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# Some Bounds on the Energy of Signed Complete Bipartite Graphs <sup>\*†</sup>

S. AKBARI<sup>a</sup>, Y. BAGHERI<sup>b</sup>, S. SAADAT AKHTAR<sup>c</sup>

<sup>a</sup>*Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran*

<sup>b</sup>*Department of Mathematics, K.N. Toosi University of Technology, Tehran, Iran*

<sup>c</sup>*Department of Mathematics, Faculty of Basic Sciences, Islamic Azad University  
Central Tehran Branch, Tehran, Iran. <sup>‡</sup>*

## Abstract

A signed graph  $G^\sigma$  is a pair  $(G, \sigma)$ , where  $G$  is a graph, and  $\sigma : E(G) \rightarrow \{-1, +1\}$  is a function. Assume that  $m \leq n$  are two positive integers. Let

$$A = \left[ \begin{array}{c|c} 0 & B \\ \hline B^t & 0 \end{array} \right]$$

is the adjacency matrix of  $K_{m,n}^\sigma$ . In this paper we show that for every sign function  $\sigma$ ,  $2\sqrt{mn} \leq E(K_{m,n}^\sigma) \leq 2m\sqrt{n}$ , where  $E(K_{m,n}^\sigma)$  is the energy of  $K_{m,n}^\sigma$ . Also it is proved that the equality holds for the upper bound if there exists a Hadamard matrix of order  $n$  for which  $B$  is an  $m$  by  $n$  submatrix of  $H$ . Also if the equality holds, then every two distinct rows of  $B$  are orthogonal. We prove that for the lower bound the equality holds if and only if  $K_{m,n}^\sigma$  is switching equivalent to  $K_{m,n}$ .

## 1 Introduction

Let  $A$  be a real symmetric matrix of order  $n$  and  $\lambda_1 \geq \dots \geq \lambda_n$  be all eigenvalues of  $A$ . The *energy* of  $A$  is defined by  $E(A) = \sum_{i=1}^n |\lambda_i|$ . A signed graph  $G^\sigma$  is a pair  $(G, \sigma)$ , where  $G$  is a graph, and  $\sigma : E(G) \rightarrow \{-1, +1\}$  is a function. We call  $\sigma$  the *sign function* of  $G^\sigma$ . The *adjacency matrix*,  $A(G^\sigma)$  corresponding to  $G^\sigma$  on the vertex set  $V = \{v_1, \dots, v_n\}$  is an  $n \times n$  matrix whose entries are

$$(A(G^\sigma))_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \text{ with positive sign.} \\ 0 & \text{if } v_i \text{ is not adjacent to } v_j. \\ -1 & \text{if } v_i \text{ is adjacent to } v_j \text{ with negative sign.} \end{cases}$$

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<sup>‡</sup>*E-mail addresses:* s.akbari@sharif.edu, yousefbagherizz@gmail.com, simasaadatzz3@gmail.com.

The *spectrum* of  $G^\sigma$  is the set of all eigenvalues of the adjacency matrix corresponding to the signed graph  $G^\sigma$  with multiplicities and denoted by  $\text{Spec}(G^\sigma) = \{\lambda_1^{m_1}, \dots, \lambda_k^{m_k}\}$ , where  $m_i$  is the multiplicity of  $\lambda_i$ . The *energy* of  $G^\sigma$  denoted by  $E(G^\sigma)$ , is the energy of the adjacency matrix of  $G^\sigma$ . A *switching* of a signed graph  $G^\sigma$  at a vertex  $v$  is changing the sign of each edge incident with  $v$ . It is easy to see that a switching at a vertex  $v$  implies a new sign function  $\sigma'$  and new signed graph  $G^{\sigma'}$  which is cospectral with  $G^\sigma$ . We say that  $G^{\sigma'}$  is a *switching* of  $G^\sigma$  if it is obtained from  $G^\sigma$  by finitely many switchings at some vertices of  $G^\sigma$ . Switching defines an equivalence relation on the set of all signed graphs with underlying graph  $G$ . It is easy to check that if  $G$  has  $m$  edges,  $n$  vertices and  $c$  components, then there are  $2^{m-n+c}$  distinct equivalence classes. It immediately implies that the signed cycles have just two classes and it is not hard to see that every class has at most one negative edge. We denote the cycle of order  $n$ , by  $C_n$ . The *sign of a cycle* in a signed graph is the product of the signs of its edges. Thus a cycle is *positive* if and only if it contains an even number of negative edges. A signed graph is said to be *balanced* if all of its cycles are positive otherwise unbalanced. In this paper  $K_{m,n}$  denotes the complete bipartite graph with part sizes  $m$  and  $n$ . We denote a signed complete bipartite graph with underlying graph of  $K_{m,n}$  by  $K_{m,n}^\sigma$ . A *conference matrix* is a square matrix  $C$  of order  $n$  with zero diagonal, such that  $CC^t = (n-1)I$ . If  $C$  is the adjacency matrix of a signed graph, then the signed graph is called a *conference graph*. A *Hadamard matrix*  $H$  is an  $n$  by  $n$  matrix whose entries are either  $-1$  or  $+1$  and whose distinct rows are mutually orthogonal. It implies that  $HH^t = H^tH = nI$ . Haemers in [8] provided some bounds for the energy of signed complete graphs of order  $n$  and proved that  $E(K_n^\sigma) \leq n\sqrt{n-1}$  and the equality holds if and only if  $K_n^\sigma$  is a *conference graph*. Also he conjectured that  $E(K_n^\sigma) \geq 2n-2$ . This conjecture was proved for  $n \leq 12$  but it is still open in general. For more information on the energy of graphs, see [1], [2], [3], [5], [6], [7], [9], [10], [11], [12], [13], [15] and [17]. In this paper we would like to obtain some results for the energy of signed complete bipartite graphs.

## 2 Bounds for the Energy of Signed Complete Bipartite Graphs

In this section we wish to obtain a sharp upper bound for the energy of signed complete bipartite graphs.

**Theorem 1.** *If  $m \leq n$  are two positive integers and*

$$A = \left[ \begin{array}{c|c} 0 & B \\ \hline B^t & 0 \end{array} \right]$$

*is the adjacency matrix of  $K_{m,n}^\sigma$  for an arbitrary sign function  $\sigma$ , then  $E(K_{m,n}^\sigma) \leq 2m\sqrt{n}$ . Moreover, if there exists a Hadamard matrix of order  $n$ , then there is a sign function  $\sigma$  for  $K_{m,n}$  such that  $E(K_{m,n}^\sigma) = 2m\sqrt{n}$ . If the equality holds, then every two distinct rows of  $B$  are orthogonal.*

**Proof.** Suppose that

$$A = \left[ \begin{array}{c|c} 0 & B \\ \hline B^t & 0 \end{array} \right],$$

is the adjacency matrix of  $K_{m,n}^\sigma$ , where  $B$  is a  $(-1,1)$ -matrix of size  $m \times n$ . Let  $A$  have the eigenvalues  $\lambda_1 \geq \dots \geq \lambda_{m+n}$ . Thus the eigenvalues of  $A^2$  are  $\lambda_1^2 \geq \dots \geq \lambda_{m+n}^2$ . Since

$$A^2 = \left[ \begin{array}{c|c} BB^t & 0 \\ \hline 0 & B^t B \end{array} \right],$$

and all non-zero eigenvalues of  $BB^t$  and  $B^t B$  are the same (see [16]), the eigenvalues of  $BB^t$  have the following form,

$$\lambda_{i_1}^2 \geq \lambda_{i_2}^2 \geq \dots \geq \lambda_{i_m}^2.$$

Now,  $tr(BB^t) = \sum_{j=1}^m \lambda_{i_j}^2$ . On the other hand,  $2mn = tr(A^2) = \sum_{i=1}^{m+n} \lambda_i^2 = 2tr(BB^t)$ . It implies that  $tr(BB^t) = mn$ . By Cauchy-Schwartz Inequality we have,

$$\left( \sum_{j=1}^m |\lambda_{i_j}| \right)^2 \leq \left( \sum_{j=1}^m \lambda_{i_j}^2 \right) \left( \sum_{j=1}^m 1 \right) = (mn)m,$$

and so  $\left( \frac{E(K_{m,n}^\sigma)}{2} \right)^2 \leq m^2 n$ . This yields that  $E(K_{m,n}^\sigma) \leq 2m\sqrt{n}$ .

Let  $H$  be a Hadamard matrix of order  $n$  and  $B$  be a submatrix of  $H$  of size  $m \times n$ . Clearly,

$$A^2 = \left[ \begin{array}{c|c} nI_m & 0 \\ \hline 0 & B^t B \end{array} \right].$$

It is not hard to see that the energy of signed complete bipartite graph corresponding to  $A$  has energy  $2m\sqrt{n}$ . Now, assume that  $E(K_{m,n}^\sigma) = 2m\sqrt{n}$ . By the equality case of Cauchy-Schwartz Inequality, all eigenvalues of  $BB^t$  are  $n$ . Thus  $BB^t = nI_m$ . Therefore every two distinct rows of  $B$  are orthogonal.  $\square$

If we consider the positive  $K_{m,n}$ , then its energy is equal to  $2\sqrt{mn}$ . Now, we state the following lemma.

**Lemma 1.**[2] *The eigenvalues of signed  $C_n$  are*

$$\lambda_k = \begin{cases} 2\cos\frac{2k\pi}{n} & \text{if } C_n \text{ is balanced.} \\ 2\cos\frac{(2k+1)\pi}{n} & \text{if } C_n \text{ is unbalanced.} \end{cases} \quad (1)$$

for  $k = 1, \dots, n$ .

**Theorem 2.**[14] *Two signed graphs  $G^{\sigma_1}$  and  $G^{\sigma_2}$  are switching equivalent if and only if they have the same set of unbalanced cycles. Subsequently,  $G^\sigma$  is switching equivalent to  $G$  if and only if all of its cycles are balanced.*

Now, we are ready to prove the following theorem.

**Theorem 3.** *Let  $m \leq n$  be two positive integers. Then for any sign function  $\sigma$ ,  $E(K_{m,n}^\sigma) \geq 2\sqrt{mn}$ , and the equality holds if and only if  $K_{m,n}^\sigma$  is the switching equivalent to  $K_{m,n}$ .*

**Proof.** By a suitable labeling of vertices, the adjacency matrix of  $K_{m,n}^\sigma$  has the following form:

$$A(K_{m,n}^\sigma) = \left[ \begin{array}{c|c} 0 & B \\ \hline B^t & 0 \end{array} \right].$$

Thus we have

$$(A(K_{m,n}^\sigma))^2 = \left[ \begin{array}{c|c} BB^t & 0 \\ \hline 0 & B^t B \end{array} \right].$$

The non-zero eigenvalues of  $BB^t$  and  $B^t B$  are the same. Obviously, every eigenvalue of  $B^t B$  is non-negative. Assume that  $\lambda_1 \geq \dots \geq \lambda_m$  are the first largest eigenvalues of  $B^t B$ . Thus  $A(K_{m,n}^\sigma)$  has at least  $n - m$  eigenvalues 0. Hence we find that  $E(K_{m,n}^\sigma) = 2 \sum_{i=1}^m \sqrt{\lambda_i}$ . Obviously,  $(\sum_{i=1}^m \sqrt{\lambda_i})^2 \geq \sum_{i=1}^m \lambda_i = \text{tr}(B^t B) = mn$ . This implies that  $E(K_{m,n}^\sigma) \geq 2\sqrt{mn}$ . Now, if  $B^t B$  has at least two positive eigenvalues, then clearly,  $E(K_{m,n}^\sigma) > 2\sqrt{mn}$ . Therefore the multiplicity of zero in  $\text{Spec}(E(K_{m,n}^\sigma))$  is  $m + n - 2$ . Since  $\sum_{i=1}^{m+n} \lambda_i^2 = 2mn$  and  $\sum_{i=1}^{m+n} \lambda_i = 0$ , the eigenvalues of  $K_{m,n}^\sigma$  are as follows:

$$\sqrt{mn}, 0, \dots, 0, -\sqrt{mn}.$$

Suppose that  $K_{m,n}^\sigma$  contains a negative cycle. By switching one can assume that this cycle has exactly one negative edge. First we show that there exists a negative 4-cycle. To see this let  $v_1, v_2, \dots, v_r$  be a negative cycle such that  $v_1 v_2$  is negative. If  $v_3 v_r$  is positive, then  $v_1 v_2 v_3 v_r$  is a negative 4-cycle, as desired. So  $v_3 v_r$  is negative. Similarly,  $v_4 v_{r-1}$  is a negative edge. By continuing this procedure we find that  $v_{\frac{r}{2}} v_{\frac{r}{2}+1} v_{\frac{r}{2}+2} v_{\frac{r}{2}+3}$  is a negative 4-cycle, as desired. The eigenvalues of a negative 4-cycle are:  $-\sqrt{2}, -\sqrt{2}, \sqrt{2}, \sqrt{2}$ . So by Interlacing Theorem, (see [4]),  $K_{m,n}^\sigma$  has at least two positive eigenvalues, a contradiction. Thus every cycle of  $K_{m,n}^\sigma$  is positive. So by Theorem 2,  $K_{m,n}^\sigma$  is switching equivalent of  $K_{m,n}$ . Other side of theorem is clear and the proof is complete.  $\square$

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