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SEMI-CLASSICAL RESOLVENT ESTIMATES FOR SHORT-RANGE L^∞ POTENTIALS

GEORGI VODEV

ABSTRACT. We prove semi-classical resolvent estimates for real-valued potentials $V \in L^\infty(\mathbb{R}^n)$, $n \geq 3$, satisfying $V(x) = \mathcal{O}(\langle x \rangle^{-\delta})$ with $\delta > 3$.

1. INTRODUCTION AND STATEMENT OF RESULTS

Our goal in this note is to study the resolvent of the Schrödinger operator

$$P(h) = -h^2 \Delta + V(x)$$

where $0 < h \ll 1$ is a semi-classical parameter, Δ is the negative Laplacian in \mathbb{R}^n , $n \geq 3$, and $V \in L^\infty(\mathbb{R}^n)$ is a real-valued potential satisfying

$$(1.1) \quad |V(x)| \leq C \langle x \rangle^{-\delta}$$

with some constants $C > 0$ and $\delta > 3$. More precisely, we are interested in bounding from above the quantity

$$g_s^\pm(h, \varepsilon) := \log \left\| \langle x \rangle^{-s} (P(h) - E \pm i\varepsilon)^{-1} \langle x \rangle^{-s} \right\|_{L^2 \rightarrow L^2}$$

where $L^2 := L^2(\mathbb{R}^n)$, $0 < \varepsilon < 1$, $s > 1/2$ and $E > 0$ is a fixed energy level independent of h . Such bounds are known in various situations. For example, for long-range real-valued C^1 potentials it is proved in [4] when $n \geq 3$ and in [8] when $n = 2$ that

$$(1.2) \quad g_s^\pm(h, \varepsilon) \leq Ch^{-1}$$

with some constant $C > 0$ independent of h and ε . Previously, the bound (1.2) was proved for smooth potentials in [2] and an analog of (1.2) for Hölder potentials was proved in [10]. A high-frequency analog of (1.2) on more complex Riemannian manifolds was also proved in [1] and [3]. In all these papers the regularity of the potential (and of the perturbation in general) plays an essential role. Without any regularity the problem of bounding g_s^\pm from above by an explicit function of h gets quite tough. Nevertheless, it has been recently shown in [9] that for real-valued compactly supported L^∞ potentials one has the bound

$$(1.3) \quad g_s^\pm(h, \varepsilon) \leq Ch^{-4/3} \log(h^{-1})$$

with some constant $C > 0$ independent of h and ε . The bound (1.3) has been also proved in [7] still for real-valued compactly supported L^∞ potentials but with the weight $\langle x \rangle^{-s}$ replaced by a cut-off function. When $n = 1$ it was shown in [6] that we have the better bound (1.2) instead of (1.3). When $n \geq 2$, however, the bound (1.3) seems hard to improve without extra conditions on the potential. The problem of showing that the bound (1.3) is optimal is largely open. In contrast, it is well-known that the bound (1.2) cannot be improved in general (e.g. see [5]).

In this note we show that the bound (1.3) still holds for non-compactly supported L^∞ potentials when $n \geq 3$. Our main result is the following

Theorem 1.1. *Under the condition (1.1), there exists $h_0 > 0$ such that for all $0 < h \leq h_0$ the bound (1.3) holds true.*

Remark. It is easy to see from the proof (see the inequality (4.2)) that the bound (1.3) holds also for a complex-valued potential V satisfying (1.1), provided that its imaginary part satisfies the condition

$$\mp \operatorname{Im} V(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^n.$$

To prove this theorem we adapt the Carleman estimates proved in [9] simplifying some key arguments as for example the construction of the phase function φ . This is made possible by defining the key function F in Section 3 differently, without involving the second derivative φ'' . The consequence is that we do not need to seek φ' as a solution to a differential equation as done in [9], but it suffices to define it explicitly. Note also that similar (but simpler) Carleman estimates have been used in [11] to prove high-frequency resolvent estimates for the magnetic Schrödinger operator with large L^∞ magnetic potentials.

2. CONSTRUCTION OF THE PHASE AND WEIGHT FUNCTIONS

We will first construct the weight function. We begin by introducing the continuous function

$$\mu(r) = \begin{cases} (r+1)^2 - 1 & \text{for } 0 \leq r \leq a, \\ (a+1)^2 - 1 + (a+1)^{-2s+1} - (r+1)^{-2s+1} & \text{for } r \geq a, \end{cases}$$

where

$$(2.1) \quad \frac{1}{2} < s < \frac{\delta - 2}{2}$$

and $a = h^{-m}$ with some parameter $m > 0$ to be fixed in the proof of Lemma 2.3 below depending only on δ and s . Clearly, the first derivative (in sense of distributions) of μ satisfies

$$\mu'(r) = \begin{cases} 2(r+1) & \text{for } 0 \leq r < a, \\ (2s-1)(r+1)^{-2s} & \text{for } r > a. \end{cases}$$

The main properties of the functions μ and μ' are given in the following

Lemma 2.1. *For all $r > 0$, $r \neq a$, we have the inequalities*

$$(2.2) \quad 2r^{-1}\mu(r) - \mu'(r) \geq 0,$$

$$(2.3) \quad \mu'(r) \geq C_1(r+1)^{-2s},$$

$$(2.4) \quad \frac{\mu(r)^2}{\mu'(r)} \leq C_2 a^4 (r+1)^{2s}$$

with some constants $C_1, C_2 > 0$.

Proof. For $r < a$ the left-hand side of (2.2) is equal to 2, while for $r > a$ it is bounded from below by

$$2r^{-1}(a^2 - s) > 0$$

provided a is taken large enough. Furthermore, we clearly have (2.3) for $r < a$ with $C_1 = 2$, while for $r > a$ it holds with $C_1 = 2s - 1$. Therefore, (2.3) holds with $C_1 = \min\{2, 2s - 1\}$. The bound (2.4) follows from the observation that $\mu(r) \leq a^2$ for all r , and $\mu'(r) \geq 2$ for $r < a$. Thus we get that (2.4) holds with $C_2 = \max\{2^{-1}, (2s - 1)^{-1}\}$. \square

We now turn to the construction of the phase function $\varphi \in C^1([0, +\infty))$ such that $\varphi(0) = 0$ and $\varphi(r) > 0$ for $r > 0$. We define the first derivative of φ by

$$\varphi'(r) = \begin{cases} \tau(r+1)^{-1} - \tau(a+1)^{-1} & \text{for } 0 \leq r \leq a, \\ 0 & \text{for } r \geq a, \end{cases}$$

where

$$(2.5) \quad \tau = \tau_0 h^{-1/3}$$

with some parameter $\tau_0 \gg 1$ independent of h to be fixed in Lemma 2.3 below. Clearly, the first derivative of φ' satisfies

$$\varphi''(r) = \begin{cases} -\tau(r+1)^{-2} & \text{for } 0 \leq r < a, \\ 0 & \text{for } r > a. \end{cases}$$

Lemma 2.2. *For all $r \geq 0$ we have the bound*

$$(2.6) \quad h^{-1}\varphi(r) \lesssim h^{-4/3} \log \frac{1}{h}.$$

Proof. We have

$$\max \varphi = \int_0^a \varphi'(r) dr \leq \tau \int_0^a (r+1)^{-1} dr = \tau \log(a+1)$$

which clearly implies (2.6) in view of the choice of τ and a . □

For $r \neq a$, set

$$A(r) = (\mu\varphi'^2)'(r)$$

and

$$B(r) = \frac{(\mu(r) (h^{-1}(r+1)^{-\delta} + |\varphi''(r)|))^2}{h^{-1}\varphi'(r)\mu(r) + \mu'(r)}.$$

The following lemma will play a crucial role in the proof of the Carleman estimates in the next section.

Lemma 2.3. *Given any $C > 0$ independent of the variable r and the parameters h , τ and a , there exist $\tau_0 = \tau_0(C) > 0$ and $h_0 = h_0(C) > 0$ so that for τ satisfying (2.5) and for all $0 < h \leq h_0$ we have the inequality*

$$(2.7) \quad A(r) - CB(r) \geq -\frac{E}{2}\mu'(r)$$

for all $r > 0$, $r \neq a$.

Proof. For $r < a$ we have

$$\begin{aligned} A(r) &= -(\varphi'^2)'(r) + \tau^2 \partial_r (1 - (r+1)(a+1)^{-1})^2 \\ &= -2\varphi'(r)\varphi''(r) - 2\tau^2(a+1)^{-1} (1 - (r+1)(a+1)^{-1}) \\ &\geq 2\tau(r+1)^{-2}\varphi'(r) - 2\tau^2(a+1)^{-1} \\ &\geq 2\tau(r+1)^{-2}\varphi'(r) - \tau^2 a^{-1}\mu'(r) \\ &\geq 2\tau(r+1)^{-2}\varphi'(r) - \mathcal{O}(h^{m-1})\mu'(r) \end{aligned}$$

where we have used that $\mu'(r) \geq 2$. Taking $m > 2$ we get

$$(2.8) \quad A(r) \geq 2\tau(r+1)^{-2}\varphi'(r) - \mathcal{O}(h)\mu'(r)$$

for all $r < a$. We will now bound the function B from above. Let first $0 < r \leq \frac{a}{2}$. Since in this case we have

$$\varphi'(r) \geq \frac{\tau}{3}(r+1)^{-1}$$

we obtain

$$\begin{aligned} B(r) &\lesssim \frac{\mu(r) (h^{-2}(r+1)^{-2\delta} + \varphi''(r)^2)}{h^{-1}\varphi'(r)} \\ &\lesssim (\tau h)^{-1} \frac{\mu(r)(r+1)^{2-2\delta}}{\varphi'(r)^2} \tau(r+1)^{-2}\varphi'(r) + h \frac{\mu(r)\varphi''(r)^2}{\mu'(r)\varphi'(r)} \mu'(r) \\ &\lesssim \tau^{-3}h^{-1}(r+1)^{6-2\delta} \tau(r+1)^{-2}\varphi'(r) + \tau h \mu'(r) \\ &\lesssim \tau_0^{-3} \tau(r+1)^{-2}\varphi'(r) + h^{2/3} \mu'(r) \end{aligned}$$

where we have used that $\delta > 3$. This bound together with (2.8) clearly imply (2.7), provided τ_0^{-1} and h are taken small enough depending on C .

Let now $\frac{a}{2} < r < a$. Then we have the bound

$$\begin{aligned} B(r) &\leq \left(\frac{\mu(r)}{\mu'(r)} \right)^2 \left(h^{-1}(r+1)^{-\delta} + |\varphi''(r)| \right)^2 \mu'(r) \\ &\lesssim \left(h^{-2}(r+1)^{2-2\delta} + \tau^2(r+1)^{-2} \right) \mu'(r) \\ &\lesssim \left(h^{-2}a^{2-2\delta} + \tau^2a^{-2} \right) \mu'(r) \\ &\lesssim \left(h^{2m(\delta-1)-2} + h^{2m-2/3} \right) \mu'(r) \lesssim h \mu'(r) \end{aligned}$$

provided m is taken large enough. Again, this bound together with (2.8) imply (2.7).

It remains to consider the case $r > a$. Using that $\mu = \mathcal{O}(a^2)$ together with (2.3) and taking into account that s satisfies (2.1), we get

$$\begin{aligned} B(r) &= \frac{(\mu(r) (h^{-1}(r+1)^{-\delta}))^2}{\mu'(r)} \\ &\lesssim h^{-2}a^4(r+1)^{4s-2\delta} \mu'(r) \lesssim h^{-2}a^{4+4s-2\delta} \mu'(r) \\ &\lesssim h^{2m(\delta-2-2s)-2} \mu'(r) \lesssim h \mu'(r) \end{aligned}$$

provided that m is taken large enough. Since in this case $A(r) = 0$, the above bound clearly implies (2.7). \square

3. CARLEMAN ESTIMATES

Our goal in this section is to prove the following

Theorem 3.1. *Suppose (1.1) holds and let s satisfy (2.1). Then, for all functions $f \in H^2(\mathbb{R}^n)$ such that $\langle x \rangle^s (P(h) - E \pm i\varepsilon)f \in L^2$ and for all $0 < h \ll 1$, $0 < \varepsilon \leq ha^{-2}$, we have the estimate*

$$(3.1) \quad \|\langle x \rangle^{-s} e^{\varphi/h} f\|_{L^2} \leq Ca^2 h^{-1} \|\langle x \rangle^s e^{\varphi/h} (P(h) - E \pm i\varepsilon)f\|_{L^2} + Ca\tau(\varepsilon/h)^{1/2} \|e^{\varphi/h} f\|_{L^2}$$

with a constant $C > 0$ independent of h , ε and f .

Proof. We pass to the polar coordinates $(r, w) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$, $r = |x|$, $w = x/|x|$, and recall that $L^2(\mathbb{R}^n) = L^2(\mathbb{R}^+ \times \mathbb{S}^{n-1}, r^{n-1} dr dw)$. In what follows we denote by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ the norm and the scalar product in $L^2(\mathbb{S}^{n-1})$. We will make use of the identity

$$(3.2) \quad r^{(n-1)/2} \Delta r^{-(n-1)/2} = \partial_r^2 + \frac{\tilde{\Delta}_w}{r^2}$$

where $\tilde{\Delta}_w = \Delta_w - \frac{1}{4}(n-1)(n-3)$ and Δ_w denotes the negative Laplace-Beltrami operator on \mathbb{S}^{n-1} . Set $u = r^{(n-1)/2} e^{\varphi/h} f$ and

$$\begin{aligned} \mathcal{P}^\pm(h) &= r^{(n-1)/2} (P(h) - E \pm i\varepsilon) r^{-(n-1)/2}, \\ \mathcal{P}_\varphi^\pm(h) &= e^{\varphi/h} \mathcal{P}^\pm(h) e^{-\varphi/h}. \end{aligned}$$

Using (3.2) we can write the operator $\mathcal{P}^\pm(h)$ in the coordinates (r, w) as follows

$$\mathcal{P}^\pm(h) = \mathcal{D}_r^2 + \frac{\Lambda_w}{r^2} - E \pm i\varepsilon + V$$

where we have put $\mathcal{D}_r = -ih\partial_r$ and $\Lambda_w = -h^2 \tilde{\Delta}_w$. Since the function φ depends only on the variable r , this implies

$$\mathcal{P}_\varphi^\pm(h) = \mathcal{D}_r^2 + \frac{\Lambda_w}{r^2} - E \pm i\varepsilon - \varphi'^2 + h\varphi'' + 2i\varphi' \mathcal{D}_r + V.$$

For $r > 0$, $r \neq a$, introduce the function

$$F(r) = -\langle (r^{-2} \Lambda_w - E - \varphi'(r)^2) u(r, \cdot), u(r, \cdot) \rangle + \|\mathcal{D}_r u(r, \cdot)\|^2$$

and observe that its first derivative is given by

$$\begin{aligned} F'(r) &= \frac{2}{r} \langle r^{-2} \Lambda_w u(r, \cdot), u(r, \cdot) \rangle + ((\varphi')^2)' \|u(r, \cdot)\|^2 \\ &\quad - 2h^{-1} \operatorname{Im} \langle \mathcal{P}_\varphi^\pm(h) u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle \\ &\quad \pm 2\varepsilon h^{-1} \operatorname{Re} \langle u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle + 4h^{-1} \varphi' \|\mathcal{D}_r u(r, \cdot)\|^2 \\ &\quad + 2h^{-1} \operatorname{Im} \langle (V + h\varphi'') u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle. \end{aligned}$$

Thus, if μ is the function defined in the previous section, we obtain the identity

$$\begin{aligned} \mu' F + \mu F' &= (2r^{-1} \mu - \mu') \langle r^{-2} \Lambda_w u(r, \cdot), u(r, \cdot) \rangle + (E\mu' + (\mu(\varphi')^2)') \|u(r, \cdot)\|^2 \\ &\quad - 2h^{-1} \mu \operatorname{Im} \langle \mathcal{P}_\varphi^\pm(h) u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle \\ &\quad \pm 2\varepsilon h^{-1} \mu \operatorname{Re} \langle u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle + (\mu' + 4h^{-1} \varphi' \mu) \|\mathcal{D}_r u(r, \cdot)\|^2 \\ &\quad + 2h^{-1} \mu \operatorname{Im} \langle (V + h\varphi'') u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle. \end{aligned}$$

Using that $\Lambda_w \geq 0$ together with (2.2) we get the inequality

$$\begin{aligned} \mu' F + \mu F' &\geq (E\mu' + (\mu(\varphi')^2)') \|u(r, \cdot)\|^2 + (\mu' + 4h^{-1} \varphi' \mu) \|\mathcal{D}_r u(r, \cdot)\|^2 \\ &\quad - \frac{3h^{-2} \mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h) u(r, \cdot)\|^2 - \frac{\mu'}{3} \|\mathcal{D}_r u(r, \cdot)\|^2 \\ &\quad - \varepsilon h^{-1} \mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) \\ &\quad - 3h^{-2} \mu^2 (\mu' + 4h^{-1} \varphi' \mu)^{-1} \|(V + h\varphi'') u(r, \cdot)\|^2 - \frac{1}{3} (\mu' + 4h^{-1} \varphi' \mu) \|\mathcal{D}_r u(r, \cdot)\|^2 \\ &\geq \left(E\mu' + (\mu(\varphi')^2)' - C\mu^2 (\mu' + h^{-1} \varphi' \mu)^{-1} (h^{-1}(r+1)^{-\delta} + |\varphi''|^2) \right) \|u(r, \cdot)\|^2 \\ &\quad - \frac{3h^{-2} \mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h) u(r, \cdot)\|^2 - \varepsilon h^{-1} \mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) \end{aligned}$$

with some constant $C > 0$. Now we use Lemma 2.3 to conclude that

$$\begin{aligned} \mu' F + \mu F' &\geq \frac{E}{2} \mu' \|u(r, \cdot)\|^2 - \frac{3h^{-2}\mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 \\ &\quad - \varepsilon h^{-1} \mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2). \end{aligned}$$

We now integrate this inequality with respect to r and use that, since $\mu(0) = 0$, we have

$$\int_0^\infty (\mu' F + \mu F') dr = 0.$$

Thus we obtain the estimate

$$\begin{aligned} \frac{E}{2} \int_0^\infty \mu' \|u(r, \cdot)\|^2 dr &\leq 3h^{-2} \int_0^\infty \frac{\mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 dr \\ (3.3) \quad &+ \varepsilon h^{-1} \int_0^\infty \mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) dr. \end{aligned}$$

Using that $\mu = \mathcal{O}(a^2)$ together with (2.3) and (2.4) we get from (3.3)

$$\begin{aligned} \int_0^\infty (r+1)^{-2s} \|u(r, \cdot)\|^2 dr &\leq C a^4 h^{-2} \int_0^\infty (r+1)^{2s} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 dr \\ (3.4) \quad &+ C \varepsilon h^{-1} a^2 \int_0^\infty (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) dr \end{aligned}$$

with some constant $C > 0$ independent of h and ε . On the other hand, we have the identity

$$\operatorname{Re} \int_0^\infty \langle 2i\varphi' \mathcal{D}_r u(r, \cdot), u(r, \cdot) \rangle dr = \int_0^\infty h\varphi'' \|u(r, \cdot)\|^2 dr$$

and hence

$$\begin{aligned} \operatorname{Re} \int_0^\infty \langle \mathcal{P}_\varphi^\pm(h)u(r, \cdot), u(r, \cdot) \rangle dr &= \int_0^\infty \|\mathcal{D}_r u(r, \cdot)\|^2 dr + \int_0^\infty \langle r^{-2} \Lambda_w u(r, \cdot), u(r, \cdot) \rangle dr \\ &\quad - \int_0^\infty (E + \varphi'^2) \|u(r, \cdot)\|^2 dr + \int_0^\infty \langle V u(r, \cdot), u(r, \cdot) \rangle dr. \end{aligned}$$

This implies

$$\begin{aligned} \int_0^\infty \|\mathcal{D}_r u(r, \cdot)\|^2 dr &\leq \mathcal{O}(\tau^2) \int_0^\infty \|u(r, \cdot)\|^2 dr \\ (3.5) \quad &+ \gamma \int_0^\infty (r+1)^{-2s} \|u(r, \cdot)\|^2 dr + \gamma^{-1} \int_0^\infty (r+1)^{2s} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 dr \end{aligned}$$

for every $\gamma > 0$. We take now γ small enough, independent of h , and recall that $\varepsilon h^{-1} a^2 \leq 1$. Thus, combining the estimates (3.4) and (3.5), we get

$$\begin{aligned} \int_0^\infty (r+1)^{-2s} \|u(r, \cdot)\|^2 dr &\leq C a^4 h^{-2} \int_0^\infty (r+1)^{2s} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 dr \\ (3.6) \quad &+ C \varepsilon h^{-1} a^2 \tau^2 \int_0^\infty \|u(r, \cdot)\|^2 dr \end{aligned}$$

with a new constant $C > 0$ independent of h and ε . It is an easy observation now that the estimate (3.6) implies (3.1). \square

4. RESOLVENT ESTIMATES

In this section we will derive the bound (1.3) from Theorem 3.1. Indeed, it follows from the estimate (3.1) and Lemma 2.2 that for $0 < h \ll 1$, $0 < \varepsilon \leq ha^{-2}$ and s satisfying (2.1) we have

$$(4.1) \quad \|\langle x \rangle^{-s} f\|_{L^2} \leq M \|\langle x \rangle^s (P(h) - E \pm i\varepsilon) f\|_{L^2} + M\varepsilon^{1/2} \|f\|_{L^2}$$

where

$$M = \exp\left(C h^{-4/3} \log(h^{-1})\right)$$

with a constant $C > 0$ independent of h and ε . On the other hand, since the operator $P(h)$ is symmetric, we have

$$(4.2) \quad \begin{aligned} \varepsilon \|f\|_{L^2}^2 &= \pm \operatorname{Im} \langle (P(h) - E \pm i\varepsilon) f, f \rangle_{L^2} \\ &\leq (2M)^{-2} \|\langle x \rangle^{-s} f\|_{L^2}^2 + (2M)^2 \|\langle x \rangle^s (P(h) - E \pm i\varepsilon) f\|_{L^2}^2. \end{aligned}$$

We rewrite (4.2) in the form

$$(4.3) \quad M\varepsilon^{1/2} \|f\|_{L^2} \leq \frac{1}{2} \|\langle x \rangle^{-s} f\|_{L^2} + 2M^2 \|\langle x \rangle^s (P(h) - E \pm i\varepsilon) f\|_{L^2}.$$

We now combine (4.1) and (4.3) to get

$$(4.4) \quad \|\langle x \rangle^{-s} f\|_{L^2} \leq 4M^2 \|\langle x \rangle^s (P(h) - E \pm i\varepsilon) f\|_{L^2}.$$

It follows from (4.4) that the resolvent estimate

$$(4.5) \quad \|\langle x \rangle^{-s} (P(h) - E \pm i\varepsilon)^{-1} \langle x \rangle^{-s}\|_{L^2 \rightarrow L^2} \leq 4M^2$$

holds for all $0 < h \ll 1$, $0 < \varepsilon \leq ha^{-2}$ and s satisfying (2.1). On the other hand, for $\varepsilon \geq ha^{-2}$ the estimate (4.5) holds in a trivial way. Indeed, in this case, since the operator $P(h)$ is symmetric, the norm of the resolvent is upper bounded by $\varepsilon^{-1} = \mathcal{O}(h^{-2m-1})$. Finally, observe that if (4.5) holds for s satisfying (2.1), it holds for all $s > 1/2$.

REFERENCES

- [1] N. BURQ, *Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel*, Acta Math. **180** (1998), 1-29.
- [2] N. BURQ, *Lower bounds for shape resonances widths of long-range Schrödinger operators*, Amer. J. Math. **124** (2002), 677-735.
- [3] F. CARDOSO AND G. VODEV, *Uniform estimates of the resolvent of the Laplace-Beltrami operator on infinite volume Riemannian manifolds*, Ann. Henri Poincaré **4** (2002), 673-691.
- [4] K. DATCHEV, *Quantative limiting absorption principle in the semiclassical limit*, Geom. Funct. Anal. **24** (2014), 740-747.
- [5] K. DATCHEV, S. DYATLOV AND M. ZWORSKI, *Resonances and lower resolvent bounds*, J. Spectral Theory **5**, (2015), 599-615.
- [6] S. DYATLOV AND M. ZWORSKI, *The mathematical theory of scattering resonances*, <http://math.mit.edu/~dyatlov/res/res.20170323.pdf>.
- [7] F. KLOPP AND M. VOGEL, *On resolvent estimates and resonance free regions for semiclassical Schrödinger operators with bounded potentials*, Pure and Applied Analysis, to appear.
- [8] J. SHAPIRO, *Semiclassical resolvent bounds in dimension two*, Proc. Amer. Math. Soc., to appear.
- [9] J. SHAPIRO, *Semiclassical resolvent bound for compactly supported L^∞ potentials*, J. Spectral Theory, to appear.
- [10] G. VODEV, *Semi-classical resolvent estimates and regions free of resonances*, Math. Nachr. **287** (2014), 825-835.
- [11] G. VODEV, *Resolvent estimates for the magnetic Schrödinger operator*, Anal. PDE **7** (2014), 1639-1648.

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