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Abstract

A parametric framework is proposed to model both attritional and atypical claims for insurance pricing. This model relies on a classical Generalized Linear Model for attritional claims and a non-standard Generalized Pareto distribution regression model for atypical claims. Maximum likelihood estimators (closed-form for the Generalized Linear Model part and computed with Iterated Weighted Least Square procedure for the Generalized Pareto distribution regression part) are proposed to calibrate the model.

Two premium principles (expected value principle and standard deviation principle) are computed on a real data set of fire warranty of a corporate line-of-business. In our methodology, the tuning of the safety loading in the two premium principles is performed to meet a solvency constraint so that the premium caps a high-level quantile of the aggregate annual claim distribution over a reference portfolio.

Keywords: commercial lines, non-life insurance, pricing, composite distribution, solvency criterion.

1 Introduction

Non-life insurance pricing relies on the appropriate usage of actuarial models on policy and claim datasets. For most personal lines of business, individual information are (massively) available in order to fit a regression model, unlike commercial lines where aggregated datasets might be the only ingredient to compute a premium. In this paper, we have the opportunity to use individual information for pricing the fire guarantee of a corporate line of business. Therefore, pricing non-life insurance consists in estimating the claim frequency and the claim severity distributions at individual level.

The claim frequency can generally be estimated parametrically (i.i.d. model or Poisson regression model) or non-parametrically (through regression trees or more advanced models).

For modeling claim severity, the claim dataset is commonly split between attritional and atypical claims. Namely, a threshold μ is chosen either from the extreme value theory or by expert judgments. Then, on the one hand, a classical Generalized Linear Model (GLM) such as gamma or inverse-Gaussian is fitted on attritional claim amounts below μ (see e.g. Ohlsson and Johansson [2010] for a recent book). On the other hand, atypical claim amounts above μ are not necessarily modeled at all. An empirical and pragmatic rule for the insurance pricing used by many actuaries is to mutualize atypical claims over the portfolio. That is, the aggregate sum of atypical claims is shared equally among all policies.

In this paper, we focus on the construction of risk classes for both attritional and atypical claims for a given insurance product. The risk features (explanatory variables) used to determines risk classes are the type of industry, the category of maximum probable loss and the total insured area for instance.

A parametric framework is provided to model attritional and atypical claims. Let $A \subset A_1 \times A_2$ be the set of risk classes (A_1 for attritional claims risk classes and A_2 for atypical claims risk classes). At each risk class $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in A$ corresponds policies having the same attributes with respect to explanatory variables considered. We consider a collective model for the annual amount of a policy of class $\boldsymbol{\alpha}$. That is to say, we assume a compound distribution for the annual aggregate claim of a policy by

$$X_{\boldsymbol{\alpha}} = \sum_{i=1}^{2} \sum_{k=1}^{M_{\alpha_i}^{(i)}} Y_{\alpha_i,k}^{(i)} \tag{1}$$

where $M_{\alpha_i}^{(i)}$ and $(Y_{\alpha_i,k}^{(i)})_k$ are respectively the random claim numbers by policy and the claim amounts for attritional claim when $i = 1$ and atypical claim when $i = 2$. We assume that $(Y_{\alpha_i,k}^{(i)})_k$ are independent and identically distributed variables and independent of $M_{\alpha_i}^{(i)}$ for $i = \{1, 2\}$. We also suppose finite first two moments for all aforementioned random variables. Consequently from e.g. Klugman et al. [2008], it is well known that the first two moments are

$$\mathbf{E}(X_{\boldsymbol{\alpha}}) = \sum_{i=1}^{2} \mathbf{E}(M_{\alpha_i}^{(i)}) \mathbf{E}(Y_{\alpha_i,1}^{(i)}) \tag{2}$$

and

$$\mathbf{Var}(X_{\boldsymbol{\alpha}}) = \sum_{i=1}^{2} \mathbf{E}(M_{\alpha_i}^{(i)}) \mathbf{Var}(Y_{\alpha_i,1}^{(i)}) + \mathbf{Var}(M_{\alpha_i}^{(i)}) (\mathbf{E}(Y_{\alpha_i,1}^{(i)}))^2. \tag{3}$$

Recently, a large series of actuarial papers studied the use of composite distributions for loss modeling. A composite or a spliced distribution is a probability distribution such that the density is different below and above the threshold μ. Cooray and Ananda [2005], Scollnik [2007] consider a lognormal - Pareto composite models, whereas Teodorescu and Panaitescu [2009], Scollnik and Sun [2012] and Nadarajah and Bakar [2014] consider Weibull - Pareto composite models. More recently, Bakar et al. [2015] focus on Weibull - transformed beta models and Reynkens et al. [2016] propose a mixed Erlang - Pareto composite model.

A first goal of this paper is to compute different premium principles which includes a safety loading $\rho > 0$. We investigate two premium principles based on the insurer practice: the expected value principle

$$\pi^1_{\boldsymbol{\alpha},\rho} = \mathbf{E}(X_{\boldsymbol{\alpha}})(1+\rho) \tag{4}$$

and the standard deviation principle

$$\pi^2_{\boldsymbol{\alpha},\rho} = \mathbf{E}(X_{\boldsymbol{\alpha}}) + \rho\sqrt{\mathbf{Var}(X_{\boldsymbol{\alpha}})}. \tag{5}$$

These are two well-known principles, see Teugels and Sundt [2004] for details. It is worth mentioning that the so-called pure premium is obtained for $\pi^1_{\boldsymbol{\alpha},0} = \pi^2_{\boldsymbol{\alpha},0} = \mathbf{E}(X_{\boldsymbol{\alpha}})$. The computations of $\pi^1_{\boldsymbol{\alpha},\rho}$ and $\pi^2_{\boldsymbol{\alpha},\rho}$ requires a proper computation of the first moment and the first two moments of $X_{\boldsymbol{\alpha}}$ respectively and the choice of the safety loading. The model is presented in Section 2 and the two premium principles are computed for a real dataset in Section 3.1.

Expert judgements and intensive use of actuaries make them set the safety loading ρ to an arbitrary value. A second goal of this paper consists in providing a solvency criterion to find the appropriate value of ρ. Here, the value of safety loading is set so that the premium caps a high-level quantile of the aggregate claim distribution over a reference portfolio. This quantile is determined by a simulation approach in Section 3.2.

2 Framework

We consider the annual aggregate claim of a policy

$$X_{\boldsymbol{\alpha}} = \sum_{i=1}^{2} \sum_{k=1}^{M^{(i)}_{\alpha_i}} Y^{(i)}_{\alpha_i,\ell} \tag{6}$$

where $M^{(i)}_{\alpha_i}$ and $(Y^{(i)}_{\alpha_i,k})_k$ are respectively the random claim numbers by policy and the claim amounts for attritional claim when $i = 1$ and atypical claim when $i = 2$. Here $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in A \subset A_1 \times A_2$ where A_1 and A_2 are fixed sets of risk classes for attritional and atypical claims respectively. In the following, it is assumed that,

- For any fixed $i \in \{1, 2\}$, the random variables $(M^{(i)}_{\alpha_i}, \alpha_i \in A_i)$ are independent;
- For any fixed $i \in \{1, 2\}$, for all $\alpha_i \in A_i$, the sequence $(Y^{(i)}_{\alpha_i,k}, k = 1, \ldots, M^{(i)}_{\alpha_i})$ are independent and identically distributed variables conditionally to $M^{(i)}_{\alpha_i}$;

- In order to get a fair approximation of the aggregate claim amount $S = \sum_{\boldsymbol{\alpha} \in A} X_{\boldsymbol{\alpha}}$, we also assume independence between attritional numbers and amounts claims and atypical number and amounts claims. Moreover, the independence between claim amounts of two different classes $X_{\boldsymbol{\alpha}}, X_{\boldsymbol{\alpha}'}$ for $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in A$ is also supposed.

2.1 Claim frequency modeling

Let us first explain our modeling of the claim frequency for attritional claims (when $i = 1$) and atypical claims (when $i = 2$).

In our model, we assume that the number of claims follows an homogeneous Poisson process with intensity λ_{α_i}. Since the annual aggregate claim is considered,

$$\mathbf{E}(M_{\alpha_i}^{(i)}) = \mathbf{Var}(M_{\alpha_i}^{(i)}) = \lambda_{\alpha_i}^{(i)}. \tag{7}$$

Generally, some policies are not observed during the whole year, i.e. the insured period is a sub-period of the year. Consequently, for any $i \in \{1, 2\}$, the random variable $M_{\alpha_i}^{(i)}$ is not observed, nor all the $Y_{\alpha_i,\ell}^{(i)}$.

For the estimation of $\lambda_{\alpha_i}^{(i)}$, the insurance company can rely on $n_{\alpha_i}^{(i)}$ policies for the risk class α_i. We index this policies by $j = 1, \ldots, n_{\alpha_i}^{(i)}$. We denote $N_{\alpha_i,1}^{(i)}, \ldots, N_{\alpha_i,n_{\alpha_i}^{(i)}}^{(i)}$ the corresponding number of claims observed on the subperiods $d_{\alpha_i,1}^{(i)}, \ldots, d_{\alpha_i,n_{\alpha_i}^{(i)}}^{(i)}$ respectively.

The Poisson process structure gives a natural estimation of λ_{α_i} (see e.g. Klugman et al. [2008]) with

$$\widehat{\lambda}_{\alpha_i}^{(i)} = \frac{\sum_{j=1}^{n_{\alpha_i}^{(i)}} N_{\alpha_i,j}^{(i)}}{\sum_{j=1}^{n_{\alpha_i}^{(i)}} d_{\alpha_i,j}^{(i)}}. \tag{8}$$

Consequently, this estimation gives the necessary estimations $\widehat{\mathbf{E}(M_{\alpha_i}^{(i)})} = \widehat{\mathbf{Var}(M_{\alpha_i}^{(i)})} = \widehat{\lambda}_{\alpha_i}^{(i)}$ in Equations (2) and (3) for the computation of premium principles (4) and (5).

2.2 Claim severity modeling

Now we turn our attention to the claim severity. The methodology proposed in this paper use the Generalized Linear models, introduced by Nelder and Wedderburn [1972] and popularized in McCullagh and Nelder [1989].The GLMs relax the assumption of identical distributions for random variables by considering explanatory variables and suppose the distributions belonging to the exponential family. Consistency and asymptotic normality of sequences of maximum likelihood estimators (MLE) for GLMs were be studied by Fahrmeir and Kaufmann [1985].

2.2.1 Attritional claims

As it has been mentioned in the introduction, a classical GLM such as gamma or inverse-Gaussian is fitted on attritional claim amounts below μ, see e.g. Ohlsson and Johansson

[2010] for a recent book. In our application, we fit on atypical claims for a given risk class $\alpha_1 \in A_1$, the gamma probability density function

$$g_{\alpha_1}(x) = \frac{\gamma_{\alpha_1}^{\nu}}{\Gamma(\nu)} x^{\nu-1} \exp(-\gamma_{\alpha_1} x) \mathbb{1}_{x\geq 0} \tag{9}$$

where $\gamma_{\alpha_1} > 0$ the rate parameter and $\nu > 0$ the shape parameter. In a GLM setting, the rate parameter γ_{α_1} is a function of explanatory variables as follows

$$\gamma_{\alpha_1} = \gamma_{\alpha_1}(\boldsymbol{\vartheta}_1) = \exp(\vartheta_{1,\alpha_1}), \tag{10}$$

where $\boldsymbol{\vartheta}_1 = (\vartheta_{1,1}, \ldots, \vartheta_{1,p_1}) \in \mathbb{R}^{p_1}$ and $p_1 = \#(A_1)$ is the number of risk classes for atypical claims.

In this model, the expectation and the variance of $Y_{\alpha_1,1}^{(1)}$ are given by

$$\mathbf{E}_{\boldsymbol{\vartheta}_1}\left(Y_{\alpha_1,1}^{(1)}\right) = \frac{\nu}{\gamma_{\alpha_1}(\boldsymbol{\vartheta}_1)} \tag{11}$$

and

$$\mathbf{Var}_{\boldsymbol{\vartheta}_1}\left(Y_{\alpha_1,1}^{(1)}\right) = \frac{\nu}{\gamma_{\alpha_1}^2(\boldsymbol{\vartheta}_1)}. \tag{12}$$

In order to compute the two premium principles defined in (4) and (5), the estimation of the parameter $\boldsymbol{\vartheta}_1$ is done with the maximum likelihood estimator $\widehat{\boldsymbol{\vartheta}}_1$ (MLE). Consistency and asymptotical normality of the MLE are given by classical theorems Fahrmeir and Kaufmann [1985] and closed-form estimators for categorical variables are given in Brouste et al. [2018]. In order to compute two aforementioned principles, we estimate the first two moments by the plug-in estimators

$$\widehat{\mathbf{E}_{\boldsymbol{\vartheta}_1}\left(Y_{\alpha_1,1}^{(1)}\right)} = \mathbf{E}_{\widehat{\boldsymbol{\vartheta}}_1}\left(Y_{\alpha_1,1}^{(1)}\right) \quad \text{and} \quad \widehat{\mathbf{Var}_{\boldsymbol{\vartheta}_1}\left(Y_{\alpha_1,1}^{(1)}\right)} = \mathbf{Var}_{\widehat{\boldsymbol{\vartheta}}_1}\left(Y_{\alpha_1,1}^{(1)}\right). \tag{13}$$

The parameter ν is calibrated by maximizing the likelihood of the estimated model (or in a equivalent way by minimizing the Akaike Information Criterion (AIC).

2.2.2 Atypical claims

In the assurance loss modeling context, sever claims or the exceedances over a large threshold are usually modeled by heavy-tail distributions, typically Generalized Pareto Distribution, see for example, Embrechts P. [1997], Chavez-Demoulin et al. [2015], Hambuckers et al. [2016]. The two last ones propose semi-parametric regression models where the explanatory variables are time or known factor levels. Davison and Smith [1990] studied a least square estimation procedure and model checking method for the GPD regression model. Regression model for generalized extreme value distributions have been broadly analyzed in Beirlant et al. [2004]. Recently, in the case of categorical explanatory variables, heavy-tailed distributions regression model have been studied in Brouste et al. [2018], and explicit solution of the Maximum Likelihood Estimator have been given. Rigby and Stasinopoulos [2005] are focused on general regression framework where all parameters are modeled by explanatory

variables and distribution is outside the exponential family. The authors impose then the twice differentiability of the probability density function with respect to the parameters.

In this paper we propose a regression model for GPD, with a Iterative Weighted Least Square (IWLS) procedure to estimate parameters of the GPD. It is not a classical GLM since the probability density function is not of exponential type.

More precisely, we fit the following probability density function on atypical claims for a given risk class $\alpha_2 \in A_2$.

$$f_{\alpha_2}(x) = \beta_{\alpha_2} \left(1 + \beta_{\alpha_2} \xi (x - \mu)\right)^{-\frac{1}{\xi}-1} \mathbb{1}_{\{x \geq \mu\}}, \tag{14}$$

where $\mu \in \mathbb{R}$ is the known threshold (or location parameter), $\beta_{\alpha_2} > 0$ is the scale parameter and $\xi > 0$ is the shape parameter. In our model, the scale parameter β_{α_2} is a function of explanatory variables as follows

$$\beta_{\alpha_2} = \beta_{\alpha_2}(\boldsymbol{\vartheta}_2) = \ell(\vartheta_{2,\alpha_2}), \tag{15}$$

where $p_2 = \#(A_2)$ is the number of risk classes for atypical claims and $\boldsymbol{\vartheta}_2 = (\vartheta_{2,1}, \ldots, \vartheta_{2,p_2}) \in \mathbb{R}^{p_2}$ is the unknown vector parameter and ℓ a well-chosen injective link function, such that $\ell(\eta) > 0$ for all η. Typically we choose $\ell = \exp$. The expectation and the variance of $Y^{(2)}_{\alpha_2,1}$ are given by

$$\mathbf{E}_{\boldsymbol{\vartheta}_2}\left(Y^{(2)}_{\alpha_2,1}\right) = \mu + \frac{1}{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)(1-\xi)} \quad \text{for } \xi < 1 \tag{16}$$

and

$$\mathbf{Var}_{\boldsymbol{\vartheta}_2}\left(Y^{(2)}_{\alpha_2,1}\right) = \frac{1}{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)^2(1-\xi)^2(1-2\xi)} \quad \text{for } \xi < 1/2. \tag{17}$$

The parameter ξ is calibrated by maximizing the likelihood of the estimated model (or in a equivalent way by minimizing the Akaike Information Criterion (AIC).

In order to compute the different premium principles, the estimation of the parameter $\boldsymbol{\vartheta}_2$ is done with the maximum likelihood estimator $\widehat{\boldsymbol{\vartheta}}_2$ (MLE). The proof of the consistency and asymptotical normality of the MLE and the presentation of the IWLS algorithm used to compute the MLE are presented in Appendix A. In order to compute two aforementioned principles, we estimate the first two moments by the plug-in estimators

$$\widehat{\mathbf{E}_{\boldsymbol{\vartheta}_2}\left(Y^{(2)}_{\alpha_2,1}\right)} = \mathbf{E}_{\widehat{\boldsymbol{\vartheta}}_2}\left(Y^{(2)}_{\alpha_2,1}\right) \quad \text{and} \quad \widehat{\mathbf{Var}_{\boldsymbol{\vartheta}_2}\left(Y^{(2)}_{\alpha_2,1}\right)} = \mathbf{Var}_{\widehat{\boldsymbol{\vartheta}}_2}\left(Y^{(2)}_{\alpha_2,1}\right). \tag{18}$$

Nevertheless, the parameter ξ can be bigger than $1/2$ (resp. 1) and the variance (resp. the mean) can be infinite. Consequently, the different premium principles won't be computable. For this reason, we consider the upper-truncated random variables $Y^{(2),G}_{\alpha_2,k} = \min(Y^{(2)}_{\alpha_2,k}, G)$ for a well chosen limit G corresponding the maximal guarantee of the contract. The expressions of the first two raw moments $\mathbf{E}_{\boldsymbol{\vartheta}_2}(Y^{(2),G}_{\alpha_2,1})$ and $\mathbf{E}_{\boldsymbol{\vartheta}_2}\left(\left(Y^{(2),G}_{\alpha_2,1}\right)^2\right)$ are given in Appendix B.

3 Numerical illustration to the fire warranty

In this paper, the developed method is illustrated with real insurance dataset. We consider a corporate portfolio on which data are observed between 2000 and 2015 from fire

warranty. The ultimate loss is adjusted to obtain a consistent view of expenses, because of the important historical observation period. The threshold between attritional and atypical claims is fixed to 150k EUR. To guarantee anonymization, the numerical values presented in the article have been modified and risk classes definitions are hidden.

The dataset consists of 1 414 688 assurance policies including 1006 large atypical claims. For simplicity, the sets of risk classes for attritional and atypical claims are similar. Namely, the set $A_1 = A_2$ is composed of $p_1 = p_2 = 11$ risk class and risk class are of the form $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ with $\alpha_1 = \alpha_2$. In other words $A \equiv A_1$ and $p = 11$.

3.1 Computation of premium principles

In order to estimate the premium principles $\pi^1_{\boldsymbol{\alpha},\rho}$ and $\pi^2_{\boldsymbol{\alpha},\rho}$ defined in Equations (4) and (5), the two first moments of $Y^i_{\alpha_i,1}$ (or $Y^{(2),G}_{\alpha_2,1}$ in the infinite expectation or variance setting for atypical claims) and $M^{(i)}_{\alpha_i}$ should be estimated for the attritional claims (when $i = 1$) and the atypical claims (when $i = 2$).

Firstly, the first two moments of $M^{(i)}_{\alpha_i}$ are estimated with (8). Secondly, we consider the claim amount defined by the models described in Sections 2.2.1 and 2.2.2 for attritional claims (GLM with gamma distribution) and atypical claims (GPD regression model) respectively. Hence estimators are given by

$$\widehat{\mathbf{E}(Y^{(i)}_{\alpha_i,1})} = \mathbf{E}_{\widehat{\boldsymbol{\vartheta}}}(Y^{(i)}_{\alpha_i,1})$$

and

$$\widehat{\mathbf{Var}(Y^{(i)}_{\alpha_i,1})} = \mathbf{Var}_{\widehat{\boldsymbol{\vartheta}}}(Y^{(i)}_{\alpha_i,1})$$

where $\widehat{\boldsymbol{\vartheta}}_1$ is the closed-form MLE for the GLM with gamma distribution defined in Brouste et al. [2018] and $\widehat{\boldsymbol{\vartheta}}_2$ is the MLE computed by the IWLS algorithm for the GPD regression model described on Appendix A. Finally estimation of the premium principles $\pi^1_{\boldsymbol{\alpha},\theta}$ and $\pi^2_{\boldsymbol{\alpha},\rho}$ are given by

$$\widehat{\pi}^1_{\boldsymbol{\alpha},\rho} = (1+\rho)\sum_{i=1}^{2} \widehat{\mathbf{E}(M^{(i)}_{\alpha_i})}\widehat{\mathbf{E}(Y^{(i)}_{\alpha_i,1})}$$

and

$$\widehat{\pi}^2_{\boldsymbol{\alpha},\rho} = \sum_{i=1}^{2} \widehat{\mathbf{E}(M^{(i)}_{\alpha_i})}\widehat{\mathbf{E}(Y^{(i)}_{\alpha_i,1})} + \rho\left(\sum_{i=1}^{2} \widehat{\mathbf{E}(M^{(i)}_{\alpha_i})}\widehat{\mathbf{Var}(Y^{(i)}_{\alpha_i,1})} + \widehat{\mathbf{Var}(M^{(i)}_{\alpha_i})}(\widehat{\mathbf{E}(Y^{(i)}_{\alpha_i,1})})^2\right)^{1/2}.$$

The non-parametric and parametric estimates of $\pi^1_{\boldsymbol{\alpha},0}$, $\pi^1_{\boldsymbol{\alpha},1}$ and $\pi^2_{\boldsymbol{\alpha},0.03}$ are reported in Table 1. In the case of the GPD regression, the shape parameter ξ is calibrated by AIC and equals $\xi = 0.6$. Consequently, the finite variance random variables $Y^{(2),G}_{\alpha_2,k}$ has been used with maximal guarantee $G = 25.10^6$. We also present the respective contributions of the attritional claims and the atypical claims in the pure premium in Table 2.

$\boldsymbol{\alpha}$	$\widehat{\pi}^1_{\boldsymbol{\alpha},0}$	$\widehat{\pi}^1_{\boldsymbol{\alpha},1}$	$\widehat{\pi}^2_{\boldsymbol{\alpha},0.03}$
1	3704	7408	7343
2	5751	11503	9246
3	1285	2570	3152
4	1792	3584	4346
5	6568	13136	12199
6	6413	12827	12227
7	2265	4529	5206
8	4756	9512	8834
9	4143	8287	7553
10	24631	49263	33174
11	935	1870	2456

Table 1: Estimation of the two premium principles (expected value and standard deviation) with fixed safety loading ρ in the GLM + GPD approach.

$\boldsymbol{\alpha}$	$\widehat{\pi}^1_{\boldsymbol{\alpha},0}$	Attritional (%)	Atypical (%)
1	3704	16	84
2	5751	15	85
3	1285	17	83
4	1792	14	86
5	6568	9	91
6	6413	5	95
7	2265	12	88
8	4756	12	88
9	4143	17	83
10	24631	15	85
11	935	23	77

$\boldsymbol{\alpha}$	$\widehat{\pi}^1_{\boldsymbol{\alpha},0}$	$\widehat{\pi}^1_{\boldsymbol{\alpha},\rho'}$	$\widehat{\pi}^2_{\boldsymbol{\alpha},\rho^*}$
1	3704	7860	9548
2	5751	12205	11363
3	1285	2727	4283
4	1792	3803	5893
5	6568	13938	15610
6	6413	13610	15749
7	2265	4806	6988
8	4756	10093	11305
9	4143	8793	9619
10	24631	52269	38350
11	935	1984	3377

Table 2: Respective contributions of the attritional and atypical claims in the pure premium for the different risk class in the GLM-GPD model (on the left). Estimation of the different premium principles; for the expectation principle with a global safety loading calibrated to $\rho' = 1.122$ and for the standard deviation principle with a global safety loading calibrated to $\rho^* = 0.048$.

3.2 Calibration of the tuning parameter

In the section, we focus on the calibration of the safety loading ρ in the expected value principle (4) and the standard deviation principle (5). For simplicity, we recall that the sets of risk classes for attritional and atypical claims are similar. Namely, the set $A_1 = A_2$ is composed of $p_1 = p_2 = 11$ risk class and risk class are of the form $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ with $\alpha_1 = \alpha_2$. In other words $A \equiv A_1$ and $p = 11$.

The mean claim amount over the insurer portfolio on a reference year is used to tune the safety margin ρ in the following way

$$P\left(\sum_{\boldsymbol{\alpha}\in A} \frac{1}{n^o_{\boldsymbol{\alpha}}} \sum_{j=1}^{n^o_{\boldsymbol{\alpha}}} X_{\boldsymbol{\alpha},j} < \sum_{\boldsymbol{\alpha}\in A} \widehat{\pi}^r_{\boldsymbol{\alpha},\rho}\right) \geq 1-\epsilon, \quad r = 1 \text{ or } 2, \tag{19}$$

where $1-\epsilon$ is a high probability such as 99% or 99.5%, n^o_α is the number of policies in class $\boldsymbol{\alpha} = (\alpha_1, \alpha_1)$ on a reference year and the random variables $X_{\boldsymbol{\alpha},j}$, $k = 1, \ldots, n^o_{\boldsymbol{\alpha}}$ are independent and identically distributed similarly to $X_{\boldsymbol{\alpha}}$. The safety loading is calibrated so as to satisfy Equation (19). Quantiles of the distribution are evaluated with 5000 Monte Carlo siimulations.

For $1-\epsilon = 99.5\%$, the value of ρ is calibrated to 1.0372 for the expected value principle and 0.0445 for the standard deviation principle, The values of these premium principles for calibrated safety loading are reported in Table 2 for each class of risk $\boldsymbol{\alpha}$.

An other possibility is to consider individual safety loading $\rho_{\boldsymbol{\alpha}}$ for each $\boldsymbol{\alpha} \in A$, i.e. the values $\rho_{\boldsymbol{\alpha}}$ are respectively solution of equation

$$P\left(\frac{1}{n^o_{\boldsymbol{\alpha}}} \sum_{j=1}^{n^o_{\boldsymbol{\alpha}}} X_{\boldsymbol{\alpha},j} < \widehat{\pi}^r_{\boldsymbol{\alpha},\rho_{\boldsymbol{\alpha}}}\right) \geq 1-\epsilon, \quad \boldsymbol{\alpha} \in A, \quad r = 1 \text{ or } 2.$$

The quantiles are evaluated with a Monte Carlo procedure. For $1-\epsilon = 99.5\%$, the values of $\rho_{\boldsymbol{\alpha}}$ and the corresponding premium are reported in Table 3.

$\boldsymbol{\alpha}$	$\widehat{\pi}^1_{\boldsymbol{\alpha},0}$	$\widehat{\pi}^1_{\boldsymbol{\alpha},\rho_{\boldsymbol{\alpha}}}$	$\rho_{\boldsymbol{\alpha}}$	$\widehat{\pi}^2_{\boldsymbol{\alpha},\rho_{\boldsymbol{\alpha}}}$	$\rho_{\boldsymbol{\alpha}}$
1	3704	7391	0.996	7391	0.030
2	5751	36836	5.405	36836	0.267
3	1285	9822	6.643	9822	0.137
4	1792	3965	1.212	3965	0.026
5	6568	21195	2.227	21195	0.078
6	6413	22372	2.488	22372	0.082
7	2265	10826	3.780	10826	0.087
8	4756	16005	2.365	16005	0.083
9	4143	9983	1.409	9983	0.051
10	24631	73232	1.973	73232	0.171
11	935	2109	1.256	2109	0.023

Table 3: Estimation of the premia for the expectation principle and for the standard deviation principle with individual safety loadings $\rho_{\boldsymbol{\alpha}}$.

4 Conclusions

In this paper, a parametric framework has been proposed to model both attritional and atypical claims. The attritional claims are modeled with a classical Generalized Linear Model (GLM) whereas the atypical claims are modeled with a non-standard Generalized Pareto regression. It is worth mentioning that the Generalized Pareto distribution (GPD) is not of exponential type and does not enter in the GLM theory.

Closed-form maximum likelihood estimators can be elicited for the GLM with categorical explanatory variables (see Brouste et al. [2018] for instance) in order to calibrate the model for attritional claims. We have proposed an Iterated Weighed Least Square procedure to compute the maximum likelihood estimators to calibrate the GPD regression model.

Two premium principles (expected value principle and standard deviation principle) are computed on a real data set of fire warranty on a corporate line-of-business. In our methodology, the tuning of the safety loading in the two premium principles is performed to meet a solvency constraint so that the gross written premium caps a high-level quantile of the aggregate claim distribution over a reference portfolio.

We can notice that the GPD could exhibit infinite expectation or infinite variance and no standard deviation premium principle can be computed. Consequently, a maximal guarantee has been introduced to compute the first two moments. Such maximal guarantee depending on the risk class could be studied in a further work.

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A IWLS algorithm for Generalized Pareto regression model

Let $(X_1, \dots, X_n)$ be a sample of independent random variables with Generalized Pareto distribution of known threshold $\mu \in \mathbb{R}$ (location parameter), shape parameter $\xi > 0$ and respective positive scale parameter $\beta_1, \dots, \beta_n$. More precisely, we fit the i-th atypical claim with the following probability density function

$$f_i(x) = \beta_i \left(1 + \beta_i \xi (x - \mu)\right)^{-\frac{1}{\xi} - 1}, \qquad x \in [\mu, \infty). \tag{20}$$

In the following, the link function is denoted ℓ such that

$$\beta_i = \beta_i(\boldsymbol{\vartheta}) = \ell(\eta_i), \quad \eta_i = \langle \boldsymbol{y}_i, \boldsymbol{\vartheta} \rangle \tag{21}$$

where $\boldsymbol{\vartheta} \in \mathbb{R}^p$ is the unknown regression parameter and $\boldsymbol{y}_i \in \mathbb{R}^p$ is the vector of covariates, $i = 1, \dots, n$. For $\xi > 0$ the log-likelihood with respect to $\boldsymbol{\vartheta}$ is

$$\log L(x, \boldsymbol{\vartheta}) = \log \beta_i(\boldsymbol{\vartheta}) - \left(\frac{1}{\xi} + 1\right) \log(1 + \beta_i(\boldsymbol{\vartheta}) \xi (x - \mu)), \quad x \in [\mu, \infty),$$

and $-\infty$ elsewhere. Let $Z_i = \xi(X_i - \mu)$. The log-likelihood L is rewritten as

$$\log L(z, \boldsymbol{\vartheta}) = \gamma(\beta_i(\boldsymbol{\vartheta}) z) - \varphi(\beta_i(\boldsymbol{\vartheta})) + \log h(z), \quad z \in [0, +\infty)$$

and $-\infty$ elsewhere, with $\varphi(\beta) = -\log(\beta)$, $\gamma(z) = -\left(\frac{1}{\xi} + 1\right) \log(1 + z)$ and $h(z) = \frac{1}{\xi}$. Typically, for $\xi > 0$, the GPD distribution does not belong to the exponential family for which $\gamma \equiv id$. Note that we don't have the classical GLM representation

$$g(\mathbf{E}_{\boldsymbol{\vartheta}}(Z_i)) = \eta_i, \quad \text{with} \quad g = \ell^{-1} \circ (\varphi')^{-1},$$

because $\varphi'(\ell(\eta_i))$ fails to be $\mathbf{E}_{\boldsymbol{\vartheta}}(Z_i)$.

With the previous notations, the log likelihood applied on $\boldsymbol{z} = (z_1, \dots, z_n)$ is

$$\log L(\boldsymbol{z}, \boldsymbol{\vartheta}) = \sum_{i=1}^{n} \left(\gamma(\ell(\eta_i) z_i) - \varphi(\ell(\eta_i))\right) \tag{22}$$

Hence the score vector is written as $\boldsymbol{S}(\boldsymbol{\vartheta}) = (S_j(\boldsymbol{\vartheta}))_{j=1,\dots,p}$ with

$$\begin{aligned} S_j(\boldsymbol{\vartheta}) &= \frac{\partial}{\partial \vartheta_j} \log L(\boldsymbol{z}, \boldsymbol{\vartheta}) \\ &= \sum_{i=1}^{n} y_i^{(j)} \ell'(\eta_i) \left(z_i \gamma'\left(\ell(\eta_i) z_i\right) - \varphi'\left(\ell(\eta_i)\right)\right), \end{aligned} \tag{23}$$

and the Hessian matrix is $\mathcal{H}(\boldsymbol{\vartheta}) = \left(\mathcal{H}(\boldsymbol{\vartheta})\right)_{j,k=1,\dots,p}$, with

$$\begin{aligned} \mathcal{H}_{j,k}(\boldsymbol{\vartheta}) &= \frac{\partial^2}{\partial \vartheta_j \partial \vartheta_k} \log L(\boldsymbol{z}, \boldsymbol{\vartheta}) \\ &= \sum_{i=1}^{n} y_i^{(j)} y_i^{(k)} \ell'(\eta_i)^2 \left(z_i^2 \gamma''(\ell(\eta_i) z_i) - \varphi''(\ell(\eta_i))\right) \\ &+ \sum_{i=1}^{n} y_i^{(j)} y_i^{(k)} \ell''(\eta_i) \left(z_i \gamma'\left(\ell(\eta_i) z_i\right) - \varphi'\left(\ell(\eta_i)\right)\right). \end{aligned}$$

The Information matrix is given by $\mathcal{I}^{\xi}(\boldsymbol{\vartheta}) = \left(\mathcal{I}_{j,k}(\boldsymbol{\vartheta})\right)_{j,k\in\{1,\dots,p\}}$, with

$$\mathcal{I}^{\xi}_{j,k}(\boldsymbol{\vartheta}) = -\mathbf{E}\mathcal{H}_{j,k}(\boldsymbol{\vartheta}) = \frac{1}{1+2\xi}\sum_{i=1}^{n} y_i^{(j)} y_i^{(k)} \frac{\ell'(\eta_i)^2}{\ell(\eta_i)^2}. \tag{24}$$

In the case of canonical link function $\ell = id$, we have

$$\mathcal{I}^{\xi}_{j,k}(\boldsymbol{\vartheta}) = -\mathcal{H}_{j,k}(\boldsymbol{\vartheta}) = \frac{1}{1+2\xi}\sum_{i=1}^{n} y_i^{(j)} y_i^{(k)} \frac{1}{\langle \boldsymbol{y}_i, \boldsymbol{\vartheta}\rangle^2}.$$

As soon as the first and second derivatives of ℓ are bounded and the second derivative is uniformly continuous and it asymptotically exists a consistent MLE estimator $\hat{\boldsymbol{\vartheta}}$ of $\boldsymbol{\vartheta}$ for $\xi = 0$, Fahrmeir and Kaufmann [1985, Theorem 4 and 5] imply than for $\xi > 0$ there asymptotically also is a consistent MLE estimator $\hat{\boldsymbol{\vartheta}}$ of $\boldsymbol{\vartheta}$. Indeed $\mathcal{I}^{\xi}(\boldsymbol{\vartheta}) = \frac{1}{1+2\xi}\mathcal{I}^{0}(\boldsymbol{\vartheta})$, where $\mathcal{I}^0$ is the Information matrix for the classical exponential distribution.

Generalized linear models are generally fitted using a Newton-type method, which reduces to an iteratively reweighted least square (IWLS) algorithm. We detail the derivation of this algorithm for the Generalized Pareto regression.

When minimizing the log-likelihood function $\log L(z, \boldsymbol{\vartheta})$ with respect to $\boldsymbol{\vartheta}$, the exact Newton method consists in computing the following iteration scheme (until convergence)

$$\boldsymbol{\vartheta}^{(r)} = \boldsymbol{\vartheta}^{(r-1)} - \left(\mathcal{H}(\boldsymbol{\vartheta}^{(r-1)})\right)^{-1} \boldsymbol{S}(\boldsymbol{\vartheta}^{(r-1)}), \tag{25}$$

where $\mathcal{H}$ and $\boldsymbol{S}$ denote the previous Hessian and score functions.

In Fisher-Scoring, since the score is centered, we do not use the exact random Hessian matrix but its expectation $E_{\vartheta}(\mathcal{H}(\boldsymbol{\vartheta}))$. The expected Hessian can be rewritten as $-E_{\vartheta}(\mathcal{H}(\boldsymbol{\theta})) = Y^T W(\boldsymbol{\vartheta}) Y$, with

$$Y = \begin{pmatrix} y_1^{(1)} & \dots & y_1^{(p)} \\ & \vdots & \\ y_n^{(1)} & \dots & y_n^{(p)} \end{pmatrix}, W = \begin{pmatrix} \frac{(\ell'(\eta_1))^2}{\ell^2(\eta_1)(1+2\xi)} & 0 & \dots \\ & \ddots & \\ \dots & 0 & \frac{(\ell'(\eta_n))^2}{\ell^2(\eta_n)(1+2\xi)} \end{pmatrix}.$$

Let w_{ij} be the general term of W. Rewriting S as

$$S_j(\boldsymbol{\vartheta}) = \sum_{i=1}^{n} y_i^{(j)} w_{ii} \frac{\ell^2(\eta_i)(1+2\xi)}{\ell'(\eta_i)} \left[z_i \gamma'(z_i \ell(\eta_i)) + \frac{1}{\ell(\eta_i)} \right],$$

The score can also be rewritten matricially as $\boldsymbol{S}(\boldsymbol{\vartheta}) = Y^T W(\boldsymbol{\vartheta}) \tilde{V}(\boldsymbol{\vartheta})$ with

$$\tilde{V} = \begin{pmatrix} \frac{\ell^2(\eta_1)(1+2\xi)}{\ell'(\eta_1)} \left[z_1 \gamma'(z_1 \ell(\eta_1)) + \frac{1}{\ell(\eta_1)} \right] \\ \vdots \\ \frac{\ell^2(\eta_n)(1+2\xi)}{\ell'(\eta_n)} \left[z_n \gamma'(z_n \ell(\eta_n)) + \frac{1}{\ell(\eta_n)} \right] \end{pmatrix}.$$

The Newton recurrence scheme (25) simplifies to

$$\begin{aligned}\boldsymbol{\vartheta}^{(r)} &= \boldsymbol{\vartheta}^{(r-1)} + \left(\mathcal{I}(\boldsymbol{\vartheta}^{(r-1)})\right)^{-1} \boldsymbol{S}\left(\boldsymbol{\vartheta}^{(r-1)}\right) \\ &= \boldsymbol{\vartheta}^{(r-1)} + \left(Y^T W\left(\boldsymbol{\vartheta}^{(r-1)}\right) Y\right)^{-1} Y^T W\left(\boldsymbol{\vartheta}^{(r-1)}\right) \tilde{V}\left(\boldsymbol{\vartheta}^{(r-1)}\right),\end{aligned}$$

This scheme can be reformulated as an IWLS algorithm

$$\boldsymbol{\vartheta}^{(r)} = \left(Y^T W\left(\boldsymbol{\vartheta}^{(r-1)}\right) Y\right)^{-1} Y^T W\left(\boldsymbol{\vartheta}^{(r-1)}\right) V\left(\boldsymbol{\vartheta}^{(r-1)}\right),$$

with $V(\boldsymbol{\vartheta}) = Y\boldsymbol{\vartheta} + \tilde{V}(\boldsymbol{\vartheta})$. W is called the working weight matrix and V the working response vector. Thus the fitting algorithm is

1. Initialization: $\boldsymbol{\vartheta}^{(0)} = (1, 0, \ldots, 0)$
2. Iteration: for $r = 1, \ldots, M$ do
   (a) Compute linear predictors $Y\boldsymbol{\vartheta}^{(r)} = (\eta_i^{(r)})_i$;
   (b) Compute working responses $V^{(r)} = (v_i)_i$
   (c) Compute working weights $W^{(r)} = \mathrm{diag}(w_1, \ldots, w_n)$
   (d) Solve the system $X^T W^{(r)} X \boldsymbol{\vartheta}^{(r+1)} = X^T W^{(r)} V^{(r)}$;
   (e) For a fixed small $\epsilon > 0$ and a distance d of $\mathbb{R}^p$, verify convergence of the sequence of estimators: $d(\boldsymbol{\vartheta}^{(r+1)}, \boldsymbol{\vartheta}^{(r)}) \leq \epsilon$.

B Expression of the moment of $\min(Y^{(2)}_{\alpha_2,1}, G)$

$$\begin{aligned}\mathbf{E}_{\boldsymbol{\vartheta}_2}(\min(Y^{(2)}_{\alpha_2,1}, G)) &= \mathbf{E}_{\boldsymbol{\vartheta}_2}(Y^{(2)}_{\alpha_2,1}\mathbf{1}(Y^{(2)}_{\alpha_2,1} \leq G)) + G\mathbf{E}_{\boldsymbol{\vartheta}_2}(\mathbf{1}(Y^{(2)}_{\alpha_2,1} \geq G)) \\ &= \int_{\mu}^{G} x f_{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)}(x)dx + G(1 - F_{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)}(G),\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}_{\boldsymbol{\vartheta}_2}((\min(Y^{(2)}_{\alpha_2,1}, G))^2) &= \mathbf{E}_{\boldsymbol{\vartheta}_2}((Y^{(2)}_{\alpha_2,1})^2\mathbf{1}(Y^{(2)}_{\alpha_2,1} \leq G)) + G^2\mathbf{E}_{\boldsymbol{\vartheta}_2}(\mathbf{1}(Y^{(2)}_{\alpha_2,1} \geq G) \\ &= \int_{\mu}^{G} x^2 f_{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)}(x)dx + G^2(1 - F_{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)}(G)),\end{aligned}$$

where f_β stands for the probability density function of the Generalized Pareto distribution defined in (20) and F_β his cumulative distribution function given by

$$F_\beta(x) = 1 - (1 + \beta\xi(x - \mu))^{-\frac{1}{\xi}} \mathbb{1}_{\{x \geq \mu\}}. \tag{26}$$

Direct computations lead to

$$\mathbf{E}_{\boldsymbol{\vartheta}_2}(\min(Y^{(2)}_{\alpha_2,1}, G)) = Gh_{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)}(G) + \mu - H_{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)}(G) + \frac{1}{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)(1 - \xi)} + G(1 - F_{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)}(G))$$

and

$$\mathbf{E}_{\boldsymbol{\vartheta}_2}((\min(Y^{(2)}_{\alpha_2,1}, G))^2) = G^2 h_{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)}(G) + \mu^2 - 2GH_{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)}(G) + 2\mu\frac{1}{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)(1-\xi)}$$
$$+ 2J_{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)}(G) - 2\frac{1}{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)^2(1-\xi)(2\xi-1)} + G^2(1 - F_{\beta_{\alpha_2}(\boldsymbol{\vartheta}_2)}(G)).$$

where

$$h_\beta(x) = -\left(1+\beta\xi(x-\mu)\right)^{-\frac{1}{\xi}} \quad \text{and} \quad H_\beta(x) = \frac{1}{\beta(1-\xi)}\left(1+\beta\xi(x-\mu)\right)^{\frac{\xi-1}{\xi}},$$

and

$$J_\beta(x) = \frac{1}{\beta^2(1-\xi)(2\xi-1)}\left(1+\beta\xi(x-\mu)\right)^{\frac{2\xi-1}{\xi}}.$$