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A VARIATIONAL FORMULATION FOR COMPUTING SHAPE DERIVATIVES OF GEOMETRIC CONSTRAINTS ALONG RAYS

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ABSTRACT. In the formulation of shape optimization problems, multiple geometric constraint functionals involve the signed distance function to the optimized shape Ω . The numerical evaluation of their shape derivatives requires to integrate some quantities along the normal rays to Ω , a task that is usually achieved thanks to the method of characteristics. The goal of the present paper is to propose an alternative, variational approach for this purpose. Our method amounts, in full generality, to compute integral quantities along the characteristic curves of a given velocity field without requiring the explicit knowledge of these curves on the spatial discretization; it rather relies on a variational problem which can be solved conveniently by the finite element method. The well-posedness of this problem is established thanks to a detailed analysis of weighted graph spaces of the advection operator $\beta \cdot \nabla$ associated to a C^1 velocity fields β . One novelty of our approach is the ability to handle velocity fields with possibly unbounded divergence: we do not assume $\operatorname{div}(\beta) \in L^\infty$. Our working assumptions are fulfilled in the context of shape optimization of C^2 domains Ω , where the velocity field $\beta = \nabla d_\Omega$ is an extension of the unit outward normal vector to the optimized shape. The efficiency of our variational method with respect to the direct integration of numerical quantities along rays is evaluated on several numerical examples. Classical albeit important implementation issues such as the calculation of a shape's curvature and the detection of its skeleton are discussed. Finally, we demonstrate the convenience and potential of our method when it comes to enforcing maximum and minimum thickness constraints in structural shape optimization.

Keywords. Advection operator, shape and topology optimization, level set method, signed distance function, thickness constraints.

AMS Subject classifications. 65K10, 49Q10.

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1. INTRODUCTION

The recent achievements of shape and topology optimization techniques in predicting very efficient designs beyond the reach of the intuition of engineers have raised a tremendous enthusiasm in industry. One key issue about the practical use of these techniques is that they often lead to shapes which are too complicated to be assembled by traditional milling or casting processes, to the point that one of the greatest challenges faced by shape and topology optimization is about the understanding and the implementation of manufacturing constraints. Such constraints generally bring into play the geometry of shapes; for instance, minimum thickness constraints, maximum thickness constraints, minimum distance between members, casting constraints have been considered; see [61, 16, 37, 49] or the recent survey [45].

Among the variety of methods proposed in the literature to enforce these constraints, a body of work has recently been devoted to formulate them by means of integral functionals involving the signed distance function d_Ω to the optimized domain Ω [5, 6, 21, 22, 49]. More precisely, the shape Ω is sought among all possible subsets of a fixed ‘hold-all’ domain $D \subset \mathbb{R}^n$ ($n = 2, 3$ in applications) as the solution of a constrained minimization problem of the form:

$$\min_{\Omega \subset D} J(\Omega), \text{ s.t. } P(\Omega) \leq 0, \text{ where } P(\Omega) := \int_D j(d_\Omega(x)) dx, \quad (1.1)$$

where $J(\Omega)$ is a measure of the performance of Ω (e.g. its elastic compliance), $j : \mathbb{R} \rightarrow \mathbb{R}$ is a given, smooth function, and $P(\Omega)$ is a geometric constraint. The above framework is quite appealing insofar as the signed distance function d_Ω naturally lends itself to clear mathematical formulations of maximum, minimum thickness constraint functionals (and the other aforementioned geometric criteria), thus making (1.1) amenable to an efficient numerical treatment by means of standard (e.g. steepest-descent) optimization algorithms. However, several technical stages in its numerical implementation pose difficulties; the leading motivation of the present article is to introduce a new variational method that make these substantially simpler to implement.

Let us provide a little more details about the main numerical issues raised by the implementation of (1.1), while staying at an explanatory level; see Sections 3.1 and 4 below for full details. The use of traditional optimization methods for the program (1.1) relies (in particular) on the knowledge of the *shape derivative* of the considered constraint functional $\Omega \mapsto P(\Omega)$. Throughout this article, this notion is understood in the framework of the Hadamard method: the shape derivative corresponds to the Fréchet derivative of $\theta \mapsto P(\Omega_\theta)$ at $\theta = 0$, for local variations of Ω of the form $\Omega_\theta = (\text{Id} + \theta)(\Omega)$ and for vector fields $\theta \in W^{1,\infty}(D, \mathbb{R}^n)$ with sufficiently small norm (see Section 4 hereafter). It was shown in our previous work [2] that the shape derivative of $P(\Omega)$ as in (1.1) has the form

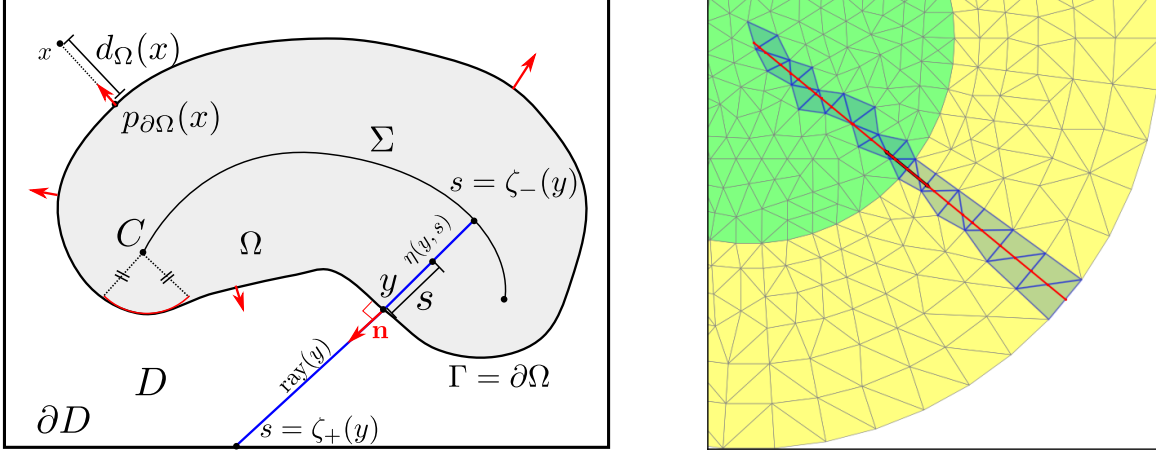
$$P'(\Omega)(\theta) = \int_{\partial\Omega} u \theta \cdot \mathbf{n} \, dy, \quad (1.2)$$

where dy refers to the surface measure on $\partial\Omega$, and the function $u \in L^1(\partial\Omega)$ is given by the formula

$$u(y) = - \int_{x \in \text{ray}(y)} j'(d_\Omega(x)) \prod_{1 \leq i \leq n-1} (1 + \kappa_i(y) d_\Omega(x)) dx, \quad \forall y \in \partial\Omega. \quad (1.3)$$

In (1.3), \mathbf{n} and $\kappa_i(y)$ stand for the unit normal vector to $\partial\Omega$ pointing outward Ω and the $(n - 1)$ principal curvatures of $\partial\Omega$ respectively; the set $\text{ray}(y)$ is the normal ray that originates from $y \in \partial\Omega$ and that stops at either the boundary ∂D of the hold-all domain, or at the skeleton (or medial axis) Σ of Ω ; see Section 3.1 for a short review of these notions and Figure 1a for an illustration. The normal velocity $\theta = -u\mathbf{n}$ is then used as a steepest-descent direction to update the shape Ω [1], which raises the need to calculate the quantity $u : \partial\Omega \rightarrow \mathbb{R}$ defined in (1.3).

In principle, the formula (1.3) featuring integration along the normal rays to $\partial\Omega$ can be implemented efficiently. In particular, its numerical evaluations at several points $y \in \partial\Omega$ can be performed in parallel. However, the practical implementation when all the functions involved in the integrand of (1.3) are discretized on a computational mesh is not trivial: it requires the computation of (i) the ray trajectories on the discrete mesh (for instance by some variant of the method of characteristics), as exemplified on Figure 1b, and (ii) the principal curvatures κ_i of the boundary $\partial\Omega$ from the datum of its numerical definition. As we shall illustrate in our numerical investigations, the accurate calculation of these quantities —even when $\partial\Omega$ is endowed with



(A) The shape optimization setting: shape $\Omega \subset D$, skeleton Σ with normal rays $(\text{ray}(y))_{y \in \partial\Omega}$, local coordinate change η , outward normal vector \mathbf{n} , center of curvature C .

(B) Integration along rays requires detecting the skeleton of the shape and computing the triangle path at the mesh level, that is the intersection points of each triangle with the ray.

FIGURE 1. Definition of geometric quantities and integration along normal rays.

a discretization as an explicit triangulated surface—is not trivial and is still an active research direction, already in the case of two space dimensions, and especially in the three-dimensional context [48, 54, 30].

The objective of this paper is to introduce a robust and easy to implement method that allows to compute integral quantities such as (1.3) by solving a variational problem for advection-like operators which alleviates the need for resorting to direct integration along rays. This yields an efficient numerical method to calculate shape derivatives of geometric constraints, which is very simple to implement using a standard finite element software, and which does not raise any additional algorithmic difficulty in 3-d than in 2-d. We shall actually perform the mathematical analysis of our method in a more general setting, that we now describe and which is subject to offer wider applications than shape optimization.

Let β be a \mathcal{C}^1 vector field on a bounded (possibly non smooth) open domain $U \subset \mathbb{R}^n$ and let $\Gamma \subset \bar{U}$ be a \mathcal{C}^1 codimension-one surface (with boundary if $\Gamma \neq \bar{\Gamma}$); two examples of admissible settings are represented on Figure 2. We assume that Γ is a Poincaré section or a *stream surface* for β , meaning that for any $y \in \Gamma$, there is a unique characteristic curve $s \mapsto \eta(y, s)$ passing through $y = \eta(y, 0)$ at time $s = 0$, and that lives in the domain U on some maximal interval $s \in (\zeta_-(y), \zeta_+(y))$. In other words, for any $y \in \Gamma$, $(\zeta_-(y), \zeta_+(y))$ is the maximum existence interval such that the solution $s \mapsto \eta(y, s)$ of the ordinary differential equation

$$\begin{cases} \frac{d}{ds} \eta(y, s) = \beta(\eta(y, s)), \\ \eta(0, y) = y, \end{cases} \quad (1.4)$$

remains in the domain U (note that by definition, $\zeta_- \leq 0 \leq \zeta_+$). A few more technical but not restrictive assumptions on Γ , ζ_- and ζ_+ shall be stated in the beginning of Section 2. We then require the vector field β to be *U-filling*, meaning that characteristics leaving from the surface Γ fill the whole domain U , *i.e.* the defined flow map η is surjective from $W = \{(y, s) \mid y \in \Gamma, s \in (\zeta_-(y), \zeta_+(y))\}$ onto U . Finally, we denote by $|D\eta|$ the Jacobian of the local coordinate change η :

$$\forall y \in \Gamma, \forall s \in (\zeta_-(y), \zeta_+(y)), \quad |D\eta|(y, s) = |\det(\nabla\eta)|, \quad (1.5)$$

where the Jacobian matrix of η reads $\nabla\eta = \left[\partial_y \eta \mid \partial_s \eta \right]$ and ∂_y denotes the collection of derivatives with respect to the $(n-1)$ tangential coordinates of Γ .

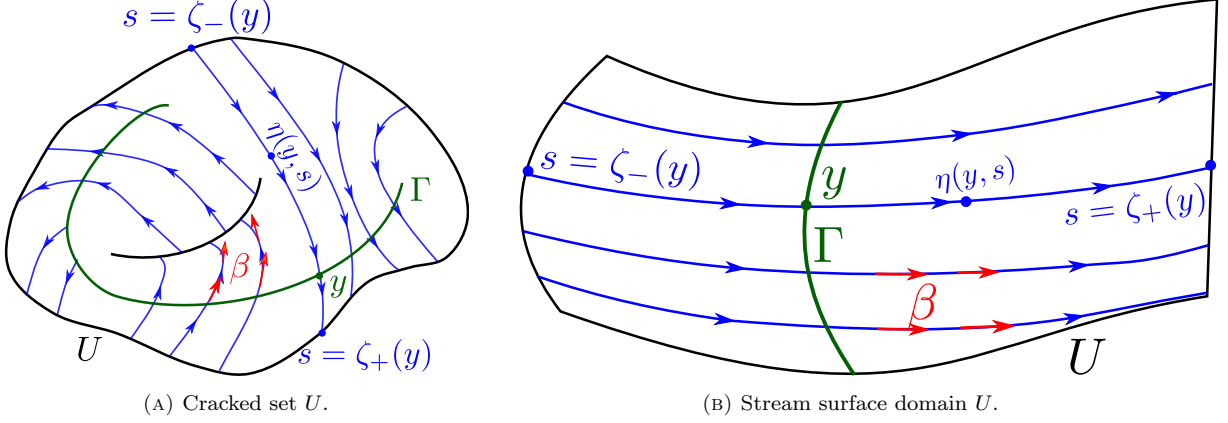


FIGURE 2. Two different types of admissible settings.

The main point of this article is to propose a numerical method to compute integrals of an arbitrary function $f : U \rightarrow \mathbb{R}$ along the characteristic curves of β , namely:

$$u(y) = \int_{\zeta_-(y)}^{\zeta_+(y)} (f \circ \eta |D\eta|)(y, s) ds, \quad \forall y \in \Gamma. \quad (1.6)$$

The shape optimization setting of Figure 1a reduces to the particular case where

$$U = D \setminus \bar{\Sigma}, \quad \Gamma = \partial\Omega, \quad \beta = \nabla d_\Omega, \quad (1.7)$$

and $\zeta_-(y)$ and $\zeta_+(y)$ are the distances at which $\text{ray}(y)$ hits either the skeleton Σ or the boundary ∂D . Furthermore, in this situation, the local coordinate change η and its Jacobian $|D\eta|$ are explicitly given by (see [13] and Section 3.1).

$$\eta(y, s) = y + s(\nabla d_\Omega) \circ \eta(y, s), \quad |D\eta|(y, s) = \prod_{1 \leq i \leq n-1} (1 + \kappa_i(y)s), \quad \forall y \in \partial\Omega, \forall s \in (\zeta_-(y), \zeta_+(y)), \quad (1.8)$$

so that the function u of (1.3) coincides with the expression (1.6) for $f = -j'$. Let us emphasize that, in this context, the open set U is not smooth; in particular, it is not located on one side of its boundary—a common assumption for many useful results in functional analysis [58]—but it is rather a “cracked domain”; see Figure 1a. This lack of smoothness raises the need for several technical ingredients in the sequel, which otherwise would have been fairly classical.

Our numerical method for the calculation of (1.6) is based on the following variational problem, set over a suitable Hilbert space V_ω :

$$\text{Find } u \in V_\omega \text{ such that } \forall v \in V_\omega, \quad \int_\Gamma uv ds + \int_U \omega(\beta \cdot \nabla u)(\beta \cdot \nabla v) dx = \int_U f v dx, \quad (1.9)$$

where $\omega > 0$ is a rather arbitrary weight function, over which relevant assumptions shall be stated later on. Indeed, our key result is to show that (1.9) is well-posed, and that the trace of its unique solution u on Γ is exactly (1.6).

A formal insight that allows to see this is the following: assume that for any $v_0 \in \mathcal{C}^0(\Gamma)$, the function v satisfying $v = v_0$ on Γ and taking constant values along the characteristic curves of β (i.e. $\beta \cdot \nabla v = 0$ in U) belongs to the space V_ω . Using v as a test function in the variational formulation (1.9) then yields:

$$\int_\Gamma uv_0 ds = \int_\Gamma uv ds + \int_U \omega(\beta \cdot \nabla u)(\beta \cdot \nabla v) dx = \int_U f v dx = \int_\Gamma \left(\int_{\zeta_-(y)}^{\zeta_+(y)} f \circ \eta |D\eta| ds \right) v_0 dy, \quad (1.10)$$

where the last equality follows from the change of variable formula (see (2.3) below). Since $v_0 \in \mathcal{C}^0(\Gamma)$ can be chosen arbitrarily in (1.10), the identity (1.6) follows. These considerations are made rigorous in Section 2.5.

The variational formulation (1.9) makes it possible to compute integrals (1.6) along the characteristic curves of β without the need to calculate these curves (or the Jacobian $|D\eta|$) explicitly on a spatial discretization. This feature is especially convenient for shape optimization applications relying on the level set method, as described in Section 4; there, the vector field $\beta = \nabla d_\Omega$ is easy to calculate on an unstructured mesh of the considered hold-all domain D from a \mathbb{P}_1 approximation of the signed distance function. The variational formulation (1.9) can then be readily implemented in a finite element framework, even if the boundary $\Gamma = \partial\Omega$ is not meshed explicitly. This method does not require accurate estimates of the principal curvatures κ_i involved in the definition (1.8) of the Jacobian $|D\eta|$, but only the correct localization of the skeleton set Σ , which, as detailed further in our numerical Section 3, influences the choice of the weight ω .

The present article is organized as follows. In Section 2, we carefully discuss the mathematical setting that guarantees the existence and uniqueness of a solution to the variational problem (1.9), and the justification of the key identity (1.6) satisfied by the trace of its solution. For that purpose, we provide a variational theory for the advection operator $\beta \cdot \nabla$ associated to arbitrary \mathcal{C}^1 vector fields β on the weighted graph space V_ω ; in particular, we examine the existence of traces on Γ and we prove an adapted Poincaré inequality for functions $v \in V_\Omega$. Let us stress that related works about these matters (e.g. [28, 31]) usually require less regularity on the vector fields β and do not rely on the existence of a flow η , but at the cost of rather strong boundedness assumptions on the divergence of β (typically $\text{div}(\beta) \in L^\infty(U)$), which do not hold for our shape optimization applications (1.7) and (1.8): indeed, in the latter situation where $\beta = \nabla d_\Omega$, the divergence $\text{div}(\nabla d_\Omega)$ typically blows up near Σ ; hence the need for our different approach. Likewise, we do not assume strong smoothness assumptions on the domain U , whose regularity shall be determined only by that of the functions ζ_-, ζ_+ (assumed continuous only).

In Section 3, we investigate the numerical accuracy of our variational method for calculating integrals along characteristic curves in the shape optimization setting (1.7) where $\beta = \nabla d_\Omega$. After a short review of the properties of the signed distance function, we compare the direct numerical integration along rays with the use of our variational method on several numerical examples where the value of (1.6) is analytically known. We also consider “practical” cases where some of the required regularity assumptions by our framework are not fulfilled. Most importantly, we illustrate how the selection of an appropriate weight ω in the variational formulation (1.9) allows to deal with the presence of cracks in the domain U —or of the skeleton Σ in shape optimization—when those are not explicitly meshed.

Eventually, in Section 4, we turn to practical shape optimization applications, in the particular case of two space dimensions, which is sufficient to demonstrate the simplicity and effectiveness of our method. Indeed, the variational approach (1.9) is easy to implement in any space dimension while the 3-d implementation of geometric integration along rays requires much more efforts than in 2-d. Elaborating on our previous works [3, 6, 22] we demonstrate that the method proposed in this paper allows for a convenient and efficient implementation of maximum and minimum thickness constraints in structural design.

2. WEIGHTED GRAPH SPACE OF THE ADVECTION OPERATOR $\beta \cdot \nabla$ FOR VELOCITY FIELDS OF CLASS \mathcal{C}^1

This mathematical section investigates the well-posedness of the variational problem (1.9) as well as the trace identity (1.6); this involves in particular the definition and properties of suitable functional spaces V_ω . After setting notations and discussing a few preliminaries in Section 2.1, we define in Section 2.2 the weighted graph space V_ω for the directional derivative $\beta \cdot \nabla$, as well as its counterpart \tilde{V}_α obtained after changing variables. Then, we show in Section 2.3 that under suitable hypotheses, \mathcal{C}^1 functions are dense in V_ω . In Section 2.4, we establish a trace theorem for functions in V_ω on Γ and we provide a Poincaré-type inequality. Finally, we prove in Section 2.5 that the variational problem (1.9) is well-posed on V_ω and properly discuss the identity (1.6).

2.1. Preliminaries, notations and assumptions

Let $U \subset \mathbb{R}^n$ be a (possibly non smooth) bounded open set and let $\Gamma \subset \bar{U}$ be an hypersurface of class \mathcal{C}^1 . Γ may not be a compact manifold (e.g. if $\Gamma \neq \bar{\Gamma}$ as in Figure 2b); in that case we require the closure $\bar{\Gamma}$ to be a manifold with boundary, which in particular prevents Γ from showing spiralling patterns near its ends. Such assumption is needed for technical reasons e.g. in the proof of Lemma 3.

Let $\beta : U \rightarrow \mathbb{R}^n$ be a vector field of class \mathcal{C}^1 on U as in the introduction. The flow map η of β is defined by (1.4); for any $y \in \Gamma$, the curve $s \mapsto \eta(y, s)$ passes through the surface Γ at $s = 0$, and it is defined on the interval $(\zeta_-(y), \zeta_+(y))$. We assume that ζ_- and ζ_+ are continuous, bounded functions on Γ , satisfying the following separation condition:

$$\exists \epsilon > 0, \forall y \in \Gamma, \zeta_+(y) - \zeta_-(y) > \epsilon. \quad (2.1)$$

We require the mapping η to be U -filling; hence, it realizes a \mathcal{C}^1 diffeomorphism from the tensor product set

$$W = \{(y, s) \mid y \in \Gamma, s \in (\zeta_-(y), \zeta_+(y))\}, \quad (2.2)$$

onto U (see [43], Chap. IV), where W is a manifold obtained as an open subset of the tensor product of Γ with the real line \mathbb{R} (note that the ‘‘open’’ character of W comes from the continuity assumption on ζ_- and ζ_+).

Remark 1. If $\Gamma \subset \bar{U}$ is taken to be the inlet boundary of the flow β (in this case, $\Gamma \subset \partial U$), the separation condition (2.1) becomes then equivalent to saying that the inflow and outflow boundaries are separated by a positive minimum distance, which is a rather classical assumption in the study of the advection operator, see e.g. Section 2.1.3 in [27]. In our case, this assumption is required in the proof of Proposition 2.

Throughout the article, the considered measure on W is that induced by the standard product measure of the surface measure dy on Γ and of the Lebesgue measure ds on \mathbb{R} . Thus, the space $L^1(W)$ of integrable functions on W is defined as the space of measurable functions $f : W \rightarrow \mathbb{R}$ such that $\int_{\Gamma} \int_{\zeta_-(y)}^{\zeta_+(y)} |f(y, s)| ds dy < +\infty$. The integral of f over W is then defined by:

$$\int_W f(y, s) ds dy := \int_{\Gamma} \int_{\zeta_-(y)}^{\zeta_+(y)} f(y, s) ds dy.$$

Under these notations, the classical change of variable formula over manifolds reads (see [43], Chap. XVI):

$$\forall f \in L^1(U), \int_U f dx = \int_W f \circ \eta |D\eta| ds dy, \quad (2.3)$$

where the Jacobian $|D\eta|$ is defined by (1.5).

Let us now consider a weight function $\omega : U \rightarrow \mathbb{R}$ involved in our variational formulation (1.9) and the following related assumptions:

(H1) $\omega \in \mathcal{C}^0(U, \mathbb{R}_+^*)$ is a positive, continuous function on U .

(H2) The transported weight $\alpha := \omega \circ \eta |D\eta| \in \mathcal{C}^0(W, \mathbb{R}_+^*)$ over W is such that the function

$$\begin{aligned} g_\alpha &: \Gamma \longrightarrow \mathbb{R}^+ \cup \{+\infty\} \\ y &\longmapsto \int_{\zeta_-(y)}^{\zeta_+(y)} \alpha(y, s) ds \end{aligned} \quad (2.4)$$

is uniformly bounded, i.e. $g_\alpha \in L^\infty(\Gamma)$.

(H3) The transported weight $\alpha = \omega \circ \eta |D\eta|$ over W is such that the function

$$\begin{aligned} h_\alpha &: \Gamma \longrightarrow \mathbb{R}^+ \cup \{+\infty\} \\ y &\longmapsto \int_{\zeta_-(y)}^{\zeta_+(y)} \left[\alpha(y, s) \int_0^s \alpha^{-1}(y, t) dt \right] ds \end{aligned} \quad (2.5)$$

is uniformly bounded: $h_\alpha \in L^\infty(\Gamma)$.

Above and in the sequel, the notations $\alpha^{-1}(y, s) = 1/\alpha(y, s)$ and $\omega^{-1}(y, s) = 1/\omega(y, s)$ are used for the inverses of the scalar weights α and ω respectively, whereas the notation $\eta^{-1} : U \mapsto W$ stands for the reciprocal mapping of the diffeomorphism η .

Hypothesis (H1) is essentially a regularity assumption, positivity being no surprise for ω to be an admissible weight in the variational formulation (1.9). Notice that no assumption is made about the behavior of ω near the boundary of U ; in particular, $\omega(x)$ may tend to 0 as x approaches ∂U . Hypothesis (H2) is an upper-boundedness assumption for α . For the practical weights that we are going to consider for our

applications in [Section 3.2](#), [\(H1\)](#) and [\(H2\)](#) are going to be satisfied almost automatically. Finally, [\(H3\)](#) is roughly a monotonicity constraint for the decay of α towards 0 as $s \rightarrow \zeta_{\pm}(y)$. In practice, we will rely on the following lemma which provides a simple monotonicity condition under which the condition [\(H3\)](#) is automatically fulfilled, indicating that the class of weights satisfying [\(H3\)](#) is large enough.

Lemma 1. *Let $\alpha \in \mathcal{C}^0(W, \mathbb{R}_+^*)$ be a weight of the product form:*

$$\forall (y, s) \in W, \alpha(y, s) = f(y, s)g(y, s) \quad (2.6)$$

where f and g are two positive functions on W satisfying:

- (i) *There exist two constants $g_-, g_+ \in \mathbb{R}$ such that for all $(y, s) \in W$, $0 < g_- \leq g(y, s) \leq g_+$.*
- (ii) *For any $y \in \Gamma$, the real function $s \mapsto f(y, s)$ is decreasing on $(0, \zeta_+(y))$ and increasing on $(\zeta_-(y), 0)$.*

Then α satisfies the condition [\(H3\)](#).

Proof. Under the above conditions, it holds, for any $y \in \Gamma$ and for $s \in (0, \zeta_+(y))$:

$$\left| \alpha(y, s) \int_0^s \alpha^{-1}(t, y) dt \right| \leq \frac{g_+}{g_-} f(y, s) \int_0^s f^{-1}(y, t) dt \leq \frac{g_+}{g_-} \zeta_+(y) \leq \frac{g_+}{g_-} \|\zeta_+ - \zeta_-\|_{L^\infty(\Gamma)}.$$

Arguing in the same fashion allows to prove a similar estimate when $s \in (\zeta_-(y), 0)$, which finally implies that $\|h_\alpha\|_{L^\infty(\Gamma)} \leq \frac{g_+}{g_-} \|\zeta_+ - \zeta_-\|_{L^\infty(\Gamma)}$; this allows to conclude. \square

Remark 2. The statement of [Lemma 1](#) does not require f or g to be continuous on W .

Remark 3. From the Liouville theorem for ordinary differential equations [\[18\]](#), it holds

$$\forall (y, s) \in W, |\mathrm{D}\eta|(y, s) = \exp\left(\int_0^s \mathrm{div}(\beta) \circ \eta(y, t) dt\right). \quad (2.7)$$

Therefore, it is straightforward to verify that [\(H1\)](#) to [\(H3\)](#) are satisfied for the constant weight $\omega = 1$ whenever $\mathrm{div}(\beta) \in L^\infty(U)$ —a customary assumption in the study of advection operators; see *e.g.* [\[11, 40, 27, 51, 14\]](#).

Our setting, based on [\(H1\)](#) to [\(H3\)](#), allows to handle more general velocity fields β , with unbounded divergence, which leaves room for the Jacobian $|\mathrm{D}\eta|$ to vanish on the boundary ∂U . This feature is crucial to deal with the shape optimization setting [\(1.7\)](#); in there, the divergence $\mathrm{div}(\beta)$ of $\beta = \nabla d_\Omega$ blows up near *centers of curvatures* $C \in \bar{\Sigma}$ (see [Figure 1a](#) and [Section 3.1](#) below) where one of the principal curvatures κ_i is such that $\kappa_i(p_{\partial\Omega}(x))d_\Omega(x) \rightarrow 1$ as $x \rightarrow C$. For instance, consider the following very simple situation where $\Omega = D$ is the unit ball in two space dimensions. Its skeleton reduces to a single point, its center. Then it is easy to see that $\Delta d_\Omega(y, s) = \frac{1}{1+s}$ for all $y \in \partial\Omega$ and $s \in (-1, 0)$, which blows up at the center for $s = -1$, so that $\mathrm{div}(\nabla d_\Omega) \notin L^\infty(D \setminus \bar{\Sigma})$. However, in this case, [\(H1\)](#) to [\(H3\)](#) hold for $\omega = 1$ since $\alpha(y, s) = 1 + s$ and $\alpha(y, s) \int_0^s \alpha^{-1}(y, t) = (1 + s) \log(1 + s)$ are uniformly bounded on $W = \partial\Omega \times (-1, 0)$.

In the following, for a weight ω satisfying [\(H1\)](#), we denote by $L_\omega^2(U)$ the weighted space

$$L_\omega^2(U) = \left\{ v \text{ measurable} \mid \int_U \omega v^2 dx < +\infty \right\}, \quad (2.8)$$

and by $\|v\|_{L_\omega^2(U)} = (\int_U \omega v^2 dx)^{1/2}$ the corresponding L^2 norm.

2.2. Definition of the graph space V_ω of the advection operator $\beta \cdot \nabla$

We recall that the assumption $\beta \in \mathcal{C}^1(U, \mathbb{R}^n)$ implies that $\mathrm{div}(\beta)$ belongs to $L_{loc}^\infty(U)$.

Definition 1 (Derivative along characteristic curves). Let $v \in L_{loc}^1(U)$ be a locally integrable function on U . The directional derivative $\beta \cdot \nabla v \in \mathcal{D}'(U)$ is the distribution on U defined by

$$\forall \phi \in \mathcal{C}_c^\infty(U), \int_U (\beta \cdot \nabla v) \phi dx = \int_U (-\beta \cdot \nabla \phi) v - \mathrm{div}(\beta) \phi v dx. \quad (2.9)$$

Remark 4. The definition [\(2.9\)](#) of $\beta \cdot \nabla$ mimics the integration by part formula that holds for functions v of class \mathcal{C}^1 . Considering test functions $\phi \in \mathcal{C}_c^\infty(U)$ allows to avoid imposing classical regularity requirements on the open domain U such as that being locally located on one side of its boundary [\[58\]](#).

Definition 2 (Graph space of the operator $\beta \cdot \nabla$). Let $\omega \in \mathcal{C}^0(U, \mathbb{R}_+^*)$ be a positive weight on U . The weighted graph space V_ω of the advection operator $\beta \cdot \nabla$ is defined by:

$$V_\omega = \{v \in L_\omega^2(U) \mid \beta \cdot \nabla v \in L_\omega^2(U)\}. \quad (2.10)$$

V_ω is a Hilbert space when it is endowed with the norm

$$\|v\|_{V_\omega} := \left(\int_D \omega v^2 dx + \int_D \omega |\beta \cdot \nabla v|^2 dx \right)^{1/2}.$$

Remark 5. The completeness of V_ω follows from very standard arguments, see e.g. [59, 27]. It is also easy to see from the assumption $\omega \in \mathcal{C}^0(U, \mathbb{R}_+^*)$ that the inclusion $L_\omega^2(U) \subset L_{loc}^1(U)$ holds.

Let us then introduce a corresponding notion of derivative with respect to the s variable and graph space on the set W :

Definition 3 (Graph space of the derivative ∂_s). Let $\alpha \in \mathcal{C}^0(W, \mathbb{R}_+^*)$ be a positive weight on W . The weak derivative $\partial_s \tilde{v} \in \mathcal{D}'(W)$ of a function $\tilde{v} \in L_{loc}^1(W)$ is the distribution defined by

$$\forall \phi \in \mathcal{C}_c^\infty(W), \int_W \partial_s \tilde{v} \phi dy ds = - \int_W \tilde{v} \partial_s \phi dy ds. \quad (2.11)$$

The weighted graph space of the operator ∂_s is defined by:

$$\tilde{V}_\alpha = \{\tilde{v} \in L_\alpha^2(W) \mid \partial_s \tilde{v} \in L_\alpha^2(W)\}. \quad (2.12)$$

This space is a Hilbert space when it is equipped with the norm

$$\|\tilde{v}\|_{\tilde{V}_\alpha} := \left(\int_W \alpha \tilde{v}^2 ds dy + \int_W \alpha |\partial_s \tilde{v}|^2 ds dy \right)^{1/2}.$$

Remark 6. The defining identity (2.9) of the weak derivative $\beta \cdot \nabla v$ also holds for test functions $\phi \in C_c^1(U)$, as follows from a standard density argument. Likewise, (2.11) holds for test functions ϕ which are only continuously differentiable with respect to the s variable.

The motivation behind the introduction of the spaces \tilde{V}_α is that they allow to transfer the difficulty of studying the ‘‘curvilinear’’ directional derivative $\beta \cdot \nabla$ over the domain U , to that of the more standard, ‘‘flat’’ derivative ∂_s over W . The price to pay is the need to account for the weight of the Jacobian $|\mathbf{D}\eta|$ resulting from the change of variables. This point is made precise by the following technical lemma.

Lemma 2. *Let $\omega \in \mathcal{C}^0(U, \mathbb{R}_+^*)$ be a positive weight on U , and let $\alpha = \omega \circ \eta |\mathbf{D}\eta|$. The following equivalence holds true:*

$$v \in V_\omega \Leftrightarrow v \circ \eta \in \tilde{V}_\alpha.$$

Additionally, for any $v \in V_\omega$, the function $\tilde{v} = v \circ \eta$ satisfies:

- (i) $\partial_s \tilde{v} = (\beta \cdot \nabla v) \circ \eta$,
- (ii) $\|v\|_{V_\omega} = \|v \circ \eta\|_{\tilde{V}_\alpha}$.

Proof. Let $v \in V_\omega$; it follows from the change of variables (2.3) that $\tilde{v} := v \circ \eta$ belongs to $L_\alpha^2(W)$. Using the definition (2.9) of the weak derivative along β with a test function $\phi \in \mathcal{C}_c^1(U)$ of the form $\phi = \tilde{\phi} \circ \eta^{-1}$ for an arbitrary $\tilde{\phi} \in \mathcal{C}_c^1(W)$ (see Remark 6), then using (2.3), we obtain:

$$\forall \tilde{\phi} \in \mathcal{C}_c^1(W), \int_W (\beta \cdot \nabla v) \circ \eta \tilde{\phi} |\mathbf{D}\eta| ds dy = \int_W (-\partial_s \tilde{\phi} \tilde{v} - \operatorname{div}(\beta) \circ \eta \tilde{\phi} \tilde{v}) |\mathbf{D}\eta| ds dy, \quad (2.13)$$

where we have used the equality $\partial_s \tilde{\phi} = (\beta \cdot \nabla \phi) \circ \eta$. Now using that $|\mathbf{D}\eta|$ is continuously differentiable with respect to s with $\partial_s |\mathbf{D}\eta| = \operatorname{div}(\beta) \circ \eta |\mathbf{D}\eta|$, as a consequence of the Liouville formula (2.7), (2.13) rewrites:

$$\forall \tilde{\phi} \in \mathcal{C}_c^1(W), \int_W (\beta \cdot \nabla v) \circ \eta \tilde{\phi} |\mathbf{D}\eta| ds dy = \int_W (-\partial_s \tilde{\phi} \tilde{v} |\mathbf{D}\eta| - \partial_s |\mathbf{D}\eta| \tilde{\phi} \tilde{v}) ds dy = - \int_W \tilde{v} \partial_s (\tilde{\phi} |\mathbf{D}\eta|) ds dy.$$

By a standard density argument, the above equality holds more generally for functions $\tilde{\phi}$ that are only continuously differentiable with respect to s on W . Therefore, taking $\tilde{\phi} = \tilde{\psi} / |\mathbf{D}\eta|$ as a test function in

the above equation for arbitrary $\tilde{\psi} \in \mathcal{C}_c^1(W)$, it follows that $\tilde{v} = v \circ \eta$ has a weak derivative $\partial_s \tilde{v}$ given by $\partial_s \tilde{v} = (\beta \cdot \nabla v) \circ \eta$. Then the change of variables (2.3) yields $\|v\|_{V_\omega} = \|v \circ \eta\|_{\tilde{V}_\alpha}$, which implies $v \circ \eta \in \tilde{V}_\alpha$.

Conversely, one proves in a similar way that if \tilde{v} is in \tilde{V}_α , the function $v = \tilde{v} \circ \eta^{-1}$ belongs to V_ω , which terminates the proof. \square

Remark 7. In the following we prove a density result, a trace theorem and a Poincaré inequality in \tilde{V}_α in order to obtain the direct transposition of these results to the graph space V_ω thanks to the above Lemma 2. There exists actually a wide literature about weighted Sobolev spaces such as \tilde{V}_α [33, 41, 42, 59], in which analogous results are proved for weights α in the so-called Muckenhoupt class A_2 (see e.g. [29] for a definition of the class A_p). Our working assumptions however are of a different nature: for instance, the hypothesis (H3) essentially requires that the inverse weight α^{-1} belong to the Muckenhoupt class A_1 .

2.3. Density of functions of class \mathcal{C}^1 in the weighted space V_ω

We now examine the density of $\mathcal{C}^\infty(W) \cap \tilde{V}_\alpha$ in \tilde{V}_α , whence we shall infer the density of $\mathcal{C}^1(U) \cap V_\omega$ in V_ω —the loss of regularity between both statements coming from the fact that the coordinate change η is only of class \mathcal{C}^1 on W .

Our study classically involves mollifying functions; since the space \tilde{V}_α of interest contains functions defined on the manifold W , a little treatment is in order. In particular, we shall need the so-called *Ahlfors regularity* of Γ (see [24]); this is the purpose of the next lemma, whose proof is outlined for the convenience of the reader. Here and throughout the article, $B(y, h) \subset \mathbb{R}^n$ stands for the ball with center y and radius h .

Lemma 3 (Area covered by extrinsic balls on an embedded manifold). *Let $\Gamma \subset \mathbb{R}^n$ be a \mathcal{C}^1 hypersurface such that either Γ is compact ($\Gamma = \bar{\Gamma}$) or $\bar{\Gamma}$ is a compact manifold with boundary. Then there exists $h_0 > 0$ and constants $m > 0$ and $M > 0$ depending only on Γ such that for any $0 < h < h_0$,*

$$\forall y \in \Gamma, \quad mh^{n-1} \leq |\Gamma \cap B(y, h)| \leq Mh^{n-1}. \quad (2.14)$$

Proof. Since $\bar{\Gamma}$ is compact, there exist finitely many open subset $V_i \subset \mathbb{R}^n$, $i = 1, \dots, N$, such that $\bar{\Gamma} \subset \bigcup_i V_i$, and for each $i = 1, \dots, N$, there exists a local coordinate chart $\phi_i : U_i \subset \mathbb{R}^{n-1} \rightarrow \phi_i(U_i) \subset \bar{\Gamma}$ such that $\phi_i(U_i) = V_i \cap \bar{\Gamma}$. The set U_i is a convex open subset of \mathbb{R}^{n-1} if $V_i \cap \partial\Gamma = \emptyset$; when the latter intersection is not empty, U_i is of the form $U_i = \tilde{U}_i \cap \mathbb{R}_+^{n-1}$, where $\tilde{U}_i \subset \mathbb{R}^{n-1}$ is a convex open subset, and \mathbb{R}_+^{n-1} is the upper half-space of \mathbb{R}^{n-1} . Let $h_0 > 0$ be a Lebesgue's number associated with this cover, that is:

$$\forall y \in \Gamma, \exists i \in \{1, \dots, N\}, \quad B(y, h_0) \subset \subset V_i. \quad (2.15)$$

Now, given $y \in V_i$, one has for any $0 < h < h_0$:

$$|\Gamma \cap B(y, h)| = \int_\Gamma \mathbf{1}_{B(y, h)} dz = \int_{\phi_i(U_i)} \mathbf{1}_{B(y, h)} dz = \int_{U_i} \mathbf{1}_{\phi_i^{-1}(B(y, h))} |\mathrm{D}\phi_i| dx,$$

where i is the index supplied by (2.15) and $\mathbf{1}_A$ denotes the characteristic function of a subset $A \subset \mathbb{R}^n$.

Let $\sigma_{n-1}(\nabla\phi_i(x)) \leq \sigma_1(\nabla\phi_i(x))$ be respectively the smallest and the largest singular values of the $n \times (n-1)$ Jacobian matrix $\nabla\phi_i(x)$, and let σ_- and σ_+ be defined by:

$$\sigma_- = \min_{i=1, \dots, N} \inf_{x \in U_i} \sigma_{n-1}(\nabla\phi_i(x)), \quad \sigma_+ = \max_{i=1, \dots, N} \sup_{x \in U_i} \sigma_1(\nabla\phi_i(x)).$$

Applying the Taylor formula to ϕ_i yields:

$$\forall i = 1, \dots, N, \quad \forall (x_0, x_1) \in U_i, \quad \sigma_- \|x_1 - x_0\| \leq \|\phi_i(x_1) - \phi_i(x_0)\| \leq \sigma_+ \|x_1 - x_0\|.$$

Therefore, possibly shrinking the value of the constant h_0 supplied by (2.15) and taking $x_0 = \phi_i^{-1}(y)$, we obtain:

$$\forall 0 < h < h_0, \quad B\left(x_0, \frac{h}{\sigma_+}\right) \subset \phi_i^{-1}(B(y, h)) \subset B\left(x_0, \frac{h}{\sigma_-}\right) \subset U_i.$$

Finally, denoting by $|B_{n-1}|$ the volume of the unit ball in \mathbb{R}^{n-1} , we obtain:

$$\begin{aligned} \frac{1}{2}|B_{n-1}| \left(\frac{\sigma_-}{\sigma_+}\right)^{n-1} h^{n-1} &\leq \int_{U_i} \mathbf{1}_{B(x_0, h/\sigma_+)} |D\phi_i| dx \\ &\leq \int_{U_i} \mathbf{1}_{\phi_i^{-1}(B(y, h))} |D\phi_i| dx = |\Gamma \cap B(y, h)| \\ &\leq \int_{U_i} \mathbf{1}_{B(x_0, h/\sigma_-)} |D\phi_i| dx \leq |B_{n-1}| \left(\frac{\sigma_+}{\sigma_-}\right)^{n-1} h^{n-1}, \end{aligned}$$

which completes the proof. \square

The purpose of the next lemma is to construct the kernels ρ_h and ξ_h which shall be used for mollification purposes in W .

Lemma 4. *For any $h > 0$, there exist two positive, smooth functions $\rho_h \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ and $\xi_h \in C_c^\infty(\mathbb{R}, \mathbb{R})$ satisfying the conditions:*

- (i) *For all points $y \in \mathbb{R}^n$, $\text{supp}(\rho_h(y, \cdot)) \subset B(y, h)$.*
- (ii) *For any $y \in \mathbb{R}^n$, $\int_\Gamma \rho_h(y, z) dz = 1$.*
- (iii) *There exist constants $C > 0$ and $h_0 > 0$ depending only on Γ such that*

$$\forall 0 < h < h_0, \forall z \in \Gamma, \int_\Gamma \rho_h(y, z) dy \leq C. \quad (2.16)$$

- (iv) *$\text{supp}(\xi_h) \subset [-h, h]$ and $\int_{\mathbb{R}} \xi_h(s) ds = 1$.*

Proof. Let $\rho \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ be a smooth, positive function such that $\text{supp}(\rho) \subset B(0, 1)$, $\rho \leq 1$ on \mathbb{R}^n and $\rho|_{B(0, 1/2)} = 1$; we define ρ_h by:

$$\forall h > 0, \forall (y, z) \in \mathbb{R}^n \times \mathbb{R}^n, \rho_h(y, z) := \frac{\rho\left(\frac{y-z}{h}\right)}{\int_\Gamma \rho\left(\frac{y-z}{h}\right) dz}; \quad (2.17)$$

note that $\rho_h(y, z)$ is not a function of $(y - z)$. The conditions (i) and (ii) of the statement are obviously satisfied by (2.17). The condition (iii) is a consequence of Lemma 3 whence the following inequalities are inferred:

$$\begin{aligned} \forall 0 < h < h_0, \forall y \in \Gamma, \int_\Gamma \rho\left(\frac{y-z}{h}\right) dz &\geq \left| \Gamma \cap B\left(y, \frac{h}{2}\right) \right| \geq m \left(\frac{h}{2}\right)^{n-1}, \\ \forall 0 < h < h_0, \forall z \in \Gamma, \int_\Gamma \rho\left(\frac{y-z}{h}\right) dy &\leq |\Gamma \cap B(z, h)| < Mh^{n-1}. \end{aligned}$$

Therefore, (2.16) holds with $C = 2^{n-1}M/m$.

Eventually, a function ξ_h satisfying (iv) is constructed from any positive function $\xi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ with compact support inside $[-1, 1]$ and unit integral over \mathbb{R} , by setting $\xi_h = h^{-1}\xi(\cdot/h)$. \square

Definition 4 (Mollification on the tensor product manifold W). For $h > 0$, let ρ_h and ξ_h be two functions as in the statement of Lemma 4. For any $u \in L_{loc}^1(W)$, $(y, s) \in W$ and $h > 0$ sufficiently small (depending on (y, s)), the mollification of u is the function $u_h = \rho_h \xi_h * u$ defined by

$$(\rho_h \xi_h * u)(y, s) = \int_\Gamma \int_{\mathbb{R}} \rho_h(y, z) \xi_h(s - t) u(z, t) dt dz. \quad (2.18)$$

Note that for a given open subdomain $W_1 \subset\subset W$ and because $\bar{\Gamma}$ is compact, there exists $h_0 > 0$, depending on W_1 , sufficiently small so that (2.18) makes sense for any $(y, s) \in W_1$ and $0 < h < h_0$.

Lemma 5. *The following properties hold true:*

- (i) *If $u \in L_{loc}^1(W)$, for any subdomain $W_1 \subset\subset W$ and for $h > 0$ sufficiently small, the convolution $\rho_h \xi_h * u$ is of class C^∞ on W_1 .*
- (ii) *If $u \in \tilde{V}_\alpha$, then for any subdomain $W_1 \subset\subset W$ and for h sufficiently small, $\partial_s(\rho_h \xi_h * u) = \rho_h \xi_h * \partial_s u$.*
- (iii) *If $\phi \in C^0(W, \mathbb{R})$ then $\rho_h \xi_h * \phi \rightarrow \phi$ in $L_{loc}^\infty(W)$ as $h \rightarrow 0$.*

Proof.

- (i) This results from the Lebesgue dominated convergence theorem and the smoothness of ρ_h and ξ_h .
- (ii) This is again a consequence of the Lebesgue dominated theorem and of an integration by parts:

$$\begin{aligned} \forall (y, s) \in W_1, (\rho_h \xi_h * \partial_s u)(y, s) &= \int_{\Gamma} \int_{\mathbb{R}} \rho_h(y, z) \xi_h(s-t) \partial_s u(z, t) dt dz \\ &= \int_{\Gamma} \int_{\mathbb{R}} \rho_h(y, z) \partial_s \xi_h(s-t) u(z, t) dt dz = \partial_s (\rho_h \xi_h * u). \end{aligned}$$

- (iii) For a given subset $W_1 \subset\subset W$, let $\epsilon > 0$ and $h_1 > 0$ be uniform continuity constants for ϕ on W_1 , i.e.

$$\forall 0 < h < h_1, (y, s) \in W_1, (z, t) \in W_1, (||y - z|| < h \text{ and } |s - t| < h) \Rightarrow |\phi(z, t) - \phi(y, s)| < \epsilon.$$

Then for any $0 < h < h_1$:

$$\begin{aligned} \forall (y, s) \in W_1, |\rho_h \xi_h * \phi(y, s) - \phi(y, s)| &= \left| \int_{\Gamma} \int_{\mathbb{R}} \rho_h(y, z) \xi_h(s-t) (\phi(z, t) - \phi(y, s)) dt dz \right| \\ &\leq \epsilon \int_{\Gamma} \int_{\mathbb{R}} \rho_h(y, z) \xi_h(s-t) dt dz = \epsilon. \end{aligned}$$

□

Proposition 1. *Let $\alpha \in \mathcal{C}^0(W, \mathbb{R}_+^*)$. Then for any function $u \in L_{\alpha, \text{loc}}^2(W)$, $u_h = \rho_h \xi_h * u \in L_{\alpha, \text{loc}}^2(W) \cap \mathcal{C}^\infty(W)$ and $u_h \rightarrow u$ in $L_{\alpha, \text{loc}}^2(W)$.*

Proof. We adapt the proof of Prop. 1.18, p.30 in [52]. Let $W_1 \subset\subset W$ be an open subset of W and consider another open subset W_2 such that $W_1 \subset\subset W_2 \subset\subset W$. For $\epsilon > 0$, let $\phi \in \mathcal{C}^0(W)$ be such that (see [53]):

$$||u - \phi||_{L_{\alpha}^2(W_2)} < \epsilon. \quad (2.19)$$

From Lemma 5, $\rho_h \xi_h * \phi \rightarrow \phi$ in $L^\infty(W_1)$ as $h \rightarrow 0$ and so, for h small enough, $||\rho_h \xi_h * \phi - \phi||_{L_{\alpha}^2(W_1)} \leq \epsilon$. Then,

$$||u - u_h||_{L_{\alpha}^2(W_1)} \leq ||u - \phi||_{L_{\alpha}^2(W_1)} + ||\phi - \rho_h \xi_h * \phi||_{L_{\alpha}^2(W_1)} + ||\rho_h \xi_h * \phi - \rho_h \xi_h * u||_{L_{\alpha}^2(W_1)}. \quad (2.20)$$

The first two terms in the right-hand side of (2.20) are controlled by ϵ owing to the previous discussion; as for the last term, we get from the Cauchy-Schwarz inequality:

$$\begin{aligned} \forall (y, s) \in W_1, |\rho_h \xi_h * (\phi - u)(y, s)|^2 &= \left| \int_{\Gamma} \int_{\mathbb{R}} \rho_h(y, z) \xi_h(s-t) (\phi(z, t) - u(z, t)) dt dz \right|^2 \\ &\leq \left(\int_{\Gamma} \int_{\mathbb{R}} \rho_h(y, z) \xi_h(s-t) \alpha^{-1}(z, t) dt dz \right) \left(\int_{\Gamma} \int_{\mathbb{R}} \rho_h(y, z) \xi_h(s-t) \alpha(z, t) |\phi(z, t) - u(z, t)|^2 dt dz \right). \end{aligned}$$

Multiplying both sides by α and integrating over W_1 yields:

$$\begin{aligned} ||\rho_h \xi_h * (\phi - u)||_{L_{\alpha}^2(W_1)}^2 &\leq ||\alpha(\rho_h \xi_h * \alpha^{-1})||_{L^\infty(W_1)} \int_{\Gamma} \int_{\mathbb{R}} \int_{\Gamma} \int_{\mathbb{R}} \rho_h(y, z) \xi_h(s-t) \alpha(z, t) |\phi(z, t) - u(z, t)|^2 dt dz ds dy \\ &\leq C ||\alpha(\rho_h \xi_h * \alpha^{-1})||_{L^\infty(W_1)} ||\phi - u||_{L_{\alpha}^2(W_2)}^2, \end{aligned}$$

where C is the constant supplied by the condition (iii) in the statement of Lemma 4. By assumption, α^{-1} is a continuous function on W_2 , which implies by Lemma 5 that $\alpha(\rho_h \xi_h * \alpha^{-1}) \rightarrow 1$ in $L^\infty(W_1)$. Hence $||\alpha(\rho_h \xi_h * \alpha^{-1})||_{L^\infty(W_1)}$ is bounded by some constant C' . Finally, using (2.19), we obtain from (2.20) that for $h > 0$ small enough:

$$||u - u_h||_{L_{\alpha}^2(W_1)} \leq (CC' + 2)\epsilon,$$

which is the desired result. □

We conclude this subsection with the desired density result of \mathcal{C}^1 functions in V_ω :

Corollary 1.

- (i) *Let $\alpha \in \mathcal{C}^0(W, \mathbb{R}_+^*)$ be a positive weight on W ; the space $\mathcal{C}^\infty(W) \cap \tilde{V}_\alpha$ is dense in \tilde{V}_α .*

(ii) Let $\omega \in \mathcal{C}^0(U, \mathbb{R}_+^*)$ be a positive weight on U ; the space $\mathcal{C}^1(U) \cap V_\omega$ is dense in V_ω .

Proof. The proof of the density (i) of $\mathcal{C}^\infty(W) \cap \tilde{V}_\alpha$ in \tilde{V}_α relies on a partition of unity argument and on the properties of [Lemma 5](#) and [Proposition 1](#), exactly along the lines of the proof of Theorem 5.15 in [52], to which the reader is referred for details.

The density (ii) of $\mathcal{C}^1(U)$ in V_ω follows then from the density of $\mathcal{C}^\infty(W) \cap \tilde{V}_\alpha$ in \tilde{V}_α with $\alpha = \omega \circ \eta|D\eta|$ and by composition with the \mathcal{C}^1 diffeomorphism η . \square

Remark 8. [Corollary 1](#) does not preclude anything regarding the density of $\mathcal{C}^1(\bar{U})$ in V_ω . A result of this kind would require careful regularity assumptions on ∂U and on the behavior of ω near ∂U .

2.4. Trace theorem and Poincaré inequality in V_ω

In this section, the trace operator on Γ is defined and studied for functions in the weighted space V_ω , or equivalently for functions in \tilde{V}_α on $\eta(\Gamma \times \{0\}) = \{(y, 0) \mid y \in \Gamma\}$. In the sequel, with a little abuse of notations, the latter set $\eta(\Gamma \times \{0\})$ is identified with Γ .

Proposition 2 (Trace theorem). *Let $\omega \in \mathcal{C}^0(U, \mathbb{R}_+^*)$ be a positive weight on U . The trace operator*

$$\begin{aligned} \gamma &: \mathcal{C}^1(U) &\rightarrow & L^2(\Gamma) \\ v &\mapsto \gamma(v) &= & v|_\Gamma \end{aligned} \tag{2.21}$$

induces a bounded operator $V_\omega \rightarrow L^2(\Gamma)$; there exists a constant $C > 0$ (possibly depending on the weight ω) such that

$$\forall v \in V_\omega, \|\gamma(v)\|_{L^2(\Gamma)} \leq C\|v\|_{V_\omega}. \tag{2.22}$$

Proof. Introducing $\alpha = \omega \circ \eta|D\eta|$, using the change of variables [\(2.3\)](#) and the density result of [Corollary 1](#), it is enough to prove that there exists a constant $C > 0$ such that:

$$\forall \tilde{v} \in \mathcal{C}^\infty(W) \cap \tilde{V}_\alpha, \|\gamma(\tilde{v})\|_{L^2(\Gamma)} \leq C\|\tilde{v}\|_{\tilde{V}_\alpha}. \tag{2.23}$$

Let us consider the following partition of Γ : $\Gamma = \Gamma_+ \cup \Gamma_-$, where $\Gamma_+ = \{y \in \Gamma \mid \zeta_-(y) \geq -\epsilon/2\}$, $\Gamma_- = \Gamma \setminus \Gamma_+$, and ϵ is the parameter featured in separation condition [\(2.1\)](#). Then

$$\begin{aligned} \forall y \in \Gamma_+, & 0 < \epsilon/2 < \zeta_+(y), \\ \forall y \in \Gamma_-, & \zeta_-(y) < -\epsilon/2 < 0. \end{aligned}$$

Now, let $a = \epsilon/2$ and $\xi \in \mathcal{C}_c^\infty(\mathbb{R})$ be such that $\xi(0) = 1$ and $\xi(-a) = \xi(a) = 0$. Let K be the bounded measurable set defined by $K := (\Gamma_- \times [-a, 0]) \cup (\Gamma_+ \times [0, a]) \subset W$. Then for any function $\tilde{v} \in \mathcal{C}^\infty(W) \cap \tilde{V}_\alpha$, it holds:

$$\begin{aligned} \int_\Gamma |\tilde{v}(y, 0)|^2 dy &= \int_\Gamma |\tilde{v}(y, 0)|^2 \xi(0) dy = \int_{\Gamma_-} \int_{-a}^0 \partial_s(\tilde{v}^2 \xi) ds dy + \int_{\Gamma_+} \int_a^0 \partial_s(\tilde{v}^2 \xi) ds dy \\ &= \int_K (2\tilde{v} \partial_s \tilde{v} \xi + \tilde{v}^2 \partial_s \xi) ds dy \\ &\leq 2\|\alpha^{-1} \xi\|_{L^\infty(K)} \|\tilde{v}\|_{L^2_\alpha(W)} \|\partial_s \tilde{v}\|_{L^2_\alpha(W)} + \|\alpha^{-1} \partial_s \xi\|_{L^\infty(K)} \|\tilde{v}\|_{L^2_\alpha(W)}^2 \\ &\leq (2\|\alpha^{-1} \xi\|_{L^\infty(K)} + \|\alpha^{-1} \partial_s \xi\|_{L^\infty(K)}) \|\tilde{v}\|_{\tilde{V}_\alpha}^2, \end{aligned}$$

which implies [\(2.23\)](#) and therefore terminates the proof of [Proposition 2](#). \square

Remark 9. The proof of [Proposition 2](#) supplies the existence of the trace on Γ of an arbitrary function $\tilde{v} \in \tilde{V}_\alpha$, which we shall also denote by $\tilde{v}|_\Gamma$.

For later purposes (see [Section 2.5](#)), we shall need the surjectivity of the above trace operator; this is the purpose of the next proposition.

Proposition 3 (Surjectivity of traces). *Let $\omega \in \mathcal{C}^0(U, \mathbb{R}_+^*)$ be a positive continuous weight.*

(i) *The trace operator defined by [\(2.21\)](#) is surjective from V_ω onto $L^2(\Gamma)$:*

$$L^2(\Gamma) = \{v|_\Gamma \mid v \in V_\omega\}.$$

(ii) If ω additionally satisfies (H2), then any function $v_0 \in L^2(\Gamma)$ can be extended constantly along the characteristics of β : there exists $v \in V_\omega$ such that $v|_\Gamma = v_0$ and $\beta \cdot \nabla v = 0$.

Proof.

(i) We rather prove that $L^2(\Gamma) = \{\tilde{v}|_\Gamma \mid \tilde{v} \in \tilde{V}_\alpha\}$, where $\alpha = \omega \circ \eta |D\eta|$; see Remark 9. To this end, consider ξ and K be as in the proof of Proposition 2. For an arbitrary function $v_0 \in L^2(\Gamma)$, we define \tilde{v} by the formula

$$\tilde{v}(y, s) := v_0(y)\xi(s) \text{ a.e. in } W.$$

Obviously, $\tilde{v}(y, 0) = v_0(y)$ and $\partial_s \tilde{v}(y, s) = v_0(y)\partial_s \xi(s)$, whence the Cauchy-Schwarz inequality yields:

$$\int_W \alpha \tilde{v}^2 ds dy = \int_\Gamma \int_{\zeta_-(y)}^{\zeta_+(y)} \alpha(y, s) \xi(s)^2 v_0(y)^2 ds dy < \|\alpha\|_{L^\infty(K)} \|\xi\|_{L^2(\mathbb{R})} \|v_0\|_{L^2(\Gamma)}^2 < +\infty,$$

$$\int_W \alpha |\partial_s \tilde{v}|^2 ds dy = \int_\Gamma \int_{\zeta_-(y)}^{\zeta_+(y)} \alpha(y, s) v_0(y)^2 |\partial_s \xi(s)|^2 ds dy \leq \|\alpha\|_{L^\infty(K)} \|\partial_s \xi\|_{L^2(\mathbb{R})} \|v_0\|_{L^2(\Gamma)}^2 < +\infty.$$

Hence \tilde{v} is a function in \tilde{V}_α such that $\tilde{v}|_\Gamma = v_0$, which is the desired result.

(ii) If (H2) is satisfied, then for an arbitrary function $v_0 \in L^2(\Gamma)$, we simply define \tilde{v} by the formula:

$$\tilde{v}(y, s) := v_0(y) \text{ a.e. in } W.$$

Then, clearly $\partial_s \tilde{v} = 0$; what's more, one has $\tilde{v} \in L^2_\alpha(W)$ as follows from the following estimate:

$$\int_W \alpha \tilde{v}^2 ds dy = \int_\Gamma \int_{\zeta_-(y)}^{\zeta_+(y)} \alpha(y, s) v_0(y)^2 ds dy \leq \|g_\alpha\|_{L^\infty(\Gamma)} \|v_0\|_{L^2(\Gamma)}^2, \quad (2.24)$$

where g_α is as in the statement of (H2). Hence the function \tilde{v} belongs to \tilde{V}_α , and so $v := \tilde{v} \circ \eta^{-1}$ is an element of V_ω which has the desired properties owing to Lemma 2. \square

We now prove a Poincaré-type inequality in the spaces V_ω under the additional assumptions (H2) and (H3) about the weight ω .

Proposition 4 (Poincaré type inequality on V_ω). *Let $\omega \in \mathcal{C}^0(U, \mathbb{R}_+^*)$ be a weight satisfying the assumptions (H1) to (H3). Then there exists a constant $C > 0$ (depending on ω) such that:*

$$\forall v \in V_\omega, \int_U \omega v^2 dx \leq C \left[\int_\Gamma v^2 dy + \int_U \omega |\beta \cdot \nabla v|^2 dx \right]. \quad (2.25)$$

Proof. Introducing again $\alpha = \omega \circ \eta |D\eta| \in \mathcal{C}^0(W, \mathbb{R}_+^*)$ and using the change of variables (2.3), we rather prove the analogous Poincaré inequality in W , that is:

$$\forall \tilde{v} \in \tilde{V}_\alpha, \int_W \alpha \tilde{v}^2 ds dy \leq C \left[\int_\Gamma \tilde{v}^2 ds + \int_W \alpha |\partial_s \tilde{v}|^2 ds dy \right]. \quad (2.26)$$

Furthermore, since $\mathcal{C}^\infty(W) \cap \tilde{V}_\alpha$ is dense in \tilde{V}_α , it is enough to prove that (2.26) holds for $\tilde{v} \in \mathcal{C}^\infty(W) \cap \tilde{V}_\alpha$, which we now do. To this end, for arbitrary $\tilde{v} \in \mathcal{C}^\infty(W) \cap \tilde{V}_\alpha$, a use of Taylor's formula yields:

$$\forall (y, s) \in W, \tilde{v}(y, s) = \tilde{v}(y, 0) + \int_0^s \partial_s \tilde{v}(y, t) dt. \quad (2.27)$$

In (2.27), the Cauchy-Schwarz inequality implies that, for $(y, s) \in W$:

$$\int_0^s |\partial_s \tilde{v}(y, t)| dt \leq \left(\int_0^s \alpha^{-1}(y, t) dt \right)^{1/2} \left(\int_{\zeta_-(y)}^{\zeta_+(y)} \alpha |\partial_s \tilde{v}|^2(y, t) dt \right)^{1/2}. \quad (2.28)$$

Now multiplying both sides of (2.27) by α and using the Young inequality ($\forall a, b \in \mathbb{R}, (a + b)^2 \leq 2a^2 + 2b^2$) together with (2.28), we obtain:

$$\forall (y, s) \in W, \alpha(y, s) |\tilde{v}(y, s)|^2 \leq 2\alpha(y, s) |\tilde{v}(y, 0)|^2 + 2\alpha(y, s) \int_0^s \alpha^{-1}(y, t) dt \int_{\zeta_-(y)}^{\zeta_+(y)} \alpha(y, t) |\partial_s \tilde{v}|^2(y, t) dt. \quad (2.29)$$

Integrating (2.29) over W now results in:

$$\begin{aligned} \int_W \alpha \tilde{v}^2 ds dy &\leq 2 \int_{\Gamma} \tilde{v}^2(y, 0) \left(\int_{\zeta_-(y)}^{\zeta_+(y)} \alpha(y, s) ds \right) dy + 2 \int_{\Gamma} h_{\alpha}(y) \left(\int_{\zeta_-(y)}^{\zeta_+(y)} \alpha |\partial_s \tilde{v}|^2 ds \right) dy \\ &\leq 2 \|g_{\alpha}\|_{L^{\infty}(\Gamma)} \int_{\Gamma} \tilde{v}^2 dy + 2 \|h_{\alpha}\|_{L^{\infty}(\Gamma)} \int_W \alpha |\partial_s \tilde{v}|^2 ds dy, \end{aligned}$$

where g_{α} and h_{α} are the functions featured in (H2) and (H3). This completes the proof of (2.26), and so that of Proposition 4. \square

Remark 10. This Poincaré type inequality is close in spirit to the “curvilinear” Poincaré inequality of Azerad [11], who considered the weight $\omega = 1$ and vector fields β satisfying $\operatorname{div} \beta = 0$.

2.5. Well-posedness of the variational problem (1.9)

We are now in a position to state and prove the main result of this section.

Proposition 5. *Let $\omega \in C^0(U, \mathbb{R}_+^*)$ be a positive weight on U , satisfying the assumptions (H1) to (H3). Then:*

(i) *For any function $f \in L_{\omega^{-1}}^2(U)$, there exists a unique solution $u \in V_{\omega}$ to the variational problem*

$$\text{Find } u \in V_{\omega} \text{ such that } \forall v \in V_{\omega}, \int_{\Gamma} uv ds + \int_U \omega(\beta \cdot \nabla u)(\beta \cdot \nabla v) dx = \int_U f v dx. \quad (2.30)$$

(ii) *The trace $u|_{\Gamma} \in L^2(\Gamma)$ of the solution u to (2.30), is independent of the weight ω as long as it fulfills (H1) to (H3); it is given by:*

$$u(y) = \int_{\zeta_-(y)}^{\zeta_+(y)} f \circ \eta |D\eta| ds \quad \text{a.e. on } \Gamma. \quad (2.31)$$

Proof.

(i) The assumption $f \in L_{\omega^{-1}}^2(U)$ ensures that $v \mapsto \int_U f v dx$ is a continuous linear form on V_{ω} owing to the Cauchy-Schwartz inequality:

$$\forall v \in V_{\omega}, \left| \int_U f v dx \right| \leq \|f\|_{L_{\omega^{-1}}^2(U)} \|v\|_{L_{\omega}^2(U)}.$$

Moreover, the Poincaré inequality of Proposition 4 ensures the coercivity on V_{ω} of the bilinear form

$$(u, v) \mapsto \int_{\Gamma} uv dy + \int_U \omega(\beta \cdot \nabla u)(\beta \cdot \nabla v) dx.$$

Hence, the classical Lax-Milgram theorem (see e.g. [31]) yields the existence and uniqueness of a solution $u \in V_{\omega}$ to (2.30).

(ii) Since ω satisfies (H2), it follows from Proposition 3 (ii) that for any $v_0 \in C^0(\Gamma)$, there exists a function $v \in V_{\omega}$ such that $v|_{\Gamma} = v_0$ and $\beta \cdot \nabla v = 0$. Taking v as an admissible test function in (2.30) then yields:

$$\forall v_0 \in C^0(\Gamma), \int_{\Gamma} uv_0 dy = \int_U f v dx = \int_{\Gamma} v_0(y) \left(\int_{\zeta_-(y)}^{\zeta_+(y)} f \circ \eta |D\eta| ds \right) dy, \quad (2.32)$$

whence (2.31). \square

Remark 11. Problem (1.9) or (2.30) can be solved, for an arbitrary choice of weight ω satisfying (H1) to (H3), if the right hand side satisfies $f \in L_{\omega^{-1}}^2(U)$. This holds true, for example when f belongs to $L^{\infty}(U)$, as soon as the weight ω satisfies: $\omega^{-1} \in L^1(U)$. This property is not a consequence of (H1) to (H3) as explained in Remark 14.

Remark 12. If (H2) is not satisfied, which is the case if for example ω blows up “too fast” near some part of the boundary of U , functions $v \in V_{\omega}$ are expected to vanish near this part of ∂U and then V_{ω} may not contain all functions which are constant along the characteristic curves of β .

3. NUMERICAL METHODS FOR INTEGRATION ALONG NORMAL RAYS

We now instantiate the general setting of [Section 2](#) to the shape optimization context [\(1.7\)](#) outlined in the introduction. As we have mentioned, it relies heavily on the notion of signed distance function, which we briefly recall in the next [Section 3.1](#) for the convenience of the reader. We then discuss in [Section 3.2](#) the construction of suitable weights ω that allow to solve accurately the variational problem [\(1.9\)](#) by means of \mathbb{P}_1 conforming finite elements. After clarifying some of the algorithmic stages required by the direct integration of [\(1.6\)](#) along rays in [Section 3.3](#)—such as the numerical computation of the principal curvatures κ_i of the considered shapes, and the detection of their skeleton on unstructured meshes—we perform several experiments as far as the choice of the weight ω in [\(1.9\)](#) is concerned in [Section 3.4](#). Finally, in [Section 3.5](#), we compare on several 2-d analytical examples the numerical calculations of [\(1.6\)](#) by means of our variational formulation [\(1.9\)](#) with those produced by direct integration of this formula along rays.

3.1. A short reminder about the signed distance function

In this section, we consider a fixed, bounded and Lipschitz ‘hold-all’ open domain $D \subset \mathbb{R}^n$, as well as a bounded Lipschitz open subdomain $\Omega \subset D$. We recall the main definitions and properties of the signed distance function that will be relevant for the shape optimization applications of [Section 4](#), referring to [\[13, 25, 26\]](#) for further details.

Definition 5 (Signed distance function). For any $x \in D$, let $d(x, \partial\Omega)$ be the usual Euclidean distance from x to $\partial\Omega$:

$$d(x, \partial\Omega) := \min_{y \in \partial\Omega} \|x - y\|, \quad (3.1)$$

where $\|\cdot\|$ denotes the Euclidean norm of \mathbb{R}^n . The signed distance function $d_\Omega : D \rightarrow \mathbb{R}$ to the domain Ω is defined by:

$$\forall x \in D, \quad d_\Omega(x) = \begin{cases} -d(x, \partial\Omega) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \partial\Omega, \\ d(x, \partial\Omega) & \text{if } x \in D \setminus \bar{\Omega}. \end{cases} \quad (3.2)$$

We then recall the definitions of the projection mapping onto $\partial\Omega$ and of the skeleton of Ω , which are illustrated on [Figure 1a](#).

Definition 6 (Skeleton and projection).

- (i) The (non empty) *set of projections* $\Pi_{\partial\Omega}(x)$ of a point $x \in D$ onto $\partial\Omega$ is the set of points $y \in \partial\Omega$ achieving the minimum in [\(3.1\)](#), that is:

$$\Pi_{\partial\Omega}(x) = \{y \in \partial\Omega \mid \|x - y\| = d(x, \partial\Omega)\}.$$

- (ii) The *skeleton* $\Sigma \subset D$ of Ω (relative to D) is the set of points $x \in D$ such that $\Pi_{\partial\Omega}(x)$ is not a singleton.
- (iii) For $x \in D \setminus \Sigma$, the unique element in $\Pi_{\partial\Omega}(x)$ is denoted by $p_{\partial\Omega}(x)$ and is called the *projection* of x onto $\partial\Omega$.

In order to describe further the connections between the signed distance function d_Ω and the geometry of Ω , we thenceforth suppose Ω to be a domain of class \mathcal{C}^2 ; note however that this assumption may not be minimal for some of the forthcoming properties.

Proposition 6 (Differentiability of d_Ω). *The signed distance function d_Ω is differentiable at any point $x \in D \setminus \Sigma$, and it is not differentiable on Σ . The gradient ∇d_Ω is an extension of the unit normal vector \mathbf{n} to $\partial\Omega$ pointing outward Ω , in the sense that it satisfies the properties:*

$$\forall y \in \partial\Omega, \quad \nabla d_\Omega(y) = \mathbf{n}(y) \text{ and } \forall x \in D \setminus \Sigma, \quad \|\nabla d_\Omega\| = 1. \quad (3.3)$$

Definition 7 (Normal rays). For $y \in \partial\Omega$, the ray emerging from y is the one-dimensional segment

$$\text{ray}(y) := \{x \in D \setminus \bar{\Sigma}, \quad p_{\partial\Omega}(x) = y\} = \{y + s\nabla d_\Omega(y) \mid s \in (\zeta_-(y), \zeta_+(y))\}. \quad (3.4)$$

where $\zeta_-(y)$ and $\zeta_+(y)$ are the maximum distances at which $\text{ray}(y)$ hits either the skeleton Σ or the boundary ∂D of the hold-all domain:

$$\forall y \in \partial\Omega, \quad \zeta_+(y) = \sup\{s \geq 0 \mid \{y + t\nabla d_\Omega(y) \mid t \in [0, s]\} \cap (\bar{\Sigma} \cup \partial D) = \emptyset\}, \quad (3.5)$$

$$\forall y \in \partial\Omega, \zeta_-(y) = \inf\{s \leq 0 \mid \{y + t\nabla d_\Omega(y) \mid t \in (s, 0]\} \cap (\bar{\Sigma} \cup \partial D) = \emptyset\}. \quad (3.6)$$

The functions ζ_- and ζ_+ are continuous on $\partial\Omega$ (see [15] or [44]).

Let us now turn to properties related to the second-order regularity of the signed distance function d_Ω , still under the assumption that the domain Ω is at least of class \mathcal{C}^2 . In all what follows, we denote for any $y \in \partial\Omega$ by $\mathbf{n}(y)$ the outward normal to Ω . Its tangential gradient $\partial_y \mathbf{n}(y)$ (with respect to the coordinates of $\partial\Omega$) is a symmetric tensor whose eigendecomposition

$$\partial_y \mathbf{n}(y) = \sum_{i=1}^{n-1} \kappa_i(y) \boldsymbol{\tau}_i(y) \boldsymbol{\tau}_i(y)^T, \quad (3.7)$$

involves the principal curvatures $(\kappa_i(y))_{1 \leq i \leq n-1}$ and the associated principal directions $(\boldsymbol{\tau}_i(y))_{1 \leq i \leq n-1}$ of $\partial\Omega$ at y . Note that for any unit extension $\tilde{\mathbf{n}} : \mathcal{V} \rightarrow \mathbb{R}^n$ of \mathbf{n} to some tubular neighborhood $\mathcal{V} \subset D$ of $\partial\Omega$ satisfying $\tilde{\mathbf{n}}(y) = \mathbf{n}(y)$ for any $y \in \partial\Omega$ and $\|\tilde{\mathbf{n}}(x)\| = 1$ for any $x \in \mathcal{V}$, it holds that $\nabla \tilde{\mathbf{n}}(y) = \partial_y \mathbf{n}(y)$.

The next proposition is related to the differentiability properties of the projection $p_{\partial\Omega}$, or equivalently to the second-order differentiability of d_Ω (see [8, 13, 15, 36, 47]).

Proposition 7 (Differentiability of $p_{\partial\Omega}$). *The following properties hold true:*

(1) *For any point $x \in D \setminus \bar{\Sigma}$, one has:*

$$1 + \kappa_i(p_{\partial\Omega}(x)) d_\Omega(x) > 0, \quad i = 1, \dots, n-1.$$

(2) *The projection $p_{\partial\Omega}$ is differentiable on $D \setminus \bar{\Sigma}$ and*

$$\forall x \in D \setminus \bar{\Sigma}, \quad \nabla p_{\partial\Omega}(x) = \sum_{i=1}^{n-1} \frac{1}{1 + \kappa_i(p_{\partial\Omega}(x)) d_\Omega(x)} \boldsymbol{\tau}_i(p_{\partial\Omega}(x)) \boldsymbol{\tau}_i(p_{\partial\Omega}(x))^T. \quad (3.8)$$

Remark 13. The projection $p_{\partial\Omega}$ is not differentiable on the boundary $\partial\Sigma$ of the skeleton Σ . Indeed (see Figure 1a and [15, 36]), a point $C \in D$ belongs to the closure $\bar{\Sigma}$ either because it lies on Σ , or because it is a *center of curvature* of $\partial\Omega$, i.e. there exists $y \in \Pi_{\partial\Omega}(C)$ and $i = 1, \dots, n-1$ such that

$$1 + d_\Omega(C) \kappa_i(y) = 0.$$

Since d_Ω is Lipschitz, it follows from Rademacher's theorem (see [32], §3.1.2) that the skeleton Σ has zero Lebesgue measure. However, if Ω is only supposed to be of class \mathcal{C}^2 , the closure $\bar{\Sigma}$ may have non zero Lebesgue measure (see [47] for an example). For $\bar{\Sigma}$ to be also of null Lebesgue measure, a little more regularity is required about Ω , namely, that it be at least of class \mathcal{C}^3 [15, 47].

From (3.7) and (3.8), we may calculate the Hessian matrix of d_Ω and the Jacobian $|\mathrm{D}\eta|$. The last proposition of this section sums up the characteristics of the shape optimization setting in the perspective of the previous Section 2.

Proposition 8 (Shape optimization setting). *Let $\Omega \subset D$ be a domain of class \mathcal{C}^2 ; then the signed distance function d_Ω is of class \mathcal{C}^2 on the open set $U := D \setminus \bar{\Sigma}$. Hence $\beta := \nabla d_\Omega$ is a vector field of class \mathcal{C}^1 on U ; the associated flow map $\eta : W \rightarrow D \setminus \bar{\Sigma}$ is a diffeomorphism of class \mathcal{C}^1 , whose expression reads:*

$$\forall (y, s) \in W, \quad \eta(y, s) = y + s \nabla d_\Omega(y), \quad (3.9)$$

where W is the set defined by (2.2). The inverse flow mapping $\eta^{-1} : U \rightarrow W$ is given by

$$\forall x \in U, \quad \eta^{-1}(x) = (p_{\partial\Omega}(x), d_\Omega(x)). \quad (3.10)$$

The divergence of the vector field $\beta = \nabla d_\Omega$ and the Jacobian $|\mathrm{D}\eta|$ of the flow map η are respectively given by

$$\forall x \in D \setminus \bar{\Sigma}, \quad \mathrm{div}(\nabla d_\Omega) = \Delta d_\Omega(x) = \sum_{i=1}^{n-1} \frac{\kappa_i(p_{\partial\Omega}(x))}{1 + d_\Omega(x) \kappa_i(p_{\partial\Omega}(x))}, \quad (3.11)$$

$$\forall (y, s) \in W, \quad |\mathrm{D}\eta| = |\det(\nabla \eta)| = \prod_{i=1}^{n-1} (1 + s \kappa_i(y)). \quad (3.12)$$

3.2. Admissible weights built upon the signed distance function

Let us now discuss the numerical resolution of the variational problem (1.9) in the shape optimization context (1.7) and (1.8) (see also Proposition 8). For our numerical applications, the hold-all domain D is equipped with a simplicial mesh \mathcal{T} (i.e. composed of triangles in 2-d, or tetrahedra in 3-d) with maximum element size h , and we rely on the Lagrange \mathbb{P}_1 finite element method for the discretization of (1.9) on account of its robustness and ease of implementation. In other terms, the approximation u_h to the solution $u \in V_\omega$ of (1.9) is sought in the space V_h of continuous piecewise linear functions on each simplex of \mathcal{T} .

As already mentioned, our final target is to calculate the trace (1.6) on $\Gamma = \partial\Omega$ of u ; in the continuous setting of Section 2, this trace does not depend on the choice of the weight ω as long as it fulfills (H1) to (H3), so that in principle, any such weight could be used. However, our commitment to use Lagrange \mathbb{P}_1 finite elements for the resolution of (1.9) makes some weights ω more adapted than others in the numerical context.

For instance, if we choose $\omega = 1$ on $U = D \setminus \bar{\Sigma}$ (which is an admissible weight on account of Lemma 6 below), the discretization of (1.9) with \mathbb{P}_1 finite elements is expected to be not very accurate if the skeleton Σ is not explicitly discretized in \mathcal{T} (as is the case in our applications; see Section 4). Indeed, functions in V_ω are in general discontinuous across Σ , and in order to solve (1.9) using the space V_h , the weight ω should be chosen in such a way that arbitrary functions $v \in V_\omega$ are well approximated (in the norm $\|\cdot\|_{V_\omega}$) by functions $v_h \in V_h$, as is reflected by the classical C ea's lemma (see e.g. [31]):

$$\|u - u_h\|_{V_\omega} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{V_\omega}, \quad (3.13)$$

for a constant $C > 0$ (which possibly depends on ω). Heuristically, and without looking for a very precise statement, these considerations call for the choice of a weight ω almost vanishing on $\bar{\Sigma}$, so that the approximation error $\|u - v_h\|_{V_\omega}$ in (3.13) attributes a lesser weight to a neighborhood of $\bar{\Sigma}$ where u is expected to be discontinuous while the functions $v_h \in V_h$ are continuous.

We now provide explicit candidates for weights ω which fulfill the conditions (H1) to (H3), take small values near the skeleton, and that we are going to use in our practical implementations.

Lemma 6. *For any real number $p \geq 0$, the weight*

$$\omega = \frac{1}{1 + |\Delta d_\Omega|^p} \quad (3.14)$$

satisfies the conditions (H1) to (H3) (this includes in particular the constant weight $\omega = 1/2$ for $p = 0$).

Proof. At first, it follows readily from the definition that ω belongs to $\mathcal{C}^0(U, \mathbb{R}_+^*)$, and is uniformly bounded on U , so that (H1) and (H2) are trivially satisfied. We then define $\kappa_-(y)$ and $\kappa_+(y)$ by:

$$\kappa_-(y) = \min(0, \min_i \kappa_i(y)), \quad \kappa_+(y) = \max(0, \max_i \kappa_i(y)), \quad (3.15)$$

as well as the corresponding multiplicities $m_-(y)$ and $m_+(y)$:

$$m_\pm(y) = \text{Card}(\{i \in \{1, \dots, n-1\} \mid \kappa_i(y) = \kappa_\pm(y)\}).$$

Using formula (3.11) for Δd_Ω , the weight $\alpha = \omega \circ \eta |D\eta|$ is decomposed as

$$\alpha(y, s) = \frac{\prod_{i=1}^{n-1} (1 + \kappa_i(y)s)}{1 + \left| \sum_i \frac{\kappa_i(y)}{1 + \kappa_i(y)s} \right|^p} = \frac{\prod_{i=1}^{n-1} (1 + \kappa_i(y)s)^{p+1}}{\prod_{i=1}^{n-1} (1 + \kappa_i(y)s)^p + \left| \sum_i \kappa_i(y) \prod_{j \neq i} (1 + \kappa_j(y)s) \right|^p} = f(y, s)g(y, s) \quad (3.16)$$

where f and g are the following functions:

$$f(y, s) = (1 + s\kappa_-(y))^{m_-(y)(p+1)} \mathbf{1}_{s \geq 0}(s) + (1 + s\kappa_+(y))^{m_+(y)(p+1)} \mathbf{1}_{s < 0}(s), \quad (3.17)$$

$$g(y, s) = \frac{\prod_{i, \kappa_i \neq \kappa_-} (1 + \kappa_i(y)s)^{p+1} \mathbf{1}_{s \geq 0}(s) + \prod_{i, \kappa_i \neq \kappa_+} (1 + \kappa_i(y)s)^{p+1} \mathbf{1}_{s < 0}(s)}{\prod_{i=1}^{n-1} (1 + \kappa_i(y)s)^p + \left| \sum_i \kappa_i(y) \prod_{j \neq i} (1 + \kappa_j(y)s) \right|^p}. \quad (3.18)$$

Then, f satisfies clearly the monotonicity condition (ii) in the statement of Lemma 1 and g is a continuous function on each of the domains $W_- = \{(y, s) \in W \mid s \leq 0\}$ and $W_+ = \{(y, s) \in W \mid s \geq 0\}$, that does not vanish on the compact sets \bar{W}_- and \bar{W}_+ . The assumptions of Lemma 1 are therefore fulfilled, so that ω satisfies (H3). This terminates the proof. \square

Remark 14.

- When it comes to solving (2.30) by relying on Proposition 5 with some data $f \in L^\infty(U)$, it is useful to observe that f belongs to $L^2_{\omega^{-1}}(U)$ as soon as the weight ω satisfies $\omega^{-1} \in L^1(U)$ (see Remark 11), which is the case if it is of the form (3.14) with $p < 2$. In the following numerical experiments, we shall see however that using values for p which are larger than 2 still provides good results.
- The weight (3.14) with $p > 0$ does not vanish on points $x \in \bar{\Sigma}$ that are not centers of curvatures. However it still takes small values on $\bar{\Sigma}$ and it is convenient to use in the implementation. Variants can easily be imagined, for example weights based on d_Ω itself or its Hessian $\nabla^2 d_\Omega$.
- For the most general setting where U is an arbitrary open set, the methodology of this section extends naturally by considering weights ω which vanish on the cracked parts of ∂U that are not explicitly meshed.

3.3. Computing weights, curvature and skeleton detection

Before going to numerical examples we clarify important practical details which are required in the numerical resolution of the variational problem (1.9) or in the implementation of the direct integration along characteristics (1.6).

The numerical setting at stake is the same as that of the previous Section 3.2: the hold-all domain D is equipped with a simplicial mesh \mathcal{T} , and the space V_h of Lagrange \mathbb{P}_1 finite element functions on \mathcal{T} is considered. The shape Ω is solely known via an accurate approximation d_h of the signed distance function d_Ω as an element in V_h ; multiple efficient methods are available to achieve such a calculation [56, 62], and we rely on the numerical algorithm from the previous work [23].

In Section 3.3.1, we discuss the issue of detecting triangles crossing the skeleton when traveling along the rays (note that this step is not required by the variational method). Then, we detail in Section 3.3.2 the method we used to compute the Laplacian Δd_Ω of the signed distance function involved in the weights (3.14) used in the variational method (in any dimension) and by the integration along rays (only in 2-d).

3.3.1. Detection of the skeleton Σ of Ω and identification of normal rays

The numerical detection of Σ in the course of the identification of the set $\text{ray}(y)$ for some given point $y \in \partial\Omega$ is achieved by assessing the following criterion (independently on the dimension), holding at the continuous level (see [26]):

$$\forall y \in \partial\Omega, \forall z_0, z_1 \in \text{ray}(y), \frac{d_\Omega(z_1) - d_\Omega(z_0)}{(z_1 - z_0) \cdot \nabla d_\Omega(y)} = 1.$$

In our implementation, when computing the ray emerging at some point $y \in \partial\Omega$ (which is detected by the fact that $d_h(y) = 0$), we travel the triangles in the mesh \mathcal{T} in the normal direction $\mathbf{n} = \nabla d_h(y)$, and we stop the calculation of the ray in the triangle $T \in \mathcal{T}$ where the entering and exiting points z_0 and z_1 satisfy:

$$\left| \frac{d_h(z_1) - d_h(z_0)}{\|z_1 - z_0\|} - \text{sign}(d_h(z_0)) \right| \geq \text{tolRay}, \quad (3.19)$$

where tolRay a small tolerance (set to 0.3 in our implementation). This provides meanwhile an approximate location of the skeleton Σ , up to a tolerance of the order of the mesh size.

Our criterion (3.19) differs from that used in the related works [6, 49]. In there, the authors detect Σ by looking at changes in the monotonicity of the signed distance function d_h along the ray, i.e. they rely on the following property of the (continuous) signed distance function d_Ω :

$$\forall y \in \partial\Omega, \forall z_0, z_1 \in \text{ray}(y), (d_\Omega(z_1) - d_\Omega(z_0))((z_1 - z_0) \cdot \nabla d_\Omega(y)) \geq 0. \quad (3.20)$$

Our personal experiment with the above criterion suggests that it may sometimes fail to detect the skeleton Σ accurately, because such a change in monotonicity may simply not occur when the ray is supposed to cross Σ in the neighborhood of center of curvatures (see Remark 13). Our criterion (3.19) may also fail depending on the chosen tolerance parameter, but it offered visible improvements (Figure 3) in our academic test-cases. Note that when integrating along the ray, the last triangle, where the skeleton is hit, is included in the integration.

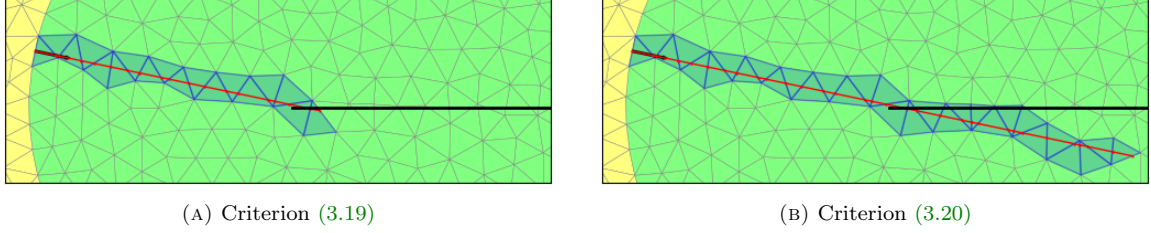


FIGURE 3. Comparison between the two criteria of Section 3.3.1 for the detection of Σ when travelling along rays in an unstructured mesh (Skeleton displayed in the black line).

3.3.2. Computation of the curvature of Ω and of the Laplacian Δd_Ω of the signed distance function

The numerical approximation of Δd_Ω for the computation of the weight ω of (3.14) is not completely straightforward and is detailed now. Note that

- (i) such step is required only to avoid the need for elaborate shock capturing schemes (e.g. discontinuous finite elements or mesh truncations) for the discrete test functions near the skeleton (see Section 3.4 below), which allows to solve the variational formulation (1.9) with simple \mathbb{P}_1 finite elements.
- (ii) even when using a \mathbb{P}_1 discretization as in all the examples following, a very accurate estimation of the Laplacian Δd_Ω for computing the weight ω of (3.14) is not needed, as long as ω vanishes near the skeleton Σ .

In contrast, integration along rays via the formula (2.31) with the Jacobian (3.12) requires accurate estimates of the principal curvatures κ_i . In 2-d, there is only one principal curvature, hence we estimate it directly from the value of Δd_Ω on $\partial\Omega$ with the method of this subsection. In 3-d, this estimation is more complicated to implement because the single knowledge of the mean curvature does not yield the values of both principal curvatures κ_1 and κ_2 : these could e.g. be computed from the eigenvalues of the Hessian $\nabla^2 d_\Omega$, however with a possibly poor accuracy.

Let us now detail our method for the numerical approximation of Δd_Ω , or equivalently that of the curvature κ of $\partial\Omega$ in 2-d when d_Ω is only known via its approximation d_h at the nodes of the mesh \mathcal{T} . Our first step consists in calculating a \mathbb{P}_1 interpolation $g_h \in V_h \times V_h$ of the piecewise constant gradient ∇d_h by solving the following variational problem:

$$\forall \psi_h \in V_h \times V_h, \int_D g_h \cdot \psi_h dx = \int_D \nabla d_h \cdot \psi_h dx. \quad (3.21)$$

The approximation of the divergence $\text{div}(\nabla d_\Omega)$ is then calculated as the (piecewise constant) divergence of the reconstructed field g_h . Unfortunately, this procedure generally produces a very noisy approximation characterized by a lot of spurious oscillations when the mesh size decreases (see Figure 4). In order to overcome this difficulty, we calculate a regularization κ_h of this noisy estimation with a Laplace kernel, namely we solve:

$$\text{Find } \kappa_h \in V_h \text{ such that } \forall \psi_h \in V_h, \int_D (\gamma_h^2 \nabla \kappa_h \cdot \nabla \psi_h + \kappa_h \psi_h) dx = \int_D \text{div}(g_h) \psi_h dx, \quad (3.22)$$

where $\gamma_h > 0$ is a regularization length scale (equal to $3h_{\max}$ where h_{\max} is the maximum edge length in the mesh).

Let us now illustrate our method by considering the example of an ellipse Ω inside a square-shaped hold-all domain D : let D and Ω be defined by

$$D = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| < 2 \text{ and } |x_2| < 2\}, \quad \Omega = \left\{ (x_1, x_2) \in D \mid \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} < 1 \right\}, \quad (3.23)$$

where $a = 1.5$ and $b = 1$. The skeleton of Ω is explicitly known in this case: $\Sigma = \{(x_1, 0) \mid |x_1| < a - b^2/a\}$ and the curvature κ of $\partial\Omega$ at a point $y = (y_1, y_2)$ is given by $\kappa(y) = ab/\gamma^3$ with $\gamma = \sqrt{\frac{b^2}{a^2} y_1^2 + \frac{a^2}{b^2} y_2^2}$. The difference between the exact curvature $\kappa(y)$ of $\partial\Omega$ and its reconstruction κ_h (at the boundary nodes discretizing $\partial\Omega$) using both procedures (3.21) and (3.22) is represented in Figure 4.

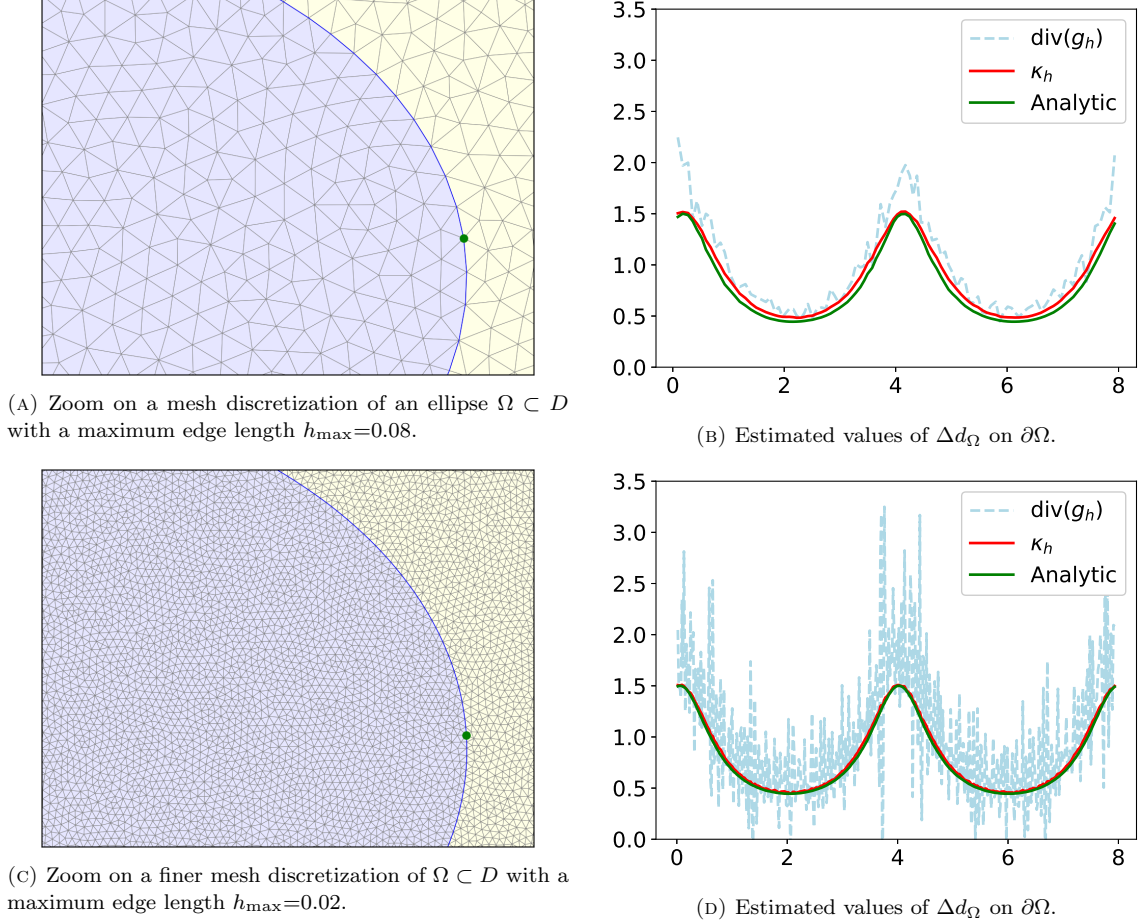


FIGURE 4. Estimation of Δd_{Ω} on the mesh \mathcal{T} for the shape Ω in (3.23) and for two different mesh resolutions. The x coordinate on the right-hand graphs represents the arc length coordinate on $\partial\Omega$ when the starting point is the green reference point. Estimates of the mean curvature of $\partial\Omega$, $\text{div}(g_h)$ and κ_h (see (3.21) and (3.22) for the definitions) are compared to the analytical value $\kappa(y)$.

3.4. Choice of the weight ω when using a \mathbb{P}_1 discretization

In this section, we perform several numerical experiments about the influence of the choice of the weight function ω in the resolution of the variational problem (1.9) using the Lagrange \mathbb{P}_1 finite element method. With this perspective in mind, and in the 2-d numerical setting described in Sections 3.2 and 3.3, we consider the issue of generating numerical functions $v \in V_{\omega}$ which are constants along the normal rays to the shape Ω . Namely, we solve the boundary-value problem

$$\begin{cases} \nabla d_{\Omega} \cdot \nabla v = 0 & \text{in } D \setminus \overline{\Sigma}, \\ v = v_0 & \text{on } \partial\Omega, \end{cases} \quad (3.24)$$

for given data $v_0 \in L^2(\partial\Omega)$.

Using the variational setting of Section 2, and in particular the Poincaré inequality of Proposition 4 and the surjectivity of traces of Proposition 3, it is not difficult to infer the existence of weak solutions to (3.24).

Proposition 9. *Let $\omega \in C^0(D \setminus \overline{\Sigma}, \mathbb{R}_+^*)$ be a weight satisfying (H1) to (H3) and let $V_{\omega,0} = \{v \in V_{\omega} \mid v = 0 \text{ on } \partial\Omega\}$ be the subspace of functions of V_{ω} vanishing on $\partial\Omega$. There exists a unique solution $v \in V_{\omega}$ to the*

variational problem:

$$\text{Find } v \in V_\omega \text{ such that } v = v_0 \text{ on } \partial\Omega \text{ and } \forall w \in V_{\omega,0}, \int_{D \setminus \bar{\Sigma}} \omega (\nabla d_\Omega \cdot \nabla v) (\nabla d_\Omega \cdot \nabla w) dx = 0. \quad (3.25)$$

The solution v is independent of the choice of ω as long as (H1) to (H3) are satisfied, and it is given by the formula:

$$v(x) = v_0(p_{\partial\Omega}(x)), \text{ a.e. } x \in D \setminus \bar{\Sigma}. \quad (3.26)$$

The formulation (3.25) is to be compared to the so-called Galerkin Least Square formulation and SUPG methods for advection-reaction problems [31], with the difference that usual assumptions of uniformly bounded divergence, which do not hold, are replaced with the regularity assumptions of Section 2.1.

Various numerical techniques may be thought of in order to solve (3.25): one could for instance use a finite element space of discrete functions vanishing on the boundary nodes of $\partial\Omega$ when the latter boundary is discretized in the computational mesh \mathcal{T} of D . For our applications, we use a boundary penalization approach which is commonly used in finite element software packages (such as `FreeFem++` [38]): a numerical approximation of the solution v to (3.24) is obtained as the solution to the approximate problem:

$$\text{Find } v_\epsilon \in V_\omega \text{ such that } \forall w \in V_\omega, \frac{1}{\epsilon} \int_{\partial\Omega} v_\epsilon w dy + \int_{D \setminus \bar{\Sigma}} \omega (\nabla v_\epsilon \cdot \nabla d_\Omega) (\nabla w \cdot \nabla d_\Omega) dx = \frac{1}{\epsilon} \int_{\partial\Omega} v_0 w dy, \quad (3.27)$$

for a very small value ϵ ($\epsilon = 1e-16$ in our implementation). If sufficient regularity holds for the true solution v of (3.25), arguments similar to those of [12] would show the convergence $v_\epsilon \rightarrow v$ when $\epsilon \rightarrow 0$.

Remark 15. The problem of building constant functions along normal rays of the form (3.24) may be solved on unstructured meshes owing to variants of the fast marching algorithm; see e.g. [19].

We now verify that a good approximation of the solution v to (3.25) (or more precisely (3.27)), which is in particular discontinuous across the skeleton Σ of Ω , can be obtained either by truncating manually the skeleton from the computational mesh \mathcal{T} and taking the weight $\omega = 1$, or by selecting a weight ω taking “small” values near Σ . For the latter experiment, we use the weight $\omega = 1/(1 + |\Delta d_\Omega|^p)$ with $p = 2$ (see also Remark 14 about the “small values of ω near Σ ”). Of course, removing the skeleton from the mesh is a not straightforward task in full generality but it is performed here for the sake of comparison.

Let us consider again the ellipse example of (3.23). We consider two different computational meshes \mathcal{T} and \mathcal{T}' . The former is a triangular mesh of D , and the latter \mathcal{T}' is a triangular mesh of $D \setminus \bar{\Sigma}$ (i.e. the skeleton $\bar{\Sigma}$ has been manually removed). In both \mathcal{T} and \mathcal{T}' , the considered shape Ω is explicitly discretized as a submesh; see Figure 5 for an illustration. The variational problem (3.27) is numerically solved for a boundary datum $v_0 \in L^2(\partial\Omega)$ given by (see Figure 6d):

$$\forall (y_1, y_2) \in \partial\Omega, v_0(y_1, y_2) = \cos(3y_1)^2 + 20y_2,$$

and the computed finite element solution is plotted on Figure 6 in the following three situations.

- The mesh \mathcal{T}' is used, in which D and $\Omega \subset D$ are meshed explicitly, and where a thin layer around $\bar{\Sigma}$ has been manually removed (see Figure 5c). The solution v to (3.27) is computed using the constant weight $\omega = 1$ and the result is displayed on Figure 6a. As expected, the fact that $\bar{\Sigma}$ be absent from \mathcal{T}' allows the numerical solution v to have very different values on either sides of $\bar{\Sigma}$.
- The mesh \mathcal{T} of D (where $\bar{\Sigma}$ has not been removed) is used (see Figure 5b), and v is computed with the constant weight $\omega = 1$; the result is represented on Figure 6b. The formulation (3.27) proves numerically stable with the choice $\omega = 1$, but it tends to over smoothen the sharp discontinuities of v near the skeleton Σ , which results in a loss of accuracy for the extension problem (3.24).
- The mesh \mathcal{T} of D is used again, but the solution v to (3.27) is now computed by using the weight $\omega = 2/(1 + |\Delta d_\Omega|^2)$; the result is represented on Figure 6c. Obviously, the obtained numerical solution is much closer to the expected result (3.26); the numerical procedure in this case seems to achieve the same order of accuracy than in the experiment using the truncated mesh \mathcal{T}' .

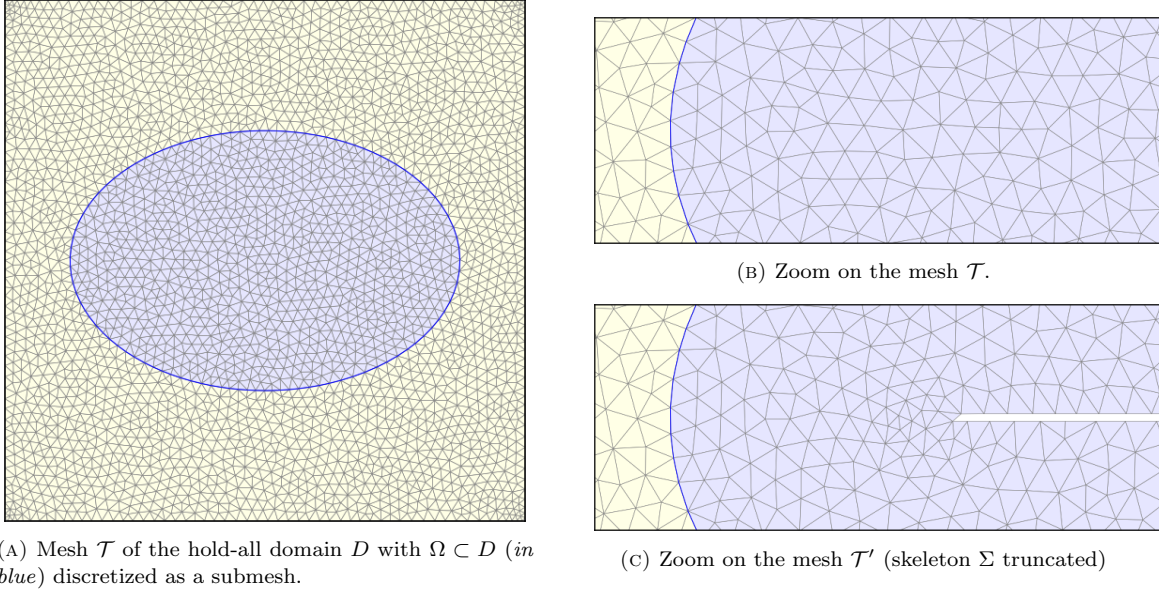


FIGURE 5. The two meshes \mathcal{T} and \mathcal{T}' used for the test-case of Section 3.4.

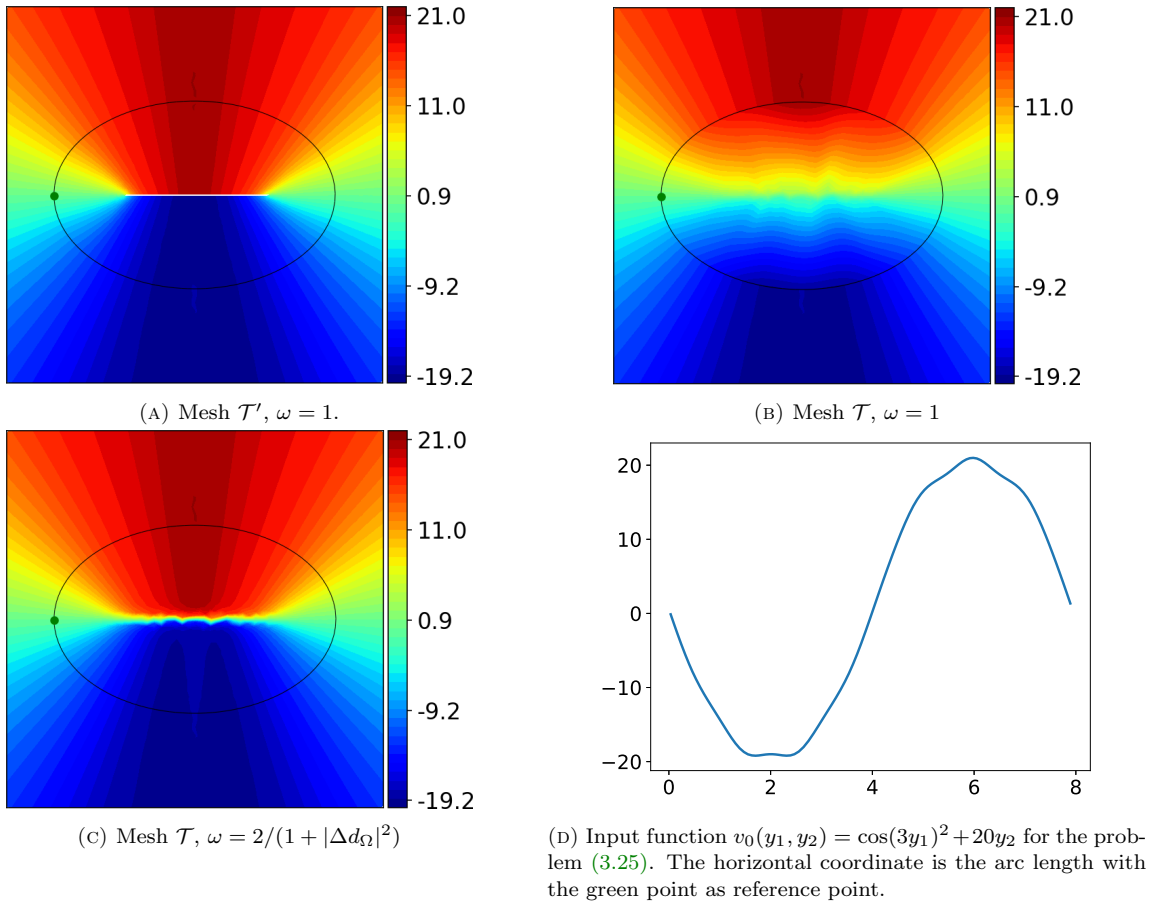


FIGURE 6. Numerical resolution of the problem (3.24) using the variational problem (3.27) for various weights, with or without removing the skeleton Σ from the computational mesh.

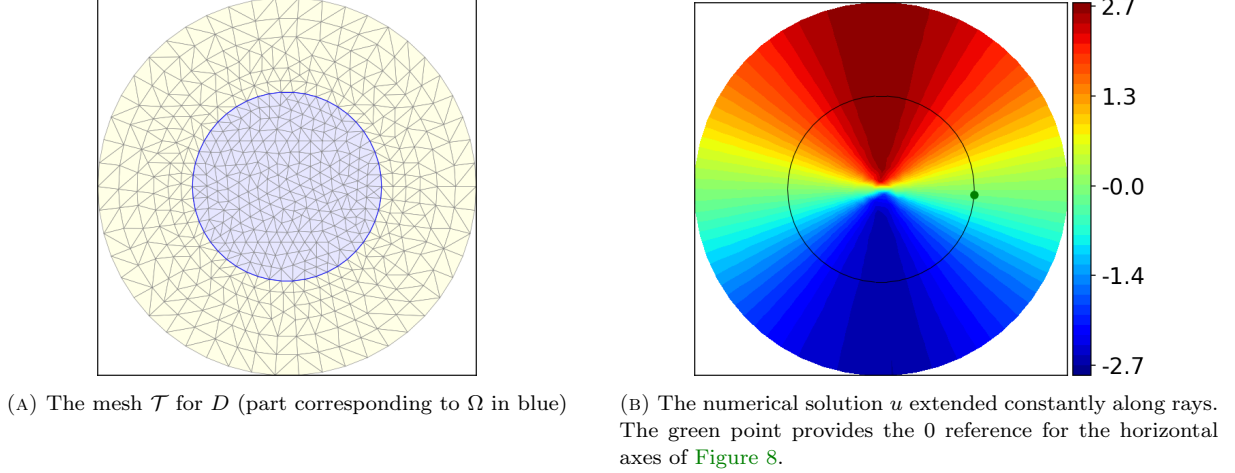


FIGURE 7. Setting of Section 3.5.1

3.5. Numerical comparisons between the variational method and direct integration along rays

We now investigate the numerical evaluation of the function $u \in L^2(\partial\Omega)$ in (1.6) for several 2-d academic configurations as far as D , Ω and the function f are concerned. In particular, we compare the evaluation of u obtained by direct integration along rays (*i.e.* implementing directly the formula (1.6)) to that obtained by solving the variational formulation (1.9) on meshes \mathcal{T} (resp. \mathcal{T}') of D in which Ω is explicitly discretized and the skeleton Σ of Ω is not removed (resp. is removed).

3.5.1. A domain with trivial skeleton: the case of a circle

We first consider the case where Ω is a disk enclosed in a larger disk D :

$$D = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 4\}, \text{ and } \Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}; \quad (3.28)$$

see Figure 7. In this case, the skeleton Σ is reduced to the point 0. The considered function f is:

$$\forall x = (x_1, x_2) \in D, \quad f(x_1, x_2) = x_2,$$

which belongs to the finite element space V_h , and is therefore amenable to an exact integration when the travel procedure along rays of Section 3.3.1 is used.

In this situation, the sought function u , given by (1.6), is known analytically; a calculation in polar coordinates indeed yields:

$$\forall (y_1, y_2) \in \partial\Omega, \quad u(y_1, y_2) = \frac{8}{3}y_2. \quad (3.29)$$

Comparisons are displayed on Figure 8 between the version of u obtained after direct integration along rays, and the numerical solutions of (1.9) obtained for various choices of weight functions ω on the mesh \mathcal{T} of D (where Σ has not been removed). We observe a good match between the variational and the direct method. As expected, the solutions computed thanks to our variational method are less accurate when the constant weight $\omega = 1$ is chosen. A significant increase in accuracy is achieved by selecting a weight ω vanishing at the center of the circle, *i.e.* $\omega = 1/(1 + |\Delta d_\Omega|^{3.5})$.

3.5.2. A C^2 domain with non trivial skeleton

We take on the example (3.23) where Ω is an ellipse inside a square-shaped hold-all domain D , reusing both meshes \mathcal{T} and \mathcal{T}' depicted on Figure 5. To emphasize the influence of the skeleton on our numerical method, we compute the quantity (1.9) for a function f which is supported in Ω only (see Figure 9):

$$f(x_1, x_2) = (1 + x_2)\mathbf{1}_\Omega(x_1, x_2). \quad (3.30)$$

The presence of the constant 1 in (3.30) avoids the “simplification” of having f vanishing on Σ . After an

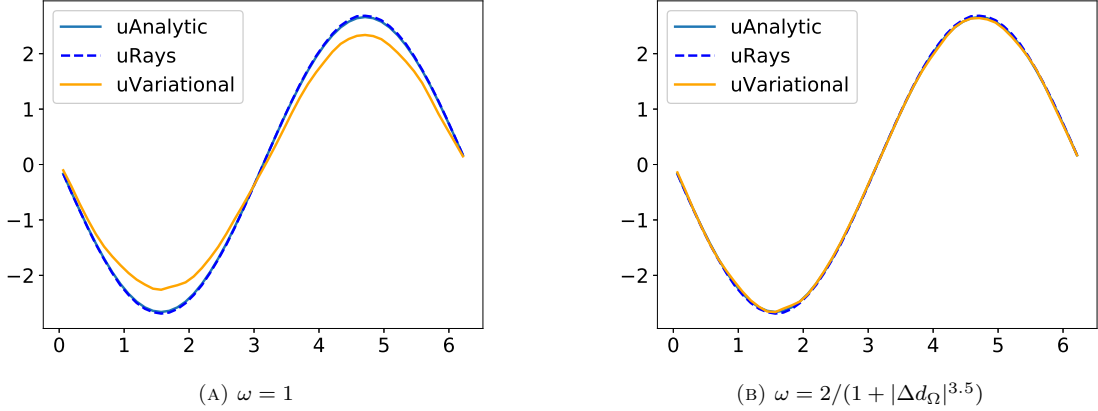


FIGURE 8. Comparison between direct integration along ray, analytical formula (1.3), and finite element solution of (1.9) for two choices of weights ω for the example of Section 3.5.1. In the legends of this figure and all those to follow, `uAnalytic`, `uRays` and `uVariational` refer respectively to the analytical value of u , its numerical estimation using (2.31), and the trace of the variational solution of (2.30).

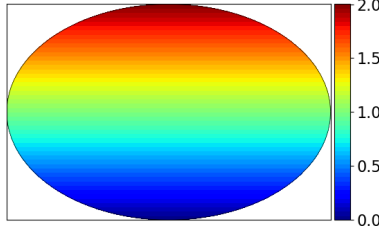


FIGURE 9. The function f for the identification problem (1.6) for the test case of Section 3.5.2.

explicit calculation based on (1.6), the exact solution is given by:

$$\forall x = (y_1, y_2) \in \partial\Omega, \quad u(y) = (1 + y_2)\zeta_- - \frac{\zeta_-^2}{2} \left(\frac{y_2}{\zeta_-} + \kappa(1 + y_2) \right) + \frac{y_2}{3} \kappa \zeta_-^2, \quad (3.31)$$

with $\kappa = \frac{ab}{\gamma^3}$, $\zeta_- = \frac{\gamma b}{a}$, $\gamma = \sqrt{\frac{b^2}{a^2} y_1^2 + \frac{a^2}{b^2} y_2^2}$.

The numerical trace u (which is extended along rays using (3.27) to ease the visualization) obtained by solving the variational formulation (1.9) on \mathcal{T} or \mathcal{T}' , and for various choices of weights, is represented on Figure 10. On Figure 11, we compare for each of these strategies the boundary values of the numerical trace u to that computed with (1.3) by direct integration along rays, and to the analytical expression (3.31).

We note that the solutions obtained by integrating along rays are generally characterized by small spurious oscillations. We explain this error by the fact that our criterion (3.19) detects the skeleton up to an error proportional to the mesh size (the actual location of the skeleton is not estimated within the last travelled triangle; see Figure 12). Such a procedure could be improved by using more systematic skeleton detection methods; see for instance the works [9, 10] in the field of computer graphics. We verify that the amplitude of these oscillations decreases with the size of the mesh, as shown on Figure 11d.

3.5.3. A C^1 domain, with discontinuous curvature

We now consider the case where Ω is a stadium, i.e. the union of a rectangle and two half-disks. Define (see Figure 13)

$$D = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \leq 2 \text{ and } |x_2| < 2\} \quad (3.32)$$

and

$$\Omega = \{(x_1, x_2) \in D \mid (|x_1| < 0.5 \text{ and } |x_2| < 0.5) \text{ or } (x_1 - 0.5)^2 + x_2^2 < 0.25 \text{ or } (x_1 + 0.5)^2 + x_2^2 < 0.25\}. \quad (3.33)$$

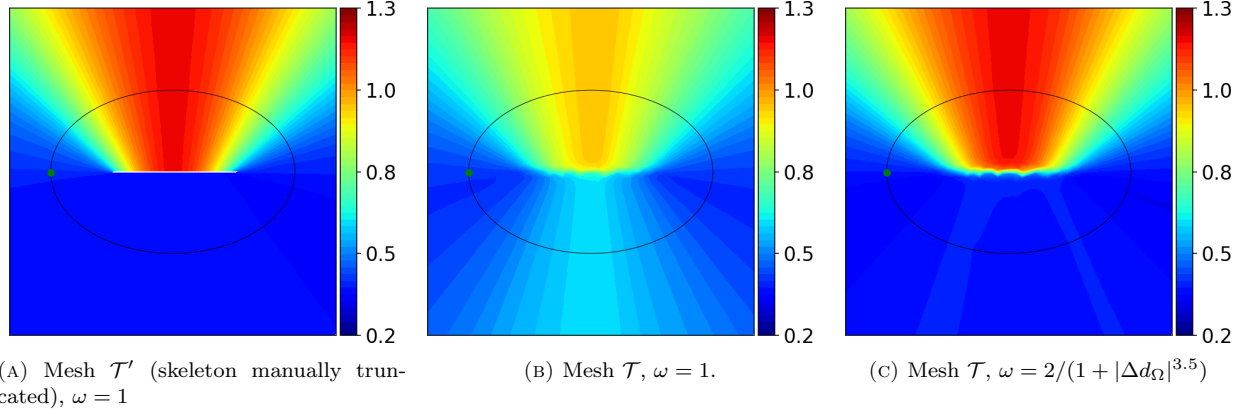


FIGURE 10. The numerical solution u of the variational problem (1.9) extended constantly along rays (for visualization purpose). Inaccuracies occur if the skeleton Σ is not removed from the mesh or if the weight ω does not vanish in its vicinity.

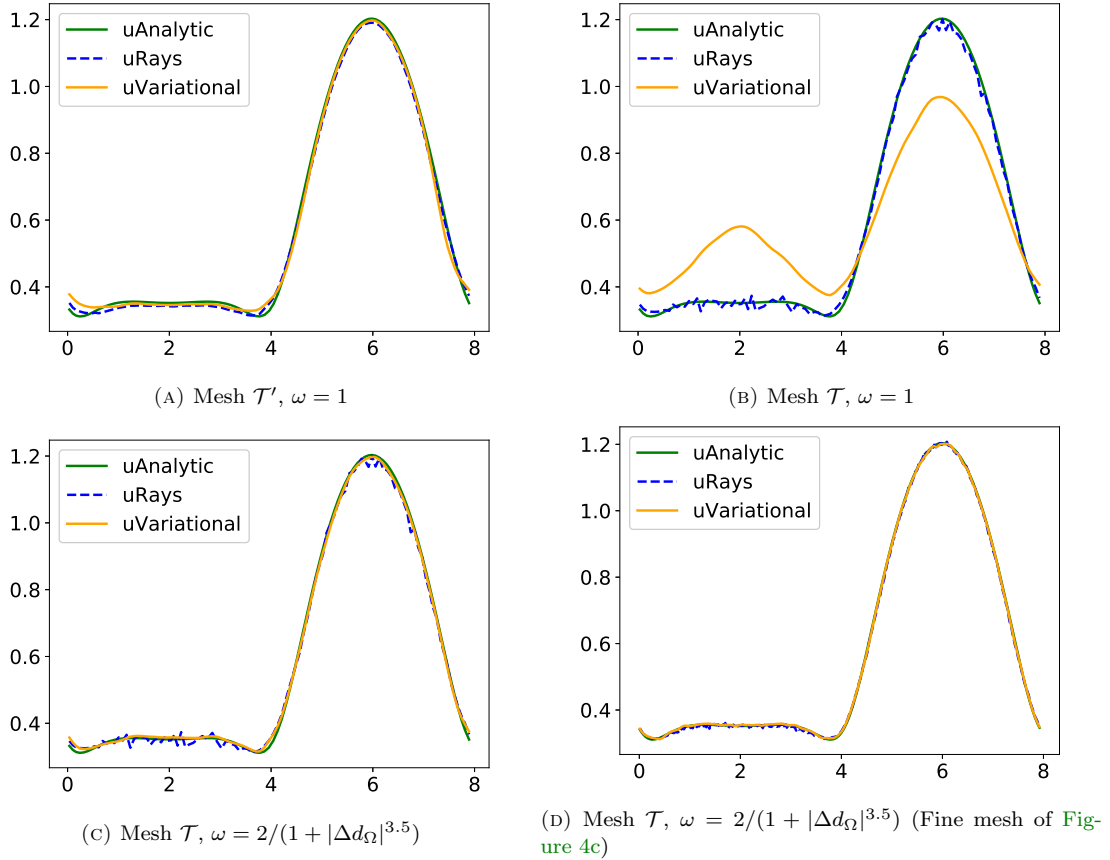


FIGURE 11. Comparison between the results of a direct integration along rays and our variational method for the example of Section 3.5.2.

The domain Ω is not of class \mathcal{C}^2 , and the curvature κ of the boundary $\partial\Omega$ is discontinuous at the points where $x_1 = \pm 0.5$. Hence, this example does not fall into the admissible setting of Proposition 8, and there is, in principle, no guarantee that our variational method based on (1.9) should still work.

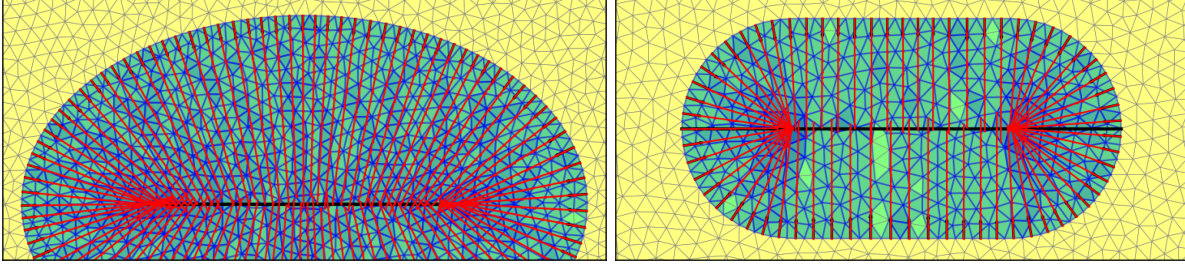


FIGURE 12. Numerical rays for the example of Sections 3.5.2 and 3.5.3. The inaccuracy of the direct integration along rays is related to that of the skeleton detection within the last triangle. Triangle's paths are colored in transparent blue (the darker, the more visited); depicting that some mesh triangles might not be crossed by all the rays or some triangles may be visited more than others.

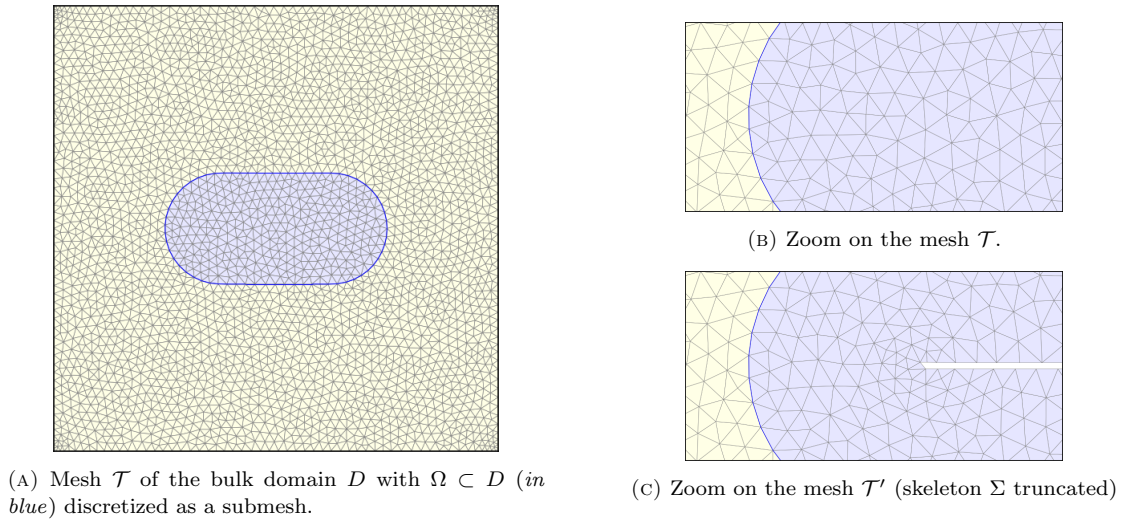


FIGURE 13. Meshes used for the test-case of Section 3.5.3.

In this example, the skeleton Σ is Ω is explicitly given by $\Sigma = \{(x_1, 0) \mid |x_1| \leq 0.5\}$. We calculate the function u in (1.6) associated to the datum function f defined by (3.30); the analytical solution u for this problem is given by

$$u(y_1, y_2) = \begin{cases} \frac{1}{2} + \frac{1}{8} \text{sign}(y_1) & \text{if } |y_1| < 0.5, \\ \frac{1}{4} + \frac{y_2}{6} & \text{if } |y_1| > 0.5. \end{cases}$$

In particular, u is ill-defined at the points $x \in \partial\Omega$ where the curvature is discontinuous.

Both methods for the numerical evaluation of u (i.e. the direct integration along rays and our variational method) are applied, and the results are represented on Figure 14. Even though this example does not fit into the admissible setting of Section 2, the results indicate that our numerical method still works, up to over-smoothing inaccuracies of the computed solution u near the discontinuities of the curvature κ .

3.5.4. A test case involving a Lipschitz domain Ω with angular corners

In this last example, we consider the following situation:

$$D = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \leq 2 \text{ and } |y| \leq 2\}, \quad \Omega = \{(x_1, x_2) \in D \mid \max(x_1^2/2 - x_2 - 1, x_2) \leq 0\}, \quad (3.34)$$

together with the datum function f :

$$f(x_1, x_2) = \cos(6x_1)^2,$$

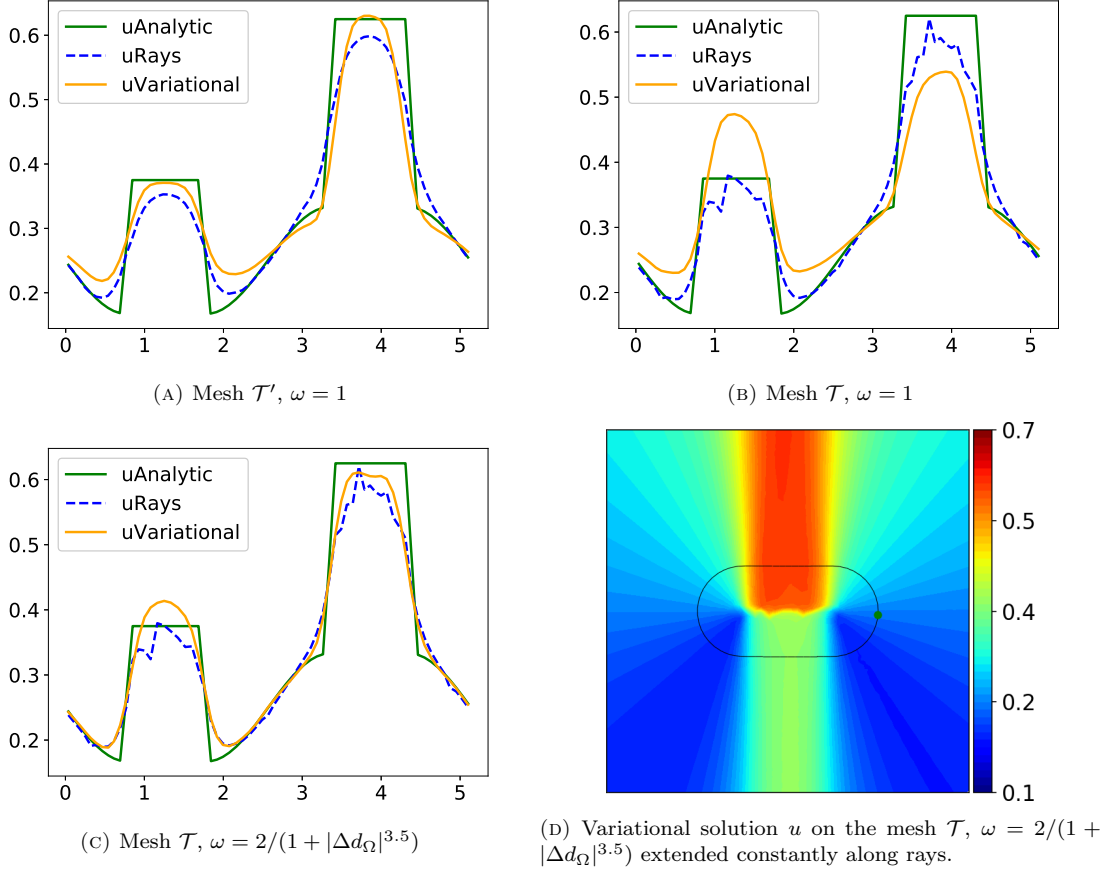


FIGURE 14. Comparison between the direct integration along rays and our variational method for the example of Section 3.5.3.

as illustrated on Figure 15. Again, the theoretical framework of Section 2 does not apply to the present situation because:

- The vector field ∇d_Ω is not of class C^1 on U and is not U -filling: the reunion of all the rays emerging from points $y \in \partial\Omega$ does not cover the whole set $U = D \setminus \bar{\Sigma}$;
- The flow η of this vector field is not of class C^1 , because the curvature κ of the boundary (and even the normal vector) is discontinuous at the corners. Numerically, $|\Delta d_\Omega|$ blows up in the whole area filled by the skeleton and the set of points that are not covered by normal rays.

Still, the quantity u given by (1.6) can be calculated analytically wherever it makes sense. A few elementary, albeit technical computations yield:

$$u(y_1, y_2) = \begin{cases} \cos(6y_1)^2(2 - \lambda_-(s^{-1}(y_1))) & \text{if } y_2 \geq 0, \\ \phi(\lambda_+(y_1)) - \phi(\lambda_-(y_1)) & \text{if } y_2 \leq 0 \end{cases} \quad (3.35)$$

where

$$\lambda_-(t) = \frac{t^2/2 - 1}{1 + 1/\sqrt{1+t^2}}, \quad \lambda_+(t) = \min((2/|t| - 1), t^2/2)\sqrt{1+t^2}, \quad (3.36)$$

s^{-1} is the reciprocal function of $t \mapsto s(t) = \left(1 + \frac{\lambda_-(t)}{\sqrt{1+t^2}}\right)t$ and ϕ is a primitive function of $\lambda \mapsto \cos(6t(1 + \lambda/\sqrt{1+t^2}))^2(1 + \lambda/(1+t^2)^{3/2})$.

A numerical approximation to this function u is computed by using either a direct integration along rays or our variational method based on (1.9), on a single mesh \mathcal{T} where the skeleton Σ is not removed, and using three possible choices of the weight ω . The results are represented on Figure 16. In all cases, the observed

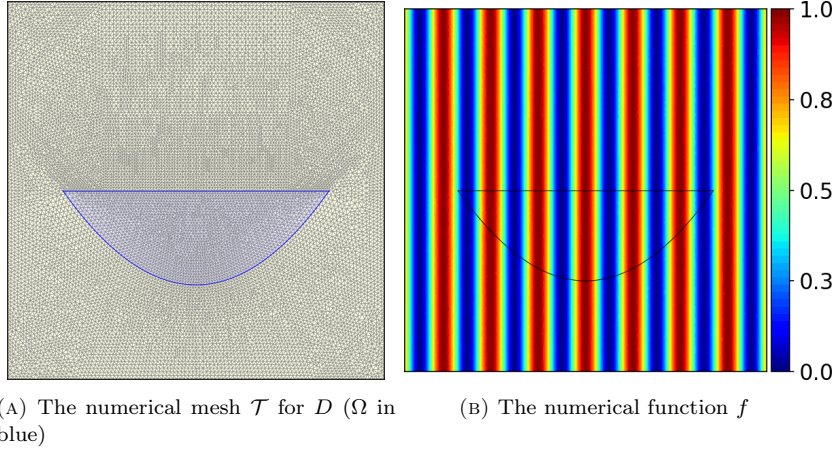


FIGURE 15. Setting for the example of Section 3.5.4.

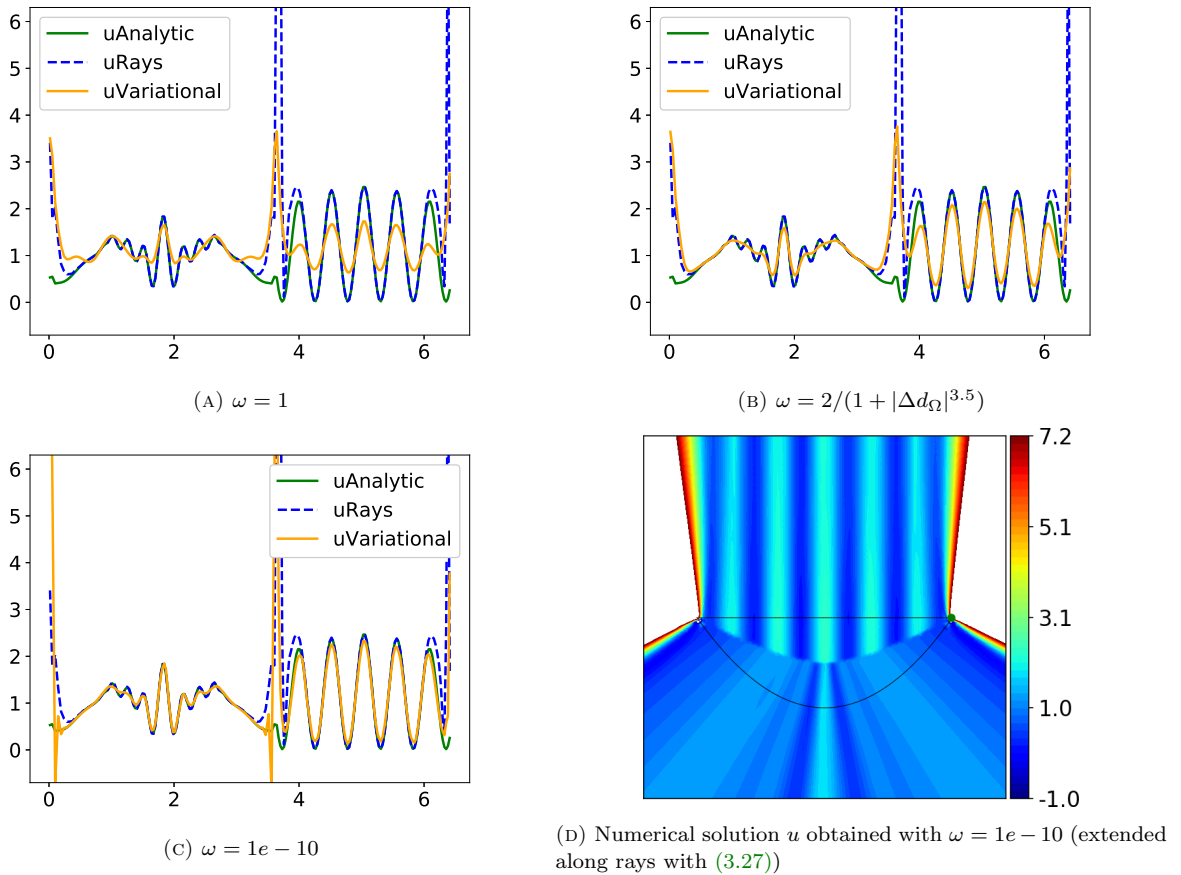


FIGURE 16. Comparison between direct integration along normal rays and our variational method for the example of Section 3.5.4.

numerical inaccuracies, characterized by very high values, are concentrated near the angular corners of Ω , while the method yields satisfying accuracy on the remaining smooth parts of $\partial\Omega$. Again, we observe the poor accuracy associated to the choice of a constant weight $\omega = 1$. For this example, a uniformly small weight $\omega = 1e^{-10}$ seems to yield a better accuracy than the previous choice $\omega = 2/(1 + |\Delta d_\Omega|^{3.5})$.

4. APPLICATIONS TO MAXIMUM AND MINIMUM THICKNESS IN SHAPE OPTIMIZATION

In the shape optimization setting which is the original motivation for our work, we show how the general method proposed in [Section 2](#) and the inferred numerical methods in [Section 3](#) allows to efficiently implement geometric constraints, namely maximum and minimum thickness constraints.

4.1. Shape optimization of linearly elastic structures

We consider *shapes*, that is, smooth bounded domains Ω enclosed in a fixed, bounded and Lipschitz hold-all domain $D \subset \mathbb{R}^n$. Any such shape Ω is clamped on a part Γ_D of its boundary, and traction loads $\mathbf{g} \in L^2(\Gamma_N)$ are applied on a disjoint region of $\partial\Omega$; the complement $\Gamma := \partial\Omega \setminus (\overline{\Gamma_D} \cup \overline{\Gamma_N})$ is traction-free and body forces are omitted for simplicity; Γ is the only region of the boundary $\partial\Omega$ which is subject to optimization.

In this situation, the displacement \mathbf{u}_Ω of the shape is characterized as the unique solution in $H^1(\Omega, \mathbb{R}^n)$ to the linearized elasticity system:

$$\begin{cases} -\operatorname{div}(Ae(\mathbf{u}_\Omega)) = 0 & \text{in } \Omega, \\ Ae(\mathbf{u}_\Omega) \cdot \mathbf{n} = \mathbf{g} & \text{on } \Gamma_N, \\ Ae(\mathbf{u}_\Omega) \cdot \mathbf{n} = 0 & \text{on } \Gamma, \\ \mathbf{u}_\Omega = 0 & \text{on } \Gamma_D, \end{cases} \quad (4.1)$$

where $e(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the strain tensor associated to the displacement \mathbf{u} and A is the Hooke's law, defined for any symmetric $n \times n$ matrix by

$$Ae(\mathbf{u}) = 2\mu e(\mathbf{u}) + \lambda \operatorname{Tr}(e(\mathbf{u}))\mathbf{I},$$

involving the Lamé coefficients λ, μ which characterize the physical properties of the constituent material.

In this context, we consider structural optimization problems of the form

$$\min_{\Omega \subset D} J(\Omega), \quad \text{s.t. } P(\Omega) \leq 0, \quad (4.2)$$

where $J(\Omega)$ is a performance criterion, which will typically be the volume $\operatorname{Vol}(\Omega)$ or the compliance $C(\Omega)$ of shapes:

$$\operatorname{Vol}(\Omega) = \int_{\Omega} dx, \quad C(\Omega) = \int_{\Omega} Ae(\mathbf{u}_\Omega) : e(\mathbf{u}_\Omega) dx, \quad (4.3)$$

and $P(\Omega)$ is a geometric constraint

$$P(\Omega) = \int_D j(d_\Omega) dx, \quad (4.4)$$

involving the signed distance function d_Ω to Ω and a given smooth function $j : \mathbb{R} \rightarrow \mathbb{R}$.

In order to achieve the numerical resolution of [\(4.2\)](#), we rely on Hadamard's method of boundary variations (see for instance [[1](#), [4](#), [39](#), [50](#), [57](#)]), whereby variations of a given shape Ω are considered under the form

$$\Omega_\theta := (\operatorname{Id} + \boldsymbol{\theta})(\Omega), \quad \boldsymbol{\theta} \in W^{1,\infty}(D, \mathbb{R}^n), \quad \|\boldsymbol{\theta}\|_{W^{1,\infty}(D, \mathbb{R}^n)} < 1. \quad (4.5)$$

Definition 8. A function $F(\Omega)$ of the domain is *shape differentiable* at a shape Ω if the underlying mapping $\boldsymbol{\theta} \mapsto F(\Omega_\theta)$, from $W^{1,\infty}(D, \mathbb{R}^n)$ into \mathbb{R} , is Fréchet differentiable at $\boldsymbol{\theta} = 0$. The corresponding derivative $\boldsymbol{\theta} \mapsto F'(\Omega)(\boldsymbol{\theta})$ is the shape derivative of F at Ω and the following expansion holds in the vicinity of $\boldsymbol{\theta} = 0$:

$$F(\Omega_\theta) = F(\Omega) + F'(\Omega)(\boldsymbol{\theta}) + o(\boldsymbol{\theta}), \quad \text{where } \frac{|o(\boldsymbol{\theta})|}{\|\boldsymbol{\theta}\|_{W^{1,\infty}(D, \mathbb{R}^n)}} \xrightarrow{\boldsymbol{\theta} \rightarrow 0} 0.$$

Remark 16. In our setting, the set of admissible deformation fields $\boldsymbol{\theta}$ used in [\(4.5\)](#) is restricted to smooth vector fields, which vanish on the non optimizable subset Γ_N and Γ_D of the boundaries of shapes.

The shapes derivatives of the volume and compliance functions defined in [\(4.3\)](#) are classically given by (see for instance [[7](#)]):

$$\operatorname{Vol}'(\Omega)(\boldsymbol{\theta}) = \int_{\Gamma} \boldsymbol{\theta} \cdot \mathbf{n} dy, \quad \text{and } C'(\Omega)(\boldsymbol{\theta}) = - \int_{\Gamma} Ae(\mathbf{u}_\Omega) : e(\mathbf{u}_\Omega)(\boldsymbol{\theta} \cdot \mathbf{n}) dy. \quad (4.6)$$

where we recall that dy stands for the surface measure on Γ . As far as the shape derivative of the geometric constraint $P(\Omega)$ in (4.4) is concerned, the following (equivalent) surface and volumetric expressions were obtained in our previous work [2]:

$$P'(\Omega)(\boldsymbol{\theta}) = \int_{D \setminus \bar{\Sigma}} j'(d_\Omega(x)) d'_\Omega(\boldsymbol{\theta})(x) dx = \int_\Gamma u \boldsymbol{\theta} \cdot \mathbf{n} dy \quad (4.7)$$

where the ‘‘Eulerian’’ derivative $d'_\Omega(\boldsymbol{\theta})$ of the signed distance function d_Ω is defined on $U = D \setminus \bar{\Sigma}$ by:

$$\forall x \in D \setminus \bar{\Sigma}, \quad d'_\Omega(\boldsymbol{\theta})(x) = -\boldsymbol{\theta}(p_{\partial\Omega}(x)) \cdot \mathbf{n}(p_{\partial\Omega}(x)), \quad (4.8)$$

and the scalar function $u : \Gamma \rightarrow \mathbb{R}$ reads:

$$u(y) = - \int_{s \in \text{ray}(y)} j'(d_\Omega(s)) \prod_{1 \leq i \leq n-1} (1 + \kappa_i(y) d_\Omega(s)) ds,$$

as follows from an application of the coarea formula with the help of the material recalled in Section 3.1.

Using the conclusions of Section 2, the function u in (4.7) may be conveniently evaluated by solving the variational problem:

$$\text{Find } u \in V_\omega \text{ such that } \forall v \in V_\omega, \quad \int_{\partial\Omega} uv dy + \int_{D \setminus \bar{\Sigma}} \omega (\nabla d_\Omega \cdot \nabla u) (\nabla d_\Omega \cdot \nabla v) dx = - \int_{D \setminus \bar{\Sigma}} j'(d_\Omega(x)) v dx, \quad (4.9)$$

for a suitable weight ω satisfying (H1) to (H3), as discussed in Section 3. Note how easily (4.7) is retrieved by taking the test function $v = -d'_\Omega(\boldsymbol{\theta})$ in (4.9). Since $\boldsymbol{\theta} \cdot \mathbf{n} \in L^\infty(\partial\Omega)$, the derivative given by (4.8) is indeed an admissible test function $d'_\Omega(\boldsymbol{\theta}) \in V_\omega$, for it belongs to $L^\infty(D \setminus \bar{\Sigma})$ and is constant along normal rays.

In our numerical implementation, we set $\omega = 1/(1 + 100|d_\Omega \Delta d_\Omega|^{3.5})$ in order to solve (4.9) with \mathbb{P}_1 finite elements. This weight is not exactly of the form of Lemma 6, but works well in numerical practice. Very similar arguments to those of the proof of Lemma 6 would in fact show the validity of assumptions (H1) to (H3). Note, however, that $\omega^{-1} \in L^1(U)$ may not rigorously hold as in the numerical examples of the previous section (see Remark 14).

Let us now outline briefly the numerical implementation of the above shape optimization framework. All the involved finite element computations are performed within the **FreeFem++** environment [38]—whether they are related to the resolution of the physical system (4.1) or to that of our variational problem (4.9). The resolution of the constrained optimization program (4.2) is carried out by the discretization of a specific gradient flow in the spirit of [55], see the details of our algorithm in our recent work [35]. When it comes to representing shapes and their evolution in the course of the iterative resolution of (4.2), the level set based mesh evolution method of [3, 34] is used, as a convenient combination of the ‘classical’ level set method for shape and topology optimization [7, 60] and the **mng** open-source mesh modification algorithm [20].

4.2. Shape optimization under a maximum thickness constraint

We first consider structural optimization problems featuring a maximum thickness constraint. Following the work in [6, 49], a shape $\Omega \subset D$ is said to have maximum thickness lower than $d_{\max} > 0$ provided:

$$\forall x \in \Omega, \quad d_\Omega(x) \geq -d_{\max}/2. \quad (4.10)$$

Loosely speaking, this amounts to saying that the skeleton Σ of Ω lies at a distance of at most d_{\max} from the boundary $\partial\Omega$. Following closely [6, 49], the pointwise constraint (4.10) is relaxed into a single integral constraint formulated in terms of the following penalty functional $P_{\text{MaxT}}(\Omega)$:

$$P_{\text{MaxT}}(\Omega) \leq \frac{d_{\max}}{2}, \quad \text{where } P_{\text{MaxT}}(\Omega) := \left(\frac{\int_\Omega h(d_\Omega) d_\Omega^2 dx}{\int_\Omega h(d_\Omega) dx} \right)^{1/2}, \quad (4.11)$$

and h is a regularized the Heaviside function centered on $d_{\max}/2$:

$$\forall x \in \mathbb{R}, \quad h(x) = \frac{1}{2} \left(1 + \tanh \left(\frac{x - d_{\max}/2}{\alpha_f d_{\max}/2} \right) \right). \quad (4.12)$$

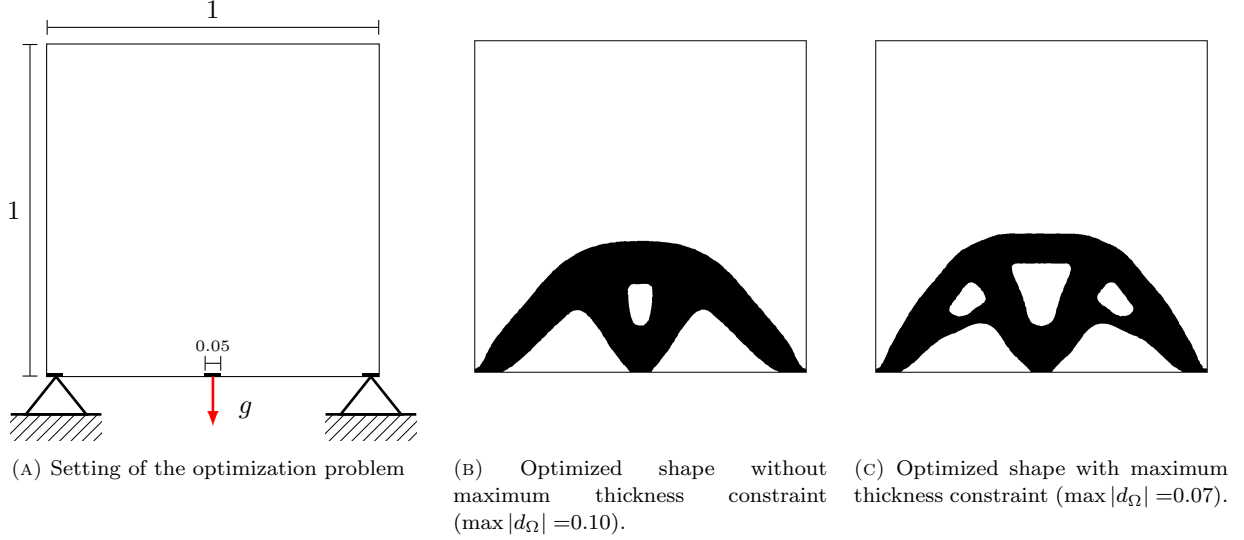


FIGURE 17. Physical setting and optimized shapes obtained in the two-dimensional arch optimization test-case of Section 4.2.1.

The parameter α_f in (4.12) tunes the level of regularization; in our context, it is set to $\alpha_f = \frac{4h_{\max}}{5d_{\max}}$, where h_{\max} is the maximum edge length of the computational mesh of D .

A simple calculation yields the shape derivative of $P_{\text{MaxT}}(\Omega)$:

$$\begin{aligned}
 P'_{\text{MaxT}}(\Omega)(\boldsymbol{\theta}) = & -\frac{1}{2P_{\text{MaxT}}(\Omega)} \frac{\int_{\Omega} h(d_\Omega) d_\Omega^2 dx}{\left(\int_{\Omega} h(d_\Omega) dx\right)^2} h(0) \int_{\Gamma} \boldsymbol{\theta} \cdot \mathbf{n} dy \\
 & + \frac{1}{2P_{\text{MaxT}}(\Omega)} \int_{\Omega} \left[\frac{h'(d_\Omega) d_\Omega^2 + 2h(d_\Omega) d_\Omega}{\int_{\Omega} h(d_\Omega) dz} - \frac{\int_{\Omega} h(d_\Omega) d_\Omega^2 dz}{\left(\int_{\Omega} h(d_\Omega) dz\right)^2} h'(d_\Omega) \right] d'_\Omega(\boldsymbol{\theta}) dx, \quad (4.13)
 \end{aligned}$$

where the term involving $d'_\Omega(\boldsymbol{\theta})$ is then computed using the variational formulation (4.9).

In the forthcoming examples of Sections 4.2.1 and 4.2.2, we solve the optimization problem of minimizing the volume $\text{Vol}(\Omega)$ of the structure while imposing that the compliance $C(\Omega)$ do not exceed a given threshold g_{\max} , as well as the maximum thickness constraint (4.11), namely:

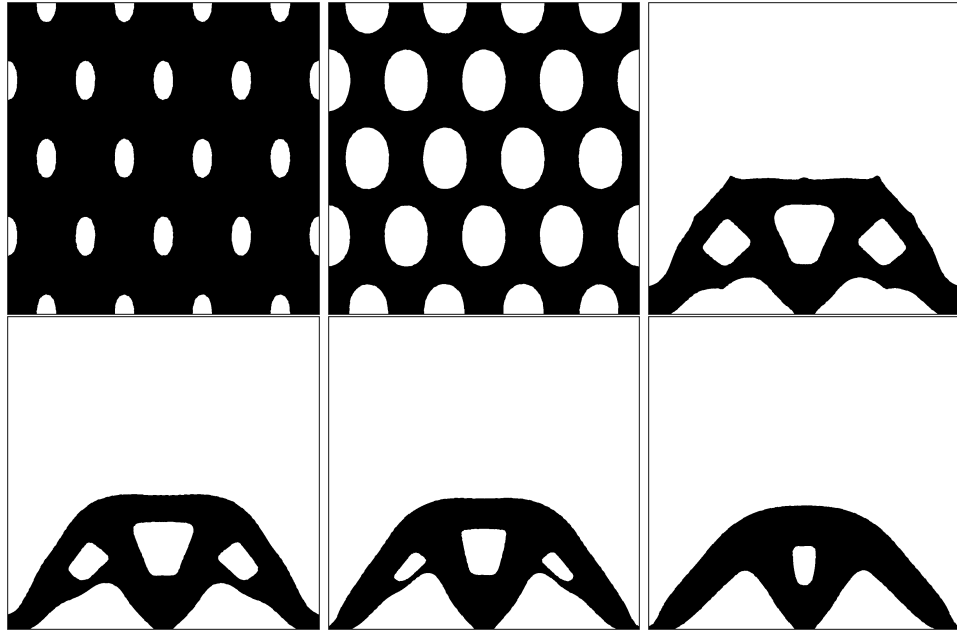
$$\begin{aligned}
 & \min \text{Vol}(\Omega) \\
 & \text{s.t.} \begin{cases} C(\Omega) \leq g_{\max} \\ P_{\text{MaxT}}(\Omega) \leq \frac{d_{\max}}{2}. \end{cases} \quad (4.14)
 \end{aligned}$$

4.2.1. Optimization of the shape of a two-dimensional arch

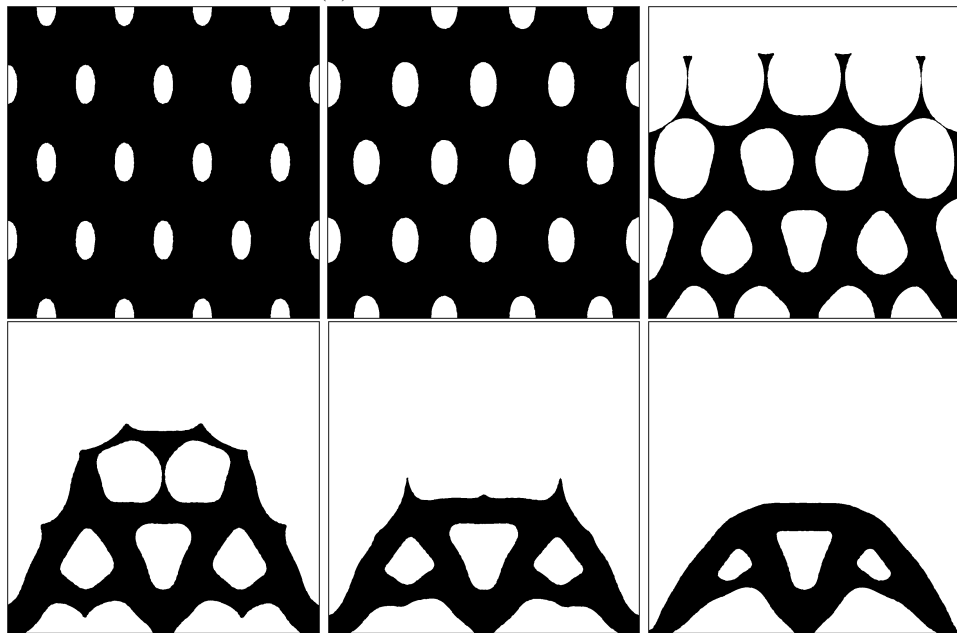
Our first test-case reproduces that considered in §8.1.1 of [6], whose setting is displayed on Figure 17a: in a square-shaped domain D , the considered shapes are clamped on both their bottom left and bottom right corners, and a vertical load $\mathbf{g} = (0, -20)$ is applied at the middle of their bottom side.

Starting from an initial shape arbitrarily perforated with several holes, the optimization problem (4.14) is solved with and without including the maximum thickness constraint, using the numerical values $g_{\max} = 7.00$ and $d_{\max} = 0.12$ when they are relevant.

The resulting optimized shapes in both cases are displayed on Figs. 17b and 17c. Several intermediate shapes as well as the convergence histories of the computation are displayed on Figs. 18 to 20. The obtained



(A) Without thickness constraint.



(B) With maximum thickness constraint.

FIGURE 18. Iterations 0, 5, 40, 65, 100 and 200 in the shape optimization example of a 2-d arch of Section 4.2.1.

shapes are quite analogous to those obtained by [6, 49] where the calculation of the shape derivative of $P_{\text{MaxT}}(\Omega)$ relied on a direct numerical integration along rays.

4.2.2. Optimization of a two-dimensional MBB-Beam

We consider now the classical MBB-Beam test-case depicted on Figure 21a: in a box D with dimensions 3×1 , a material shape Ω is constrained to no horizontal motion on the left boundary, and to no vertical motion on the bottom right corner. A vertical load $\mathbf{g} = (0, -10)$ is applied on the top left corner, and the

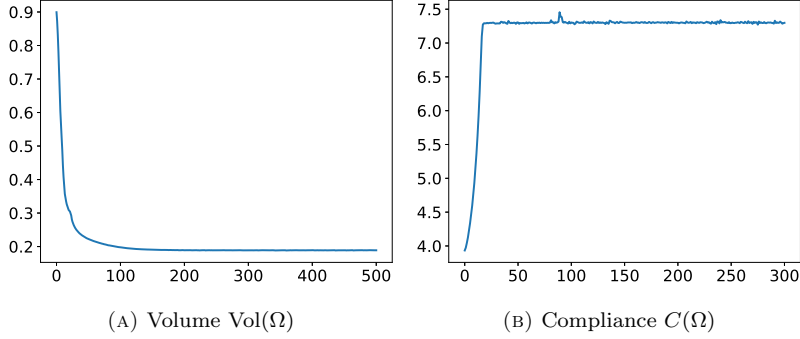


FIGURE 19. Optimization histories for the optimization problem (4.14) *without* maximum thickness constraint. Final values: volume $\text{Vol}(\Omega) = 0.19$, and compliance $C(\Omega) = 7.00$.

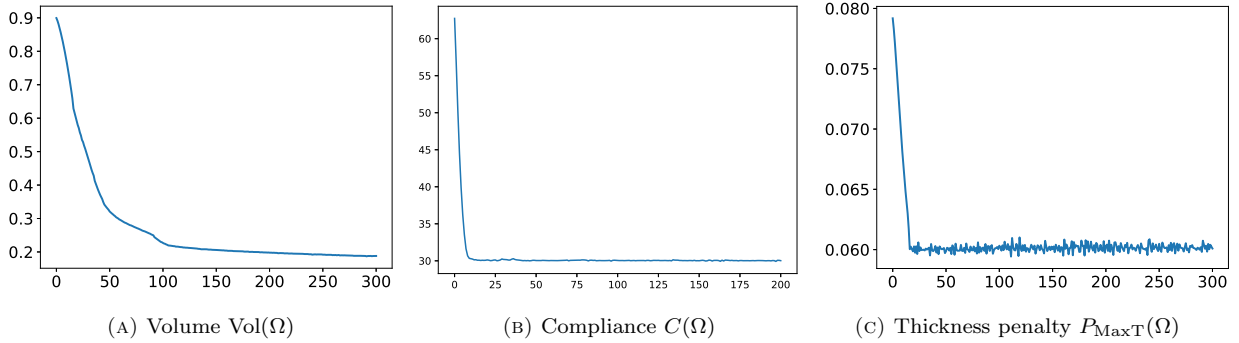


FIGURE 20. Optimization histories for the optimization problem (4.14) *with* maximum thickness constraint. Final values: volume $\text{Vol}(\Omega) = 0.19$, compliance $C(\Omega) = 7.30$, and penalty $P_{\text{MaxT}}(\Omega) = 0.06$.

optimization problem (4.14) is considered again, with the numerical values for the thresholds $g_{\text{max}} = 30.00$ and $d_{\text{min}} = 0.16$.

The resulting optimized shapes with and without including the maximum thickness constraint in (4.14) are represented on Figs. 21b and 21c and the convergence histories of the computation are shown on Figs. 22 to 23.

4.3. Shape optimization examples under a minimum thickness constraint

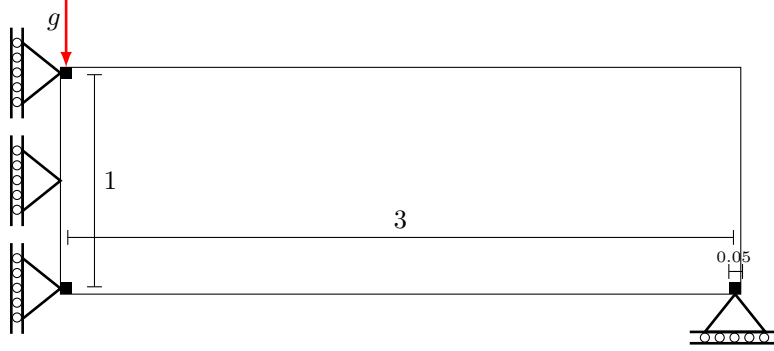
We now turn to the implementation of a minimum thickness constraint thanks to our variational method. Following [6, 49], we say a shape Ω has a thickness greater than d_{min} if

$$\forall y \in \partial\Omega, \zeta_-(y) < -d_{\text{min}}/2. \quad (4.15)$$

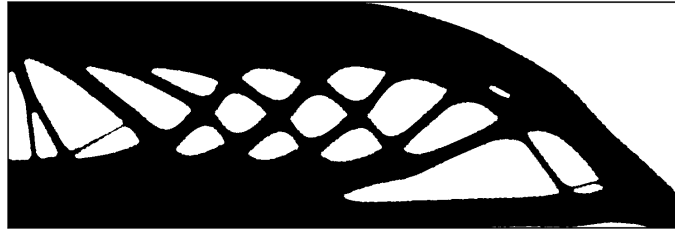
In other words, the boundary $\partial\Omega$ is at a minimum distance $d_{\text{min}}/2$ of the part $\Sigma \cap \Omega$ of the skeleton located inside the shape.

Enforcing a minimum thickness as a hard constraint (*i.e.* rather than a penalty term in the objective function) is by no means a straightforward task, because:

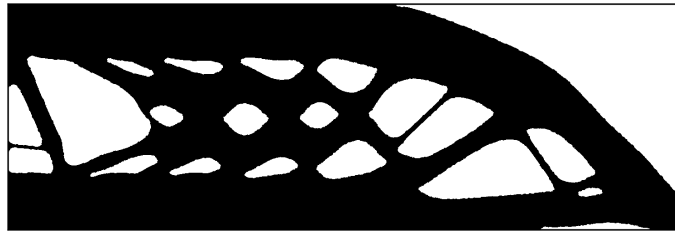
- (1) our definition (4.15) of minimum thickness involve the distance to the skeleton ζ_- , which is not differentiable with respect to the shape,
- (2) it is not clear how to formulate (4.15) by mean of a penalty functional such as (4.11) to penalize localizations on the shape that do not meet the thickness requirement,
- (3) even if we were able to enforce the constraint at each iteration, such would prevent topology changes to occur naturally, which would be an issue.



(A) Setting of the optimization problem. Small black areas correspond to non optimizable parts of the design domain.



(B) Optimized shape without maximum thickness constraint ($\max d_\Omega = 0.36$).



(C) Optimized shape with maximum thickness constraint.

FIGURE 21. Physical setting and obtained optimized shapes in the 2-d MBB beam optimization test-case of Section 4.2.2 ($\max |d_\Omega| = 0.26$).

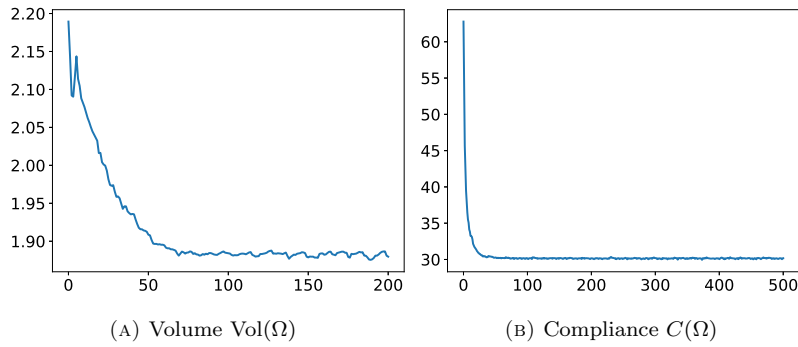


FIGURE 22. Optimization histories for the shape optimization problem (4.14) of the 2-d MBB-Beam of Section 4.2.2 *without* including a maximum thickness constraint. Final values: volume $\text{Vol}(\Omega) = 1.88$, and compliance $C(\Omega) = 29.96$.

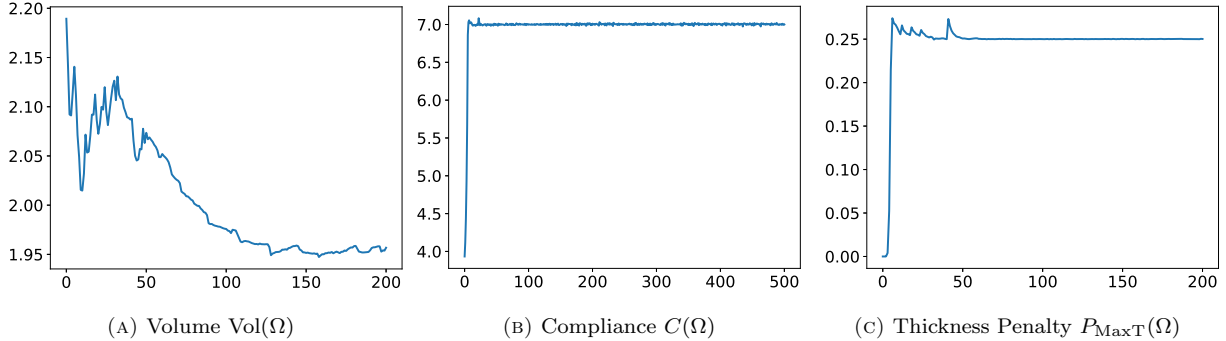


FIGURE 23. Optimization histories for the shape optimization problem (4.14) of the 2-d MBB-Beam of Section 4.2.2 with maximum thickness constraint. Final values: $\text{Vol}(\Omega) = 1.96$, compliance: $C(\Omega) = 30.03$, and penalty $P_{\text{MaxT}}(\Omega) = 0.25$.

We propose in the following a more flexible setting to enforce such a requirement in a structural optimization problem. Elaborating on ideas proposed in [49, 16, 17, 46], the minimum thickness requirement is implemented in the objective function rather than in the constraints: we minimize a penalty functional $P_{\text{MinT}}(\Omega)$ for the minimum thickness under constraints on the volume and compliance of shapes, i.e. we solve:

$$\begin{aligned} \min \quad & P_{\text{MinT}}(\Omega) \\ \text{s.t.} \quad & \begin{cases} C(\Omega) \leq g_{\max} \\ \text{Vol}(\Omega) \leq V_{\max}. \end{cases} \end{aligned} \quad (4.16)$$

This strategy is expected to work because our optimization algorithm is designed to satisfy violated constraints *first*, before then attempting to reduce the objective function while maintaining the constraints respected.

The penalty functional $P_{\text{MinT}}(\Omega)$ we considered in our case when solving (4.16) is taken from [49] and is specially designed to have a zero derivative when the constraint (4.15) is satisfied:

$$P_{\text{MinT}}(\Omega) = - \int_{\Omega} d_{\Omega}^2 \max(d_{\Omega} + d_{\min}/2, 0)^2 dx. \quad (4.17)$$

The shape derivative of $P_{\text{MinT}}(\Omega)$ is given by

$$P'_{\text{MinT}}(\Omega)(\boldsymbol{\theta}) = - \int_{\Omega} 2(d_{\Omega} \max(d_{\Omega} + d_{\min}/2, 0)^2 + d_{\Omega}^2 \max(d_{\Omega} + d_{\min}/2, 0)) d'_{\Omega}(\boldsymbol{\theta}) dx. \quad (4.18)$$

Note that an increase of perimeter entails a decrease in the value of $P_{\text{MinT}}(\Omega)$, but this behavior is tempered in our case by the volume constraint. Variants can be considered to address such an issue, and we refer to [49] for the details.

4.3.1. Optimization of the shape of a 2-d cantilever beam

We first consider the classical two-dimensional cantilever benchmark example, as depicted on Figure 24: in a box D with size 2×1 , shapes Ω are clamped on their left-hand side, and a vertical load $\mathbf{g} = (0, -10)$ is applied on the middle of their right-hand side.

The optimization problem (4.16) is solved with the parameter values $g_{\max} = 70.00$ and $V_{\max} = 0.80$. The resulting optimized shapes are displayed on Figure 25 without including minimum thickness constraint (we minimize the volume $V(\Omega)$ subject to the compliance constraint as in (4.14)), and with the minimum thickness for two different values of d_{\min} . The corresponding convergence histories are shown on Figs. 26 to 28.

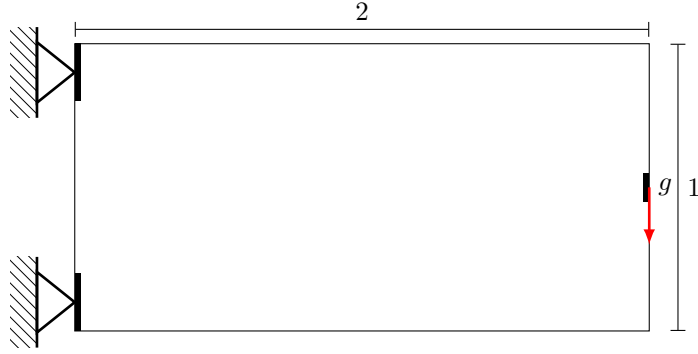
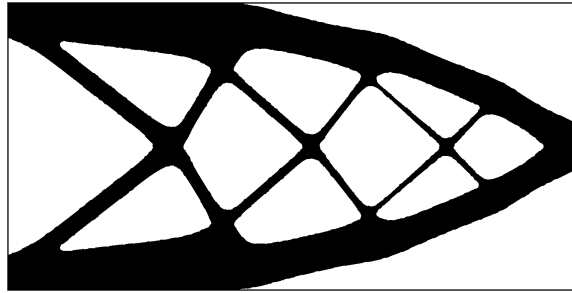
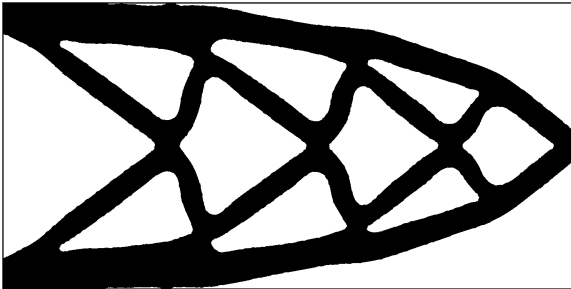


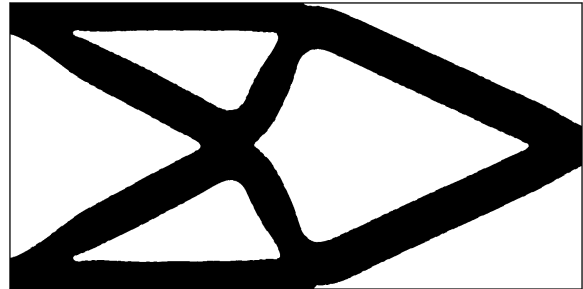
FIGURE 24. Setting of the cantilever test case of Section 4.3.1.



(A) Optimized shape without including minimum thickness constraint.



(B) Optimized shape with minimum thickness constraint ($d_{\min} = 0.1$).



(c) Optimized shape with minimum thickness constraint ($d_{\min} = 0.2$).

FIGURE 25. Optimization of the shape of the 2-d cantilever of Section 4.3.1 under minimum thickness constraint.

4.3.2. Shape optimization of a 2-d MBB Beam under a minimum thickness constraint

We now apply the same methodology on the MBB beam test-case of Section 4.2.2. Optimized shapes are compared on Figure 29 without minimum thickness (the result being that of Figure 21b), and for two values of d_{\min} . The corresponding convergence histories are shown on Figs. 22, 30 and 31. Finally, some intermediate shapes of the optimization process are reprinted on Figure 32.

One observes that for the last case of Figure 29c with $d_{\min} = 0.2$, the minimum thickness constraint is not satisfied everywhere but a substantial improvement is visible over the first design of Figure 29a. Notably, this approach is sufficiently flexible to guide the optimization path towards shapes with very different topologies.

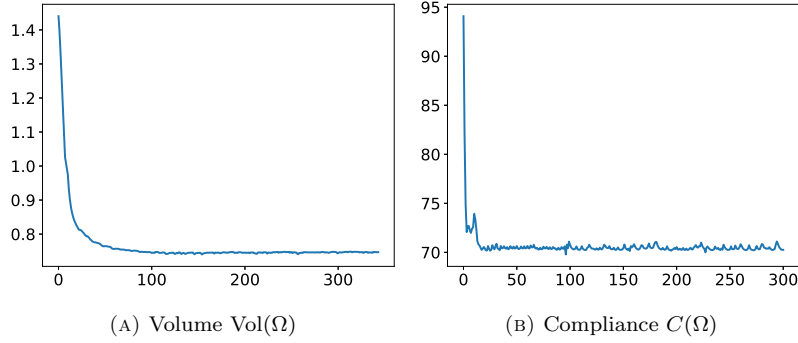


FIGURE 26. Optimization history for the 2-d cantilever optimization problem (4.14) of Section 4.3.1 *without* minimum thickness constraint. Final values: volume $\text{Vol}(\Omega) = 0.75$, and compliance $C(\Omega) = 70.29$.

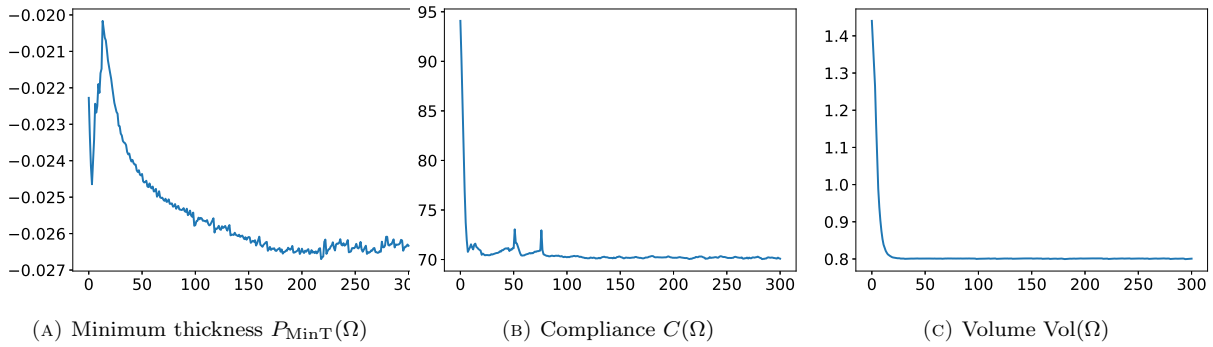


FIGURE 27. Optimization history for the 2-d cantilever optimization problem (4.14) of Section 4.3.1 *with* minimum thickness constraint. ($d_{\text{min}} = 0.1$) Final values : volume $\text{Vol}(\Omega) = 0.80$, compliance: $C(\Omega) = 70.26$.

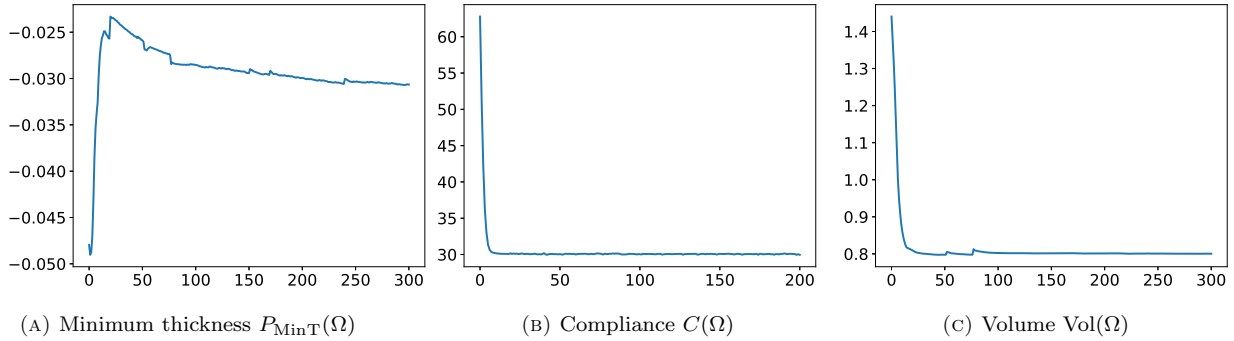
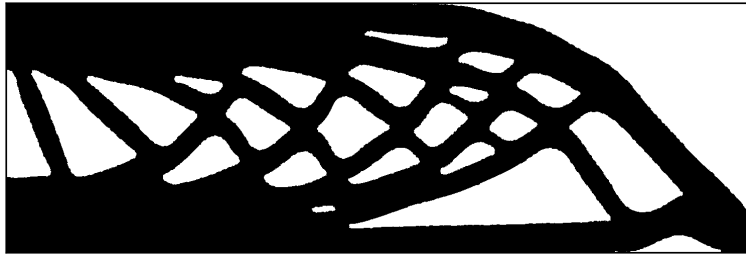


FIGURE 28. Optimization history for the 2-d cantilever optimization problem (4.14) of Section 4.3.1 *with* minimum thickness constraint. ($d_{\text{min}} = 0.2$). Final values: volume $\text{Vol}(\Omega) = 0.80$, and compliance: $C(\Omega) = 70.09$.



(A) Optimized shape without minimum thickness constraint.



(B) Optimized shape with minimum thickness constraint ($d_{\min} = 0.1$).



(C) Optimized shape with minimum thickness constraint ($d_{\min} = 0.2$).

FIGURE 29. Minimum thickness optimization for a cantilever test case of Section 4.3.1.

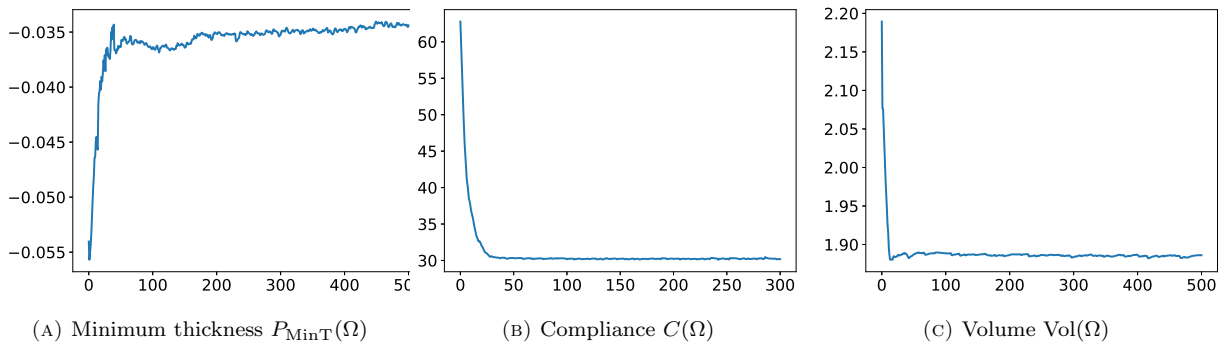


FIGURE 30. Convergence histories for the MBB-Beam test-case of Section 4.3.1 with minimum thickness constraint. ($d_{\min} = 0.1$). Final values: volume $\text{Vol}(\Omega) = 1.89$, and compliance $C(\Omega) = 30.17$.

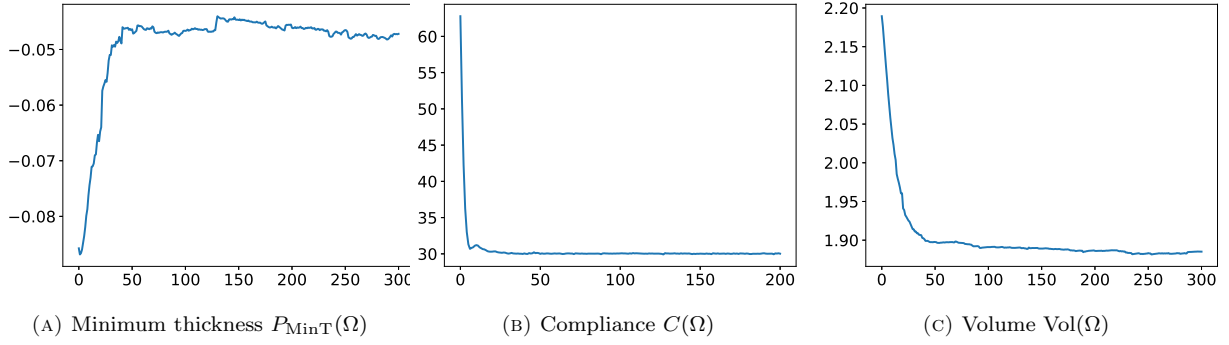
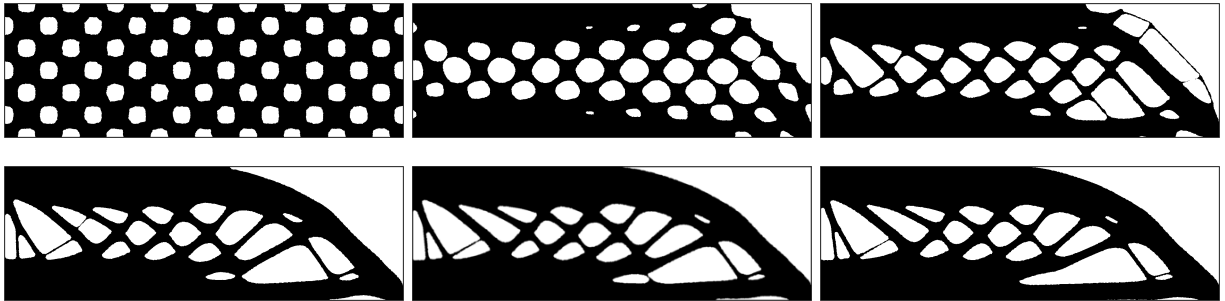
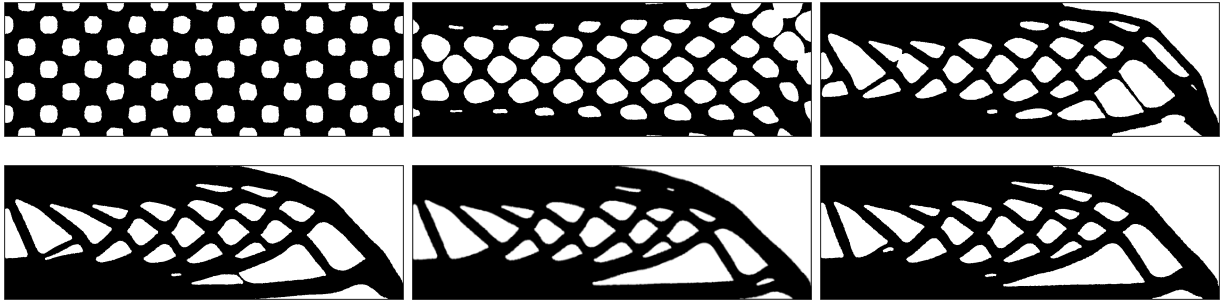


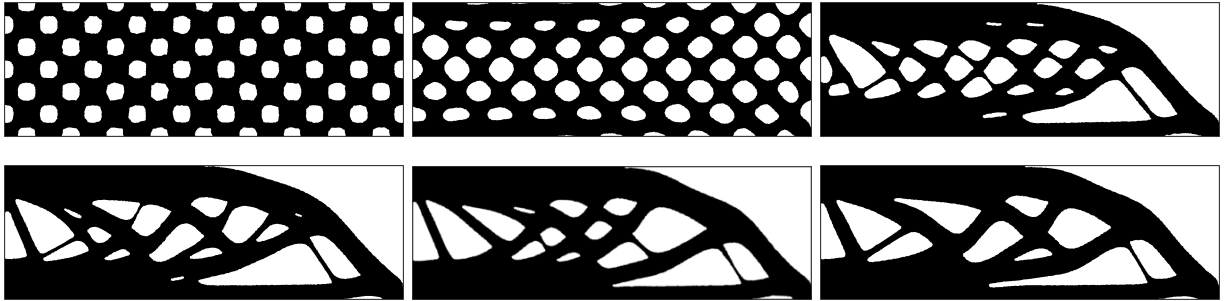
FIGURE 31. Convergence curves for the MBB-Beam test-case of Section 4.3.1 with minimum thickness constraint ($d_{\text{min}} = 0.2$). Final values: volume $\text{Vol}(\Omega) = 1.89$, and compliance $C(\Omega) = 30.18$.



(A) No minimum thickness constraint (setting of Figure 21b).



(B) Minimum thickness constraint with $d_{\text{min}} = 0.1$



(C) Minimum thickness constraint with $d_{\text{min}} = 0.2$

FIGURE 32. Optimization histories for the MBB beam test-case with minimum thickness constraints of Section 4.3.2: Iterations 0, 10, 40, 100, 150 and 200.

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