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# The Case of Neumann, Robin, and Periodic Lateral Conditions for the Semi-infinite Generalized Graetz Problem and Applications

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1           **THE CASE OF NEUMANN, ROBIN AND PERIODIC LATERAL**  
2           **CONDITION FOR THE SEMI INFINITE GENERALIZED GRAETZ**  
3           **PROBLEM AND APPLICATIONS.**

4           VALENTION DEBARNOT, JÉRÔME FEHRENBACH, FRÉDÉRIC DE GOURNAY, LÉO  
5           MARTIRE \*

6           **Abstract.** The Graetz problem is a convection-diffusion equation in a pipe invariant along a di-  
7           rection. The contribution of the present work is to propose a mathematical analysis of the Neumann,  
8           Robin and periodic boundary condition on the boundary of a semi-infinite pipe. The solution in the  
9           3D space of the original problem is reduced to eigenproblems in the 2D section of the pipe. The set of  
10           solutions is described, its structure depends on the type of boundary condition and of the sign of the  
11           total flow of the fluid. This analysis is the cornerstone of numerical methods to solve Graetz problem  
12           in finite pipes, semi infinite pipes and exchangers of arbitrary cross section. Numerical test-cases  
13           illustrate the capabilities of these methods to provide solutions in various configurations.

14           **1. Introduction.**

15           **1.1. Context.** The seminal work of Graetz in the late 19th century addressed a  
16           stationnary convection-diffusion problem inside an axi-symmetrical cylindrical pipe  
17           [5], where the regime was supposed to be convection-dominated which means that  
18           the longitudinal diffusion was neglected. It was the first contribution to the mod-  
19           elling of convective transport coupled with diffusion, with important applications  
20           nowadays as the parallel convective exchangers involved in heating or cooling systems  
21           [16], haemodialysis [1], and heat exchangers [7]. The first extension to the Graetz  
22           problem, known as the “extended Graetz problem” takes into account longitudinal  
23           diffusion [10, 3, 18, 9]. Papoutsakis *et al.* in [12, 11] introduced a symmetric oper-  
24           ator acting on a two-components space that solves the extended Graetz problem  
25           in axi-symmetrical configurations. The so-called “conjugated Graetz problem” where  
26           multiple solid or fluid phases are taken into account was proposed in [13, 14] in the case  
27           of an axi-symmetrical configuration. These successive models aimed at taking into ac-  
28           count more and more complex and realistic situations, and when only axi-symmetrical  
29           configurations were considered the equations boiled down to one-dimensional prob-  
30           lems. The adaptation to parallel plates heat exchangers of these one-dimensional  
31           models, together with a parametric study was proposed in [6]. The reader may also  
32           consult [2] for a review on the conjugated Graetz problem.

33           The work on non-axisymmetrical configurations was initiated in [15] where the  
34           operator was proved to be self-adjoint with compact resolvent when Dirichlet bound-  
35           ary conditions are applied on the boundary of the domain. In the case of a single fluid  
36           stream the negative eigenvalues correspond to downstream propagation, and positive  
37           eigenvalues to upstream propagation. The main novelty was that arbitrary geometries  
38           were adressed, and a detailed mathematical analysis of the Dirichlet problem was pro-  
39           posed. **The authors of [15] coined this problem as the “generalized Graetz problem”.**  
40           Numerical methods for the approximation of this operator and error estimates were  
41           provided in [4].

42           The objective of the present work is to extend the work of [4] and provide explicit  
43           methods with general lateral boundary conditions, beyond the Dirichlet case. The  
44           cross section of the domain has an arbitrary geometry and can incorporate different

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45 fluid domains, possibly with opposite signs of the velocity. The lateral boundary  
 46 conditions that we address can be Dirichlet, Neumann, Robin, periodic or a mixture  
 47 of these different cases on different parts of the boundary. The periodic boundary  
 48 conditions with rectangular or hexagonal cell are adapted to the analysis of micro-  
 49 exchangers, where a design pattern is repeated.

**1.2. Setting.** In convection-dominated heat or mass transfer, we address the generalized Graetz problem which occurs in a cylinder of arbitrary section  $\Omega$  and of length  $I$ , possibly  $I = \mathbb{R}^+$ , see Figure 1.1. The diffusion coefficient is supposed to be invariant by translation along  $e_z$  the axis of the cylinder. Similarly, the velocity vector  $v$  is supposed to be oriented in the direction of the axis of the cylinder, that is  $v = he_z$  with  $h \in L^\infty(\Omega)$ . The equation for the temperature  $T$  inside the domain is then

$$c\partial_{zz}T + \operatorname{div}(\sigma\nabla T) - h\partial_zT = 0 \text{ on } \Omega \times I, \quad (\text{E})$$

50 with diffusion coefficients  $c, \sigma > 0$  bounded in  $\Omega$  with bounded inverse. In (E), it is  
 51 implicitly supposed that the heat capacity  $C$  and the density  $\rho$  of the fluid satisfy  
 52  $\rho C = 1$ . If one has to handle several fluids with different physical properties, the  
 53 choice of an adequate normalization leads to (E). In this case, at each point  $x \in \Omega$ ,  
 54  $h(x)$  represents the velocity multiplied by  $\rho(x)C(x)$ . For simplicity reasons, we refer  
 55 in the sequel to  $h$  as the *velocity*.

The lateral homogeneous boundary conditions (LBC) may be of Neumann, Dirichlet, Robin and periodic type, respectively on  $\Gamma_N, \Gamma_D, \Gamma_R, \Gamma_\# \subset \partial\Omega$  given by

$$\begin{cases} \sigma\nabla T \cdot n = 0 \text{ on } \Gamma_N \times I: \text{ Neumann, and/or} \\ T = 0 \text{ on } \Gamma_D \times I: \text{ Dirichlet, and/or} \\ \sigma\nabla T \cdot n + aT = 0 \text{ on } \Gamma_R \times I: \text{ Robin, and/or} \\ T \text{ is periodic on } \Gamma_\# \times I: \text{ periodic} \end{cases} \quad (\text{LBC})$$

where  $a > 0$  in the Robin condition, and  $\Gamma_\#$  must be tailored to support periodic conditions (e.g.  $\Omega$  is the unit square,  $\Gamma_\# = (\{x = 0\} \cup \{x = 1\}) \cap \partial\Omega$  and the boundary condition is  $T(0, y) = T(1, y)$ ). As usual, the  $\Gamma$ 's involved in the definition of the boundary condition must form a partition of  $\partial\Omega$ . Note that the Neumann (resp. Dirichlet) boundary conditions are degenerate cases of the Robin condition corresponding to  $a = 0$  (resp.  $a = +\infty$ ). The Inlet/Outlet boundary condition (I/OBC) is of Dirichlet and/or of Neumann type and is given by

$$T = T_D \text{ on } \Omega_D \text{ and } \partial_z T = S_N \text{ on } \Omega_N \text{ with } \Omega_D \cup \Omega_N = \Omega \times \partial I. \quad (\text{I/OBC})$$

56 In the case  $I = \mathbb{R}^+$ , we intentionally stay vague about the definition of  $\partial I$ , it is one  
 57 of the results of this work to determine whether an (I/OBC) is needed on  $z = +\infty$ .

58 A more realistic model in regimes of high velocities takes into account a viscosity  
 59 term, see e.g. [8] where a study in a microchannel including viscous effects and longi-  
 60 tudinal conduction is performed. Our approach can also account for viscosity, the  
 61 details are presented in Section 2.3.

62 **1.3. Lax-Milgram.** Note that the equation (E) is an elliptic equation with an  
 63 additionnal convective term. It is possible to use Lax-Milgram's theorem [17] under  
 64 the hypothesis that the Inlet/Outlet boundary condition is Dirichlet in the region  
 65 where the flow is incoming. More precisely:

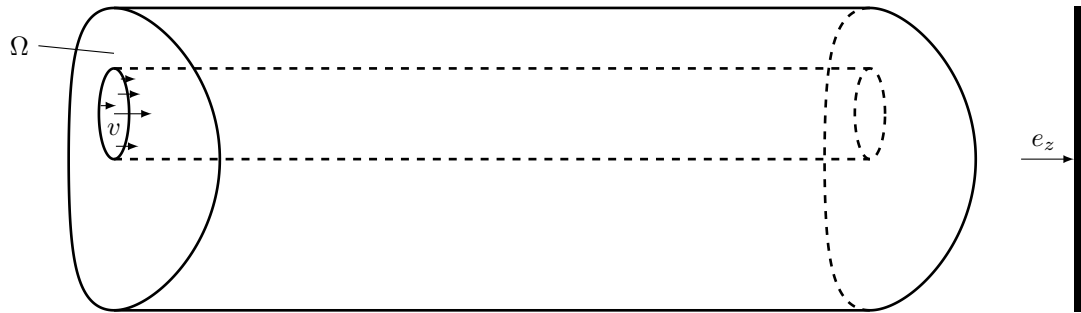


FIGURE 1.1. The domain  $\Omega \times I$  where the Graetz problem is posed.

PROPOSITION 1. Let  $I = [z_1, z_2]$  and  $\omega_{\pm} = \{x \text{ s.t. } \pm h(x) > 0\}$ . If

$$\omega_+ \times \{z_1\} \subset \Omega_D \text{ and } \omega_- \times \{z_2\} \subset \Omega_D$$

66 and if  $T_D$  and  $S_N$  are regular enough, then there exists a unique solution to (E) with  
 67 the boundary conditions (LBC) and (I/OBC).

The proof is only sketched here for the sake of completeness. Denote  $\mathcal{X}$  the natural space of elements where the solution is sought, that is

$$\mathcal{X} = \{T \in H^1(\Omega \times I) \text{ s.t. } T = 0 \text{ on } (\Gamma_D \times I) \cup \Omega_D \text{ and } T \text{ periodic on } \Gamma_{\#} \times I\}.$$

Non-homogeneous Dirichlet boundary conditions of (I/OBC) are solved using a lift of  $T_D$ , still denoted  $T_D$  that satisfies the lateral boundary conditions (LBC) with  $\partial_z T_D = 0$  on  $\Omega_N$  and denote

$$f_D = c\partial_{zz}T_D + \text{div}(\sigma\nabla T_D) - h\partial_z T_D.$$

The change of unknown  $\tilde{T} = T - T_D$  where  $T$  solves (E) and (LBC), leads to the following variational formulation: find  $\tilde{T} \in \mathcal{X}$  such that for every  $\phi \in \mathcal{X}$ :

$$\underbrace{\int_{\Omega \times I} c\partial_z \tilde{T} \partial_z \phi + \sigma \nabla \tilde{T} \cdot \nabla \phi + h\partial_z \tilde{T} \phi + \int_{\Gamma_R} a\tilde{T} \phi}_{b(\tilde{T}, \phi)} + \underbrace{\int_{\Omega_N} S_N \phi - \int_{\Omega \times I} f_D \phi}_{\ell(\phi)} = 0.$$

The term  $b(T, \phi)$  is bilinear in  $(T, \phi)$  and continuous for the standard norm of  $\mathcal{X}$ , the term  $\ell(\phi)$  is linear continuous if  $T_D$  and  $S_N$  are regular enough. It remains to study the coercivity of  $b$ .

$$b(T, T) = \int_{\Omega \times I} c\partial_z T \cdot \partial_z T + \sigma \nabla T \cdot \nabla T + h\partial_z T \cdot T = \int_{\Omega \times I} \|\nabla_{3D} T\|_{\kappa}^2 + \frac{1}{2} \int_{\Omega \times I} h\partial_z(T^2),$$

where  $\kappa$  is a positive matrix with diagonal entries  $(\sigma, \sigma, c)$  in the basis  $(e_x, e_y, e_z)$ . The first term is coercive. The second term is

$$\frac{1}{2} \int_{\Omega} hT^2 \Big|_{z=z_1}^{z=z_2} = \frac{1}{2} \int_{\Omega \times \{z_2\}} hT^2 - \frac{1}{2} \int_{\Omega \times \{z_1\}} hT^2.$$

68 It is nonnegative for all  $T \in \mathcal{X}$  if and only if the Inlet/Outlet condition is of Dirichlet  
 69 type at the boundary where the flow is entering the domain ( $z = z_1$  if  $h > 0$ , and  
 70  $z = z_2$  if  $h < 0$ ).

71 **1.4. Presentation of the paper.** The objective of the present paper is to  
72 provide a general framework that allows to solve (E) with any type of boundary  
73 condition beyond the case where Lax-Milgram's theorem can be used. Section 2  
74 details the notation and the main properties of the operator involved in the solution,  
75 as well as the modifications required to take into account a viscosity term. The main  
76 results, namely Theorems 4 and 5, are detailed in Section 3, their proof is postponed  
77 to the Appendix. In Section 4 we solve the problem in a semi-infinite domain and  
78 show that depending on the case the temperature at infinity  $T_\infty$  can either be a  
79 free parameter of the problem or be imposed by the other condition. In Section 5 we  
80 address the case of a domain of finite length, and numerical strategies are detailed in the  
81 different cases depending on the lateral boundary condition and on the Inlet/Outlet  
82 condition. Test cases are presented in Section 6.

**2. State of the art and position of the problem.** The equation (E) may be interpreted as an evolution equation in the variable  $z$  if it is cast into

$$\partial_z \begin{pmatrix} \partial_z T \\ T \end{pmatrix} = \mathcal{A} \begin{pmatrix} \partial_z T \\ T \end{pmatrix} \text{ on } \Omega \times I, \text{ with } \mathcal{A} \begin{pmatrix} u \\ s \end{pmatrix} = \begin{pmatrix} hc^{-1}u - c^{-1} \operatorname{div} \sigma \nabla s \\ u \end{pmatrix}. \quad (2.1)$$

83 The goal of this section is to guide the reader to the analysis of (2.1) that was proposed  
84 in [4], to enlarge the frame to Neumann and periodic lateral boundary condition, and  
85 to define the notation and state the results that will be used in the sequel. Since  $\mathcal{A}$   
86 is a symmetric operator with a compact resolvent, classical eigendecomposition leads  
87 to an explicit representation of the solution of (2.1) in the basis of eigenvectors (see  
88 e.g. [17]).

89 **DEFINITION 2.** *We say that “the constants are not controlled” when  $\Gamma_D \cup \Gamma_R = \emptyset$ ,*  
90 *in other words when there is no Dirichlet or Robin condition on the lateral part of*  
91 *the boundary of the domain. The case where the constants are not controlled and in*  
92 *addition  $\int_\Omega h = 0$  is called the “balanced case”.*

93 **From the engineering point of view, the balanced case is a special instance of**  
94 **counter-current configuration, where the integral of the velocities in the 2 directions**  
95 **have the same magnitude and the boundary of the domain  $\Omega$  is perfectly insulating or**  
96 **periodic.** As we prove in this section, the case where the constants are not controlled  
97 is a case where the constants are a solution of (E) and the balanced case is a case  
98 where  $\mathcal{A}$  admits a non-trivial kernel.

99 **2.1. Study of the operator  $\mathcal{A}$ .** In this section we detail the Hilbert space, the  
100 scalar product, the kernel, range and pseudo-inverse of the symmetric operator  $\mathcal{A}$ .

**Hilbert space and scalar product.** First, introduce the space  $H$  that encodes the lateral boundary condition. When the constants are controlled define:

$$H = \{s \in H^1(\Omega), \text{ such that } s = 0 \text{ on } \Gamma_D \text{ and } s \text{ periodic on } \Gamma_\sharp\}.$$

If there is no Dirichlet or Robin boundary condition, hence no control on the constants, quotient by the constants and define:

$$H = \{s \in H^1(\Omega)/\mathbb{R}, \text{ such that } s \text{ periodic on } \Gamma_\sharp\}.$$

Then, define the Hilbert space  $\mathcal{H}$  as

$$\mathcal{H} = \{(u, s) \mid u \in L^2(\Omega), s \in H\}$$

which is endowed with the scalar product:

$$((u, s) | (u', s'))_{\mathcal{H}} = \int_{\Omega} cuu' + \sigma \nabla s \cdot \nabla s' + \int_{\Gamma_R} ass'.$$

101 The crucial step in showing that  $\mathcal{H}$  is a Hilbert space is to show that the scalar product  
 102 is definite. Setting  $((u, s) | (u, s))_{\mathcal{H}} = 0$  immediatly gives  $u = 0$  and  $\nabla s = 0$ , hence  $s$   
 103 is a constant. If the constants are controlled, then  $\Gamma_D \cup \Gamma_R \neq \emptyset$  and  $s = 0$ , whereas  
 104 if the constants are not controlled then  $s$  is a constant and  $s = 0$  in  $H$ .

The domain of the operator  $\mathcal{A}$  is:

$$\mathcal{D}(\mathcal{A}) = \{(u, s) \in \mathcal{H}, u \in H^1(\Omega), \operatorname{div}(\sigma \nabla s) \in L^2(\Omega) + \text{boundary conditions (LBC)}\},$$

105 where the boundary conditions are  $u \in H$ , and  $\sigma \nabla s \cdot n$  is equal to 0 on  $\Gamma_N$ , is equal  
 106 to  $-as$  on  $\Gamma_R$  and is periodic on  $\Gamma_{\sharp}$ . On  $\mathcal{D}(\mathcal{A})$ , the operator is symmetric as we prove  
 107 now. Let  $\phi = (u, s)$  and  $\phi' = (u', s') \in \mathcal{D}(\mathcal{A})$ :

$$\begin{aligned} (\mathcal{A}\phi | \phi')_{\mathcal{H}} &= \int_{\Omega} (hu - \operatorname{div} \sigma \nabla s)u' + \sigma \nabla u \cdot \nabla s' + \int_{\Gamma_R} aus' \\ &= \int_{\Omega} hu u' + \sigma \nabla u \cdot \nabla s' + \underbrace{\sigma \nabla u' \cdot \nabla s + \int_{\partial \Omega} (-\sigma \nabla s \cdot n)u' + \int_{\Gamma_R} aus'}_{(1)}, \end{aligned}$$

108 and the term (1) is symmetric thanks to (LBC) on  $\mathcal{D}(\mathcal{A})$ .

**Inverse of the Laplacian** Define the inverse of the Laplace operator as:

$$u = \Delta_{\sigma}^{-1} f \text{ iff } \begin{cases} \operatorname{div}(\sigma \nabla u) = f, \text{ and} \\ u \in H \\ + \text{boundary conditions,} \end{cases}$$

109 where the boundary conditions are  $\sigma \nabla u \cdot n = 0$  on  $\Gamma_N$  and  $\sigma \nabla u \cdot n + au = 0$  on  $\Gamma_R$ .  
 110 If the constants are controlled, then  $\Delta_{\sigma}^{-1}$  is well defined on  $L^2(\Omega)$ , whereas if there  
 111 is only Neumann or periodic boundary conditions (no control of the constants), the  
 112 operator  $\Delta_{\sigma}^{-1}$  is only defined if  $f \in L_m^2(\Omega)$ , the subspace of  $L^2(\Omega)$  with null average.

**Kernel of  $\mathcal{A}$**  Following from the definition of  $\mathcal{A}$  in (2.1), the kernel of  $\mathcal{A}$  is the set of  $(u, s)$  in  $\mathcal{D}(\mathcal{A})$  such that

$$u = 0 \text{ in } H \text{ and } hu - \operatorname{div}(\sigma \nabla s) = 0.$$

When the constants are controlled, both  $u$  and  $s$  are then equal to 0. When the constants are not controlled, since  $u$  is a constant, then  $s = u \Delta_{\sigma}^{-1} h$  in  $\Omega$  which admits a solution if and only if  $\int_{\Omega} h = 0$ . To summarize the kernel of  $\mathcal{A}$  is:

$$\mathcal{K}(\mathcal{A}) = \begin{cases} \operatorname{Vect}(\phi_0 = (1, \Delta_{\sigma}^{-1} h)) \text{ in the balanced case ,} \\ \{0\} \text{ in the other cases} \end{cases}$$

**Range and inverse of  $\mathcal{A}$**  The range of  $\mathcal{A}$ , denoted  $\mathcal{R}(\mathcal{A})$ , is defined as the orthogonal of  $\mathcal{K}(\mathcal{A})$  in  $\mathcal{H}$  and the inverse of  $\mathcal{A}$  is an operator from  $\mathcal{R}(\mathcal{A})$  to  $\mathcal{D}(\mathcal{A})$ , defined as follows:

$$\forall \phi = (u, s) \in \mathcal{R}(\mathcal{A}), \mathcal{A}^{-1} \phi = \begin{cases} (s, \Delta_{\sigma}^{-1}(hs - cu)) \text{ if the constants are controlled} \\ (s + k, \Delta_{\sigma}^{-1}(hs - cu + hk)), k \in \mathbb{R} \text{ if not.} \end{cases} .$$

When the constants are not controlled, the constant  $k \in \mathbb{R}$  is chosen so that

$$\begin{cases} \int_{\Omega} hs - cu + hk = 0 & \text{in the non-balanced case,} \\ (\mathcal{A}^{-1}\phi, \phi_0)_{\mathcal{H}} = 0 & \text{in the balanced case.} \end{cases}$$

It is easily checked that for all  $\phi \in \mathcal{R}(\mathcal{A})$ ,  $\mathcal{A}^{-1}\phi \in \mathcal{D}(\mathcal{A})$  and that  $\mathcal{A}\mathcal{A}^{-1}\phi = \phi$ . The operator  $\mathcal{A}^{-1}$  is then symmetric (as a consequence of the symmetry of  $\mathcal{A}$ ). Note also that in the balanced case, one can also write:

$$\forall \phi = (u, s) \in \mathcal{R}(\mathcal{A}), \quad \mathcal{A}^{-1}\phi = (s, \Delta_{\sigma}^{-1}(hs - cu)) + k\phi_0.$$

**Eigenvalue decomposition of  $\mathcal{A}$**  The operator  $\mathcal{A}^{-1}$  is a compact self-adjoint operator on  $\mathcal{R}(\mathcal{A})$ . To prove this let  $\phi_n = (u_n, s_n)$  be a bounded sequence in  $\mathcal{H}$ . Then up to a subsequence it is a weakly convergent sequence and  $s_n$  converges strongly in  $L^2(\Omega)$ . Using the fact that  $\Delta_{\sigma}^{-1}$  is a compact operator from  $L^2$  to  $H$  finishes the proof. We denote by  $\lambda_i$  the non-zero ordered eigenvalues of  $\mathcal{A}$  and by  $\phi_i = (U_i, \lambda_i^{-1}U_i)$  the corresponding eigenvectors. By convention,  $\lambda_i$  is of the sign of  $i$  so that

$$-\infty \leftarrow \lambda_{-n} \leq \lambda_{-n-1} \leq \dots \leq \lambda_{-1} < 0 < \lambda_1 \leq \dots \leq \lambda_{n-1} \leq \lambda_n \rightarrow +\infty.$$

113 In the balanced case, we add to the family  $(\phi_i)_i$  the vector  $\phi_0 = (1, \Delta_{\sigma}^{-1}h)$ , so  
114 that the Hilbert space  $\mathcal{H}$  is the space spanned by the eigenvectors  $(\phi_i)_{i \in \mathbb{Z}}$ .

**2.2. Solution of the evolution equation.** The diagonalization of the operator  $\mathcal{A}$  allows to solve the evolution equation (E):

$$c\partial_{zz}T + \operatorname{div}(\sigma\nabla T) - h\partial_z T = 0 \text{ on } \Omega \times I. \quad (\text{E})$$

Let  $T \in C^1(I, L^2(\Omega)) \cap C^0(I, H)$  be a solution of this equation with corresponding lateral boundary conditions (LBC). If we denote  $\phi : z \mapsto (\partial_z T(z), T(z))$  in  $C^0(\mathcal{H})$ , then the equation (E) is equivalent to  $\partial_z \phi = \mathcal{A}\phi$ , and the solution  $\phi$  is given by

$$\phi(z) = \sum_{i \in \mathbb{Z}} \frac{(\phi(0)|\phi_i)_{\mathcal{H}}}{\|\phi_i\|_{\mathcal{H}}^2} e^{\lambda_i z} \phi_i. \quad (2.2)$$

One can either identify the first coordinate and integrate w.r.t.  $z$  or identify the second coordinate and denote

$$\psi = \sum_{i \in \mathbb{Z}^*} \lambda_i^{-1} \frac{(\phi(0)|\phi_i)_{\mathcal{H}}}{\|\phi_i\|_{\mathcal{H}}^2} \phi_i,$$

115 to obtain

$$T(z) = \sum_{i \in \mathbb{Z}^*} (\psi|\phi_i)_{\mathcal{H}} U_i e^{\lambda_i z} \text{ if the constants are controlled, i.e. } \Gamma_D \cup \Gamma_R \neq \emptyset,$$

$$T(z) = \sum_{i \in \mathbb{Z}^*} (\psi|\phi_i)_{\mathcal{H}} U_i e^{\lambda_i z} + a_0 \text{ with } a_0 \in \mathbb{R} \text{ if } \Gamma_D \cup \Gamma_R = \emptyset \text{ and } \int_{\Omega} h \neq 0,$$

$$T(z) = \sum_{i \in \mathbb{Z}^*} (\psi|\phi_i)_{\mathcal{H}} U_i e^{\lambda_i z} + a_0 + a_1(z + \Delta_{\sigma}^{-1}h) \text{ in the balanced case}$$

$$\text{with } a_0 \in \mathbb{R} \text{ and } a_1 = \frac{(\phi(0)|\phi_0)_{\mathcal{H}}}{\|\phi_0\|^2}.$$

116 If  $\partial_z T$  and  $T$  are given at  $z = 0$  such that  $\phi(0) = (\partial_z T(0), T(0))$  belongs to  $\mathcal{H}$ ,  
 117 then  $\psi$  is uniquely determined. Moreover the constant  $a_0$  is also determined by  $T|_{z=0}$   
 118 (and also  $a_1$  in the balanced case). We stress that this solution may not be defined  
 119 everywhere, indeed the series on the right-hand side of (2.2) has to be convergent in  
 120 some sense and the convergence of the series for  $z = 0$  is not sufficient to ensure the  
 121 convergence for  $z \neq 0$  due to the multiplication by  $e^{\lambda_i z}$  for non-zero  $\lambda_i$ 's. The set of  
 122 initial datum  $\phi$  that allows this series to exist is known as the set of compatible initial  
 123 condition for the Cauchy problem.

**2.3. Including a viscous term.** Let us consider the following modification of  
 the equation (E) where a viscous term is added:

$$c\partial_{zz}T + \operatorname{div}(\sigma\nabla T) - h\partial_z T = \mu|\nabla h|^2. \quad (2.3)$$

124 **PROPOSITION 3.** *Let  $T$  be the solution of the Graetz equation with viscosity (2.3).  
 125 Then there exists an explicit change of unknown function that transforms the problem  
 126 with viscosity into a problem without viscosity of the form (E). Therefore the solution  
 127 of the problem (2.3) reduces to the solution of the original problem (E).*

128 *Proof.* Once a particular solution  $\tilde{T}$  is found, the change of variable  $\widehat{T} = T - \tilde{T}$   
 129 transforms by linearity the problem with viscosity (2.3) into the problem without  
 130 viscosity. We distinguish different cases, depending on if the constants are controlled  
 131 or not, and in the case the constants are not controlled we treat separately the non-  
 132 balanced and the balanced case. In each case we provide an explicit particular solution  
 133  $\tilde{T}$ .

a) If the constants are controlled, a particular solution is given by

$$\tilde{T} = \Delta_\sigma^{-1}(\mu|\nabla h|^2).$$

b) If the constants are not controlled, in the non-balanced case

$$\tilde{T} = \alpha z + \Delta_\sigma^{-1}(\mu|\nabla h|^2 + \alpha h),$$

where  $\alpha \in \mathbb{R}$  satisfies

$$\int_\Omega (\mu|\nabla h|^2 + \alpha h) = 0.$$

c) If the constants are not controlled, in the balanced case, the particular solu-  
 tion is given by  $\tilde{T} = \alpha(\frac{z^2}{2} + z\Delta_\sigma^{-1}h) + \Delta_\sigma^{-1}\gamma$ , with  $\alpha \in \mathbb{R}$  and  $\gamma \in L^2(\Omega)$  such  
 that:

$$\begin{cases} \alpha \left( \int_\Omega c - h\Delta_\sigma^{-1}h \right) = \int_\Omega \mu|\nabla h|^2 \\ \gamma = \mu|\nabla h|^2 - \alpha(c - h\Delta_\sigma^{-1}h) \end{cases} \quad (2.4)$$

The choice of  $\alpha$  ensures that  $\gamma$  has zero average so that  $\Delta_\sigma^{-1}\gamma$  is well defined.  
 Note that  $\alpha$  is well defined since

$$\int_\Omega c - h\Delta_\sigma^{-1}h = \int_\Omega c + \int_\Omega \sigma|\nabla h|^2 > 0.$$

134 Note that the last term is equal to  $\|\phi_0\|_{\mathcal{H}}^2$ .  $\square$



135 **3. Main decomposition theorem.** In this section, the decomposition of a  
 136 temperature field on the non-positive eigenspace is studied. The result stated in  
 137 Theorem 5 considers different cases depending on the control of constants and the  
 138 sign of the total flow.

**3.1. Notation and statement of the problem.** The  $D(\mathcal{A}^\alpha)$  norm or “ $\alpha$ -norm” in short is defined by

$$\|\phi\|_\alpha^2 = \sum_{i \in \mathbb{Z}} \lambda_i^{2\alpha} \frac{(\phi_i|\phi)_{\mathcal{H}}^2}{\|\phi_i\|_{\mathcal{H}}^2} \quad \forall \phi \in \mathcal{H}.$$

The space  $D(\mathcal{A}^\alpha)$  is the set of  $\phi \in \mathcal{R}(\mathcal{A})$  whose  $\alpha$ -norm is  $< +\infty$ . It is easy to check that  $D(\mathcal{A}^1) = D(\mathcal{A})$  and that  $D(\mathcal{A}^0) = \mathcal{R}(\mathcal{A})$ . Define  $P$  an orthogonal projection on  $\mathcal{H}$  as:

$$\forall \phi = (u, s) \in \mathcal{H}, \quad P\phi = (u, 0). \quad (3.1)$$

For any  $I$  subset of  $\mathbb{Z}$  define  $\pi_I$  the orthogonal projection

$$\pi_I \phi = \sum_{i \in I} \frac{(\phi_i|\phi)_{\mathcal{H}}}{\|\phi_i\|_{\mathcal{H}}^2} \phi_i \quad (3.2)$$

139 We denote  $\pi_+ = \pi_{\mathbb{N}^*}$ ,  $\pi_- = \pi_{-\mathbb{N}^*}$ ,  $\pi_0 = \pi_{\{0\}}$  and  $\mathcal{R}(\pi_I) = \pi_I(\mathcal{H})$ .

140 The problem of decomposition of a temperature field on the non-positive eigenspace  
 141 is stated as follows:

*For any  $\phi \in \mathcal{H}$ , find  $\psi$  such that*

$$P\psi = P\phi \text{ and } \pi_+\psi = 0. \quad (3.3)$$

142 A similar problem of decomposition on the non-negative eigenspace is obtained by  
 143 replacing  $\pi_+$  by  $\pi_-$ . All the results of the present section have a counterpart obtained  
 144 by changing the sign of  $z$ .

**3.2. Necessary and sufficient condition.** In order to tackle problem (3.3), we first consider the following related problem:

$$\text{Find } \psi \in \mathcal{R}(\pi_-) \text{ such that } \pi_- P \pi_- \psi = \pi_- P \phi. \quad (3.4)$$

145 Indeed if  $\psi$  solves (3.3), then multiplying the equation by  $\pi_-$  and assuming that the  
 146 kernel of  $\mathcal{A}$  is reduced to the nullspace (which is true except in the balanced case),  
 147 one derives equation (3.4). Such a problem admits a unique solution, given by the  
 148 following theorem:

**THEOREM 4.** *The operator  $\pi_- P \pi_-$  is invertible on  $\mathcal{R}(\pi_-)$ . Define  $B_-$  the self-adjoint operator of  $\mathcal{H}$  as*

$$B_- \phi = \pi_- (\pi_- P \pi_-)^{-1} \pi_- \phi.$$

*Moreover it holds*

$$\|B_- \phi\|_{\mathcal{H}} \leq C \|\pi_- \phi\|_{\mathcal{H}} \quad \text{and} \quad \|B_- \phi\|_{1/2} \leq C \|\pi_- \phi\|_{1/2}.$$

149 *One can similarly define an operator  $B_+$ , obtained by replacing  $\pi_-$  by  $\pi_+$ .*

150 The result is proved in [4] for the full-Dirichlet case, that is  $\Gamma_D = \partial\Omega$ . The proof  
 151 can be adapted without major changes to the case in consideration. It is reproduced

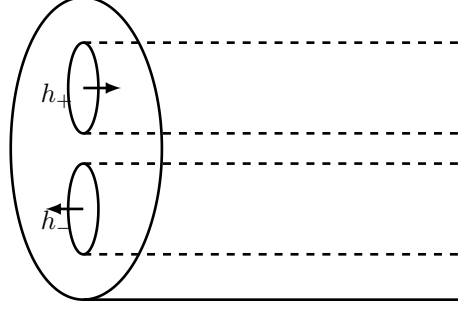


FIGURE 4.1. Example of a semi-infinite cylinder, with two fluid domains.

152 in Appendix B for the convenience of the reader. Problem (3.3) is then solved in the  
 153 next theorem.

THEOREM 5. Let  $\phi \in \mathcal{H}$ , define  $\Phi = (1, 0) \in \mathcal{H}$  and consider problem (3.3) of finding  $\psi$  a solution of

$$P\psi = P\phi \text{ and } \pi_+\psi = 0. \quad (3.3)$$

- If the constants are not controlled and  $\int_{\Omega} h > 0$ , there exists a solution if and only if

$$(\Phi - PB_-\Phi|\phi)_{\mathcal{H}} = 0.$$

154 In this case, the solution is unique and given by  $\psi = B_-P\phi$ .

- If the constants are not controlled and  $\int_{\Omega} h = 0$  (balanced case), then  $(\Phi - PB_-\Phi|\Phi)_{\mathcal{H}} \neq 0$  and there exists a unique solution given by

$$\psi = B_-P\phi + \frac{(\Phi - PB_-\Phi|\phi)_{\mathcal{H}}}{(\Phi - PB_-\Phi|\Phi)_{\mathcal{H}}} (B_-\Phi - \phi_0).$$

- In every other case then  $\psi = B_-P\phi$  is the unique solution.

156 The proof of this result is given in Appendix C.

**4. Resolution of the semi-infinite problem.** In the semi-infinite problem, the equation is set on the cylinder  $\Omega \times \mathbb{R}^+$ , see Figure 4.1. The equation (E) becomes:

$$\partial_{zz}T + \Delta T + h\partial_zT = 0 \text{ on } \Omega \times \mathbb{R}^+.$$

157 In this section we address different cases depending on the type of the Inlet/Outlet  
 158 condition, namely either Dirichlet or Neumann.

159 In order to ensure uniqueness of the solution, we add the extra hypothesis that  
 160 the temperature does not grow exponentially. We will say that the temperature has  
 161 subexponential growth if and only if for every  $\lambda > 0$ , then  $T(z) = o(e^{\lambda z})$  as  $z$  goes to  
 162  $+\infty$ .

**4.1. Semi-infinite problem, Dirichlet Inlet/Outlet Condition.** We consider the Dirichlet (I/OBC) condition:

$$T|_{z=0} = T_0 \text{ on } \Omega. \quad (4.1)$$

163 Denote  $\phi_{\mathcal{D}} = (T_0, 0) \in \mathcal{H}$ .

164 PROPOSITION 6. Consider the Graetz problem (E) on the semi-infinite cylinder  
 165  $\Omega \times [0, +\infty)$ , with subexponential growth together with Dirichlet Inlet/Outlet condition  
 166 (4.1).

a) If the constants are controlled, then there exists a unique solution given by

$$T = \sum_{i < 0} e^{\lambda_i z} (B_- \phi_{\mathcal{D}} | \phi_i)_{\mathcal{H}} U_i.$$

167

In this case the temperature at infinity is 0.

b) If the constants are not controlled and  $\int_{\Omega} h \neq 0$ , then there exists a unique solution given by

$$T = \sum_{i < 0} e^{\lambda_i z} (B_- (\phi_{\mathcal{D}} - T_{\infty} \Phi) | \phi_i)_{\mathcal{H}} U_i + T_{\infty},$$

168

where  $T_{\infty}$  is an arbitrary constant in the case  $\int_{\Omega} h < 0$  and  $T_{\infty} = (\Phi - PB_- \Phi | \phi_{\mathcal{D}})_{\mathcal{H}} (\Phi - PB_- \Phi | \Phi)_{\mathcal{H}}^{-1}$  in the case  $\int_{\Omega} h > 0$ . In this case the temperature at infinity is the constant  $T_{\infty}$ . Note that if  $\int_{\Omega} h > 0$ , the temperature at infinity is determined by  $\phi_{\mathcal{D}}$  whereas in the case  $\int_{\Omega} h < 0$ , it is a free parameter of the problem.

169

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c) In the balanced case, the set of solutions is given by:

$$T(z) = \sum_{i < 0} (B_- (\phi_{\mathcal{D}} - c_1 \Phi - c_2 (\Delta_{\sigma}^{-1} h, 0)) | \phi_i)_{\mathcal{H}} U_i e^{\lambda_i z} + c_1 + c_2 (z + \Delta_{\sigma}^{-1} h),$$

where  $c_2$  is an arbitrary constant and

$$c_1 = (\Phi - PB_- \Phi | \phi_{\mathcal{D}} - c_2 (\Delta_{\sigma}^{-1} h, 0))_{\mathcal{H}} (\Phi - PB_- \Phi | \Phi)_{\mathcal{H}}^{-1}.$$

173

In this case the temperature at infinity has the linear growth rate  $T(z) \simeq c_2 z + (c_2 \Delta_{\sigma}^{-1} h + c_1) + o(z)$ . If the temperature is not allowed to have a linear growth rate, then  $c_2 = 0$  and the temperature at infinity is  $c_1$  which is determined by the initial conditions.

174

175

176

*Proof.* We use the result of Section 2.2 on the solution of the evolution equation, that is  $T$  solves (E) if and only if there exists  $\psi \in \mathcal{R}(\mathcal{A})$  and constants  $c_1$  and  $c_2$  such that:

$$T(z) = \sum_{i \in \mathbb{Z}^*} (\psi | \phi_i)_{\mathcal{H}} U_i e^{\lambda_i z} + c_1 + c_2 (z + \Delta_{\sigma}^{-1} h),$$

where  $c_1 = c_2 = 0$  if the constants are controlled and  $c_2 = 0$  in the non-balanced case. The subexponential growth condition ensures that  $\pi_+ \psi = 0$ . The condition  $(T(z=0), 0) = \phi_{\mathcal{D}}$  yields

$$P\phi_{\mathcal{D}} = P\psi + c_1 \Phi + c_2 (\Delta_{\sigma}^{-1} h, 0). \quad (4.2)$$

Using Theorem 5 leads to distinguishing the following cases:

a) If the constants are controlled then  $c_1 = c_2 = 0$  and the equation  $P\phi_{\mathcal{D}} = P\psi$  with  $\pi_+ \psi = 0$  has the unique solution  $\psi = B_- \phi_{\mathcal{D}}$ .

b) In the non-balanced case,  $c_2 = 0$  and  $P(\phi_{\mathcal{D}} - c_1 \Phi) = P\psi$  together with  $\pi_+ \psi = 0$  implies  $\psi = B_- (\phi_{\mathcal{D}} - c_1 \Phi)$  without any additional assumption in the case  $\int_{\Omega} h < 0$ . In the case where  $\int_{\Omega} h > 0$  the compatibility condition is

$$(\Phi - PB_- \Phi | \phi_{\mathcal{D}} - c_1 \Phi) = 0, \text{ which gives } c_1 = (\Phi - PB_- \Phi | \phi_{\mathcal{D}}) (\Phi - PB_- \Phi | \Phi)^{-1}.$$

c) Finally, in the balanced case, let us fix an arbitrary value  $c_2$ . The conditions  $\psi \in \mathcal{R}(\mathcal{A})$  and  $\pi_+\psi = 0$  are equivalent to  $\psi \in \mathcal{R}(\pi_-)$ . Denoting  $\tilde{\Phi} = (\Delta_\sigma^{-1}h, 0)$  we re-cast (4.2) into

$$P\phi_{\mathcal{D}} = P\psi + c_1\tilde{\Phi} + c_2\tilde{\Phi}, \quad \psi \in \mathcal{R}(\pi_-)$$

or equivalently

$$P(\phi_{\mathcal{D}} - c_2\tilde{\Phi}) = P(\psi + c_1\phi_0), \quad \psi \in \mathcal{R}(\pi_-).$$

In view of Theorem 5 with  $\phi = \phi_{\mathcal{D}} - c_2\tilde{\Phi}$ , there is a unique solution to the above equation given by

$$c_1 = (\Phi - PB_- \Phi | \phi_{\mathcal{D}} - c_2\tilde{\Phi})_{\mathcal{H}} (\Phi - PB_- \Phi | \Phi)_{\mathcal{H}}^{-1}, \quad \psi = B_-(\phi_{\mathcal{D}} - c_1\tilde{\Phi} - c_2\tilde{\Phi}).$$

177  $\square$

**4.2. Semi-infinite problem, Neumann Inlet/Outlet Condition.** We consider the Neumann (I/OBC) condition:

$$\partial_z T|_{z=0} = S_0. \tag{4.3}$$

178 Denote  $\phi_{\mathcal{N}} = (S_0, 0) \in \mathcal{H}$ .

179 **PROPOSITION 7.** Consider the Graetz problem (E) on the semi-infinite cylinder  
180  $\Omega \times [0, +\infty)$ , with subexponential growth together with Neumann Inlet/Outlet condition  
181 (4.3).

a) If the constants are controlled, then there exists a unique solution given by

$$T = \sum_{i < 0} e^{\lambda_i z} (\mathcal{A}^{-1} B_- \phi_{\mathcal{N}} | \phi_i)_{\mathcal{H}} U_i.$$

182 In this case the temperature at infinity is 0.

b) If the constants are not controlled and  $\int_{\Omega} h \neq 0$ . If  $\int_{\Omega} h > 0$  there always exists a solution, if  $\int_{\Omega} h < 0$ , there exists a solution if and only if

$$(\Phi - PB_- \Phi | \phi_{\mathcal{N}}) = 0.$$

When the solution exists, it is of the form

$$T = \sum_{i < 0} e^{\lambda_i z} (\mathcal{A}^{-1} B_- (\phi_{\mathcal{N}} - T_{\infty} \Phi) | \phi_i)_{\mathcal{H}} U_i + T_{\infty},$$

183 where the temperature at infinity  $T_{\infty}$  is a free parameter of the problem.

c) In the balanced case, the set of solutions is given by

$$T(z) = \sum_{i < 0} (\mathcal{A}^{-1} B_- (\phi_{\mathcal{N}} + c_2\tilde{\Phi}) | \phi_i)_{\mathcal{H}} U_i e^{\lambda_i z} + c_1 + c_2(z + \Delta_\sigma^{-1}h),$$

where  $c_1$  is an arbitrary constant and  $c_2$  is given by:

$$c_2 = -(\Phi - PB_- \Phi | \phi_{\mathcal{N}}) (\Phi - PB_- \Phi | \Phi)^{-1}.$$

184 In this case the temperature at infinity has the linear growth rate  $T(z) \simeq$   
185  $c_2 z + (c_2 \Delta_\sigma^{-1} h + c_1) + o(z)$ .

*Proof.* We proceed as in the previous section. It follows from the result of Section 2.2 on the solution of the evolution equation that  $T$  solves (E) if and only if there exists  $\psi \in \mathcal{R}(\mathcal{A})$  and constants  $c_1$  and  $c_2$  such that:

$$T(z) = \sum_{i \in \mathbb{Z}^*} (\psi | \phi_i)_{\mathcal{H}} U_i e^{\lambda_i z} + c_1 + c_2(z + \Delta_{\sigma}^{-1} h),$$

where  $c_1 = c_2 = 0$  if the constants are controlled and  $c_2 = 0$  in the non-balanced case. Derivating w.r.t.  $z$  one finds

$$P\phi_{\mathcal{N}} = P\mathcal{A}\psi + c_2\Phi, \quad \pi_+\psi = 0. \quad (4.4)$$

Using Theorem 5 leads to distinguishing the following cases:

a) If the constants are controlled then  $c_1 = c_2 = 0$  and equation (4.4) admits a unique solution  $\mathcal{A}\psi = B_-\phi_{\mathcal{N}}$ . The invertibility of  $\mathcal{A}$  gives the result.

b) If the constants are not controlled then  $c_2 = 0$ . If  $\int_{\Omega} h > 0$  there is always a solution  $\mathcal{A}\psi$  to equation (4.4) and the operator  $\mathcal{A}$  is invertible, hence there exists a unique solution  $\psi$  to (4.4) given by  $\psi = \mathcal{A}^{-1}B_-\phi_{\mathcal{N}}$ . The constant  $c_1$  is then a free parameter of the problem. If  $\int_{\Omega} h < 0$  then the condition for equation (4.4) to admit a solution is:

$$(\Phi - PB_-\Phi | \phi_{\mathcal{N}}) = 0.$$

If this condition is met, by the invertibility of  $\mathcal{A}$ ,  $\psi = \mathcal{A}^{-1}B_-\phi_{\mathcal{N}}$  is the unique solution to (4.4) and  $c_1$  is a free parameter of the problem.

c) In the balanced case let  $c_2$  be an arbitrary constant. It follows from Theorem 5 that  $\mathcal{A}\psi$  satisfies (4.4) if and only if

$$\mathcal{A}\psi = B_-\psi + \frac{(\Phi - PB_-\Phi | \phi)_{\mathcal{H}}}{(\Phi - PB_-\Phi | \Phi)_{\mathcal{H}}} (B_-\Phi - \phi_0)_{\mathcal{H}} \quad \text{where } \phi = \phi_{\mathcal{N}} + c_2\Phi. \quad (4.5)$$

For  $\psi$  to exist, the right hand side must belong to the range of  $\mathcal{A}$ , i.e. be orthogonal to  $\phi_0$ . Performing the scalar product of the left hand side of (4.5) with  $\phi_0$  and recalling that the range of  $B_-$  is orthogonal to  $\mathcal{K}(\mathcal{A})$  we obtain the following necessary condition:

$$(\Phi - PB_-\Phi | \phi_{\mathcal{N}} + c_2\Phi)_{\mathcal{H}} = 0,$$

which is equivalent to:

$$c_2 = -(\Phi - PB_-\Phi | \phi_{\mathcal{N}})(\Phi - PB_-\Phi | \Phi)^{-1}.$$

186 Conversely, if the above condition is met, then equation (4.5) admits a unique inverse  
187 in  $\mathcal{R}(\mathcal{A})$  and  $c_1$  is a free parameter of the problem.  $\square$

**5. Resolution of the problem in a finite domain.** We aim to solve Graetz equation in a domain of finite length  $\Omega \times [-L, L]$ :

$$\begin{cases} c\partial_{zz}T + \text{div } \sigma \nabla T - h\partial_z T = 0 & \Omega \times [-L, L], \\ \text{(LBC)} & (\partial\Omega) \times [-L, L], \\ \text{Inlet/Outlet condition} & \Omega \times \{-L, L\}, \end{cases} \quad (5.1)$$

where the I/O conditions can be of Neumann or Dirichlet type. According to Section 2.2, the solutions may be sought in the form

$$T(z) = \sum_{i < 0} (\psi | \phi_i) e^{\lambda_i(z+L)} U_i + \sum_{i > 0} (\psi | \phi_i) e^{\lambda_i(z-L)} U_i + c_1 + c_2(z + \Delta_{\sigma}^{-1} h),$$

188 with  $c_1 = c_2 = 0$  if the constants are controlled and  $c_2 = 0$  in the non-balanced case.

189 The unknowns in this equation are  $(\psi|\phi_i)$  for  $i < 0$  and  $i > 0$ , plus possibly  
 190 (depending on the case)  $c_1$  and  $c_2$ . Note that  $\sum_{i<0}(\psi|\phi_i)U_i = P\pi_-\psi$ , and therefore  
 191 if  $P\pi_-\psi$  is known it suffices to decompose this vector on the basis of  $L^2(\Omega)$  given by  
 192  $(U_i)_{i<0}$  to obtain the desired coefficients for  $i < 0$ . Similarly the coefficients  $(\psi|\phi_i)$  for  
 193  $i > 0$  are obtained by considering the coefficients of  $P\pi_+\psi$  on the basis composed of  
 194 the  $(U_i)_{i>0}$ . Therefore the unknowns to be determined are  $P\pi_-\psi, P\pi_+\psi$  plus possibly  
 195  $c_1$  and  $c_2$ .

196 Let  $X$  be the vector composed of all the unknowns. Then satisfying the I/O  
 197 conditions amounts to solving a linear system for  $X$ . In the rest of this section we  
 198 detail the linear system in each case, but beforehand we focus on a linear operator  
 199 involved in the system.

200 **5.1. Study of the linear operator  $M$ .** We define and study a linear operator  
 201 that will be involved in the solution of the problem in a cylinder of length  $2L$ .

PROPOSITION 8. *Let  $M_{\pm}$  be the operators from  $\mathcal{R}(P)$  to  $\mathcal{R}(P)$  and  $M$  be given by:*

$$M_{\pm} = Pe^{\mp 2LA}B_{\pm} \quad \text{and} \quad M = \begin{pmatrix} 0 & M_+ \\ M_- & 0 \end{pmatrix}.$$

202 *Then*

a) *there exists a constant  $C$  such that*

$$\|M\| \leq Ce^{-2\lambda L}, \quad \text{where } \lambda = \min(\lambda_1, -\lambda_{-1}).$$

203 *As a consequence  $\|M\| < 1$  for sufficiently large  $L$ .*

204 b) *If the constants are controlled, then for  $L$  positive sufficiently small,  $\|M^2\| < 1$ .*

205 c) *It follows that  $Id + M$  is invertible on  $\mathcal{R}(P) \times \mathcal{R}(P)$  for large  $L$  and for small positive  $L$ .*

207 *Proof.*

208 a) Since  $M_+ = Pe^{-2LA}B_+$  we have

$$\|M_+\| \leq \|B_+\|e^{-2L\lambda_1}.$$

209 A similar upper bound for  $M_-$  gives the result.

b) Define

$$J(L) = \sup_{\|(\phi_1, \phi_2)\|=1} \|M^2(\phi_1, \phi_2)\| < 1$$

Since  $J(0) = 1$ , it is sufficient to prove that  $J'(0) < 0$ . Note that

$$M^2 = \begin{pmatrix} M_+M_- & 0 \\ 0 & M_-M_+ \end{pmatrix}.$$

210 Let us fix  $\phi \in \mathcal{R}(P)$ , and define  $j(L) = \|M_+(L)M_-(L)\phi\|^2$ . Then  $j(0) = \|\phi\|^2$  and it  
 211 remains to prove that  $j'(0) \leq -C\|\phi\|^2$  with a positive constant  $C$  independent of  $\phi$ .

212 The derivative of  $j$  is:

$$\begin{aligned} j'(L) &= (M'_+(L)M_-(L)\phi + M_+(L)M'_-(L)\phi|M_+(L)M_-(L)\phi) \\ &= (-2PAe^{-2LA}B_+Pe^{2LA}B_-\phi + 2Pe^{-2LA}B_+PAe^{2LA}B_-\phi|Pe^{-2LA}B_+Pe^{2LA}B_-\phi) \end{aligned}$$

213 hence

$$j'(0) = -2(PAB_+\phi|\phi) + 2(PAB_-\phi|\phi).$$

214 But since  $P\phi = \phi$ ,  $PB_+P = P$  and  $PA = \mathcal{A} + PAP - \mathcal{A}P$  we have

$$\begin{aligned} (PAB_+\phi|\phi) &= (PAB_+\phi|B_+\phi) \\ &= ((\mathcal{A} + PAP - \mathcal{A}P)B_+\phi|B_+\phi) \\ &= (\mathcal{A}B_+\phi|B_+\phi) + (PAPB_+\phi|B_+\phi) - (\mathcal{A}PB_+\phi|B_+\phi) \\ &= (\mathcal{A}B_+\phi|B_+\phi) + (\mathcal{A}\phi|\phi) - (\mathcal{A}\phi|B_+\phi). \end{aligned}$$

This proves that

$$(PAB_+\phi|\phi) = (\mathcal{A}\phi|B_+\phi) = \frac{1}{2}((\mathcal{A}B_+\phi|B_+\phi) + (\mathcal{A}\phi|\phi)).$$

Similarly we obtain that

$$(PAB_-\phi|\phi) = \frac{1}{2}((\mathcal{A}B_-\phi|B_-\phi) + (\mathcal{A}\phi|\phi)).$$

As a summary we find that

$$j'(0) = -(\mathcal{A}B_+\phi|B_+\phi) + (\mathcal{A}B_-\phi|B_-\phi) < \lambda_{-1}\|B_-\phi\|^2 - \lambda_1\|B_+\phi\|^2 < (\lambda_{-1} - \lambda_1)\|\phi\|^2.$$

215 c) The operator  $Id + M$  is invertible for large  $L$  by a). Note that  $M_{\pm}$  as endomorphism  
 216 of  $\mathcal{R}(P)$  are compact for  $L > 0$  and equal to identity for  $L = 0$ . As a result  $Id + M$   
 217 is invertible for small  $L > 0$  if and only if there is no eigenvector associated to the  
 218 value  $-1$ . A sufficient condition for invertibility is then that  $M^2$  does not admit 1 as  
 219 eigenvalue, which is proved in b) for  $L$  sufficiently small.  $\square$

220 **5.2. The Dirichlet case.** The different cases for Dirichlet I/O condition are  
 221 summarized in the following

PROPOSITION 9. *The Dirichlet I/O condition  $T|_{z=-L} = T_{-L}$  and  $T|_{z=L} = T_{+L}$  are equivalent to the following linear system*

$$ZX = b,$$

222 where  $Z$ ,  $X$  and  $b$  are defined depending of the (LBC) and given in the table below.

	constants controlled	constants not controlled	
		unbalanced ( $\int_{\Omega} h > 0$ )	balanced
$Z$	$\begin{pmatrix} Id & M_+ \\ M_- & Id \end{pmatrix}$	$\begin{pmatrix} Id & M_+ & \Phi \\ M_- & Id & \Phi \\ 0 & M_+^* u_-^T & (\Phi u_-) \end{pmatrix}$	$\begin{pmatrix} Id & M_+ & \Phi & -L\Phi + \tilde{\Phi} \\ M_- & Id & \Phi & L\Phi + \tilde{\Phi} \\ 0 & M_+^* u_-^T & (\Phi u_-) & (-L\Phi + \tilde{\Phi} u_-) \\ M_-^* u_+^T & 0 & (\Phi u_+) & (L\Phi + \tilde{\Phi} u_+) \end{pmatrix}$
$X$	$\begin{pmatrix} P\pi_-\psi \\ P\pi_+\psi \end{pmatrix}$	$\begin{pmatrix} P\pi_-\psi \\ P\pi_+\psi \\ c_1 \end{pmatrix}$	$\begin{pmatrix} P\pi_-\psi \\ P\pi_+\psi \\ c_1 \\ c_2 \end{pmatrix}$
$b$	$\begin{pmatrix} \phi_{-L} \\ \phi_{+L} \end{pmatrix}$	$\begin{pmatrix} \phi_{-L} \\ \phi_{+L} \\ (\phi_{-L} u_-) \end{pmatrix}$	$\begin{pmatrix} \phi_{-L} \\ \phi_{+L} \\ (\phi_{-L} u_-) \\ (\phi_L u_+) \end{pmatrix}$

224 where we recall that  $\tilde{\Phi} = (\Delta_\sigma^{-1}h, 0)$  and we define  $\phi_{\pm L} = (T_\pm, 0)$ ,  $u_- = \Phi - PB_- \Phi$   
 225 and  $u_+ = \Phi - PB_+ \Phi$ .

226 Moreover, for sufficiently large  $L$  this system is invertible.

227 Note 1: Thanks to the Lax-Milgram theorem in  $3D$  (see Section 1.3), we know  
 228 before hand that there exists a unique solution to the system  $ZX = b$ .

229 Note 2: In the case when the constants are not controlled and  $\int_\Omega h < 0$  it suffices  
 230 to change the sign of  $z$ , or equivalently to remplace the  $-$  by  $+$ .

*Proof.* The (I/OBC) are equivalent to the following

$$\begin{cases} P\pi_- \psi = P\theta_- & \text{with } \theta_- = \phi_{-L} - e^{-2LA}\pi_+ \psi - c_1 \Phi - c_2(-L\Phi + \tilde{\Phi}), \\ P\pi_+ \psi = P\theta_+ & \text{with } \theta_+ = \phi_{+L} - e^{2LA}\pi_- \psi - c_1 \Phi - c_2(L\Phi + \tilde{\Phi}). \end{cases} \quad (5.2)$$

Combining

$$Pe^{-2LA}\pi_+ \psi = Pe^{-2LA}B_+ P\pi_+ \psi = M_+ P\pi_+ \psi,$$

231 and the similar version when the roles of  $+$  and  $-$  are interchanged with equations (5.2)  
 232 we obtain the first two rows of the matrix  $Z$ .

233 a) when the constants are controlled,  $c_1 = c_2 = 0$  and (5.2) reads  $ZX = b$ .

b) when the constants are not controlled and  $\int_\Omega h > 0$  then  $c_2 = 0$ . Theorem  
 5 requires an additional compatibility condition to solve the first equation. This  
 condition reads

$$(\Phi - PB_- \Phi | \theta_-) = 0,$$

234 which is the additional equation in the system  $ZX = b$ .

c) In the balanced case, after the change of variable  $\tilde{\psi} = \pi_- \psi$  the first equation  
 in (5.2)  $P\pi_- \psi = P\theta_-$  is equivalent to

$$\begin{cases} P\tilde{\psi} = P\theta_- \\ \pi_+ \tilde{\psi} = 0 \end{cases} \quad \text{and} \quad (\tilde{\psi} | \phi_0) = 0.$$

235 Theorem 5 gives an explicit expression for the solution of the system on the left, and  
 236 the condition on the right becomes  $(\theta_- | \Phi - PB_- \Phi) = 0$ , which is the third row of the  
 237 system  $ZX = b$ . The last row is obtained using the second equation in (5.2).

238 When  $L$  becomes large,  $M_+$  and  $M_-$  are exponentially small and in each case  
 239 the matrix  $Z$  is asymptotic to an invertible matrix. The sole non-obvious case is the  
 240 balanced case, where one can observe that the  $2 \times 2$  lower right block is asymptotically

241 equivalent to  $\begin{pmatrix} (\Phi | u_-) & (-L\Phi | u_-) \\ (\Phi | u_+) & (L\Phi | u_+) \end{pmatrix}$  which has a determinant  $2L(\Phi | u_-)(\Phi | u_+) \neq 0$ .

242 When  $L$  is large,  $Z$  can be rewritten as  $Z = A + B$  with  $B$  small and  $A$  easily inverted.  
 243 One can use a Neumann series strategy to solve  $Zx = b$ .  $\square$

244 **5.3. The Neumann Inlet/Outlet case.** The different cases for Neumann I/O  
 245 condition are summarized in the following

PROPOSITION 10. *The Neumann I/O condition  $\partial_z T|_{z=-L} = S_{-L}$  and  $\partial_z T|_{z=L} = S_{+L}$  are equivalent to the following linear system*

$$ZX = b,$$



246 where  $Z$ ,  $X$  and  $b$  are defined depending of the (LBC) and given in the table below.

	constants controlled	constants not controlled	
		unbalanced ( $\int_{\Omega} h > 0$ )	balanced
247 $Z$	$\begin{pmatrix} Id & M_+ \\ M_- & Id \end{pmatrix}$	$\begin{pmatrix} Id & M_+ \\ M_- & Id \\ 0 & M_+^* u_-^T \end{pmatrix}$	$\begin{pmatrix} Id & M_+ & \Phi \\ M_- & Id & \Phi \\ 0 & M_+^* u_-^T & (\Phi u_-) \\ M_-^* u_+^T & 0 & (\Phi u_+) \end{pmatrix}$
$X$	$\begin{pmatrix} P\mathcal{A}\pi_- \psi \\ P\mathcal{A}\pi_+ \psi \end{pmatrix}$	$\begin{pmatrix} P\mathcal{A}\pi_- \psi \\ P\mathcal{A}\pi_+ \psi \end{pmatrix}$	$\begin{pmatrix} P\mathcal{A}\pi_- \psi \\ P\mathcal{A}\pi_+ \psi \\ c_2 \end{pmatrix}$
$b$	$\begin{pmatrix} \phi_{-L} \\ \phi_{+L} \end{pmatrix}$	$\begin{pmatrix} \phi_{-L} \\ \phi_{+L} \\ (\phi_{-L} u_-) \end{pmatrix}$	$\begin{pmatrix} \phi_{-L} \\ \phi_{+L} \\ (\phi_{-L} u_-) \\ (\phi_L u_+) \end{pmatrix}$

248 where we define  $\phi_{\pm L} = (S_{\pm}, 0)$ ,  $u_- = \Phi - PB_- \Phi$  and  $u_+ = \Phi - PB_+ \Phi$ .

249 Note 1: when the constants are not controlled the value of  $c_1$  is arbitrary. In these  
 250 cases the linear systems are rectangular and the existence of the solution depends on  
 251 a compatibility conditions that expresses that  $b$  is in the range of  $Z$ .

252 Note 2: once the quantities  $P\mathcal{A}\pi_{\pm}\psi$  are known, then the  $(\mathcal{A}\psi|\phi_i)$  for  $i > 0$  and  
 253  $i < 0$  can be computed as explained above, and  $(\psi|\phi_i)$  is obtained by dividing by  $\lambda_i$ .

*Proof.* The (I/OBC) are equivalent to the following

$$\begin{cases} P\mathcal{A}\pi_- \psi = P\theta_- & \text{with } \theta_- = \phi_{-L} - e^{-2L\mathcal{A}}\mathcal{A}\pi_+ \psi - c_1\Phi - c_2(-L\Phi + \tilde{\Phi}), \\ P\mathcal{A}\pi_+ \psi = P\theta_+ & \text{with } \theta_+ = \phi_{+L} - e^{2L\mathcal{A}}\mathcal{A}\pi_- \psi - c_1\Phi - c_2(L\Phi + \tilde{\Phi}). \end{cases} \quad (5.3)$$

254 A discussion similar to the Dirichlet case leads to the result.  $\square$

## 255 6. Numerical tests.

256 **6.1. First test case: a domain of finite length.** The section of the domain  
 257 of the first test-case is the square  $\Omega = [-5, 5]^2$  with a circular fluid subdomain of  
 258 radius 2 centered at the origin. The velocity and eigenvalues of the operator  $\mathcal{A}$  are  
 computed with  $P1$  finite element methods on the mesh of Figure 6.1 . The velocity has



FIGURE 6.1. The mesh for the first test case is composed of 13589 vertices and 26776 triangles. The solid domain is in white and the fluid domain in grey.

259

260 a parabolic profile (Poiseuille flow) with prescribed total flow  $Q \in \{1, 10, 100, 1000\}$ .  
 261 The lateral boundary conditions are of Robin type with parameter  $a$ . The thermal  
 262 conductivities are equal to  $c = \sigma = 1$ . In total 100 eigenvalues / eigenvectors of  $\mathcal{A}$   
 263 are computed.

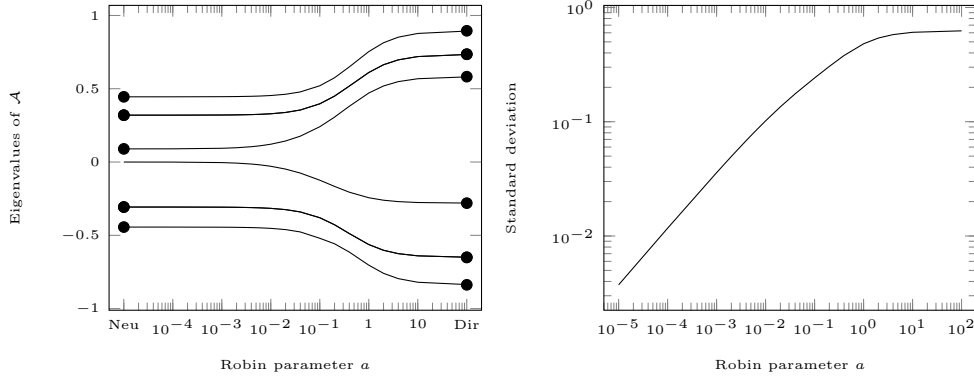


FIGURE 6.2. On the left: Evolution of the eigenvalues of  $\mathcal{A}$  of smallest magnitude for varying parameter of the Robin lateral boundary condition. Eigenvalues for the Neumann (resp. Dirichlet) boundary conditions are shown as bullet on the left (resp. right) of the curves. On the right: relative  $L^2$  difference between the eigenvector with largest negative eigenvalue and its mean as the Robin parameter varies.

264 We first set  $Q = 10$  and vary the Robin parameter  $a$ . When  $a = 0$ , one retrieves  
 265 the Neumann case and when  $a = +\infty$ , one retrieves the Dirichlet case. In order to  
 266 emphasize this fact we plot in Figure 6.2 (left) the eigenvalues of smallest magnitude  
 267 for different values of  $a$ . We also plot with dots the eigenvalues associated to the  
 268 Neumann problem (on the left of the curves) and the one associated to the Dirichlet  
 269 case (on the right of the curves). The smooth transition from Neumann to Dirichlet  
 270 as the Robin parameter varies is striking except from the fact that there exists an  
 271 eigenvalue that goes to zero as  $a$  goes to zero even if the Neumann problem does not  
 272 have zero as eigenvalue. We claim that this behavior is consistent with theory. First  
 273 0 is not an eigenvalue of the Neumann case since the total flow is non-zero (hence  
 274 we are not in a balanced case even if the constants are not controlled). Second, we  
 275 remark that the zero eigenvalue is the limit of a negative eigenvalue. Remember from  
 276 Proposition 6 that it is always possible to decompose a temperature field on the set  
 277 of negative eigenvectors in the Robin case (part a), but for the Neumann case it  
 278 is necessary to add a constant (part b). In other words, in the Neumann case the  
 279 constant must be added to the negative eigenvectors to obtain a Hilbert basis of  $H$ ,  
 280 while the set of positive eigenvectors form a Hilbert basis on their own. This explains  
 281 why the constant emerges as the limit of a negative eigenvector, see Figure 6.2 (right)  
 282 where the convergence of the eigenvector to the constant is numerically demonstrated.

In a second parametric study, we fix  $a = 1$  and we let both  $Q$  and  $L$  vary. First we plot the spectral radius of the matrix  $M$  defined in Proposition 8 versus the exchanger length  $L$  for the different values of the total flow  $Q$  in Figure 6.3 (left). Figure 6.3 (center) shows the evolution of the 5 smallest positive and 5 largest negative eigenvalues of  $M$  for a fixed total flow  $Q = 20$ . This test-case shows that, apart from the case  $L = 0$ , the spectral radius of the matrix  $M$  is always smaller than

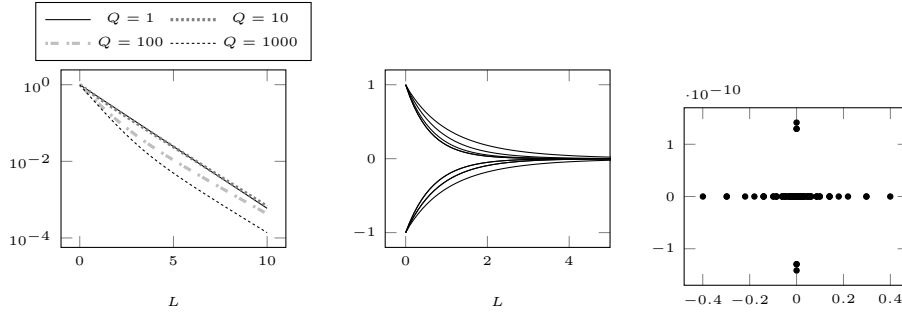


FIGURE 6.3. Eigenvalues of  $M$  for a Robin test case with  $a = 1$ . Left: evolution of the spectral radius for different values of the total flow  $Q$ . Center: the evolution of the five largest positive and first smallest negative eigenvalue for a total flow of  $Q = 20$ . Right: eigenvalues in the complex plane for  $L = 1$  and  $Q = 20$ .

one, so that the matrix  $Id + M$  is indeed always invertible. The exponential decrease for large  $L$  and the decrease at the origin follows from Proposition 8. Moreover, since the spectral radius of  $M$  is strictly smaller than one, a Neumann series strategy to solve

$$(Id + M)^{-1}b = \sum_k (-M)^k b$$

is legitimate. In Figure 6.3 (right), the whole spectrum of  $M$  is shown in the complex plane. Although the spectrum seems real, we do not have mathematical proof of this fact.

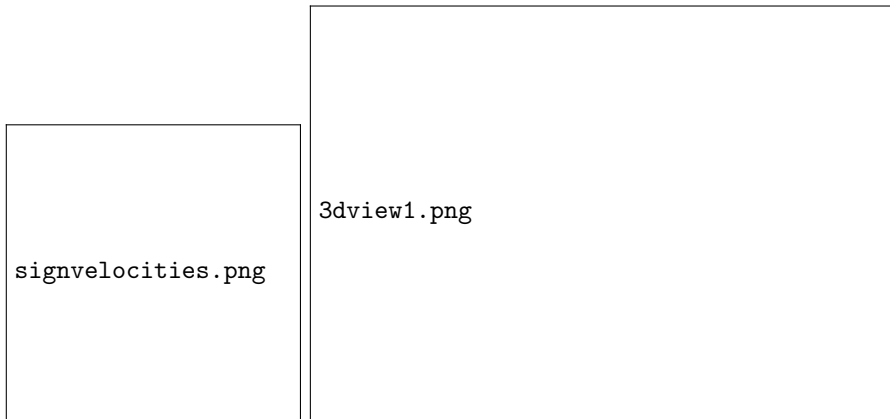


FIGURE 6.4. Geometry of the periodic exchanger. On the left, a cut inside the exchanger with the sign of the fluid velocities. On the right, a 3D representation of the exchanger. The tubes where the temperature is set at  $\infty$  are colored accordingly to their temperature.

**6.2. Second test-case: a periodic exchanger.** The second test case consists of a heat exchanger with periodic boundary conditions. The whole device consists of one solid exchanger through which pass four tubes containing fluids. A cut along the middle of the exchanger is shown in Figure 6.4 (left) where the sign of the velocity of the fluid in the inner tubes is displayed. The fluids are assumed to obey a Poiseuille

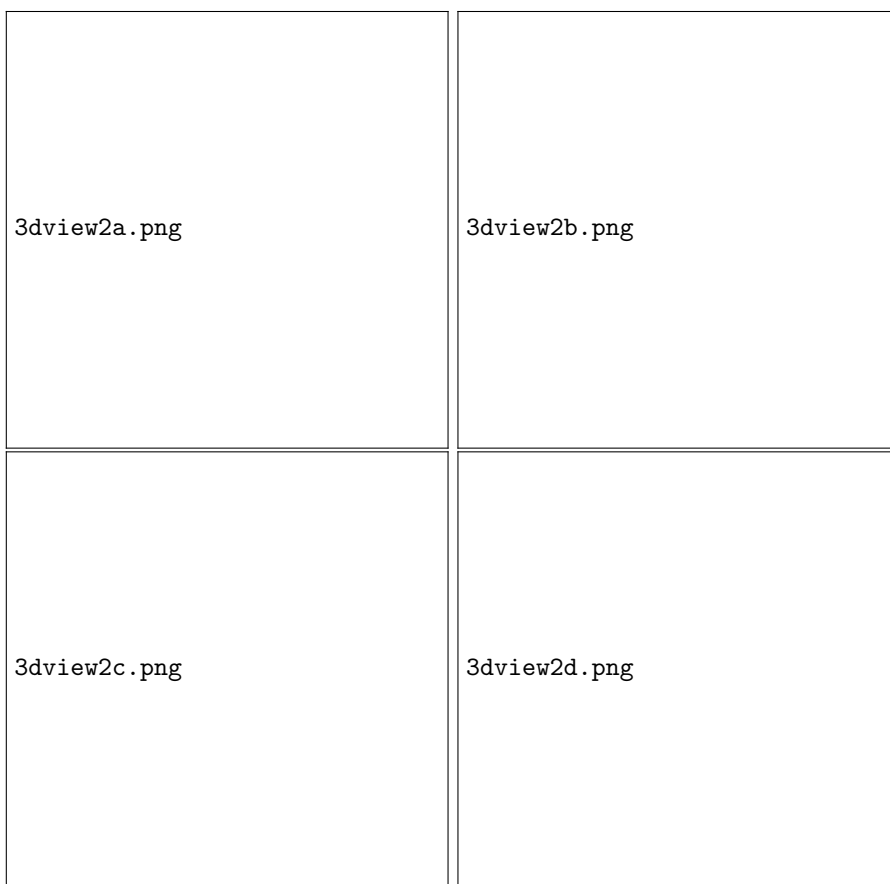


FIGURE 6.5. *Four different solutions of the periodic exchanger, with different length and total flow. The length of the exchanger is set to  $L = 10$  on top and  $L = 20$  on bottom. The total flow is set to  $Q = 10$  on left and  $Q = 30$  on right.*

291 flow, the velocities are then quadratic in the radial coordinates of their corresponding  
 292 tubes. The length of the exchanger is denoted  $L$ , the section of the exchanger is the  
 293 square  $[-4, 4]^2$ , the radii of the inner tubes are fixed to 1 and the distance of the  
 294 center of the inner tubes to the center of the exchanger is  $\sqrt{2^2 + 2^2}$ . The conductivity  
 295 in both the fluid and solid part is set to 1. The temperature is fixed for the four  
 296 tubes with incoming flow (two at each side) on the exchanger, the warm temperature  
 297 is set to  $+1$  and the cold temperature to  $-1$ , see Figure 6.4 (left). In what follows,  
 298  $Q$  denotes the total flow of fluid in one tube. We glue together the different Graetz  
 299 problems using the methodology developed in [4].

300 In Figure 6.5, four solutions are shown for different values of the length  $L$  and  
 301 the flow  $Q$ .

302 Figure 6.6 displays the efficiency and the total exchange for different values of  
 303  $Q$  and  $L$ . For a tube containing fluid whose velocity is positive (resp. negative), the  
 304 temperature at  $-\infty$  (resp.  $+\infty$ ) is set to 1 (resp  $-1$ ), the efficiency of the exchanger  
 305 is then defined by  $-T_{+\infty}/T_{-\infty}$  (resp  $-T_{-\infty}/T_{+\infty}$ ), where  $T_{\pm\infty}$  is the temperature at  
 306 infinity. This efficiency is between  $-1$  and 1. The exchange is simply the total amount  
 307 of heat exchanged and is equal to  $Q$  times the efficiency. The aim of this test-case is

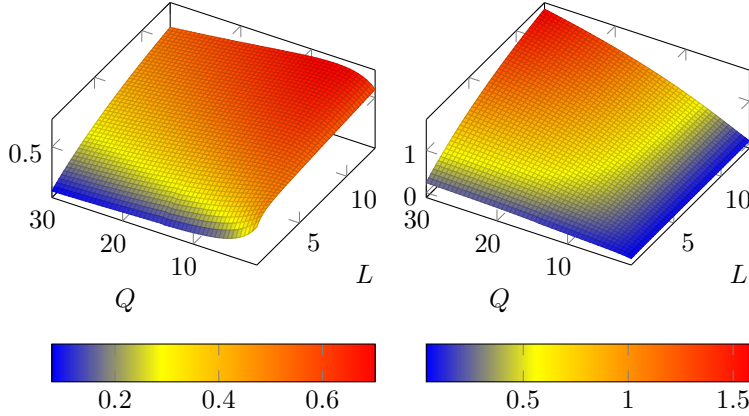


FIGURE 6.6. Values of the efficiency (left) and the exchange (right) for different values of the length  $L$  and the total flow  $Q$  of the periodic exchanger.  $L$  ranges from 0.5 to 13 and  $Q$  ranges from 1 to 30. Each direction has been sampled 50 times for a total of 2500 exchanger computations.

308 to document the fact that our method is able to deal with any boundary conditions  
 309 and type of exchanger. It is well suited for parametric studies.

310 **7. Conclusion.** In the present work we have proposed a general framework ded-  
 311 icated to the resolution of the generalized Graetz problem in arbitrary geometry, in-  
 312 volving any type of boundary conditions. The main novelty is the introduction of  
 313 insulating boundary condition (Neumann or periodic) that allow to model realistic  
 314 heat exchangers. Our study highlighted a special case that we call the balanced case,  
 315 when  $\int_{\Omega} h = 0$  (together with Neumann or periodic boundary condition) where the  
 316 solution is different than in the general case. We have also proposed a number of  
 317 numerical illustration in various test cases.

318 **Appendix A. Technical lemmas.** We prove here results that will be used in  
 319 the sequel.

320 LEMMA 11. For each  $\phi = (u, s) \in \mathcal{D}(\mathcal{A})$ ,  $\tilde{\phi} = (\tilde{u}, \tilde{s}) \in \mathcal{H}$ , we have

$$(Id - P)\mathcal{A}(Id - P)\phi = 0. \quad (\text{A.1})$$

$$(P\mathcal{A}P\phi|\tilde{\phi})_{\mathcal{H}} = \int_{\Omega} hu\tilde{u} \leq \|h\|_{L^{\infty}(\Omega)} \|P\phi\|_{\mathcal{H}} \|P\tilde{\phi}\|_{\mathcal{H}}. \quad (\text{A.2})$$

321 *Proof.* This results from elementary calculations using the definition of  $\mathcal{A}$  (2.1),  
 322 and the definition of  $P$  (3.1).  $\square$

324 LEMMA 12. Let  $\Phi = (1, 0) \in \mathcal{H}$ . Let  $\phi \in \mathcal{D}(\mathcal{A}^{-1/2})$  such that  $P\phi = \phi$ , then

$$(\mathcal{A}^{-1}\phi|\phi)_{\mathcal{H}} = \left( \int_{\Omega} h \right)^{-1} (\phi|\Phi)_{\mathcal{H}}^2 \text{ in the non-balanced case}$$

$$(\mathcal{A}^{-1}\phi|\phi)_{\mathcal{H}} = 0 \text{ in the "balanced" or "constant controlled" case}$$

325 *Proof.* The expression of  $\mathcal{A}^{-1}$  is given in section 2.1 for the various cases. Let  
 $\phi \in \mathcal{D}(\mathcal{A}^{-1/2})$  such that  $P\phi = \phi$ , hence there exists  $u \in L^2(\Omega)$  such that  $\phi = (u, 0)$ .  
 If the constants are controlled then  $\mathcal{A}^{-1}\phi = (0, \Delta_{\sigma}^{-1}(-cu))$  and  $(\mathcal{A}^{-1}\phi|\phi)_{\mathcal{H}} = 0$ . If

the constants are not controlled and  $\int_{\Omega} h \neq 0$ , then

$$\mathcal{A}^{-1}\phi = (k, \Delta_{\sigma}^{-1}(-cu + hk)) \text{ with } k \int_{\Omega} h = \int_{\Omega} cu,$$

hence

$$(\mathcal{A}^{-1}\phi|\phi)_{\mathcal{H}} = \int_{\Omega} cku = \left(\int_{\Omega} h\right)^{-1} \left(\int_{\Omega} cu\right)^2.$$

326 Finally, in the balanced case, since  $\phi \in \mathcal{D}(\mathcal{A}^{-1/2})$ , then  $(\phi|\phi_0)_{\mathcal{H}} = 0$  and  $\mathcal{A}^{-1}\phi =$   
 327  $(0, \Delta_{\sigma}^{-1}(-cu)) + k\phi_0$  exists and  $(\mathcal{A}^{-1}\phi|\phi)_{\mathcal{H}} = 0$ .  $\square$

LEMMA 13. *In the balanced case,*

$$(\Phi - PB_{-}\Phi|\Phi)_{\mathcal{H}} \neq 0.$$

328

329 *Proof.* Suppose the contrary and set  $\theta = \Phi - PB_{-}\Phi$ , we have  $P\theta = \theta$  and by  
 330 definition of  $B_{-}$ , we have  $\pi_{-}\theta = 0$ . Moreover, we have

$$\begin{aligned} (\theta|\phi_0)_{\mathcal{H}} &= (\Phi - PB_{-}\Phi|\phi_0)_{\mathcal{H}} = (P\Phi - PB_{-}\Phi|\phi_0)_{\mathcal{H}} = (\Phi - B_{-}\Phi|P\phi_0)_{\mathcal{H}} \\ &= (\Phi - B_{-}\Phi|P\Phi)_{\mathcal{H}} = (\Phi - PB_{-}\Phi|\Phi)_{\mathcal{H}} = 0 \end{aligned}$$

Lemma 12 ensures that  $\mathcal{A}^{-1}\theta$  exists and that

$$(\mathcal{A}^{-1}\theta|\theta)_{\mathcal{H}} = 0$$

Since  $\theta$  belongs to  $\mathcal{R}(\pi_{+})$  and all the eigenvalues of  $\mathcal{A}$  are positive on this space, this implies that  $\theta = 0$  and then  $\Phi = PB_{-}\Phi$ . Hence there exists  $s$  such that  $B_{-}\Phi = (1, s)$  and

$$(\mathcal{A}B_{-}\Phi|B_{-}\Phi)_{\mathcal{H}} = ((h - \Delta_{\sigma}s, 0) | (1, s))_{\mathcal{H}} = 0.$$

331 But  $B_{-}\Phi$  belongs to  $\mathcal{R}(\pi_{-})$  and since all the eigenvalues of  $\mathcal{A}$  are negative on  $\mathcal{R}(\pi_{-})$ ,  
 332  $(\mathcal{A}B_{-}\Phi|B_{-}\Phi)_{\mathcal{H}} = 0$  implies that  $B_{-}\Phi = 0$  which is in violation of  $\Phi = PB_{-}\Phi$ . Hence  
 333  $(\Phi - B_{-}\Phi|\Phi)_{\mathcal{H}} \neq 0$ .  $\square$

**Appendix B. Proof of Theorem 4.** Let  $M \in \mathbb{N}^*$  and denote for short  $\pi = \pi_{[-M, -1]}$ . The operator  $\pi P \pi$  is a symmetric operator on a finite-dimensional space, hence it is diagonalisable in an orthonormal basis. The first step is to prove that this operator is definite positive with a lower bound on its eigenvalues that is independent of  $M$ . Let  $\rho$  be an eigenvalue of  $\pi P \pi$  and  $\mathbf{v}$  an associated normalized eigenvector:  $\pi P \pi \mathbf{v} = \rho \mathbf{v}$ ,  $(\mathbf{v}|\mathbf{v})_{\mathcal{H}} = 1$  and  $\pi \mathbf{v} = \mathbf{v}$ . Since

$$\rho = (\pi P \pi \mathbf{v}|\mathbf{v})_{\mathcal{H}} = (P \pi \mathbf{v}|\pi \mathbf{v})_{\mathcal{H}} = (P \pi \mathbf{v}|P \pi \mathbf{v})_{\mathcal{H}} = \|P \mathbf{v}\|_{\mathcal{H}}^2 \leq \|\mathbf{v}\|_{\mathcal{H}}^2 = 1,$$

then  $0 \leq \rho \leq 1$ . Using (A.2) gives

$$|(P \mathcal{A} P \mathbf{v}|\mathbf{v})_{\mathcal{H}}| \leq \|h\|_{L^{\infty}(\Omega)} \|P \mathbf{v}\|_{\mathcal{H}}^2.$$

It follows from (A.1) that  $((Id - P)\mathcal{A}(Id - P)\mathbf{v}|\mathbf{v})_{\mathcal{H}} = 0$  and  $\pi \mathcal{A} = \mathcal{A} \pi$ , we have

$$(P \mathcal{A} P \mathbf{v}|\mathbf{v})_{\mathcal{H}} = (2\rho - 1)(\mathcal{A} \mathbf{v}|\mathbf{v})_{\mathcal{H}}$$

Since  $|(\mathcal{A}\mathbf{v}|\mathbf{v})_{\mathcal{H}}| = |\sum_{i \in I} \lambda_i \frac{(\mathbf{v}|\phi_i)_{\mathcal{H}}^2}{\|\phi_i\|_{\mathcal{H}}^2}| \geq |\lambda_{-1}| \|\mathbf{v}\|_{\mathcal{H}}^2 = |\lambda_{-1}|$ , we have

$$|\lambda_{-1}(2\rho - 1)| \leq \|h\|_{L^\infty(\Omega)} \|P\mathbf{v}\|_{\mathcal{H}}^2 = \|h\|_{L^\infty(\Omega)} \rho. \quad (\text{B.1})$$

This in turn implies that  $\rho \geq \frac{|\lambda_{-1}|}{2|\lambda_{-1}| + \|h\|_{L^\infty(\Omega)}}$ , hence there exists  $C$  independent of  $M$  such that

$$(\pi P \pi \phi | \phi)_{\mathcal{H}} \geq C \|\pi \phi\|_{\mathcal{H}} \quad \forall \phi \in \mathcal{H}(\mathcal{A}). \quad (\text{B.2})$$

334 Since  $\pi_- \phi$  is the strong  $\mathcal{H}$ -limit of  $\pi \phi$  as  $M$  goes to infinity and the constant  $C$  does  
 335 not depend on  $M$ . Passing to the limit, we recover equation (B.2) with  $\pi$  replaced by  
 336  $\pi_-$ . The Lax-Milgram theorem applies and  $\pi_- P \pi_-$  is a bijection from  $\mathcal{R}(\pi_-)$  onto  
 337  $\mathcal{R}(\pi_-)$  with a continuous inverse bounded by a constant in  $\mathcal{H}$ -norm.

We turn our interest to the bound in  $1/2$  norm of  $B_-$ . Let  $\phi \in \mathcal{D}(\mathcal{A}^{1/2})$ , for any  $M \in \mathbb{N}^*$  denote  $\pi = \pi_{[-M, -1]}$ , and let  $\mathbf{v} = \pi B_- \phi$ . We have  $\pi \mathbf{v} = \mathbf{v}$  and  $\mathbf{v} \in \mathcal{D}(\mathcal{A})$ . Recalling (A.1) and  $\pi \mathcal{A} = \mathcal{A} \pi$ , we have

$$(P \mathcal{A} P \mathbf{v} | \mathbf{v})_{\mathcal{H}} = ((\mathcal{A} P + P \mathcal{A} - \mathcal{A}) \mathbf{v} | \mathbf{v})_{\mathcal{H}} = 2(P \mathbf{v}, \mathcal{A} \mathbf{v})_{\mathcal{H}} - (\mathcal{A} \mathbf{v}, \mathbf{v})_{\mathcal{H}}.$$

Hence, since  $\pi \mathbf{v} = \mathbf{v}$  and  $\pi$  is a projection on negative eigenvalues of  $\mathcal{A}$  only, then  $\|\mathbf{v}\|_{1/2}^2 = -(\mathcal{A} \mathbf{v} | \mathbf{v})_{\mathcal{H}}$  and

$$\|\mathbf{v}\|_{1/2}^2 = (P \mathcal{A} P \mathbf{v} | \mathbf{v})_{\mathcal{H}} - 2(P \mathbf{v} | \mathcal{A} \mathbf{v})_{\mathcal{H}} \leq \|h\|_{L^\infty(\Omega)} \|\mathbf{v}\|_{\mathcal{H}}^2 + 2\|\pi P \mathbf{v}\|_{1/2} \|\mathbf{v}\|_{1/2} \quad (\text{B.3})$$

Using the bound on the  $\mathcal{H}$ -norm of  $B_-$ , we have

$$\|\mathbf{v}\|_{\mathcal{H}} = \|\pi B_- \phi\|_{\mathcal{H}} \leq C \|\pi_- \phi\|_{\mathcal{H}} \leq C \|\pi_- \phi\|_{1/2}. \quad (\text{B.4})$$

We infer from (B.3) and (B.4) that  $\|\mathbf{v}\|_{1/2} \leq C(\|\pi_- \phi\|_{1/2} + \|\pi P \mathbf{v}\|_{1/2})$ . We let  $M$  go to infinity, then  $\pi P \mathbf{v} = \pi P \pi B_- \phi$  goes to  $\pi_- \phi$  and  $\mathbf{v}$  goes to  $B_- \phi$ , we obtain:

$$\|B_- \phi\|_{1/2} \leq C \|\pi_- \phi\|_{1/2},$$

338 which finishes the proof.

### 339 **Appendix C. Proof of Theorem 5.**

340 *First case:  $\mathcal{K}(\mathcal{A}) = \{0\}$ , i.e. every case but the balanced case*

341 In this case the condition  $\pi_+ \psi = 0$  is then equivalent to  $\psi = \pi_- \psi$ .

After multiplication of (3.3) by  $B_- \pi_-$ , one obtains the following necessary condition for (3.3) to hold, which proves uniqueness:

$$\psi = B_- P \phi.$$

342 Denote  $\theta = P B_- P \phi - P \phi$ , the question of the existence of the solution is reduced to  
 343 studying under which condition  $\theta = 0$ .

344 We have  $P \theta = \theta$  and Theorem 4 states that  $\pi_- \theta = 0$ . This implies that  $\theta \in \mathcal{R}(\pi_+)$ .  
 345 The operator  $\mathcal{A}^{-1}$  is symmetric definite positive on  $\mathcal{R}(\pi_+)$ , and induces the scalar  
 346 product of the  $-1/2$ -norm. Lemma 12 states that, if the constants are controlled we  
 347 have  $(\mathcal{A}^{-1} \theta | \theta)_{\mathcal{H}} = 0$ , and it follows that  $\theta = 0$ . This proves the result when the  
 348 constants are controlled.

Assume now that  $\Gamma_D \cup \Gamma_R = \emptyset$  and  $\int_{\Omega} h \neq 0$ , Lemma 12 states that

$$\|\theta\|_{-1/2}^2 = (\mathcal{A}^{-1}\theta|\theta)_{\mathcal{H}} = \left( \int_{\Omega} h \right)^{-1} (\theta|\Phi)_{\mathcal{H}}^2. \quad (\text{C.1})$$

349 If  $\int_{\Omega} h < 0$ , the two terms have opposite signs, hence both are zero. Then  $\theta = 0$  and  
 350 this proves the result for the case  $\Gamma_D \cup \Gamma_R = \emptyset$  and  $\int_{\Omega} h < 0$ .

Let us assume now that  $\int_{\Omega} h > 0$ . Since changing the sign of  $\lambda$  amounts to study the same problem where  $h$  is replaced by  $-h$ , we deduce from the case  $\int_{\Omega} h < 0$  with  $\phi = \Phi$  and the relation  $P\Phi = \Phi$  that  $PB_+\Phi = \Phi$ . Since  $\Phi \in \mathcal{D}(A^{1/2})$ , it follows from Theorem 4 that  $B_+\Phi \in \mathcal{D}(A^{1/2})$ . Hence there exists a  $s^* \in H$  such that  $B_+\Phi = (1, s^*)$  and we have  $\mathcal{A}B_+\Phi = (c^{-1}h - c^{-1}\operatorname{div} \sigma \nabla s^*, 0)$ . This proves

$$P\mathcal{A}B_+\Phi = \mathcal{A}B_+\Phi,$$

and a simple calculation proves that

$$(\mathcal{A}B_+\Phi|\Phi)_{\mathcal{H}} = (\mathcal{A}B_+\Phi|B_+\Phi)_{\mathcal{H}} = \int_{\Omega} h \quad (\text{C.2})$$

We then compute

$$(\Phi - PB_-\Phi|\mathcal{A}B_+\Phi)_{\mathcal{H}} = (\Phi - B_-\Phi|P\mathcal{A}B_+\Phi)_{\mathcal{H}} = (\Phi - B_-\Phi|\mathcal{A}B_+\Phi)_{\mathcal{H}} \underbrace{=}_{(1)} (\Phi|\mathcal{A}B_+\Phi)_{\mathcal{H}} \neq 0,$$

351 where the equality (1) is obtained by remarking that  $\mathcal{A}B_+\Phi \in \mathcal{R}(\pi_+)$  and  $B_-\Phi \in$   
 352  $\mathcal{R}(\pi_-)$  which are orthogonal spaces. We then obtain  $\Phi - PB_-\Phi \neq 0$ .

353 It follows from (C.1) that

$$\begin{aligned} \|\theta\|_{-1/2}^2 &= \left( \int_{\Omega} h \right)^{-1} (\theta|\Phi)_{\mathcal{H}}^2 = \left( \int_{\Omega} h \right)^{-1} (\theta|PB_+\Phi)_{\mathcal{H}}^2 \\ &= \left( \int_{\Omega} h \right)^{-1} (P\theta|B_+\Phi)_{\mathcal{H}}^2 = \left( \int_{\Omega} h \right)^{-1} (\theta|B_+\Phi)_{\mathcal{H}}^2. \end{aligned}$$

Using that  $\theta$  and  $B_+\Phi$  belong to  $\mathcal{R}(\pi_+)$  on which all the eigenvalues of  $\mathcal{A}^{-1}$  are positive, the above equation implies

$$\|\theta\|_{-1/2}^2 = \left( \int_{\Omega} h \right)^{-1} (\theta|\mathcal{A}B_+\Phi)_{-1/2}^2.$$

We recall that  $\|\mathcal{A}B_+\Phi\|_{-1/2}^2 = (\mathcal{A}B_+\Phi|B_+\Phi)_{\mathcal{H}} = \int_{\Omega} h$ , and we obtain

$$\|\theta\|_{-1/2}^2 \|\mathcal{A}B_+\Phi\|_{-1/2}^2 = (\theta|\mathcal{A}B_+\Phi)_{-1/2}^2$$

which is an equality case in Cauchy-Schwarz inequality. This implies that  $\theta$  and  $\mathcal{A}B_+\Phi$  are colinear. Hence there exists some constant  $t$  such that

$$\theta = t\mathcal{A}B_+\Phi.$$

354 Performing the scalar product with  $\Phi$  and using the fact that  $(\mathcal{A}B_+\Phi|\Phi) \neq 0$   
 355 which follows from (C.2), we conclude that  $t = 0$  (hence  $\theta = 0$ ) if and only if  $(\theta|\Phi) = 0$ ,  
 356 which reads  $(\phi|\Phi - PB_-\Phi)_{\mathcal{H}} = 0$ .

357 *Second case:  $\mathcal{K}(\mathcal{A}) \neq \{0\}$ , which is the balanced case.*



In the balanced case the kernel of  $\mathcal{A}$  is  $\mathbb{R}\phi_0$ , where we recall from section 2.1 that  $P\phi_0 = \Phi$ . The condition  $\pi_+\psi = 0$  is equivalent to the existence of  $\alpha \in \mathbb{R}$  such that  $\psi = \pi_-\psi + \alpha\phi_0$ . The condition  $P\psi = P\phi$  is thus equivalent to

$$P\phi = P\pi_-\psi + \alpha\Phi. \quad (\text{C.3})$$

358

Necessary condition:

After multiplying Equation (C.3) by  $B_-$  one obtains:

$$\pi_-\psi = B_-P\phi - \alpha B_-\Phi.$$

Replacing the expression of  $\pi_-\psi$  in (C.3) yields the following necessary condition:

$$PB_-P\phi + \alpha\Phi - \alpha PB_-\Phi = P\phi,$$

which reads

$$\alpha(\Phi - PB_-\Phi) = P\phi - PB_-P\phi.$$

It follows from lemma 13 that  $(\Phi - PB_-\Phi|_{\mathcal{H}})_{\mathcal{H}} \neq 0$ , then it is necessary that

$$\alpha = \frac{(\Phi - PB_-\Phi|\phi)_{\mathcal{H}}}{(\Phi - PB_-\Phi|\Phi)_{\mathcal{H}}}.$$

$\psi$  is uniquely determined by

$$\psi = B_-P\phi + \frac{(\Phi - PB_-\Phi|\phi)_{\mathcal{H}}}{(\Phi - PB_-\Phi|\Phi)_{\mathcal{H}}}(\phi_0 - B_-\Phi) \quad (\text{C.4})$$

359

Conversely, if  $\psi$  is defined by (C.4), it is clear that  $\pi_+\psi = 0$ . Let  $\theta = P\psi - P\phi$ :

360

it suffices to prove that  $\theta = 0$  to ensure that  $\psi$  solves the problem.

$$(\theta|\phi_0)_{\mathcal{H}} = (\theta|\Phi)_{\mathcal{H}} = 0$$

by choice of  $\alpha$ . A simple calculation shows that

$$\pi_-\theta = 0.$$

361

This proves that  $\theta \in \mathcal{R}(\pi_+)$ , where  $\mathcal{A}^{-1}$  is a symmetric positive definite operator. It

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follows from lemma 12 that  $(\mathcal{A}^{-1}\theta|\theta)_{\mathcal{H}} = 0$  and hence  $\theta = 0$ . This finishes the proof.

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