_n is finite, \mathcal{H}_0 is a countable and locally finite Heyting algebra.

Proposition 3.1 *Given a Heyting algebra H , the following properties are equivalent.*

1. H is countable, locally finite, and every finite extension B of a subalgebra A of H embeds into H over A .
2. H is countable, locally finite, and has the Density and Splitting properties.
3. H is countable, locally finite and existentially closed.
4. H is \mathcal{L}_{HA} -isomorphic to \mathcal{H}_0 .

Proof: (4) \Rightarrow (1) We have to prove that \mathcal{H}_0 itself has the properties of item 1. We already know that \mathcal{H}_0 is countable and locally finite. Let B be a finite extension of a subalgebra A of H . We want to embed B in H over A . By an immediate induction we can assume that B is a minimal proper finite extension of A . Now A is contained in H_n for some integer $n \geq 0$, which we can assume to be greater than the number of join-irreducible elements of A . So A is one of the algebras A_1, \dots, A_α of the above construction of \mathcal{H}_0 , and B is isomorphic over A to some $B_{i,j}$ as above. By construction H_{n+1} contains a copy of $B_{i,j}$, hence of B , which contains A . So B embeds over A into H_{n+1} , hence *a fortiori* into \mathcal{H}_0 .

(1) \Rightarrow (4) Both H and \mathcal{H}_0 have the properties of item 1. We construct an \mathcal{L}_{HA} -isomorphism by induction. Let $(a_n)_{n \in \mathbb{N}}$ be an enumeration of H , and $(b_n)_{n \in \mathbb{N}}$ an enumeration of \mathcal{H}_0 . Assume that for some integer $n \geq 0$ we have constructed an \mathcal{L}_{HA} -isomorphism $\varphi_n : A_n \rightarrow B_n$ where A_n (resp. B_n) is a finite subalgebra of H (resp. \mathcal{H}_0) containing a_k (resp. b_k) for every $k < n$. This assumption is obviously true for $n = 0$, with φ_0 the empty map. Let k_0 be the first integer $k \geq n$ such that $a_k \notin A_n$. Let $A_{n+1/2}$ be the Heyting algebra generated in H by $A_n \cup \{a_{k_0}\}$. Since H is locally finite, $A_{n+1/2}$ is a finite extension of A_n . Composing φ_n^{-1} by the inclusion map, we get an embedding of B_k into A_{n+1} . The assumption (1) gives an embedding of A_{n+1} into \mathcal{H}_0 over B_n , or equivalently a subalgebra $B_{n+1/2}$ of \mathcal{H}_0 containing B_n and an \mathcal{L}_{HA} -isomorphism $\varphi_{n+1/2} : A_{n+1/2} \rightarrow B_{n+1/2}$ whose restriction to A_n is φ_n . Now let l_0 be the smallest integer $l \geq n$ such that $b_l \notin B_{l+1/2}$, and let B_{l+1} be the Heyting algebra generated in \mathcal{H}_0 by $B_{n+1/2} \cup \{b_{l_0}\}$. A symmetric argument gives a subalgebra A_n of H containing $A_{n+1/2}$ and an \mathcal{L}_{HA} -isomorphism $\psi_{n+1} : B_{n+1} \rightarrow A_{n+1}$ whose restriction to $A_{n+1/2}$ is $\varphi_{n+1/2}^{-1}$. We then let $\varphi_{n+1} = \psi_{n+1}^{-1}$ and let the induction go on. Finally we get an isomorphism $\varphi : H \rightarrow \mathcal{H}_0$ whose restriction to each A_n is φ_n , and we are done.

(1) \Rightarrow (3) Let $\varphi(\bar{a}, \bar{q})$ be a quantifier-free \mathcal{L}_{HA} -formula with parameters $\bar{a} \in \mathcal{H}_0^l$, such that $\exists \bar{q} \varphi(\bar{a}, \bar{q})$ is satisfied in some extension of \mathcal{H}_0 . Let A be the Heyting algebra generated by \bar{a} . Since \mathcal{H}_0 is locally finite, A is finite. By the Finite Extension Property, it follows that $\exists \bar{q} \varphi(\bar{a}, \bar{q})$ is satisfied in some finite extension B of A . By assumption (1), B embeds into \mathcal{H}_0 over A . So $\exists \bar{q} \varphi(\bar{a}, \bar{q})$ is satisfied in \mathcal{H}_0 .

(3) \Rightarrow (2) \Rightarrow (1) is just Theorem 2.5 and Theorem 2.6, which finishes the proof. \blacksquare

Remark 3.2 We have constructed \mathcal{H}_0 inside a given non-trivial existentially closed Heyting algebra \tilde{H} . The same construction in another non-trivial existentially closed Heyting algebra \tilde{H}' would give a Heyting algebra \mathcal{H}'_0 contained

in \tilde{H}' to which Proposition 3.1 would apply as well as to \mathcal{H}_0 . The characterisation given there proves that \mathcal{H}_0 and \mathcal{H}'_0 are isomorphic. As a consequence, every non-trivial existentially closed Heyting algebra contains a copy of \mathcal{H}_0 .

4 The QE property

Given an integer $n \geq 0$ and l -tuples \bar{a}, \bar{a}' in a Heyting algebra A let

$$\text{Th}_n(\bar{a}) = \{\varphi(\bar{p}) : \deg \varphi \leq n \text{ and } \varphi(\bar{a}) = \mathbf{1}\}$$

and

$$Y_n(\bar{a}) = \{y \in X_{F(\bar{p})} : \exists y' \in X_{F(\bar{p})}, y \sim_n y' \text{ and } \text{Ker}^\uparrow \pi_{\bar{a}} \subseteq y'\}.$$

We say that \bar{a} and \bar{a}' are n -**similar** if $\text{Th}_n(\bar{a}) = \text{Th}_n(\bar{a}')$, and write it $\bar{a} \approx_n \bar{a}'$. They are ω -**similar** if they are n -similar for every n , or equivalently if $\text{Ker}^\uparrow \pi_{\bar{a}} = \text{Ker}^\uparrow \pi_{\bar{a}'}$. Note that \bar{a} and \bar{a}' are ω -similar if and only if the function mapping a_i to a'_i for $1 \leq i \leq l$, extends to an \mathcal{L}_{HA} -isomorphism between the Heyting algebras generated by \bar{a} and \bar{a}' in A .

Proposition 4.1 *Let \bar{a}, \bar{a}' be two l -tuples in a Heyting algebra A . For every integer $n \geq 0$ the following assertions are equivalent.*

1. $\text{Th}_n(\bar{a}) = \text{Th}_n(\bar{a}')$.
2. $Y_n(\bar{a}) = Y_n(\bar{a}')$.
3. For every ball B in $X_{F(\bar{p})}$ with radius 2^{-n} : $\varphi_B(\bar{a}) \leq \psi_B(\bar{a})$ if and only if $\varphi_B(\bar{a}') \leq \psi_B(\bar{a}')$.

Proof: (2) \Leftrightarrow (3) Given $y \in X_{F(\bar{p})}$, let B the ball in $X_{F(\bar{p})}$ with center y and radius 2^{-n} . By Claim 2.8, $y \in Y_n(\bar{a})$ if and only if $\varphi_B(\bar{a}) \rightarrow \psi_B(\bar{a}) \neq \mathbf{1}$, that is if and only if $\varphi_B(\bar{a}) \not\leq \psi_B(\bar{a})$. The equivalence follows.

(1) \Rightarrow (3) Given a ball B in $X_{F(\bar{p})}$ with radius 2^{-n} , φ_B and ψ_B have degree $< n$ hence $\varphi_B \rightarrow \psi_B$ have degree $\leq n$. The assumption (1) then implies that $\varphi_B(\bar{a}) \rightarrow \psi_B(\bar{a}) = \mathbf{1}$ if and only if $\varphi_B(\bar{a}') \rightarrow \psi_B(\bar{a}') = \mathbf{1}$, and (3) follows.

(3) \Rightarrow (2) Assume that $\text{Th}_n(\bar{a}) \neq \text{Th}_n(\bar{a}')$, for example $\text{Th}(\bar{a}) \not\subseteq \text{Th}_n(\bar{a}')$, and pick any $\theta(\bar{p})$ in $\text{Th}_n(\bar{a}') \setminus \text{Th}_n(\bar{a})$. Then θ belongs to $\text{Ker}^\uparrow \pi_{\bar{a}'}$ but not to $\text{Ker}^\uparrow \pi_{\bar{a}}$. Fact 2.1 then gives $y \in X_{F(\bar{p})}$ such that $\text{Ker}^\uparrow \pi_{\bar{a}} \subseteq y$ and $\theta \notin y$. In particular $y \in Y_n(\bar{a})$, but for every $y' \in X_{F(\bar{p})}$ such that $y \sim_n y'$, since $\theta(\bar{p})$ has degree $\leq n$ we have

$$\theta \notin y \implies \theta \notin y' \implies \text{Ker}^\uparrow \pi_{\bar{a}'} \not\subseteq y',$$

so $y \notin Y_n(\bar{a}')$.

■

Remark 4.2 Item 3 of Proposition 4.1 gives that the relation \approx_n for l -tuples in A is definable by the \mathcal{L}_{HA} -formula with $2l$ free variables

$$\text{Equiv}_{l,n}(\bar{p}, \bar{p}') = \bigwedge_{B \in \mathcal{B}_n(\bar{p})} \left[(\varphi_B(\bar{p}) \leq \psi_B(\bar{p})) \Leftrightarrow (\varphi_B(\bar{p}') \leq \psi_B(\bar{p}')) \right].$$

Now we can introduce the corner stone of our axiomatization of existentially closed Heyting algebras. Given a Heyting algebra A , we let $h_{l,d}(A)$ be the smallest integer n such that for every $\bar{a}, \bar{a}' \in A^l$ such that $\bar{a} \approx_n \bar{a}'$: for every system $\mathcal{S}_{t,\bar{s}}(\bar{p}, q)$ of degree $\leq d$, $\mathcal{S}_{t,\bar{s}}(\bar{a}, q)$ has a solution in A if and only if $\mathcal{S}_{t,\bar{s}}(\bar{a}', q)$ has a solution in A . If no such integer exists we let $h_{l,d}(A) = +\infty$. We call $h_{l,d}(A)$ the (l, d) -**index** of A . We say that A has the **QE property** if $h_{l,d}(A)$ is finite for every (l, d) . The following characterisation motivates this terminology.

Example 4.3 If T is any theory of Heyting algebras which eliminates the quantifier in \mathcal{L}_{HA} , then every model of T has the QE property by Proposition 4.4 below. In particular:

- Every dense Boolean algebra has the QE property.
- By Pitt's result every existentially closed Heyting algebra has the QE property.

Proposition 4.4 *A Heyting algebra A has the QE property if and only if the complete theory of A in \mathcal{L}_{HA} eliminates the quantifiers.*

Proof: Assume that the theory of A in \mathcal{L}_{HA} eliminates the quantifiers. There are up to intuitionist equivalence finitely many IPC formulas in l variables with degree $\leq d$, hence finitely many systems in $l + 1$ variable of degree $\leq d$. Let $\mathcal{S}_i(\bar{p}, q)$ for $1 \leq i \leq N$ an enumeration of them. For each of them, by assumption there is a quantifier-free formula $\chi_{\mathcal{S}}(\bar{p})$ such that the theory of A proves that $\exists q, \mathcal{S}_i(\bar{p}, q)$ is equivalent to $\chi_i(\bar{p})$. Let n be the maximal degree of all the atomic formulas composing these formulas χ_i . For every $\bar{a}, \bar{a}' \in A^n$ such that $\bar{a} \approx_n \bar{a}'$, $A \models \chi_i(\bar{a})$ if and only if $A \models \chi_i(\bar{a}')$. So the system $\mathcal{S}_i(\bar{a}, q)$ has a solution in A if and only if the system $\mathcal{S}_i(\bar{a}', q)$ has a solution in A , that is $h_{l,d}(A) \leq n$.

Reciprocally, assume that $h_{l,d}(A)$ is finite for every integers l, d . Let $\varphi(\bar{p}, a)$ be a quantifier-free formula in $l + 1$ variables. It is logically equivalent to a disjunction of finitely many systems $\mathcal{S}_i(\bar{p}, q)$, for $1 \leq i \leq N$. Let d be the maximal degree of all these systems, and $n = h_{l,d}(A)$. Using again that there are, up to intuitionist equivalence, finitely many IPC formulas in l variables with degree $\leq n$, there are also finitely many equivalence classes for \approx_n in A^l . For every such equivalence class C , let $\bar{a} \in C$ and let φ_C (resp. ψ_C) be the conjunction (resp. disjunction) of the formulas $t(\bar{p}) = \mathbf{1}$ (resp $t(\bar{p}) \neq \mathbf{1}$) for $t(\bar{p})$ ranging over a finite set of representatives of the IPC formulas which belong to $\text{Th}_n(\bar{a})$ (resp. which don't belong to $\text{Th}_n(\bar{a})$). Clearly a tuple $\bar{a}' \in A^l$ belongs to C if and only if $A \models \varphi_C(\bar{a}) \wedge \neg\psi_C(\bar{a}')$. Now, for each system $\mathcal{S}_i(\bar{p}, q)$ let \mathcal{C}_i be the list of equivalence classes C for \approx_n in A^l such that $\mathcal{S}_i(\bar{a}, q)$ has a solution in A for some $\bar{a} \in C$ (hence also for every $\bar{a} \in C$ by definition of $n = h_{l,d}(A)$). By construction, for every $\bar{a} \in A^l$

$$A \models \exists q, \mathcal{S}_i(\bar{a}, q) \iff \bar{a} \in \bigcup \mathcal{C}_i \iff A \models \bigvee_{C \in \mathcal{C}_i} (\varphi_C(\bar{a}) \wedge \neg\psi_C(\bar{a})).$$

Let $\chi_i(\bar{p})$ be the IPC formula on the right. By construction the theory of A proves that $\exists q, \varphi(\bar{p}, q)$ is equivalent to $\bigvee_{i \leq N} \chi_i(\bar{p})$. The latter is quantifier-free, hence the theory of A eliminates the quantifiers by an immediate induction.

■

Proposition 4.5 *For any given $l, d, n \in \mathbf{N}$, there is a closed $\forall\exists$ -formula $\text{FC}_{l,d}^n$ in \mathcal{L}_{HA} for every Heyting algebras A*

$$h_{l,d}(A) \leq n \iff A \models \text{FC}_{l,d}^n.$$

In particular, every Heyting algebra elementarily equivalent to A has the same (l, d) -index.

Proof: For every system $\mathcal{S}_{t,\bar{s}}(\bar{p}, q)$ of degree $\leq d$ let

$$\text{FC}_{t,\bar{s}}^n(\bar{p}, \bar{p}') = \left[\left(\text{Equiv}_{l,n}(\bar{p}, \bar{q}') \bigwedge \exists q', \mathcal{S}_{t,\bar{s}}(\bar{p}', q') \right) \Rightarrow \exists b, \mathcal{S}_{t,\bar{s}}(\bar{p}, q) \right]$$

where $\text{Equiv}_{l,n}(\bar{p}, \bar{p}')$ is the quantifier-free \mathcal{L}_{HA} -formula introduced in Remark 4.2. This is a $\forall\exists$ -formula. Clearly $h_{l,d}(A) \leq n$ if and only if $A \models \text{FC}_{t,\bar{s}}^n(\bar{a}, \bar{a}')$ for every $\bar{a}, \bar{a}' \in A^l$ and every system $\mathcal{S}_{t,\bar{s}}(\bar{p}, q)$ of degree $\leq d$. There are only finitely many non-equivalent IPC formulas of degree $\leq d$, hence finitely many non-equivalent systems of degree $\leq d$. So $h_{l,d}(A) \leq n$ if and only if A satisfies the conjunction of the finitely many formulas $\forall \bar{p}, \bar{p}' \text{FC}_{t,\bar{s}}^n(\bar{p}, \bar{p}')$.
■

Let T be the model-completion of the theory of Heyting algebras. By Pitt's result the models of T are exactly the existentially closed Heyting algebras. In particular, the Heyting algebra \mathcal{H}_0 constructed in Section 3 is a model of T . So by Proposition 4.4 \mathcal{H}_0 has the QE property. Once and for all, we let $h_{l,d} = h_{l,d}(\mathcal{H}_0) < +\infty$.

Theorem 4.6 *For every Heyting algebra H the following properties are equivalent.*

1. H is existentially closed.
2. H has the Density, the Splitting and the QE properties.
3. H has the Density and Splitting properties, and $h_{l,d}(H) = h_{l,d}$ for every integers l, d .

In particular, the model-completion of the theory of Heyting algebras is axiomatised by the Density and Splitting axioms, and the formulas $\text{FC}_{l,d}^{h_{l,d}}$ for every integers l, d .

Proof: (1) \Rightarrow (3) Since H is existentially closed, it has the Density and Splitting properties by Theorem 2.5. Moreover $\mathcal{H}_0 \subseteq H$ by Remark 3.2, and \mathcal{H}_0 is existentially closed by Proposition 3.1, so $\mathcal{H}_0 \preceq H$ by Pitts' result. In particular $\mathcal{H}_0 \equiv H$ so $h_{l,d}(H) = h_{l,d}(\mathcal{H}_0)$ for every l, d by Proposition 4.5.

(3) \Rightarrow (2) is clear.

(2) \Rightarrow (1) The case of the one-point Heyting algebra being trivial, we can assume that $\mathbf{0} \neq \mathbf{1}$ in H . By model-theoretic non-sense, we can assume that H is ω -saturated and it suffices to prove that for every $\bar{a} \in H^l$ and every finitely generated Heyting algebra B containing the algebra A generated by \bar{a} in H , there is an embedding of B into H over A . By an immediate induction we are reduced to the case where B is generated over A by a single element b . Replacing if necessary H by an elementary extension, thanks to Proposition 4.5 we can assume that H is ω -saturated.

Let Σ be the set of atomic or neg-atomic \mathcal{L}_{HA} -formulas $\varphi(\bar{p}, q)$ such that $B \models \varphi(\bar{a}, b)$. We only have to prove that for some $b' \in H$, Σ is satisfied by (\bar{a}, b') (by which we mean that $H \models \varphi(\bar{a}, b')$ for every formula $\varphi(\bar{p}, q)$ in Σ). Indeed, B will then be isomorphic over A to the Heyting algebra generated in H by (\bar{a}, b') , so we are done. Since H is ω -saturated, it suffices to prove that every finite fragment of Σ is satisfied by (\bar{a}, b') for some $b' \in H$. Such a finite fragment is equivalent to a system. So we want to prove that for every system $\mathcal{S}_{t, \bar{s}}(\bar{p}, q)$, if $\mathcal{S}_{t, \bar{s}}(\bar{a}, q)$ has a solution in B then it has a solution in H .

Now let $\mathcal{S}_{t, \bar{s}}(\bar{p}, q)$ be a system such that $\mathcal{S}_{t, \bar{s}}(\bar{a}, q)$ has a solution in B , and let d be the degree of $\mathcal{S}_{t, \bar{s}}(\bar{p}, q)$. By Corollary 2.7 B embeds into H . So let $(\bar{a}_0, b_0) \in H^{l+1}$ be the image of (\bar{a}, b) by such an embedding. Then $\text{Th}_n(\bar{a}, b) = \text{Th}_n(\bar{a}_0, b_0)$ for every integer $n \geq 0$. *A fortiori* $\text{Th}_n(\bar{a}) = \text{Th}_n(\bar{a}_0)$ for every n , and in particular for $n = h_{l,d}(H)$. Now $\mathcal{S}_{t, \bar{s}}(\bar{a}_0, q)$ has solution in H , namely b_0 , and $\bar{a} \approx_n \bar{a}_0$ for $n = h_{l,d}(H)$. So $\mathcal{S}_{t, \bar{s}}(\bar{a}, q)$ has a solution in H , by definition of $h_{l,d}(H)$.

■

5 The Open Mapping Theorem

Our aim in this section is to derive the Open Mapping Theorem of [vGR18] from Theorem 4.6.

We first need a lemma. As usually we consider IPC formulas $t(\bar{p}, q)$ and $s_k(\bar{p}, q)$ as elements of $F(\bar{p}, q)$. In particular, $t(\bar{p}, q) \uparrow$ in the next result is the principal filter generated by $t(\bar{p}, q)$ in $F(\bar{p}, q)$.

Lemma 5.1 *Let $\mathcal{S}_{t, \bar{s}}(\bar{p}, q)$ be a system, with $\bar{s} = (s_1, \dots, s_\kappa)$. For every l -tuple \bar{a} in a Heyting algebra A , the following assertions are equivalent.*

1. $\mathcal{S}_{t, \bar{s}}(\bar{a}, q)$ has a solution in some extension of A .
2. $\text{Ker}^\uparrow \pi_{\bar{a}}$ contains $t(\bar{p}, q) \uparrow \cap F(\bar{p})$, and the filter generated in $F(\bar{p}, q)$ by $\text{Ker}^\uparrow \pi_{\bar{a}} \cup \{t(\bar{p}, q)\}$ does not contain any of the $s_k(\bar{p}, q)$'s.
3. For every $y \in X_{F(\bar{p})}$ containing $\text{Ker}^\uparrow \pi_{\bar{a}}$, there is $x \in \llbracket t(\bar{p}, q) \rrbracket$ such that $x \cap F(\bar{p}) = y$. Moreover, for each $k \leq \kappa$, there is at least one of these y 's for which x can be chosen outside of $\llbracket s_k(\bar{p}, q) \rrbracket$.

Proof: Let A_0 be the Heyting algebra generated in A by \bar{a} . If $\mathcal{S}_{t, \bar{s}}(\bar{a}, q)$ has a solution in some extension of A_0 , by the amalgamation property of Heyting algebras it has a solution in some extension of A , and reciprocally. Thus, replacing A by A_0 if necessary, we can assume that A is generated by \bar{a} . We let G be the filter generated in $F(\bar{p}, q)$ by $\text{Ker}^\uparrow \pi_{\bar{a}} \cup \{t\}$, or equivalently by $\text{Ker}^\uparrow \pi_{\bar{a}} \cup t \uparrow$.

(1) \Rightarrow (2) In some extension H of A , let b be such that $t(\bar{a}, b) = \mathbf{1}$ and $s_k(\bar{a}, b) \neq \mathbf{1}$ for every $k \leq \kappa$. There is a unique \mathcal{L}_{HA} -morphism $\pi_{\bar{a}, b} : F(\bar{p}, q) \rightarrow H$ mapping (\bar{p}, q) onto (\bar{a}, b) . By construction $\pi_{\bar{a}}$ is the restriction of $\pi_{\bar{a}, b}$ to $F(\bar{p})$, hence $\text{Ker}^\uparrow \pi_{\bar{a}} = \text{Ker}^\uparrow \pi_{\bar{a}, b} \cap F(\bar{p})$. We have

$$t \in \text{Ker}^\uparrow \pi_{\bar{a}, b} \Rightarrow t \uparrow \subseteq \text{Ker}^\uparrow \pi_{\bar{a}, b} \Rightarrow t \uparrow \cap F(\bar{p}) \subseteq \text{Ker}^\uparrow \pi_{\bar{a}, b} \cap F(\bar{p}).$$

So $t \uparrow \cap F(\bar{p})$ is contained in $\text{Ker}^\uparrow \pi_{\bar{a}}$. Moreover G is contained in $\text{Ker}^\uparrow \pi_{\bar{a}, b}$ and $s_k \notin \text{Ker}^\uparrow \pi_{\bar{a}, b}$, hence $s_k \notin G$, for every $k \leq \kappa$.

(2) \Rightarrow (1) Assuming (2), we are claiming that $G \cap X_{F(\bar{p})} = \text{Ker}^\uparrow \pi_{\bar{a}}$. The right-to-left inclusion is clear. Reciprocally, if $\varphi(\bar{p}) \in G$ then $\varphi(\bar{p}) \geq \varphi_0(\bar{p}) \wedge \tau(\bar{p}, q)$ for some $\varphi_0 \in \text{Ker}^\uparrow \pi_{\bar{a}}$ and $\tau \in t^\uparrow$. So $\varphi_0 \rightarrow \varphi \geq \tau$, that is $\varphi_0 \rightarrow \varphi \in t^\uparrow \cap X_{F(\bar{p})}$, hence $\varphi_0 \rightarrow \varphi \in \text{Ker}^\uparrow \pi_{\bar{a}}$ by assumption (2). It follows that $\varphi_0 \wedge (\varphi_0 \rightarrow \varphi) \in \text{Ker}^\uparrow \pi_{\bar{a}}$, that is $\varphi_0 \wedge \varphi \in \text{Ker}^\uparrow \pi_{\bar{a}}$. *A fortiori* $\varphi \in \text{Ker}^\uparrow \pi_{\bar{a}}$ which proves our claim.

Let $B = X_{F(\bar{p}, q)}/G$ and $f : X_{F(\bar{p}, q)} \rightarrow B$ be the canonical projection. $G \cap X_{F(\bar{p})}$ is obviously the filter kernel of the restriction of f to $X_{F(\bar{p})}$. Since $G \cap X_{F(\bar{p})} = \text{Ker}^\uparrow \pi_{\bar{a}}$, by Factorisation of Morphisms we then have a commutative diagram as follows (with left arrow the natural inclusion).

$$\begin{array}{ccc} F(\bar{p}, q) & \xrightarrow{f} & B \\ \uparrow & & \uparrow \\ F(\bar{p}) & \xrightarrow{\pi_{\bar{a}}} & A \end{array}$$

We have $t \in G$ by construction, and $s_k \notin G$ for every $k \leq \kappa$ by assumption (2). So, after identification of A with its image by the dashed arrow, B is an extension of A in which $f(q)$ is a solution of $\mathcal{S}_{t, \bar{s}}(\bar{a}, q)$.

(2) \Rightarrow (3) Given $y \in X_{F(\bar{p})}$ containing $\text{Ker}^\uparrow \pi_{\bar{a}}$, assume for a contradiction that for every $x \in \llbracket t \rrbracket$, $x \cap F(\bar{p}) \neq y$. If $x \cap F(\bar{p}) \not\subseteq y$, pick any φ_x in $(x \cap F(\bar{p})) \setminus y$ and let $U_x = \llbracket \varphi_x \rrbracket$. If $x \cap F(\bar{p}) \not\supseteq y$, pick any ψ_x in $y \setminus (x \cap F(\bar{p}))$ and let $U_x = \llbracket \psi_x \rrbracket^c$. The family $(U_x)_{x \in \llbracket t \rrbracket}$ is an open cover of the compact space $\llbracket t \rrbracket$. A finite subcover gives $\varphi_1, \dots, \varphi_\alpha \in F(\bar{p}) \setminus y$ and $\psi_1, \dots, \psi_\beta \in y$ such that

$$\llbracket t \rrbracket \subseteq \bigcup_{i \leq \alpha} \llbracket \varphi_i \rrbracket \cup \bigcup_{j \leq \beta} \llbracket \psi_j \rrbracket^c.$$

That is $\llbracket t \rrbracket \subseteq \llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket^c$ with $\varphi = \mathbb{W}_{i \leq \alpha} \varphi_i$ and $\psi = \mathbb{A}_{j \leq \beta} \psi_j$. In particular, $t \leq \psi \rightarrow \varphi$ so

$$\psi \rightarrow \varphi \in t^\uparrow \cap F(\bar{p}) \subseteq \text{Ker}^\uparrow \pi_{\bar{a}} \subseteq y \implies \psi \rightarrow \varphi \in y.$$

On the other hand every $\psi_i \in y$ hence $\psi \in y$, and every $\varphi_i \notin y$ hence $\varphi \notin y$ (because y is a prime filter). In particular $\psi \rightarrow \varphi \notin y$, a contradiction.

We have proved that the first part of (2) implies the first part of (3). Since G is the intersection of the prime filters of $F(\bar{p}, q)$ which contain G , the second part of (2) implies that for each $k \leq \kappa$ there is $x \in X_{\bar{p}, q}$ containing $\text{Ker}^\uparrow \pi_{\bar{a}}$ and t but not s_k . Letting $y = x \cap F(\bar{p})$, we get the second part of (3).

(3) \Rightarrow (2) Let $\mathcal{Y} = \{y \in X_{F(\bar{p})} : \text{Ker}^\uparrow \pi_{\bar{a}} \subseteq y\}$ and $\mathcal{X} = \{x \cap F(\bar{p}) : x \in \llbracket t \rrbracket\}$. The first part of (3) says that $\mathcal{Y} \subseteq \mathcal{X}$, hence $\bigcap \mathcal{Y} \supseteq \bigcap \mathcal{X}$. Since every filter is the intersection of the prime filters which contain it, $\bigcap \mathcal{Y} = \text{Ker}^\uparrow \pi_{\bar{a}}$ and $\bigcap \llbracket t \rrbracket = t^\uparrow$ hence

$$\bigcap \mathcal{X} = \bigcap \llbracket t \rrbracket \cap F(\bar{p}) = t^\uparrow \cap F(\bar{p}).$$

So $\text{Ker}^\uparrow \pi_{\bar{a}} \supseteq t^\uparrow \cap F(\bar{p})$, which is the first part of (2). Now for each $k \leq \kappa$ the second part of (3) gives $x \in \llbracket t \rrbracket \setminus \llbracket s_k \rrbracket$ such that $x \cap F(\bar{p})$ contains $\text{Ker}^\uparrow \pi_{\bar{a}}$. So $G \subseteq x$, in particular $s_k \notin G$.

■

Theorem 5.2 (Open Mapping for $X_{F(\bar{p})}$) For every integer d , every $y \in X_{F(\bar{p})}$ and every $x_0 \in X_{F(\bar{p},q)}$, if $d_{\bar{p}}(y, x_0 \cap F(\bar{p})) < 2^{-h_{l,d}}$, there exists $x \in X_{F(\bar{p},q)}$ such that $y = x \cap F(\bar{p})$ and $d_{\bar{p},q}(x, x_0) < 2^{-d}$.

This result says that the dual of the natural embedding $f : F(\bar{p}) \rightarrow F(\bar{p}, q)$, is an open map $f^\dagger : X_{F(\bar{p})} \rightarrow X_{F(\bar{p},q)}$. As shown in [vGR18], it easily follows that for every \mathcal{L}_{HA} -morphism $f : A \rightarrow B$ between finitely presented Heyting algebras, the dual map $f^\dagger : X_B \rightarrow X_A$ is an open map.

Proof: Let $y_0 = x_0 \cap F(\bar{p})$ and $B = B_{\bar{p}}(y_0, 2^{-h_{l,d}}) = \llbracket \varphi_B \rrbracket \setminus \llbracket \psi_B \rrbracket$. Let $A_0 = F(\bar{p})/y_0$, $B_0 = F(\bar{p}, q)/x_0$ and (\bar{a}_0, b_0) be the image of (\bar{p}, q) by the canonical projection $\pi_{\bar{a}_0, b_0} : F(\bar{p}, q) \rightarrow B_0$. By construction A_0 is the Heyting algebra generated by \bar{a}_0 in B_0 , and the system $\mathcal{S}_{\varphi_B, \bar{\psi}_B}(\bar{a}_0, q)$ has a solution in B_0 , namely b_0 .

Let $A = F(\bar{p})/y$, and \bar{a} be the image of \bar{p} by the canonical projection $\pi_{\bar{a}} : F(\bar{p}) \rightarrow A$. By construction $d_{\bar{p}}(y_0, 2^{-h_{l,d}}) < 2^{-h_{l,d}}$, that is $y \sim_{h_{l,d}} y_0$. Hence for every IPC formula $\theta(\bar{p})$ of degree $\leq h_{l,d}$, $\theta(\bar{p}) \in y$ if and only if $\theta(\bar{p}) \in y_0$. But $\theta(\bar{p}) \in y_0$ if and only if $\theta(\bar{a}) = \mathbf{1}$, and similarly for \bar{a}_0 , so $\text{Th}_{h_{l,d}}(\bar{a}) = \text{Th}_{h_{l,d}}(\bar{a}_0)$.

Now A and A_0 are non-trivial by construction, so they contain the two-element Heyting algebra. By the amalgamation property, both of them embed in a common extension, which itself embeds in an existentially closed extension H . So $\mathcal{S}_{\varphi_B, \bar{\psi}_B}(\bar{a}_0, q)$ has a solution in A_0 , hence in H , and $\bar{a} \approx_{h_{l,d}} \bar{a}_0$. Now $h_{l,d}(H) = h_{l,d}$ by Theorem 4.6, so $\mathcal{S}_{\varphi_B, \bar{\psi}_B}(\bar{a}, q)$ has a solution b in H . The equivalence (1) \Leftrightarrow (3) of Lemma 5.1 applies: since $y = \text{Ker}^\dagger \pi_{\bar{a}}$, there is $x \in \llbracket \varphi_B \rrbracket \setminus \llbracket \psi_B \rrbracket$ (that is $x \in B$) such that $x \cap F(\bar{p}) = y$. \blacksquare

6 Discriminant and co-discriminant

Throughout this section we fix a system of degree $\leq d$

$$\mathcal{S}_{t, \bar{s}}(\bar{p}, q) = \left[t(\bar{s}, p) = \mathbf{1} \bigwedge \bigwedge_{k \leq \kappa} s_k(\bar{p}, q) \neq \mathbf{1} \right].$$

Like the usual discriminant does for polynomial equations, we want to find IPC formulas in the coefficients of the system whose values at any l -tuple \bar{a} in a Heyting algebra A decides if $\mathcal{S}_{t, \bar{s}}(\bar{a}, q)$ has a solution in some extension of A .

We define the **discriminant** of t as

$$\Delta_t(\bar{p}) = \bigwedge_{B \in \mathcal{D}_t} (\varphi_B(\bar{p}) \rightarrow \psi_B(\bar{p}))$$

where \mathcal{D}_t is the set of balls B in $X_{F(\bar{p})}$ of radius $2^{-R_{l,d}}$ such that

$$t(\bar{p}, q) \wedge \varphi_B(\bar{p}) \leq \psi_B(\bar{p}).$$

For each $k \leq \kappa$, the **co-discriminant** of (t, s_k) is

$$\nabla_{t, s_k}(\bar{p}) = \bigwedge_{B \in \mathcal{D}'_{t, s_k}} (\varphi_B(\bar{p}) \rightarrow \psi_B(\bar{p}))$$

where \mathcal{D}'_{t, s_k} is the set of balls B in $X_{F(\bar{p})}$ of radius $2^{-R_{l,d}}$ such that

$$t(\bar{p}, q) \wedge \varphi_B(\bar{p}) \not\leq s_k(\bar{p}, q) \vee \psi_B(\bar{p}).$$

Theorem 6.1 For every l -tuple \bar{a} in a Heyting algebra A , the system

$$\mathcal{S}_{t,\bar{s}}(\bar{p}, q) = \left[t(\bar{s}, p) = \mathbf{1} \bigwedge \bigwedge_{k \leq \kappa} s_k(\bar{p}, q) \neq \mathbf{1} \right].$$

has a solution in some extension of A if and only if $\Delta_t(\bar{a}) = \mathbf{1}$ and $\nabla_{t,s_k}(\bar{a}) \neq \mathbf{1}$ for every $k \leq \kappa$.

The proof is based on Lemma 5.1 and the Open Mapping Theorem.

Proof: (of Theorem 6.1) By Lemma 5.1 the system $\mathcal{S}_{t,\bar{s}}(\bar{a}, q)$ has a solution in some extension of A if and only if (i) for every $y \in X_{F(\bar{p})}$ containing $\text{Ker}^\uparrow \pi_{\bar{a}}$ there is $x \in X_{F(\bar{p},q)}$ such that $x \cap X_{F(\bar{p})} = y$ and $t \in x$, and for each $k \leq \kappa$; (ii)_k there are $y \in X_{F(\bar{p})}$ and $x \in X_{F(\bar{p},q)}$ such that $x \cap X_{F(\bar{p})} = y$ contains $\text{Ker}^\uparrow \pi_{\bar{a}}$, $t \in x$ and $s_k \notin x$.

Now for every ball B in $X_{F(\bar{p})}$ of radius $2^{-R_{i,d}}$ such that $\varphi_B(\bar{a}) \rightarrow \psi_B(\bar{a}) \neq \mathbf{1}$, by Claim 2.8 there is $y \in B$ containing $\text{Ker}^\uparrow \pi_{\bar{a}}$, so (i) implies that there is also $x \in X_{F(\bar{p},q)}$ such that $x \cap X_{F(\bar{p})} = y$ and $t \in x$. In particular x and y contain the same elements of $F(\bar{p})$, hence $\psi_B \in x$ and $\psi_B \notin x$ since $y \in B = \llbracket \varphi_B \rrbracket \setminus \llbracket \psi_B \rrbracket$. Altogether $t \wedge \varphi_B \in x$ and $\psi_B \notin x$, consequently $t \wedge \varphi_B \not\leq \psi_B$ in $F(\bar{p}, q)$, that is $B \notin \mathcal{D}_t$. Equivalently, (i) implies that $\varphi_B(\bar{a}) \rightarrow \psi_B(\bar{a}) = \mathbf{1}$ for every ball $B \in \mathcal{D}_t$, that is $\Delta_t(\bar{a}) = \mathbf{1}$.

Conversely, let us prove that $\Delta_t(\bar{a}) = \mathbf{1}$ implies (i). Pick any $y \in X_{F(\bar{p})}$ containing $\text{Ker}^\uparrow \pi_{\bar{a}}$. Let B be the unique ball in $X_{F(\bar{p})}$ of radius $2^{-R_{i,d}}$ containing y . Then $\varphi_B(\bar{a}) \rightarrow \psi_B(\bar{a}) \neq \mathbf{1}$ by Claim 2.8, so $B \notin \mathcal{D}_t$ because $\Delta_t(\bar{a}) = \mathbf{1}$. By definition of \mathcal{D}_t we have $t \wedge \varphi_B \not\leq \psi_B$, so there is a point $x' \in X_{F(\bar{p},q)}$ such that $t \wedge \varphi_B \in x'$ and $\psi_B \notin x'$. Let $y' = x' \cap X_{F(\bar{p})}$, since y' and x' contain the same elements in $F(\bar{p})$ we have $\varphi_B \in y'$ and $\psi_B \notin y'$, that is $y' \in \llbracket \varphi_B \rrbracket \setminus \llbracket \psi_B \rrbracket = B$. Now, since $y \in B$ and B has radius $2^{-R_{i,d}}$, the Open Mapping Theorem gives $x \in X_{F(\bar{p},q)}$ such that $x \sim_d x'$ and $x \cap X_{F(\bar{p})} = y$. In particular $t \in x'$ implies that $t \in x$, that is $x \in \llbracket t \rrbracket$, which proves (i).

Fix now some $k \leq K$ and assume that $\nabla_{t,s}(\bar{a}) \neq \mathbf{1}$, so there is a ball B in $X_{F(\bar{p})}$ of radius $2^{-R_{i,d}}$ such that: (a)_k $t \wedge \varphi_B \not\leq s_k \vee \psi_B$, and; (b) $\varphi_B(\bar{a}) \rightarrow \psi_B(\bar{a}) \neq \mathbf{1}$. Then (a)_k gives some $x' \in X_{F(\bar{p},q)}$ which contains t and φ_B but neither s_k nor ψ_B . Let $y' = x' \cap F(\bar{p})$, then $\varphi_B \in y'$ and $\psi_B \notin y'$, because y' and x' contain the same elements in $F(\bar{p})$. So y' belongs to $\llbracket \varphi_B \rrbracket \setminus \llbracket \psi_B \rrbracket = B$. By (b) and Claim 2.8 there is some $y \in B$ which contains $\text{Ker}^\uparrow \pi_{\bar{a}}$. Since $y' = x' \cap F(\bar{p})$ and $y, y' \in B$, the Open Mapping Theorem then gives $x \in X_{F(\bar{p},q)}$ such that $x \sim_n x'$ and $x \cap F(\bar{p}) = y$. In particular $t \in x$ and $s_k \notin x$ because t, s_k have degree $\leq d$ and $x \sim_d x'$, which proves (ii)_k.

It only remains to check that (ii)_k implies that $\nabla_{t,s_k}(\bar{a}) \neq \mathbf{1}$. By assumption (ii)_k there are some $y \in X_{F(\bar{p})}$ and $x \in X_{F(\bar{p},q)}$ such that $t \in x$ and $s_k \notin x$, and such that $y = x \cap F(\bar{p})$ contains $\text{Ker}^\uparrow \pi_{\bar{a}}$. Let B be the unique ball in $X_{F(\bar{p})}$ of radius $2^{-R_{i,d}}$ containing y . Then $\varphi_B \in x$ and $\psi_B \notin x$ because x and y contain the same elements of $F(\bar{p})$. Altogether $t \wedge \varphi_B \in x$ and $s_k \vee \psi_B \notin x$, so $t \wedge \varphi_B \not\leq s_k \vee \psi_B$, that is $B \in \mathcal{D}'_{t,s_k}$. Moreover y belongs to B and contains $\text{Ker}^\uparrow \pi_{\bar{a}}$ so $\varphi_B(\bar{a}) \rightarrow \psi_B(\bar{a}) \neq \mathbf{1}$, and *a fortiori* $\nabla_{t,s_k}(\bar{a}) \neq \mathbf{1}$.

■

7 Appendix: the Finite Extension Property

We collect here all the results from [DJ11] needed for the proof of the Finite Extension Property (Theorem 7.6 below). The classical Finite Model Property only concerns satisfiability of formulas $\exists \bar{p}, t(\bar{p}) \neq \mathbf{1}$ with $t(\bar{p})$ an IPC-formula. The slight improvement below is Proposition 8.1 in [DJ11].

Proposition 7.1 *Let \mathcal{V} be a variety of Heyting algebras having the Finite Model Property. If an existential \mathcal{L}_{HA} -formula is satisfied in some \mathcal{V} -algebra, then it is satisfied in a finite \mathcal{V} -algebra.*

Recall that the **height** of a poset (E, \leq) is the maximal integer n such that there exists $a_0 < a_1 < \dots < a_n$ in E . If no such maximum exists we say that E has infinite length. The **dimension** of a lattice L , denoted $\dim L$, is the height of its prime filter spectrum (or equivalently of its prime ideal spectrum). This is also the classical Krull dimension of $\text{Spec}^\uparrow L$ and of $\text{Spec}_\downarrow L$.

We introduced in [DJ11] a notion of dimension and co-dimension for the elements of a lattice, which turned out to have better properties in co-Heyting algebras. Given an element a of a Heyting algebra A , we define the **dual co-dimension** of a in A to be the co-dimension of a in A^{op} , the co-Heyting algebra obtained by reversing the order of A . Equivalently, by Theorem 3.8 in [DJ11], it is the greatest integer d such that

$$\exists x_0, \dots, x_d \in A, a \geq x_d \gg \dots \gg x_0.$$

Remark 7.2 An immediate consequence of this characterisation is that if B is a Heyting algebra such that $a \in B \subseteq A$, the dual co-dimension of a in A is greater than or equal to its dual co-dimension in B .

We let dA denote² the set of elements in A with dual co-dimension $> d$. Let us recall some of the properties of the dual co-dimension proved, after reversing the order, in [DJ11].

(CD₁) *A Heyting algebra A has dimension $\leq d$ if and only if every $a \in A \setminus \{\mathbf{1}\}$ has dual co-dimension $\leq d$ in A , that is $(d+1)A = \{\mathbf{1}\}$ (Remark 2.1).*

(CD₂) *There is an IPC formula $\varepsilon_{l,d}(\bar{p})$ such that for every Heyting algebra A generated by an l -tuple \bar{a} , dA is the filter generated in A by $\varepsilon_{l,d}(\bar{a})$ (Proposition 8.2).*

(CD₃) *For every finitely generated Heyting algebra A , A/dA is finite (Corollary 5.5 and Corollary 7.5).*

Remark 7.3 From (CD₁) and Remark 7.2 it follows that if a Heyting algebra A has finite dimension d , then every Heyting algebra contained in A has dimension $\leq d$. This also follows from [Hos67] and [Ono71], who proved *mutatis mutandis*, that the class of Heyting algebras of dimension $\leq d$ is a variety (a direct proof of this is given in [DJ11], Proposition 3.12).

²This is slightly different from our notation in [DJ11], where we let dA be the set of elements of co-dimension $\geq d$.

Lemma 7.4 *Given an integer $d \geq 0$ there is an integer $n_d \geq 0$ such that for every Heyting algebra H of dimension $\leq d$, every $\bar{a} \in H^l$ and every l -tuple \bar{a}' in a Heyting algebra H' , if $\text{Th}_n(\bar{a}) = \text{Th}_n(\bar{a}')$ then $\text{Ker}^\uparrow \pi_{\bar{a}} = \text{Ker}^\uparrow \pi_{\bar{a}'}$.*

Proof: By Remark 7.3, for every Heyting algebra H of dimension $\leq d$, and every $\bar{a} \in H^l$, the Heyting algebra A generated by \bar{a} in H has dimension $\leq d$. Then $(d+1)A = \mathbf{1}$ by (CD₁), that is $\varepsilon_{l,d}(\bar{a}) = \mathbf{1}$ by (CD₂). In particular, $\varepsilon_{l,d}(\bar{p}) \in \text{Ker}^\uparrow \pi_{\bar{a}}$ hence $(d+1)F(\bar{p})$ is contained in $\text{Ker}^\uparrow \pi_{\bar{a}}$. By (CD₃), $F(\bar{p})/(d+1)F(\bar{p})$ is finite. Hence there exists only finitely many prime filters y in $F(\bar{p})$ which contain $(d+1)F(\bar{p})$, and since the filter $(d+1)F(\bar{p})$ is principal, every such y is principal (here we use Fact 2.2).

So let $\xi_1(\bar{p}), \dots, \xi_\alpha(\bar{p})$ be a list of IPC formulas such that every $y \in X_{F(\bar{p})}$ containing $(d+1)F(\bar{p})$ is generated by $\xi_i(\bar{p})$ for some $i \leq \alpha$. Let n_d be the maximum of the degrees of the $\xi_i(\bar{p})$'s and of $\varepsilon_{l,d}(\bar{p})$ and assume that $\bar{a}' \approx_n \bar{a}$. Then $\varepsilon_{l,d}(\bar{a}') = \mathbf{1}$, that is $\varepsilon_{l,d}(\bar{p}) \in \text{Ker}^\uparrow \pi_{\bar{a}'}$ so $(d+1)F(\bar{p})$ is contained in $\text{Ker}^\uparrow \pi_{\bar{a}'}$. Thus, for every $y \in X_{F(\bar{p})}$, y contains $\text{Ker}^\uparrow \pi_{\bar{a}}$ (resp. $\text{Ker}^\uparrow \pi_{\bar{a}'}$) if and only if y is generated by $\xi_i(\bar{p})$ for some $i \leq \alpha$ such that $\xi_i(\bar{a}) = \mathbf{1}$ (resp. $\xi_i(\bar{a}') = \mathbf{1}$). Since $\text{Th}_n(\bar{a}) = \text{Th}_n(\bar{a}')$ by assumption, $\xi_i(\bar{a}) = \mathbf{1}$ if and only if $\xi_i(\bar{a}') = \mathbf{1}$. This ensures that $\text{Ker}^\uparrow \pi_{\bar{a}}$ and $\text{Ker}^\uparrow \pi_{\bar{a}'}$ are contained in the same prime filters of $F(\bar{p})$. By Fact 2.1 it follows that $\text{Ker}^\uparrow \pi_{\bar{a}} = \text{Ker}^\uparrow \pi_{\bar{a}'}$. ■

Remark 7.5 In [Bel86] an explicit formula $\xi(\bar{p})$, with degree $\leq 2d+1$, is given for every join-irreducible element of $F(\bar{p})$ of foundation rank $\leq d$ (see also Theorem 3.3 in [DJ10]). A prime filter y of $F(\bar{p})$ contains $(d+1)F(\bar{p})$ if and only if is generated by such a join-irreducible element $\xi(\bar{p})$, so we can take $n_d = 2d+1$ in Lemma 7.4.

Theorem 7.6 (Finite Extension Property) *Let \mathcal{V} be a variety of Heyting algebras having the Finite Model Property. If an existential \mathcal{L}_{HA} -formula with parameters in a finite \mathcal{V} -algebra A is satisfied in a \mathcal{V} -algebra containing A , then it is satisfied in some finite \mathcal{V} -algebra containing A .*

Proof: It suffices to prove the result for a formula $\exists \bar{q} \mathcal{S}_{t,\bar{s}}(\bar{a}, \bar{q})$ with $\bar{a} \in A^l$ generating A . Let d be the degree of $\mathcal{S}_{t,\bar{s}}(\bar{p}, \bar{q})$. Let $n \in \mathbf{N}$ be given by Lemma 7.4 applied to the integer $\dim A$ (since A is finite, A is also finite dimensional). Up to intuitionist equivalence there are finitely many IPC formulas $\varphi(\bar{p})$ of degree $\leq n$. Let u be the conjunction of those which belong to $\text{Th}_n(\bar{a})$, and $\bar{v} = (v_1, \dots, v_\alpha)$ be an enumeration of the others. Now consider the system $\mathcal{S}_{t',\bar{s}'}(\bar{p}, \bar{q})$ where $t' = t \wedge u$ and $\bar{s}' = (s_1, \dots, s_\kappa, v_1, \dots, v_\alpha)$.

By assumption $\mathcal{S}_{t,\bar{s}}(\bar{a}, \bar{q})$ has a solution \bar{b} in some extension of A . Then (\bar{a}, \bar{b}) is a solution of $\mathcal{S}_{t',\bar{s}'}(\bar{p}, \bar{q})$. By Proposition 7.1, $\mathcal{S}_{t',\bar{s}'}(\bar{p}, \bar{q})$ then has a solution (\bar{a}', \bar{b}') in a finite Heyting algebra H' in \mathcal{V} . In particular \bar{b}' is a solution $\mathcal{S}_{t,\bar{s}}(\bar{a}', \bar{q})$. Moreover, $u(\bar{a}') = \mathbf{1}$ and $v_i(\bar{a}') \neq \mathbf{1}$ for every $i \leq \alpha$, hence $\text{Th}_n(\bar{a}') = \text{Th}_n(\bar{a})$. By assumption on n , this implies that $\text{Ker}^\uparrow \pi_{\bar{a}} = \text{Ker}^\uparrow \pi_{\bar{a}'}$. So there is an isomorphism from A to the Heyting algebra A' generated by \bar{a}' which maps \bar{a} onto \bar{a}' . By embedding A into H' via this isomorphism, we get a finite \mathcal{V} -algebra containing A in which $\mathcal{S}_{t,\bar{s}}(\bar{a}, \bar{q})$ has a solution. ■

Acknowledgement I would like to thank Sam van Gool and Luca Reggio for helpful discussions during the TOLO VI conference a Tbilissi, Georgia, and to express my gratitude to the organizers of this meeting hosting these discussions.

References

- [BD74] Raymond Balbes and Philip Dwinger. *Distributive lattices*. University of Missouri Press, Columbia, Mo., 1974.
- [Bel86] Fabio Bellissima. Finitely generated free Heyting algebras. *J. Symbolic Logic*, 51(1):152–165, 1986.
- [Bil07] Marta Bílková. Uniform interpolation and propositional quantifiers in modal logics. *Studia Logica*, 85(1):1–31, 2007.
- [Dar] Luck Darnière. Model completion of scaled lattices and lattices of closed p -adic semi-algebraic sets. Soon on arxiv!
- [DJ10] Luck Darnière and Markus Junker. On Bellissima’s construction of the finitely generated free Heyting algebras, and beyond. *Arch. Math. Logic*, 49(7-8):743–771, 2010.
- [DJ11] Luck Darnière and Markus Junker. Codimension and pseudometric in co-Heyting algebras. *Algebra Universalis*, 64(3):251–282, 2011.
- [DJ18] Luck Darnière and Markus Junker. Model completion of varieties of co-Heyting algebras. *Houston J. Math.*, 44(1):49–82, 2018.
- [Esa74] L. L. Esakia. Topological Kripke models. *Dokl. Akad. Nauk SSSR*, 214:298–301, 1974.
- [Ghi99] Silvio Ghilardi. Unification in intuitionistic logic. *J. Symbolic Logic*, 64(2):859–880, 1999.
- [GZ97] Silvio Ghilardi and Marek Zawadowski. Model completions and r-Heyting categories. *Ann. Pure Appl. Logic*, 88(1):27–46, 1997.
- [Hos67] Tsutomu Hosoi. On intermediate logics. I. *J. Fac. Sci. Univ. Tokyo Sect. I*, 14:293–312, 1967.
- [Kre97] Philip Kremer. Propositional quantification in the topological semantics for S4. *Notre Dame J. Formal Logic*, 38(2):295–313, 1997.
- [Ono71] Hiroakira Ono. Kripke models and intermediate logics. *Publ. Res. Inst. Math. Sci.*, 6:461–476, 1970/71.
- [Pit92] Andrew M. Pitts. On an interpretation of second-order quantification in first-order intuitionistic propositional logic. *J. Symbolic Logic*, 57(1):33–52, 1992.
- [Poł98] Tomasz Połacik. Pitts’ quantifiers are not topological quantification. *Notre Dame J. Formal Logic*, 39(4):531–544, 1998.
- [vGR18] Samuel J. van Gool and Luca Reggio. An open mapping theorem for finitely copresented Esakia spaces. *Topology Appl.*, 240:69–77, 2018.