Reachability based Model Predictive Control for Semi-active Suspension System

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Abstract— **This paper proposes a Model Predictive Control (MPC) framework for control of a quarter car semi-active suspension system based on reachable sets approach. The proposed approach provides the flexibility of systematically including the information on the bounds of future road disturbances over the prediction horizon onto the MPC problem formulation by means of reachable sets. This inclusion of future bounds into control design leverages the efficiency of the operation of suspension system. Firstly, a unique Linear Parameter Varying (LPV) model is formulated for control design, which implicitly accounts for the dissipativity constraint of semi-active suspension system. Secondly, an efficient online computation of the reachable sets with the available disturbance information at the current instant for the MPC problem is developed. The proposed methodology is an integration of the preceding steps into a single MPC problem. The effectiveness of the proposed methodology is validated through simulations and the results exhibit better performance of the proposed controller compared to skyhook controller in terms of satisfaction of objective and constraint requirements.**

I. INTRODUCTION

The vehicle suspension system for road vehicles is one of the most vital automotive components which guarantees safety and comfort for on-board passengers. The prime functionality of the suspension system is to mitigate the effects of the road disturbances of the moving vehicle and thereby reducing the effects of chassis vibration (for comfort) and also, to ensure the condition that the wheels of the vehicle are in contact with the road to provide sufficient traction for control of lateral and longitudinal dynamics of the vehicle (for safety). Under the basis of operation, the suspension systems are broadly classified into a) Passive b) Semi-active and c) Active. Under the ambit of the conflicting objectives/functionality of the suspension system i.e. comfort and road holding, it behooves to look for a suspension system that adapts its characteristics such that the conflict is resolved in a conducive manner. This flexibility of adaptation of system characteristics is void in passive suspension system and thus, confines its range of operation to a narrow region. To overcome this limitation, semi-active and active suspension systems provide the provision to adapt the damping characteristics in real-time by means of appropriate control methodology. In comparison between the semi-active and active suspension system, the former fares better to the latter on a relative scale in terms of performance, cost, safety and negligible power demand [1].

Although there are several advantages of the semi-active suspension system, concomitantly, there exist various intricacies in dealing with efficient design of control algorithm such as region of operation, physical constraints, dissipativity constraints etc. Amongst these difficulties one of the most onerous task is to account for the dissipativity constraint over the entire duration of operation. There have been several research contributions for design of control strategies such as Skyhook [2], Mixed Skyhook and Acceleration driven damping control (ADD) [3] and more recently, over the last decade, there has been a lot of attention for LPV/H-∞ based methods [4]–[7] due to its enhanced performance and efficiency in terms of dealing with nonlinearities and objective satisfaction. Several variants of MPC based methods such as Implicit-MPC [8], [9], Explicit-MPC [10], [11], Fast MPC [12] methods have been proposed in the past to address this issue in a predictive control framework.

This paper for the first time, proposes a reachability based MPC approach for control of semi-active suspension system for a quarter car model. The prime motivation for the approach is due to the fact that the dynamics of the quarter car model involves the road disturbance term which plays an important role in the performance of the system. This was not explicitly included in the previous MPC design in [10]–[12]. Utilizing the future disturbance sequences would incorporate more information into the MPC problem and would yield better results [9]. However, in practice, it is seldom possible to obtain an exact road model or future disturbances. In [8], a constant disturbance model is assumed over the horizon with the current disturbance measured from a disturbance observer for the road profile. Yet, this is not guaranteed to provide best results if the road profile is too capricious over the prediction horizon. Thus, from the above instantiations, it motivates to utilize bounded disturbance sets for the future values of road profile. This information is included into the MPC formulation by means of reachability sets for the future states of the system.

The paper is organized as follows. Section II provides an overview of the required preliminaries. Section III discusses the dynamics of quarter car and the LPV reformulation of the model subsumed with the dissipativity constraint. Section IV discusses the objective and constraint requirements for the MPC problem. Section V expounds the proposed approach in detail. Section VI discusses the results and simulation and finally, the paper is concluded with Section VII with conclusions and future works.

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II. PRELIMINARIES

A. Notations

Let X, Y define two polytopic sets in $\mathbb{R}^n$. The $\mathcal{H}$ representation of the set X is defined as $\mathcal{H}_X = \{x \in \mathbb{R}^n | A_X x \leq b_X\}$, where $\{A_X, b_X\}$ represents the half space structure for the underlying set. The mapping of the set X under a function f is succinctly represented with the composite operator $f \circ X$. **card**(X) represents the cardinality of the set X. **co**$\{X, Y\}$ represents the convex hull between the sets X and Y. **vert**$\{X\}$ represents the vertices of set X. Minkowski sum between X and Y is defined as $X \oplus Y = \{x + y \in \mathbb{R}^n | x \in X, y \in Y\}$. The operator $\otimes$ represents the Kronecker product. The operator $\text{proj}_\theta(X)$ represents the projection of the set X over a lower dimensional space $\theta \subseteq \mathbb{R}^p$, where $1 \leq p \leq n$. $\mathbb{E}_{x\sim\mathbb{P}}$ represents the expectation operator under a probability distribution $\mathbb{P}$.

B. Reachable set description

Consider a dynamical system defined with the discrete time state transition equation

$$x^+ = f(x, u) \tag{1}$$

where $x \in \mathbb{R}^n$ represents the system state and $u \in \mathcal{U} \subset \mathbb{R}^m$ represents a bounded input to the system. Consider an initial set $\mathcal{X}_0 \subset \mathbb{R}^n$, then the one-step reachability set and N-step reachability set for the given system are defined as follows:

Definition 1: For the given system (1), the one-step reachable set [13] from the initial set $\mathcal{X}_0$ is defined as

$$\begin{aligned}\mathcal{R}(\mathcal{X}_0) = \{y \in \mathbb{R}^n |\ \exists x(0) \in \mathcal{X}_0 \subset \mathbb{R}^n,\ \exists u(0) \in \mathcal{U} \subset \mathbb{R}^m \\ s.t.\ y = f(x(0), u(0))\}\end{aligned} \tag{2}$$

Definition 2: For the given system (1) and an initial set $\mathcal{X}_0$ the N-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ [13] is defined in a recursive form of one-step reachable set with

$$\mathcal{R}_{i+1} = \mathcal{R}(\mathcal{R}_i(\mathcal{X}_0)),\ \mathcal{R}_0(\mathcal{X}_0) = \mathcal{X}_0,\ i = 0, \dots N-1 \tag{3}$$

Remark 1: Under the special case of linear dynamical systems, i.e. $x^+ = Ax + Bu$, the one-step reachable set is compactly expressed using the Minkowski representation with

$$\mathcal{R}(\mathcal{X}_0) = (A \circ \mathcal{X}_0) \oplus (B \circ \mathcal{U}) \tag{4}$$

The N-step reachable set is computed by recursively propagating the set operation defined in (4).

III. MODEL FORMULATION

A. System description

Consider a quarter car model, illustrated in Fig. 1, which consists of two mass elements which are the sprung mass (chassis) and the unsprung mass element (wheel). The vertical dynamics model for the system around equilibrium is expressed with

$$\begin{aligned} m_s \ddot{z}_s &= -k_s(z_s - z_{us}) + u \\ m_{us}\ddot{z}_{us} &= k_s(z_s - z_{us}) - u - k_t(z_{us} - z_r) \end{aligned} \tag{5}$$

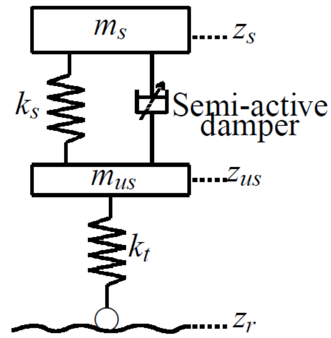

Fig. 1. Quarter car vehicle model

where, m_s and m_{us} are the sprung and unsprung masses respectively, k_s and k_{us} are the stiffness coefficients of the damper system and wheel respectively. z_s, $\dot{z}_s$, z_{us} and $\dot{z}_{us}$ are the sprung mass position, velocity and unsprung mass position, velocity respectively, z_r is the vertical road displacement. u is the force exerted due to the ER semi-active damper system which can be described with $u(t) = c(t)\dot{z}_{def}$, where $c(t)$ is the variable damper coefficient and is controlled with appropriate driving voltage. $\dot{z}_{def} = \dot{z}_{us} - \dot{z}_s$ and $z_{def} = z_{us} - z_s$ are the deflection velocity and deflection position between the sprung and unsprung mass. The dynamics equation (5) is compactly expressed in state space form with

$$\dot{x} = A_c x(t) + B_c u(t) + B_c^{dist} d(t) \tag{6}$$

where, $x = [z_s\ z_{us}\ \dot{z}_s\ \dot{z}_{us}]^T$ are the system states, $d(t) = z_r$ is the disturbance input from the road profile. $A_c \in \mathbb{R}^{4\times4}$, $B_c \in \mathbb{R}^{4\times1}$ and $B_c^{dist} \in \mathbb{R}^{4\times1}$ are the system matrix, input matrix and disturbance matrix respectively. The continuous time state space model is transformed to discrete time system with a zero order hold (ZOH) sampling method with a sampling period of T_s. Thus, the discrete time model obtained from (6) is expressed with

$$x^+ = A_d x_k + B_d u_k + B_d^{dist} d_k \tag{7}$$

where, A_d, B_d and B_d^{dist} are the discrete time system matrix, input matrix and disturbance matrix respectively with appropriate dimensions.

B. Dissipativity constraint

The dissipativity constraint arises due the inherent dissipative nature of the semi-active damper system. The constraint can be mathematically expressed with

$$\begin{aligned} u_{min} \leq u_k \leq u_{max},\ \text{if } \dot{z}_{def} \geq 0 \\ u_{max} \leq u_k \leq u_{min},\ \text{if } \dot{z}_{def} < 0 \end{aligned} \tag{8}$$

where, $u_{min} = c_{min}\dot{z}_{def}$ and $u_{max} = c_{max}\dot{z}_{def}$ are the minimum and maximum range of forces exerted by damper system and c_{min} and c_{max} are the minimum and maximum damping coefficients. The dissipativity constraint (8) is non-convex due to the mixed integer nature of the constraint, illustrated in Fig. 2.

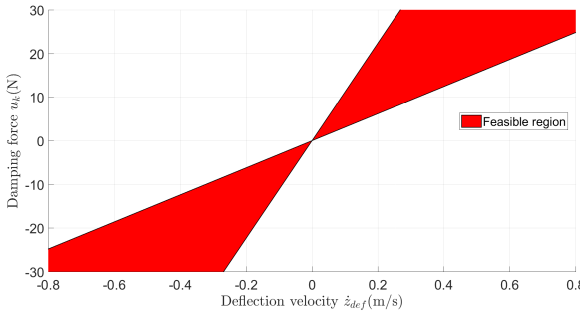

Fig. 2. Damper force vs deflection velocity plot

Fig. 3. SOBEN Semi-active suspension platform

TABLE I

MODEL PARAMETERS FOR MPC DESIGN FOR QUARTER CAR MODEL

Parameter	Symbol	Value (SI unit)
Chassis quarter car mass	m_s	9.08(kg)
Unsprung mass	m_{us}	0.32(kg)
Suspension stiffness	k	1396(N/m)
Tyre stiffness	k_t	18097.6(N/m)
Damping minimum coefficient	c_{min}	31(Ns/m)
Damping maximum coefficient	c_{max}	110.729(Ns/m)
Maximum force exerted by damper	$\overline{u}$	18(N)
Minimum force exerted by damper	$\underline{u}$	$-18(N)$
Maximum deflection	z_{def}^{max}	0.025(m)
Maximum deflection velocity	$\dot{z}_{def}^{max}$	0.5806(m/s)
Sampling period	T_s	5 ms
Horizon length	N	7

C. Reformulation to LPV model

The dissipativity constraint (8) is reformulated in a convex combination form expressed with

$$u_k = (\frac{1-\alpha_k}{2})u_{min} + (\frac{1+\alpha_k}{2})u_{max}, \text{ if } \dot{z}_{def} \geq 0$$
$$u_k = (\frac{1+\alpha_k}{2})u_{min} + (\frac{1-\alpha_k}{2})u_{max}, \text{ if } \dot{z}_{def} < 0 \tag{9}$$

where, $\alpha_k \in [-1, 1]$ defines the convex weights for (9). The switch case over the constraint (9), which is premised upon $\dot{z}_{def}$, is expressed as a switch variable $\delta_k \in \{-1, 1\}$ such that

$$\delta_k = \begin{cases} +1, \text{ if } \dot{z}_{def} \geq 0 \\ -1, \text{ if } \dot{z}_{def} < 0 \end{cases} \tag{10}$$

Substituting switch condition δ_k from (10) into equation (9) yields the input u_k with

$$u_k = \Big[(\frac{1-\alpha_k}{2})u_{min} + (\frac{1+\alpha_k}{2})u_{max}\Big](\frac{1+\delta_k}{2}) + \Big[(\frac{1+\alpha_k}{2})u_{min} + (\frac{1-\alpha_k}{2})u_{max}\Big](\frac{1-\delta_k}{2}) \tag{11}$$

Plugging u_k from (11) into discrete time state space equation (7) yields a new LPV discrete time system

$$x^+ = \overline{A_d}x_k + \overline{B_d(\rho_k)}\alpha_k + B_d^{dist}d_k \tag{12}$$

with $\overline{A_d} = A_d + c_{nom}B_dC_{\dot{z}_{def}}$, $\overline{B_d(\rho_k)} = c_{mid}B_d\rho_k$, where $c_{nom} = \frac{c_{min}+c_{max}}{2}$ and $c_{mid} = \frac{c_{max}-c_{min}}{2}$, the time varying parameter is ρ_k and α_k is the new input for the LPV system (12). Since, δ_k is premised on $\dot{z}_{def}$, the logical operation $\rho_k = \dot{z}_{def} \wedge \delta_k$ results in a non-negative value defined in the interval $\rho_k \in [0, |\dot{z}_{def}|]$. The row vectors $C_{\dot{z}_{def}} = [0\ 0\ -1\ 1]$ and $C_{z_{def}} = [-1\ 1\ 0\ 0]$ extracts the deflection velocity and position from the state vector respectively. The damper input force from (11) is reformulated and expressed with $u_k = c_{nom}C_{\dot{z}_{def}}x_k + c_{mid}\rho_k\alpha_k$. The numerical values of the parameters is listed in Table I utilized from the INOVE test platform model discussed in [14]. The INOVE platform is shown in Fig. 3.

IV. MPC DESIGN REQUIREMENTS

A. Objective requirements

The chosen objective design for the semi-active suspension is classified into a) Comfort objective and b) Road Holding objective (see [1] for more details).

- **Comfort objective**: The prime goal of the comfort based objective design is to minimize the vertical acceleration of the chassis ($\ddot{z}_s$). The comfort objective for the given horizon N is expressed as

$$J_{0\to N}^{com} = \sum_{k=0}^{N-1}(\ddot{z}_s(k))^2 \tag{13}$$

- **Road holding objective**: The prime goal of the road holding based objective design is to minimize the displacement between the road and the wheel ($z_{us} - z_r$). The road holding objective for the given horizon N is expressed as

$$J_{0\to N}^{rh} = \sum_{k=0}^{N-1}(z_{us}(k) - z_r(k))^2 \tag{14}$$

B. Constraint requirements

The constraints incorporated into MPC problem are

- **Input constraints**:
  - Maximum damper force constraint: This forms a mixed state-input constraint from (11), such that $u_k \in [\underline{u}, \overline{u}]$, where $\underline{u}$ and $\overline{u}$ are the minimum and maximum saturation forces for the semi-active suspension system.
  - Convex constraint for the new input: $\alpha_k \in [-1, 1]$.
- **State constraints**:
  - Maximum deflection between the chassis and wheel position: $|z_{def}| \leq z_{def}^{max}$, where z_{def}^{max} is the maximum deflection between the chassis and the wheel.

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  - Maximum deflection velocity between the chassis and wheel position: $|\dot{z}_{def}| \leq \dot{z}_{def}^{max}$, where $\dot{z}_{def}^{max}$ is the maximum deflection velocity. This constraint also bounds the time varying parameter $\rho \in [0, \rho^{max}]$, where $\rho^{max} = \dot{z}_{def}^{max}$.
  - Reachable sets that bound the future states of the system for the entire prediction horizon i.e $x_i \in \mathcal{X}_i$, where $\mathcal{X}_i$ represents the reachable set at time instant $i = 1 \dots N$, discussed in detail in Section V-E.
- **Dynamics constraint**: The equality constraints due to dynamics of the system defined in (12).

V. Reachability based MPC

A. Problem formulation

The proposed MPC approach utilizes the reachable sets to bound the future states of the system under the presence of bounded road disturbances. These reachable sets are incorporated into the MPC formulation as state constraints and the MPC design variables are computed such that it satisfies the imposed objective and input/state constraints for the system.

B. Disturbance rejection controller (K) design

As discussed previously, the MPC controller incorporates the evolution of system states by means of reachable sets by utilizing the future road disturbance bounds. Strictly under this condition, the MPC controller at all times presumes the worst case scenario and computes the control inputs for the defined reachable sets. This consequently would lead to conservativeness of the control design. To reduce the conservativeness, the control input can be pre-composed by means of an affine mapping defined by $\alpha_k = Kx_k + v_k$, where K is the disturbance rejection controller which acts as a stabilizing closed loop feedback controller over the prediction horizon. This reduces the influence of the road disturbance over the prediction horizon and also reduces the proliferation of the reachable sets due to its stabilizing nature. v_k is the new MPC control input for the system which satisfies the objective/constraints requirements.

Some of the key aspects of designing such a controller are a) it must satisfy the constraints imposed on the system b) the controller must be stabilizing and c) a poised trade off between the disturbance rejection controller and the MPC controller is achieved. Conditioned with the aforementioned statements, the approach adopted to design the controller is an ellipsoidal invariant set with maximal contraction factor. The imposed state and input constraints mentioned in Section IV-B on the system are collectively expressed with

$$g(x, \alpha) \leq 0 \tag{15}$$

where, $g : \mathbb{R}^5 \rightarrow \mathbb{R}^5_-$. The individual constraints on the states and inputs are computed using projection operation over (15), i.e. $\mathtt{X} = \text{proj}_x(g(x, \alpha))$ and $\mathtt{U} = \text{proj}_\alpha(g(x, \alpha))$. The given state and input constraints $\mathtt{X}$ and $\mathtt{U}$ are symmetric polytopic sets and can be expressed with normalized $\mathcal{H}$-representation, $\mathtt{X} = \{x \in \mathbb{R}^4 | \ |Fx| \leq 1\}$ and $\mathtt{U} = \{\alpha \in \mathbb{R} | \ |\alpha| \leq 1\}$.

The controller K is computed by solving the Semi-Definite Programming (SDP) problem for ellipsoidal invariant set with a maximal contraction factor. The Linear Matrix Inequalities (LMI) listed in the SDP problem (16) signifies a) Lyapunov's stability criterion b) satisfaction of state constraints with a contraction factor λ i.e. the ellipsoid $E(P) \subseteq \sqrt{\lambda}\mathtt{X}$ and c) input constraints ($\mathtt{U}$). The SDP problem is defined by

$$\begin{aligned} \min_{P,Y,\lambda} \quad & \lambda \\ \text{s.t.} \quad & \begin{bmatrix} P & \overline{A_d}P + \overline{B_{d}}_i Y \\ P\overline{A_d}^T + Y^T \overline{B_{d}}_i^T & P \end{bmatrix} \succ 0 \\ & \begin{bmatrix} \lambda & F_j^T P \\ PF_j & P \end{bmatrix} \succ 0 \\ & \begin{bmatrix} 1 & Y \\ Y^T & P \end{bmatrix} \succ 0 \end{aligned} \tag{16}$$

where, the matrix P represents the Lyapunov matrix, $Y = KP$ which is obtained from the congruence transformation [15] and $\lambda \in (0, 1]$ is the contraction factor. The LMI is solved for all vertex of the polytopic model, i.e. $\forall i \in$ $\textbf{vert}\{\textbf{co}\{[\overline{A_d}\ \overline{B_d(0)}], [\overline{A_d}\ \overline{B_d(\rho^{max})}]\}\}$ and $\forall j \in \textbf{card}(\mathtt{X})$. The disturbance rejection controller is obtained from the relation $K = YP^{-1}$.

C. Reachable set input condition

The MPC controller is precomposed with a disturbance rejection controller (discussed in Section V-B) i.e. $\alpha_k = Kx_k + v_k$, $k = 0, \dots, N-1$ and the new MPC control variable v_k is switched off.

D. Information availability at initial time

The initial reachable set is a singleton $\mathcal{X}_0 = x(0)$ is known and the disturbance bounds over the horizon is known a priori, i.e. $|d_k| \leq \gamma_k$, where γ_k is the disturbance bound at time instant $0 \leq k \leq N-1$. This case stems from the scenario where the current estimate of the road disturbance is unknown or unobservable, however the future bounds of the road disturbance is known.

E. Online computation of reachable sets

The reachable set for the non-input system defined at time k is computed using (4) by recursively performing the following set operation

$$\mathcal{X}_{k+1} = (\Psi \circ \mathcal{X}_k) \oplus (B_d^{dist} \circ \gamma_k \mathcal{Z}_k) \tag{17}$$

where Ψ is the system dynamics matrix for the reachable set input condition as mentioned Section V-C, i.e. $\Psi = \overline{A_d} + \overline{B_d(\rho^{max})}K$ and the disturbance sets are $\gamma_k \mathcal{Z}_k$, where $\mathcal{Z}_k = \{z_k \in \mathbb{R} | \ |z_k| \leq 1\}$, which represents a unit cube in $\mathbb{R}$.

Despite the above method computes the reachable sets, a verbatim implementation is not pragmatically doable under online conditions due to the fact that Minkowski sums are computationally taxing and it is to be computed at every time instant. However, the problem is recasted into a different form and reachable sets are computed with the aid of the telescopic matrices with

$$\begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ \vdots \\ \mathcal{X}_N \end{bmatrix} = \Phi^{\{\Psi\}} \mathcal{X}_0 \oplus \Gamma^{\{\Psi, B_d^{dist}\}} \begin{bmatrix} \mathcal{Z}_0 \\ \mathcal{Z}_1 \\ \vdots \\ \mathcal{Z}_{N-1} \end{bmatrix} \tag{18}$$

where $\Phi^{\{\Psi\}}$ and $\Gamma^{\{\Psi, B_d^{dist}\}}$ are described with

$$\Phi^{\{\Psi\}} = \begin{bmatrix} \Psi \\ \Psi^2 \\ \vdots \\ \Psi^N \end{bmatrix} \quad \Gamma^{\{\Psi, B_d^{dist}\}} = \begin{bmatrix} B_d^{dist} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \Psi^{N-1} B_d^{dist} & \cdots & B_d^{dist} \end{bmatrix} \tag{19}$$

For a given particular time instant $1 \leq j \leq N$, the reachable set is computed from the j^{th} row from (18), i.e.

$$\mathcal{X}_j = \Phi_j^{\{\Psi\}} \mathcal{X}_0 \oplus \Lambda_j \tag{20}$$

where,

$$\Lambda_j = \Gamma_j^{\{\Psi, B_d^{dist}\}} \begin{bmatrix} \mathcal{Z}_0 \\ \mathcal{Z}_1 \\ \vdots \\ \mathcal{Z}_{j-1} \end{bmatrix} \tag{21}$$

The projected hypercube (21) are represented in $\mathcal{H}$ representation with $\mathcal{H}_{\Lambda_j}$ with the half space structure pair $\{A_{\Lambda_j}, b_{\Lambda_j}\}$. The reachable sets for the states in (18) are also described in $\mathcal{H}$-representation with $\mathcal{H}_{\mathcal{X}_j}$ and the half space structure pair is represented with $\{A_{\mathcal{X}_j}, b_{\mathcal{X}_j}\}$. The $A_{\mathcal{X}_j}$ parameters are computed offline from (21) e.i. $A_{\mathcal{X}_j} = A_{\Lambda_j}$ and the online computation involves the manipulation of the vectors $b_{\mathcal{X}_j}$, which depends upon the disturbance bounds as mentioned in Section V-D. The vectors $b_{\mathcal{X}_j}$ are computed with

$$b_{\mathcal{X}_j} = b_{\Lambda_j} \otimes \gamma_j + A_{\Lambda_j} \Psi^j \mathcal{X}_0 \tag{22}$$

F. The finite time optimal control problem (FTOCP)

The FTOCP is casted with respect to the required performance objectives i.e. either comfort or road holding as mentioned in Section IV-A.

$$\begin{aligned} \min_{\alpha_{0\to N-1}} \quad & J(x_0, \alpha_{0\to N-1}) = \theta_1 J_{0\to N}^{com} + \theta_2 J_{0\to N}^{rh} \\ \text{s.t.} \quad & x^+ = \Psi x_k + \overline{B(\rho_k)} \alpha_k, \ \rho^+ = \rho_k \\ & c_{mid} C_{\dot{z}_{def}} x_k + c_{nom} \rho_k \alpha_k \in [\underline{u}, \overline{u}] \\ & \alpha_k \in [-1, 1], \ x_0 = x(0) \\ & |C_{z_{def}} x_k| \leq z_{def}^{max}, \ |C_{\dot{z}_{def}} x_k| \leq \dot{z}_{def}^{max} \\ & x_1 \in \mathcal{X}_1, \ x_2 \in \mathcal{X}_2, \ \ldots x_N \in \mathcal{X}_N \\ & \mathcal{X}_0 = \{x_0\}, \ \rho_0 = \{|C_{\dot{z}_{def}} x_0|\} \end{aligned} \tag{23}$$

The dynamics of LPV parameter is assumed to be constant over the prediction horizon. The parameters θ_1 and θ_2 (fixed a priori) are weighting coefficients between comfort and road holding objective and also, it form a convex combination between the two objectives i.e. $\theta_1 + \theta_2 = 1$ and $\theta_1, \theta_2 \geq 0$. Once, the optimization routine is solved, the optimal input at $k = 0$ i.e. α_0^* is applied to system by computing the actual damper force from (11) and the problem is repeated in a receding horizon manner.

VI. Results and Simulation

A. Skyhook controller

Skyhook controller is one of the most prominent and has been the de facto controller for semi-active suspension system [2]. The controller swings between minimum and maximum damper coefficient depending upon a switch condition. Mathematically, the controller is expressed with

$$u_k = \begin{cases} \min\{c_{max} \dot{z}_{def}, \overline{u}\}, & \text{if } \dot{z}_s \dot{z}_{def} \geq 0 \\ \max\{c_{min} \dot{z}_{def}, \underline{u}\}, & \text{if } \dot{z}_s \dot{z}_{def} < 0 \end{cases} \tag{24}$$

B. Metric computation

The metric utilized to define the quality of performance are a) normalized RMS value of sprung mass acceleration ($\ddot{z}_s^{RMS}$) and b) normalized dynamic wheel forces (F_{us}^{RMS}) [12], which are defined by

$$\begin{aligned} \ddot{z}_s^{RMS} &= \sqrt{\frac{1}{\tau} \int_0^\tau \left(\frac{\ddot{z}_s(t)}{g} \right)^2 dt} \\ F_{us}^{RMS} &= \sqrt{\frac{1}{\tau} \int_0^\tau \left(\frac{F_{z_{us}}(t)}{F_{z_{us}}^{stat}} \right)^2 dt} \end{aligned} \tag{25}$$

where, $g = 9.81\, m/s^2$ is the acceleration due to gravity and $F_{z_{us}}^{stat} = (m_s + m_{us}) g$ is the static normal force and τ is the duration of simulation. The road profile to excite the system is an uniform random sequence with bounds $\pm$1mm, i.e. $d_k \sim \mathrm{U}[-0.001, 0.001]$. Since the input road profile is a random process, the entire system is indirectly driven by a shockwave and thus, the metric is transformed into a stochastic variable. The modified metric in stochastic sense is given by the relation $\mathbb{E}_{d_k \sim \mathrm{U}}[\ddot{z}_s^{RMS}]$ and $\mathbb{E}_{d_k \sim \mathrm{U}}[F_{us}^{RMS}]$.

The analytical computation of the expectation is cumbersome and not tractable, thus it is approximated using Monte-Carlo simulations, which are defined by the relations

$$\begin{aligned} \mathbb{E}_{d_k \sim \mathrm{U}}[\ddot{z}_s^{RMS}] &\approx \frac{1}{N_{MC}} \sum_{i=0}^{N_{MC}-1} \ddot{z}_s^{RMS}(i) \\ \mathbb{E}_{d_k \sim \mathrm{U}}[F_{us}^{RMS}] &\approx \frac{1}{N_{MC}} \sum_{i=0}^{N_{MC}-1} F_{z_{us}}^{RMS}(i) \end{aligned} \tag{26}$$

where, $N_{MC} = 50$ is the number of Monte-Carlo simulations and the duration of each simulation is $10s$.

C. Simulation implementation

The proposed method is implemented in MATLAB/Simulink environment. The reachable set offline parameters are computed using Multiparametric toolbox (MPT) [16], the LMI problem for the disturbance rejection controller is computed using CVX toolbox [17]. The MPC problem is programmed in CVXGEN interface [18] and the optimization C-code is patched to Simulink by means of S-function builder.

D. Simulation results

The proposed method is compared against skyhook controller for the two different objective requirements. The following enlisted tables displays the performance obtained for different simulation trials.

- Comfort oriented design

Metric	Proposed method	Skyhook	Gain (%)
$\ddot{z}_s^{RMS}$	0.0653	0.1249	47.72%
F_{us}^{RMS}	0.2025	0.2038	0.64%

- Road holding oriented design

Metric	Proposed method	Skyhook	Gain (%)
$\ddot{z}_s^{RMS}$	0.0958	0.1250	23.36%
F_{us}^{RMS}	0.1878	0.2028	7.41%

From the simulations, it is evident that the proposed method fares well compared to the skyhook controller as the performance gains are higher for the enumerated cases.

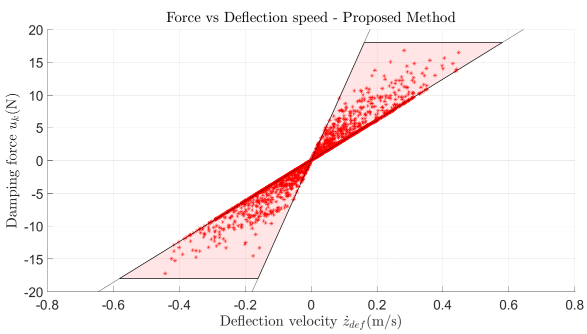

Fig. 4. Damping force vs Deflection speed - Proposed method

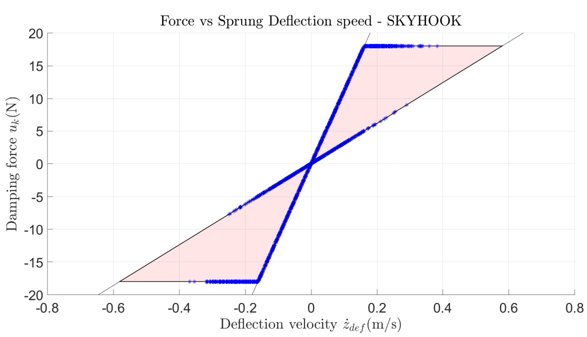

Fig. 5. Damping force vs Deflection speed - Skyhook

Fig. 4 and Fig. 5 illustrates the dissipativity constraint for a particular objective setting (applies to other objective settings as well). From the plots, it is evident that the proposed method utilizes the control authority of the semi-active suspension system in a judicious manner and also respects the constraints whereas the skyhook controller slides over the perimeter of the constraints and does not fully utilize the potential of the system.

VII. Conclusion and Future Works

The proposed method of MPC proves to be a viable option for improvement of performance and MATLAB/Simulink results expounds the fact that the proof of concept works well in simulation. There are further more room for improvement in the proposed method, to name a few a) outer approximation of reachable sets for longer horizon, b) improved objective function for both comfort and road holding and c) better design of the disturbance rejection controller K. The future line of research may involve nonlinear control design using similar approach and real-time implementation on SOBEN test bench at GIPSA-lab, Grenoble.

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