Complete symmetry classification and compact matrix representations for 3D strain gradient elasticity

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COMPLETE SYMMETRY CLASSIFICATION AND COMPACT MATRIX REPRESENTATIONS FOR 3D STRAIN GRADIENT ELASTICITY

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Abstract. Strain Gradient Elasticity (SGE) is now often used in mechanics and physics, owing to its capability to model some non-classical phenomena, such as size effects, in materials and structures. However, certain fundamental questions about it have not yet received complete responses. In its linear setting, the constitutive law of SGE is characterized by a fourth-order elasticity tensor, a fifth-order one and a sixth-order one. Even if the matrix representations for the 3D fourth- and sixth-order elasticity tensors are available for all possible symmetry classes, the counterparts for the 3D fifth-order tensor, whose presence is unavoidable for materials with non-centrosymmetric microstructure, are still lacking. In addition, although the symmetry classes for each of the fourth-, fifth- and sixth-order elasticity tensor spaces are known, the symmetry classes for these tensor spaces as a whole have never been reported and clarified in the literature. The present work solves these two fundamental problems preventing the full understanding and exploitation of the linear constitutive law of SGE. Precisely, the matrix representations of the fifth-order tensor for all of its 29 symmetry classes are provided in a compact and well-structured way. Further, linear SGE is shown to possess 48 symmetry classes, and for each of these symmetry classes, the matrix representations of its fourth-, fifth- and sixth-order tensors are now available.

1. Introduction

In its classical (or standard) setting, continuum mechanics [Truesdell and Toupin, 1960, Truesdell and Noll, 1965] resorts only to the first displacement gradient in describing deformations. Even if this simple geometrical (or kinematical) framework is satisfactory in most situations of practical interest for classical bulk materials and at the usual engineering scale, its validity for emergent materials becomes questionable. For example, consequent electric polarization can be induced even in non-piezoelectric nanomaterials due to high strain-gradients. This phenomenon, known as flexoelectricity, cannot be modelled in the realm of classical continuum mechanics [Cross, 2006, Le Quang and He, 2011, Zubko et al., 2013, Yvonnet and Liu, 2017]. Another example concerns architectured materials whose microstructure spans several scales [Brechet and Embury, 2013]. This lack of scale separation prohibits the use of classical continuum mechanics in their overall macroscopic description. Last, but not least, specific microstructures can be designed to maximize higher order effects. The pantographic architecture is a well-known example of such a situation [dell'Isola and Steigmann, 2015, dell'Isola et al., 2016]. For the last decade, non-standard effects in architectured materials have been experimentally and numerically evidenced [Alibert et al., 2003, Liu et al., 2012, Liu and Hu, 2016, Bacigalupo and Gambarotta, 2014a,b, 2017, Rosi and Auffray, 2016, Poncelet et al., 2018, Reda et al., 2018]. Two examples of such non-classical effects occurring in architectured materials are illustrated on the figures below. The first example, investigated by Rosi and Auffray

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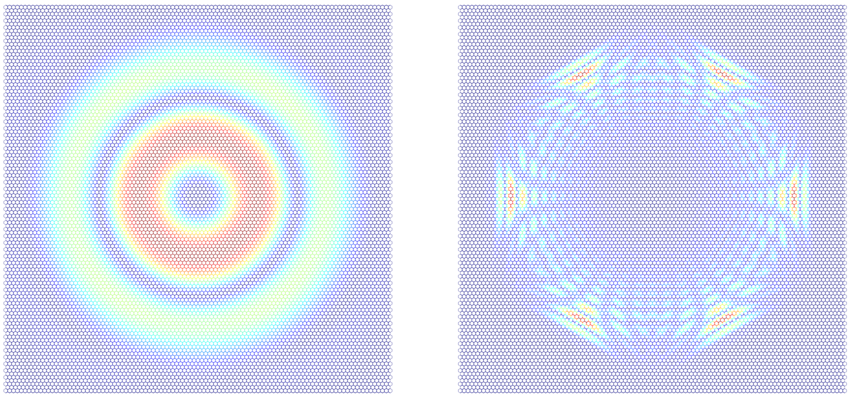
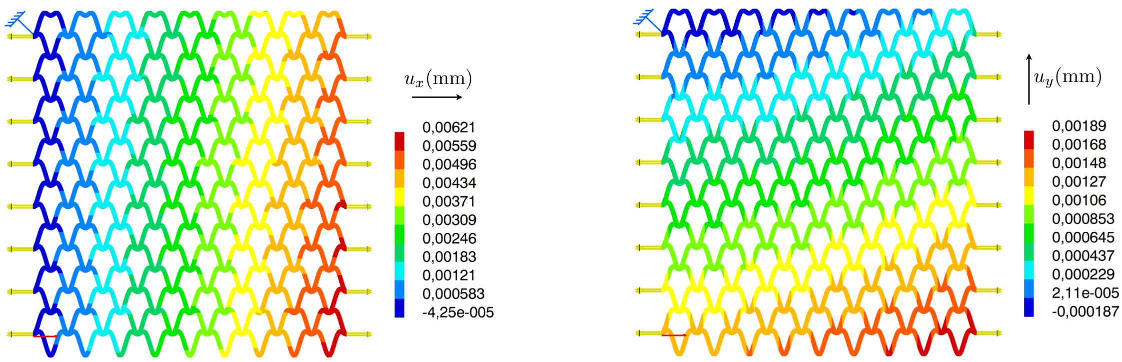
[2016], consists in the time domain analysis of the propagation of a shear pulses in a honeycomb lattice. On Figure 1 the total displacement is depicted for two pulse having different spectral contents. For a low frequency pulse, the propagation is isotropic (Figure 1a) while for a high frequency pulse the propagation becomes anisotropic (Figure 1b)



(A) Low Frequency pulse (B) High Frequency pulse

FIGURE 1. Propagation of a shear pulse in a honeycomb lattice. Total displacement is displayed.

The second example is a numerical experiement extracted from Poncelet et al. [2018]. In this experiment a sample having a non-centrosymmetric inner architecture is loaded in uniaxial tension. The resulting displacement fields are depicted on Figure 2a and Figure 2b. The result shows a flexure-like displacement superimposed to the classical elongation. Such a behaviour reveals that, due to the non-centro symmetric unit cell, a coupling between the stress and the strain-gradient takes place within the sample. The aforementioned two effects are scale-depend and tend to vanish as the unit cell size decreases with respect to the sample one.



(A) Displacement field u_x for Z_2^π pattern (B) Displacement field u_y for Z_2^π pattern

FIGURE 2. Displacement fields under 1 N uniform tensile force along x-direction for a non centrosymmetric pattern specimen.

For the continuum modelling of the aforementioned emergent materials, two options are at hand: (a) to remain in the realm of classical continuum mechanics by explicitly describing the complex microstructure; (b) to get rid of the microstructure by incorporating its effects into a non-standard continuum formulation. From the numerical point of view, and especially for optimization perspectives, the option (a) is often prohibitively expensive. The option (b), belonging to what is called *Generalized Continuum Mechanics* [Maugin, 2010], is the route here chosen because it is much less costly.

The first Strain-Gradient Elasticity (SGE) proposed by Mindlin [1964] is among the most important generalized continuum theories. In its linear setting, the infinitesimal strain tensor ε and its gradient $\eta = \varepsilon \otimes \nabla$ are linearly related to the second-order Cauchy stress tensor σ and the third-order hyperstress tensor τ by Equation 1 where a fourth-order tensor $\mathbf{C}$, a fifth-order tensor $\mathbf{M}$ and a sixth-order tensor $\mathbf{A}$ are involved and verify the index permutation symmetry properties specified in Equation 2. In the foregoing examples, the sixth-order tensor $\mathbf{A}$ is involved in the continuum description of the hexagonal wave propagation (c.f. Figure 1), while the fifth-order one $\mathbf{M}$ intervenes in the coupling between stress and strain-gradient (c.f. Figure 2).

The fourth-order tensor $\mathbf{C}$ defines the conventional elastic properties of a material. Its study had experienced a long history [Love, 1944] before a complete understanding was achieved quite recently [Boehler et al., 1994, Forte and Vianello, 1996, Olive et al., 2017]. Concerning the fifth- and sixth-order tensors, $\mathbf{M}$ and $\mathbf{A}$, their investigation were far from being complete. In addition, studies in strain-gradient elasticity had been almost exclusively focused on isotropic materials and their extension to anisotropic materials is quite recent [Auffray et al., 2009, 2013, 2015, Lazar and Po, 2015, Placidi et al., 2016, Mousavi et al., 2016, Yaghoubi et al., 2017, Reda et al., 2018]. First results were obtained in the 2D context by Auffray et al. [2009] who derived all anisotropic matrices of $\mathbf{A}$; these results were then extended to $\mathbf{M}$ [Auffray et al., 2015].

The 3D case is far more complex. A first result was given by dell'Isola et al. [2009] who provided an explicit matrix representation of the isotropic sixth-order tensor $\mathbf{A}$. Papanicolopulos [2011] investigated features of the tensor $\mathbf{M}$ with respect to the symmetry group SO(3) (hemitropic). Some analytical solutions to problems of hemitropic (SO(3)-invaraint) strain-gradient elasticity have been obtained by Ieşan [2013, 2014], Ieşan and Quintanilla [2016]. But the classification of anisotropic systems was absent. In Olive and Auffray [2013, 2014a], theoretical results about the number and types of symmetry classes for $\mathbf{A}$ and $\mathbf{M}$ in 3D were obtained. It is hence know that the space of sixth-order tensors $\mathbf{A}$ is divided into 17 different symmetry classes, while the space of fifth-order tensors $\mathbf{M}$ is partitioned into 29 different symmetry classes. The matrix representations of the sixth-order tensor $\mathbf{A}$ for all its symmetry classes were specified for the first time by Auffray et al. [2013] in a compact and well-structured way. These matrix representations have been shown to be useful for analysing the results of higher-order homogenization in architectured materials [Bacigalupo and Gambarotta, 2014a], understanding wave propagation in lattice materials [Bacigalupo and Gambarotta, 2014b, Rosi and Auffray, 2016] and, more recently, determining second order elasticity by molecular dynamics [Admal et al., 2017]. Note also that simplified anisotropic constitutive laws have recently been proposed in [Polizzotto, 2017, 2018].

In SGE, the fifth-order coupling tensor $\mathbf{M}$ of a material is null or non-null according as its microstructure is centro-symmetric or not [Lakes and Benedict, 1982, Lakes, 2001]. In the 3D case, the complete matrix representations of $\mathbf{M}$ are still unknown. In 3D, apart from the work of Papanicolopulos [2011] and Ieşan and Quintanilla [2016], little has been done about the strain-gradient elasticity of materials with non-centro-symmetric microstructure. However, as proved by Boutin [1996] via asymptotic analysis, the effect of $\mathbf{M}$ may be dominant over the one of $\mathbf{A}$. In statics, some recent experiments on an architectured beam [Poncelet et al., 2018] have evidenced the necessity of involving $\mathbf{M}$ in modelling its overall behaviour. Further, $\mathbf{M}$ is necessary to modelling interesting physical phenomena of materials with non-centro-symmetric microstructure. In dynamics, the coupling described by $\mathbf{M}$ induces changes in wave polarization and is associated to the acoustic activity of crystals and to so-called gyrotropic effects Toupin [1962], Portigal and Burstein [1968], Maranganti and Sharma [2007].

A comprehensive understanding of $\mathbf{C}$ (resp. $\mathbf{A}$) is available in the sense that the definite answers to the following three fundamental questions have been provided: (i) How many symmetry classes and which symmetry classes has $\mathbf{C}$ (resp. $\mathbf{A}$)? (ii) For every given symmetry class, how many independent material parameters has $\mathbf{C}$ (resp. $\mathbf{A}$)? (iii) For each given symmetry class, what is the explicit matrix form of $\mathbf{C}$ (resp. $\mathbf{A}$) relative to an orthonormal basis? At the present time, answers to questions (i) and (ii) concerning the static $\mathbf{M}$ tensor and its dynamic counterpart $\mathbf{M}^{\star}$ are available while the explicit matrix representations of $\mathbf{M}$ and $\mathbf{M}^{\star}$ have not been provided so far. For practical applications, the last lacking result is the most important one.

In the present work, attention will be focused on studying the fifth-order coupling tensor $\mathbf{M}$ in the context of strain-gradient elastostatics. Questions related to its dynamic counterpart $\mathbf{M}^{\star}$ constitute an independent study and the answers to them are postponed to another contribution. The present paper aims at obtaining the explicit matrix representations of $\mathbf{M}$ for all its 29 symmetry classes in a compact and well-structured way. As will be seen, the complexity and richness of $\mathbf{M}$ make that a proper solution to this problem is not straightforward at all.

The matrices obtained in the present work for the fifth-order tensor $\mathbf{M}$ complement those for the classical fourth-order tensor $\mathbf{C}$ and the sixth-order tensor $\mathbf{A}$ by Auffray et al. [2013]. The combination of these different results leads to the important conclusion that there are exactly 48 different symmetry classes for 3D gradient-strain elasticity. This number means that SGE is much more complex and richer than the classical elasticity which has only 8 classes of symmetry. Finally, the results of this work allow us to have compact and well-structured matrix representations of $\mathbf{C}$, $\mathbf{M}$ and $\mathbf{A}$ for each of the possible symmetry classes.

The next sections of the paper are organized as follows. In section 2, the constitutive law of strain gradient elasticity is recalled. The section 3 is devoted to introducing the most important concepts concerning symmetry classes in O(3) and to recapitulating theorems about the symmetry classes of $\mathbf{M}$ obtained by Olive and Auffray [2014a]. The main results of the present work are given in section 4 and section 5. In section 4, the explicit matrix representations of $\mathbf{M}$ for its 29 symmetry classes are provided in a compact and well-structured form. The matrix representations of $\mathbf{M}$ are presented in such a manner that they can be directly used without resorting to group theory. In section 5, the results obtained in the present paper are combined with those of Auffray et al. [2013] for the sixth-order elasticity tensor $\mathbf{A}$. It follows from this combination that there exists 48 symmetry classes in O(3) for linear strain gradient elasticity. In section 6, a few concluding remarks are drawn.

There may be two ways to read this paper. If one likes to know not only the main results of the paper but also how to obtain them, the whole paper should be read. If one likes just to know and use the main results of the paper but is not necessarily interested in the way of obtaining them, section 3 can be skipped, apart from the symmetry class notations and the simple geometrical figures giving a physical interpretation of them, subsection 4.1 and all the appendices can also skipped.

1.1. **Notations.** Throughout this paper, the Euclidean space $\mathcal{E}^3$ is equipped with a Cartesian coordinates system associated to an orthonormal basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Concerning notation, the following convention is retained:

- Blackboard fonts will denote tensor spaces : $\mathbb{T}$;
- Tensors of order > 1 will be denoted using uppercase Roman Bold fonts : $\mathbf{T}$;
- Vectors will be denoted by lowercase Roman Bold fonts : $\mathbf{t}$.

The following matrix spaces will be used:

- $\mathcal{M}(n)$ is the n^2-D space of n dimensional square matrices ;
- $\mathcal{M}(n,m)$ is the nm-D space of $n \times m$ rectangular matrices.

The orthogonal group in $\mathbb{R}^3$ is defined as $\mathrm{O}(3) = \{\mathbf{Q} \in \mathrm{GL}(3) | \mathbf{Q}^T = \mathbf{Q}^{-1}\}$, in which GL(3) denotes the set of invertible transformations acting on $\mathbb{R}^3$. The following elements of O(3) will be used in this study:

- $\mathbf{Q}$ a generic orthogonal transformation;
- $\mathbf{R}(\mathbf{v}; \theta) \in \mathrm{O}(3)$ the rotation about $\mathbf{v} \in \mathbb{R}^2$ through an angle $\theta \in [0; 2\pi)$;
- $\mathbf{P}_\mathbf{n} \in \mathrm{O}(3)\backslash\mathrm{SO}(3)$ the reflection through the line normal to $\mathbf{n}$ ($\mathbf{P}_\mathbf{n} = \mathbb{1} - 2\mathbf{n} \otimes \mathbf{n}$).

2. Strain-Gradient Elasticity

In this section, the constitutive law of Strain Gradient Elasticity (SGE) is introduced. In this introduction neither the momentum equation, nor the boundary conditions will be detailed. The readers interested in these points can refer to the following references for a more detailed presentation [Mindlin, 1964, Mindlin and Eshel, 1968, Germain, 1973, dell'Isola et al., 2009, Bertram, 2016].

In SGE the constitutive law gives the symmetric Cauchy stress tensor σ and the hyperstress tensor τ in terms of:

- the infinitesimal strain tensor ε;
- the strain-gradient tensor $\eta = \varepsilon \otimes \nabla$ which, using index notation, reads $\eta_{ijk} = \varepsilon_{ij,k}$ with the comma denoting derivation.

Recall that ε and σ are second order symmetric tensors, while η and τ are third order tensors symmetrical with respect to permutation of their first two indices. The constitutive law is specified by

$$\begin{cases} \sigma_{ij} = C_{ijlm}\varepsilon_{lm} + M_{ijlmn}\eta_{lmn}, \\ \tau_{ijk} = M_{lmijk}\varepsilon_{lm} + A_{ijklmn}\eta_{lmn}. \end{cases} \tag{1}$$

Above,

- $\mathbf{C}$ is the classical fourth-order elastic tensor;
- $\mathbf{M}$ is the fifth-order coupling elastic (CE) tensor;
- $\mathbf{A}$ is the sixth-order elastic (SOE) tensor.

These tensors satisfy the following index permutation symmetries:

$$C_{\underline{(ij)}\ \underline{(lm)}} \quad ; \quad M_{(ij)(kl)m} \quad ; \quad A_{\underline{(ij)k}\ \underline{(lm)n}} \tag{2}$$

where (..) stands for the minor symmetries whereas $\underline{..}$ symbolises the major one. The associated vector spaces are defined accordingly:

$$\mathbb{E}\mathrm{la} := \{\mathbf{C} \in \otimes^4(\mathbb{R}^3) | C_{\underline{(ij)}\ \underline{(lm)}}\}\ ,\ \mathbb{M} := \{\mathbf{M} \in \otimes^5(\mathbb{R}^3) | M_{(ij)(kl)m}\}\ ,\ \mathbb{A} := \{\mathbf{A} \in \otimes^6(\mathbb{R}^3) | A_{\underline{(ij)k}\ \underline{(lm)n}}\}$$

As a consequence the strain gradient elastic behaviour is characterized by a triplet of tensors

$$\mathcal{L} = (C_{ijkl}, M_{ijklm}, A_{ijklmn}) \in \mathbb{E}\mathrm{la} \times \mathbb{M} \times \mathbb{A}.$$

The space of strain gradient elastic tensors is hence defined as

$$\mathbb{S}\mathrm{gr} = \mathbb{E}\mathrm{la} \times \mathbb{M} \times \mathbb{A}.$$

Until now, $\mathbb{E}\mathrm{la}$ and $\mathbb{A}$, the vector spaces of $\mathbf{C}$ and $\mathbf{A}$, have been investigated, both in the 2D and 3D cases [Mehrabadi and Cowin, 1990, Forte and Vianello, 1996, Auffray et al., 2009, 2013]. The answers to the following three questions have been provided:
(a) How many symmetry classes and which symmetry classes do $\mathbb{E}\mathrm{la}$ and $\mathbb{A}$ have?
(b) For every given symmetry class, how many independent material parameters do $\mathbb{E}\mathrm{la}$ and $\mathbb{A}$ possess?
(c) For each given symmetry class, what are the explicit matrix forms of $\mathbf{C}$ and $\mathbf{A}$ relative to an adapted orthonormal basis?

In the 2D case, for $\mathbb{E}\mathrm{la}$, He and Zheng [1996] demonstrated that the space of classical fourth-order tensors is partitioned into 4 classes. This result was also obtained in a different way by Vianello [1997]. For $\mathbb{A}$ the question was solved in 2D by Auffray et al. [2009], the space of sixth-order tensors is more complex since it is divided into 8 classes. The space $\mathbb{M}$ has been studied recently in 2D [Auffray et al., 2015, 2016], and shown to be divided into 6 classes.

In the 3D situation, the number of symmetry classes increases importantly since $\mathbb{E}\mathrm{la}$ is now divided into 8 classes [Forte and Vianello, 1996], and $\mathbb{A}$ into 17 classes [Olive and Auffray, 2013, Auffray et al., 2013]. At the present time, these questions remain open for the fifth-order tensor space in 3D. Some theoretical results are available [Olive and Auffray, 2014b, Auffray, 2013], but without explicit construction. In order to have a complete anisotropic SGE theory, answering the aforementioned three questions for $\mathbb{M}$ is indispensable.

3. Symmetry classes

3.1. **Material symmetry & Physical symmetry.** Let consider a body as a compact subset $\mathcal{D}_0$ of $\mathcal{E}^3$ having a microstructure $\mathcal{M}$ attached to any of its material points $P \in \mathcal{D}_0$. Those points are located with respect to a reference frame $(\mathcal{R})$. The microstructure describes the local organisation of the matter at scales below the one used for the continuous description (see Figure 3). As discussed in section 2 the elastic behaviour is assumed to be described at the macrolevel by a SGE constitutive law formulated by Equation 1.

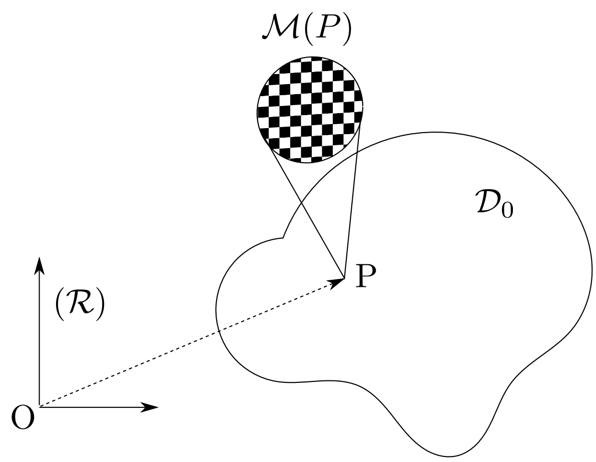


Figure 3. What is hidden below a material point

As for crystals, microstructures can possess invariance properties with respect to orthogonal transformations $\mathbf{Q} \in \mathrm{O}(3)$. Hence at each material point P, the set of such transformations forms a point group $\mathrm{G}_{\mathcal{M}(P)} \subseteq \mathrm{O}(3)$ which describes the local material symmetries, formally

$$\mathrm{G}_{\mathcal{M}(P)} := \{\mathbf{Q} \in \mathrm{O}(3), \quad \mathbf{Q} \cdot \mathcal{M}(P) = \mathcal{M}(P)\}$$

At the continuous macroscopic scale the detailed description of the microstructure is lost, and information on the microstructure is contained in $\mathrm{G}_{\mathcal{M}}(P)$. In the case of an homogeneous medium the point dependence vanishes and $\mathrm{G}_{\mathcal{M}(P)} = \mathrm{G}_{\mathcal{M}}$. This hypothesis of material homogeneity will be assumed for the rest of the paper.

Linear constitutive laws are encoded by tensors. For strain-gradient linear elasticity, the behaviour is described by a triplet of tensors $\mathcal{L} = (\mathbf{C}, \mathbf{M}, \mathbf{A})$. As the material element is transformed by a punctual isometry $\mathbf{Q}$, the physical property of the strain gradient elastic material is defined by another triplet $\mathcal{L}^{\star} = (\mathbf{C}^{\star}, \mathbf{M}^{\star}, \mathbf{A}^{\star})$. The link between the two set of tensors is given by:

$$\begin{cases} C^{\star}_{ijkl} = Q_{io}Q_{jp}Q_{kq}Q_{lr}C_{opqr}; \\ M^{\star}_{ijklm} = Q_{io}Q_{jp}Q_{kq}Q_{lr}Q_{ms}M_{opqrs}; \\ A^{\star}_{ijklmn} = Q_{io}Q_{jp}Q_{kq}Q_{lr}Q_{ms}Q_{nt}A_{opqrst}. \end{cases}$$

The notion of physical symmetry group has to be introduced. The symmetry group of $\mathcal{L}$ is defined as[1]:

$$\mathrm{G}_{\mathcal{L}} = \mathrm{G}_{\mathbf{A}} \cap \mathrm{G}_{\mathbf{M}} \cap \mathrm{G}_{\mathbf{C}}. \tag{3}$$

In which

$$\begin{cases} \mathrm{G}_{\mathbf{C}} := \{\mathbf{Q} \in \mathrm{O}(3) | Q_{io}Q_{jp}Q_{kq}Q_{lr}C_{opqr} = C_{ijkl}\}; \\ \mathrm{G}_{\mathbf{M}} := \{\mathbf{Q} \in \mathrm{O}(3) | Q_{io}Q_{jp}Q_{kq}Q_{lr}Q_{ms}M_{opqrs} = M_{ijklm}\}; \\ \mathrm{G}_{\mathbf{A}} := \{\mathbf{Q} \in \mathrm{O}(3) | Q_{io}Q_{jp}Q_{kq}Q_{lr}Q_{ns}Q_{mt}A_{opqrst} = A_{ijklmn}\}. \end{cases}$$

The link between these two notions is given by the Curie principle which states that the material symmetry group (cause) is included in the physical symmetry group (consequence):

$$\mathrm{G}_{\mathcal{M}} \subseteq \mathrm{G}_{\mathbf{C}}$$

More details concerning this principle can be be found in Zheng and Boehler [1994].

3.2. **Symmetry group and symmetry class.** Let $\mathbf{Q}$ be an element of the 3D orthogonal group $\mathrm{O}(3)$. A fifth-order tensor $\mathbf{M}$ is said to be invariant under the action of $\mathbf{Q}$ if and only if

$$Q_{io}Q_{jp}Q_{kq}Q_{lr}Q_{ms}M_{opqrs} = M_{ijklm}. \tag{4}$$

The symmetry group of $\mathbf{M}$ is defined as the subgroup $\mathrm{G}_{\mathbf{M}}$ of $\mathrm{O}(3)$ constituted of all the orthogonal transformations leaving $\mathbf{M}$ invariant[2]

$$\mathrm{G}_{\mathbf{M}} = \{\mathbf{Q} \in \mathrm{O}(3) \, | Q_{io}Q_{jp}Q_{kq}Q_{lr}Q_{ms}M_{opqrs} = M_{ijklm}\}. \tag{5}$$

Physically, the operations contained in $\mathrm{O}(3)$ are:

[1]It should be noted that an apparently different definition is provided for the symmetry group of a strain-gradient material by Bertram [2016]. Their definition is, in fact, more general since isochoric transformations preserving the elastic energy are considered. But, once restricted to isometric transformations, their definition coincides with ours.

[2]In odd dimension, the inversion belongs to the symmetry group of all even-order tensors, hence the problem can be reduced to $\mathrm{SO}(3)$. This is why in [Forte and Vianello, 1996, Auffray et al., 2013] the classification of symmetry has been made with respect to $\mathrm{SO}(3)$. In the present situation the full orthogonal group has to be considered. The link between the two classifications is as follows, if $[H]$ is the symmetry class of an even order tensor with respect to $\mathrm{SO}(3)$, its symmetry class with respect to $\mathrm{O}(3)$ will be $[H \oplus \mathrm{Z}_2^c]$, with Z_2^c denoting the group associated to the inversion operation.

- rotations (proper transformations of determinant 1), such elements form a subgroup $\mathrm{SO}(3)$ of the full orthogonal group;
- mirrors and inversion (improper transformations of determinant -1).

As proposed by Forte and Vianello [1996] two fifth-order tensors $\mathbf{M}$ and $\mathbf{N}$ exhibits symmetries of the same kind if and only if their symmetry groups are conjugate in the sense that

$$\exists \mathbf{Q} \in \mathrm{O}(3), \mathrm{G}_{\mathbf{N}} = \mathbf{Q}\mathrm{G}_{\mathbf{M}}\mathbf{Q}^T. \tag{6}$$

Thus, the symmetry class $[\mathrm{G}_{\mathbf{M}}]$ of $\mathbf{M}$ corresponding to the conjugacy class of $\mathrm{G}_{\mathbf{M}}$ in $\mathrm{O}(3)$ is defined by

$$[\mathrm{G}_{\mathbf{M}}] = \{\mathrm{G} \subseteq \mathrm{O}(3) | \mathrm{G} = \mathbf{Q}\mathrm{G}_{\mathbf{M}}\mathbf{Q}^T,\ \mathbf{Q} \in \mathrm{O}(3)\}. \tag{7}$$

In other words, the symmetry class of $\mathbf{M}$ corresponds to its symmetry group modulo its orientation. This idea is illustrated Figure 4.

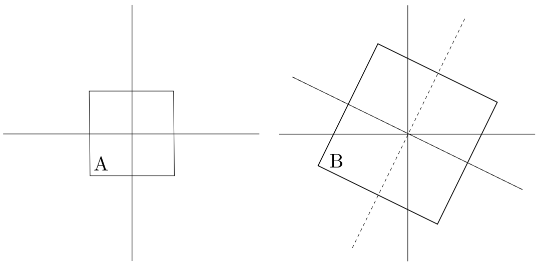


FIGURE 4. Figures A and B have different but conjugate symmetry groups. Hence they belong to the same symmetry class.

To carry out the classification of odd order tensors, we should, in a first time, describe the $\mathrm{O}(3)$-closed subgroups. Throughout this paper, mathematical group notations will be used; the equivalence between this system and the classical crystallographic ones (Hermann-Mauguin, Schoenflies) is recapitulated in Appendix B.

3.3. $\mathrm{O}(3)$**-closed subgroups.** Classification of $\mathrm{O}(3)$-closed subgroups is a classical theorem that can be found in many references (see, e.g., [Ihrig and Golubitsky, 1984, Sternberg, 1994]):

Lemma 3.1. *Every closed subgroup of* $\mathrm{O}(3)$ *is conjugate to one group of the following list, which has been divided into three classes:*

I. *Closed subgroups of* $\mathrm{SO}(3)$.
II. $\tilde{K} := K \oplus \mathrm{Z}_2^c$, *where* K *is a closed subgroup of* $\mathrm{SO}(3)$ *and* $\mathrm{Z}_2^c = \{\mathbb{1}, -\mathbb{1}\}$;
III. *Closed subgroups neither comprising* $-\mathbb{1}$ *nor contained in* $\mathrm{SO}(3)$.

Above, $\mathbb{1}$ *denotes the identity transformation, and* $-\mathbb{1}$ *stands for the inversion transformation with respect to the origin.*

In words, the subgroups of Type I consist of only rotations while those of Type II contain in addition the inversion (or central symmetry)[3]. The type III subgroups contain symmetry planes but theirs combinations do not generate the inversion[4]. Before detailing the structure of classes, let us make a terminologic remark. A subgroup will be said to be

Centrosymmetric: if it contains the inversion; hence the subgroups of Type II subgroups are centrosymmetric;

[3]Type II subgroups also contain symmetry planes, but their constitutive feature is to possess the inversion.

[4]As soon a group possesses three mutual orthogonal symmetry planes it possesses the inversion.

Chiral: if all its operations are orientation-preserving; hence those of Type I are chiral;

Polar: if it contains a single rotational axis; polar subgroups can be found in Type I and Type III subgroups.

These characteristics can be combined, and the different situations are summed up on Figure 5. In Appendix B, the characteristics of each crystallographic subgroup of O(3) are detailed.

Remark 3.2. To understand precisely what is meant by chirality it is important to distinguish between :

(1) Physical rigid transformations of a body. As clever as he is, an experimenter can only act on a specimen through rotation, its action belongs to SO(3).
(2) Mathematical rigid transformations of a body. Mathematical transformations can also involve improper operations, the action belongs to O(3).

Consider a body Ω. This object will be said achiral if its image under any mathematical transformation can be realized by a physical transformation. If this property failed, Ω will be said to be chiral. It can be proved that Ω is chiral if and only if $G_{\mathbf{M}}$ is conjugate to a subgroup of Type I. Further it should be noted that the *chirality* of an object is not intrinsic and depends on the dimension of the space in which this object is embedded.

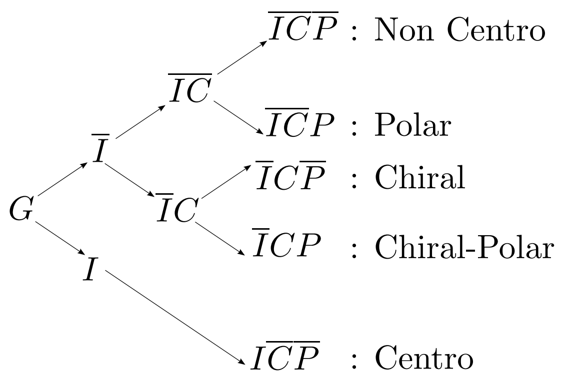


FIGURE 5. Physical properties of O(3) subgroups. I, C and P indicate, respectively, the Centro-symmetric, Chiral and Polar properties of a group G. An overline indicates that the associated property is lacking.

Subgroups of Type I. As previously said, they contain nothing else than rotations. Hence these subgroups are chiral subgroups of O(3). Among them we can distinguish:

Planar groups: : $\{\mathbb{1}, \mathrm{Z}_{n\geq 2}, \mathrm{D}_{n\geq 2}, \mathrm{SO}(2), \mathrm{O}(2)\}$, which are O(2)-closed subgroups;

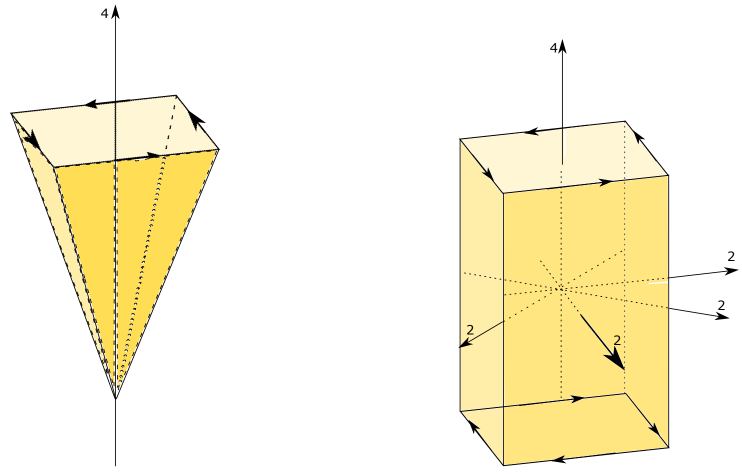
Exceptional groups: : $\{\mathcal{T}, \mathcal{O}, \mathcal{I}, \mathrm{SO}(3)\}$, which are symmetry groups of chiral Platonic polyhedrons completed by the rotation group of the sphere.

Let us first detail the set of planar subgroups:

- $\mathbb{1}$, the identity;
- Z_n $(n \geq 2)$, the cyclic group of order n, generated by the n-fold rotation $\mathbf{R}\left(\mathbf{e}_3; \theta = \dfrac{2\pi}{n}\right)$. Figures having these symmetry groups are said to be polar;
- D_n $(n \geq 2)$, the dihedral group of order $2n$ generated by Z_n and $\mathbf{R}(\mathbf{e}_1; \pi)$. Figures having these symmetry groups are non-polar;
- SO(2), the subgroup of rotations $\mathbf{R}(\mathbf{e}_3; \theta)$ with $\theta \in [0; 2\pi)$;
- O(2), the subgroup generated by SO(2) and $\mathbf{R}(\mathbf{e}_1; \pi)$.

The classes of exceptional subgroups are: $\mathcal{T}$ the tetrahedral group of order 12 which fixes a tetrahedron, $\mathcal{O}$ the octahedral group of order 24 which fixes an octahedron (or a cube), and $\mathcal{I}$ the subgroup of order 60 which fixes an icosahedron (or a dodecahedron). Exceptional subgroups are non-polar. On Figure 6, examples of Z_4- and D_4-invariant objects are provided. On these figures:

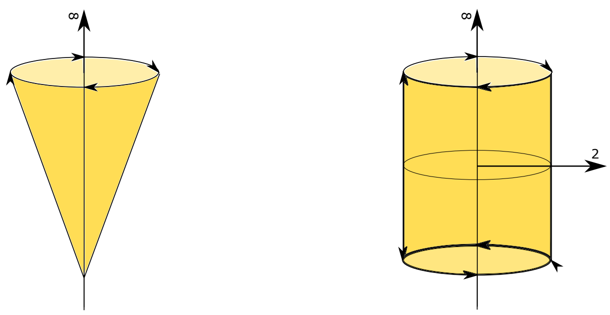
- the order of the invariance is indicated on the rotational axis (depicted with an arrow);
- arrows drawn on the figures indicate the *spin* of the object. This spin is due to the lack of mirror symmetry and, in these cases, associated to chirality;
- the difference between a polar group (Z_4) and a non-polar (D_4) is clear. In the polar situation there is an orientation along the axis of rotation.



(A) (B)

FIGURE 6. Different invariant figures of Type I: (A) is Z_4-invariant, while (B) is D_4-invariant. Arrows labelled with a small n indicate a n-fold axis of rotation.

We also consider the infinite subgroups associated to the former examples:



(A) (B)

FIGURE 7. Different invariant figures of Type I: (A) is SO(2)-invariant, while (B) is O(2)-invariant.

The invariance SO(2) corresponds to the symmetry group of a helix. Depending on its spin, this helix can be right- or left-handed. If a rotation in the plane of order 2 is added to the set of generators, we obtain the group O(2), which is, in particular, the symmetry group of the torsion loading.

Subgroups of Type II. These subgroups are the centro-symmetric subgroups of $O(3)$.

The subgroups of type $Z_n \oplus Z_2^c$ posses a plane of symmetry normal to the axis of the generator of Z_n. As illustrated Figure 8 (A), a figure having invariance of this type can possess an overall spin. It should be noted that this does not make those subgroups chiral. Considering Figure 8 (A), its image by a mirror operation, lets say $\mathbf{P}_{\mathbf{e}_2}$, has its overall spin reversed, but, up to a rotation this image can be superposed to the original. Hence the figure is not chiral.

Remark 3.3. In the literature of lattice materials, and especially when investigating their phononic (dynamics) or auxetic (statics) properties, *tetrachiral* or *hexachiral* patterns are studied. The term *chiral* in their denomination refers to in-plane chirality. The 3D chiral nature of the pattern depends on how the 3D structure is made out from the bidimensional one. If the volume is extruded along a direction normal to the 2D plane (which is generally the case) the ponctual group of the resulting lattice will, respectively, be $Z_4 \oplus Z_2^c$ or $Z_6 \oplus Z_2^c$ for a *tetrachiral* or a *hexachiral* materials. Hence, usually *tetrachiral* or *hexachiral* materials are, in fact, 3D achiral.

The subgroups of type $D_n \oplus Z_2^c$ are invariant with respect to mirror symmetries. This property is due to the fact that:

$$\mathbf{R}(\mathbf{e}_1;\pi) \cdot -\mathbb{1} = -\mathbb{1} \cdot \mathbf{R}(\mathbf{e}_1;\pi) = \mathbf{P}_{\mathbf{e}_1}.$$

Symmetry planes (in solid lines and without arrow) are indicated on the example in Figure 8 (B) .

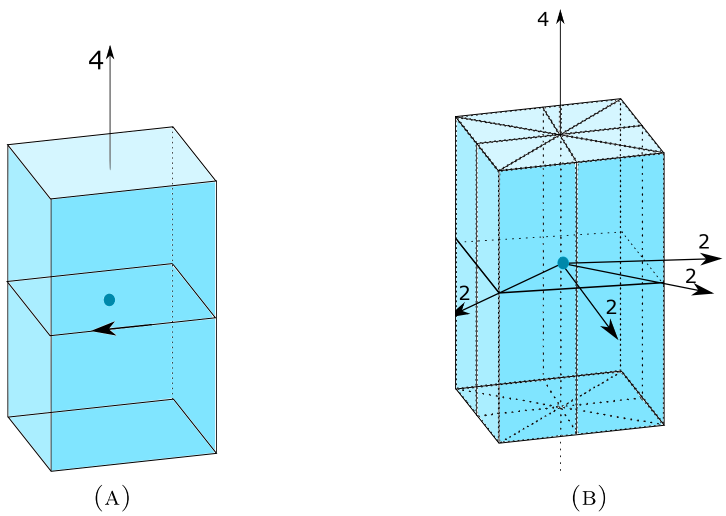
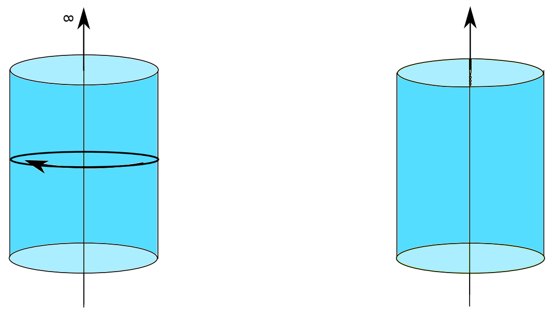


FIGURE 8. Different invariant figures of Type II: (A) is $Z_4 \oplus Z_2^c$-invariant, while (B) is $D_4 \oplus Z_2^c$-invariant. The central inversion is indicated by a dot.

Let us also consider the infinite subgroups associated to the former examples:



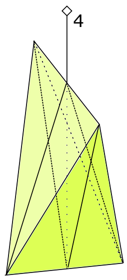
(A) (B)

FIGURE 9. Different invariant figures of Type II: (A) is $\mathrm{SO}(2) \oplus \mathrm{Z}_2^c$-invariant, while (B) is $\mathrm{O}(2) \oplus \mathrm{Z}_2^c$-invariant.

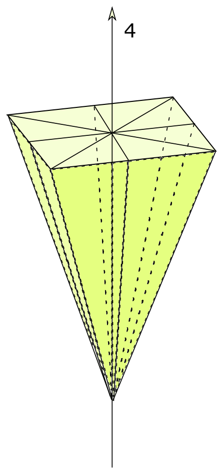
Subgroups of Type III. The construction of these subgroups is more involved, and a short description of their structure is provided in [Sternberg, 1994, Olive and Auffray, 2014a]. Basically these groups are not centrosymmetric but contain mirror inversions with respect to some planes. As a consequence those groups are not chiral.

The collection in question is detailed as follows:

- Z_2^- is the reflection group of order 2 generated by $\mathbf{P_n}$;
- Z_{2n}^- $(n \geq 2)$ is the group of order $2n$, generated by the $2n$-fold rotoreflection $\mathbf{R}\left(\mathbf{e}_3; \theta = \dfrac{\pi}{n}\right) \cdot \mathbf{P}_{\mathbf{e}_3}$;
- D_{2n}^h $(n \geq 2)$ represents the prismatic group of order $4n$ generated by Z_{2n}^- and $\mathbf{R}(\mathbf{e}_1, \pi)$. In the denomination h indicates the presence of horizontal mirrors. When n is odd it is the symmetry group of a regular prism, and when n is even it is the symmetry group of a regular antiprism;
- D_n^v $(n \geq 2)$ denotes the pyramidal group of order $2n$ generated by Z_n and $\mathbf{P}_{\mathbf{e}_1}$, which is the symmetry group of a regular pyramid. In the denomination v indicates the presence of vertical mirrors. Those groups are polar ;
- $\mathrm{O}(2)^-$ symbolizes the limit group of D_n^v for continuous rotation, which is therefore generated by $\mathbf{R}(\mathbf{e}_3; \theta)$ and $\mathbf{P}_{\mathbf{e}_1}$. It is the symmetry group of a cone;



(A)



(B)

Figure 10. Different invariant figures of Type III: (A) is Z_4^--invariant figure (which is not a regular tetrahedron due to having edges with different lengths), while (B) is D_4^v-invariant. As clear from the figure, D_4^v is polar. The diamond shape indicates an axis of rotoinversion.

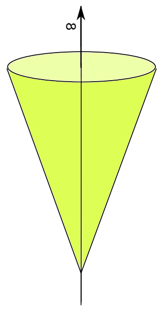


Figure 11. $O(2)^-$-invariant figure

These planar subgroups are completed by the achiral tetrahedral symmetry $\mathcal{O}^-$ which is of order 24. This group has the same rotation axes as $\mathcal{T}$, but with six mirror planes, each through two 3-fold axes.

Remark 3.4. It can be observed that odd-order tensors that are invariant with respect to the inversion are null. The symmetry class of a null tensor is $[O(3)]$. Hence non-vanishing odd-order tensors have their symmetry group conjugate to a closed subgroup of type I or III. In the literature these situations are sometimes referred to as being chiral. This denomination is not correct owing to the fact that subgroups of type III are non chiral. The coupling in SGE is due to the lack of centrosymmetry.

3.4. **Symmetry classes.** Let $\mathfrak{I}(\mathbb{T})$ denote the set of symmetry classes of the space $\mathbb{T}$. Using theorems established in Olive and Auffray [2014a] we have the following result:

Theorem 3.5. *The space* $\mathbb{M}$ *is partitioned into 29 symmetry classes:*

$$\begin{aligned}\mathfrak{I}(\mathbb{M}) \quad = \quad & \{[\mathbb{1}], [Z_2], \cdots, [Z_5], [D_2^v], \cdots, [D_5^v], [Z_2^-], [Z_4^-], \cdots, [Z_8^-], [D_2], \cdots, [D_5], \\ & [D_4^h], [D_6^h], \cdots, [D_{10}^h], [SO(2)], [O(2)], [O(2)^-], [\mathcal{T}], [\mathcal{O}^-], [\mathcal{O}], [SO(3)], [O(3)]\}\end{aligned}$$

This result motivates two comments:

(1) In the former result the class $[\mathrm{Z}_{10}^-]$ is missing. Indeed this means that there exists a rotation that brings any Z_{10}^--invariant tensor into a D_{10}^h-invariant one. A similar situation was yet observed for $\mathbb{E}\mathrm{la}$ where, for the same reason, the classes $[\mathrm{Z}_3]$ and $[\mathrm{Z}_4]$ are empty.

(2) The number of symmetry classes of $\mathbf{M}$ has to be compared to the one of $\mathbf{A}$. In the first case we have a fifth-order tensor and 29 classes, while for the latter, which is a sixth-order one, there are "only" 17 classes. The large number of classes is due to the oddity of the tensor which increases by far the complexity.

From the harmonic structure[5] of $\mathbb{M}$, the number of independent components in each symmetry class can be determined using trace formula [Auffray, 2014]:

Name	Triclinic	Monoclinic	Trigonal	Tetragonal	Pentagonal	∞-gonal
$[\mathrm{G}_\mathbb{M}]$	$\mathbb{1}$	$[\mathrm{Z}_2]$	$[\mathrm{Z}_3]$	$[\mathrm{Z}_4]$	$[\mathrm{Z}_5]$	$[\mathrm{SO}(2)]$
$\#_{\mathrm{indep}}(\mathbb{M})$	108	52	36	26	22	20
$[\mathrm{G}_\mathbb{M}]$		$[\mathrm{D}_2^v]$	$[\mathrm{D}_3^v]$	$[\mathrm{D}_4^v]$	$[\mathrm{D}_5^v]$	$[\mathrm{O}^-(2)]$
$\#_{\mathrm{indep}}(\mathbb{M})$		28	20	15	13	12
$[\mathrm{G}_\mathbb{M}]$	$[\mathrm{Z}_2^-]$	$[\mathrm{Z}_4^-]$	$[\mathrm{Z}_6^-]$	$[\mathrm{Z}_8^-]$		$[\mathrm{O}(3)]$
$\#_{\mathrm{indep}}(\mathbb{M})$	56	26	16	6		0
$[\mathrm{G}_\mathbb{M}]$		$[\mathrm{D}_4^h]$	$[\mathrm{D}_6^h]$	$[\mathrm{D}_8^h]$	$[\mathrm{D}_{10}^h]$	$[\mathrm{O}(3)]$
$\#_{\mathrm{indep}}(\mathbb{M})$		13	8	3	1	0
$[\mathrm{G}_\mathbb{M}]$		$[\mathrm{D}_2]$	$[\mathrm{D}_3]$	$[\mathrm{D}_4]$	$[\mathrm{D}_5]$	$[\mathrm{O}(2)]$
$\#_{\mathrm{indep}}(\mathbb{M})$		24	16	11	9	8
$[\mathrm{G}_\mathbb{M}]$		$[\mathcal{T}]$	$[\mathcal{O}^-]$	$[\mathcal{O}]$		$[\mathrm{SO}(3)]$
$\#_{\mathrm{indep}}(\mathbb{M})$		8	5	3		1

Table 1. The names, the sets of subgroups $[\mathrm{G}_\mathbb{M}]$ and the numbers of independent components $\#_{\mathrm{indep}}(\mathbb{M})$ for the symmetry classes of $\mathbb{M}$.

4. Matrix representations of fifth-order elasticity tensors

The goal of the present section is to determine, for each symmetry class, the *explicit matrix form* of $\mathbf{M} \in \mathbb{M}$ relative to an appropriate orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. To achieve this aim we follow a strategy introduced for classical elasticity by Mehrabadi and Cowin [1990] and extended to strain-gradient elasticity in [Auffray et al., 2009] and [Auffray et al., 2013]. This approach is summarized hereafter.

4.1. **Fixed point set.** Recall first that for any subgroup H of $\mathrm{O}(3)$ we can define the associated fixed point set as

$$\mathrm{Fix}(H) := \{\mathbf{M} \in \mathbb{M}, \quad \mathbf{Q} \star \mathbf{M} = \mathbf{M}, \quad \forall \mathbf{Q} \in H\}$$

so that for all $\mathbf{M} \in \mathrm{Fix}(H)$ the symmetry group $\mathrm{G}_\mathbf{M}$ contains H. In the following an element $\mathbf{M} \in \mathrm{Fix}(H)$ will be denoted $\mathbf{M}_H$.

Remark 4.1. The fact that $\mathrm{Fix}(H) \neq \{0\}$ for some subgroup H does not mean that $[H]$ is the symmetry class of some tensor $\mathbf{M}$. For instance we have $\mathrm{Fix}(\mathrm{Z}_{10}^-) \neq \{0\}$ but if $\mathbf{M} \neq \mathbf{0}$ in $\mathrm{Fix}(\mathrm{Z}_{10}^-)$ then $\mathrm{G}_\mathbf{M}$ is conjugate to D_{10}^h.

[5] It can be established that

$$\mathbb{M} \simeq \mathbb{H}^5 \oplus 2\mathbb{H}^{4,\sharp} \oplus 5\mathbb{H}^3 \oplus 5\mathbb{H}^{2,\sharp} \oplus 6\mathbb{H}^1 \oplus \mathbb{H}^{0,\sharp}$$

In which $\mathbb{H}^k$, and $\mathbb{H}^{k,\sharp}$ indicates the space of kth-order harmonic tensor endowed respectively with the standard action and the sign action. Further details can be found in Olive and Auffray [2014a].

To obtain the normal forms for the different classes the generators provided in the following table have been used :

Group	Generators
Z_2^-	$\mathbf{P}_{\mathbf{e}_3}$
Z_n	$\mathbf{R}\left(\mathbf{e}_3; \frac{2\pi}{n}\right)$
D_n	$\mathbf{R}\left(\mathbf{e}_3; \frac{2\pi}{n}\right),\ \mathbf{R}(\mathbf{e}_1; \pi)$
$\mathrm{Z}_{2n}^-,\ n \geq 2$	$\mathbf{R}\left(\mathbf{e}_3; \frac{\pi}{n}\right) \cdot \mathbf{P}_{\mathbf{e}_3}$
$\mathrm{D}_{2n}^h\ n \geq 2$	$\mathbf{R}\left(\mathbf{e}_3; \frac{\pi}{n}\right) \cdot \mathbf{P}_{\mathbf{e}_3},\ \mathbf{R}(\mathbf{e}_1, \pi)$
D_n^v	$\mathbf{R}\left(\mathbf{e}_3; \frac{2\pi}{n}\right),\ \mathbf{P}_{\mathbf{e}_1}$
$\mathcal{T}$	$\mathbf{R}(\mathbf{e}_3; \pi),\ \mathbf{R}(\mathbf{e}_1; \pi),\ \mathbf{R}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3; \frac{2\pi}{3})$
$\mathcal{O}$	$\mathbf{R}(\mathbf{e}_3; \frac{\pi}{2}),\ \mathbf{R}(\mathbf{e}_1; \pi),\ \mathbf{R}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3; \frac{2\pi}{3})$
$\mathcal{O}^-$	$\mathbf{R}(\mathbf{e}_3; \pi) \cdot \mathbf{P}_{\mathbf{e}_3},\ \mathbf{P}_{\mathbf{e}_2+\mathbf{e}_3}$

TABLE 2. The set of group generators used to construct matrix representation for each symmetry class.

The choice of the generators indicated in (2) has been made in order to have the following relation:

$$\mathbb{M} = \mathrm{Fix}(\mathrm{D}_2) \oplus \mathrm{Fix}(\mathrm{D}_2^v) \oplus \mathrm{Fix}(\mathrm{Z}_2^-)$$

This relation means that any triclinic tensors $\mathbf{M}$ is the sum of three tensors $\mathbf{M}$ of higher symmetry. The consequence of this decomposition is used in the following to provide results in a condensed form.

4.2. **Orthonormal basis and matrix component ordering.** Let be defined the following spaces:

$$\mathbb{T}_{(ij)} = \{\mathrm{T} \in \otimes^2(\mathbb{R}^3) | T_{(ij)}\} \quad ; \quad \mathbb{T}_{(ij)k} = \{\mathrm{T} \in \otimes^3(\mathbb{R}^3) | T_{(ij)k}\}$$

which are, in 3D, respectively, 6- and 18-dimensional vector spaces. Therefore

- the fourth-order elasticity tensor $\mathbf{C}$ is a self-adjoint endomorphism of $\mathbb{T}_{(ij)}$;
- the fifth-order coupling elasticity tensor $\mathbf{M}$ is a linear application from $\mathbb{T}_{(ij)k}$ to $\mathbb{T}_{(ij)}$;
- the sixth-order elasticity tensor $\mathbf{A}$ is a self-adjoint endomorphism of $\mathbb{T}_{(ij)k}$.

In order to express the Cauchy-stress tensor σ, the strain tensor ε, the strain-gradient tensor η and the hyperstress tensor τ as 6- and 18-dimensional vectors and write $\mathbf{C}$, $\mathbf{M}$ and $\mathbf{A}$ as, respectively: a 6×6, 6×18 and 18×18 matrices, we introduce the following orthonormal basis vectors:

$$\begin{aligned}
\widetilde{\mathbf{e}}_I &= \left(\frac{1-\delta_{ij}}{\sqrt{2}} + \frac{\delta_{ij}}{2}\right)(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i), \quad 1 \leq I \leq 6, \\
\widehat{\mathbf{e}}_\alpha &= \left(\frac{1-\delta_{ij}}{\sqrt{2}} + \frac{\delta_{ij}}{2}\right)(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i) \otimes \mathbf{e}_k, \quad 1 \leq \alpha \leq 18,
\end{aligned}$$

where the summation convention for a repeated subscript does not apply. Then, the aforementioned tensors can be expressed as:

$$\widetilde{\varepsilon} = \sum_{I=1}^{6} \widetilde{\varepsilon}_I \widetilde{\mathbf{e}}_I, \quad \widetilde{\sigma} = \sum_{I=1}^{6} \widetilde{\sigma}_I \widetilde{\mathbf{e}}_I, \quad \widehat{\eta} = \sum_{\alpha=1}^{18} \widehat{\eta}_\alpha \widehat{\mathbf{e}}_\alpha, \quad \widehat{\tau} = \sum_{\alpha=1}^{18} \widehat{\tau}_\alpha \widehat{\mathbf{e}}_\alpha \tag{8}$$

$$\widetilde{\mathbf{C}} = \sum_{I,J=1,1}^{6,6} \widetilde{C}_{IJ}\widetilde{\mathbf{e}}_I \otimes \widetilde{\mathbf{e}}_\mathbf{J} \quad \overline{\mathbf{M}} = \sum_{I,\alpha=1,1}^{6,18} \overline{M}_{I\alpha}\widetilde{\mathbf{e}}_I \otimes \widehat{\mathbf{e}}_\alpha, \quad \widehat{\mathbf{A}} = \sum_{\alpha,\beta=1,1}^{18,18} \widehat{A}_{\alpha\beta}\widehat{\mathbf{e}}_\alpha \otimes \widehat{\mathbf{e}}_\beta, \tag{9}$$

so that the relations in (1) can be written in the matrix form

$$\begin{cases} \widetilde{\sigma}_I = \widetilde{C}_{IJ}\widetilde{\varepsilon}_J + \overline{M}_{I\alpha}\widehat{\eta}_\alpha, \\ \widehat{\tau}_\alpha = \overline{M}_{\alpha J}\widetilde{\varepsilon}_J + \widetilde{A}_{\alpha\beta}\widehat{\eta}_\beta. \end{cases} \tag{10}$$

The relationships between the matrix components $\widetilde{\varepsilon}_I$ and ε_{ij}, and between $\widehat{\eta}_\alpha$ and η_{ijk} are

$$\widetilde{\varepsilon}_I = \begin{cases} \varepsilon_{ij} & \text{if } \ i = j, \\ \sqrt{2}\varepsilon_{ij} & \text{if } \ i \neq j; \end{cases} \qquad \widehat{\eta}_\alpha = \begin{cases} \eta_{ijk} & \text{if } \ i = j, \\ \sqrt{2}\eta_{ijk} & \text{if } \ i \neq j; \end{cases} \tag{11}$$

and, obviously, the same relations between $\widetilde{\sigma}_I$ and σ_{ij} and $\widehat{\tau}_\alpha$ and τ_{ijk} hold. For the constitutive tensors we have the following correspondences:

$$\widetilde{C}_{IJ} = \begin{cases} C_{ijkl} & \text{if } \ i = j \text{ and } k = l, \\ \sqrt{2}C_{ijkl} & \text{if } \ i \neq j \text{ and } k = l \text{ or } i = j \text{ and } k \neq l, \\ 2C_{ijkl} & \text{if } \ i \neq j \text{ and } k \neq l; \end{cases} \tag{12}$$

$$\overline{M}_{I\alpha} = \begin{cases} M_{ijklm} & \text{if } \ i = j \text{ and } k = l, \\ \sqrt{2}M_{ijklm} & \text{if } \ i \neq j \text{ and } k = l \text{ or } i = j \text{ and } k \neq l, \\ 2M_{ijklm} & \text{if } \ i \neq j \text{ and } k \neq l; \end{cases} \tag{13}$$

$$\widehat{A}_{\alpha\beta} = \begin{cases} A_{ijklmn} & \text{if } \ i = j \text{ and } l = m, \\ \sqrt{2}A_{ijklmn} & \text{if } \ i \neq j \text{ and } l = m \text{ or } i = j \text{ and } l \neq m, \\ 2A_{ijklmn} & \text{if } \ i \neq j \text{ and } l \neq m. \end{cases} \tag{14}$$

It remains to choose appropriate two-to-one and three-to-one subscript correspondences between ij and I, on one hand, and ijk and α, on the other hand. For the classical variables the standard two-to-one subscript correspondence is used, i.e:

I or J	1	2	3	4	5	6
ij	11	22	33	23	13	12

TABLE 3. The two-to-one subscript correspondences for 3D strain/stress tensors

The three-to-one subscript correspondence for strain-gradient/hyperstress tensor, specified in Table 5, is adopted so as to make the 6th-order tensor $\mathbf{A}$ block-diagonal for dihedral classes[6].

[6]Further comments on the reason of such a choice can be found in Auffray et al. [2013].

α or β	1	2	3	4	5	Type of mechanism
ijk	111	221	122	331	133	X-Interaction
α or β	6	7	8	9	10	
ijk	222	112	121	332	233	Y-Interaction
α or β	11	12	13	14	15	
ijk	333	113	131	223	232	Z-Interaction
α	16	17	18			
ijk	231	132	123			Coupling

TABLE 4. The three-to-one subscript correspondences for 3D strain-gradient/hyperstress tensors

The matrix representations of first- and second-order elasticity tensors have already been investigated. Hence, in the remaining subsection, attention will be limited to the tensor $\mathbf{M}$.

4.3. **Transformation matrix.** Using the introduced orthogonal bases and the subscript correspondences, the action of an orthogonal tensor $\mathbf{Q} \in \mathrm{O}(3)$ on $\mathbf{M}$ can be represented by two different matrices: a 6×6 matrix $\widetilde{\mathbf{Q}}$, and a 18×18 matrix $\widehat{\mathbf{Q}}$ in a way such that

$$Q_{io}Q_{jp}Q_{kq}Q_{lr}Q_{ms}M_{opqrs} = \widetilde{Q}_{IJ}\overline{M}_{J\alpha}\widehat{Q}_{\alpha\beta} \tag{15}$$

where

$$\widetilde{Q}_{IJ} = \frac{1}{2}(Q_{io}Q_{jp} + Q_{ip}Q_{jo}) \quad ; \quad \widehat{Q}_{\alpha\beta} = \frac{1}{2}(Q_{io}Q_{jp} + Q_{ip}Q_{jo})Q_{kq} \tag{16}$$

with I and J being associated to ij and op, and α and β being associated to ijk and opq respectively. Thus, formula (4) expressing the invariance of $\mathbf{M}$ under the action of $\mathbf{Q}$ is equivalent to

$$\widetilde{\mathbf{Q}}\overline{\mathbf{M}}\widehat{\mathbf{Q}}^T = \overline{\mathbf{M}} \tag{17}$$

where $\overline{\mathbf{M}}$ stands for the 6×18 matrix of components $\overline{M}_{J\alpha}$.

The matrices provided hereafter are obtained by solving the linear system (17) for the generators associated to the symmetry classes previously identified (c.f. Theorem 3.5).

4.4. **Matrix representations for all symmetry classes.**

4.4.1. *Class symmetry characterized by* $\mathbb{1}$. In this case, the material in question is totally anisotropic and the SGE matrix $\overline{\mathbf{M}}$ comprises 108 independent components. The explicit expression of $\overline{\mathbf{M}}$ as a full 6×18 matrix is

$$\overline{\mathrm{M}}_{\mathbf{I}} = \begin{pmatrix} A^{(15)} & B^{(15)} & C^{(15)} & D^{(9)} \\ E^{(5)} & F^{(5)} & G^{(5)} & H^{(3)} \\ I^{(5)} & J^{(5)} & K^{(5)} & L^{(3)} \\ M^{(5)} & N^{(5)} & O^{(5)} & P^{(3)} \end{pmatrix}$$

To lighten the notations, in the following, the overline will be abandoned. The elementary matrices are generic elements of the following spaces:

- $A^{(15)},\ B^{(15)},\ C^{(15)} \in \mathcal{M}(3,5)$;
- $D^{(9)} \in \mathcal{M}(3)$;
- $E^{(5)},\ F^{(5)},\ G^{(5)},\ I^{(5)},\ J^{(5)},\ K^{(5)},\ L^{(5)},\ M^{(5)},\ N^{(5)},\ O^{(5)} \in \mathcal{M}(1,5)$;
- $H^{(3)},\ L^{(3)},\ P^{(3)} \in \mathcal{M}(1,3)$.

4.4.2. *Symmetry classes* $[\mathrm{Z}_n], [\mathrm{D}_n], [\mathrm{D}_n^v]$. Classes $[\mathrm{D}_n]$ are chiral, classes $[\mathrm{D}_n^v]$ are polar while classes $[\mathrm{Z}_n]$ has these two features. This fact is reflected, with the chosen generators, by the relation :

$$\mathrm{Fix}(\mathrm{Z}_n) = \mathrm{Fix}(\mathrm{D}_n) \oplus \mathrm{Fix}(\mathrm{D}_n^v) \tag{18}$$

So only the matrices of $\mathbf{M}_{\mathrm{D}_n}$ and $\mathbf{M}_{\mathrm{D}_n^v}$ will be detailed, the remaining one being deduced from the above relation.

Symmetry classes $[\mathrm{Z}_2], [\mathrm{D}_2], [\mathrm{D}_2^v]$.

$$\mathrm{M}_{\mathrm{D}_2} = \begin{pmatrix} 0 & 0 & 0 & D^{(9)} \\ E^{(5)} & 0 & 0 & 0 \\ 0 & J^{(5)} & 0 & 0 \\ 0 & 0 & O^{(5)} & 0 \end{pmatrix}, \quad \mathrm{M}_{\mathrm{D}_2^v} = \begin{pmatrix} 0 & 0 & C^{(15)} & 0 \\ 0 & F^{(5)} & 0 & 0 \\ I^{(5)} & 0 & 0 & 0 \\ 0 & 0 & 0 & P^{(3)} \end{pmatrix}$$

$$\mathbf{M}_{\mathrm{Z}_2} = \mathbf{M}_{\mathrm{D}_2} + \mathbf{M}_{\mathrm{D}_2^v}$$

It should be emphasized that these matrices are specific to the particular choice of symmetry elements (generators) specified in Table 2. A different choice of generators will change the location of the non-zero components.

Remark 4.2. It appears clearly that matrices $\mathrm{M}_{\mathrm{D}_2}$ and $\mathrm{M}_{\mathrm{D}_2^v}$ can be made block diagonal and block anti-diagonal by choosing a three-to-one subscript correspondence that differs from the one indicated in Table 4. But doing so the well-structured shape of matrices representing the sixth-order elasticity tensor $\mathbf{A}$ is lost [Auffray et al., 2013]. In the present paper, and in order to easily combining the new results with those already obtained in Auffray et al. [2013] we choose to keep the convention defined and used in this former publication. An alternative subscript correspondence can be used but, in this case, the matrices provided in Auffray et al. [2013] have to be permuted to be consistent.

Symmetry classes $[\mathrm{Z}_3], [\mathrm{D}_3], [\mathrm{D}_3^v]$.

$$\mathrm{M}_{\mathrm{D}_3} = \begin{pmatrix} A^{(6)} & 0 & 0 & D^{(4)} \\ E^{(4)} & 0 & 0 & H^{(2)} \\ 0 & -E^{(4)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & f(H^{(2)}) & 0 \\ 0 & f(A^{(6)}) & f(D^{(4)}) & 0 \end{pmatrix}$$

$$\mathrm{M}_{\mathrm{D}_3^v} = \begin{pmatrix} 0 & B^{(6)} & C^{(8)} & 0 \\ 0 & F^{(4)} & 0 & 0 \\ F^{(4)} & 0 & 0 & L^{(2)} \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -f(L^{(2)}) & 0 \\ 0 & 0 & 0 & 0 \\ -f(B^{(6)}) & 0 & 0 & f(C^{(8)}) \end{pmatrix}$$

$$\mathbf{M}_{\mathrm{Z}_3} = \mathbf{M}_{\mathrm{D}_3} + \mathbf{M}_{\mathrm{D}_3^v}$$

Symmetry classes $[\mathrm{Z}_4], [\mathrm{D}_4], [\mathrm{D}_4^v]$.

$$\mathrm{M}_{\mathrm{D}_4} = \begin{pmatrix} 0 & 0 & 0 & D^{(4)} \\ E^{(5)} & 0 & 0 & 0 \\ 0 & -E^{(5)} & 0 & 0 \\ 0 & 0 & O^{(2)} & 0 \end{pmatrix}, \quad \mathrm{M}_{\mathrm{D}_4^v} = \begin{pmatrix} 0 & 0 & C^{(8)} & 0 \\ 0 & F^{(5)} & 0 & 0 \\ F^{(5)} & 0 & 0 & 0 \\ 0 & 0 & 0 & P^{(2)} \end{pmatrix}$$

$$\mathbf{M}_{\mathrm{Z}_4} = \mathbf{M}_{\mathrm{D}_4} + \mathbf{M}_{\mathrm{D}_4^v}$$

Symmetry classes $[\mathrm{Z}_5], [\mathrm{D}_5], [\mathrm{D}_5^v]$.

$$\mathrm{M}_{\mathrm{D}_5} = \begin{pmatrix} A^{(1)} & 0 & 0 & D^{(4)} \\ E^{(4)} & 0 & 0 & 0 \\ 0 & -E^{(4)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & f(A^{(1)}) & f(D^{(4)}) & 0 \end{pmatrix}$$

$$\mathrm{M}_{\mathrm{D}_5^v} = \begin{pmatrix} 0 & B^{(1)} & C^{(8)} & 0 \\ 0 & F^{(4)} & 0 & 0 \\ F^{(4)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f(B^{(1)}) & 0 & 0 & f(C^{(8)}) \end{pmatrix}$$

$$\mathbf{M}_{\mathrm{Z}_5} = \mathbf{M}_{\mathrm{D}_5} + \mathbf{M}_{\mathrm{D}_5^v}$$

Symmetry classes $[\mathrm{SO}(2)], [\mathrm{O}(2)], [\mathrm{O}^-(2)]$.

$$\mathrm{M}_{\mathrm{O}(2)} = \begin{pmatrix} 0 & 0 & 0 & D^{(4)} \\ E^{(4)} & 0 & 0 & 0 \\ 0 & -E^{(4)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & f(D^{(4)}) & 0 \end{pmatrix}$$

$$\mathrm{M}_{\mathrm{O}^-(2)} = \begin{pmatrix} 0 & 0 & C^{(8)} & 0 \\ 0 & F^{(4)} & 0 & 0 \\ F^{(4)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f(C^{(8)}) \end{pmatrix}$$

$$\mathbf{M}_{\mathrm{SO}(2)} = \mathbf{M}_{\mathrm{O}(2)} + \mathbf{M}_{\mathrm{O}^-(2)}$$

4.4.3. *Symmetry Classes* $[\mathrm{Z}_{2p}^-], [\mathrm{D}_{2p}^h]$.

Symmetry Classes $[\mathrm{Z}_2^-]$.

$$\mathrm{M}_{\mathrm{Z}_2^-} = \begin{pmatrix} A^{(15)} & B^{(15)} & 0 & 0 \\ 0 & 0 & G^{(5)} & H^{(3)} \\ 0 & 0 & K^{(5)} & L^{(3)} \\ M^{(5)} & N^{(5)} & 0 & 0 \end{pmatrix}$$

Symmetry Classes $[\mathrm{Z}_4^-], [\mathrm{D}_4^h]$.

$$\mathrm{M}_{\mathrm{Z}_4^-} = \begin{pmatrix} 0 & 0 & C^{(7)} & D^{(5)} \\ E^{(5)} & F^{(5)} & 0 & 0 \\ -F^{(5)} & E^{(5)} & 0 & 0 \\ 0 & 0 & O^{(3)} & P^{(1)} \end{pmatrix}, \ \mathrm{M}_{\mathrm{D}_4^h} = \begin{pmatrix} 0 & 0 & C^{(7)} & 0 \\ 0 & F^{(5)} & 0 & 0 \\ -F^{(5)} & 0 & 0 & 0 \\ 0 & 0 & 0 & P^{(1)} \end{pmatrix}$$

$$\mathrm{M}_{\mathrm{Z}_4^-} = \mathrm{M}_{\mathrm{D}_4^h} + \begin{pmatrix} 0 & 0 & 0 & D^{(5)} \\ E^{(5)} & 0 & 0 & 0 \\ 0 & E^{(5)} & 0 & 0 \\ 0 & 0 & O^{(3)} & 0 \end{pmatrix}$$

Symmetry Classes $[\mathrm{Z}_6^-], [\mathrm{D}_6^h]$.

$$\mathrm{M}_{\mathrm{D}_6^h} = \begin{pmatrix} A^{(6)} & 0 & 0 & 0 \\ 0 & 0 & 0 & H^{(2)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & f(H^{(2)}) & 0 \\ 0 & f(A^{(6)}) & 0 & 0 \end{pmatrix}$$

$$\mathrm{M}_{\mathrm{Z}_6^-} = \mathrm{M}_{\mathrm{D}_6^h} + \begin{pmatrix} 0 & B^{(6)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L^{(2)} \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -f(L^{(2)}) & 0 \\ 0 & 0 & 0 & 0 \\ -f(B^{(6)}) & 0 & 0 & 0 \end{pmatrix}$$

Symmetry Classes $[\mathrm{Z}_8^-], [\mathrm{D}_8^h]$.

$$\mathrm{M}_{\mathrm{D}_8^h} = \begin{pmatrix} 0 & 0 & 0 & D^{(2)} \\ E^{(1)} & 0 & 0 & 0 \\ 0 & -E^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & f(D^{(2)}) & 0 \end{pmatrix}$$

$$\mathrm{M}_{\mathrm{Z}_8^-} = \mathrm{M}_{\mathrm{D}_8^h} + \begin{pmatrix} 0 & 0 & C^{(2)} & 0 \\ 0 & F^{(1)} & 0 & 0 \\ F^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f(C^{(2)}) \end{pmatrix}$$

Symmetry Class $[\mathrm{D}_{10}^h]$.

$$\mathrm{M}_{\mathrm{D}_{10}^h} = \begin{pmatrix} A^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & f(A^{(1)}) & 0 & 0 \end{pmatrix}$$

4.4.4. *Symmetry classes* $[\mathcal{T}], [\mathcal{O}], [\mathcal{O}^-]$.

$$\mathrm{M}_{\mathcal{T}} = \begin{pmatrix} 0 & 0 & 0 & D^{(3)} \\ O^{(5)} & 0 & 0 & 0 \\ 0 & O^{(5)} & 0 & 0 \\ 0 & 0 & O^{(5)} & 0 \end{pmatrix}, \ \mathrm{M}_{\mathcal{O}^-} = \begin{pmatrix} 0 & 0 & 0 & D_2^{(2)} \\ O^{(3)} & 0 & 0 & 0 \\ 0 & O^{(3)} & 0 & 0 \\ 0 & 0 & O^{(3)} & 0 \end{pmatrix}, \ \mathrm{M}_{\mathcal{O}} = \begin{pmatrix} 0 & 0 & 0 & D^{(1)} \\ O^{(2)} & 0 & 0 & 0 \\ 0 & -O^{(2)} & 0 & 0 \\ 0 & 0 & O^{(2)} & 0 \end{pmatrix}$$

4.4.5. *Symmetry class* $[\mathrm{SO}(3)]$.

$$\mathrm{M}_{\mathrm{SO}(3)} = \begin{pmatrix} 0 & 0 & 0 & D^{(1)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ f(D^{(1)}) & 0 & 0 & 0 \\ 0 & -f(D^{(1)}) & 0 & 0 \\ 0 & 0 & f(D^{(1)}) & 0 \end{pmatrix}$$

Independent parameter matrices:

$$A^{(6)}, B^{(6)} \in \begin{pmatrix} a_{11} & a_{12} & \frac{\sqrt{2}}{2}(a_{11}+a_{12}+2a_{22}) & a_{14} & a_{15} \\ -(a_{11}+a_{12}+a_{22}) & a_{22} & -\frac{\sqrt{2}}{2}(a_{11}-a_{22}) & -a_{14} & -a_{15} \\ -\frac{\sqrt{2}}{2}a_{33} & \frac{\sqrt{2}}{2}a_{33} & a_{33} & 0 & 0 \end{pmatrix};$$

$$A^{(1)}, B^{(1)} \in \begin{pmatrix} a_{11} & -a_{11} & -\sqrt{2}a_{11} & 0 & 0 \\ -a_{11} & a_{11} & \sqrt{2}a_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C^{(8)} \in \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{22} & c_{23} \\ c_{11} & c_{22} & c_{23} & c_{12} & c_{13} \\ c_{31} & c_{32} & c_{33} & c_{32} & c_{33} \end{pmatrix} ; \ C^{(7)} \in \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ -c_{11} & -c_{14} & -c_{15} & -c_{12} & -c_{13} \\ 0 & c_{32} & c_{33} & -c_{32} & -c_{33} \end{pmatrix}$$

$$C^{(2)} \in \begin{pmatrix} 0 & c_{12} & c_{13} & -c_{12} & -c_{13} \\ 0 & -c_{12} & -c_{13} & c_{12} & c_{13} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D^{(5)} \in \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{11} & d_{13} \\ d_{31} & d_{31} & d_{33} \end{pmatrix} \;;\; D^{(4)} \in \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ -d_{12} & -d_{11} & -d_{13} \\ d_{31} & -d_{31} & 0 \end{pmatrix} \;;\; D^{(3)} \in \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{13} & d_{11} & d_{12} \\ d_{12} & d_{13} & d_{11} \end{pmatrix}$$

$$D^{(2)} \in \begin{pmatrix} d_{11} & d_{11} & d_{13} \\ -d_{11} & -d_{11} & -d_{13} \\ 0 & 0 & 0 \end{pmatrix} \;;\; D_2^{(2)} \in \begin{pmatrix} d_{11} & d_{12} & d_{12} \\ d_{12} & d_{11} & d_{12} \\ d_{12} & d_{12} & d_{11} \end{pmatrix} \;;\; D^{(1)} \in \begin{pmatrix} 0 & d_{12} & -d_{12} \\ -d_{12} & 0 & d_{12} \\ d_{12} & -d_{12} & 0 \end{pmatrix}$$

$$E^{(4)}, F^{(4)} \in \begin{pmatrix} e_{11} & e_{12} & \frac{\sqrt{2}}{2}(e_{11}-e_{12}) & e_{14} & e_{15} \end{pmatrix} \;;\; E^{(1)}, F^{(1)} \in \begin{pmatrix} e_{11} & -e_{11} & -\sqrt{2}e_{11} & 0 & 0 \end{pmatrix}$$

$$H^{(2)}, L^{(2)}, P^{(2)} \in \begin{pmatrix} h_{11} & h_{11} & h_{13} \end{pmatrix}$$

$$O^{(3)} \in \begin{pmatrix} o_{11} & o_{12} & o_{13} & o_{12} & o_{13} \end{pmatrix} \;;\; O^{(2)} \in \begin{pmatrix} 0 & o_{12} & o_{13} & -o_{12} & -o_{13} \end{pmatrix}$$

Non-independent parameter matrices:

$$f(A^{(6)}), f(B^{(6)}) = \begin{pmatrix} -\frac{\sqrt{2}}{2}(2a_{11}+a_{12}+a_{22}) & -\frac{\sqrt{2}}{2}(a_{12}-a_{22}) & -(a_{11}+a_{22}) & -\sqrt{2}a_{14} & -\sqrt{2}a_{15} \end{pmatrix}$$

$$f(A^{(1)}), f(B^{(1)}) \in \begin{pmatrix} \sqrt{2}a_{11} & -\sqrt{2}a_{11} & 2a_{11} & 0 & 0 \end{pmatrix}$$

$$f(C^{(2)}) \in \begin{pmatrix} -\sqrt{2}c_{13} & -\sqrt{2}c_{13} & -2c_{12} \end{pmatrix}$$

$$f(C^{(8)}) = \begin{pmatrix} \frac{\sqrt{2}}{2}(c_{13}-c_{23}) & \frac{\sqrt{2}}{2}(c_{13}-c_{23}) & c_{12}-c_{22} \end{pmatrix}$$

$$f(D^{(2)}) \in \begin{pmatrix} 0 & d_{13} & \sqrt{2}d_{11} & -d_{13} & -\sqrt{2}d_{11} \end{pmatrix}$$

$$f(D^{(4)}) = \begin{pmatrix} 0 & -d_{13} & -\frac{\sqrt{2}}{2}(d_{11}+d_{12}) & d_{13} & \frac{\sqrt{2}}{2}(d_{11}+d_{12}) \end{pmatrix}$$

$$f(D^{(1)}) \in \begin{pmatrix} 0 & d_{12} & -\frac{\sqrt{2}}{2}d_{12} & -d_{12} & \frac{\sqrt{2}}{2}d_{12} \end{pmatrix}$$

$$f(H^{(2)}), f(L^{(2)}) = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2}h_{13} & -h_{12} & \frac{\sqrt{2}}{2}h_{13} & h_{12} \end{pmatrix}$$

5. Complete symmetry classes

This last section is devoted to combining the classifications of $\mathbf{C}$, $\mathbf{M}$ and $\mathbf{A}$ so as to obtain all the symmetry classes of linear SGE. In section 3 the symmetry group of $\mathcal{L}$ had been defined as:

$$\mathrm{G}_{\mathcal{L}} = \mathrm{G}_{\mathbf{A}} \cap \mathrm{G}_{\mathbf{M}} \cap \mathrm{G}_{\mathbf{C}}$$

and the corresponding symmetry class is defined as

$$[\mathrm{G}_{\mathcal{L}}] = \{\mathrm{G} \subseteq \mathrm{O}(3) | \mathrm{G} = \mathbf{Q}\mathrm{G}_{\mathcal{L}}\mathbf{Q}^T, \mathbf{Q} \in \mathrm{O}(3)\}. \tag{19}$$

The symmetry class of a strain gradient material is conjugate to a closed subgroup of O(3).

The classifications of $\mathbf{C}$ and $\mathbf{A}$, which are even-order tensors, have been made with respect to SO(3), while that of $\mathbf{M}$, which is an odd-order tensor, was done relative to O(3). To propose a global classification for $\mathcal{L}$ these different classifications have to be unified. The way to connect these results is detailed in Appendix A, and the results are summarized in the following table:

Symmetry class for the constitutive law	Symmetry class for the even-order tensors
$[\mathrm{Z}_n]$	$[\mathrm{Z}_n \oplus \mathrm{Z}_2^c]$
$[\mathrm{D}_n]$	$[\mathrm{D}_n \oplus \mathrm{Z}_2^c]$
$[\mathrm{Z}_2^c]$	$[\mathrm{Z}_2 \oplus \mathrm{Z}_2^c]$
$[\mathrm{Z}_2^-]$	$[\mathrm{Z}_2 \oplus \mathrm{Z}_2^c]$
$[\mathrm{Z}_{2n}^-]$	$[\mathrm{Z}_{2n} \oplus \mathrm{Z}_2^c]$
$[\mathrm{D}_{2n}^h]$	$[\mathrm{D}_{2n} \oplus \mathrm{Z}_2^c]$
$[\mathrm{D}_n^v]$	$[\mathrm{D}_n \oplus \mathrm{Z}_2^c]$
$[\mathrm{O}(2)^-]$	$[\mathrm{O}(2) \oplus \mathrm{Z}_2^c]$
$[\mathcal{O}^-]$	$[\mathcal{O} \oplus \mathrm{Z}_2^c]$
$[\mathcal{P}]$	$[\mathcal{P} \oplus \mathrm{Z}_2^c]$

TABLE 5. How a symmetry group in O(3) is viewed by an even-order tensor. In this table $\mathcal{P}$ stands for a exceptional group.

As a result, the 3D strain-gradient elasticity is divided into 48 symmetry classes[7]. The nature of these classes and the associated number of independent parameters are provided in the two following tables:

Name	Triclinic	Monoclinic	Orthotropic	Trigonal	Tetragonal	Pentagonal	Hexagonal	∞-gonal
$[\mathrm{G}_\mathcal{L}]$	$\mathbb{1}$	$[\mathrm{Z}_2]$		$[\mathrm{Z}_3]$	$[\mathrm{Z}_4]$	$[\mathrm{Z}_5]$	$[\mathrm{Z}_6]$	$[\mathrm{SO}(2)]$
$\#_{\mathrm{indep}}(\mathbb{L})$	300	156		99	77	62	58	56
$[\mathrm{G}_\mathcal{L}]$			$[\mathrm{D}_2]$	$[\mathrm{D}_3]$	$[\mathrm{D}_4]$	$[\mathrm{D}_5]$	$[\mathrm{D}_6]$	$[\mathrm{O}(2)]$
$\#_{\mathrm{indep}}(\mathbb{L})$			84	56	45	37	35	34
$[\mathrm{G}_\mathcal{L}]$	$[\mathrm{Z}_2^c]$	$[\mathrm{Z}_2 \oplus \mathrm{Z}_2^c]$		$[\mathrm{Z}_3 \oplus \mathrm{Z}_2^c]$	$[\mathrm{Z}_4 \oplus \mathrm{Z}_2^c]$	$[\mathrm{Z}_5 \oplus \mathrm{Z}_2^c]$	$[\mathrm{Z}_6 \oplus \mathrm{Z}_2^c]$	$[\mathrm{SO}(2) \oplus \mathrm{Z}_2^c]$
$\#_{\mathrm{indep}}(\mathbb{L})$	192	104		63	51	40	38	36
$[\mathrm{G}_\mathcal{L}]$			$[\mathrm{D}_2 \oplus \mathrm{Z}_2^c]$	$[\mathrm{D}_3 \oplus \mathrm{Z}_2^c]$	$[\mathrm{D}_4 \oplus \mathrm{Z}_2^c]$	$[\mathrm{D}_5 \oplus \mathrm{Z}_2^c]$	$[\mathrm{D}_6 \oplus \mathrm{Z}_2^c]$	$[\mathrm{O}(2) \oplus \mathrm{Z}_2^c]$
$\#_{\mathrm{indep}}(\mathbb{L})$			60	40	34	28	27	26
$[\mathrm{G}_\mathcal{L}]$		$[\mathrm{Z}_2^-]$	$[\mathrm{Z}_4^-]$	$[\mathrm{Z}_6^-]$	$[\mathrm{Z}_8^-]$			
$\#_{\mathrm{indep}}(\mathbb{L})$		160	77	54	42			
$[\mathrm{G}_\mathcal{L}]$		$[\mathrm{D}_2^v]$	$[\mathrm{D}_3^v]$	$[\mathrm{D}_4^v]$	$[\mathrm{D}_5^v]$			$[\mathrm{O}^-(2)]$
$\#_{\mathrm{indep}}(\mathbb{L})$		88	60	49	41			38
$[\mathrm{G}_\mathcal{L}]$			$[\mathrm{D}_4^h]$	$[\mathrm{D}_6^h]$	$[\mathrm{D}_8^h]$	$[\mathrm{D}_{10}^h]$		
$\#_{\mathrm{indep}}(\mathbb{L})$			47	35	29	27		

TABLE 6. The names, the sets of subgroups $[\mathrm{G}_\mathcal{L}]$ and the numbers of independent components $\#_{\mathrm{indep}}(\mathcal{L})$ for the plane symmetry classes of SGE.

[7]This result is found through inspection by combining the results of our classification. More precisely, for each possible material symmetry class, we determine the least symmetric triplet of physical symmetry classes compatible with the material symmetries.

Name	Tetrahedral	Cubic	Icosahedral	$\infty\infty$
$[\mathrm{G}_{\mathcal{L}}]$	$[\mathcal{T}]$	$[\mathcal{O}]$	$[\mathcal{I}]$	$[\mathrm{SO}(3)]$
$\#_{\text{indep}}(\mathbb{M})$	28	17	9	8
$[\mathrm{G}_{\mathcal{L}}]$	$[\mathcal{T}\oplus \mathrm{Z}_2^c]$	$[\mathcal{O}\oplus \mathrm{Z}_2^c]$	$[\mathcal{I}\oplus \mathrm{Z}_2^c]$	$[\mathrm{O}(3)]$
$\#_{\text{indep}}(\mathbb{M})$	20	14	8	7
$[\mathrm{G}_{\mathcal{L}}]$	$[\mathcal{O}^-]$			
$\#_{\text{indep}}(\mathbb{M})$	19			

TABLE 7. The names, the sets of subgroups $[\mathrm{G}_{\mathcal{L}}]$ and the numbers of independent components $\#_{\text{indep}}(\mathcal{L})$ for the spatial symmetry classes of SGE.

The matrix structure in each case is detailed below.

5.1. **Type I : Chiral classes.** In each symmetry class, the constitutive law has the following synthetic form:

Polar & Chiral.

$$\mathcal{L}_{\mathbb{1}} = \begin{pmatrix} \mathbf{C}_{\mathrm{Z}_2^c} & \mathbf{M}_{\mathbb{1}} \\ \mathbf{M}_{\mathbb{1}}^T & \mathbf{A}_{\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{Z}_2} = \begin{pmatrix} \mathbf{C}_{\mathrm{Z}_2\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{Z}_2} \\ \mathbf{M}_{\mathrm{Z}_2}^T & \mathbf{A}_{\mathrm{Z}_2\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{20}$$

$$\mathcal{L}_{\mathrm{Z}_3} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_3\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{Z}_3} \\ \mathbf{M}_{\mathrm{Z}_3}^T & \mathbf{A}_{\mathrm{Z}_3\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{Z}_4} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_4\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{Z}_4} \\ \mathbf{M}_{\mathrm{Z}_4}^T & \mathbf{A}_{\mathrm{Z}_4\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{21}$$

$$\mathcal{L}_{\mathrm{Z}_5} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{Z}_5} \\ \mathbf{M}_{\mathrm{Z}_5}^T & \mathbf{A}_{\mathrm{Z}_5\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{Z}_6} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{SO}(2)} \\ \mathbf{M}_{\mathrm{SO}(2)}^T & \mathbf{A}_{\mathrm{Z}_6\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{22}$$

$$\mathcal{L}_{\mathrm{SO}(2)} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{SO}(2)} \\ \mathbf{M}_{\mathrm{SO}(2)}^T & \mathbf{A}_{\mathrm{SO}(2)\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{23}$$

The notation $\mathbf{T}_H$ indicates a H-invariant tensor. In this notation, H is a group not a class; the choice and the compatibility of the generators are essential. For example, let consider $\mathcal{L}_{\mathrm{Z}_4}$, it is the same group Z_4 (same orientation of the axis of rotation) which is contained in the symmetry group of each tensor. If not due to the relation Equation 3 the symmetry group of the constitutive law would be $\mathbb{1}$.

Chiral.

$$\mathcal{L}_{\mathrm{D}_2} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_2\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{D}_2} \\ \mathbf{M}_{\mathrm{D}_2}^T & \mathbf{A}_{\mathrm{D}_2\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_3} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_3\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{D}_3} \\ \mathbf{M}_{\mathrm{D}_3}^T & \mathbf{A}_{\mathrm{D}_3\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{24}$$

$$\mathcal{L}_{\mathrm{D}_4} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_4\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{D}_4} \\ \mathbf{M}_{\mathrm{D}_4}^T & \mathbf{A}_{\mathrm{D}_4\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_5} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{D}_5} \\ \mathbf{M}_{\mathrm{D}_5}^T & \mathbf{A}_{\mathrm{D}_5\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{25}$$

$$\mathcal{L}_{\mathrm{D}_6} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{O}(2)} \\ \mathbf{M}_{\mathrm{O}(2)}^T & \mathbf{A}_{\mathrm{D}_6\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{O}(2)} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{O}(2)} \\ \mathbf{M}_{\mathrm{O}(2)}^T & \mathbf{A}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{26}$$

$$\mathcal{L}_{\mathcal{T}} = \begin{pmatrix} \mathbf{C}_{\mathcal{O}\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathcal{T}} \\ \mathbf{M}_{\mathcal{T}}^T & \mathbf{A}_{\mathcal{T}\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathcal{O}} = \begin{pmatrix} \mathbf{C}_{\mathcal{O}\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathcal{O}} \\ \mathbf{M}_{\mathcal{O}}^T & \mathbf{A}_{\mathcal{O}\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{27}$$

$$\mathcal{L}_{\mathcal{I}} = \begin{pmatrix} \mathbf{C}_{\mathrm{SO}(3)\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{SO}(3)} \\ \mathbf{M}_{\mathrm{SO}(3)}^T & \mathbf{A}_{\mathcal{I}\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{SO}(3)} = \begin{pmatrix} \mathbf{C}_{\mathrm{SO}(3)\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{SO}(3)} \\ \mathbf{M}_{\mathrm{SO}(3)}^T & \mathbf{A}_{\mathrm{SO}(3)\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{28}$$

5.2. **Type II : Centro-symmetric classes.** In each symmetry class, the constitutive law has the following synthetic form:

Centro-Symmetric.

$$\mathcal{L}_{\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathrm{Z}_2^c} & 0 \\ 0 & \mathbf{A}_{\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{Z}_2\oplus\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathrm{Z}_2\oplus\mathrm{Z}_2^c} & 0 \\ 0 & \mathbf{A}_{\mathrm{Z}_2\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{29}$$

$$\mathcal{L}_{\mathrm{Z}_3\oplus\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_3\oplus\mathrm{Z}_2^c} & 0 \\ 0 & \mathbf{A}_{\mathrm{Z}_3\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{Z}_4\oplus\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_4\oplus\mathrm{Z}_2^c} & 0 \\ 0 & \mathbf{A}_{\mathrm{Z}_4\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{30}$$

$$\mathcal{L}_{\mathrm{Z}_5\oplus\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & 0 \\ 0 & \mathbf{A}_{\mathrm{Z}_5\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{Z}_6\oplus\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & 0 \\ 0 & \mathbf{A}_{\mathrm{Z}_6\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{31}$$

$$\mathcal{L}_{\mathrm{SO}(2)\oplus\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & 0 \\ 0 & \mathbf{A}_{\mathrm{SO}(2)\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \tag{32}$$

$$\mathcal{L}_{\mathrm{D}_2\oplus\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_2\oplus\mathrm{Z}_2^c} & 0 \\ 0 & \mathbf{A}_{\mathrm{D}_2\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_3\oplus\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_3\oplus\mathrm{Z}_2^c} & 0 \\ 0 & \mathbf{A}_{\mathrm{D}_3\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{33}$$

$$\mathcal{L}_{\mathrm{D}_4\oplus\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_4\oplus\mathrm{Z}_2^c} & 0 \\ 0 & \mathbf{A}_{\mathrm{D}_4\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_5\oplus\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & 0 \\ 0 & \mathbf{A}_{\mathrm{D}_5\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{34}$$

$$\mathcal{L}_{\mathrm{D}_6\oplus\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & 0 \\ 0 & \mathbf{A}_{\mathrm{D}_6\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & 0 \\ 0 & \mathbf{A}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{35}$$

$$\mathcal{L}_{\mathcal{T}\oplus\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathcal{O}\oplus\mathrm{Z}_2^c} & 0 \\ 0 & \mathbf{A}_{\mathcal{T}\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathcal{O}\oplus\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathcal{O}\oplus\mathrm{Z}_2^c} & 0 \\ 0 & \mathbf{A}_{\mathcal{O}\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{36}$$

$$\mathcal{L}_{\mathcal{I}\oplus\mathrm{Z}_2^c} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(3)} & 0 \\ 0 & \mathbf{A}_{\mathcal{I}\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{O}(3)} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(3)} & 0 \\ 0 & \mathbf{A}_{\mathrm{O}(3)} \end{pmatrix} \tag{37}$$

5.3. **Type III : Non Centro-symmetric classes.** In each symmetry class, the constitutive law takes the following synthetic form:

Polar.

$$\mathcal{L}_{\mathrm{Z}_2^-} = \begin{pmatrix} \mathbf{C}_{\mathrm{Z}_2\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{Z}_2^-} \\ \mathbf{M}^T_{\mathrm{Z}_2^-} & \mathbf{A}_{\mathrm{Z}_2\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{38}$$

$$\mathcal{L}_{\mathrm{D}_2^v} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_2\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{D}_2^v} \\ \mathbf{M}^T_{\mathrm{D}_2^v} & \mathbf{A}_{\mathrm{D}_2\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_3^v} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_3\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{D}_3^v} \\ \mathbf{M}^T_{\mathrm{D}_3^v} & \mathbf{A}_{\mathrm{D}_3\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{39}$$

$$\mathcal{L}_{\mathrm{D}_4^v} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_4\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{D}_4^v} \\ \mathbf{M}^T_{\mathrm{D}_4^v} & \mathbf{A}_{\mathrm{D}_4\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_5^v} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{D}_5^v} \\ \mathbf{M}^T_{\mathrm{D}_5^v} & \mathbf{A}_{\mathrm{D}_5\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{40}$$

$$\mathcal{L}_{\mathrm{O}(2)^-} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{O}(2)^-} \\ \mathbf{M}^T_{\mathrm{O}(2)^-} & \mathbf{A}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \tag{41}$$

Non-centrosymmetric.

$$\mathcal{L}_{\mathrm{Z}_4^-} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_4\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{Z}_4^-} \\ \mathbf{M}^T_{\mathrm{Z}_4^-} & \mathbf{A}_{\mathrm{Z}_4\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{Z}_6^-} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{Z}_6^-} \\ \mathbf{M}^T_{\mathrm{Z}_6^-} & \mathbf{A}_{\mathrm{Z}_6\oplus\mathrm{Z}_2^c} \end{pmatrix} \tag{42}$$

$$\mathcal{L}_{\mathrm{Z}_8^-} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2)\oplus\mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{Z}_8^-} \\ \mathbf{M}^T_{\mathrm{Z}_8^-} & \mathbf{A}_{\mathrm{SO}(2)\oplus\mathrm{Z}_2^c} \end{pmatrix} \quad ; \tag{43}$$

$$\mathcal{L}_{\mathrm{D}_4^h} = \begin{pmatrix} \mathbf{C}_{\mathrm{D}_4 \oplus \mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{D}_4^h} \\ \mathbf{M}^T_{\mathrm{D}_4^h} & \mathbf{A}_{\mathrm{D}_4 \oplus \mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_6^h} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2) \oplus \mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{D}_6^h} \\ \mathbf{M}^T_{\mathrm{D}_6^h} & \mathbf{A}_{\mathrm{D}_6 \oplus \mathrm{Z}_2^c} \end{pmatrix} \tag{44}$$

$$\mathcal{L}_{\mathrm{D}_8^h} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2) \oplus \mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{D}_8^h} \\ \mathbf{M}^T_{\mathrm{D}_8^h} & \mathbf{A}_{\mathrm{O}(2) \oplus \mathrm{Z}_2^c} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_{10}^h} = \begin{pmatrix} \mathbf{C}_{\mathrm{O}(2) \oplus \mathrm{Z}_2^c} & \mathbf{M}_{\mathrm{D}_{10}^h} \\ \mathbf{M}^T_{\mathrm{D}_{10}^h} & \mathbf{A}_{\mathrm{O}(2) \oplus \mathrm{Z}_2^c} \end{pmatrix} \tag{45}$$

$$\mathcal{L}_{\mathcal{O}^-} = \begin{pmatrix} \mathbf{C}_{\mathcal{O} \oplus \mathrm{Z}_2^c} & \mathbf{M}_{\mathcal{O}^-} \\ \mathbf{M}^T_{\mathcal{O}^-} & \mathbf{A}_{\mathcal{O} \oplus \mathrm{Z}_2^c} \end{pmatrix} \tag{46}$$

With the results of this section, the symmetry classes of linear SGE have been clarified and the compact and well-structured matrix representations of the three elasticity tensors are now available for use.

6. Conclusions

The development and application of Strain Gradient Elasticity (SGE) have been almost exclusively confined to the isotropic case. The complexity and richness of anisotropic SGE is far from having been exploited. In particular, the development of architectured materials, on one hand, and the increasing importance of nanomaterials, on the other hand, have led to a paradigm shift. At the present time, in materials and mechanical sciences, the anisotropic features of generalized continua becomes a topic of increasing interest. In the present work, and in its previous companion one [Auffray et al., 2013], we have studied the anisotropic features of Strain Gradient Elasticity in its full setting : in 3D and without assuming the centrosymmetry of the matter. As a result we obtain, for the first time, a complete and explicit picture of the modelling possibilities opened by linear SGE. By taking the gradient of strain into account in the linear constitutive law, the number of symmetry classes grows from 8 systems to 48. In itself, this result illustrates the richness and the complexity of linear SGE. These results will be with no doubt necessary for the experimental identification, theoretical investigation and numerical implementation of linear SGE. In particular, that those results will be useful for the continuum description of architectured materials and especially for the modelling of waves propagation in them. The study of the 3D strain-gradient elastodynamics will be the object of a forthcoming work, in which the properties of the dynamic tensor $\mathbf{M}^\star$ evoked in the introduction will be investigated.

Appendix A. Classification of the strain gradient elasticity law.

The classifications of $\mathbf{C}$ and $\mathbf{A}$, which are even-order tensors, have been made with respect to SO(3), while that of $\mathbf{M}$, which is an odd-order tensor, has been done relative to O(3). To propose a global classification for $\mathcal{L}$ these different classifications have to be unified. First, observe that if $[H]$ is the symmetry class of an even order tensor with respect to SO(3), its symmetry class relative to O(3) corresponds $[H \oplus \mathrm{Z}_2^c]$ with Z_2^c denoting the group associated with the inversion operation. The symmetry class of an even order tensor is therefore conjugate to a closed subgroup of O(3) of type II. So if $[\mathrm{G}_{\mathcal{L}}]$ is conjugate to a closed subgroup of O(3), the symmetry class of an even order tensor will be $[H \oplus \mathrm{Z}_2^c]$. If the conjugacy group is of type I or II, the determination of H is straightforward; for the remaining case its determination is less direct and will be detailed. Consider an element $\mathbf{Q} \in \mathrm{O}(3)\backslash\mathrm{SO}(3)$. By definition $\det(\mathbf{Q}) = -1$ and $\mathbf{Q}$ can be written:

$$\mathbf{Q} = -\mathbb{1} \cdot \mathbf{R}, \quad \mathbf{R} \in \mathrm{SO}(3)$$

This relation will be referred to as the characteristic decomposition. The tensorial action implies that

$$(\mathbf{Q} \star \mathrm{T})_{i_1 i_2 \dots i_n} \to Q_{i_1 j_1} Q_{i_2 j_2} \dots Q_{i_n j_n} T_{j_1 j_2 \dots j_n}.$$

Hence, according to the characteristic decomposition, for $\mathbf{Q} \in \mathrm{O}(3)/\mathrm{SO}(3)$:

$$(\mathbf{Q} \star \mathrm{T})_{i_1 i_2 \dots i_n} \to (-1)^n R_{i_1 j_1} R_{i_2 j_2} \dots R_{i_n j_n} T_{j_1 j_2 \dots j_n}.$$

In the case of even order tensor,

$$(\mathbf{Q} \star \mathrm{T})_{i_1 i_2 \dots i_n} \to R_{i_1 j_1} R_{i_2 j_2} \dots R_{i_n j_n} T_{j_1 j_2 \dots j_n}.$$

Thus, when $\mathbf{Q} \in \mathrm{O}(3)/\mathrm{SO}(3)$ acts on an even-order tensor, what is "viewed" by the tensor is the action of $\mathbf{R} \in \mathrm{SO}(3)$. Consider type III subgroups. They contain, in different ways, plane mirror symmetries. In this case the characteristic decomposition gives

$$\mathbf{P}_{\mathbf{n}} = -\mathbb{1} \cdot \mathbf{R}(\mathbf{n}; \pi).$$

As a consequence, Z_2^- generated by $\mathbf{P}_{\mathbf{n}}$ is seen as Z_2 and Z_{2n}^- ($n \geq 2$) generated by $\mathbf{R}\left(\mathbf{e}_3; \frac{\pi}{n}\right) \cdot \mathbf{P}_{\mathbf{e}_3}$ is seen as Z_{2n}. The other situations can be deduced.

Appendix B. Dictionary

Type I subgroups

System	Hermann-Maugin	Schonflies	Group	Nature
Triclinic	1	Z_1	$\mathbb{1}$	$\overline{\mathrm{I}}\mathrm{CP}$
Monoclinic	2	Z_2	Z_2	$\overline{\mathrm{I}}\mathrm{CP}$
Orthotropic	222	D_2	D_2	$\overline{\mathrm{I}}\mathrm{C}\overline{\mathrm{P}}$
Trigonal	3	Z_3	Z_3	$\overline{\mathrm{I}}\mathrm{CP}$
Trigonal	32	D_3	D_3	$\overline{\mathrm{I}}\mathrm{C}\overline{\mathrm{P}}$
Tetragonal	4	Z_4	Z_4	$\overline{\mathrm{I}}\mathrm{CP}$
Tetragonal	422	D_4	D_4	$\overline{\mathrm{I}}\mathrm{C}\overline{\mathrm{P}}$
Hexagonal	6	Z_6	Z_6	$\overline{\mathrm{I}}\mathrm{CP}$
Hexagonal	622	D_6	D_6	$\overline{\mathrm{I}}\mathrm{C}\overline{\mathrm{P}}$
	∞	Z_∞	$\mathrm{SO}(2)$	$\overline{\mathrm{I}}\mathrm{CP}$
	$\infty 2$	D_∞	$\mathrm{O}(2)$	$\overline{\mathrm{I}}\mathrm{C}\overline{\mathrm{P}}$
Cubic	23	T	$\mathcal{T}$	$\overline{\mathrm{I}}\mathrm{C}\overline{\mathrm{P}}$
Cubic	432	O	$\mathcal{O}$	$\overline{\mathrm{I}}\mathrm{C}\overline{\mathrm{P}}$
	532	I	$\mathcal{I}$	$\overline{\mathrm{I}}\mathrm{C}\overline{\mathrm{P}}$
	$\infty\infty$		$\mathrm{SO}(3)$	$\overline{\mathrm{I}}\mathrm{C}\overline{\mathrm{P}}$

Table 8. Dictionary between different group notations for Type I subgroups. The last column indicates the nature of the group: C = Chiral, P=Polar, I = Centrosymetric, and overline indicates that the property is missing.

Type II subgroups

System	Hermann-Maugin	Schonflies	Group	Nature
Triclinic	$\bar{1}$	Z_i	Z_2^c	$\mathrm{I}\overline{\mathrm{CP}}$
Monoclinic	$2/m$	Z_{2h}	$Z_2 \oplus Z_2^c$	$\mathrm{I}\overline{\mathrm{CP}}$
Orthotropic	mmm	D_{2h}	$D_2 \oplus Z_2^c$	$\mathrm{I}\overline{\mathrm{CP}}$
Trigona	$\bar{3}$	$S_6,\ Z_{3i}$	$Z_3 \oplus Z_2^c$	$\mathrm{I}\overline{\mathrm{CP}}$
Trigonal	$\bar{3}m$	D_{3d}	$D_3 \oplus Z_2^c$	$\mathrm{I}\overline{\mathrm{CP}}$
Tetragonal	$4/m$	Z_{4h}	$Z_4 \oplus Z_2^c$	$\mathrm{I}\overline{\mathrm{CP}}$
Tetragonal	$4/mmm$	D_{4h}	$D_4 \oplus Z_2^c$	$\mathrm{I}\overline{\mathrm{CP}}$
Hexagonal	$6/m$	Z_{6h}	$Z_6 \oplus Z_2^c$	$\mathrm{I}\overline{\mathrm{CP}}$
Hexagonal	$6/mmm$	D_{6h}	$D_6 \oplus Z_2^c$	$\mathrm{I}\overline{\mathrm{CP}}$
	∞/m	$Z_{\infty h}$	$SO(2) \oplus Z_2^c$	$\mathrm{I}\overline{\mathrm{CP}}$
	∞/mm	$D_{\infty h}$	$O(2) \oplus Z_2^c$	$\mathrm{I}\overline{\mathrm{CP}}$
Cubic	$m\bar{3}$	T_h	$\mathcal{T} \oplus Z_2^c$	$\mathrm{I}\overline{\mathrm{CP}}$
Cubic	$m\bar{3}m$	O_h	$\mathcal{O} \oplus Z_2^c$	$\mathrm{I}\overline{\mathrm{CP}}$
	$\bar{5}\bar{3}m$	I_h	$\mathcal{I} \oplus Z_2^c$	$\mathrm{I}\overline{\mathrm{CP}}$
	$\infty/m\infty/m$		$O(3)$	

Table 9. Dictionary between different group notations for Type II subgroups. The last column indicates the nature of the group: C = Chiral, P=Polar, I = Centrosymetric, and overline indicates that the property is missing.

Type III subgroups

System	Hermann-Maugin	Schonflies	Group	Nature
Monocinic	m	Z_s	Z_2^-	$\overline{\mathrm{IC}}\mathrm{P}$
Orthotropic	$2mm$	Z_{2v}	D_2^v	$\overline{\mathrm{IC}}\mathrm{P}$
Trigonal	$3m$	Z_{3v}	D_3^v	$\overline{\mathrm{IC}}\mathrm{P}$
Tetragonal	$\bar{4}$	S_4	Z_4^-	$\overline{\mathrm{ICP}}$
Tetragonal	$4mm$	Z_{4v}	D_4^v	$\overline{\mathrm{IC}}\mathrm{P}$
Tetragonal	$\bar{4}2m$	D_{2d}	D_4^h	$\overline{\mathrm{ICP}}$
Hexagonal	$\bar{6}$	Z_{3h}	Z_6^-	$\overline{\mathrm{ICP}}$
Hexagonal	$6mm$	Z_{6v}	D_6^v	$\overline{\mathrm{IC}}\mathrm{P}$
Hexagonal	$\bar{6}2m$	D_{3h}	D_6^h	$\overline{\mathrm{ICP}}$
Cubic	$\bar{4}3m$	T_d	$\mathcal{O}^-$	$\overline{\mathrm{ICP}}$
	∞m	$Z_{\infty v}$	$O(2)^-$	$\overline{\mathrm{IC}}\mathrm{P}$

Table 10. Dictionary between different group notations for Type III subgroups. The last column indicates the nature of the group: C = Chiral, P=Polar, I = Centrosymetric, and overline indicates that the property is missing.

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