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# Strong solution for Korteweg system in $\mathrm{bmo}^{-1}\left(\mathbb{R}^{N}\right)$ with initial density in $L^{\infty}$ 

Boris Haspot * ${ }^{*}$


#### Abstract

In this paper we investigate the question of the existence of strong solution in finite time for the Korteweg system for small initial data provided that the initial momentum $\rho_{0} u_{0}$ belongs to $\mathrm{bmo}_{T}^{-1}\left(\mathbb{R}^{N}\right)$ for $T>0$ and the initial density $\rho_{0}$ is in $L^{\infty}\left(\mathbb{R}^{N}\right)$ with $N \geq 1$ and far away from the vacuum. This result extends the so called Koch-Tataru theorem for the Korteweg system. It is also interesting to observe that any initial shock on the density is instantaneously regularized inasmuch as the density becomes Lipschitz for any $\rho(t, \cdot)$ with $t>0$. We also prove the existence of global strong solution for initial data $\left(\rho_{0}-1, \rho_{0} u_{0}\right) \in\left(B_{2, \infty}^{\frac{N}{2}-1}\left(\mathbb{R}^{N}\right) \cap\right.$ $\left.B_{2, \infty}^{\frac{N}{2}}\left(\mathbb{R}^{N}\right) \cap L^{\infty}\left(\mathbb{R}^{N}\right)\right) \times\left(B_{2, \infty}^{\frac{N}{2}-1}\left(\mathbb{R}^{N}\right)\right)^{N}$. This result allows in particular to extend the notion of Oseen solution (corresponding to particular solution of the incompressible Navier Stokes system in dimension $N=2$ ) to the Korteweg system provided that the vorticity of the momentum $\rho_{0} u_{0}$ is a Dirac mass $\alpha \delta_{0}$ with $\alpha$ sufficiently small. IHowever unlike the Navier Stokes equations the property of self similarity is not conserved for the Korteweg system since there is no invariance by scaling because the term of pressure.


## 1 Introduction

We are concerned with compressible fluids endowed with internal capillarity. The model we consider originates from the XIXth century work by J. F. Van der Waals and D. J. Korteweg $[25,20]$ and was actually derived in its modern form in the 1980s using the second gradient theory (see [8, 18, 24]). Korteweg-type models are based on an extended version of nonequilibrium thermodynamics, which assumes that the energy of the fluid not only depends on standard variables but also on the gradient of the density.
We are now going to consider the so-called local Korteweg system which is a compressible capillary fluid model, it can be derived from a Cahn-Hilliard like free energy (see the pioneering work by J.- E. Dunn and J. Serrin in [8] and also [1, 4, 13]). The conservation of mass and of momentum write:

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} \rho+\operatorname{div}(\rho u)=0  \tag{1.1}\\
\frac{\partial}{\partial t}(\rho u)+\operatorname{div}(\rho u \otimes u)-\operatorname{div}(2 \mu \mathrm{D}(\mathrm{u}))+\nabla \mathrm{P}(\rho)=\operatorname{divK}
\end{array}\right.
$$

[^0]Here $u=u(t, x) \in \mathbb{R}^{N}$ stands for the velocity field with $N \geq 1, \rho=\rho(t, x) \in \mathbb{R}^{+}$is the density, $D(u)=\frac{1}{2}\left(\nabla u+{ }^{t} \nabla u\right)$ is the strain tensor and $P(\rho)$ is the pressure (we will only consider regular pressure law). We denote by $\mu$ the viscosity coefficients of the fluid. We supplement the problem with initial condition $\left(\rho_{0}, u_{0}\right)$.
The Korteweg tensor reads as:

$$
\begin{equation*}
\operatorname{div} K=2 \kappa^{2} \rho \nabla\left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}\right) \tag{1.2}
\end{equation*}
$$

where $\kappa(\rho)=\frac{\kappa^{2}}{\rho}$ is the coefficient of capillarity with $\kappa^{2} \in \mathbb{R}^{+}$. The term $\operatorname{div} K$ allows to describe the variation of density at the interfaces between two phases, generally a mixture liquid-vapor. In our case it is also called quantum pressure.
We briefly mention that the existence of global strong solutions for the system (1.1) with small initial data for $N \geq 2$ is known since the works by Hattori and Li [17] in the case of constant capillary coefficient $\kappa(\rho)$. Danchin and Desjardins in [7] improved this result by working with initial data $\left(\rho_{0}-1, \rho_{0} u_{0}\right)$ belonging to the homogeneous Besov spaces ${ }^{1}$ $B_{2,1}^{\frac{N}{2}} \times B_{2,1}^{\frac{N}{2}-1}$ (it is important to point out that $B_{2,1}^{\frac{N}{2}}$ is embedded in $L^{\infty}$ which allows to control the $L^{\infty}$ norm of the density, it is even better since it implies that $\rho_{0}$ is necessary a continuous function). In [16], it is proved the existence of global strong solution with small initial data provided that $\left(\rho_{0}-1, \rho_{0} u_{0}\right)$ is in $\left(B_{2,2}^{\frac{N}{2}} \cap L^{\infty}\right) \times B_{2,2}^{\frac{N}{2}-1}$. This result extends [7] but does not allow to deal with general shocks on the initial density.

We wish now rewrite the system (1.1) using the formulation introduced in [2] (see also the existence of global energy weak solution), we consider the following effective velocity with $c>0$ :

$$
\begin{equation*}
v_{c}=u+c \nabla \ln \rho, \tag{1.3}
\end{equation*}
$$

which enables us to rewrite the system (1.1) as follows when $\kappa^{2} \leq \mu^{2}{ }^{2}$ using the fact that $\operatorname{div}(\rho \nabla \nabla \ln \rho)=2 \rho \nabla\left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}\right)$ :

$$
\left\{\begin{array}{l}
\partial_{t} \rho-c \Delta \rho+\operatorname{div}\left(\rho v_{c}\right)=0  \tag{1.4}\\
\rho \partial_{t} v+\rho u \cdot \nabla v-\mu \operatorname{div}(\rho \nabla v)-(\mu-c) \operatorname{div}\left(\rho^{t} \nabla v\right)+\nabla P(\rho)-2 \kappa_{1}^{2} \rho \nabla\left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}\right)=0
\end{array}\right.
$$

with $\kappa_{1}^{2}=\kappa^{2}-2 \mu c+c^{2}$. We specify now the value of $c$ and we want to deal with a $c$ such that $\kappa_{1}^{2}=0$, we take now:

$$
c_{1}=\mu-\sqrt{\mu^{2}-\kappa^{2}} \text { and } c_{2}=\mu+\sqrt{\mu^{2}-\kappa^{2}}
$$

[^1]Setting now $v_{1}=u+c_{1} \nabla \ln \rho$ and $v_{2}=u+c_{2} \nabla \ln \rho$, we have:

$$
\left\{\begin{align*}
& \partial_{t} \rho-c_{1} \Delta \rho+\operatorname{div}\left(\rho v_{1}\right)=0  \tag{1.5}\\
& \begin{array}{rl}
\partial_{t}\left(\rho v_{1}\right)+\frac{1}{2} \operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)+\frac{1}{2} & \operatorname{div}\left(\rho v_{2} \otimes v_{1}\right)-\mu \Delta\left(\rho v_{1}\right) \\
& \quad-\sqrt{\mu^{2}-\kappa^{2}} \nabla \operatorname{div}\left(\rho v_{1}\right)+\nabla P(\rho)=0
\end{array} \\
& \begin{array}{rl}
\partial_{t}\left(\rho v_{2}\right)+\frac{1}{2} \operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)+\frac{1}{2} & \operatorname{div}\left(\rho v_{2} \otimes v_{1}\right)-\mu \Delta\left(\rho v_{2}\right) \\
& +\sqrt{\mu^{2}-\kappa^{2}} \nabla \operatorname{div}\left(\rho v_{2}\right)+\nabla P(\rho)=0
\end{array}
\end{align*}\right.
$$

In this paper our main goal consists in proving the existence of global strong solution for the system (1.1) with minimal regularity assumption on the initial data. More precisely since this system models a mixture liquid vapor with different density, we wish to show the existence of strong solution for initial density $\rho_{0}$ belonging only to the set $L^{\infty}\left(\mathbb{R}^{N}\right)$ (and in addition for initial density far away from the vacuum). In particular it implies that our functional setting will include the case of initial density admitting shocks what is also fundamental both in the physical theory of non-equilibrium thermodynamics as well as in the mathematical study of inviscid models for compressible flow.
In addition we wish also to deal with momentum $\rho_{0} u_{0}$ exhibiting specific structure, typically we have in mind the case of initial vorticity belonging to the set of finite measure in dimension $N=2$ (we will recall later that there is such a theory for Navier-Stokes equations and even explicit such solutions, the so called Oseen tourbillon). In order to obtain such results, it seems necessary to work in space with minimal regularity assumptions. Typically the space $B_{p, r}^{\frac{N}{p}-1}$ with third index $r=+\infty$ is a good candidate for the initial momentum $\rho_{0} u_{0}$.
To do this, let us now recall the notion of scaling for the Korteweg's system (1.1). Such an approach is now classical for incompressible Navier-Stokes equation and yields local well-posedness (or global well-posedness for small initial data) in spaces with minimal regularity. In our situation we can easily check that, if $(\rho, u)$ solves (1.1), then $\left(\rho_{\lambda}, u_{\lambda}\right)$ solves also this system:

$$
\begin{equation*}
\rho_{\lambda}(t, x)=\rho\left(\lambda^{2} t, \lambda x\right), u_{\lambda}(t, x)=\lambda u\left(\lambda^{2} t, \lambda x\right) \tag{1.6}
\end{equation*}
$$

provided the pressure laws $P$ have been changed to $\lambda^{2} P$.
Remark 1 It is very important to point out that since there is only a scaling invariance up to the pressure term, we can not hope to show the existence of global strong self similar solution for the Korteweg system. Indeed in comparison with the Navier-Stokes system, we know that it exists global in time self similar solution provided that the initial velocity $u_{0}$ is homogeneous of degree -1 (such initial data exists for example in the set $B_{p, \infty}^{\frac{N}{p}-1}$ with $1 \leq p<+\infty)$.
In particular in the sequel we will able to deal with initial data $\left(\rho_{0}, \rho_{0} u_{0}\right)$ such that $\rho_{0}$ is homogeneous of degree 0 and $\rho_{0} u_{0}$ of degree -1 , however the associated strong solution will be a priori not self similar.

The previous transformation (1.6) suggests us however to choose initial data $\left(\rho_{0}, u_{0}\right)$ in spaces whose norm is invariant for all $\lambda>0$ by the transformation $\left(\rho_{0}, u_{0}\right) \longrightarrow$
$\left(\rho_{0}(\lambda \cdot), \lambda u_{0}(\lambda \cdot)\right)$. A natural candidate is for example the space $L^{\infty}\left(\mathbb{R}^{N}\right) \times \dot{H}^{\frac{N}{2}-1}$. As we mentioned previously this invariance was also used for the first time by Kato [19] to prove that the Navier-Stokes equation is locally well-posed for arbitrary data in $L^{N}\left(\mathbb{R}^{N}\right)$ and globally well-posed for small initial data. Kato's result was extended to larger scale invariant function spaces (one interest of dealing with larger function spaces is that they may contain initial data which are homogeneous of degree -1 and therefore give rise to self-similar solutions), in particular in [5] Cannone, Meyer and Planchon proved the existence of global strong solution with small initial data in $B_{p, \infty}^{\frac{N}{p}-1}$ with $p<+\infty$. A similar analysis was carried out for the vorticity equation in Morrey spaces by Giga and Miyakawa [12]. Finally this approach has been generalized by Koch and Tataru in [21] when the initial data is small in $B M O^{-1}\left(\mathbb{R}^{N}\right)$.
As explained these results allow to obtain the existence of global self similar solution for small initial data when $N \geq 2$ for the incompressible Navier-Stokes equations. In dimension $N=2$, it allows in particular to prove the existence and the uniqueness of the solution for initial data $u_{0}$ verifying $\operatorname{curl} u_{0}=\alpha \delta_{0}$ or $u_{0}=\alpha \frac{x^{\perp}}{|x|^{2}}$ with $\alpha$ small enough. These solutions are the so-called Lamb-Oseen solutions which are self-similar. However we have even an explicit formula for these solutions even when $|\alpha|$ is large, the LambOseen vortex are given by:

$$
\begin{equation*}
\operatorname{curl} u_{\alpha}(t, x)=\frac{\alpha}{t} G\left(\frac{x}{\sqrt{t}}\right), u_{\alpha}(t, x)=\frac{\alpha}{\sqrt{t}} v_{G}\left(\frac{x}{\sqrt{t}}\right), x \in \mathbb{R}^{2}, t>0 \tag{1.7}
\end{equation*}
$$

where:

$$
G(\xi)=\frac{1}{4 \pi} e^{-\frac{|\xi|^{2}}{4}}, v_{G}(\xi)=\frac{1}{2 \pi} \frac{\xi^{\perp}}{|\xi|^{2}}\left(1-e^{-\frac{|\xi|^{2}}{4}}\right), \xi \in \mathbb{R}^{2}
$$

with $\xi^{\perp}=\left(-\xi_{2}, \xi_{1}\right)$. In passing, we mention that the existence of global solution with initial vorticity in the set $\mathcal{M}\left(\mathbb{R}^{2}\right)$ of all finite measures on $\mathbb{R}^{2}$ was first proved by Cottet [6] and independently by Giga, Miyakawa and Osada [12]. In [12], the authors proved also the uniqueness when the atomic part of the initial vorticity is sufficiently small. The uniqueness for any $\operatorname{curl} u_{0} \in \mathcal{M}\left(\mathbb{R}^{2}\right)$ is proved in [10], it allows in particular to obtain the existence and the uniqueness of global self similar solution for large initial data when $N=2$.
In this paper we are interested in adapting the technics of Koch-Tataru to the case of the Korteweg system (1.5). More precisely, we wish to prove the existence of strong solution in finite time for $(1.1)$ provided that $\rho_{0} v_{1}(0, \cdot)$ and $\rho_{0} v_{2}(0, \cdot)$ are sufficiently small in norm $\|\cdot\|_{b m o_{T}^{-1}\left(\mathbb{R}^{N}\right)}$ for $T>0$ and when the initial density is far away from the vacuum and is bounded in $L^{\infty}$ norm. We recall that $b m o^{-1}\left(\mathbb{R}^{N}\right)$ is the set of temperated distribution $u_{0}$ for which for all $T>0$ we have:

$$
\sup _{x \in \mathbb{R}^{N}, 0<t<T}\left(t^{-\frac{N}{2}} \int_{0}^{t} \int_{B(x, \sqrt{t})}\left|e^{s \Delta} u_{0}(y)\right|^{2} d y d s\right)^{\frac{1}{2}}<+\infty
$$

We define the norm of $\|\cdot\|_{b m o_{T}^{-1}\left(\mathbb{R}^{N}\right)}$ by:

$$
\left\|u_{0}\right\|_{b m o^{-1}\left(\mathbb{R}^{N}\right)}=\sup _{x \in \mathbb{R}^{N}, 0<t<T}\left(t^{-\frac{N}{2}} \int_{0}^{t} \int_{B(x, \sqrt{t})}\left|e^{s \Delta} u_{0}(y)\right|^{2} d y d s\right)^{\frac{1}{2}}
$$

with $e^{t \Delta} u_{0}$ be the solution to the heat equation with initial data $u_{0}$ :

$$
e^{t \Delta} u_{0}=u_{0} * \phi_{\sqrt{4 t}} \text { with } \phi(x)=\pi^{-\frac{N}{2}} e^{-|x|^{2}} \text { and } \phi_{t}(x)=t^{-N} \phi\left(\frac{x}{t}\right) .
$$

In the sequel we will denote by $\mathcal{E}_{T}$ the space of temperated distribution associated to the following norm:

$$
\begin{equation*}
\|u\|_{\mathcal{E}_{T}}=\sup _{0<t \leq T} \sqrt{t}\|u(t, \cdot)\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\left(\sup _{x \in \mathbb{R}^{N}, 0<t \leq T} t^{-\frac{N}{2}} \int_{0}^{t} \int_{B(x, \sqrt{t})}|u|^{2}(s, y) d t d y\right)^{\frac{1}{2}}, \tag{1.8}
\end{equation*}
$$

More precisely we will show $\rho v_{1}$ and $\rho v_{2}$ belong to $\mathcal{E}_{T}$ for $T>0$ the time of existence and that $\left(\rho, \frac{1}{\rho}\right)$ is in $L_{T}^{\infty}\left(L^{\infty}\left(\mathbb{R}^{N}\right)\right)$. We recall that $\rho v_{1}=\rho u+c_{1} \nabla \rho$ and $\rho v_{2}=\rho u+c_{2} \nabla \rho$, it implies in particular that $\nabla \rho=\frac{1}{2 \sqrt{\mu^{2}-\kappa^{2}}}\left(\rho v_{2}-\rho v_{1}\right)$ and we deduce from the definition of $\mathcal{E}_{T}$ that it exists $C>0$ independent on $t$ such that for all $0<t \leq T$ we have:

$$
\begin{equation*}
\|\nabla \rho(t, \cdot)\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq \frac{C}{\sqrt{t}} \tag{1.9}
\end{equation*}
$$

Combining (1.9) and the fact that $\rho$ remains in $L_{T}^{\infty}\left(L^{\infty}\left(\mathbb{R}^{N}\right)\right)$ it implies that instantaneously the density $\rho$ is regularized and becomes necessary Lipschitz even if the initial density $\rho_{0}$ admits shocks.
The second point is that with our choice on the initial data, it implies that $\rho_{0} u_{0}$ is in $b m o_{T}^{-1}$. In other way we can work with Dirac mass $\alpha \delta_{0}$ for the initial velocity associated to the momentum $\rho_{0} u_{0}$ in dimension $N=2$ provided that $\alpha$ is sufficiently small, it consists to prescribe the initial momentum as follows $\rho_{0} u_{0}(x)=\alpha \frac{x^{\perp}}{|x|^{2}}$ with $x \in \mathbb{R}^{2}$. It means that we can extend the notion of Lam-Oseen tourbillon to the case of the Korteweg system at least when $\alpha$ is sufficiently small. It is obvious that in this compressible framework the divergence of such Lamb-Oseen tourbillon does not remain null all along the time, this is due to the coupling between vorticity and divergence of the velocity.
Similarly we can combine vorticity vortex and divergnece vortex if we take the following initial momentum $\rho_{0} u_{0}(x)=\alpha \frac{x^{\perp}}{|x|^{2}}+\alpha_{1} \frac{x^{\perp}}{|x|^{2}}$ with $x \in \mathbb{R}^{2}$ and $|\alpha|,\left|\alpha_{1}\right|$ sufficiently small.

### 1.1 Mathematical results

In this section we state our main result.
Theorem 1.1 Let $0<\kappa^{2} \leq \mu^{2}$ and $N \geq 1$. Let $\rho_{0} \in L^{\infty}\left(\mathbb{R}^{N}\right)$ with $\rho_{0} \geq c>0$ and $\rho_{0} v_{1}(0, \cdot), \rho_{0} v_{2}(0, \cdot) \in b m o^{-1}\left(\mathbb{R}^{N}\right)$. Then there exists $T>0$ sufficiently small such that there exists a unique solution $(\rho, \rho u)$ of the system (1.1) on $[0, T]$ provided that for $\varepsilon_{1}>0$ we have :

$$
\begin{equation*}
\left\|\rho_{0} v_{1}(0, \cdot)\right\|_{b m o_{T}^{-1}}+\left\|\rho_{0} v_{2}(0, \cdot)\right\|_{b m o_{T}^{-1}} \leq \varepsilon_{1} \tag{1.10}
\end{equation*}
$$

In addition it exists $C>0$ such that:

$$
\begin{equation*}
\left\|\rho v_{1}\right\|_{\mathcal{E}_{T}}+\left\|\rho v_{2}\right\|_{\mathcal{E}_{T}}+\|\rho\|_{L_{T}^{\infty}\left(L^{\infty}\right)}+\left\|\frac{1}{\rho}\right\|_{L_{T}^{\infty}\left(L^{\infty}\right)} \leq C \tag{1.11}
\end{equation*}
$$

Remark 2 It is important to observe that the condition (1.10) implies that $\nabla \rho_{0}$ is small in $b m o^{-1}\left(\mathbb{R}^{N}\right)$ with the norm $\|\cdot\|_{b m o_{T}^{-1}}$ when $0<\kappa^{2}<\mu^{2}$. It is also interesting to observe that there is no smallness assumption on $\rho_{0}$ when $\kappa^{2}=\mu^{2}$ since in this case $v \rho v_{1}(0, \cdot)=\rho v_{2}(0, \cdot)$. We deduce also on this case that $\rho u(0, \cdot)$ is not necessary small in $b m o^{-1}$ but the smallness carries on the momentum of the effective velocity $\rho v_{1}(0, \cdot)$ which describes the coupling between the velocity $u_{0}$ and the density $\rho_{0}$.

Corollary 1 Let $0<\kappa^{2} \leq \mu^{2}$ and $N \geq 1$. Let $\rho_{0} \in L^{\infty}\left(\mathbb{R}^{N}\right)$ with $\rho_{0} \geq c>0$, assume that $\nabla \rho_{0}$ and $\rho_{0} u_{0}$ are in $\overline{\mathcal{D}\left(\mathbb{R}^{N}\right)}{ }^{\text {bmo }}$-1 then there exists $T>0$ such that there exists $a$ unique solution $(\rho, \rho u)$ of the system (1.1). We have in addition the estimate (1.11).

We prove now a result of global strong solution with small initial data.
Theorem 1.2 Let $0<\kappa^{2} \leq \mu^{2}, \bar{\rho}>0, N \geq 2$ and $P^{\prime}(\bar{\rho})>0$. We assume that $\left(\rho_{0}-\bar{\rho}, u_{0}\right) \in\left(B_{2, \infty}^{\frac{N}{2}-1} \cap B_{2, \infty}^{\frac{N}{2}} \cap L^{\infty}\right) \times\left(B_{2, \infty}^{\frac{N}{2}-1}\right)^{N}$. There exists $\varepsilon_{0}$ such that if:

$$
\begin{equation*}
\left\|\rho_{0}-\bar{\rho}\right\|_{B_{2, \infty}}{ }^{\frac{N}{2}-1} \cap B_{2, \infty}^{\frac{N}{2}} \cap L^{\infty}+\left\|u_{0}\right\|_{B_{2, \infty}^{2}}^{\frac{N}{2}-1} \leq \varepsilon_{0} \tag{1.12}
\end{equation*}
$$

then there exists a global strong solution $(\rho, u)$ of the system (1.1). In addition it exists $C>0$ such that for $s_{1} \in\left(\frac{3}{4}, 1\right)$ :
$\left\{\begin{array}{l}\left\|\left(\rho v_{1}, \rho v_{2}\right)\right\|_{\widetilde{L}^{\infty}\left(\mathbb{R}^{+}, B_{2, \infty}^{\frac{N}{2}-1}\right) \cap \widetilde{L}^{1}\left(\mathbb{R}^{+}, B_{2, \infty}^{\frac{N}{2}+1}\right)}+\|\rho-\bar{\rho}\|_{\widetilde{L}^{\infty}\left(\mathbb{R}^{+}, B_{2, \infty}^{\frac{N}{2}-1} \cap B_{2, \infty}^{\frac{N}{2}}\right) \cap \widetilde{L}^{1}\left(\mathbb{R}^{+}, B_{2, \infty}^{\frac{N}{2}+1} \cap B_{2, \infty}^{\frac{N}{2}+2}\right)} \leq C, \\ \|\rho-\bar{\rho}\|_{L^{\infty}\left(\mathbb{R}^{+}, L^{\infty}\left(\mathbb{R}^{N}\right)\right)} \leq C \varepsilon_{0} \\ \sup _{t \in \mathbb{R}^{+}} t^{\frac{s_{1}}{2}} \|\left(\rho(t, \cdot)-\bar{\rho}, \rho v_{1}(t, \cdot), \rho v_{2}(t, \cdot) \|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}} \leq C .\right.\end{array}\right.$

Remark 3 As previously, this theorem allows to prove the existence of global strong Oseen solution provided that $\operatorname{curl}\left(\rho_{0} u_{0}\right)=\alpha \delta_{0}$ with $|\alpha|$ small enough in dimension $N=2$. When $\kappa^{2}<\mu^{2}$ it would be possible to extend this result by working with Besov spaces constructed on $L^{p}$ Lebesgue spaces in high frequencies as in [14] for compressible NavierStokes equations.

## 2 Proof of the Theorem 1.1 and the Corollary 1

We are going now to prove the Theorem 1.1 in the case $0<\kappa^{2}<\mu^{2}$. The case $\kappa^{2}=\mu^{2}$ is similar except that $v=v_{1}=v_{2}$ and we apply the estimates in $b m o_{T}^{-1}$ on $\rho v$.

From (1.5), we observe that we have for $t>0, \mathbb{Q}=\nabla(\Delta)^{-1}$ div and $\mathbb{P}=I d-\mathbb{P}$ :

$$
\left\{\begin{array}{l}
\rho(t, \cdot)=e^{c_{1} t \Delta} \rho_{0}-\int_{0}^{t} e^{c_{1}(t-s) \Delta} \operatorname{div}\left(\rho v_{1}\right) d s \\
\mathbb{P} \rho v_{1}(t, \cdot)=e^{\mu t \Delta} \mathbb{P}\left(\rho v_{1}(0, \cdot)\right)-\frac{1}{2} \int_{0}^{t} e^{\mu(t-s) \Delta} \mathbb{P}\left(\operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)+\operatorname{div}\left(\rho v_{2} \otimes v_{1}\right)\right) d s \\
\mathbb{P} \rho v_{2}(t, \cdot)=e^{\mu t \Delta} \mathbb{P}\left(\rho v_{2}(0, \cdot)\right)-\frac{1}{2} \int_{0}^{t} e^{\mu(t-s) \Delta} \mathbb{P}\left(\operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)+\operatorname{div}\left(\rho v_{2} \otimes v_{1}\right)\right) d s \\
\mathbb{Q} \rho v_{1}(t, \cdot)=e^{c_{2} t \Delta} \mathbb{Q}\left(\rho v_{1}(0, \cdot)\right)-\int_{0}^{t} e^{c_{2}(t-s) \Delta} \mathbb{Q}\left(F\left(\rho, v_{1}, v_{2}\right)(s)\right) d s \\
\mathbb{Q} \rho v_{2}(t, \cdot)=e^{c_{1} t \Delta} \mathbb{Q}\left(\rho v_{2}(0, \cdot)\right)-\int_{0}^{t} e^{c_{1}(t-s) \Delta} \mathbb{Q}\left(F\left(\rho, v_{1}, v_{2}\right)(s)\right) d s
\end{array}\right.
$$

with:

$$
F\left(\rho, v_{1}, v_{2}\right)(s)=\frac{1}{2}\left(\operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)+\operatorname{div}\left(\rho v_{2} \otimes v_{1}\right)\right)(s, \cdot)+\nabla P(\rho(s, \cdot))
$$

We are going now to work in the following space:

$$
\left\|\left(\rho, v_{1}, v_{2}\right)\right\|_{X_{T}}=\left\|\rho v_{1}\right\|_{\mathcal{E}_{T}}+\left\|\rho v_{2}\right\|_{\mathcal{E}_{T}}+\left\|\left(\rho, \frac{1}{\rho}\right)\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}
$$

for $T$ small enough. We observe in particular that for any $\rho \in L_{T}^{\infty}\left(L^{\infty}\left(\mathbb{R}^{N}\right)\right)$ we have:

$$
\begin{equation*}
\|\rho m\|_{\mathcal{E}_{T}} \leq\|\rho\|_{L_{T}^{\infty}\left(L^{\infty}\left(\mathbb{R}^{N}\right)\right)}\|m\|_{\mathcal{E}_{T}} . \tag{2.13}
\end{equation*}
$$

We shall use in the sequel a contracting mapping argument for the function $\psi$ defined as follows:

$$
\begin{align*}
& \psi\left(\rho, \rho v_{1}, \rho v_{2}\right)=\left(\begin{array}{c}
e^{c_{1} t \Delta} \rho_{0} \\
e^{\mu t \Delta} \mathbb{P}\left(\rho v_{1}(0, \cdot)\right)+e^{c_{2} t \Delta} \mathbb{Q}\left(\rho v_{1}(0, \cdot)\right) \\
e^{\mu t \Delta} \mathbb{P}\left(\rho v_{2}(0, \cdot)\right)+e^{c_{1} t \Delta} \mathbb{Q}\left(\rho v_{2}(0, \cdot)\right)
\end{array}\right) \\
& -\int_{0}^{t}\left(\begin{array}{c}
e_{1}(t-s) \Delta \\
\operatorname{div}\left(\rho v_{1}\right) \\
\frac{1}{2} e^{\mu(t-s) \Delta} \mathbb{P}\left(\operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)+\operatorname{div}\left(\rho v_{2} \otimes v_{1}\right)\right)+e^{c_{2}(t-s) \Delta} \mathbb{Q}\left(F\left(\rho, v_{1}, v_{2}\right)\right) \\
\frac{1}{2} e^{\mu(t-s) \Delta} \mathbb{P}\left(\operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)+\operatorname{div}\left(\rho v_{2} \otimes v_{1}\right)\right)+e^{c_{1}(t-s) \Delta} \mathbb{Q}\left(F\left(\rho, v_{1}, v_{2}\right)\right)
\end{array}\right) d s . \tag{2.14}
\end{align*}
$$

More precisely we want to prove that $\psi$ is a contractive map from $E_{R, M, T}$ to $E_{R, M, T}$ with $R, M>0$ :

$$
\begin{align*}
& E_{R, M, \alpha, T}=\left\{\left(\rho, \rho v_{1}, \rho v_{2}\right) \in L_{T}^{\infty}\left(L^{\infty}\left(\mathbb{R}^{N}\right)\right) \times \mathcal{E}_{T} \times \mathcal{E}_{T} /\right. \\
& \left.\rho \geq \frac{1}{M} \text { on }[0, T] \times \mathbb{R}^{N},\left\|\rho v_{1}\right\|_{\mathcal{E}_{T}}+\left\|\rho v_{2}\right\| \mathcal{E}_{T} \leq 2 R,\|\rho\|_{L_{T}^{\infty}\left(L^{\infty}\left(\mathbb{R}^{N}\right)\right.} \leq 2\left\|\rho_{0}\right\|_{L^{\infty}}\right\} \tag{2.15}
\end{align*}
$$

$E_{R, M, T}$ is endowed with the following norm:

$$
\begin{equation*}
\left\|\left(\rho, \rho v_{1}, \rho v_{2}\right)\right\|_{E_{R, M, \alpha, T}}=\left\|\left(\rho v_{1}, \rho v_{2}\right)\right\|_{\mathcal{E}_{T}}+\frac{1}{\beta}\|\rho\|_{L_{T}^{\infty}\left(L^{\infty}\right)} \tag{2.16}
\end{equation*}
$$

with $\beta>0$ sufficiently large that we will defined later and $E_{R, M, T}$ is a Banach space. Let $c>0$, we know that it exists $C>0$ (depending on $c$ ) such that for all $0<t \leq T$ we
have (see [22] p163):

$$
\begin{align*}
& \left\|\int_{0}^{t} e^{c(t-s) \Delta} \mathbb{P} \operatorname{div}(v \otimes w) d s\right\|_{\mathcal{E}_{T}} \leq C\|v\|_{\mathcal{E}_{T}}\|w\|_{\mathcal{E}_{T}} \\
& \left\|\int_{0}^{t} e^{c(t-s) \Delta} \mathbb{Q} \operatorname{div}(v \otimes w) d s\right\|_{\mathcal{E}_{T}} \leq C\|v\|_{\mathcal{E}_{T}}\|w\|_{\mathcal{E}_{T}} \tag{2.17}
\end{align*}
$$

Similarly we have (see [21]) for $C, C_{1}>0$ :

$$
\begin{align*}
& \left\|\int_{0}^{t} e^{c(t-s) \Delta} \nabla P(\rho)(s, \cdot) d s\right\|_{\mathcal{E}_{T}} \\
& \leq C\left(\sup _{0<t \leq T} t\|P(\rho)(t, \cdot)\|_{L^{\infty}}+\sup _{0<t \leq T, x \in \mathbb{R}^{N}} \frac{1}{t^{\frac{N}{2}}} \int_{0}^{t} \int_{B(x, \sqrt{t})}|P(\rho)(s, y)| d s d y\right)  \tag{2.18}\\
& \leq C_{1} T\|P(\rho)\|_{L_{T}^{\infty}\left(L^{\infty}\left(\mathbb{R}^{N}\right)\right)}
\end{align*}
$$

We deduce from (2.17), (2.13) and (2.18) that it exists $C>0$ such that for any $T>0$ we have:

$$
\begin{gather*}
\left\|\left(\psi_{2}\left(\rho, \rho v_{1}, \rho v_{2}\right), \psi_{3}\left(\rho, \rho v_{1}, \rho v_{2}\right)\right)\right\|_{\mathcal{E}_{T}^{2}} \leq C\left(\left\|\rho_{0} v_{1}(0, \cdot)\right\|_{b m o_{T}^{-1}}+\left\|\rho_{0} v_{2}(0, \cdot)\right\|_{b m o_{T}^{-1}}\right. \\
\left.+\left\|\frac{1}{\rho}\right\|_{L_{T}^{\infty}\left(L^{\infty}\right)}\left\|\rho v_{1}\right\|\left\|_{\mathcal{E}_{T}}\right\| \rho v_{2}\left\|_{\mathcal{E}_{T}}+T\right\| P(\rho) \|_{L_{T}^{\infty}\left(L^{\infty}\left(\mathbb{R}^{N}\right)\right)}\right) . \tag{2.19}
\end{gather*}
$$

Let us estimate now the $L^{\infty}$ norm of $\left(\psi_{1}\left(\rho, \rho v_{1}, \rho v_{2}\right), \frac{1}{\psi_{1}\left(\rho, \rho v_{1}, \rho v_{2}\right)}\right)$, we have then for any $t \in[0, T]$ and using the maximum principle:

$$
\left\{\begin{array}{l}
\left\|\psi_{1}\left(\rho, \rho v_{1}, \rho v_{2}\right)(t, \cdot)\right\|_{L^{\infty}} \leq\left\|\rho_{0}\right\|_{L^{\infty}}+\left\|\int_{0}^{t} e^{c_{1}(t-s) \Delta} \operatorname{div}\left(\rho v_{1}\right) d s\right\|_{L^{\infty}}  \tag{2.20}\\
\psi_{1}\left(\rho, \rho v_{1}, \rho v_{2}\right)(t, \cdot) \geq \min _{x \in \mathbb{R}^{N}} \rho_{0}(x)-\left\|\int_{0}^{t} e^{c_{1}(t-s) \Delta} \operatorname{div}\left(\rho v_{1}\right) d s\right\|_{L^{\infty}}
\end{array}\right.
$$

We have then using integration by parts and for $0<t \leq T$ it exists $C, C_{1}, C_{2}, C_{3}>0$ such that:

$$
\begin{align*}
& \left|\int_{0}^{t} e^{c_{1}(t-s) \Delta} \operatorname{div}(\rho v) d s\right| \leq\left|\int_{0}^{t} \int_{\mathbb{R}^{N}} \frac{1}{\left(4 c_{1} \pi(t-s)\right)^{\frac{N}{2}}} e^{-\frac{|x-y|^{2}}{4 c_{1}(t-s)}} \operatorname{div}\left(\rho v_{1}\right)(s, y) d y d s\right| \\
& \left.\leq C \int_{0}^{t} \int_{\mathbb{R}^{N}} \frac{1}{\left(4 \pi c_{1}(t-s)\right)^{\frac{N}{2}+\frac{1}{2}}} \frac{|x-y|}{\sqrt{4 c_{1}(t-s)}} e^{-\frac{|x-y|^{2}}{4 c_{1}(t-s)}}\left|\rho v_{1}(s, y)\right| d y d s \right\rvert\, \\
& \leq C_{1} \int_{0}^{t} \frac{1}{\sqrt{t-s}}\left\|\frac{1}{(4 \pi(t-s))^{\frac{N}{2}}} \frac{|\cdot|}{\sqrt{4(t-s)}} e^{-\frac{|\cdot|^{2}}{4(t-s)}}\right\|_{L^{1}}\left\|\rho v_{1}(s, \cdot)\right\|_{L^{\infty} d s}  \tag{2.21}\\
& \leq C_{2}\left\|\rho v_{1}\right\|_{\mathcal{E}_{T}} \int_{0}^{t} \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}} d s \\
& \leq C_{3}\left\|\rho v_{1}\right\|_{\mathcal{E}_{T}}
\end{align*}
$$

We deduce now from (2.20) and (2.21) that for $C>0$ large enough we have for $t \in[0, T]$ :

$$
\left\{\begin{array}{l}
\left\|\psi_{1}\left(\rho, \rho v_{1}, \rho v_{2}\right)(t, \cdot)\right\|_{L^{\infty}} \leq\left\|\rho_{0}\right\|_{L^{\infty}}+C\left\|\rho v_{1}\right\|_{\mathcal{E}_{T}}  \tag{2.22}\\
\psi_{1}\left(\rho, \rho v_{1}, \rho v_{2}\right)(t, \cdot) \geq \min _{x \in \mathbb{R}^{N}} \rho_{0}(x)-C\left\|\rho v_{1}\right\|_{\mathcal{E}_{T}}
\end{array}\right.
$$

Let us prove now that for $M, T, R$ well chosen, we have:

$$
\begin{equation*}
\psi\left(E_{R, M, T}\right) \subset E_{R, M, T} \tag{2.23}
\end{equation*}
$$

Taking $\frac{1}{M}=\frac{c}{2}$ and $R$ such that $2 C R \leq \min \left(\frac{c}{2},\left\|\rho_{0}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)$ (with $C>0$ defined in (2.22)) and where $\min _{x \in \mathbb{R}^{N}} \rho_{0}(x) \geq c>0$, we deduce from (2.22) that for any $t \in[0, T]$ and for any $\left(\rho, \rho v_{1}, \rho v_{2}\right) \in E_{R, M, \alpha, T}$ :

$$
\left\{\begin{array}{l}
\left\|\psi_{1}\left(\rho, \rho v_{1}, \rho v_{2}\right)(t, \cdot)\right\|_{L^{\infty}} \leq 2\left\|\rho_{0}\right\|_{L^{\infty}}  \tag{2.24}\\
\psi_{1}\left(\rho, \rho v_{1}, \rho v_{2}\right)(t, \cdot) \geq \frac{c}{2}=\frac{1}{M} .
\end{array}\right.
$$

Now we deduce from (2.19) that we have for any $\left(\rho, \rho v_{1}, \rho v_{2}\right) \in E_{R, M, T}$ and $C_{1}>0$ large enough:

$$
\begin{align*}
\|\left(\psi_{2}\left(\rho, \rho v_{1}, \rho v_{2}\right),\right. & \left.\psi_{3}\left(\rho, \rho v_{1}, \rho v_{2}\right)\right) \|_{\mathcal{E}_{T}^{2}} \leq C_{1}\left(\left\|\rho_{0} v_{1}(0, \cdot)\right\|_{b m o_{T}^{-1}}+\left\|\rho_{0} v_{2}(0, \cdot)\right\|_{b m o_{T}^{-1}}\right.  \tag{2.25}\\
& \left.+4 M R^{2}+T M_{1}\right)
\end{align*}
$$

We set now $R=4 C_{1}\left(\left\|\rho_{0} v_{1}(0, \cdot)\right\|_{b m o_{T}^{-1}}+\left\|\rho_{0} v_{2}(0, \cdot)\right\|_{b m o_{T}^{-1}}\right)$ and $M_{1}=\sup _{x \in\left[0,2\left\|\rho_{0}\right\|_{L^{\infty}}\right]} P(x)$. Now choosing $R$ such that $4 C_{1} M R^{2} \leq \frac{R}{2}$ and $T$ such that $C_{1} M T \leq \frac{R}{4}$ we deduce from (2.25) that we have for any $\left(\rho, \rho v_{1}, \rho v_{2}\right) \in E_{R, M, T}$ :

$$
\begin{equation*}
\left\|\left(\psi_{2}\left(\rho, \rho v_{1}, \rho v_{2}\right), \psi_{3}\left(\rho, \rho v_{1}, \rho v_{2}\right)\right)\right\|_{\mathcal{E}_{T}^{2}} \leq 2 R \tag{2.26}
\end{equation*}
$$

Fr In conclusion we have chosen $R$ and $T$ such that:

$$
\begin{equation*}
R \leq \frac{\min \left(\frac{c}{2},\left\|\rho_{0}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)}{2 C}, R \leq \frac{c^{2}}{32 C_{1}} \quad \text { and } \quad T \leq \frac{R}{4 C_{1} M} \tag{2.27}
\end{equation*}
$$

with $R=4 C_{1}\left(\left(\left\|\rho_{0} v_{1}(0, \cdot)\right\|_{b m o_{T}^{-1}}+\left\|\rho_{0} v_{2}(0, \cdot)\right\|_{b m o_{T}^{-1}}\right)\right.$. We deduce (2.23) from (2.24) and (2.26). It suffices then in practice to take $T \leq \frac{R}{4 C_{1} M}$ with $R=\min \left(\frac{c}{4 C}, \frac{c^{2}}{32 C_{1}}\right)$.

Let us prove now that $\psi$ is a contractive map. More precisely let us estimate $\| \psi\left(\rho, \rho v_{1}, \rho v_{2}\right)-$ $\psi\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right) \|_{E_{R, M, T}}$ with $\left(\left(\rho, \rho v_{1}, \rho v_{2}\right),\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)\right) \in E_{R, M, T}$. We have then:

$$
\begin{align*}
& \psi\left(\rho, \rho v_{1}, \rho v_{2}\right)-\psi\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)= \\
& \int_{0}^{t}\left(\begin{array}{c}
e^{c_{1}(t-s) \Delta} \operatorname{div}\left(\rho_{1} w_{1}-\rho v_{1}\right) \\
e^{\mu(t-s) \Delta} \mathbb{P}\left(F_{1}\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)-F_{1}\left(\rho, v_{1}, v_{2}\right)\right)+e^{c_{2}(t-s) \Delta} \mathbb{Q}\left(F\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)-F\left(\rho, v_{1}, v_{2}\right)\right) \\
e^{\mu(t-s) \Delta} \mathbb{P}\left(F_{1}\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)-F_{1}\left(\rho, v_{1}, v_{2}\right)\right)+e^{c_{1}(t-s) \Delta} \mathbb{Q}\left(F\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)-F\left(\rho, v_{1}, v_{2}\right)\right)
\end{array}\right) d s . \tag{2.28}
\end{align*}
$$

with:

$$
F_{1}\left(\rho, v_{1}, v_{2}\right)=\frac{1}{2}\left(\operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)+\operatorname{div}\left(\rho v_{2} \otimes v_{1}\right)\right)
$$

Now we have:

$$
\begin{align*}
& \operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)-\operatorname{div}\left(\rho_{1} w_{1} \otimes w_{2}\right)= \\
& \operatorname{div}\left(\frac{1}{\rho}\left(\rho v_{1}-\rho_{1} w_{1}\right) \otimes \rho v_{2}\right)+\operatorname{div}\left(\rho_{1} w_{1} \otimes\left(\rho v_{2}-\rho_{1} w_{2}\right) \frac{1}{\rho}\right)+\operatorname{div}\left(\rho_{1} w_{1} \otimes \rho_{1} w_{2} \frac{\left(\rho_{1}-\rho\right)}{\rho \rho_{1}}\right) \tag{2.29}
\end{align*}
$$

From (2.17) and (2.37), we deduce that for $C, C_{1}>0$ large enough and for any $\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)$, $\left.\left(\rho, v_{1}, v_{2}\right)\right)$ in $E_{R, M, T}$ we have for $C, C_{1}>0$ large enough:

$$
\begin{align*}
& \left\|\int_{0}^{t} e^{\mu(t-s) \Delta} \mathbb{P}\left(F_{1}\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)-F_{1}\left(\rho, v_{1}, v_{2}\right)\right) d s\right\|_{\mathcal{E}_{T}} \\
& \leq C\left\|\frac{1}{\rho}\right\|_{L_{T}^{\infty}\left(L^{\infty}\right)}\left(\left\|\rho v_{1}-\rho_{1} w_{1}\right\|_{\mathcal{E}_{T}}\left\|\rho v_{2}\right\|_{\mathcal{E}_{T}}+\left\|\rho v_{2}-\rho_{1} w_{2}\right\|_{\mathcal{E}_{T}}\left\|\rho_{1} w_{1}\right\|_{\mathcal{E}_{T}}\right. \\
& +\left\|\rho_{1} w_{1}\right\|\left\|_{\mathcal{E}_{T}}\right\| \rho_{1} w_{2}\left\|_{\mathcal{E}_{T}}\right\| \frac{1}{\rho_{1}}\left\|_{L_{T}^{\infty}\left(L^{\infty}\right)}\right\| \rho-\rho_{1}\left\|_{L_{T}^{\infty}\left(L^{\infty}\right)}+\right\| \rho v_{2}-\rho_{1} w_{2}\left\|_{\mathcal{E}_{T}}\right\| \rho v_{1} \|_{\mathcal{E}_{T}}  \tag{2.30}\\
& \left.+\left\|\rho v_{1}-\rho_{1} w_{1}\right\|_{\mathcal{E}_{T}}\left\|\rho_{1} w_{2}\right\|_{\mathcal{E}_{T}}\right) \\
& \leq C_{1} M\left(R+M \beta R^{2}\right)\left\|\left(\rho, \rho v_{1}, \rho v_{2}\right)-\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)\right\|_{E_{R, M, T}}
\end{align*}
$$

We proceed similarly for the part $\left\|\int_{0}^{t} e^{c_{i}(t-s) \Delta} \mathbb{Q}\left(F_{1}\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)-F_{1}\left(\rho, v_{1}, v_{2}\right)\right) d s\right\|_{\mathcal{E}_{T}}$ with $i=1$, 2. Similarly from (2.18), we have for $c=c_{1}, c_{2}$ and $C>0$ large enough:

$$
\begin{align*}
& \left\|\int_{0}^{t} e^{c_{1}(t-s) \Delta} \nabla\left(P(\rho)-P\left(\rho_{1}\right)\right) d s\right\|_{\mathcal{E}_{T}} \leq C T\left\|P(\rho)-P\left(\rho_{1}\right)\right\|_{L_{T}^{\infty}\left(L^{\infty}\left(\mathbb{R}^{N}\right)\right)}  \tag{2.31}\\
& \leq C M_{2} T \beta\left\|\left(\rho, \rho v_{1}, \rho v_{2}\right)-\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)\right\|_{E_{R, M, T}}
\end{align*}
$$

with $M_{2}=\sup _{x \in\left[0,2\left\|\rho_{0}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right]}\left|P^{\prime}(x)\right|$. From (2.28), (2.37), (2.30) and (2.31) we deduce that it exists $C>0$ sufficiently large such that:

$$
\begin{align*}
& \left\|\psi_{2}\left(\rho, \rho v_{1}, \rho v_{2}\right)-\psi_{2}\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right), \psi_{3}\left(\rho, \rho v_{1}, \rho v_{2}\right)-\psi_{3}\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)\right\|_{\mathcal{E}_{T}}  \tag{2.32}\\
& \quad \leq C\left[M\left(R+M \beta R^{2}\right)+M_{2} \beta T\right]\left\|\left(\rho, \rho v_{1}, \rho v_{2}\right)-\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)\right\|_{E_{R, M, T}}
\end{align*}
$$

From (2.21), it yields that for $C>0$ large enough:

$$
\begin{equation*}
\frac{1}{\beta}\left\|\psi_{1}\left(\rho, \rho v_{1}, \rho v_{2}\right)-\psi_{1}\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)\right\|_{L_{T}^{\infty}\left(L^{\infty}\left(\mathbb{R}^{N}\right)\right)} \leq \frac{C}{\beta}\left\|\left(\rho v_{1}-\rho_{1} w_{1}\right)\right\|_{\mathcal{E}_{T}} \tag{2.33}
\end{equation*}
$$

Combining (2.32) and (2.33), we deduce that we have for $C>0$ large enough:

$$
\begin{align*}
& \left\|\psi\left(\rho, \rho v_{1}, \rho v_{2}\right)-\psi\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)\right\|_{\mathcal{E}_{T}} \\
& \leq C\left(M\left(R+M \beta R^{2}\right)+M_{2} \beta T+\frac{1}{\beta}\right)\left\|\left(\rho, \rho v_{1}, \rho v_{2}\right)-\left(\rho_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)\right\|_{E_{R, M, T}} \tag{2.34}
\end{align*}
$$

It suffices now to choose $\beta, T$ and $R$ such that:

$$
\begin{equation*}
C\left(M\left(R+M \beta R^{2}\right)+M_{2} \beta T+\frac{1}{\beta}\right)<1 \tag{2.35}
\end{equation*}
$$

It proves in particular that the map $\psi$ is contractive and it concludes the proof of the theorem 1.2.
Let us prove now the corollary 1, if $\rho_{0} v_{1}(0, \cdot) \in \overline{\left(\mathcal{D}\left(\mathbb{R}^{N}\right)\right)^{N}}\left(b m o^{-1}\left(\mathbb{R}^{N}\right)\right)^{N}$ then we can observe that:

$$
\begin{equation*}
\lim _{T \rightarrow 0}\left\|e^{t \Delta} \rho_{0} v_{1}(0, \cdot)\right\|_{\mathcal{E}_{T}}=0 \tag{2.36}
\end{equation*}
$$

Indeed assume that $w_{0} \in\left(\mathcal{D}\left(\mathbb{R}^{N}\right)\right)^{N}$ then we have using the maximum principle for the heat equation:

$$
\begin{aligned}
\left\|e^{t \Delta} w\right\|_{\mathcal{E}_{T}}^{2} & \leq\|w\|_{L^{\infty}} \sup _{0<t<T, x_{0} \in \mathbb{R}^{N}} \frac{1}{t^{\frac{N}{2}}} \int_{0}^{t} \int_{B\left(x_{0}, \sqrt{t}\right)} d x d s \\
& \leq T\|w\|_{L^{\infty}}
\end{aligned}
$$

By density we can conclude. In particular it implies that for $T>0$ small enough we have:

$$
\begin{equation*}
\left\|e^{t \Delta} \rho_{0} v_{1}(0, \cdot)\right\|_{\mathcal{E}_{T}}+\left\|e^{t \Delta} \rho_{0} v_{2}(0, \cdot)\right\|_{\mathcal{E}_{T}}<\varepsilon_{1} \tag{2.37}
\end{equation*}
$$

Using the proof of the Theorem 1.1), we conclude that there exists a strong solution ( $\rho, \rho u$ ) on $[0, T]$. Indeed it we follow the previous proof, it suffices to fix $R$ and $T$ sufficiently small and verifying the previous estimate. By density it exist $w_{1} \in\left(\mathcal{D}\left(\mathbb{R}^{N}\right)\right)^{N}$ such that:

$$
\left\|e^{t \Delta}\left(w_{1}-\rho_{0} v_{1}(0, \cdot)\right)\right\|_{\mathcal{E}_{T}} \leq\left\|w_{1}-\rho_{0} v_{1}(0, \cdot)\right\|_{b m o_{T}^{-1}} \leq \frac{R}{2}
$$

Now we choose $T_{1} \leq T$ sufficiently small such that:

$$
\left\|e^{t \Delta} w_{1}\right\|_{\mathcal{E}_{T}} \leq \frac{R}{2}
$$

We have now:

$$
\left.\| e^{t \Delta} \rho_{0} v_{1}(0, \cdot)\right) \|_{\mathcal{E}_{T_{1}}} \leq R
$$

and $R, T_{1}$ verifies the estimates of the proof of the Theorem 1.1.

## 3 Proof of the Theorem 1.2

In order to prove the theorem 1.2, we are going to start by studying the linear system associated to (1.5):

$$
\left\{\begin{array}{l}
\partial_{t} q-c \Delta q+\operatorname{div} m=F  \tag{3.38}\\
\partial_{t} m-\mu \Delta m-\alpha \nabla \operatorname{div} m+\beta \nabla q=G
\end{array}\right.
$$

with $c>0, \mu>0, \mu+\alpha>0$ and $\beta>0$. In the sequel we will note $W_{c, \mu, \alpha, \beta}$ the semigroup associated to the previous system and we have in particular from the Duhamel formula:

$$
\begin{equation*}
\binom{q}{m}(t, \cdot)=W_{c, \mu, \alpha, \beta}(t)\binom{q_{0}}{m_{0}}+\int_{0}^{t} W_{c, \mu, \alpha, \beta}(t-s)\binom{F}{G}(s) d s \tag{3.39}
\end{equation*}
$$

This system has been studied by Bahouri et al (see [3]) in the framework of the Besov space $B_{2,1}^{s}$ when $c=0$. We are going now to extend this study to the case of general Besov space of the form $B_{2, r}^{s}$. Setting:

$$
d=\Lambda^{-1} \operatorname{div} m \text { and } \Omega=\Lambda^{-1} \operatorname{curlm}
$$

with by definition $\widehat{\Lambda^{s} f}(\xi)=|\xi|^{s} \widehat{f}(\xi)$ when $\alpha \in \mathbb{R}$ and for $f$ a temperated distribution, we will study the following system:

$$
\left\{\begin{array}{l}
\partial_{t} q-c \Delta q+\Lambda d=F  \tag{3.40}\\
\partial_{t} d-\bar{\nu} \Delta d-\beta \Lambda q=\Lambda^{-1} \operatorname{div} G \\
\partial_{t} \Omega-\mu \Delta \Omega=\Lambda^{-1} \operatorname{curl} G
\end{array}\right.
$$

with $\bar{\nu}=(\mu+\alpha)$. We refer to [3] for the definition of the Chemin-Lerner spaces $\widetilde{L}_{T}^{\rho}\left(B_{p, r}^{s}\right)$ with $(\rho, p, r) \in[1,+\infty]^{3}, T>0$ and $s \in \mathbb{R}$.
Proposition 3.1 Let $T \in(0,+\infty]$. We assume that $\left(q_{0}, u_{0}\right)$ belongs to $B_{2, \infty}^{s-1, s} \times\left(B_{2, \infty}^{s-1}\right)^{N}$ with the source terms $(F, G)$ in $\widetilde{L}_{T}^{1}\left(B_{2, \infty}^{s-1} \cap B_{2, \infty}^{s}\right) \times\left(\widetilde{L}_{T}^{1}\left(B_{2, \infty}^{s}\right)\right)^{N}$.
$\operatorname{Let}(q, u) \in\left(\widetilde{L}_{T}^{\infty}\left(B_{2, \infty}^{s-1} \times B_{2, \infty}^{s}\right) \cap \widetilde{L}_{T}^{1}\left(B_{2, \infty}^{s+1} \times B_{2, \infty}^{s+2}\right)\right) \times\left(\widetilde{L}_{T}^{\infty}\left(B_{2, \infty}^{s-1}\right) \cap \widetilde{L}_{T}^{1}\left(B_{2, \infty}^{s+1}\right)\right)^{N}$ be a solution of the system (3.38), then there exists a universal constant $C$ such that for any $T>0$ we have:

$$
\begin{equation*}
\|(q, \nabla q, u)\|_{\widetilde{L}_{T}^{\infty}\left(B_{2, \infty}^{s-1}\right) \cap \widetilde{L}_{T}^{1}\left(B_{2, \infty}^{s+1}\right)} \leq C\left(\left\|\left(q_{0}, \nabla q_{0}, u_{0}\right)\right\|_{B_{2, \infty}^{s-1}}+\|(F, \nabla F, G)\|_{\widetilde{L}_{T}^{1}\left(B_{2, \infty}^{s-1}\right)}\right) \tag{3.41}
\end{equation*}
$$

Proof: Let $(q, m)$ be a solution of (3.38), we are going to separate the case of the low and high frequencies, which have a different behavior concerning the control of the derivative index for the Besov spaces. Our goal consists now in studying the system (3.40) and in particular to estimate ( $q, d$ ) and $\Omega$. We observe that $\Omega$ verifies simply an heat equation and classical estimates on the heat equation in Besov spaces give (see [3]):

$$
\begin{equation*}
\|\Omega\|_{\widetilde{L}_{T}^{\infty}\left(B_{2, \infty}^{s-1}\right) \cap \widetilde{L}_{T}^{1}\left(B_{2, \infty}^{s+1}\right)} \leq C\left(\left\|\Omega_{0}\right\|_{B_{2, \infty}^{s-1}}+\|G\|_{\widetilde{L}_{T}^{1}\left(B_{2, \infty}^{s-1}\right)}\right) \tag{3.42}
\end{equation*}
$$

Let us study now the unknowns $(q, d)$.

## Case of low frequencies

We assume here that $l \in \mathbb{Z}$ with $l \leq l_{0}$ (we will determine later $l_{0} \in \mathbb{Z}$ ). Applying operator $\Delta_{l}$ (see [3] for the definition of $\Delta_{l}$ ) to the system (3.40) and denoting $g_{l}=\Delta_{l} g$, we obtain the following system:

$$
\left\{\begin{array}{l}
\frac{d}{d t} q_{l}+c \Lambda^{2} q_{l}+\Lambda d_{l}=F_{l}  \tag{3.43}\\
\frac{d}{d t} d_{l}+\bar{\nu} \Lambda^{2} d_{l}-\beta \Lambda q_{l}=G_{l}
\end{array}\right.
$$

We set:

$$
\begin{equation*}
f_{l}^{2}=\beta\left\|q_{l}\right\|_{L^{2}}^{2}+\left\|d_{l}\right\|_{L^{2}}^{2} \tag{3.44}
\end{equation*}
$$

Taking the $L^{2}$ scalar product of the first equation of (3.43) with $q_{l}$ and of the second equation with $d_{l}$, we get the following two identities:

$$
\left\{\begin{array}{l}
\frac{1}{2} \frac{d}{d t}\left\|q_{l}\right\|_{L^{2}}^{2}+\left(\Lambda d_{l}, q_{l}\right)+c\left\|\Lambda q_{l}\right\|_{L^{2}}^{2}=\left(F_{l}, q_{l}\right)  \tag{3.45}\\
\frac{1}{2} \frac{d}{d t}\left\|d_{l}\right\|_{L^{2}}^{2}+\bar{\nu}\left\|\Lambda d_{l}\right\|_{L^{2}}^{2}-\beta\left(\Lambda q_{l}, d_{l}\right)=\left(G_{l}, d_{l}\right)
\end{array}\right.
$$

We deduce that:

$$
\begin{equation*}
\frac{1}{2} \frac{d}{d t} f_{l}^{2}+\left(\beta c\left\|\Lambda q_{l}\right\|_{L^{2}}+\bar{\nu}\left\|\Lambda d_{l}\right\|_{L^{2}}^{2}\right) \leq\left\|G_{l}\right\|_{L^{2}}\left\|d_{l}\right\|_{L^{2}}+\beta\left\|F_{l}\right\|_{L^{2}}\left\|q_{l}\right\|_{L^{2}} \tag{3.46}
\end{equation*}
$$

From the definition of $f_{l}^{2}$ we deduce that it exists $\alpha_{1}, C>0$ independent on $l_{0}$ such that:

$$
\begin{equation*}
\frac{1}{2} \frac{d}{d t} f_{l}^{2}+\alpha_{1} 2^{2 l} f_{l}^{2} \leq C f_{l}\left(\left\|G_{l}\right\|_{L^{2}}+\left\|F_{l}\right\|_{L^{2}}\right) \tag{3.47}
\end{equation*}
$$

We have in particular for $l \leq l_{0}$ :

$$
\begin{equation*}
\frac{1}{2} \frac{d}{d t} f_{l}+\alpha_{1} 2^{2 l} f_{l} \leq C\left(\left\|G_{l}\right\|_{L^{2}}+\left\|F_{l}\right\|_{L^{2}}\right) \tag{3.48}
\end{equation*}
$$

## Case of high frequencies

We consider now the case where $l \geq l_{0}+1$ and we define now $f_{l}$ as follows:

$$
f_{l}^{2}=\left\|\Lambda q_{l}\right\|_{L^{2}}^{2}+A\left\|d_{l}\right\|_{L^{2}}^{2}-\frac{2}{c+\bar{\nu}}\left(\Lambda q_{l}, d_{l}\right)
$$

with $A>0$ to be determinated. We apply the operator $\Lambda \Delta_{l}$ to the first equation of (3.40), multiply by $\Lambda q_{l}$ and integrate over $\mathbb{R}^{N}$, so we obtain:

$$
\begin{equation*}
\frac{1}{2} \frac{d}{d t}\left\|\Lambda q_{l}\right\|_{L^{2}}^{2}+c\left\|\Lambda^{2} q_{l}\right\|_{L^{2}}^{2}+\left(\Lambda^{2} d_{l}, \Lambda q_{l}\right)=\left(\Lambda F_{l}, \Lambda q_{l}\right) \tag{3.49}
\end{equation*}
$$

Moreover we have in a similar way:

$$
\left\{\begin{array}{l}
\frac{1}{2} \frac{d}{d t}\left\|d_{l}\right\|_{L^{2}}^{2}+\bar{\nu}\left\|\Lambda d_{l}\right\|_{L^{2}}^{2}-\beta\left(\Lambda q_{l}, d_{l}\right)=\left(G_{l}, d_{l}\right)  \tag{3.50}\\
\frac{d}{d t}\left(\Lambda q_{l}, d_{l}\right)+\left\|\Lambda d_{l}\right\|_{L^{2}}^{2}-\beta\left\|\Lambda q_{l}\right\|_{L^{2}}^{2}+(c+\bar{\nu})\left(\Lambda^{2} d_{l}, \Lambda q_{l}\right)=\left(\Lambda F_{l}, d_{l}\right)+\left(G_{l}, \Lambda q_{l}\right)
\end{array}\right.
$$

By linear combination of (3.49)-(3.50) we have:

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} f_{l}^{2}+c\left\|\Lambda^{2} q_{l}\right\|_{L^{2}}^{2}+\left(A \bar{\nu}-\frac{1}{c+\bar{\nu}}\right)\left\|\Lambda d_{l}\right\|_{L^{2}}^{2}+\frac{\beta}{c+\bar{\nu}}\left\|\Lambda q_{l}\right\|_{L^{2}}^{2}-A \beta\left(\Lambda q_{l}, d_{l}\right)  \tag{3.51}\\
& =-\frac{1}{c+\bar{\nu}}\left(\Lambda F_{l}, d_{l}\right)-\frac{1}{c+\bar{\nu}}\left(G_{l}, \Lambda q_{l}\right)+\left(\Lambda F_{l}, \Lambda q_{l}\right)+A\left(G_{l}, d_{l}\right)
\end{align*}
$$

We have now in using Young inequalities for all $a>0$ :

$$
\begin{gather*}
\left|\left(d_{l}, \Lambda q_{l}\right)\right| \leq \frac{a}{2}\left\|\Lambda q_{l}\right\|_{L^{2}}^{2}+\frac{1}{2 a}\left\|d_{l}\right\|_{L^{2}}^{2} \\
\frac{1}{2} \frac{d}{d t} f_{l}^{2}+c\left\|\Lambda^{2} q_{l}\right\|_{L^{2}}^{2}+\left(A \bar{\nu}-\frac{1}{c+\bar{\nu}}\right)\left\|\Lambda d_{l}\right\|_{L^{2}}^{2}+\left(\frac{\beta}{c+\bar{\nu}}-\frac{A a \beta}{2}\right)\left\|\Lambda q_{l}\right\|_{L^{2}}^{2}-\frac{A \beta}{2 a}\left\|d_{l}\right\|_{L^{2}}^{2} \\
\leq-\frac{1}{c+\bar{\nu}}\left(\Lambda F_{l}, d_{l}\right)-\frac{1}{c+\bar{\nu}}\left(G_{l}, \Lambda q_{l}\right)+\left(\Lambda F_{l}, \Lambda q_{l}\right)+A\left(G_{l}, d_{l}\right) \tag{3.52}
\end{gather*}
$$

We have now since $l \geq l_{0}+1$ for $c_{1}>0$ :

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} f_{l}^{2}+c\left\|\Lambda^{2} q_{l}\right\|_{L^{2}}^{2}+\left(A \bar{\nu}-\frac{1}{c+\bar{\nu}}-2^{-2 l_{0}} \frac{A c_{1} \beta}{2 a}\right)\left\|\Lambda d_{l}\right\|_{L^{2}}^{2}+\left(\frac{\beta}{c+\bar{\nu}}-\frac{A a \beta}{2}\right)\left\|\Lambda q_{l}\right\|_{L^{2}}^{2} \\
& \leq-\frac{1}{c+\bar{\nu}}\left(\Lambda F_{l}, d_{l}\right)-\frac{1}{c+\bar{\nu}}\left(G_{l}, \Lambda q_{l}\right)+\left(\Lambda F_{l}, \Lambda q_{l}\right)+A\left(G_{l}, d_{l}\right) \tag{3.53}
\end{align*}
$$

We choose now $a, A$ and $l_{0}$ as follows:

$$
\begin{equation*}
A \bar{\nu}=\frac{2 M}{c+\bar{\nu}}, \frac{\beta}{c+\bar{\nu}}-\frac{A a \beta}{2} \geq 0, A \bar{\nu}-\frac{1}{c+\bar{\nu}}-2^{-2 l_{0}} \frac{A \beta}{2 a} \geq \frac{1}{2(c+\bar{\nu})} . \tag{3.54}
\end{equation*}
$$

with $M>1$ sufficiently large to determine later. Now from the definition of $f_{l}^{2}$ and using Young inequality, we have:
$\frac{1}{2}\left\|\Lambda q_{l}\right\|_{L^{2}}^{2}+\left(\frac{2 M}{\bar{\nu}(c+\bar{\nu})}-\frac{2}{(c+\bar{\nu})^{2}}\right)\left\|d_{l}\right\|_{L^{2}}^{2} \leq f_{l}^{2} \leq \frac{3}{2}\left\|\Lambda q_{l}\right\|_{L^{2}}^{2}+\left(\frac{2 M}{\bar{\nu}(c+\bar{\nu})}+\frac{2}{(c+\bar{\nu})^{2}}\right)\left\|d_{l}\right\|_{L^{2}}^{2}$
We choose now $M$ sufficiently large such that $\frac{2 M}{\bar{\nu}(c+\bar{\nu})}-\frac{2}{(c+\bar{\nu})^{2}} \geq \frac{1}{2}$. Using (3.53) and (3.55) we deduce that there exists constants $\alpha_{2}>0$ and $C_{1}>0$ such that for $l \geq l_{0}+1$ we have:

$$
\begin{equation*}
\frac{1}{2} \frac{d}{d t} f_{l}^{2}+\alpha_{2} 2^{2 l} f_{l}^{2} \leq C_{1} f_{l}\left(2^{l}\left\|F_{l}\right\|_{L^{2}}+\left\|G_{l}\right\|_{L^{2}}\right) \tag{3.56}
\end{equation*}
$$

It yields that:

$$
\begin{equation*}
\frac{1}{2} \frac{d}{d t} f_{l}+\alpha_{2} 2^{2 l} f_{l} \leq C_{1}\left(2^{l}\left\|F_{l}\right\|_{L^{2}}+\left\|G_{l}\right\|_{L^{2}}\right) \tag{3.57}
\end{equation*}
$$

## Final estimates

Integrating over $[0, t]$ the estimates (3.48) and (3.57) and multiplying by $2^{l(s-1)}$, we have for $C_{2}>0$ large enough and any $l \in \mathbb{Z}$ :

$$
\begin{align*}
& \frac{1}{2} 2^{l(s-1)} f_{l}(t)+\min \left(\alpha_{1}, \alpha_{2}\right) 2^{2 l s} \int_{0}^{t} f_{l}(s) d s  \tag{3.58}\\
& \quad \leq \frac{1}{2} f_{l}(0)+C_{2}\left(\left\|G_{l}\right\|_{\widetilde{L}_{t}^{1}\left(B_{2, \infty}^{s-1}\right)}+\left\|F_{l}\right\|_{\widetilde{L}_{t}^{1}\left(B_{2, \infty}^{s-1} \cap B_{2, \infty}^{s-1}\right)}\right)
\end{align*}
$$

From the definition of $f_{l}$ we deduce the estimate (3.41) using in particular (3.55). It concludes the proof of the proposition.

Let us study again the system (3.38) and applying div to the momentum equation, we have:

$$
\left\{\begin{array}{l}
\partial_{t} q-c \Delta q+\operatorname{div} m=0  \tag{3.59}\\
\partial_{t} \operatorname{div} m-\bar{\nu} \Delta \operatorname{div} m+\beta \Delta q=0
\end{array}\right.
$$

We assume now that $\mu, c, \nu, \beta>0$ and $\nu \neq c$. The only case where $\nu=c$ will be in the sequel the case $\kappa^{2}=\mu^{2}$. We will study this case later.
When we apply the Fourier transform $\mathcal{F}$, we have then:

$$
\begin{equation*}
\mathcal{F}\binom{q}{m}(t, \xi)=e^{-t A(\xi)} \mathcal{F}\binom{q_{0}}{m_{0}}(\xi) \tag{3.60}
\end{equation*}
$$

with:

$$
A(\xi)=\left(\begin{array}{cc}
c|\xi|^{2} & 1 \\
-\beta|\xi|^{2} & \bar{\nu}|\xi|^{2}
\end{array}\right)
$$

The characteristic polynom is:

$$
P_{A(\xi)}(\lambda)=\lambda^{2}-\lambda|\xi|^{2}(c+\nu)+c \nu|\xi|^{4}+\beta|\xi|^{2} .
$$

The eigenvalues are:

- if $|\xi|^{2} \geq \frac{4 \beta}{(\nu-c)^{2}}:$

$$
\begin{aligned}
& \lambda_{1 h}(\xi)=\frac{1}{2}|\xi|^{2}\left((\nu+c)\left|+|\nu-c| \sqrt{1-\frac{4 \beta}{(\nu-c)^{2}|\xi|^{2}}}\right)\right. \\
& \lambda_{2 h}(\xi)=\frac{1}{2}|\xi|^{2}\left((\nu+c)\left|-|\nu-c| \sqrt{1-\frac{4 \beta}{(\nu-c)^{2}|\xi|^{2}}}\right) .\right.
\end{aligned}
$$

- if $|\xi|^{2}<\frac{4 \beta}{(\nu-c)^{2}}$ :

$$
\begin{aligned}
& \lambda_{1 l}(\xi)=\frac{1}{2}|\xi|^{2}\left((\nu+c)|+i| \nu-c \left\lvert\, \sqrt{\frac{4 \beta}{(\nu-c)^{2}|\xi|^{2}}-1}\right.\right) \\
& \lambda_{2 l}(\xi)=\frac{1}{2}|\xi|^{2}\left((\nu+c)|-i| \nu-c \left\lvert\, \sqrt{\frac{4 \beta}{(\nu-c)^{2}|\xi|^{2}}-1} .\right.\right.
\end{aligned}
$$

## High Frequencies

When $|\xi|^{2} \geq \frac{4 \beta}{(\nu-c)^{2}}$ we have:

$$
\mathcal{F}\binom{q}{\operatorname{div} m}(t, \xi)=P(\xi)\left(\begin{array}{cc}
e^{-t \lambda_{1 h}(\xi)} & 0  \tag{3.61}\\
0 & e^{-t \lambda_{2 h}(\xi)}
\end{array}\right)(P(\xi))^{-1} \mathcal{F}\binom{q_{0}}{\operatorname{div} m_{0}}(\xi) .
$$

with:

$$
\begin{aligned}
& P(\xi)=\left(\begin{array}{cc}
\frac{-2}{|\xi|^{2}(c-\nu-|c-\nu| \alpha(\xi))} & \frac{-2}{|\xi|^{2}(c-\nu+|c-\nu| \alpha(\xi))} \\
1 & 1
\end{array}\right) \\
& (P(\xi))^{-1}=-\frac{\beta}{|c-\nu| \alpha(\xi)}\left(\begin{array}{cc}
1 & \frac{2}{|\xi|^{2}(c-\nu+|c-\nu| \alpha(\xi))} \\
-1 & \frac{-2}{|\xi|^{2}(c-\nu-|c-\nu| \alpha(\xi))}
\end{array}\right)
\end{aligned}
$$

We have denoted $\alpha(\xi)$ by $\alpha(\xi)=\sqrt{1-\frac{4 \beta}{(\nu-c)^{2}|\xi|^{2}}}$. Finally we get:

$$
\begin{align*}
& \mathcal{F}\binom{q}{m}(t, \xi)=-\frac{\beta}{|c-\nu| \alpha(\xi)} \\
& \left(\begin{array}{cc}
\frac{-2 e^{-t \lambda_{1 h}(\xi)}}{|\xi|^{2}(c-\nu-|c-\nu| \alpha(\xi))}+\frac{2 e^{-t \lambda_{2 h}(\xi)}}{|\xi|^{2}(c-\nu+|c-\nu| \alpha(\xi))} & \frac{1}{\beta|\xi|^{2}}\left(e^{-t \lambda_{2 h}(\xi)}-e^{-t \lambda_{1 h}(\xi)}\right) \\
e^{-t \lambda_{1 h}(\xi)}-e^{-t \lambda_{2 h}(\xi)} & \frac{2 e^{t t \lambda_{h}(\xi)}}{|\xi|^{2}(c-\nu+|c-\nu| \alpha(\xi))}-\frac{2 e^{-t \lambda_{2 h}(\xi)}}{|\xi|^{2}(c-\nu-|c-\nu| \alpha(\xi))}
\end{array}\right) \\
& \times \mathcal{F}\binom{q_{0}}{\operatorname{div} m_{0}}(\xi) . \tag{3.62}
\end{align*}
$$

It yields then:

$$
\begin{align*}
& \mathcal{F} q(t, \xi)=\left(e^{-t \lambda_{1 h}(\xi)}-e^{-t \lambda_{2 h}(\xi)}\right)\left(\frac{\operatorname{sgn}(c-\nu)}{2 \alpha(\xi)} \mathcal{F} q_{0}(\xi)+\frac{1}{|\xi|^{2}|c-\nu| \alpha(\xi)} \mathcal{F} \operatorname{div} m_{0}(\xi)\right) \\
& +\frac{1}{2}\left(e^{-t \lambda_{2 h}(\xi)}+e^{-t \lambda_{1 h}(\xi)}\right) \mathcal{F} q_{0}(\xi) \\
& \mathcal{F} \operatorname{div} m(t, \xi)=\left(e^{-t \lambda_{2 h}(\xi)}-e^{-t \lambda_{1 h}(\xi)}\right)\left(\frac{\beta}{|c-\nu| \alpha(\xi)} \mathcal{F} q_{0}(\xi)+\frac{\operatorname{sgn}(c-\nu)}{2 \alpha(\xi)} \mathcal{F} \operatorname{div} m_{0}(\xi)\right) \\
& +\frac{1}{2}\left(e^{-t \lambda_{2 h}(\xi)}+e^{-t \lambda_{1 h}(\xi)}\right) \mathcal{F} \operatorname{div}_{0}(\xi) \tag{3.63}
\end{align*}
$$

## Low Frequencies

When $|\xi|^{2}<\frac{4 \beta}{(\nu-c)^{2}}$ we have:

$$
\mathcal{F}\binom{q}{\operatorname{div} m}(t, \xi)=P_{1}(\xi)\left(\begin{array}{cc}
e^{-t \lambda_{1 l}(\xi)} & 0  \tag{3.64}\\
0 & e^{-t \lambda_{2 l}(\xi)}
\end{array}\right)\left(P_{1}(\xi)\right)^{-1} \mathcal{F}\binom{q_{0}}{\operatorname{div} m_{0}}(\xi)
$$

In addition if we denote $\alpha_{1}(\xi)=\sqrt{\frac{4 \beta}{(\nu-c)^{2}|\xi|^{2}}-1}$, we have:

$$
\begin{aligned}
& P_{1}(\xi)=\left(\begin{array}{cc}
\frac{-2}{|\xi|^{2}\left(c-\nu-i|c-\nu| \alpha_{1}(\xi)\right)} & \frac{-2}{|\xi|^{2}\left(c-\nu+i|c-\nu| \alpha_{1}(\xi)\right)}
\end{array}\right) \\
& \left(P_{1}(\xi)\right)^{-1}=\frac{1}{|c-\nu| \alpha_{1}(\xi)}\left(\begin{array}{cc}
1 & \frac{2}{|\xi|^{2}\left(c-\nu+i|c-\nu| \alpha_{1}(\xi)\right)} \\
-1 & \frac{-2}{|\xi|^{2}\left(c-\nu-i|c-\nu| \alpha_{1}(\xi)\right)}
\end{array}\right)
\end{aligned}
$$

Finally we have:

$$
\begin{align*}
& \mathcal{F}\binom{q}{m}(t, \xi)=\frac{\beta i}{|c-\nu| \alpha_{1}(\xi)} \\
& \left(\begin{array}{cc}
\frac{-2 e^{-t \lambda_{1 l}(\xi)}}{|\xi|^{2}\left(c-\nu-i|c-\nu| \alpha_{1}(\xi)\right)}+\frac{2 e^{-t \lambda_{2 l}(\xi)}}{|\xi|^{2}\left(c-\nu+i|c-\nu| \alpha_{1}(\xi)\right)} & \left.\frac{1}{\beta|\xi|^{2}\left(e^{-t \lambda_{2 l}(\xi)}\right.}-e^{-t \lambda_{1 l}(\xi)}\right) \\
e^{-t \lambda_{1 l}(\xi)}-e^{-t \lambda_{2 l}(\xi)} & \frac{2 e^{--\lambda_{l l}(\xi)}}{|\xi|^{2}\left(c-\nu+i|c-\nu| \alpha_{1}(\xi)\right)}-\frac{2 e^{-t \lambda_{2 l}(\xi)}}{|\xi|^{2}\left(c-\nu-i|c-\nu| \alpha_{1}(\xi)\right)}
\end{array}\right) \\
& \times \mathcal{F}\binom{q_{0}}{\operatorname{div} m_{0}}(\xi) . \tag{3.65}
\end{align*}
$$

We obtain finally:

$$
\begin{align*}
& \mathcal{F} q(t, \xi)=i\left(e^{-t \lambda_{2 l}(\xi)}-e^{-t \lambda_{1 l}(\xi)}\right)\left(\frac{\operatorname{sgn}(c-\nu)}{2 \alpha_{1}(\xi)} \mathcal{F} q_{0}(\xi)+\frac{1}{|\xi|^{2}|c-\nu| \alpha_{1}(\xi)} \mathcal{F} \operatorname{div} m_{0}(\xi)\right) \\
& +\frac{1}{2}\left(e^{-t \lambda_{2 l}(\xi)}+e^{-t \lambda_{11}(\xi)}\right) \mathcal{F} q_{0}(\xi) \\
& \mathcal{F} \operatorname{div} m(t, \xi)=i\left(e^{-t \lambda_{1 l}(\xi)}-e^{-t \lambda_{2 l}(\xi)}\right)\left(\frac{\beta}{|c-\nu| \alpha_{1}(\xi)} \mathcal{F} q_{0}(\xi)+\frac{\operatorname{sgn}(c-\nu)}{2 \alpha_{1 l}(\xi)} \mathcal{F} \operatorname{div} m_{0}(\xi)\right) \\
& +\frac{1}{2}\left(e^{-t \lambda_{2 l}(\xi)}+e^{-t \lambda_{1 l}(\xi)}\right) \mathcal{F} \operatorname{div}_{0}(\xi) \tag{3.66}
\end{align*}
$$

Proposition 3.2 Let $\phi$ be a smooth function supported in the shell $\mathcal{C}\left(0, R_{1}, R_{2}\right)$ with $0<R_{1}<R_{2}$. There exist two positive constants $\kappa$ and $C$ depending only on $\phi$ and such that for all $t \geq 0$ and $\lambda>0$, we have:

$$
\begin{equation*}
\left\|\phi\left(\lambda^{-1} D\right) W(t)\binom{q_{0}}{m_{0}}\right\|_{L^{2}} \leq C e^{-\kappa t \lambda^{2}}\left\|\phi\left(\lambda^{-1} D\right)\binom{q_{0}}{m_{0}}\right\|_{L^{2}} \tag{3.67}
\end{equation*}
$$

Proof: From (3.63), we deduce that for $|\xi|^{2} \geq \frac{4 \beta}{(\nu-c)^{2}}$ (with $R_{k}$ the Riez transform such that $\mathcal{F} R_{k} g(\xi)=\frac{i \xi_{k}}{|\xi|} \mathcal{F} g(\xi)$ for $g$ a temperated distribution):

$$
\begin{align*}
& {\left[\mathcal{F} \phi\left(\lambda^{-1} D\right) W(t)\binom{q_{0}}{m_{0}}\right]_{1}(t, \xi)} \\
& =\phi\left(\frac{|\xi|}{\lambda}\right)\left(\left(e^{-t \lambda_{1 h}(\xi)}-e^{-t \lambda_{2 h}(\xi)}\right)\left(\frac{\operatorname{sgn}(c-\nu)}{2 \alpha(\xi)} \mathcal{F} q_{0}(\xi)+\frac{1}{|\xi \| c-\nu| \alpha(\xi)} \sum_{j=1}^{N} \mathcal{F} R_{j}\left(m_{0}\right)_{j}(\xi)\right)\right. \\
& \left.+\frac{1}{2}\left(e^{-t \lambda_{2 h}(\xi)}+e^{-t \lambda_{1 h}(\xi)}\right) \mathcal{F} q_{0}(\xi)\right) \\
& {\left[\mathcal{F} \phi\left(\lambda^{-1} D\right) W(t)\binom{q_{0}}{m_{0}}\right]_{1+i}(t, \xi)=\phi\left(\frac{\xi}{\lambda}\right)\left(( e ^ { - t \lambda _ { 2 h } ( \xi ) } - e ^ { - t \lambda _ { 1 h } ( \xi ) } ) \left(-\frac{\beta}{|\xi \| c-\nu| \alpha(\xi)} \mathcal{F} R_{i} q_{0}(\xi)\right.\right.} \\
& \left.\left.+\frac{\operatorname{sgn}(c-\nu)}{2 \alpha(\xi)} \mathcal{F}\left(\mathbb{Q} m_{0}\right)_{i}(\xi)\right)+\frac{1}{2}\left(e^{-t \lambda_{2 h}(\xi)}+e^{-t \lambda_{1 h}(\xi)}\right) \mathcal{F}\left(\mathbb{Q} m_{0}\right)_{i}(\xi)+e^{-\mu t|\xi|^{2}} \mathcal{F}\left(\mathbb{P} m_{0}\right)_{i}(\xi)\right), \tag{3.68}
\end{align*}
$$

with $i \in[1, N]$. We have used the fact in the previous formula that $\mathbb{P} m$ verifies simply an heat equation. Similarly from (3.66), we have for $|\xi|^{2}<\frac{4 \beta}{(\nu-c)^{2}}$ :

$$
\begin{align*}
& {\left[\mathcal{F} \phi\left(\lambda^{-1} D\right) W(t)\binom{q_{0}}{m_{0}}\right]_{1}(t, \xi)=\phi\left(\frac{|\xi|}{\lambda}\right)\left(i ( e ^ { - t \lambda _ { 2 l } ( \xi ) } - e ^ { - t \lambda _ { 1 l } ( \xi ) } ) \left(\frac{\operatorname{sgn}(c-\nu)}{2 \alpha_{1}(\xi)} \mathcal{F} q_{0}(\xi)\right.\right.} \\
& \left.\left.+\frac{1}{|\xi \| c-\nu| \alpha_{1}(\xi)} \sum_{j=1}^{N} \mathcal{F} R_{j}\left(m_{0}\right)_{j}(\xi)\right)+\frac{1}{2}\left(e^{-t \lambda_{2 l}(\xi)}+e^{-t \lambda_{1 l}(\xi)}\right) \mathcal{F} q_{0}(\xi)\right) \\
& {\left[\mathcal{F} \phi\left(\lambda^{-1} D\right) W(t)\binom{q_{0}}{m_{0}}\right]_{1+i}(t, \xi)=\phi\left(\frac{\xi}{\lambda}\right)\left(i ( e ^ { - t \lambda _ { 1 l } ( \xi ) } - e ^ { - t \lambda _ { 2 l } ( \xi ) } ) \left(\frac{-\beta}{|c-\nu||\xi| \alpha_{1}(\xi)} \mathcal{F} R_{i} q_{0}(\xi)\right.\right.} \\
& \left.\left.\left.+\frac{\operatorname{sgn}(c-\nu)}{2 \alpha_{1}(\xi)} \mathcal{F}\left(\mathbb{Q} m_{0}\right)_{i}(\xi)\right)+\frac{1}{2}\left(e^{-t \lambda_{2 l}(\xi)}+e^{-t \lambda_{1 l}(\xi)}\right)\right) \mathcal{F}\left(\mathbb{Q} m_{0}\right)_{i}(\xi)+e^{-\mu t|\xi|^{2}} \mathcal{F}\left(\mathbb{P} m_{0}\right)_{i}(\xi)\right) \tag{3.69}
\end{align*}
$$

Applying Plancherel Theorem, we observe easily that it exists $\varepsilon>0$ small enough and $C_{\varepsilon}, \kappa_{\varepsilon}>0$ such that:

$$
\begin{equation*}
\left\|1_{\mathbb{R}^{N} / \mathcal{C}\left(0, \sqrt{\frac{4 \beta}{|\nu-c|^{2}}-\varepsilon}, \sqrt{\left.\frac{4 \beta}{|\nu-c|^{2}}+\varepsilon\right)}\right.} \mathcal{F} \phi\left(\lambda^{-1} D\right) W(t)\binom{q_{0}}{m_{0}}\right\|_{L^{2}} \leq C_{\varepsilon} e^{-\kappa_{\varepsilon} \lambda^{2} t}\left\|\phi\left(\lambda^{-1} D\right)\binom{q_{0}}{m_{0}}\right\|_{L^{2}} \tag{3.70}
\end{equation*}
$$

Indeed we use the fact that when $\xi \in \mathbb{R}^{N} / \mathcal{C}\left(0, \sqrt{\frac{4 \beta}{|\nu-c|^{2}}-\varepsilon}, \sqrt{\frac{4 \beta}{|\nu-c|^{2}}+\varepsilon}\right)$ we have:

$$
\alpha_{1}(\xi) \geq|\nu-c| \sqrt{\frac{\varepsilon}{4 \beta\left(1-\frac{\varepsilon(\nu-c)^{2}}{4 \beta}\right)}}, \alpha(\xi) \geq|\nu-c| \sqrt{\frac{\varepsilon}{4 \beta\left(1+\frac{\varepsilon(\nu-c)^{2}}{4 \beta}\right)}}, \alpha_{1}(\xi)|\xi| \geq \sqrt{\varepsilon}
$$

The only difficulty is the behavior of the solution in the region $\mathcal{C}\left(0, \sqrt{\frac{4 \beta}{|\nu-c|^{2}}-\varepsilon}, \sqrt{\frac{4 \beta}{|\nu-c|^{2}}+\varepsilon}\right)$, in particular when $\mathcal{C}\left(0, R_{1} \lambda, R_{2} \lambda\right) \cap \mathcal{C}\left(0, \sqrt{\frac{4 \beta}{|\nu-c|^{2}}-\varepsilon}, \sqrt{\frac{4 \beta}{|\nu-c|^{2}}+\varepsilon}\right) \neq \emptyset$. We have in particular when $|\xi|^{2} \in\left(\frac{4 \beta}{|\nu-c|^{2}}, \frac{4 \beta}{|\nu-c|^{2}}+\varepsilon\right)$ that $\alpha(\xi) \in(0, \sqrt{\varepsilon})$ :

$$
\left|\frac{e^{-t \lambda_{1}(\xi)}-e^{-t \lambda_{2}(\xi)}}{\alpha(\xi)}\right|=e^{-\frac{\left.t \xi\right|^{2}(\nu+c)}{2}}\left|\frac{e^{-\frac{t|\xi|^{2}|\nu-c| \alpha(\xi)}{2}}-e^{\frac{t|\xi|^{2}|\nu-c| \alpha(\xi)}{2}}}{\alpha(\xi)}\right|
$$

When $t \alpha(\xi) \leq 1$, it exists $C>0$ sufficiently large such that for any $x \in\left[0,|\nu-c|\left(\frac{4 \beta}{|\nu-c|^{2}}+\right.\right.$ $\varepsilon)$ ] we have:

$$
e^{x}-e^{-x} \leq C x
$$

It implies that we have when $t \alpha(\xi) \leq 1$, it exists $C_{1}, C_{2}>0$ large enough such that:

$$
\begin{equation*}
\left|\frac{e^{-t \lambda_{1}(\xi)}-e^{-t \lambda_{2}(\xi)}}{\alpha(\xi)}\right| \leq C_{1} t|\xi|^{2} e^{-\frac{t|\xi|^{2}(\nu+c)}{2}} \leq C_{2} e^{-\frac{t|\xi|^{2}(\nu+c)}{4}} \tag{3.71}
\end{equation*}
$$

When $\operatorname{ta}(\xi) \geq 1$ (and $\frac{1}{\alpha(\xi)} \leq t$ ), we deduce that for $C_{3}>0$ large enough and $\varepsilon \leq \frac{1}{2}$ :

$$
\begin{align*}
\left|\frac{e^{-t \lambda_{1}(\xi)}-e^{-t \lambda_{2}(\xi)}}{\alpha(\xi)}\right| & \leq \frac{|\nu-c|^{2}}{4 \beta} t|\xi|^{2} e^{-\frac{t|\xi|^{2}(\nu+c)}{2}}\left|e^{-\frac{t|\xi|^{2}|\nu-c| \alpha(\xi)}{2}}-e^{\frac{t|\xi|^{2}|\nu-c| \alpha(\xi)}{2}}\right| \\
& \leq \frac{|\nu-c|^{2}}{4 \beta} t|\xi|^{2} e^{-\frac{t|\xi|^{2}(\nu+c)}{2}}+\frac{|\nu-c|^{2}}{4 \beta} t|\xi|^{2} e^{-\frac{t|\xi|^{2}(\nu+c-\sqrt{\varepsilon}|\nu-c|)}{2}}  \tag{3.72}\\
& \leq C_{3} e^{-\frac{t|\xi|^{2}(\nu+c)}{4}}
\end{align*}
$$

From (3.68), (3.70), (3.71) and (3.72), we deduce that it exists $\varepsilon>0$ small enough and $C_{\varepsilon}, \kappa_{\varepsilon}>0$ such that:

$$
\begin{equation*}
\left\|1_{\mathbb{R}^{N} / \mathcal{C}\left(0, \frac{2 \sqrt{\beta}}{|\nu-c|}, \sqrt{\left.\frac{4 \beta}{|\nu-c|^{2}}+\varepsilon\right)}\right.} \mathcal{F} \phi\left(\lambda^{-1} D\right) W(t)\binom{q_{0}}{m_{0}}\right\|_{L^{2}} \leq C_{\varepsilon} e^{-\kappa_{\varepsilon} \lambda^{2} t}\left\|\phi\left(\lambda^{-1} D\right)\binom{q_{0}}{m_{0}}\right\|_{L^{2}} \tag{3.73}
\end{equation*}
$$

Let us deal now with the case $|\xi|^{2} \in\left(\frac{4 \beta}{|\nu-c|^{2}}, \frac{4 \beta}{|\nu-c|^{2}}-\varepsilon\right)$ whoch corresponds to $\alpha_{1}(\xi) \in$ $(0, \sqrt{\varepsilon})$, we have then:

$$
\begin{aligned}
& \left|\frac{e^{-t \lambda_{1}(\xi)}-e^{-t \lambda_{2}(\xi)}}{\alpha_{1}(\xi)}\right|=e^{-\frac{t|\xi|^{2}(\nu+c)}{2}}\left|\frac{e^{-\frac{i t|\xi|^{2}|\nu-c| \alpha_{1}(\xi)}{2}}-e^{\frac{i t|\xi|^{2}|\nu-c| \alpha(\xi)}{2}}}{\alpha_{1}(\xi)}\right| \\
& =2 e^{-\frac{t|\xi|^{2}(\nu+c)}{2}} \frac{\left|\sin \left(\frac{t|\xi|^{2}|\nu-c| \alpha_{1}(\xi)}{2}\right)\right|}{\left|\alpha_{1}(\xi)\right|}
\end{aligned}
$$

When $t \alpha_{1}(\xi) \leq 1$, it exists $C_{1}, C_{2}>0$ large enough such that:

$$
\begin{equation*}
\left|\frac{e^{-t \lambda_{1}(\xi)}-e^{-t \lambda_{2}(\xi)}}{\alpha_{1}(\xi)}\right| \leq C_{1} t|\xi|^{2} e^{-\frac{t|\xi|^{2}(\nu+c)}{2}} \leq C_{2} e^{-\frac{t|\xi|^{2}(\nu+c)}{4}} \tag{3.74}
\end{equation*}
$$

When $t \alpha_{1}(\xi) \geq 1$, we deduce that for $C_{3}>0$ large enough:

$$
\begin{align*}
\left|\frac{e^{-t \lambda_{1}(\xi)}-e^{-t \lambda_{2}(\xi)}}{\alpha_{1}(\xi)}\right| & \leq \frac{2}{\frac{4 \beta}{|\nu-c|^{2}}-\varepsilon} t|\xi|^{2} e^{-\frac{t|\xi|^{2}(\nu+c)}{2}}  \tag{3.75}\\
& \leq C_{3} e^{-\frac{t|\xi|^{2}(\nu+c)}{4}}
\end{align*}
$$

From (3.66), (3.70), (3.73), (3.74) and (3.75) we obtain finally from the Plancherel Theorem that there exist $C, \kappa>0$ such that for any $t>0$ we have:

$$
\begin{equation*}
\left\|\phi\left(\lambda^{-1} D\right) W(t)\binom{q_{0}}{m_{0}}\right\|_{L^{2}} \leq C e^{-\kappa \lambda^{2} t}\left\|\phi\left(\lambda^{-1} D\right)\binom{q_{0}}{m_{0}}\right\|_{L^{2}} \tag{3.76}
\end{equation*}
$$

It concludes the proof of the proposition 3.2.
We are going now to prove time decay estimates in Besov spaces for the semi group $W(t)$.
Proposition 3.3 Let $s \in \mathbb{R}, r \in[1,+\infty]$ and $s_{1}>s,\left(q_{0}, m_{0}\right) \in\left(B_{2, \infty}^{s}\right)^{N+1}$ then it exists $C_{s_{1}}>0$ such that for all $t>0$ we have:

$$
\begin{equation*}
\left\|W(t)\binom{q_{0}}{m_{0}}\right\|_{B_{2, r}^{s_{1}}} \leq \frac{C_{S_{1}}}{t^{\frac{s_{1}-s}{2}}}\left\|\binom{q_{0}}{m_{0}}\right\|_{B_{2, r}^{s}} \tag{3.77}
\end{equation*}
$$

Proof: From proposition 3.2, we have for $\kappa, C>0$ and any $l \in \mathbb{Z}$ :

$$
\begin{equation*}
\left\|\Delta_{l} W(t)\binom{q_{0}}{m_{0}}\right\|_{L^{2}} \leq C e^{-\kappa 2^{2 l} t}\left\|\Delta_{l}\binom{q_{0}}{m_{0}}\right\|_{L^{2}} \tag{3.78}
\end{equation*}
$$

We deduce that for any $l \in \mathbb{Z}$, we have:

$$
\begin{equation*}
2^{l s_{1}}\left\|\Delta_{l} W(t)\binom{q_{0}}{m_{0}}\right\|_{L^{2}} \leq \frac{C}{t^{\frac{s_{1}-s}{2}}}\left(2^{2 l} t\right)^{\frac{s_{1}-s}{2}} e^{-\kappa 2^{2 l}} 2^{l s}\left\|\Delta_{l}\binom{q_{0}}{m_{0}}\right\|_{L^{2}} \tag{3.79}
\end{equation*}
$$

We use now the fact that $x \frac{s_{1}-s}{2} e^{-\kappa x}$ is bounded in $L^{\infty}\left(\mathbb{R}^{+}\right)$to deduce that it exists $C_{s_{1}}>0$ such that:

$$
\begin{equation*}
\left\|2^{l s_{1}}\right\| \Delta_{l} W(t)\binom{q_{0}}{m_{0}}\left\|_{L^{2}}\right\|_{l^{r}} \leq \frac{C_{s_{1}}}{t^{\frac{s_{1}-s}{2}}}\left\|2^{l s}\right\| \Delta_{l}\binom{q_{0}}{m_{0}} \|_{l^{r}} \tag{3.80}
\end{equation*}
$$

It concludes the proof of the proposition 3.3.

### 3.1 Proof of the Theorem 1.2

We shall use a contracting mapping argument to prove the Theorem 1.2 ( for the moment we only consider the case $0<\kappa^{2}<\mu^{2}$ ) and we consider the following map $\psi_{1}$ defined as follows with $q=\rho-1$ :

$$
\begin{align*}
\psi_{1}\left(q, \rho v_{1}, \rho v_{2}\right)= & W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(t)\binom{q_{0}}{\rho v_{1}(0, \cdot)} \\
& +\int_{0}^{t} W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(t-s)\binom{0}{F\left(\rho, v_{1}, v_{2}\right)}(s) d s  \tag{3.81}\\
\psi_{2}\left(q, \rho v_{1}, \rho v_{2}\right)= & W_{c_{2}, \mu,--\sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(t)\binom{q_{0}}{\rho v_{2}(0, \cdot)} \\
& +\int_{0}^{t} W_{c_{2}, \mu,-\sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(t-s)\binom{0}{F\left(\rho, v_{1}, v_{2}\right)}(s) d s
\end{align*}
$$

with

$$
F\left(\rho, v_{1}, v_{2}\right)=-\frac{1}{2}\left(\operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)+\operatorname{div}\left(\rho v_{2} \otimes v_{1}\right)\right)-\left(P^{\prime}(\rho)-P^{\prime}(1)\right) \nabla \rho .
$$

We define finally $\psi_{3}$ as follows:

$$
\begin{equation*}
\psi_{3}\left(q, \rho v_{1}, \rho v_{2}\right)=\binom{\psi_{1}\left(q, \rho v_{1}, \rho v_{2}\right)}{\left(\psi_{2}\right)_{2}\left(q, \rho v_{1}, \rho v_{2}\right)} . \tag{3.82}
\end{equation*}
$$

Let us prove now that $\psi_{3}$ is a map from $X_{\frac{N}{2}}$ in itself with $s_{1} \in\left(\frac{3}{4}, 1\right)$ :

$$
\begin{aligned}
& E_{\frac{N}{2}}=\left(\widetilde{L}^{\infty}\left(\mathbb{R}^{+}, B_{2, \infty}^{\frac{N}{2}-1} \cap B_{2, \infty}^{\frac{N}{2}}\right) \cap \widetilde{L}^{1}\left(\mathbb{R}^{+}, B_{2, \infty}^{\frac{N}{2}+1} \cap B_{2, \infty}^{\frac{N}{2}+2}\right)\right) \times\left(\widetilde{L}^{\infty}\left(\mathbb{R}^{+}, B_{2, \infty}^{\frac{N}{2}-1}\right) \cap \widetilde{L}^{1}\left(\mathbb{R}^{+}, B_{2, \infty}^{\frac{N}{2}+1}\right)^{2 N}\right. \\
& \left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{W_{\frac{N}{2}}}=\sup _{t \in \mathbb{R}^{+}} t^{\frac{s_{1}}{2}} \|\left(q(t, \cdot), \rho v_{1}(t, \cdot), \rho v_{2}(t, \cdot) \|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}}\right. \\
& \left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{X_{\frac{N}{2}}}=\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{E_{\frac{N}{2}} \cap W_{\frac{N}{2}}}+\|q\|_{L^{\infty}\left(\mathbb{R}^{+}, L^{\infty}\left(\mathbb{R}^{N}\right)\right)} .
\end{aligned}
$$

The space $E_{\frac{N}{2}}$ in which we work is more complicated as in [7], indeed we need decay estimate in time in Besov space on the solution in order to control the $L^{\infty}$ norm of $q$. In [7], the control of the $L^{\infty}$ norm of $q$ is a direct consequence of Besov embedding $B_{2,1}^{\frac{N}{2}} \hookrightarrow L^{\infty}$ since the third index of the Besov spaces are 1.
From the proposition (3.1), we deduce that for $C>0$ large enough:

$$
\begin{align*}
&\left\|\psi_{3}\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{E_{\frac{N}{2}}} \leq C\left(\left\|q_{0}\right\|_{B_{2, \infty}}{ }^{\frac{N}{2}-1} \times B_{2, \infty}^{\frac{N}{2}}+\left\|\rho v_{1}(0, \cdot)\right\|_{B_{2, \infty}}+\left\|\rho v_{2}(0, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1}}\right.  \tag{3.83}\\
&\left.+\left\|F\left(\rho, v_{1}, v_{2}\right)\right\|_{\tilde{L}^{1}\left(\mathbb{R}^{+}, B_{2, \infty}^{2}\right)}\right)
\end{align*}
$$

Next using classical paraproduct law and composition theorems (see [3]), we get for $C_{1}>0$ large enough and a continuous function $C$ :

$$
\begin{align*}
& \left\|\operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)\right\|_{\widetilde{L}^{1}\left(\mathbb{R}^{+}, B_{2, \infty}^{\frac{N}{2}-1}\right)} \leq C_{1}\left\|\rho v_{1}\right\|_{E_{\frac{N}{2}}}\left\|\rho v_{2}\right\|_{E_{\frac{N}{2}}}\left(1+\left\|\frac{1}{\rho}-1\right\|_{L^{\infty}\left(\mathbb{R}^{+}, L^{\infty}\left(\mathbb{R}^{N}\right)\right) \cap \tilde{L}^{\infty}\left(\mathbb{R}^{+}, B_{2}, \infty\right.}^{\left.\left.\frac{N}{2}\right)\right)}\right), \\
& \left\|\left(P^{\prime}(\rho)-P^{\prime}(1)\right) \nabla q\right\|_{\tilde{L}^{1}\left(\mathbb{R}^{+}, B_{2, \infty}^{2}\right)} \leq C_{1}\|\nabla q\|_{\tilde{L}^{\frac{N}{3}}\left(\mathbb{R}^{+}, B_{2, \infty}^{\frac{N}{2}-\frac{1}{2}}\right)}\left\|P^{\prime}(1+q)-P^{\prime}(1)\right\|_{\tilde{L}^{4}\left(\mathbb{R}^{+}, B_{2, \infty}^{\frac{N}{2}-\frac{1}{2}}\right)} \\
& \leq C_{2}\|\nabla q\|_{\tilde{L}^{\frac{4}{3}}\left(\mathbb{R}^{+}, B_{2, \infty}^{\frac{N}{2}-\frac{1}{2}}\right)}\|q\|_{\tilde{L}^{4}\left(\mathbb{R}^{+}, B_{2, \infty}^{\frac{N}{2}-\frac{1}{2}}\right)} C\left(\|q\|_{L^{\infty}\left(\mathbb{R}^{+}, L^{\infty}\left(\mathbb{R}^{N}\right)\right)}\right) . \tag{3.84}
\end{align*}
$$

Combining (3.83), (3.84), interpolation and composition theorems, we obtain for $C_{1}>0$ large enough and a continuous function $C$ :

$$
\begin{align*}
& \left\|\psi_{3}\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{E_{\frac{N}{2}}} \leq C_{1}\left(\left\|q_{0}\right\|_{B_{2, \infty}^{\frac{N}{2}-1} \times B_{2, \infty}^{\frac{N}{2}}}+\left\|\rho v_{1}(0, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1}}+\left\|\rho v_{2}(0, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1}}\right. \\
& \left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{X_{\frac{N}{2}}^{2}}\left(1+C\left(\left\|\left(\rho, \frac{1}{\rho}\right)\right\|_{L^{\infty}\left(\mathbb{R}^{+}, L^{\infty}\left(\mathbb{R}^{N}\right)\right)}\left(1+\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{E_{\frac{N}{2}}}\right)\right) .\right. \tag{3.85}
\end{align*}
$$

It remains now to estimate the $\left\|\psi_{3}\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{W^{\frac{N}{2}}}$ norm, using the definition of $\psi_{1}$ and $\psi_{2}($ see $(3.81))$ we have for $t>0$ :

$$
\begin{align*}
& \left\|\psi_{1}\left(q, \rho v_{1}, \rho v_{2}\right)(t, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}} \leq\left\|W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(t)\binom{q_{0}}{\rho v_{1}(0, \cdot)}\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}} \\
& +\int_{0}^{t}\left\|W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(t-s)\binom{0}{F\left(\rho, v_{1}, v_{2}\right)}(s)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}} d s \\
& \left\|\psi_{2}\left(q, \rho v_{1}, \rho v_{2}\right)(t, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}} \leq\left\|W_{c_{1}, \mu,-\sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(t)\binom{q_{0}}{\rho v_{2}(0, \cdot)}\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}} \\
&  \tag{3.86}\\
& +\int_{0}^{t}\left\|W_{c_{1}, \mu,-\sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(t-s)\binom{0}{F\left(\rho, v_{1}, v_{2}\right)}(s)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}} d s .
\end{align*}
$$

From proposition 3.3, it yields that for $C>0$ large enough we have:

$$
\begin{align*}
& \left\|\psi_{3}\left(q, \rho v_{1}, \rho v_{2}\right)(t, \cdot)\right\|_{B_{2, \infty}} \frac{N}{\frac{N}{2}-1+s_{1}} \leq \frac{C}{t^{\frac{s_{1}}{2}}}\left\|\left(q_{0}, \rho v_{1}(0, \cdot), \rho v_{2}(0, \cdot)\right)\right\|_{B_{2, \infty}^{\frac{N}{2}-1}} \\
& +\int_{0}^{t} \frac{C}{(t-s)^{1-\frac{s_{1}}{2}}}\left\|F\left(\rho, v_{1}, v_{2}\right)(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-3+2 s_{1}}} d s \tag{3.87}
\end{align*}
$$

We have now using classical paraproduct laws for $s>0$ and $C$ a continuous function:

$$
\begin{align*}
& \left.\left\|\operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)(s, \cdot)\right\|_{B_{2, \infty}} \frac{N}{2}-3+2 s_{1} \leq \|\left(\frac{1}{\rho}-1\right) \rho v_{1} \otimes \rho v_{2}\right)(s, \cdot) \|_{B_{2, \infty}^{\frac{N}{2}-2+2 s_{1}}} \\
& \quad+\left\|\left(\rho v_{1} \otimes \rho v_{2}\right)(s, \cdot)\right\|_{B_{2, \infty}^{2}}{ }^{\frac{N}{2}-2+2 s_{1}} \\
& \leq C_{1} C\left(\left\|\frac{1}{\rho}(s, \cdot)\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)\left(1+\|q(s, \cdot)\|_{B_{2, \infty}^{\frac{N}{2}} \cap L^{\infty}}\right)\left\|\rho v_{1}(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}}\left\|\rho v_{2}(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}} \tag{3.88}
\end{align*}
$$

Similarly we have for $C>0$ large enough and $s_{1} \in\left(\frac{3}{4}, 1\right)$ :

$$
\begin{align*}
& \left\|\left(P^{\prime}(\rho)-P^{\prime}(1)\right) \nabla q(s, \cdot)\right\|_{B_{2, \infty} \frac{N}{2}-3+2 s_{1}} \leq C\|\nabla q(s, \cdot)\|_{B_{2, \infty}^{\frac{N}{2}-2+s_{1}}}\left\|\left(P^{\prime}(1+q)-P^{\prime}(1)\right)(s, \cdot)\right\|_{B_{2, \infty}}{ }^{\frac{N}{2}-1+s_{1}} \\
& \leq C\left(\|q(s, \cdot)\|_{L^{\infty}}\right)\|q(s, \cdot)\|_{B_{2, \infty}^{2}}^{2}-1+s_{1} \tag{3.89}
\end{align*}
$$

From (3.87), (3.88) and (3.89), we get for $C_{1}>0$ large enough and $C$ a continuous function:

$$
\begin{aligned}
& \left\|\psi_{3}\left(q, \rho v_{1}, \rho v_{2}\right)(t, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}} \leq \frac{C_{1}}{t^{\frac{s_{1}}{2}}}\left\|\left(q_{0}, \rho v_{1}(0, \cdot), \rho v_{2}(0, \cdot)\right)\right\|_{B_{2, \infty}^{\frac{N}{2}-1}} \\
& +\int_{0}^{t} \frac{1}{(t-s)^{1-\frac{s_{1}}{2}}}\left(C\left(\left\|\frac{1}{\rho}(s, \cdot)\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)\left(1+\|q(s, \cdot)\|_{B_{2, \infty}^{\frac{N}{2}} \cap L^{\infty}}\right) \|_{\rho v_{1}(s, \cdot) \|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}}}^{\times\left\|\rho v_{2}(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}}+C\left(\|q(s, \cdot)\|_{\left.L^{\infty}\right)}\|q(s, \cdot)\|_{B_{2, \infty}^{2}}^{2}+1+s_{1}\right.}\right) d s .
\end{aligned}
$$

We get finally:

$$
\begin{align*}
& \left\|\psi_{3}\left(q, \rho v_{1}, \rho v_{2}\right)(t, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}} \leq \frac{C_{1}}{t^{\frac{s_{1}}{2}}}\left\|\left(q_{0}, \rho v_{1}(0, \cdot), \rho v_{2}(0, \cdot)\right)\right\|_{B_{2, \infty}^{\frac{N}{2}-1}} \\
& +\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{W^{\frac{N}{2}}}^{2} \int_{0}^{t} \frac{C_{1}}{(t-s)^{1-\frac{s_{1}}{2}}} \frac{1}{s^{s_{1}}} C\left(\left\|\left(\frac{1}{\rho}(s, \cdot), \rho(s, \cdot)\right)\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)\left(1+\|q(s, \cdot)\|_{B_{2, \infty}^{\frac{N}{2}} \cap L^{\infty}}\right) d s \\
& \leq \frac{C_{1}}{t^{\frac{s_{1}}{2}}}\left\|\left(q_{0}, \rho v_{1}(0, \cdot), \rho v_{2}(0, \cdot)\right)\right\|_{B_{2, \infty}^{\frac{N}{2}-1}} \\
& +\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{W^{\frac{N}{2}}}^{2} C\left(\left\|\left(\rho, \frac{1}{\rho}\right)\right\|_{L^{\infty}\left(\mathbb{R}^{+}, L^{\infty}\right)}\right)\left(1+\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{X_{\frac{N}{2}}}\right) \int_{0}^{t} \frac{1}{(t-s)^{1-\frac{s_{1}}{2}}} \frac{1}{s^{s_{1}}} d s \tag{3.90}
\end{align*}
$$

We have now since $s_{1} \in\left(\frac{3}{4}, 1\right)$

$$
\int_{0}^{t} \frac{1}{(t-s)^{1-\frac{s_{1}}{2}}} \frac{1}{s^{s_{1}}} d s=\frac{1}{t^{\frac{s_{1}}{2}}} \int_{0}^{1} \frac{1}{(1-u)^{1-\frac{s_{1}}{2}}} \frac{1}{u^{s_{1}}} d u
$$

It comes that for $C>0$ large enough we have:

$$
\begin{align*}
& \left\|\psi_{3}\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{W_{\frac{N}{2}}} \leq C\left(\left\|\left(q_{0}, \rho v_{1}(0, \cdot), \rho v_{2}(0, \cdot)\right)\right\|_{B_{2, \infty}^{\frac{N}{2}-1}}\right. \\
& \left.\quad+\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{W_{\frac{N}{2}}}^{2} C\left(\left\|\left(\rho, \frac{1}{\rho}\right)\right\|_{L^{\infty}\left(\mathbb{R}^{+}, L^{\infty}\right)}\right)\left(1+\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{X_{\frac{N}{2}}}\right)\right) \tag{3.91}
\end{align*}
$$

It remains now to estimate the $L^{\infty}\left(\mathbb{R}^{+}, L^{\infty}\left(\mathbb{R}^{N}\right)\right)$ norm on $\left[\psi_{3}\left(q, \rho v_{1}, \rho v_{2}\right)\right]_{1}$. From (3.81) and using the fact that $B_{2,1}^{\frac{N}{2}}\left(\mathbb{R}^{N}\right)$ is embedded in $L^{\infty}\left(\mathbb{R}^{N}\right)$, we have for $C>0$ large enough:

$$
\begin{gather*}
\left\|\left[\psi_{3}\left(q, \rho v_{1}, \rho v_{2}\right)\right]_{1}(t, \cdot)\right\|_{L^{\infty}\left(\mathbb{R}^{+}, L^{\infty}\left(\mathbb{R}^{N}\right)\right)} \leq\left\|\left[W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(t)\binom{q_{0}}{\rho v_{1}(0, \cdot)}\right]_{1}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \\
+\int_{0}^{t}\left\|\left[W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(t-s)\binom{0}{F\left(\rho, v_{1}, v_{2}\right)}\right]_{1}(s)\right\|_{B_{2,1}^{\frac{N}{2}}} d s . \tag{3.92}
\end{gather*}
$$

$W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}\binom{q_{0}}{\rho v_{1}(0, \cdot)}$ is the solution of the system (3.38), we deduce from the first equation of (3.38) and using the maximum principle and the proposition 3.3 that for $t>0$ and $C, C_{1}, C_{2}, C_{3}>0$ large enough:

$$
\begin{aligned}
& \left\|\left[W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}\binom{q_{0}}{\rho v_{1}(0, \cdot)}\right]_{1}(t, \cdot)\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq\left\|q_{0}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \\
& \quad+\int_{0}^{t}\left\|e^{c_{1}(t-s) \Delta} \operatorname{div}\left[W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(s)\binom{q_{0}}{\rho v_{1}(0, \cdot)}\right]_{2}\right\|_{B_{2,1}^{\frac{N}{2}}} d s \\
& \leq\left\|q_{0}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\int_{0}^{t} \frac{C}{(t-s)^{1-\frac{s_{1}}{4}}}\left\|\left[W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(s)\binom{q_{0}}{\rho v_{1}(0, \cdot)}\right]_{2}\right\|_{B_{2,1}^{\frac{N}{2}-1+\frac{s}{2}}} d s
\end{aligned}
$$

And it yields:

$$
\begin{align*}
& \left\|\left[W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}\binom{q_{0}}{\rho v_{1}(0, \cdot)}\right]_{1}(t, \cdot)\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq\left\|q_{0}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \\
& +\int_{0}^{t} \frac{C_{1}}{(t-s)^{1-\frac{s_{1}}{4}}}\left\|\left[W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(s)\binom{q_{0}}{\rho v_{1}(0, \cdot)}\right]_{2}\right\|_{B_{2, \infty}^{2}}^{\frac{1}{2}} \\
& \quad \times\left\|\left[W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(s)\binom{q_{0}}{\rho v_{1}(0, \cdot)}\right]_{2}\right\|_{B_{2, \infty}^{2}}^{\frac{1}{2}}  \tag{3.93}\\
& \leq\left\|q_{0}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\int_{0}^{t} \frac{C_{2}}{(t-s)^{1-\frac{s_{1}}{4}}} \frac{1}{s_{1}^{\frac{1}{4}}}\left\|\binom{q_{0}}{\rho v_{1}(0, \cdot)}\right\|_{B_{B_{2, \infty}}^{\frac{N}{2}-1}} d s \\
& \leq\left\|q_{0}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+C_{3}\left\|\binom{q_{0}}{\rho v_{1}(0, \cdot)}\right\|_{B_{2, \infty}^{\frac{N}{2}-1}} .
\end{align*}
$$

The third inequality corresponds to an interpolation inequality in Besov spaces. In a similar way, we have for $C, C_{1}, C_{2}, C_{3}>0$ large enough and using proposition 3.3 and interpolation in Besov spaces (see [3]):

$$
\begin{align*}
& \int_{0}^{t}\left\|\left[W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(t-s)\binom{0}{F\left(\rho, v_{1}, v_{2}\right)}\right]_{1}(s)\right\|_{B_{2,1}^{\frac{N}{2}}} d s \\
& \leq \int_{0}^{t} \frac{C}{(t-s)^{\frac{3}{2}-\frac{3 s_{1}}{4}}}\left\|\binom{0}{F\left(\rho, v_{1}, v_{2}\right)}\right\|_{B_{2,1}^{\frac{N}{2}-3+\frac{3}{2} s_{1}}} d s \\
& \leq \int_{0}^{t} \frac{C_{1}}{(t-s)^{\frac{3}{2}-\frac{3 s_{1}}{4}}}\left(\left\|F_{2}\left(\rho, v_{1}, v_{2}\right)(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{3}-3+2 s_{1}}}^{\frac{N}{2}}\left\|F_{2}\left(\rho, v_{1}, v_{2}\right)(s, \cdot)\right\|_{B_{2, \infty}^{\frac{1}{3}}}^{\frac{N}{2}-3+\frac{s_{1}}{2}}\right.  \tag{3.94}\\
& \left.+\left\|F_{3}(\rho)(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{4}}}^{\frac{N}{2}-3+2 s_{1}}\left\|F_{3}\left(\rho, v_{1}, v_{2}\right)(s, \cdot)\right\|_{B_{2, \infty}^{\frac{3}{4}}}^{\frac{N}{2}-3+\frac{4 s_{1}}{3}}\right) d s,
\end{align*}
$$

with $F_{2}\left(\rho, v_{1}, v_{2}\right)=\frac{1}{2}\left(\operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)+\operatorname{div}\left(\rho v_{2} \otimes v_{1}\right)\right)$ and $F_{3}(\rho)=\left(P^{\prime}(\rho)-P^{\prime}(1)\right) \nabla \rho$. Using classical paraproduct law, composition Theorem and interpolation (see [3]), we have now for $s>0$, a function $C$ continuous and $C_{1}>0$ large enough:

$$
\begin{align*}
& \left.\left\|\operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)(s)\right\|_{B_{2, \infty}^{\frac{N}{2}-3+\frac{s_{1}}{2}}} \leq \|\left(\frac{1}{\rho}-1\right) \rho v_{1} \otimes \rho v_{2}\right)(s, \cdot) \|_{B_{2, \infty}}{ }^{\frac{N}{2}-2+\frac{s_{1}}{2}} \\
& +\left\|\left(\rho v_{1} \otimes \rho v_{2}\right)(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-2+\frac{s_{1}}{2}}} \\
& \leq C_{1} C\left(\left\|\frac{1}{\rho}(s, \cdot)\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)\left(1+\|q(s, \cdot)\|_{B_{2, \infty}^{\frac{N}{2}} \cap L^{\infty}}\right)\left\|\rho v_{1}(s, \cdot)\right\|_{B_{2, \infty} \frac{N}{2}-1}\left\|\rho v_{2}(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+\frac{s_{1}}{2}}} \\
& \left.\leq C_{1} C\left(\left\|\frac{1}{\rho}(s, \cdot)\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)\left(1+\|q(s, \cdot)\|_{B_{2, \infty}^{\frac{N}{2}} \cap L^{\infty}}\right)\left\|\rho v_{1}(s, \cdot)\right\|_{B_{2, \infty}}{ }^{\frac{N}{2}-1}\left\|\rho v_{2}(s, \cdot)\right\|_{B_{2, \infty}}^{\frac{1}{2}}\left\|\rho v_{2}(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1}}^{\frac{1}{2}} \|\right)^{\frac{N}{2}-1+s_{1}} \tag{3.95}
\end{align*}
$$

Similarly we have using composition, interpolation theorems and paraproduct laws (we can since $s_{1}>\frac{3}{4}$ ), it exists a continuous function $C$ and $C_{1}>0$ large enough such that:

$$
\begin{align*}
& \left\|\left(P^{\prime}(1+q)-P^{\prime}(1)\right) \nabla q(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-3+\frac{4 s_{1}}{3}}} \leq\|\nabla q(s, \cdot)\|_{B_{2, \infty}^{\frac{N}{2}-2+\frac{2 s_{1}}{3}}}\left\|\left(P^{\prime}(1+q)-P^{\prime}(1)\right)(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+\frac{2 s_{1}}{3}}} \\
& \leq C\left(\|\rho(s, \cdot)\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)\|q(s, \cdot)\|_{B_{2, \infty}^{2}}^{2}-\frac{N}{\frac{N}{2}-1+\frac{2 s_{1}}{3}} \\
& \leq C_{1} C\left(\|\rho(s, \cdot)\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)\|q(s, \cdot)\|_{B_{2, \infty}^{4}}^{\frac{4}{3}-1+s_{1}} \tag{3.96}
\end{align*}\|q(s, \cdot)\|_{B_{2, \infty}^{\frac{N}{3}}}^{\frac{2}{3}-1} .
$$

It yields using (3.88), (3.89), (3.95) and (3.96) that for $C$ a continuous function and $C_{1}>0$ sufficiently large:

$$
\begin{align*}
& +\left\|F_{3}(\rho)(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{4}-3+2 s_{1}}}^{\frac{1}{N}}\left\|F_{3}\left(\rho, v_{1}, v_{2}\right)(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{4}}}^{\frac{3}{2}-3+\frac{4 s_{1}}{3}} \\
& \leq\left(C\left(\left\|\left(\frac{1}{\rho}, \rho\right)(s, \cdot)\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)\left(1+\|q(s, \cdot)\|_{B_{2, \infty}^{\frac{N}{2}} \cap L^{\infty}}\right)\left(\left\|\rho v_{1}(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}}\left\|\rho v_{2}(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}}\right)\right)^{\frac{2}{3}} \times \\
& \left(C\left(\left\|\left(\frac{1}{\rho}, \rho\right)(s, \cdot)\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)\left(1+\|q(s, \cdot)\|_{B_{2, \infty}^{\frac{N}{2}} \cap L^{\infty}}\right)\left(\left\|\rho v_{1}(s, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1}}\left\|\rho v_{2}(s, \cdot)\right\|_{B_{2, \infty}^{\frac{1}{2}}}^{\frac{1}{2}-1}\left\|\rho v_{2}(s, \cdot)\right\|_{B_{2, \infty}^{\frac{1}{2}-1+s_{1}}}^{\frac{1}{2}}\right)\right)^{\frac{1}{3}} \\
& +\left(C\left(\left\|\left(\frac{1}{\rho}, \rho\right)(s, \cdot)\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)\left(1+\|q(s, \cdot)\|_{B_{2, \infty}^{\frac{N}{2}} \cap L^{\infty}}\right)\|q(s, \cdot)\|_{B_{2_{2, \infty}}^{\frac{N}{2}-1+s_{1}}}^{2}\right)^{\frac{1}{4}} \\
& \times\left(C_{1} C\left(\|\rho(s, \cdot)\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)\|q(s, \cdot)\|_{B_{2, \infty}}^{\frac{4}{3}}{ }_{B_{2}^{N}-1+s_{1}}\|q(s, \cdot)\|_{B_{2, \infty}^{\frac{N}{3}-1}}^{\frac{2}{3}}\right)^{\frac{3}{4}}, \\
& \leq C\left(\left\|\left(\frac{1}{\rho}, \rho\right)(s, \cdot)\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)\left(1+\|q\|_{E_{\frac{N}{2}} \cap L^{\infty}\left(\mathbb{R}^{+}, L^{\infty}\left(\mathbb{R}^{N}\right)\right)}\right)\left(\frac{1}{s^{\frac{3 s_{1}}{4}}}\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{W_{\frac{N}{2}}^{2}}^{\frac{3}{2}}\right. \\
& \left.\times\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{E_{\frac{N}{2}}}^{\frac{1}{2}}+\frac{1}{s^{\frac{3 s_{1}}{4}}}\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{W_{\frac{N}{2}}^{2}}^{\frac{3}{2}}\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{E_{\frac{N}{2}}}^{\frac{1}{2}}\right), \\
& \leq \frac{1}{s^{\frac{3 s_{1}}{4}}} C\left(\left\|\left(\frac{1}{\rho}, \rho\right)(s, \cdot)\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)\left(1+\|q\|_{E_{\frac{N}{2}} \cap L^{\infty}\left(\mathbb{R}^{+}, L^{\infty}\left(\mathbb{R}^{N}\right)\right)}\right)\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{X_{\frac{N}{2}}}^{2} . \tag{3.97}
\end{align*}
$$

Plugging (3.97) in (3.94), we obtain for a continuous fonction $C$ since $s_{1}>\frac{3}{4}$ :

$$
\begin{align*}
& \int_{0}^{t}\left\|\left[W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(t-s)\binom{0}{F\left(\rho, v_{1}, v_{2}\right)}\right]_{1}(s)\right\|_{B_{2,1}^{\frac{N}{2}}} d s \\
& \leq \frac{C_{1}}{\sqrt{t}} C\left(\left\|\left(\frac{1}{\rho}, \rho\right)(s, \cdot)\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)\left(1+\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{X_{\frac{N}{2}}}\right)\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{X_{\frac{N}{2}}^{2}}^{2} \tag{3.98}
\end{align*}
$$

Combining (3.92), (3.93) and (3.98), we have for $C_{1}>0$ sufficiently large and $C$ a continuous function we have for any $t>0$ :

$$
\begin{align*}
& \left\|\left[\psi_{3}\left(q, \rho v_{1}, \rho v_{2}\right)\right]_{1}(t, \cdot)\right\|_{L^{\infty}\left(\mathbb{R}^{+}, L^{\infty}\left(\mathbb{R}^{N}\right)\right)} \leq\left\|q_{0}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+C_{1}\left\|\binom{q_{0}}{\rho v_{1}(0, \cdot)}\right\|_{B_{2, \infty}^{\frac{N}{2}-1}} \\
& +\frac{1}{\sqrt{t}} C\left(\left\|\left(\frac{1}{\rho}, \rho\right)(s, \cdot)\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right)\left(1+\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{X_{\frac{N}{2}}}\right)\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{W_{\frac{N}{2}}}\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{E_{\frac{N}{2}}} \tag{3.99}
\end{align*}
$$

This estimate proves that we need to distinguish the short time and the long time, indeed (3.99) is interesting only for the long time.

We are going now to apply a fixed point theorem and we define the map $\psi_{4}$ as follows for any $t>0$ :

$$
\left\{\begin{array}{l}
\psi_{4}\left(q, \rho v_{1}, \rho v_{2}\right)(t, \cdot)=\psi\left(q, \rho v_{1}, \rho v_{2}\right)(t, \cdot) \quad \text { if } t \in(0, T], \\
\psi_{4}\left(q, \rho v_{1}, \rho v_{2}\right)(t, \cdot)=\psi_{3}\left(q, \rho v_{1}, \rho v_{2}\right)(t, \cdot) \quad \text { if } t>T,
\end{array}\right.
$$

with $\psi$ defined as in the previous section:

$$
\begin{aligned}
& \psi\left(q, \rho v_{1}, \rho v_{2}\right)=\left(\begin{array}{c}
e^{c_{1} t \Delta} q_{0} \\
e^{\mu t \Delta} \mathbb{P}\left(\rho v_{1}(0, \cdot)\right)+e^{c_{2} t \Delta} \mathbb{Q}\left(\rho v_{1}(0, \cdot)\right) \\
e^{\mu t \Delta} \mathbb{P}\left(\rho v_{2}(0, \cdot)\right)+e^{c_{1} t \Delta} \mathbb{Q}\left(\rho v_{2}(0, \cdot)\right)
\end{array}\right) \\
& e^{c_{1}(t-s) \Delta} \operatorname{div}\left(\rho v_{1}\right)
\end{aligned} \int_{0}^{t}\binom{\frac{1}{2} \mu(t-s) \Delta \mathbb{P}\left(\operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)+\operatorname{div}\left(\rho v_{2} \otimes v_{1}\right)\right)+e^{c_{2}(t-s) \Delta} \mathbb{Q}\left(F\left(\rho, v_{1}, v_{2}\right)\right)}{\frac{1}{2} e^{\mu(t-s) \Delta} \mathbb{P}\left(\operatorname{div}\left(\rho v_{1} \otimes v_{2}\right)+\operatorname{div}\left(\rho v_{2} \otimes v_{1}\right)\right)+e^{c_{1}(t-s) \Delta} \mathbb{Q}\left(F\left(\rho, v_{1}, v_{2}\right)\right)} d s .
$$

and $T>0$ to determine later. We are going now to prove that $\psi$ is a map from $X_{\frac{N}{2}, R} \cap$ $E_{R_{1}, M, T}$ in itself with $s_{1} \in\left(\frac{3}{4}, 1\right)$ and $0<R, R_{1}, T<\frac{1}{2}$ sufficiently small that we will define later. It is important to mention that we can work in small time on $(0, T)$ in $E_{R_{1}, M, T}$ since $\left(\rho v_{1}(0, \cdot), \rho v_{2}(0, \cdot)\right)$ are in $B M O^{-1}\left(\mathbb{R}^{N}\right)$, indeed we know that $B_{2, \infty}^{\frac{N}{2}-1}$ is embedded in $B M O^{-1}\left(\mathbb{R}^{N}\right)$. We define $X_{\frac{N}{2}, R}$ as follows:

$$
X_{\frac{N}{2}, R}=\left\{\left(q, \rho v_{1}, \rho v_{2}\right) \in X_{\frac{N}{2}},\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{X_{\frac{N}{2}}} \leq R\right\}
$$

In the sequel we will note $X_{\frac{N}{2}, T}^{+}$and $X_{\frac{N}{2}, T}^{-}$to define the subsets of $X_{\frac{N}{2}}$ where the norms are respectively only considered on the time interval $[T,+\infty[$ and $[0, T]$. We

Combining (3.85), (3.91) and (3.99) we have for $C$ a continuous function and $C_{1}>0$
large enough and for all $\left(q, \rho v_{1}, \rho v_{2}\right) \in X_{\frac{N}{2}, R} \cap E_{R_{1}, M, T}$ :

$$
\begin{align*}
& \left\|\psi_{4}\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{X_{\frac{N}{2}}^{+}} \leq\left\|\psi_{3}\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{X_{\frac{N}{2}}^{+}} \leq C_{1}\left(\left\|\left(q_{0}, \rho v_{1}(0, \cdot), \rho v_{2}(0, \cdot)\right)\right\|_{B_{2, \infty}}+\left\|q_{0}\right\|_{B_{2, \infty}^{2} \cap L^{\infty}}\right. \\
& \left.\quad+\frac{1}{\sqrt{T}}\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{X_{\frac{N}{2}}^{2}}^{2}\left(1+C\left(\left\|\rho, \frac{1}{\rho}\right\|_{L^{\infty}\left(\mathbb{R}^{+}, L^{\infty}\right)}\right)\right)\left(1+\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{X_{\frac{N}{2}}}\right)\right) \\
& \leq C_{1}\left(\left\|\left(q_{0}, \rho v_{1}(0, \cdot), \rho v_{2}(0, \cdot)\right)\right\|_{B_{2, \infty}^{\frac{N}{2}-1}}^{\frac{N}{2}}+\left\|q_{0}\right\|_{B_{2, \infty}^{\frac{N}{2}} \cap L^{\infty}}^{N^{2}}\right. \\
& \left.\quad+\frac{R^{2}}{\sqrt{T}}\left(1+\sup _{y \in\left[0, \max \left(R+1, \frac{1}{1-R}\right)\right]} C(y)\right)(1+R)\right) \tag{3.100}
\end{align*}
$$

We take now $R$ such that:

$$
R=2 C_{1}\left(\left\|\left(q_{0}, \rho v_{1}(0, \cdot), \rho v_{2}(0, \cdot)\right)\right\|_{B_{2, \infty}^{\frac{N}{2}-1}}+\left\|q_{0}\right\|_{B_{2, \infty}^{\frac{N}{2}} \cap L^{\infty}}\right) .
$$

We assume now that $0<R<\frac{1}{2}$ and then the initial data are sufficiently small such that:

$$
R^{2}\left(1+\sup _{y \in[0,2]} C(y)\right)(1+R) \leq \frac{R \sqrt{T}}{2}
$$

It gives the following condition:

$$
\begin{equation*}
\frac{3}{2} R\left(1+\sup _{y \in[0,2]} C(y)\right) \leq \frac{\sqrt{T}}{4} \tag{3.101}
\end{equation*}
$$

In a similar way we can estimate $\left\|\psi_{4}\right\|_{X_{\frac{N}{2}}^{-} \cap X_{T}}$. It suffices to apply exactly the same estimates than in the previous section for the norm $X_{T}$. For the norm $X_{\frac{N}{2}}^{-}$we have to repeat the same estimates in Besov spaces on a finite time interval $(0, T)$ and to choose $T, R_{1}>0$ sufficiently small as previously. We just say few words on the case of the norm $W_{\frac{N}{2}}^{-}$. First we have by interpolation since $s_{1} \in\left(\frac{3}{4}, 1\right)$ :

$$
\begin{aligned}
\sup _{t \in[0, T]} t^{\frac{s}{1}_{2}^{2}}\left\|\left[\psi_{4}\left(q, \rho v_{1}, \rho v_{2}\right)(t, \cdot)\right]_{1}\right\|_{B_{2, \infty}}{ }^{\frac{N}{2}-1+s_{1}} & \leq T^{\frac{s_{1}}{2}}\left\|\left[\psi_{4}\left(q, \rho v_{1}, \rho v_{2}\right)(t, \cdot)\right]_{1}\right\|_{\tilde{L}^{\infty}\left([0, T], B_{2, \infty}^{2}\right.}^{\frac{N}{2}-1} \cap B_{2, \infty}^{\left.\frac{N}{2}\right)} \\
& \leq T^{\frac{s_{1}}{2}}\left\|\psi_{4}\right\|_{E_{\frac{N}{2}}^{-}}
\end{aligned}
$$

The norm $E_{\frac{N}{2}}^{-}$is classical to estimate (we refer to [3] or to the previous estimates). We deduce now that for $C>0$ large enough:

$$
\begin{aligned}
& \sup _{t \in[0, T]} t^{\frac{s_{1}}{2}}\left\|\psi_{4}\left(q, \rho v_{1}, \rho v_{2}\right)(t, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}} \leq T^{\frac{s_{1}}{2}}\left\|\psi_{4}\right\|_{E_{\frac{N}{2}}^{-}}+C\left(\left\|\rho v_{1}(0, \cdot)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}}\right. \\
& +\left\|\rho v_{2}(0, \cdot)\right\|_{B_{2, \infty}}{ }^{\frac{N}{2}-1+s_{1}} \\
& \quad+C \int_{0}^{t} \sum_{i=1}^{3}\left\|e^{\beta_{i}(t-s) \Delta}\left[\operatorname{div}\left(\rho v_{1} \otimes v_{2}+v_{2} \otimes v_{1}\right)(s)+\nabla P(\rho)(s)\right]\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}} d s,
\end{aligned}
$$

with $\beta_{1} \in\left(c_{1}, c_{2}, \mu\right)$. We have now for $C_{1}, C_{2}>0$ large enough, $C$ a continuous function and $t \in[0, T]$ :

$$
\begin{align*}
& \int_{0}^{t}\left\|e^{\beta_{i}(t-s) \Delta} \operatorname{div}\left(\rho v_{1} \otimes v_{2}+v_{2} \otimes v_{1}\right)(s)\right\|_{B_{2, \infty}^{\left(\frac{N}{2}-1+s_{1}\right.}} d s \\
& \leq C_{1} \int_{0}^{t} \frac{1}{(t-s)^{1-\frac{s_{1}}{2}}}\left\|\rho v_{1} \otimes v_{2}(s)\right\|_{B_{2, \infty}^{\frac{N}{2}-2+2 s_{1}}} \\
& \leq \int_{0}^{t} \frac{1}{(t-s)^{1-\frac{s_{1}}{2}} \frac{1}{s^{s_{1}}}\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{W_{\frac{N}{2}}^{-}}^{2} C\left(\left\|\frac{1}{\rho}(s, \cdot)\right\|_{L^{\infty}}\right)\left(1+\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{E_{\frac{N}{2}}^{-}}\right) d s} \begin{array}{l}
\leq \frac{C_{2}}{t^{\frac{s_{1}}{2}}}\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{W_{\frac{N}{2}}^{-}}^{2} C\left(\left\|\frac{1}{\rho}(s, \cdot)\right\|_{L^{\infty}}\right)\left(1+\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{E_{\frac{N}{2}}^{-}}\right) .
\end{array} .
\end{align*}
$$

Proceeding as previously, we have also for $C_{1}, C_{2}>0$ large enough, $C$ a continuous function and $t \in[0, T]::$

$$
\begin{align*}
& \int_{0}^{t}\left\|e^{\beta_{i}(t-s) \Delta} \nabla P(\rho)(s)\right\|_{B_{2, \infty}^{\frac{N}{2}-1+s_{1}}} d s \\
& \leq C_{2} \int_{0}^{t} \frac{1}{(t-s)^{\frac{s_{1}}{2}}}\|P(1+q)-P(1)\|_{B_{2, \infty}^{\frac{N}{2}}} d s  \tag{3.103}\\
& \leq \frac{C_{2} T}{t^{\frac{s_{1}}{2}}} C\left(\|\rho(s, \cdot)\|_{L^{\infty}}\right)\left\|\left(q, \rho v_{1}, \rho v_{2}\right)\right\|_{E_{\frac{N}{2}}^{-}}
\end{align*}
$$

We obtain then the stability for the norm $X_{\frac{N}{2}}^{-}$using (3.102) and (3.103).
In a similar way, we can prove that $\psi_{4}$ is contractive. More precisely taking $\left(q, \rho v_{1}, \rho v_{2}\right)$, $\left(q_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)$ in $X_{\frac{N}{2}, R} \cap E_{R_{1}, M, T}$ we have for $t>T$ :

$$
\begin{align*}
& \psi_{1}\left(q, \rho v_{1}, \rho v_{2}\right)-\psi_{1}\left(q_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right) \\
& =\int_{0}^{t} W_{c_{1}, \mu, \sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(t-s)\binom{0}{F\left(\rho, v_{1}, v_{2}\right)-F\left(\rho_{1}, w_{1}, w_{2}\right)}(s) d s \\
& \left.\begin{array}{rl}
\psi_{2}\left(q, \rho v_{1}, \rho v_{2}\right) & -\psi_{2}\left(q_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right) \\
= & \int_{0}^{t} W_{c_{1}, \mu,-\sqrt{\mu^{2}-\kappa^{2}}, P^{\prime}(1)}(t-s)\left(F\left(\rho, v_{1}, v_{2}\right)-F\left(\rho_{1}, w_{1}, w_{2}\right)\right.
\end{array}\right)(s) d s
\end{align*}
$$

As previously we show that for $C_{1}>0$ large enough we have:

$$
\begin{align*}
\| \psi_{4}\left(q, \rho v_{1}, \rho v_{2}\right)- & \psi_{4}\left(q_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right) \|_{X_{\frac{N}{2}}^{+}}  \tag{3.105}\\
& \leq C_{1} R(1+R)\left\|\left(q, \rho v_{1}, \rho v_{2}\right)-\left(q_{1}, \rho_{1} w_{1}, \rho_{1} w_{2}\right)\right\|_{X_{\frac{N}{2}}}
\end{align*}
$$

Taking again $R$ sufficiently small we deduce that the map $\psi_{4}$ is contractive (it suffices again to repeat the same process on $[0, T]$ with $T>0$ sufficiently small). It concludes
the proof of the Theorem 1.2 in the case $0<\kappa^{2}<\mu^{2}$. The proof in the case $\kappa^{2}=\mu^{2}$ is similar except that $v_{1}=v_{2}$, in addition when we study the system (3.59) there is no distinction between high and low frequencies. The eigenvalues associated to the matrix $A(\xi)$ are always complex and the system has the same behavior than a equation mixing heat and Schrödinger equation.

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[^0]:    *Université Paris Dauphine, PSL Research University, Ceremade, Umr Cnrs 7534, Place du Maréchal De Lattre De Tassigny 75775 Paris cedex 16 (France), haspot@ceremade.dauphine.fr
    ${ }^{\dagger}$ ANGE project-team (Inria, Cerema, UPMC, CNRS), 2 rue Simone Iff, CS 42112, 75589 Paris, France.

[^1]:    ${ }^{1}$ In the sequel we will only consider homogeneous Besov space that we will note $B_{p, r}^{s}$ even if they are generally written $\dot{B}_{p, r}^{s}$ with $s \in \mathbb{R}$ and $(p, r) \in[1,+\infty]^{2}$. We refer to [3] for the definition of the homogeneous Besov spaces and for the notion of product in the Besov spaces related to the so called paraproduct. We will use the same notation as in [3].
    ${ }^{2}$ We will only consider this case in the sequel.

