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VARIATION OF SINGULAR KÄHLER-EINSTEIN METRICS: POSITIVE KODAIRA DIMENSION

by

Junyan Cao, Henri Guenancia & Mihai Păun

Abstract. — Given a Kähler fiber space $p : X \to Y$ whose generic fiber is of general type, we prove that the fiberwise singular Kähler-Einstein metric induces a semipositively curved metric on the relative canonical bundle $K_{X/Y}$ of p. We also propose a conjectural generalization of this result for relative twisted Kähler-Einstein metrics. Then we show that our conjecture holds true if the Lelong numbers of the twisting current are zero. Finally, we explain the relevance of our conjecture for the study of fiber-wise Song-Tian metrics (which represent the analogue of KE metrics for fiber spaces whose generic fiber has positive but not necessarily maximal Kodaira dimension).

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Introduction

Let $p : X \to Y$ be a Kähler fiber space. By this we mean that p is a proper, surjective holomorphic map with connected fibers such that the total space X is Kähler. Important questions in birational geometry (such as e.g. Iitaka C_{nm} conjecture) are treated by investigating the properties of direct images

$$p_\star(mK_{X/Y}) \tag{0.1}$$

where m is a positive integer, and $K_{X/Y} := K_X - p^\star(K_Y)$ is the relative canonical bundle of the map p. In other words, one considers the variation of the pluricanonical linear series $H^0(X_y, mK_{X/Y}|_{X_y})$ for $y \in Y$ and some fixed $m \gg 0$.

In this article we will adopt a slightly different point of view by working with an object which "encodes" the asymptotic behavior of the entire canonical ring $\bigoplus_m p_\star(mK_{X/Y})$. If the generic fiber of p is of general type, then this turns out to be the singular Kähler-Einstein metric. The direct image (0.1) is positively curved, and our main concern in this article is to show that the same holds true for the metric induced on $K_{X/Y}$ by fiber-wise singular Kähler-Einstein metrics.

General set-up and main results. — Let $p : X \to Y$ be a Kähler fiber space. We will systematically use the notation $Y^\circ \subset Y$ for a set contained in the regular values of the map p such that the complement $Y \setminus Y^\circ$ is analytic. Let $X^\circ := p^{-1}(Y^\circ)$ be its inverse image.

Let (L, h_L) be a $\mathbb{Q}$-line bundle endowed with a singular metric h_L whose curvature current is positive, i.e.

$$\Theta_{h_L}(L) \geqslant 0. \tag{0.2}$$

We assume for the moment that the relative adjoint bundle $K_{X/Y} + L$ is p-big. In many important geometric settings (including the case $\mathcal{I}(h_L) = \mathcal{O}_X$) for every $y \in Y$ general enough there exists a unique closed positive current $\omega_{\mathrm{KE},y} \in c_1(K_{X_y} + L)$ such that

$$\operatorname{Ric} \omega_{\mathrm{KE},y} = -\omega_{\mathrm{KE},y} + \Theta_{h_L}(L) \tag{0.3}$$

The precise framework for (0.3) to hold will become clear in Section 1. In what follows $\omega_{\mathrm{KE},y}$ will be referred to as *singular Kähler-Einstein metric* by analogy with the case L trivial and K_{X_y} ample.

The results we establish in this article are converging towards the following general problem.

Conjecture 0.1. — *In the above set-up, the relative Kähler-Einstein metrics $(\omega_{\mathrm{KE},y})_{y \in Y^\circ}$ induce a metric $e^{-\phi_{\mathrm{KE}}}$ on $K_{X^\circ/Y^\circ} + L|_{X^\circ}$ which is positively curved and which extends canonically across $X \smallsetminus X^\circ$ to a positively curved metric on $K_{X/Y} + L$.*

As consequence of important approximation results in pluripotential theory we show that a much more general form of the conjecture above would follow provided that one is able to deal with the case where $\Theta_{h_L}(L)$ equals the current of integration along a divisor with simple normal crossings support and coefficients in $(0, 1)$ plus a smooth form, cf. Theorem 1.6.

Our main theorem states the following.

Theorem A. — *Conjecture 0.1 holds true if the Lelong numbers of the curvature current corresponding to h_L are zero on the p-inverse image of a Zariski open subset of Y.*

For example, if $L = 0$ then Theorem A shows that the metric on $K_{X/Y}$ induced by the fiberwise KE current is positively curved.

One of the main motivations for Conjecture 0.1 will become clear from the context we next discuss. Let $p : X \to Y$ be a Kähler fiber space, and let B be an effective $\mathbb{Q}$-divisor on X with coefficients in $(0, 1)$ such that $B|_{X_y}$ has simple normal crossings support for y generic. Assume furthermore that $K_{X_y} + B|_{X_y}$ has *positive* Kodaira dimension. Here we use the notation "B" rather than L in order to emphasize that the metric is fixed.

There exists a relative version of the so-called *canonical metric* introduced by Song and Tian [**ST12**] and generalized by Eyssidieux, Guedj and Zeriahi [**EGZ18**]. It is defined on the base Z' of a birational model $q' : X' \to Z'$ of the relative Iitaka fibration $q : X \dashrightarrow Z$ over Y, cf. Section 4 for more details. In case of a family p whose generic fiber has maximal Kodaira dimension, the metric in [**ST12**] coincides with the singular Kähler-Einstein metric (up to a birational transformation).

Theorem B. — *Let $p : X \to Y$ be a Kähler fiber space such that for y generic, $\kappa(K_{X_y} + B_y) > 0$. Let $f : X \dashrightarrow Z$ be the relative Iitaka fibration of $K_{X/Y} + B$, and let $f' : X' \to Z'$ a birational model of f such that X' and Z' are smooth.*

$$\begin{tikzcd} X' \arrow[rr, "f'"] \arrow[d] & & Z' \arrow[d] \\ X \arrow[rr, dashed, "f"] \arrow[dr, "p"'] & & Z \arrow[dl] \\ & Y & \end{tikzcd}$$

Let $\omega_{\mathrm{can},y}$ be the canonical metric on Z'_y of the pair (X'_y, B'_y); it induces a current $\omega^{\circ}_{\mathrm{can}}$ over the smooth locus of $Z' \to Y$.
Moreover we assume that Conjecture 0.1 holds true. Then the current $\omega^{\circ}_{\mathrm{can}}$ is positive and extends canonically to a closed positive current on Z'.

Coming back to the absolute case, let (X, B) be a compact Kähler klt pair such that $\kappa(X, B) > 0$ and let $f' : X' \to Z'$ be a bimeromorphic model of the Iitaka fibration of $K_X + B$ such that X' and Z' are smooth. On top of the canonical metric ω_{can} on Z', another important metric comes into play which it related to the direct image $f'_*(m(K_{X'/Z'} + B))$ for m large and divisible enough. More precisely, there exist bimeromorphic modifications $\pi : \widehat{X} \to X'$, $\mu : \widehat{Z} \to Z'$ as well as a Kähler fiber space $\widehat{f} : \widehat{X} \to \widehat{Z}$ fitting the following commutative diagram

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{f}"] \arrow[d, "\pi"'] & \widehat{Z} \arrow[d, "\mu"] \\ X' \arrow[r, "f'"] & Z' \end{tikzcd}$$

such that there exists a "NS-type" Kähler-Einstein metric ω_{KE} on $\widehat{Z}$. This means that the Narasimhan-Simha h_{NS} on the line bundle $\widehat{L} := \frac{1}{m}\widehat{f}_*(m(K_{\widehat{X}/\widehat{Z}} + B))^{**}$ enjoys some particular integrability properties so that the equation

$$\mathrm{Ric}\,\omega_{\mathrm{KE}} = -\omega_{\mathrm{KE}} + i\Theta_{h_{\mathrm{NS}}}(\widehat{L})$$

is satisfied on $\widehat{Z}$ in the sense of Definition 1.1, cf subsection 4.1 for more details. The following result relates the "NS-type" Kähler-Einstein metric ω_{KE} on $\widehat{Z}$ and the canonical metric ω_{can} on Z', cf. Proposition 4.6.

Proposition C. — *With the notation above, the Kähler-Einstein and canonical metrics are related by the following identity holding on Z':*

$$\mu_*\omega_{\mathrm{KE}} = \omega_{\mathrm{can}}.$$

Previously known results. — There are basically two types of techniques used in order to address the questions we are interested in here, due to Schumacher and Tsuji in [**Sch08**] and [**Tsu11**], respectively. The former concerns the smooth case (e.g. K_{X_y} ample) and it is based on a maximum principle. The later consists in showing that non-singular KE metrics can be obtained by an iteration scheme involving pluricanonical sections normalized in a specific way. Both methods have their advantages and flaws. For example, it is difficult to conceive that Schumacher method can be used in the presence of base points. Also, at first sight the method of Tsuji looks very general. However, it uses in an essential manner the asymptotic expansion of Bergman kernels, which depends on at least two derivatives of the metric. This is the main reason why we cannot deal with the general case of a line bundle (L, h_L) as in Conjecture 0.1.

In the paragraph that follows we recall the definitions of relative (singular) Kähler-Einstein metrics and collect some earlier results.

Relative singular Kähler-Einstein metrics. — A singular Kähler-Einstein metric is a generic term to refer to a non-smooth, closed, positive, $(1,1)$-current ω that satisfies a Kähler-Einstein like equation in a weak sense. Among the most natural examples are: Kähler-Einstein with conic singularities, mentioned above, cf. also [**Bre13**, **CGP13**, **JMR16**], Kähler-Einstein metrics on singular varieties, cf. [**EGZ09**, **BBE+19**, **BG14**]. These metrics are obtained by solving an equation of the form

$$\operatorname{Ric}\omega = \lambda\omega + T \tag{0.4}$$

on a compact Kähler manifold X, where T is a closed $(1,1)$-current (e.g. the current of integration along a $\mathbb{R}$-divisor with coefficients in $]-\infty,1]$). The cohomology class $\alpha \in H^{1,1}(X,\mathbb{R})$ of ω is determined by the equation unless $\lambda = 0$, and it may be degenerate. That is, instead of being Kähler, α may be semipositive and big, or even merely big. The singularities of ω may then appear because of the singularities of T or the non-Kählerness of $\{\omega\}$. The singularities of the first type are rather well known when T is a current of integration along an effective divisor with snc support (one gets conic or cusp singularities, cf. e.g. [**Kob84**, **TY87**, **Gue14**]), but they are mostly mysterious in the second case with a few numbers of exceptions like when X is a resolution of singularities of a variety Y with orbifold singularities, or isolated conical singularities cf. [**HS17**], and α is the pull-back of a Kähler class on Y.

Earlier results. — If the generic fiber has ample canonical bundle, that is, if K_{X_y} is ample for any $y \in Y^\circ$, then it follows from the Aubin-Yau theorem [**Aub78**] [**Yau78**] that one can endow each smooth fiber with a Kähler-Einstein metric with $\lambda = -1$. This induces a metric on $K_{X/Y}|_{X^\circ}$ whose curvature form $\omega^\circ_{\mathrm{KE}}$ is smooth (by implicit function theorem). Moreover, the restriction $\omega^\circ_{\mathrm{KE}}|_{X_y}$ coincides with the KE metric. The surprizing important fact is that $\omega^\circ_{\mathrm{KE}} \geqslant 0$ on X°, as it has been showed by Schumacher [**Sch08**] and independently by Tsuji [**Tsu11**].

• Following Schumacher's strategy, one obtains in [**Pău17**] a generalization of this result to the Kähler setting (including the extension property) only assuming that $K_{X_y} + \{\beta\}|_{X_y}$ is relatively ample for some smooth, semipositive, closed $(1,1)$-form β on X.

Based on this approach again, the second name author studied the conic analogue of these questions, cf. [**Gue20**]: let $B = \sum b_i B_i$ is a divisor with snc support on X and coefficients in $(0,1)$ and assume that $K_{X_y} + B|_{X_y}$ is ample for $y \in Y^\circ$, the the relative conical Kähler-Einstein metric solution of $\operatorname{Ric}\omega_y = -\omega_y + [B|_{X_y}]$ induces a singular $(1,1)$ current $\omega^\circ_{\mathrm{KE}}$ on X° that is positive, and extends canonical to a positive current $\omega_{\mathrm{KE}} \in c_1(K_{X/Y} + B)$.

We refer to [**Sch12**], [**Cho15**], [**BCS20**] for other applications of this method.

• In the case of a manifold with ample canonical bundle, Tsuji observes that $\omega^\circ_{\mathrm{KE}}$ is the limit of relative Bergman kernels whose variation is known to be semipositive cf. [**BP08**]. The metric induced by fiber-wise Bergman kernels extends, cf. *loc. cit*. Therefore $\omega^\circ_{\mathrm{KE}}$ extends canonically to a current $\omega_{\mathrm{KE}} \in c_1(K_{X/Y})$ on X. See [**Tsu10**] for potential applications.

• In a more general singular case $K_{X_y} + B_y$ big, the fiberwise Kähler-Einstein metrics pick up singularities that are yet to be understood, and neither of the previous approaches seem to work.

About the proof. — The strategy of the proof of Theorem A is explained in detail at the beginning of Section 3 and consists in realizing the singular Kähler-Einstein metric as a limit of suitably chosen and renormalized Bergman kernels, as those are known to vary in a psh way by [**BP08**]. Although the global scheme of our arguments is similar to [**Tsu11**], the level of difficulties induced by the presence of base points in the problems we are treating here is far more severe. In order to overcome them, one has to resort to using numerous intricate approximation processes. Ultimately, our feeling is that the room to manoeuvre is so small that Theorem A

is probably close to the optimal result that our method can reach, aside from the orbifold case discussed in Section 3.6.

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1. Pluricanonical sections and singular Kähler-Einstein metrics

Let X be a compact Kähler manifold of dimension n. Let (L, h_L) be a $\mathbb{Q}$-line bundle endowed with a possibly singular hermitian metric $h_L = e^{-\phi_L}$ with positive curvature, that is

$$\Theta_{h_L}(L) = dd^c\phi_L \geqslant 0$$

in the sense of currents.

We now recall the definition of Kähler-Einstein metric for the pair (X, L) in case $K_X + L$ is big. This definition has been given by [BEGZ10, § 6] when $L = 0$, and can be easily adapted to our slightly more general context.

Definition-Proposition 1.1. — *Let X be a compact Kähler manifold, let (L, h_L) be a $\mathbb{Q}$-line bundle endowed with a singular hermitian metric $h_L = e^{-\phi_L}$ with positive curvature, that is, $\Theta_{h_L}(L) \geqslant 0$ in the sense of currents. We assume moreover that*

(1.1.1) The $\mathbb{Q}$-line bundle $K_X + L$ is big.
(1.1.2) The algebra $R(X, L) = \bigoplus_{m\geqslant 0} H^0(X, \lfloor m(K_X + L)\rfloor)$ is finitely generated.
(1.1.3) For every $p \in \mathbb{N}$ and every $s \in H^0(X, p(K_X + L))$, we have $\int_X |s|^{2/p} e^{-\phi_L} < +\infty$.

Then, there exists a unique closed, positive $(1, 1)$-current ω_{KE} on X which satisfies the following conditions.

(1.1.4) The current ω_{KE} belongs to the big cohomology class $c_1(K_X + L)$ and it has full mass, that is, $\int_X \langle \omega_{\mathrm{KE}}^n \rangle = \mathrm{vol}(K_X + L)$.
(1.1.5) The current ω_{KE} satisfies the following equation in the weak sense of currents

$$\mathrm{Ric}\,\omega_{\mathrm{KE}} = -\omega_{\mathrm{KE}} + \frac{i}{2\pi}\Theta_{h_L}(L).$$

Remark 1.2. — Some remarks are in order.

(a) An important feature of this definition is that it is birationally invariant. More precisely, if $(X, L, e^{-\phi_L})$ satisfies conditions (1.1.1)-(1.1.3) and if $\pi : X' \to X$ is any birational proper morphism, then so does $(X', L', e^{-\phi_{L'}})$ where $L' := \pi^*L, \phi_{L'} := \pi^*\phi_L$. Furthermore, if ω'_{KE} is the Kähler-Einstein metric of $(X', L', e^{-\phi_{L'}})$, then $\omega'_{\mathrm{KE}} = \pi^*\omega_{\mathrm{KE}} + [K_{X'/X}]$.

(b) Conditions (1.1.2) and (1.1.3) are automatically satisfied if the multiplier ideal sheaf of h_L is trivial, that is, if $\mathcal{I}(h_L) = \mathcal{O}_X$. This is clear for (1.1.3). As for (1.1.2), we use the following argument. As $K_X + L$ is big, X is automatically projective and we have

$$K_X + L \equiv_{\mathbb{Q}} A + E$$

for some ample $\mathbb{Q}$-line bundle A and an effective $\mathbb{Q}$-divisor E. From the solution of the openness conjecture, cf. [**Ber15**, **GZ15**], for some m_0 large enough, we have

$$\mathcal{I}(e^{-\phi_L - \frac{1}{m_0}\phi_E}) = \mathcal{O}_X. \tag{1.6}$$

As the question is birationally invariant, by Demailly's regularization theorem, after some birational morphism, we can suppose that

$$L + \frac{1}{m_0}A \equiv_{\mathbb{Q}} B + H$$

for some effective $\mathbb{Q}$-divisor B and a semi-ample $\mathbb{Q}$-line bundle H such that ϕ_L is more singular than ϕ_B, where ϕ_B a canonical singular weight attached to B, cf proof of Lemma 2.3 for instance. Together with (1.6), $B + \frac{1}{m_0}E$ is klt. Thanks to [**BCHM10**], the canonical ring of $K_X + (B + \frac{1}{m_0}E) + H$ is finitely generated. Combining this with the relation

$$(1 + \frac{1}{m_0})(K_X + L) \sim_{\mathbb{Q}} K_X + (B + \frac{1}{m_0}E) + H,$$

the condition (1.1.2) is proved.

(c) If L corresponds to an effective, klt $\mathbb{Q}$-divisor B and ϕ_L is the canonical singular weight on B, then one recovers the standard log Kähler-Einstein metric whose existence follows essentially from [**BEGZ10**], cf. e.g. [**Gue13**, § 2.3].

(d) The condition (1.1.5) can be rewritten in terms of non-pluripolar Monge-Ampère equations as follows:

$$\langle (dd^c\phi_{\mathrm{KE}})^n \rangle = e^{\phi_{\mathrm{KE}} - \phi_L}$$

where ϕ_{KE} is a local weight for ω_{KE} and where $\langle \cdot^n \rangle$ denotes the non-pluripolar Monge-Ampère operator, cf. [**BEGZ10**, Def 1.1 & Prop 1.6].

(e) We will see in the proof that ω_{KE} has minimal singularities in the sense of [**DPS01**, Def 1.4]. Moreover, if h_L is smooth on a non-empty Zariski open subset of X, then one can prove that ω_{KE} is smooth on a Zariski open set by reducing the problem to the semiample and big case and use [**EGZ09**]. To our knowledge, there is still no purely analytical proof of the generic smoothness as explained in the few lines following [**BEGZ10**, Thm. C].

Proof of Definition-Proposition 1.1. — Set $R(X, L) := \bigoplus_{m \geqslant 0} H^0(X, \lfloor m(K_X + L) \rfloor)$, and let us define $X_{lc} := \operatorname{Proj} R(X, L)$ to be the log canonical model of X, cf. e.g. [**BCHM10**, Def. 3.6.7]. Taking a desingularization of the graph of the natural birational map $f : X \dashrightarrow X_{lc}$, one get the following diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ X \arrow[rr, dashed, "f"] & & X_{lc} \end{tikzcd}$$

and the following formula

$$\mu^*(K_X + L) = \nu^*(K_{X_{lc}} + L_{lc}) + F \tag{1.7}$$

where $L_{lc} := f_*L$ (recall that f does not contract any divisor) and F is an effective ν-exceptional divisor. Clearly, $K_{X_{lc}} + L_{lc}$ is ample. Thus, setting $\widetilde{L} := \mu^*L, A := \nu^*(K_{X_{lc}} + L_{lc})$ and $E := F + K_{\widetilde{X}/X}$, the decomposition

$$K_{\widetilde{X}} + \widetilde{L} = A + E$$

is a Zariski decomposition of $K_{\widetilde{X}} + \widetilde{L}$, and we have $\int_{\widetilde{X}} |s|^{\frac{2}{p}} e^{-\phi_{\widetilde{L}}} < +\infty$ for any section $s \in H^0(\widetilde{X}, p(K_{\widetilde{X}} + \widetilde{L}))$ and where $\phi_{\widetilde{L}} := \mu^* \phi_L$. We want to solve the following equation

$$(dd^c \widetilde{\phi})^n = e^{\widetilde{\phi} + \phi_E - \phi_{\widetilde{L}}} \tag{1.8}$$

for $\widetilde{\phi}$ a bounded psh weight on A, where ϕ_E is the canonical singular weight attached to E. Thanks to the results in [Gue13, § 2.3], we are reduced to establishing the following property.

$$e^{\phi_E - \phi_{\widetilde{L}}} \in L^{1+\varepsilon} \quad \text{for some } \varepsilon > 0. \tag{1.9}$$

Take p large enough so that $p\widetilde{L}, pE$ are integral and $|pA|$ is basepoint free. Let $\{\tau_1, \cdots \tau_r\}$ be a basis of $H^0(\widetilde{X}, pA)$. Then $\sum_{i=1}^r |\tau_i|^2$ is non-vanishing everywhere. Let s_{pE} be the canonical section of pE. Thanks to (1.1.3), we have

$$\int_{\widetilde{X}} (\sum_{i=1}^r |\tau_i|^2)^{\frac{1}{p}} |s_{pE}|^{\frac{2}{p}} e^{-\phi_{\widetilde{L}}} < +\infty.$$

Together with the fact that $\sum_{i=1}^r |\tau_i|^2$ is non vanishing everywhere, we get

$$e^{-\phi_{\widetilde{L}} + \phi_E} \in L^1_{\mathrm{loc}}. \tag{1.10}$$

By applying the solution of the generalized openness conjecture [GZ15, Ber15] to the psh weight $\phi_{\widetilde{L}} + (1 - \frac{1}{p})\phi_{pE}$, we see that (1.9) follows from (1.10).

Now, define $\phi := \widetilde{\phi} + \phi_F$. From the Zariski decomposition (1.7), it follows that the psh weight ϕ on $\mu^*(K_X + L)$ has minimal singularities and it satisfies $\langle (dd^c \phi)^n \rangle = \langle (dd^c \widetilde{\phi})^n \rangle = e^{\phi - \phi_{\widetilde{L}}}$ as the operator $\langle \cdot^n \rangle$ puts no mass on proper analytic sets. There exists a unique psh weight ϕ_{KE} on $K_X + L$ such that $\phi = \mu^* \phi_{KE}$. It has automatically minimal singularities and satisfies $\langle (dd^c \phi_{\mathrm{KE}})^n \rangle = e^{\phi_{\mathrm{KE}} - \phi_L}$; this ends the proof. □

It will be convenient to use the following setting.

Setting 1.3. — *Let $p : X \to Y$ be a projective fibration between two smooth Kähler manifolds. Let (L, h_L) be a holomorphic singular hermitian $\mathbb{Q}$-line bundle on X such that $i\Theta_{h_L}(L) \geqslant 0$. Let $Y^\circ \subset Y$ be a Zariski open subset such that where p is smooth over Y°, and set $X^\circ := p^{-1}(Y^\circ)$. Assume that the additional two conditions are satisfied.*

(1.3.11) The $\mathbb{Q}$-line bundle $K_X + L$ is p-big and for every $y \in Y^\circ$, the algebra

$$R(X_y, L) = \bigoplus_{m \geqslant 0} H^0(X_y, \lfloor m(K_{X_y} + L_y) \rfloor)$$

is finitely generated.

(1.3.12) Let $y \in Y^\circ$. For every $m \in \mathbb{N}$ and every $s \in H^0(X_y, m(K_X + L))$, we have $\int_{X_y} |s|_{h_L}^{\frac{2}{m}} < +\infty$.

We can now state the precise form of the conjecture already mentioned in the introduction.

Conjecture 1.4. — *In the Setting 1.3 above, the Kähler-Einstein metrics $(\omega_{\mathrm{KE},y})_{y \in Y^\circ}$ on the smooth fibers in the sense of Definition-Proposition 1.1 induce a metric $e^{-\phi_{\mathrm{KE}}}$ on $K_{X/Y} + L$ over X° such that:*

(1.4.13) $i\Theta_{\phi_{\mathrm{KE}}}(K_{X/Y} + L) \geqslant 0$ on X°.

(1.4.14) The metric $e^{-\phi_{\mathrm{KE}}}$ extends canonically across $X \smallsetminus X^\circ$ and $i\Theta_{\phi_{\mathrm{KE}}}(K_{X/Y} + L) \geqslant 0$ on X.

The above conjecture is very general as it deals with a wide range of singular hermitian bundles (L, h_L). A version of this is the following.

Conjecture 1.5. — *Let $p : X \to Y$ be a Kähler fiber space. Let (L, h_L) be a holomorphic hermitian $\mathbb{Q}$-line bundle on X such that*

(1.5.15) *$(L, h_L) = (B + \Lambda, h_B h_\Lambda)$ where B is an effective $\mathbb{Q}$-divisor and the restriction on the generic fiber $B|_{X_y}$ is klt with simple normal crossings support, h_B is the canonical singular metric on B and Λ is a $\mathbb{Q}$-line bundle with a smooth hermitian metric h_Λ such that $\Theta_{h_\Lambda}(\Lambda) \geqslant 0$ on X.*

(1.5.16) *The $\mathbb{Q}$-line bundle $K_X + L$ is p-big and admits a relative Zariski decomposition, i.e. $K_X + L \equiv_{\mathbb{Q}} A + E$ for some relatively semiample and big $\mathbb{Q}$-line bundle A and an effective $\mathbb{Q}$-divisor E such that the natural map*

$$p^* p_* \mathcal{O}_X(mA) \longrightarrow p^* p_* \mathcal{O}_X(mA + mE)$$

is a sheaf isomorphism over Y° for any m divisible enough.

Then, the relative Kähler-Einstein metric $e^{-\phi_{\mathrm{KE}}}$ induced on $K_{X/Y} + L$ over X° satisfies

$$\Theta_{\phi_{\mathrm{KE}}}(K_{X/Y} + L) \geqslant 0 \quad on \quad X^\circ.$$

Our first result, proved in Section 2, is to reduce Conjecture 1.4 to Conjecture 1.5:

Theorem 1.6. — *Conjecture 1.5 implies Conjecture 1.4.*

At this stage, we are not able to prove Conjecture 1.4 (or Conjecture 1.5) in full generality but only in the particular case where the metric $h_L = e^{-\phi_L}$ on L has vanishing Lelong numbers; i.e. $\forall x \in X$, $\nu(\phi_L, x) = 0$. The proof is given in Section 3.5 and relies on the approach developed in Section 3.2, after a reduction step explained in Section 3.1.

Theorem 1.7. — *Conjecture 1.4 holds true provided that Lelong numbers of the metric $h_L = e^{-\phi_L}$ of L vanish identically.*

2. Proof of Theorem 1.6

We have organized this section in the following way: first we show that (1.4.13) implies (1.4.14). The proof of (1.4.13) will be given in the second part of our arguments.

The extension property.

Proposition 2.1. — *In the Setting 1.3, the local weights of ϕ_{KE} are locally bounded above near $X \smallsetminus X^\circ$.*

Proof. — The proof of the Proposition follows very closely [**Pău17**, §3.3], so we will mostly sketch the proof.

Let $y \in Y^\circ$ and let us pick any point x in X_y. We choose a Stein neighborhood Ω of x in X; we write $\Omega_y = \Omega \cap X_y$, choose a potential τ_y of $\omega_{\mathrm{KE},y}$ such that the equation satisfied by τ_y on Ω_y is

$$\langle (dd^c \tau_y)^n \rangle = e^{\tau_y - \varphi_L} \left| \frac{dz}{dt} \right|^2$$

where φ_L is a local weight for h_L on Ω, and the coordinates $(z_1, \ldots, z_n, t_1, \ldots, t_m)$ are chosen so that $p(\underline{z}, \underline{t}) = \underline{t}$. We set $H_{m,y} := \left\{ f \in \mathcal{O}(\Omega_y); \int_{\Omega_y} |f|^2 e^{-m\tau_y} \langle (dd^c \tau_y)^n \rangle \leqslant 1 \right\}$. Note that $e^{-m\tau_y} \langle (dd^c \tau_y)^n \rangle = e^{-(m-1)\tau_y - u} \left| \frac{dz}{dt} \right|^2$ for some psh function u on Ω. Then, thanks to Demailly's approximation theorem, one has

$$\tau(y)(x) = \lim_{m \to \infty} \sup_{f \in H_{m,y}} \frac{1}{m} \log |f(x)|$$

But for $f \in H_{m,y}$, Hölder's inequality yields

$$\int_{\Omega_y} |f|^{2/m} e^{-\tau_y} \langle (dd^c \tau_y)^n \rangle \leqslant \Big(\mathrm{vol}(K_{X_y} + L_y) \Big)^{\frac{m}{m-1}} \tag{2.1}$$

The right hand side is bounded above independently of y and m; this can be seen for instance by finding a birational model $\pi : X' \to X$ where $\pi^*(K_X + L)$ has a relative Zariski decomposition $A + E$ so that the volume of $K_{X_y} + L_y$ is simply the intersection number (A_y^n) which is independent of $y \in Y^\circ$. Furthermore, the $L^{2/m}$ version of Ohsawa-Takegoshi extension theorem [BP12] yields a holomorphic function F on Ω that extends f and such that

$$|F(x)|^{2/m} \leqslant C_\Omega \int_\Omega |F|^{2/m} |dz|^2 \leqslant C \int_{\Omega_y} |f|^{2/m} \left| \frac{dz}{dt} \right|^2 \leqslant C' \int_{\Omega_y} |f|^{2/m} e^{-\tau_y} \langle (dd^c \varphi_y)^n \rangle$$

as φ_L is bounded above on Ω. Moreover, the integral on the right hand side is bounded above uniformly in y and m by (2.1). Therefore $\sup_{X_y} \tau_y \leqslant C$ for a constant C that uniform as long as $y \in Y^\circ$ varies in compact subsets of Y. □

Regularization

Thanks to Proposition 2.1, Conjecture 1.4 reduces to its first Item (1.4.13). That property is local on the base so from now on, the base Y will be a small Stein open set. The rest of this section is devoted to showing that Conjecture 1.5 implies that Item (1.4.13) in Conjecture 1.4 holds.

Lemma 2.2. — *It is enough to prove Item (1.4.13) of Conjecture 1.4 when $K_X + L$ admits a relative Zariski decomposition, namely $K_X + L \equiv_{\mathbb{Q}} A + E$ for some relatively semiample and big $\mathbb{Q}$-line bundle A and an effective $\mathbb{Q}$-divisor E such that the natural map*

$$p^* p_* \mathcal{O}_X(mA) \longrightarrow p^* p_* \mathcal{O}_X(mA + mE)$$

is a sheaf isomorphism over Y° for any m divisible enough.

Proof. — By assumption, the $\mathcal{O}_Y$-algebra $\mathcal{E} := \bigoplus_{m \geqslant 0} p_*(m(K_{X/Y} + L))$ is finitely generated. By blowing up the base locus of $\mathcal{E}$, we can find a birational map $\mu : \widetilde{X} \to X$ such that on the generic fiber $\widetilde{X}_y$ of $\mu \circ p$, we have the Zariski decomposition of $K_{\widetilde{X}_y} + \mu^* L|_{\widetilde{X}_y}$. Therefore, there exists a Zariski dense open subset $Y_0 \subset Y$ such that $K_{\widetilde{X}} + \mu^* L$ admits a relative Zariski decomposition $K_{\widetilde{X}} + \mu^* L = A + E$ on $(\mu \circ p)^{-1}(Y_0)$ and for any m divisible enough, the natural map

$$(\mu \circ p)^* (\mu \circ p)_* \mathcal{O}_{\widetilde{X}}(mA) \longrightarrow (\mu \circ p)^* (\mu \circ p)_* \mathcal{O}_{\widetilde{X}}(mA + mE) \qquad \text{on } (\mu \circ p)^{-1}(Y_0)$$

is an isomorphism.

Now, let $\omega_{\mathrm{KE},\widetilde{X}}$ (resp. $\omega_{\mathrm{KE},X}$) be the relative Kähler-Einstein metric with respect to $(\widetilde{X}, \mu^* L, \mu^* \phi_L)$ (resp. (X, L, ϕ_L)) over Y_0. Thanks to Remark 1.2 (a), we have

$$\mu_* \omega_{\mathrm{KE},\widetilde{X}} = \omega_{\mathrm{KE},X} \qquad \text{on } p^{-1}(Y_0).$$

If we can prove that $\omega_{\mathrm{KE},\widetilde{X}} \geqslant 0$ on $(\mu \circ p)^{-1}(Y_0)$, then

$$\omega_{\mathrm{KE},X} = \mu_* \omega_{\mathrm{KE},\widetilde{X}} \geqslant 0 \qquad \text{on } p^{-1}(Y_0).$$

Together with Proposition 2.1, the lemma is proved. □

Proposition 2.3. — *It is enough to prove Item (1.4.13) of Conjecture 1.4 when $(L, h_L) = (B + \Lambda, h_B h_\Lambda)$ where B is an effective $\mathbb{Q}$-divisor and the restriction on the generic fiber $B|_{X_y}$ is klt with normal crossing support, h_B is the canonical singular metric on B and Λ is a $\mathbb{Q}$-line bundle with a smooth hermitian metric h_Λ such that $i\Theta_{h_\Lambda}(\Lambda) \geqslant 0$ on X.*

Proof. — We proceed in two steps.

Step 1. *Reduction to the case where $dd^c\phi_L$ is a Kähler current*
By Lemma 2.2 above, one can assume that $K_X + L = A + E$ is a relative Zariski decomposition of $K_X + L$ on X. As Y is Stein and $K_X + L$ is p-big, $K_X + L$ is big on X. Therefore, there exists a weight ϕ_0 with analytic singularities on $(K_X + L)$ such that $dd^c\phi_0 > 0$ on X. Let us fix some small $\delta > 0$. We set $L_\delta := L + \delta(K_X + L)$ so that the relative Zariski decomposition of $K_X + L_\delta$ is $(1+\delta)A + (1+\delta)E$. Let $\phi_L + \delta\phi_0$ be the weight on L_δ. Then $dd^c(\phi_L + \delta\phi_0)$ is a Kähler current. To finish the proof of Step 1, it remains to prove that

- (i) The triplet $(X, L_\delta, e^{-\phi_L - \delta\phi_0})$ admits a relative Kähler-Einstein metric $\omega_{\mathrm{KE},\delta}$ for $\delta > 0$ small enough.
- (ii) We have $\omega_{\mathrm{KE},\delta} \to \omega_{\mathrm{KE}}$ in the weak topology when δ approaches zero.

To make notation more tractable, we will –from now on and in this first step only – work on a fixed fiber X_y and drop all indices y.

• *Proof of* (i).
We know from (1.9) that there exists $p > 1$ such that $e^{\phi_E - \phi_L} \in L^p$. Then $e^{(1+\delta)\phi_E - (\phi_L + \delta\phi_0)} \in L^r$ for some $1 < r < p$ as long as δ is small enough. Even better,

$$||e^{(1+\delta)\phi_E - (\phi_L + \delta\phi_0)}||_{L^r(X)} \leqslant C \tag{2.2}$$

for some uniform $C > 0$. Thanks to Definition-Proposition 1.1, we get (i).

• *Proof of* (ii).
It requires more work. Let ω_A be a smooth semipositive form in $c_1(A)$, let h_L (resp. h_E) be a smooth hermitian metric on $\mathcal{O}_X(L)$ (resp. on $\mathcal{O}_X(E)$) and let ω be a reference Kähler form such that

$$i\Theta_\omega(K_X) + i\Theta_{h_L}(L) = \omega_A + i\Theta_{h_E}(E).$$

Finally, let us choose potentials $\varphi_L, \varphi_0, \varphi_E$ such that $i\Theta_{h_L}(L) + dd^c\varphi_L = dd^c\phi_L$, $i\Theta_\omega(K_X) + i\Theta_{h_L}(L) + dd^c\varphi_0 = dd^c\phi_0$ and $i\Theta_{h_E}(E) + dd^c\varphi_E = [E]$. The Kähler-Einstein metric $\omega_{\mathrm{KE},\delta}$ can be written as $\omega_{\mathrm{KE},\delta} = \omega_\delta + (1+\delta)[E]$ where $\omega_\delta = (1+\delta)\omega_A + dd^c\varphi_\delta \in c_1((1+\delta)A)$ is a positive current with bounded potentials such that

$$((1+\delta)\omega_A + dd^c\varphi_\delta)^n = e^{\varphi_\delta + (1+\delta)\varphi_E - \varphi_L - \delta\varphi_0}\omega^n \tag{2.3}$$

Let us write $d\mu := e^{\varphi_E - \varphi_L}\omega^n$ and $d\mu_\delta := e^{(1+\delta)\varphi_E - \varphi_L - \delta\varphi_0}\omega^n$.

Claim 2.4. — *There exists a constant $C > 0$ independent of δ such that*

$$||\varphi_\delta||_{L^\infty(X)} \leqslant C. \tag{2.4}$$

Proof of Claim 2.4. — A first trivial observation is that one can rewrite (2.3) as a Monge-Ampère equation in a fixed cohomology class as follows

$$(\omega_A + dd^c\frac{1}{1+\delta}\varphi_\delta)^n = e^{\varphi_\delta + (1+\delta)\varphi_E - \varphi_L - \delta\varphi_0 - n\log(1+\delta)}\omega^n$$

Thanks to the a priori estimates established in [EGZ09], the claim comes down to showing that there exists a uniform $C > 0$ such that

$$\sup_X \varphi_\delta \leqslant C \tag{2.5}$$

and that $e^{(1+\delta)\varphi_E-\varphi_L-\delta\varphi_0}$ admits uniform L^p bounds for some $p>1$; but we already know that from (2.2). Let us prove (2.5) now. As φ_δ has bounded potentials, its Bedford-Taylor Monge-Ampère has full mass, i.e.

$$\int_X e^{\varphi_\delta+(1+\delta)\varphi_E-\varphi_L-\delta\varphi_0}\omega^n=(1+\delta)^n(A^n)$$

and, in particular, the integral $\int_X e^{\varphi_\delta+(1+\delta_0)\varphi_E}dV$ is uniformly bounded above for $\delta_0>0$ fixed. An application of Jensen's inequality yields $\int_X \varphi_\delta dV\leqslant C$, and the bound (2.5) then follows from standard properties of quasi-psh functions. □

The proof of Item (ii) above now follows from the following claim.

***Claim* 2.5.** — *When δ approaches zero, the function φ_δ converges weakly to φ.*

Proof of Claim 2.5. — An equivalent formulation of the claim is that

$$\varphi_\delta-\sup_X\varphi_\delta\underset{\delta\to 0}{\longrightarrow}\widetilde{\varphi}:=\varphi-\sup_X\varphi. \tag{2.6}$$

This is consequence of [BG14, Thm. 4.5], but the bound (2.4) actually makes the arguments much easier. We will only recall the main lines. First, one chooses a sequence δ_j such that $\varphi_j:=\varphi_{\delta_j}-\sup_X\varphi_{\delta_j}$ converges weakly to some sup-normalized ω_A-psh function ψ; we want to show that $\psi=\widetilde{\varphi}$. We use the variational characterization of φ_j as the supremum of the functional $\mathcal{G}_j=\mathcal{E}_j+\mathcal{L}_j$ acting on sup-normalized $(1+\delta_j)\omega_A$-psh functions. Here, $\mathcal{E}_j$ is the usual energy functional attached to $(1+\delta_j)\omega_A$ and $\mathcal{L}_j(\bullet)=-\log\int_X e^\bullet d\mu_{\delta_j}$. Thanks to (2.2) and (2.4), the dominated convergence theorem implies

$$\lim_{j\to+\infty}\mathcal{L}_j(\varphi_j)=\mathcal{L}(\psi). \tag{2.7}$$

Moreover, [BG14, Lem. 4.6] implies that

$$\overline{\lim_{j\to+\infty}}\,\mathcal{E}_j(\varphi_j)\leqslant\mathcal{E}(\psi). \tag{2.8}$$

As $\varphi\in\mathrm{PSH}(X,(1+\delta_j)\omega_A)$, one has automatically $\mathcal{G}_j(\varphi_j)\geqslant\mathcal{G}(\widetilde{\varphi})$. Finally, as Bedford-Taylor product is continuous with respect to smooth convergence, one has $\lim_j\mathcal{E}_j(\widetilde{\varphi})=\mathcal{E}(\widetilde{\varphi})$. Putting these last two results together with (2.7) and (2.8), one finds

$$\mathcal{G}(\psi)\geqslant\overline{\lim_{j\to+\infty}}\,\mathcal{G}_j(\varphi_j)\geqslant\overline{\lim_{j\to+\infty}}\,\mathcal{G}_j(\widetilde{\varphi})=\mathcal{G}(\widetilde{\varphi})$$

hence the result. □

In conclusion, φ_δ converges weakly to φ, hence $\omega_{\mathrm{KE},\delta}$ converges to ω_{KE}. This argument was done fiberwise, but it clear that the weak convergence on the fiber implies the weak convergence in any small neighborhood of the given fiber as well. This proves (ii) and completes Step 1.

Step 2. *Reduction to the case where ϕ_L has analytic singularities*
By Step 1, one can assume that $dd^c\phi_L$ is a Kähler current. By Demailly regularization theorem [Dem92], ϕ_L is the weak, decreasing limit of strictly psh weights $\phi_{L,\varepsilon}$ on L with analytic singularities, say with singularities along the analytic set Z_ε. Taking a log resolution $\pi_\varepsilon:X_\varepsilon\to X$ of (X,Z_ε), one can assume that $\pi^*\phi_{L,\varepsilon}=\phi_{B_\varepsilon}+\phi_{A_\varepsilon}$ where ϕ_{B_ε} is the canonical singular psh weight on an effective normal crossing $\mathbb{Q}$-divisor B_ε, and $\phi_{A,\varepsilon}$ is a smooth psh weight on some $\mathbb{Q}$-line bundle A_ε with $\Theta_{\phi_{A,\varepsilon}}(A_\varepsilon)\geqslant 0$ on X_ε.

After passing to another birational model if necessary, one can assume that over a generic fiber, we have a Zariski decomposition

$$K_{(X_\varepsilon)_y} + \pi_\varepsilon^* L_\varepsilon|_{(X_\varepsilon)_y} = M_y + E_y,$$

and $B_\varepsilon|_{(X_\varepsilon)_y} + E_y$ is normal crossing. Let $\Gamma_\varepsilon := B_\varepsilon \wedge E$ be the common part of B and E. We have the following Zariski decomposition

$$K_{(X_\varepsilon)_y} + \big(B_\varepsilon|_{(X_\varepsilon)_y} - \Gamma_\varepsilon|_{(X_\varepsilon)_y}\big) + A_\varepsilon|_{(X_\varepsilon)_y} = M_y + \big(E_y - \Gamma_\varepsilon|_{(X_\varepsilon)_y}\big).$$

Furthermore, thanks to (1.3.12) and the decreasing property of $(\phi_{L,\varepsilon})$, we know that the divisor $\big(B_\varepsilon|_{(X_\varepsilon)_y} - \Gamma_\varepsilon|_{(X_\varepsilon)_y}\big)$ is klt on $(X_\varepsilon)_y$.

Let ω_ε be the relative Kähler-Einstein metric of $(X_\varepsilon \to Y, \pi_\varepsilon^* L, \pi_\varepsilon^*(\phi_{L,\varepsilon}))$ and let ω'_ε be the relative Kähler-Einstein metric of $(X_\varepsilon \to Y, (B_\varepsilon - \Gamma_\varepsilon) + A_\varepsilon, \phi_{B_\varepsilon} - \phi_{\Gamma_\varepsilon} + \phi_{A_\varepsilon})$. By definition, we have

$$\omega_\varepsilon = \omega'_\varepsilon + [\Gamma_\varepsilon]. \tag{2.9}$$

If Conjecture 1.4 holds for $(X_\varepsilon \to Y, (B_\varepsilon - \Gamma_\varepsilon) + A_\varepsilon, \phi_{B_\varepsilon} - \phi_{\Gamma_\varepsilon} + \phi_{A_\varepsilon})$, thanks to (2.9) and Remark 1.2, it holds also for $(X \to Y, L, \phi_{L,\varepsilon})$. Finally, when ε converges to 0, the relative Kähler-Einstein metric of $(X \to Y, L, \phi_{L,\varepsilon})$ converges to the relative Kähler-Einstein metric of $(X \to Y, L, \phi_L)$ as a direct consequence of the comparison principle. Therefore Theorem 1.6 is proved. □

3. Proof of Theorem 1.7

We will present our arguments in several steps, according to the following plan.

(a) *It is enough to prove Theorem 1.7 in case h_L non-singular.* This is based on two results: we first use that h_L is limit of non-singular metrics whose negative part of the curvature tends to zero. Another important fact we are using is that the algebra associated to $K_X + L + \delta H$ is finitely generated, for any H ample and for any positive rational δ.

(b) *It is enough to prove Theorem 1.7 provided that A is an ample $\mathbb{Q}$-bundle.* Remark that in general, the semi-ample part A of the Zariski decomposition is not ample. In this second step we write A as limit of ample bundles, and show that the solution of the resulting Monge-Ampère equation converges to the singular KE metric.

(c) *Reduction to the case $c_1(L) \in \mathbb{Z}$.* Let p be a positive integer such that pL is a line bundle. Then we write $p(K_X + L) = K_X + (p-1)(K_X + L) + L$ and then we replace our initial $\mathbb{Q}$-bundle L with the line bundle $L_p := (p-1)(K_X + L) + L$. The problem is that we also have to replace h_L with a positively curved metric on L_p. The metric on L is given. It is less clear what should be the metric on $K_X + L$, since it has to fulfill two conditions: its curvature must be positive, and in the relative setting (i.e. when we replace X with a fiber of p) it must induce a positively curved metric on the twisted relative canonical bundle. It seems impossible to achieve this in one single step. What *is* possible is to set up an iteration scheme so that the resulting limit coincides with the singular KE metric.

(d) *If $L(= L_p)$ is a line bundle, show that the singular Kähler-Einstein metric corresponding to (X, L) can be obtained as limit of iterated Bergman kernels.* We conclude by this fact, since the fiber-wise Bergman kernel metric has the required curvature properties specified in (c) above.

Also, at each step we establish the relevant convergence results needed to conclude at the end.

3.1. Reduction to the case h_L non-singular. — Let $(L, h_L = e^{-\phi_L})$ be a hermitian line bundle on a projective variety X such that $K_X + L$ is big and ϕ_L is a psh weight with vanishing Lelong numbers. Let ϕ be the weight on $K_X + L$ such that $\omega_\phi := dd^c\phi$ is the Kähler-Einstein metric of $(X, L, e^{-\phi_L})$, ie

$$\mathrm{Ric}\,(\omega_\phi) = -\omega_\phi + dd^c\phi_L.$$

Let H be an ample line bundle on X, and let ϕ_H be a weight on H such that $dd^c\phi_H$ is a Kähler form. Thanks to Demailly's regularization theorem, there exists a family of smooth weights $\phi_{L,\varepsilon}$ on L such that

$$\phi_{L,\varepsilon} \downarrow \phi_L \quad \text{and} \quad dd^c(\phi_{L,\varepsilon} + \varepsilon\phi_H) \geqslant 0.$$

Now, let $\delta \geqslant \varepsilon$ be a positive number, and let $\phi_{\delta,\varepsilon}$ be the suitably normalized weight on $K_X + L + \delta H$ such that $\omega_{\phi_{\delta,\varepsilon}} := dd^c\phi_{\delta,\varepsilon}$ is the Kähler-Einstein metric of $(X, L + \delta H, e^{-\phi_{L,\varepsilon} - \delta\phi_H})$, ie

$$\mathrm{Ric}\,(\omega_{\phi_{\delta,\varepsilon}}) = -\omega_{\phi_{\delta,\varepsilon}} + dd^c(\phi_{L,\varepsilon} + \delta\phi_H).$$

Proposition 3.1. — *With the notation above, there exists a family of positive numbers $(\delta_\varepsilon)_{\varepsilon>0}$ decreasing to zero such that $\omega_{\phi_{\delta_\varepsilon,\varepsilon}}$ converges weakly to ω_ϕ when ε approaches zero.*

As an consequence, one gets the following

Corollary 3.2. — *It is sufficient to prove Theorem 1.7 when h_L is smooth.*

Proof of Proposition 3.1. — Let us start by setting some additional notation. Let θ (resp θ_L) be a closed smooth $(1,1)$-form in the cohomology class $c_1(K_X + L)$ (resp. $c_1(L)$). Let $\omega_H := dd^c\phi_H$ and let dV be a smooth volume form such that $-\mathrm{Ric}\,(dV) + \theta_L = \theta$. Finally, let $\varphi_L, \varphi_{L,\varepsilon}$ be some quasi-psh functions such that $\theta_L + dd^c\varphi_L = dd^c\phi_L$ (resp. $\theta_L + dd^c\varphi_{L,\varepsilon} = dd^c\phi_{L,\varepsilon}$) and satisfying additionnally that $\varphi_{L,\varepsilon} \downarrow \varphi_L$ when $\varepsilon \downarrow 0$. Let $\varphi_{\delta,\varepsilon}$ be the unique $(\theta + \delta\omega_H)$-psh function with minimal singularities solution of

$$(\theta + \delta\omega_H + dd^c\varphi_{\delta,\varepsilon})^n = e^{\varphi_{\delta,\varepsilon} - \varphi_{L,\varepsilon}} dV$$

whose existence is guaranteed by [**BEGZ10**] (cf. also [**Gue13**, Thm. 2.2]). When $\varepsilon = 0$, one writes $\varphi_\delta := \varphi_{\delta,0}$, and one sets $\varphi := \varphi_0$. Note that $\theta + dd^c\varphi$ is the Kähler-Einstein metric of $(X, L, e^{-\phi_L})$.

For the time being, let $\delta > 0$ be fixed. As $\varphi_{L,\varepsilon}$ decreases toward φ_L and φ_L has vanishing Lelong number, the convergence $e^{-\varphi_{L,\varepsilon}} \uparrow e^{-\varphi_L}$ happens in any L^p space for $p > 0$ thanks to Skoda's integrability theorem, cf. e.g. [**Sko72**, Prop.7.1]. In particular, if follows from [**GLZ18**, Thm. 5.2] that $\varphi_{\delta,\varepsilon}$ converges weakly to $\varphi_\delta = \varphi_{\delta,0}$ when ε approaches zero.

As $\omega_H \geqslant 0$, the θ-psh function φ is also $\theta + \delta\omega_H$-psh and it is a subsolution of the equation

$$(\theta + \delta\omega_H + dd^c\psi)^n = e^{\psi - \varphi_L} dV$$

so one gets

$$\varphi \leqslant \varphi_\delta \tag{3.1}$$

for any $\delta > 0$. Moreover, the same argument shows that φ_δ decreases when $\delta \downarrow 0$. Let $\varphi^* := \lim_{\delta\to 0} \varphi_\delta$. If we can prove that $\varphi^* = \varphi$, then we will be done.

From (3.1), one can deduce two things. First, φ^* is a θ-psh function with minimal singularities. Also, the sequence $(\varphi_\delta)_{\delta>0}$ is locally bounded on the ample locus Ω of $K_X + L$. Because the Monge-Ampère operator is continuous with respect to bounded decreasing sequences, one finds that

$$(\theta + dd^c\varphi^*)^n = e^{\varphi^* - \varphi_L} dV \quad \text{on } \Omega.$$

As the non-pluripolar Monge-Ampère operator does not put any mass to analytic sets, it follows that the previous equation is satisfied on the whole X. As φ and φ^* have minimal singularities, the currents $\theta + dd^c\varphi$ and $\theta + dd^c\varphi^*$ have full mass (almost by definition, cf remarks below [BEGZ10, Def. 2.1]) and therefore

$$\begin{aligned}\int_X e^{\varphi^*-\varphi_L}dV &= \int_X (\theta + dd^c\varphi^*)^n \\ &= \int_X (\theta + dd^c\varphi)^n \\ &= \int_X e^{\varphi-\varphi_L}dV\end{aligned}$$

As $\varphi \leqslant \varphi^*$ by (3.1), we see that $\varphi = \varphi^*$ almost everywhere. As both functions are θ-psh, they must agree on X. □

3.2. The approximation of A. — A first remark is that thanks to Lemma 2.2 and Corollary 3.2, one can assume that h_L is smooth and that $K_X + L$ admits a relative Zariski decomposition over X° (which denotes here the inverse image of a well-chosen open subset of Y),

$$K_X + L = A + E$$

By Kodaira lemma, there exists an effective $\mathbb{Q}$-divisor E_X such that $A - E_X$ is p-ample. As Y is chosen to be Stein, one can assume that $A - E_X$ is globally ample. Therefore for each positive, small enough δ we have

$$\begin{aligned}K_X + L &= A + E \\ &= A_\delta + E_\delta\end{aligned} \tag{3.2}$$

where $A_\delta := (1-\delta)A + \delta(A - E_X)$ is ample and $E_\delta := E + \delta E_X$ for any $\delta > 0$.

Convention. *For the rest of this subsection our results will exclusively concern the fibers X_y of p. Since $y \in Y^\circ$ is fixed, we will denote X_y by X and drop the index y in the relevant line bundles and weights that will be considered here.*

3.2.1. *Notations.* — Let ω_A in $c_1(A)$ be a smooth, semi-positive representative. We denote by $E := \sum_{i=1}^k a_i E_i$ and let $E_X := \sum_{i=1}^k c_i E_i$ the divisors above where some of a_i, c_i could be zero. Since $\{A - E_X\}$ is a Kähler class we can fix a Kähler form $\omega_0 \in \{A - E_X\}$. For each positive δ we obtain a Kähler form

$$\omega_\delta := (1-\delta)\omega_A + \delta\omega_0 \in c_1(A_\delta)$$

where $A_\delta := A - \delta E_X$. We write

$$E_\delta = \sum_{i=1}^k a_i^\delta E_i,$$

where

$$a_i^\delta = a_i + \delta c_i \tag{3.3}$$

Let s_i be a defining section for E_i and let $h_{E_i} = e^{-\rho_i}$ be a non-singular Hermitian metric on $\mathcal{O}_X(E_i)$. We obtain the metrics $h_E = \prod h_{E_i}^{a_i}$ and $h_{E_\delta} = \prod h_{E_i}^{a_i^\delta}$ on E and E_δ respectively.

We define

$$|s_{E_\delta}|^{2\{\ell p\}} := \prod_i |s_{E_i}|^{2(\lceil \ell p a_i^\delta\rceil - \ell p a_i^\delta)}$$

where $|s_{E_i}|^2$ denotes the squared norm of s_{E_i} with respect to h_i.

The Kähler-Einstein metric $\omega_\varphi = \omega_A + dd^c\varphi$ of (X, L, ϕ_L) satisfies the following Monge-Ampère equation on X:

$$(\omega_A + dd^c\varphi)^n = |s_E|^2 e^{\varphi - f_L}\omega^n \tag{3.4}$$

where ω is a reference Kähler metric on X and f_L is the unique smooth function on X such that

$$\omega_A + \Theta(E) + dd^c f_L = \Theta_{h_L}(L) + \Theta_\omega(K_X), \qquad \int_X f dV_\omega = 0. \tag{3.5}$$

It will be convenient for later to fix some notations for the local expression of the objects above. Let $U \subset X$ be an open coordinate subset such that the $\mathbb{Q}$-bundles above are trivial when restricted to U.

Let ρ_i be the local weight of the metric h_i with respect to a trivialization of $\mathcal{O}_X(E_i)|_U$. We will use the notation

$$\rho_{E_\delta} := \sum a_i^\delta \rho_i$$

for the weight of the induced metric on E_δ.

In a similar manner we introduce ϕ_A, ϕ_0, ϕ_L on $A, A - E_X$ and L respectively, such that $dd^c\phi_A = \omega_A$ and $dd^c\phi_0 = \omega_0$. Finally we consider for $\delta > 0$

$$\phi_{A_\delta} := (1-\delta)\phi_A + \delta\phi_0. \tag{3.6}$$

We assume that the metrics h_i are chosen such that $\phi_0 = \phi_A - \delta\sum_i c_i\rho_i$.

Expressed in terms of local weights and coordinates, the equality (3.5) becomes

$$\phi_A + \phi_E + f_L = \phi_L + \log\det(\omega_{\alpha\bar\beta}) \tag{3.7}$$

modulo a pluri-harmonic function on U. We see that we are free to choose the trivialization of E and L together with a coordinate system (z_i) such that (3.7) becomes an equality by modifying the weights ϕ_A.

3.2.2. *The approximation statement.* — For $\delta, \varepsilon > 0$, Aubin-Yau theorem shows that the equation

$$(\omega_\delta + dd^c\varphi_{\delta,\varepsilon})^n = (|s_E|^2 + \varepsilon^2)e^{\varphi_{\delta,\varepsilon} - f_L}\omega^n \tag{3.8}$$

has a unique solution such that $\omega_\delta + dd^c\varphi_{\delta,\varepsilon}$ is a smooth Kähler metric. In the two equations above, $|s_E|^2$ (resp. $|s_E|^2 + \varepsilon^2$) has to be interpreted as $\prod_i |s_i|^{2a_i}$ (resp. $\prod_i (|s_i|^2 + \varepsilon^2)^{a_i}$). We have the following convergence result.

Proposition 3.3. — *There exists a family of positive numbers $(\delta_\varepsilon)_{\varepsilon>0}$ decreasing to 0 when ε approaches zero such that*

$$\lim_{\varepsilon\to 0} ||\varphi_{\delta_\varepsilon,\varepsilon} - \varphi||_{L^\infty(X)} = 0.$$

Proof. — For now, let $\delta \in (0,1)$ be fixed. By [GLZ18, Thm. 1.1], one has

$$\limsup_{\varepsilon\to 0} ||\varphi_{\delta,\varepsilon} - \varphi_{\delta,0}||_{L^\infty(X)} = 0. \tag{3.9}$$

Let

$$\psi_\delta := \frac{1}{1-\delta}\Big(\varphi_{\delta,0} - n\log(1-\delta)\Big).$$

The above function satisfies the Monge-Ampère equation

$$(\omega_A + \frac{\delta}{1-\delta}\omega_0 + dd^c\psi_\delta)^n = |s_E|^2 e^{(1-\delta)\psi_\delta - f_L}\omega^n. \tag{3.10}$$

The ω_A-psh function $\hat{\psi}_\delta := \psi_0 + \frac{\delta}{1-\delta}\cdot \inf_X \psi_0$ is a subsolution of (3.10). Indeed, one has

$$\begin{aligned}\Big(\omega_A + \frac{\delta}{1-\delta}\omega_0 + dd^c\hat{\psi}_\delta\Big)^n &\geqslant (\omega_A + dd^c\psi_0)^n \\ &= |s_E|^2 e^{(1-\delta)\hat{\psi}_\delta - f_L} e^{\delta(\psi_0 - \inf_X \psi_0)}\omega^n \\ &\geqslant |s_E|^2 e^{(1-\delta)\hat{\psi}_\delta - f_L}\omega^n.\end{aligned}$$

Therefore, one gets $\hat{\psi}_\delta \leqslant \psi_\delta$, ie

$$\psi_0 \leqslant \psi_\delta - \frac{\delta}{1-\delta}\cdot \inf_X \psi_0.$$

In particular, one finds a uniform lower bound $\psi_\delta \geqslant -C$ where $C > 0$ is independent of δ. Let $K := \min\{0, \inf_\delta \inf_X \psi_\delta\}$ where the first infimum ranges over $\delta \in [0, e^{-1}]$ say. Using the same argument as above, one concludes that for any $0 \leqslant \eta \leqslant \delta \leqslant e^{-1}$, one has

$$\psi_\eta - \frac{K\eta}{1-\eta} \leqslant \psi_\delta - \frac{K\delta}{1-\delta}.$$

That is, the family of ω_A-psh functions $(\psi_\delta - \frac{K\delta}{1-\delta})_{\delta>0}$ decreasing toward a bounded ω_A-psh $\tilde{\psi}_0$ when $\delta \downarrow 0$. The function $\tilde{\psi}_0$ satisfies the same Monge-Ampère equation (3.10) as ψ_0 thanks to the continuity of the Monge-Ampère operator with respect to bounded decreasing sequences. Therefore, one has $\tilde{\psi}_0 = \psi_0 = \varphi$.

Now, the ω_A-psh function φ is continuous. Indeed, this is because ω_A is the pull-back of a Hodge form on a (singular) space by a birational morphism, hence one can apply jointly [EGZ09, Thm. A] and [CGZ13, Cor. C]. All in all, Dini's theorem shows that the convergence $\psi_\delta - \frac{K\delta}{1-\delta} \to \varphi$ is uniform. In particular, $\varphi_{\delta,0}$ converges uniformly to φ when $\delta \to 0$. The Proposition now follows from (3.9) and a suitable diagonal process. □

3.3. Reduction to the case $c_1(L) \in H^2(X,\mathbb{Z})$. — We fix an integer $p \geqslant 1$ such that pE is integral and pL is a line bundle. The first step in the algorithm which will follow consists in solving the equation

$$(p\omega_\delta + dd^c\varphi_{1,\delta,\varepsilon})^n = e^{\varphi_{1,\delta,\varepsilon} - f_L}(|s_E|^2 + \varepsilon^2)\omega^n \tag{3.11}$$

This is very similar to (3.8). In particular, one can apply Proposition 3.3 to show that there exists a family of numbers $(\delta_\varepsilon^{(1)})_{\varepsilon>0}$ decreasing to zero when $\varepsilon \downarrow 0$ such that

$$\limsup_{\varepsilon\to 0} ||\varphi_{1,\delta_\varepsilon^{(1)},\varepsilon} - \varphi_{1,0,0}||_{L^\infty(X)} = 0.$$

One sets $\varphi_{1,\varepsilon} := \varphi_{1,\delta_\varepsilon^{(1)},\varepsilon}$ and $\varphi_1 := \varphi_{1,0,0}$. Next, one solves the equation

$$(p\omega_\delta + dd^c\varphi_{2,\delta,\varepsilon})^n = e^{\varphi_{2,\delta,\varepsilon} - \frac{p-1}{p}\varphi_{1,\varepsilon} - f_L}(|s_E|^2 + \varepsilon^2)\omega^n \tag{3.12}$$

Proposition 3.3 applies again verbatim to show that there exists a family of numbers $(\delta_\varepsilon^{(2)})_{\varepsilon>0}$ decreasing to zero when $\varepsilon \downarrow 0$ such that

$$\limsup_{\varepsilon\to 0} ||\varphi_{2,\delta_\varepsilon^{(2)},\varepsilon} - \varphi_{2,0,0}||_{L^\infty(X)} = 0.$$

We set $\varphi_{2,\varepsilon} := \varphi_{2,\delta_\varepsilon^{(2)},\varepsilon}$, $\varphi_2 := \varphi_{2,0,0}$ and repeat the procedure. The result is the following.

Proposition 3.4. — *For each integer $m \geqslant 1$ there exist a family of positive reals $(\delta_\varepsilon^{(m)})_{\varepsilon>0}$ decreasing to zero and a family of smooth strictly $p\omega_{\delta_\varepsilon^{(m)}}$-psh functions $\varphi_{m,\varepsilon}$ such that*

$$(p\omega_{\delta_\varepsilon^{(m)}} + dd^c\varphi_{m,\varepsilon})^n = e^{\varphi_{m,\varepsilon} - \frac{p-1}{p}\varphi_{m-1,\varepsilon} - f_L}(|s_E|^2 + \varepsilon^2)\omega^n \tag{3.13}$$

and

$$\limsup_{\varepsilon\to 0} ||\varphi_{m,\varepsilon} - \varphi_m||_{L^\infty(X)} = 0. \tag{3.14}$$

where φ_m are the unique $p\omega_A$-psh bounded functions such that $\varphi_0 = 0$ and

$$(p\omega_A + dd^c\varphi_m)^n = |s_E|^2 e^{\varphi_m - \frac{p-1}{p}\varphi_{m-1} - f_L}\omega^n \tag{3.15}$$

Thanks to (3.2), one gets for each integer $m \geqslant 1$ a decomposition

$$K_X + L = A_{\delta_\varepsilon^{(m)}} + E_{\delta_\varepsilon^{(m)}}. \tag{3.16}$$

and one can define the weights

$$\phi_m := p\phi_A + \varphi_m, \quad \phi_{m,\varepsilon} := p\phi_{A_{\delta_\varepsilon^{(m)}}} + \varphi_{m,\varepsilon} \tag{3.17}$$

on pA and $pA_{\delta_\varepsilon^{(m)}}$ respectively, cf (3.6). Let ϕ_E be a singular weight on E such that $dd^c\phi_E = [E]$, and let ϕ_L a smooth weight on L such that $dd^c\phi_L = i\Theta_{h_L}(L)$. The expressions $e^{\phi_m - \frac{p-1}{p}\phi_{m-1} + \phi_E - \phi_L}$ define a global volume form which we normalize (by adding a constant to ϕ_L) such that

$$\int_X e^{\phi_m - \frac{p-1}{p}\phi_{m-1} + \phi_E - \phi_L} d\lambda = (A^n). \tag{3.18}$$

If follows from (3.15) combined with the definition of f_L cf. (3.5) that ϕ_m solves

$$(dd^c\phi_m)^n = e^{\phi_m - \frac{p-1}{p}\phi_{m-1} + \phi_E - \phi_L} d\lambda. \tag{3.19}$$

3.4. Convergence of the Ricci iteration. — The current $\omega_m := dd^c\phi_m = p\omega_A + dd^c\varphi_m$, satisfies the following twisted Kähler-Einstein-like equation:

$$\mathrm{Ric}\,\omega_m = -\omega_m + \frac{p-1}{p}\omega_{m-1} - [E] + i\Theta_{h_L}(L). \tag{3.20}$$

Its behavior when $m \to +\infty$ is given by the following result.

Proposition 3.5. — *When m tends to $+\infty$, the current $\frac{1}{p}\omega_m$ converges weakly to the (unique) twisted Kähler-Einstein metric $\omega_\infty \in c_1(A)$ solution of*

$$-\mathrm{Ric}\,\omega_\infty + i\Theta_{h_L}(L) = \omega_\infty + [E].$$

Remark 3.6. — We see that ω_∞ is equal to the Kähler-Einstein metric ω_{KE} of (X, L, ϕ_L) on $X \setminus E$. More precisely we have $\omega_{\mathrm{KE}} = \omega_\infty + [E]$.

Proof. — Recall that $\omega_m = p\omega_A + dd^c\varphi_m$ is solution of the Monge-Ampère equation

$$(p\omega_A + dd^c\varphi_m)^n = e^{\varphi_m - \frac{p-1}{p}\varphi_{m-1}} d\mu$$

where $d\mu = |s_E|^2 \cdot \omega^n$. We aim to show that for each $m \geqslant 2$, one has

$$||\varphi_m - \varphi_{m-1}||_{L^\infty(X)} \leqslant \frac{p-1}{p}||\varphi_{m-1} - \varphi_{m-2}||_{L^\infty(X)} \tag{3.21}$$

Let $C_m := \sup_X(\varphi_{m-1} - \varphi_{m-2})$, and let $U_m = \{\varphi_m > \varphi_{m-1} + \frac{p-1}{p}C_m\}$. An application of the comparison principle yields:

$$\int_{U_m} e^{\varphi_m - \frac{p-1}{p}\varphi_{m-1}} d\mu \leqslant \int_{U_m} e^{\varphi_{m-1} - \frac{p-1}{p}\varphi_{m-2}} d\mu. \tag{3.22}$$

On U_m, one has:

$$\begin{aligned} \varphi_m - \frac{p-1}{p}\varphi_{m-1} &> \frac{1}{p}\varphi_{m-1} + \frac{p-1}{p}C_m \\ &= (\varphi_{m-1} - \frac{p-1}{p}\varphi_{m-2}) + \frac{p-1}{p}[C_m - (\varphi_{m-1} - \varphi_{m-2})] \\ &\geqslant \varphi_{m-1} - \frac{p-1}{p}\varphi_{m-2}. \end{aligned}$$

Together with (3.22), we know that U_m has measure zero with respect to $d\mu$, hence also with respect to $(p\omega_A + dd^c\varphi_m)^n$. By the domination principle, cf. e.g. [**BEGZ10**, Cor. 2.5], we see U_m is empty, hence $\varphi_m - \varphi_{m-1} \leqslant \frac{p-1}{p}\sup_X(\varphi_{m-1} - \varphi_{m-2})$. Using an analogous argument, one can show that $\varphi_m - \varphi_{m-1} \geqslant \frac{p-1}{p}\inf_X(\varphi_{m-1} - \varphi_{m-2})$, which proves (3.21). It follows by iteration that

$$||\varphi_m - \varphi_{m-1}||_{L^\infty(X)} \leqslant (\frac{p-1}{p})^{m-1}||\varphi_1||_{L^\infty(X)}$$

and therefore the sequence $(\varphi_m)_{m\geqslant 1}$ converges uniformly to a $p\omega_A$-psh function φ_∞. As Bedford-Taylor product is continuous with respect to uniform convergence, φ_∞ satisfies:

$$(p\omega_A + dd^c\varphi_\infty)^n = e^{\frac{1}{p}\varphi_\infty} d\mu$$

which proves the proposition. □

3.5. Convergence of the Bergman kernel iteration. — In this paragraph we fix an integer $m \geqslant 1$ and we prove that the twisted Kähler-Einstein metric ω_m is the weak limit of iterated Bergman kernels.
Consider the line bundle

$$L_p := (p-1)(K_X + L) + L, \tag{3.23}$$

where p is a positive integer such that pL is a line bundle and pE has integer coefficients. We recall that the triple (X, L, h_L) satisfies the following.

- $K_X + L = A + E$ is a Zariski decomposition of the big line bundle $K_X + L$.
- The hermitian metric $h_L = e^{-\phi_L}$ on the $\mathbb{Q}$-line bundle L is a smooth and has semipositive curvature, i.e. $dd^c\phi_L \geqslant 0$.

We then endow L_p with the metric given by the weights

$$\tau_m := (p-1)\left(\frac{\phi_{m-1}}{p} + \phi_E\right) + \phi_L \tag{3.24}$$

where ϕ_{m-1} is the weight corresponding to the metric on pA defined by (3.17), ϕ_E is a singular weight on $\mathcal{O}_X(E)$ such that $dd^c\phi_E = [E]$ and finally ϕ_L is the smooth metric on L satisfying $dd^c\phi_L = \Theta_{h_L}(L)$ such that we have the equality (3.19).

We write

$$(\ell+1)(K_X + L_p) = K_X + \ell(K_X + L_p) + L_p$$

and then we can define a singular metric h_ℓ on the line bundle $\ell(K_X + L_p)$ by induction on ℓ in the following manner:

$$h_{\ell+1} := K_{\ell+1}^{-1} \tag{3.25}$$

where

$$K_{\ell+1} := K(X, (\ell+1)(K_X + L_p), h_\ell \cdot e^{-\tau_m})$$

is the Bergman kernel of $(\ell+1)(K_X + L_p)$ endowed with the metric above. Of course that it depends on m and p, even if our notation does not reflects this.

In the current subsection we are aiming at the following result, from which Theorem 1.7 will follow easily.

Theorem 3.7. — *Under the assumptions above, the renormalized Bergman kernels* $\left(n!^\ell \ell!^{-n} K_\ell\right)^{1/\ell}$ *converge to* $e^{\phi_m + p\phi_E}$ *as* $\ell \to +\infty$.

Prior to the proof of this result we are making a few preliminary remarks concerning the singularities of h_ℓ. Since $K_X + L = A + E$ is a Zariski decomposition, one knows that if p is divisible enough, the multiplication by s_{pE} induces an isomorphism $H^0(X, pA) \to H^0(X, p(K_X + L))$. Therefore all sections $s \in H^0(X, \ell(K_X + L_p))$ vanish along pE at order at least ℓ. Since A is semi-ample, there exists a section whose vanishing order along pE is exactly ℓ. A quick induction shows that every section $s \in H^0(X, (\ell+1)(K_X + L_p))$ is square integrable with respect to $h_\ell \cdot e^{-\tau_p}$.

The MA equation for ϕ_m is as follows

$$(dd^c \phi_m)^n = e^{\phi_m + \phi_E - \phi_L - \frac{p-1}{p}\phi_{m-1}} d\lambda \tag{3.26}$$

given (3.19). The solution ϕ_m is not regular enough for what is needed in the arguments to follow, so we also consider the approximation obtained in Proposition 3.4 for which we have

$$(dd^c \phi_{m,\varepsilon})^n = e^{\phi_{m,\varepsilon} - \frac{p-1}{p}\phi_{m-1,\varepsilon} - \phi_L} (\varepsilon^2 e^{\rho_E} + e^{\phi_E}) d\lambda \tag{3.27}$$

where $e^{-\rho_E}$ is the smooth metric on E we fixed in section 3.2.
The weights

$$\ell\left(\phi_{m,\varepsilon} + p\phi_{E_\varepsilon}\right) \tag{3.28}$$

are defining a metric on $\ell(K_X + L_p)$ and therefore the quantity

$$C_\ell := \inf_X \frac{K_\ell}{|s_{E_\varepsilon}|^{2\{\ell p\}} e^{\ell(\phi_{m,\varepsilon} + p\phi_{E,\varepsilon})}} \tag{3.29}$$

is a strictly positive real number, cf. the discussion above.

The proof of Theorem 1.7 relies heavily on the following statement.

Proposition 3.8. — *For every m fixed, there exists $\kappa_{\varepsilon,\ell} > 0$ such that:*

$$C_\ell \geqslant \kappa_{\varepsilon,\ell} \cdot \frac{\ell^n}{n!} \cdot C_{\ell-1}$$

and

$$\lim_{\varepsilon \to 0} \lim_{\ell \to +\infty} \left(\prod_{k=1}^{\ell} \kappa_{\varepsilon,k} \right)^{1/\ell} = 1.$$

Proof. — We have organized our arguments in four main steps.

Step 1. Choice of an appropriate local section u.

Let $x_0 \in X$ be an arbitrary point. Let $x_0 \in U \subset X$ be an open subset of X such that the restriction to U of all our bundles (i.e. pL, pE...) is trivial. We consider the local weights ρ_i for h_i cf. Section 3.2.1 such that $\rho_i(x_0) = 0$ for all i. We take a coordinate system $(z_i)_{i=1,\dots,n}$ on U, centered at x_0, and we assume that (3.7) holds. All the local computations to follow are done with respect to this data.

We introduce the quadratic function

$$h(z) := \phi_{m,\varepsilon}(x_0) + +2\sum_i \partial_i\left(\phi_{m,\varepsilon}\right)(x_0)z_i + 4\sum_{i,j}\partial^2_{i,j}\left(\phi_{m,\varepsilon}\right)(x_0)z_iz_j \tag{3.30}$$

and the holomorphic section of $(1+\ell)(K_X + L_p)|_U$

$$u := \prod f_i^{\lceil p(1+\ell)a_i^\varepsilon\rceil}\cdot e^{\frac{1+\ell}{2}h}dz \otimes e^{\otimes\ell}_{K_X+L_p}\otimes e_{L_p} \tag{3.31}$$

written as $(n,0)$-form with values in $\ell(K_X+L_p)+L_p|_U$. Note that $a_i^\varepsilon > a_i$ for each index i.

We denote by h_ε the metric on $\ell(K_X+L_p)+L_p$ defined by the weights

$$\ell\left(\phi_{m,\varepsilon} + p\phi_{E_\varepsilon}\right) + \tau_m + \log|s_{E_\varepsilon}|^{2\{\ell p\}} \tag{3.32}$$

where we recall that $\tau_m = (p-1)\left(\dfrac{\phi_{m-1}}{p} + \phi_E\right) + \phi_L$ was introduced in (3.24). The measure induced by the point-wise norm of u with respect to the metric (3.32) is equal to

$$|u|^2_{h_\varepsilon} = e^{(1+\ell)\Re(h)}\cdot e^{-\ell(\phi_{m,\varepsilon}+p\phi_{E,\varepsilon})-\tau_m}\prod|f_i|^{2\lceil p(1+\ell)a_i^\varepsilon\rceil}\frac{d\lambda}{|s_{E_\varepsilon}|^{2\{\ell p\}}} \tag{3.33}$$

and it can be reorganized as follows

$$|u|^2_{h_\varepsilon} = e^{(1+\ell)(\Re(h)-\phi_{m,\varepsilon}))}\cdot\prod|f_i|^{2\mu_i}\cdot e^{\{\ell pa_i^\varepsilon\}\rho_i}e^{\phi_{m,\varepsilon}-\frac{p-1}{p}\phi_{m-1}+\phi_E-\phi_L}d\lambda \tag{3.34}$$

where $\mu_i := \lceil p(1+\ell)a_i^\varepsilon\rceil - p\ell a_i^\varepsilon - pa_i - \{\ell pa_i^\varepsilon\}$.

We therefore have the point-wise inequality

$$|u|^2_{h_\varepsilon} \leqslant \gamma_\varepsilon F e^{(1+\ell)(\Re(h)-\phi_{m,\varepsilon})}(dd^c\phi_{m,\varepsilon})^n \tag{3.35}$$

on the set U, where the positive function F and the constant γ_ε are as follows.

(i) We define $\gamma_\varepsilon := \sup\limits_U e^{\frac{p-1}{p}|\phi_{m-1,\varepsilon}-\phi_{m-1}|}$. By Proposition 3.4, we have $\gamma_\varepsilon \to 0$.

(ii) Let $F := \prod|f_i|^{2\mu_i}\cdot e^{\{\ell pa_i^\varepsilon\}\rho_i}$ be the function corresponding to the product in (3.34). We have

$$F = \prod|f_i|^{2(\lceil p(1+\ell)\varepsilon c_i\rceil - \lceil p\ell\varepsilon c_i\rceil)}\cdot e^{\{\ell pa_i^\varepsilon\}\rho_i}$$

hence $F(0) \leqslant 1$, and $\sup_U(F) - 1$ is smaller than the diameter of U multiplied with a bounded constant.

Step 2. Estimate of the L^2 norm of u.

Let $B(r_\varepsilon)$ be the Euclidean ball centered at x_0 of radius r_ε with respect to the fixed Kähler metric ω. We have the following inequalities, which will be proved by a direct computation at the end of this section.

Claim 3.9. — *For every $\varepsilon > 0$ fixed, there exists a radius r_ε and a sequence a_ℓ converging to 0 (independent of $x_0 \in X$), such that*

$$\frac{(\ell+1)^n}{n!}\int_{B(r_\varepsilon)} Fe^{-(\ell+1)(\phi_{m,\varepsilon}-\mathrm{Re}(h))}(dd^c\phi_{m,\varepsilon})^n \leqslant 1 + a_\ell \qquad \textit{for every } \ell \in \mathbb{N}, \tag{3.36}$$

and

$$(\ell+1)^n\int_{B(r_\varepsilon)\setminus B(\frac{r_\varepsilon}{2})} Fe^{-(\ell+1)(\phi_{m,\varepsilon}-\mathrm{Re}(h))}(dd^c\phi_{m,\varepsilon})^n \leqslant a_\ell \qquad \textit{for every } \ell \in \mathbb{N}. \tag{3.37}$$

An important point of the claim is that the sequence $\{a_\ell\}$ is independent of x_0. Combined with the inequality (3.35) we obtain

$$\frac{(\ell+1)^n}{n!}\int_{B(r_\varepsilon)} |u|^2_{h_\varepsilon} \leqslant 1 + a_\ell \qquad \text{for every } \ell \in \mathbb{N}, \tag{3.38}$$

and

$$(\ell+1)^n\int_{B(r_\varepsilon)\setminus B(\frac{r_\varepsilon}{2})} |u|^2_{h_\varepsilon} \leqslant a_\ell \qquad \text{for every } \ell \in \mathbb{N}. \tag{3.39}$$

Step 3. Construction of a global section.

For every $\varepsilon > 0$ fixed, we will construct in this step a section $v_{\ell,\varepsilon} \in H^0(X, (\ell+1)p(K_X + L))$ such that $v_{\ell,\varepsilon}(x_0) = u(x_0)$ together with an estimate for its L^2 norm

$$\frac{(\ell+1)^n}{n!}\cdot\int_X |v_{\ell,\varepsilon}|^2_{h_\ell} e^{-\tau_m} \tag{3.40}$$

uniform with respect to the point $x_0 \in X$.

Let ρ be a smooth function on $X \setminus \{x_0\}$ which equals $n \log |x - x_0|^2$ near x_0. For every $\varepsilon > 0$, we can find a cut-off function χ_ε for $B(r_\varepsilon)$, namely $\chi_\varepsilon \equiv 1$ on $B(r_\varepsilon/2)$ and $\chi_\varepsilon \equiv 0$ on $X \smallsetminus B(r_\varepsilon)$ such that

$$e^{-\rho}|\bar\partial\chi_\varepsilon|^2_{\omega_{m,\varepsilon}} \leqslant M_\varepsilon \qquad \text{on } B(r_\varepsilon) \smallsetminus B(r_\varepsilon/2) \tag{3.41}$$

for some constant M_ε independent of $x_0 \in X$. One can easily check that for ℓ large enough (depending on ε), we have

$$\ell dd^c(\phi_{m,\varepsilon} + p\phi_{E,\varepsilon}) + dd^c\tau_{m,\varepsilon} + dd^c\big(\rho + \log|s_{E_\varepsilon}|^{2\{(1+\ell)p\}}\big) \geqslant dd^c\phi_{m,\varepsilon} \tag{3.42}$$

on X, since $dd^c\phi_{m,\varepsilon} = \omega_{m,\varepsilon}$ is a Kähler metric for each m, ε. Thanks to (3.38) we have

$$\frac{(\ell+1)^n}{n!}\int_X |\chi_\varepsilon u|^2_{h_\varepsilon} \leqslant 1 + a_\ell.$$

By the inequality (3.39) and the construction of χ_ε, we have

$$\int_X |\bar\partial(\chi_\varepsilon u)|^2_{h_\varepsilon,\omega_{m,\varepsilon}} \leqslant \int_{B(r_\varepsilon)\setminus B(\frac{r_\varepsilon}{2})} e^{-\rho}|\bar\partial\chi_\varepsilon|^2_{\omega_{m,\varepsilon}}\cdot e^{-(\ell+1)(\phi_{m,\varepsilon}-\mathrm{Re}(h))}(dd^c\phi_{m,\varepsilon})^n.$$

Together with (3.39) and (3.41), we get

$$(\ell+1)^n\int_X |\bar\partial(\chi_\ell u)|^2_{h_\varepsilon,\omega_{m,\varepsilon}} e^{-\rho} \leqslant a_\ell \cdot M_\varepsilon.$$

Thanks to (3.42), one can solve the $\bar\partial$-equation and apply Hörmander estimates (see e.g. [BDIP96, Cor. 14.3 on p.86]) for ℓ large enough (independent of x) to the $\bar\partial$-closed form

$$\bar\partial(\chi_\varepsilon u) \in \mathcal{C}^\infty(X, \Lambda^{n,1}T^*_X \otimes E)$$

where $E = \ell(K_X + L_p) + L_p$ is endowed with the hermitian metric $h_{\ell,\varepsilon}$. This yields a global, smooth section $u_{\ell,\varepsilon}$ of $\Lambda^{n,0}T^*_X \otimes E = (\ell+1)(K_X + L_p)$ such that

$$\bar\partial u_{\ell,\varepsilon} = \bar\partial(\chi_\varepsilon u) \quad \text{and} \quad (\ell+1)^n \int_X |u_{\ell,\varepsilon}|^2_{h_\varepsilon} \leqslant a_\ell \cdot M_\varepsilon.$$

Because of the non-integrability of $e^{-\rho}$ at x_0, one has $u_\ell(x_0) = 0$. As a consequence, the section $v_{\ell,\varepsilon} := \chi_\varepsilon u - u_{\ell,\varepsilon} \in H^0(X, (\ell+1)(K_X + L_p))$ satisfies the inequality

$$\frac{(\ell+1)^n}{n!} \cdot \int_X |v_{\ell,\varepsilon}|^2_{h_\varepsilon} \leqslant 1 + a_\ell \tag{3.43}$$

and we also have $v_{\ell,\varepsilon}(x_0) = u(x_0)$. By definition of C_ℓ we have

$$\frac{(\ell+1)^n}{n!} \cdot \int_X |v_\ell|^2_{h_\ell} e^{-\tau_m} \leqslant \gamma_\varepsilon \frac{1+a_\ell}{C_\ell}. \tag{3.44}$$

Step 4. Conclusion.

Thanks to (3.44) we obtain that the following inequality

$$K_{\ell+1} \geqslant \frac{C_\ell}{\gamma_\varepsilon(1+a_\ell)} \frac{(\ell+1)^n}{n!} \cdot e^{(\ell+1)(\phi_{m,\varepsilon} + p\phi_{E_\varepsilon})} |s_{E_\varepsilon}|^{2\{(1+\ell)p\}}$$

at x_0. Therefore

$$C_{\ell+1} \geqslant \kappa_{\varepsilon,\ell} \cdot \frac{(\ell+1)^n}{n!} \cdot C_\ell,$$

where $\kappa_{\varepsilon,\ell} = (\gamma_\varepsilon(1+a_\ell))^{-1}$. Although the sequence $\{a_\ell\}$ depends on ε we have

$$\lim_{\ell\to+\infty} \left(\prod_{k=1}^{\ell} \kappa_{\varepsilon,k} \right)^{1/\ell} = \gamma_\varepsilon$$

since it tends to 0. Therefore

$$\lim_{\varepsilon\to 0} \lim_{\ell\to+\infty} \left(\prod_{k=1}^{\ell} \kappa_{\varepsilon,k} \right)^{1/\ell} = 1,$$

and Proposition 3.8 is proved. □

It remains to prove Claim 3.9 stated in Step 3.

Proof of Claim 3.9. — Up to replacing $\phi_{m,\varepsilon}$ by $\phi_{m,\varepsilon} - \mathrm{Re}(h)$ (which does not change the metric $dd^c\phi_{m,\varepsilon}$), one can assume that $\phi_{m,\varepsilon}$ has no polyharmonic terms of order two or less in its expansion near x_0, and $F(x_0) = 1$. With respect to local coordinates (z_i) centered at x, one has

$$\phi_{m,\varepsilon} = \sum_{j,k} a_{j,k} z_j \bar z_k + R(z)$$

where $R(z) = O(|z|^3)$ and the matrix $A = (a_{j,k})$ is positive definite. These quantities are depending on ε but the important point is that, when $\varepsilon > 0$ is fixed, one can find a constant $C_\varepsilon > 0$ independent of the chosen point x such that

$$|R(z)| \leqslant C_\varepsilon |z|^3, \quad C_\varepsilon^{-1} I_n < A < C_\varepsilon I_n.$$

The constant C_ε can be chosen to be commensurable to $\sup_X(|\omega_{m,\varepsilon}|_\omega + |\nabla^\omega \omega_{m,\varepsilon}|)$.

After the change of variable $w := \sqrt{\ell+1}\sqrt{A} \cdot z$ and up to increasing the constant C_ε a little, the integral we have to bound is dominated by

$$\int_{|w|^2 \leqslant (\ell+1) r_\varepsilon^2} e^{-|w|^2(1 - C_\varepsilon \ell^{-1/2}|w|)} (1 + C_\varepsilon \ell^{-1/2}|w|) \cdot (dd^c|w|^2)^n / n!.$$

Now, if one chooses $r_\varepsilon < 1/(2C_\varepsilon)$, one sees that the integrand is less than $2e^{-|w|^2/2}$, hence one can apply the dominated convergence theorem to conclude that our integral is asymptotically dominated by

$$\int_{\mathbb{C}^n} e^{-|w|^2}\cdot (dd^c|w|^2)^n/n! = \frac{1}{\pi^n}\int_{\mathbb{C}^n} e^{-|w|^2} idw_1\wedge d\bar{w}_1\wedge\cdots\wedge idw_n\wedge d\bar{w}_n = 1$$

which concludes the proof of (3.36).

As for (3.37), the same change of variable reduces our integral to

$$\int_{\frac{1}{4}(\ell+1)r_\varepsilon^2\leqslant|w|^2\leqslant(\ell+1)r_\varepsilon^2} e^{-|w|^2(1-C_\varepsilon\ell^{-1/2}|w|)}(1+C_\varepsilon\ell^{-1/2}|w|)\cdot(dd^c|w|^2)^n/n!$$

which, up to increasing C_ε, is dominated by

$$C_\varepsilon\int_{\frac{1}{4}(\ell+1)r_\varepsilon^2\leqslant|w|^2\leqslant(\ell+1)r_\varepsilon^2} e^{-\ell r_\varepsilon^2/8}\cdot(dd^c|w|^2)^n = O(\ell^n e^{-\ell r_\varepsilon^2/8}).$$

The estimate (3.37) follows. □

We will also need an integral upper estimate of K_ℓ; it follows easily from the definition of the Bergman kernel.

Proposition 3.10. — *One has the following upper bound:*

$$\limsup_{\ell\to+\infty}\int_X n!(\ell!^{-n}K_\ell)^{1/\ell}e^{-\tau_p}\leqslant(pA)^n$$

Proof. — First, observe that $(\ell!^{-n}K_\ell)^{1/\ell}e^{-\tau_p}$ is a volume form, so that the claim is licit. Let $(u_1,\dots,u_{N_\ell})$ be an orthonormal basis of $H^0(X,\ell(K_X+L_p))$ with respect to the Bergman L^2 metric

$$K_{\ell-1}^{-1}e^{-\tau_p}$$

Since $\ell p(K_X+L) = \ell pA+\ell pE$ is the Zariski decomposition of $\ell p(K_X+L)$, every (pluri)-section is L^2 with respect to the Bergman metric. In particular we have

$$N_\ell := \dim H^0(X,\ell pA) = \frac{(pA)^n}{n!}\cdot\ell^n(1+O(\ell^{-1})) \tag{3.45}$$

by Riemann-Roch formula. One has

$$\int_X K_\ell\cdot K_{\ell-1}^{-1}e^{-\tau_p} = N_\ell.$$

Therefore, applying Hölder's inequality with $p=\ell$ and $q=\frac{\ell}{\ell-1}$, one gets

$$\begin{aligned}\int_X K_\ell^{\frac{1}{\ell}}\cdot e^{-\tau_p} &\leqslant \left(\int_X K_\ell\cdot K_{\ell-1}^{-1}e^{-\tau_p}\right)^{\frac{1}{\ell}}\cdot\left(\int_X K_{\ell-1}^{\frac{1}{\ell-1}}e^{-\tau_p}\right)^{\frac{\ell-1}{\ell}}\\ &\leqslant N_\ell^{1/\ell}\cdot\left(\int_X K_{\ell-1}^{\frac{1}{\ell-1}}e^{-\tau_p}\right)^{\frac{\ell-1}{\ell}}.\end{aligned}$$

By induction, one gets:

$$\int_X K_\ell^{\frac{1}{\ell}}\cdot e^{-\tau_p}\leqslant\left(\prod_{i=1}^{\ell}N_i\right)^{\frac{1}{\ell}}.$$

Now, thanks to (3.45), the right-hand side of this inequality is equal to

$$\frac{(pA)^n}{n!}\cdot(\ell!)^{\frac{n}{\ell}}\left(1+O\left(\frac{\log\ell}{\ell}\right)\right)$$

which concludes the proof of the proposition.

□

After all these preliminary statements we can prove the main result of this subsection.

Proof of Theorem 3.7. — By Proposition 3.10 combined with Jensen inequality, $\{(\ell!^{-n}K_\ell)^{1/\ell}\}_{\ell=1}^{+\infty}$ is a family of upper bounded psh weights. Therefore, to prove the Theorem, it is sufficient to prove that any convergent subsequence of $\{n!\,(\ell!^{-n}K_\ell)^{1/\ell}\}_{\ell=1}^{+\infty}$ converges to $e^{\phi_m+p\phi_E}$.
Let $\{n!\,(\ell_s!^{-n}K_{\ell_s})^{1/\ell_s}\}_{s=1}^{+\infty}$ be a convergent subsequence of $\{n!\,(\ell!^{-n}K_\ell)^{1/\ell}\}_{\ell=1}^{+\infty}$ and let Γ be the limit. Thanks to Proposition 3.8, one infers:

$$n!\,(\ell!^{-n}K_\ell)^{1/\ell} \geqslant \left(\prod_{k=1}^{\ell}\kappa_{\varepsilon,k}\right)^{1/\ell}\cdot|s_{E_\varepsilon}|^{\frac{2\{\ell p\}}{\ell}}e^{\phi_{m,\varepsilon}+p\phi_{E_\varepsilon}}$$

Letting ℓ tend to $+\infty$, and then letting ε tend to zero, item (3.14) in Proposition 3.25 and Proposition 3.8 yield:

$$\liminf_{\ell\to+\infty} n!\,(\ell!^{-n}K_{\ell,m})^{1/\ell} \geqslant e^{\phi_m+p\phi_E}. \tag{3.46}$$

Therefore, we have

$$\Gamma \geqslant e^{\phi_m+p\phi_E}. \tag{3.47}$$

Note that

$$\int_X \Gamma e^{-(p-1)\left(\frac{\phi_{m-1}}{p}+\phi_E\right)}e^{-\phi_L} = \lim_{\ell\to+\infty}\int_X n!(\ell_s!^{-n}K_{\ell_s,m})^{1/\ell_s}e^{-(p-1)\left(\frac{\phi_{m-1}}{p}+\phi_E\right)}e^{-\phi_L}.$$

Combining this with Proposition 3.10, we get

$$\int_X \Gamma e^{-(p-1)\left(\frac{\phi_{m-1}}{p}+\phi_E\right)}e^{-\phi_L} \leqslant (pA)^n = \int_X e^{\phi_m+p\phi_E}\cdot e^{-(p-1)\left(\frac{\phi_{m-1}}{p}+\phi_E\right)}e^{-\phi_L},$$

where the last equality comes from (3.18). Together with (3.47), we get $\Gamma = e^{\phi_m+p\phi_E}$ and the theorem is proved. □

Remark 3.11. — The convergence $(\ell!^{-n}K_\ell)^{1/\ell} \to \frac{e^{\phi_m+p\phi_E}}{n!}$ has been proved for a fixed fiber X_y, but it readily implies convergence in $L^1_{\mathrm{loc}}(X^\circ)$. Indeed, as (A_y^n) is independent of $y \in Y^\circ$, Proposition 3.10 coupled with Jensen inequality show that the weights of the metric $(\ell!^{-n}K_\ell)^{-1/\ell}$ are uniformly bounded above locally near $X \smallsetminus X_0$, hence the pointwise convergence almost everywhere on X° implies convergence in $L^1_{\mathrm{loc}}(X^\circ)$.

Now, we can finally give the proof of Theorem 1.7.

Proof of Theorem 1.7. — Thanks to the reduction steps, we can suppose that on the fibers over $y \in Y^0$, we have a Zariski decomposition

$$(K_{X/Y}+L)|_{X_y} = A_y + E_y,$$

where A is semi-ample and big, E_y has snc support and h_L is smooth with semipositive curvature.

We first prove by induction that for every $m \in \mathbb{N}$, $\phi_m + p\phi_E$ is a psh weight on X°.

For $m = 1$: We get a sequence of metrics $(h_{\ell,1})_{\ell\geqslant 1}$ on $p\ell(K_X+L)$ defined by (3.25). Recall that $\phi_0 = \phi_A$ is the weight of ω_A. Thanks to [**BP08**, Thm. 0.1], $h_{1,1}$ has positive curvature on the total space X. We suppose by induction that $h_{\ell,1}$ has positive curvature. Then $h_{\ell,1}\cdot e^{-(p-1)\left(\frac{\phi_A}{p}+\phi_E\right)}e^{-\phi_L}$ has also positive curvature. By applying [**BP08**, Thm. 0.1] again, $h_{\ell+1,1}$ has

positive curvature on the total space X. As a consequence, $h_{\ell,1}$ has positive curvature on the total space X for all ℓ. Together with Theorem 3.7, the limit $\phi_1 + p\phi_E$ is a psh weight on X^0.

Then we apply the same process again to $m = 2$, and get a sequence of metrics $(h_{\ell,2})_{\ell \geqslant 1}$ on $p\ell(K_X + L)$ with positive curvature, and therefore the limit $\phi_2 + p\phi_E$ is psh. By induction on m, we know that $\phi_m + p\phi_E$ is psh for any $m \geqslant 1$.

We can now prove the Theorem. Thanks to Proposition 3.5 and Remark 3.6, $\phi_m + p\phi_E$ converges to the relative Kähler-Einstein metric of (X°, L, ϕ_L). As $\phi_m + p\phi_E$ has positive curvature, the relative Kähler-Einstein metric has also positive curvature. □

3.6. The orbifold case. — In this paragraph, we would like to discuss an extension of Theorem 1.7 to a particular case where the weight ϕ_L could have positive Lelong numbers. More precisely, we have the following

Proposition 3.12. — *Let $p : X \to Y$ be a Kähler fiber space and let $B = \sum_{i \in I} \left(1 - \frac{1}{m_i}\right) B_i$ be an effective divisor such that:*

- *The $\mathbb{Q}$-line bundle $K_{X/Y} + B$ is p-ample.*
- *For $y \in Y$ generic, $B|_{X_y}$ has snc support.*
- *The numbers $m_i \geqslant 1$ are integers such that $(m_i, m_j) = 1$ whenever $B_i \cap B_j \neq \emptyset$.*

Then, the relative Kähler-Einstein metric is positively curved and extends canonically through the singular locus of p.

Proof. — The same strategy as for the proof of Theorem 1.7 applies, modulo the fact that $\phi_{m,\varepsilon}$ from (3.27) are to be replaced by their orbifold counterparts, so that we have

$$(dd^c\phi_{m,\varepsilon})^n = e^{\phi_{m,\varepsilon} - \frac{p-1}{p}\phi_{m-1,\varepsilon} - \phi_B}(\varepsilon^2 e^{\rho_E} + e^{\phi_E})d\lambda \tag{3.48}$$

where ϕ_B is the canonical *singular* weight on the $\mathbb{Q}$-line bundle $\mathcal{O}_X(B)$. The current $dd^c\phi_{m,\varepsilon}$ defines an *orbifold Kähler metric*, that is, its pull back to local uniformizing charts near the support of B becomes a genuine Kähler metric. In this setting, a new problem arise (due to the presence of singularities): the peak sections from Proposition 3.8 have to be replaced by orbifold peak sections. One way to bypass this is to use the orbifold Bergman kernel expansion due to Ross-Thomas [**RT11**], see also Dai-Liu-Ma [**DLM12**]. Instead of considering the Bergman kernels K_ℓ on $\ell(K_{X/Y} + B)$, one considers suitable linear combinations of the Bergman kernels of the form $\sum_{\alpha=0}^{m_i-1} K_{\ell+\alpha}$. These combinations, unlike K_ℓ alone, turn out to admit the same expansion as in the smooth case at order zero, when $\ell \to +\infty$. This is where the assumption on the arithmetic relation between the $m_i's$ is important. The rest of the proof of Theorem 1.7 can be applied almost without any change to conclude. □

Remark 3.13. — *It is likely that combining the ideas above with the techniques in* [**RT11**] *may allow us to weaken the assumption that $K_{X/Y} + B$ is p-ample and only assume that $K_{X/Y} + B$ is p-big admits a relative Zariski decomposition on X.*

4. The case of intermediate Kodaira dimension

4.1. Iitaka fibration and associated Kähler-Einstein metric. — We will first consider the absolute case. Let X be a compact Kähler manifold and let B be a $\mathbb{Q}$-effective divisor such that the pair (X, B) is klt and such that $\kappa(K_X + B) > 0$. Let Z be the canonical model of (X, B). We consider $f : X \dashrightarrow Z$ the Iitaka fibration induced by the linear system $|m(K_X + B)|$ for m large and divisible enough. Thanks to [**BCHM10**] in the projective case and [**Fuj15**] for the Kähler case the space Z is normal. After desingularisation f induces a fibration between two compact Kähler manifolds. For simplicity we will denote the new map by $f : X \to Z$. In general the

torsion-free sheaf $f_\star(m(K_{X/Z}+B))$ is not locally free. This is the case for the reflexive hull $(f_\star(m(K_{X/Z}+B)))^{\star\star}$.

We now recall the definition of the Narashimhan-Simha metric on the line bundle $(f_\star(m(K_{X/Z}+B)))^{\star\star}$.

Definition 4.1. — *Let $Z_0 \subset Z$ be a locus such that f is smooth over Z_0 and $B|_{X_z}$ is klt for every $z \in Z_0$. Let $z \in Z_0$ and let $s \in (f_\star(m(K_{X/Z}+B)))_z = H^0(X_z, mK_{X_z}+mB)$. We define the Narashimhan-Simha metric*

$$\|s\|^2_{h_m} := (\int_{X_z} |s|^{\frac{2}{m}}_{h_B})^m,$$

where h_B is the canonical singular hermitian metric with respect to the divisor B. Thanks to [**BP08**]*, h_m can be canonically extended as a possible singular metric on*

$$(Z, (f_\star(m(K_{X/Z}+B)))^{\star\star}).$$

We call it the m-th Narashimhan-Simha metric.

Remark 4.2. — *We can easily check that the weight of h_m is locally integrable over the locus where $f_\star(m(K_{X/Z}+B))$ is locally free. Moreover, the pair $(Z, \frac{1}{m}(f_\star(m(K_{X/Z}+B)))^{\star\star}, h_m^{\frac{1}{m}})$ is independent of the choice of m, namely for any two m_1, m_2 large and sufficiently divisible, we have an isometry*

$$(\frac{1}{m_1}(f_\star(m_1(K_{X/Z}+B)))^{\star\star}, h_{m_1}^{\frac{1}{m_1}}) \equiv_{\mathbb{Q}} (\frac{1}{m_2}(f_\star(m_2(K_{X/Z}+B)))^{\star\star}, h_{m_2}^{\frac{1}{m_2}}).$$

By construction, the pair $(Z, \frac{1}{m}(f_\star(m(K_{X/Z}+B)))^{\star\star}, h_m^{\frac{1}{m}})$ satisfies (1.1.1) and (1.1.2). However, it does not satisfy in general (1.1.3), roughly because of the codimension two subsets of the base Z whose f-inverse image have codimension one. This situation can be improved by a trick due to [**Vie83**] which we now recall. By Hironaka's flattening theorem cf. [**Vie83**, Lemma 7.3], we can find a morphism $f' : X' \to Z'$ between two compact Kähler manifolds which satisfies the following commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Z' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\mu} \\ X & \xrightarrow{f} & Z \end{array}$$

such that the morphisms π and μ are bimeromorphic, and moreover, each hypersurface $W \subset X'$ such that $\mathrm{codim}_{Y'} f'(W) \geqslant 2$ is π-contractible, i.e., $\mathrm{codim}_X \pi(W) \geqslant 2$.

We denote by $\hat{B}$ the strict transform of B by π, and write $K_{X'} + \hat{B} = \pi^*(K_X + B) + \sum a_i E_i$. We set $B' := \hat{B} + \sum_{a_i<0}(-a_i)E_i$. Then (X', B') is klt. Let us choose m large enough so that $\mathcal{F}_m := f'_\star(m(K_{X'/Z'}+B'))$ is non-zero. Then, $\mathcal{F}_m$ is a torsion free sheaf of rank one on Z' and its reflexive hull $\mathcal{F}_m^{\star\star}$ is a line bundle that we can equip with the m-th Narashimhan-Simha metric h_m. Thanks to Remark 4.2, the following $\mathbb{Q}$-line bundle and the metric

$$L := \frac{1}{m} f'_\star(m(K_{X'/Z'}+B'))^{\star\star}, \qquad h := h_m^{\frac{1}{m}}$$

are independent of the choice of m. Let ϕ be the weight of h.

We have the following statement, which connects our current setting with Definition/Proposition 1.1.

Proposition 4.3. — *In the above setting, the following holds.*

(4.3.1) $i\Theta_\phi(L) \geqslant 0$ *on* Z'.

(4.3.2) $K_{Z'} + L$ is a big $\mathbb{Q}$-line bundle and for any $m \in \mathbb{N}$ sufficiently divisible, the algebra

$$\bigoplus_{p \geqslant 0} H^0(Z', pm(K_{Z'} + L))$$

is finitely generated.

(4.3.3) For every $p \in \mathbb{N}$ sufficiently divisible and every $s \in H^0(Z', p(K_{Z'} + L))$, we have

$$\int_{Z'} |s|^{\frac{2}{p}} e^{-\phi} < +\infty.$$

Proof. — The first item is a direct consequence of [BP08].

For the second term, let $m \in \mathbb{N}$ sufficiently divisible such that for every $p \in \mathbb{N}$, $H^0(X', pm(K_{X'} + B'))$ is generated by $\otimes^p H^0(X', m(K_{X'} + B'))$. By the construction of L, there exist two effective divisors E_+ and E_- on X' such that

$$m(K_{X'} + B') + E_- = m \cdot (f')^*(K_{Z'} + L) + E_+, \tag{4.4}$$

and for every $\tau \in H^0(X', m(K_{X'} + B'))$, τ vanishes over E_+. As a consequence, for every $s \in H^0(X', pm(K_{X'} + B'))$, s vanishes over $p[E_+]$. Note that $(f')_*(E_-)$ is supported in the non locally free locus of $f'_\star(m(K_{X'/Z'} + B'))$. Then E_- is π-contractible. Together with the above argument, for every $p \in \mathbb{N}$, we have the natural isomorphisms

$$H^0(Z', pm(K_{Z'} + L)) = H^0(X', pm(K_{X'} + B') + pE_-) = H^0(X, pm(K_X + B)).$$

As a consequence, $K_{Z'} + L$ is big and the algebra

$$\bigoplus_{p \geqslant 0} H^0(Z', pm(K_{Z'} + L))$$

is finitely generated.

For the third term, let s_{E_+} be the canonical section of E_+ and let $h_{B'}$ (resp. h_{E_-}) be the canonical singular metric on B' (resp. E_-). As p is sufficient divisible, we can assume that $p_1 := \frac{p}{m} \in \mathbb{N}$. Thanks to (4.4), we have

$$(f')^*(s) \otimes s_{E_+}^{\otimes p_1} \in H^0(X', p_1(mK_{X'} + mB' + E_-)).$$

By the definition of the Narashimhan-Simha metric, we have

$$\int_{Z'} |s|^{\frac{2}{p}} e^{-\phi} = \int_{X'} |(f')^*(s) \otimes s_{E_+}^{\otimes p_1}|^{\frac{2}{p}}_{h_{B'}, h_{E_-}}.$$

Note that E_- is π-contractible, $(f')^*(s) \otimes s_{E_+}^{\otimes p_1}$ vanishes along E_- of order at least $p_1 E_-$. Together with the fact that B' is klt, we have

$$\int_{X'} |(f')^*(s) \otimes s_{E_+}^{\otimes p_1}|^{\frac{2}{p}}_{h_{B'}, h_{E_-}} < +\infty.$$

The proposition is proved. □

Remark 4.4. — *Using the above argument, we know that $e^{-\phi}$ is in $L^1_{\mathrm{loc}}(Z' \setminus f'(E_-))$.*

Together with Proposition 4.3, we get

Corollary 4.5. — *With the above notation, Z' admits a natural Kähler-Einstein metric ω_{KE} in the sense of Definition 1.1. This metric satisfies*

$$\operatorname{Ric} \omega_{\mathrm{KE}} = -\omega_{\mathrm{KE}} + \Theta_\phi(L) \quad \text{on } Z'. \tag{4.5}$$

We call ω_{KE} the Kähler-Einstein metric associated to the Iitaka fibration of (X, B).

4.2. Relation with the canonical metrics. — Let X be a compact Kähler manifold and let B be a $\mathbb{Q}$-divisor with snc support such that (X, B) is klt. We suppose that $\kappa(K_X + B) \geqslant 1$. Thanks to [BCHM10, Fuj15], the canonical model Z of (X, B) is normal. After blowing up the indeterminacy locus of the Iitaka fibration, we can suppose that the Iitaka fibration of $K_X + B$ induces a morphism $f : X \to Z$ and there is an ample $\mathbb{Q}$-line bundle A on Z such that

$$K_X + B = f^*A + E_X$$

is a Zariski decomposition for some effective $\mathbb{Q}$-divisor E_X with normal crossing support on X. In that context, the analogue of Kähler-Einstein metrics for the pair (X, B, E_X), sometimes called canonical metrics, are objects that are singular metrics ω_{can} on Z satisfying a "canonical" Monge-Ampère equation. They were first introduced by Song-Tian when $B = E_X = 0$ [ST12] and later generalized by Eyssidieux-Guedj-Zeriahi [EGZ18, Definition 2.2, 2.7].

Let us recall the definition of the canonical metric in this setting. One first picks a smooth hermitian $h_A = e^{-\phi_A}$ on A with positive curvature $\chi := dd^c\phi_A$. Then, one introduces a measure μ_{h_A,h_E} on X by setting

$$\mu_{h_A,h_E} := \frac{(\sigma \wedge \bar{\sigma})^{\frac{1}{N}} e^{-\phi_B}}{|\sigma|^{2/N}_{f^*h_A,h_E}}$$

where σ is a local trivialization of $N(K_X + B)$ for N divisible enough, ϕ_B is the canonical singular weight on B and h_E is the canonical singular metric on E. Finally, one defines $\omega_{\text{can}} := \chi + dd^c\varphi_{\text{can}}$ as the unique positive current on Z with bounded potentials such that

$$(\chi + dd^c\varphi_{\text{can}})^{\dim Z} = e^{\varphi_{\text{can}}} f_*\mu_{h_A,h_E}. \tag{4.6}$$

Note that the singularity of h_E gives rise the zero locus of μ_{h_A,h_E}. One can check that the measure $f_*\mu_{h_A,h_E}$ has $L^{1+\varepsilon}$ density with respect to a smooth volume form, cf. [EGZ18, Lemma 2.1]. Moreover, the canonical metric ω_{can} is independent of the choice the hermitian metric h_A, cf. [EGZ18, Lemma 2.4].

By applying the construction in subsection 4 to $f : X \to Z$, we can find a morphism $f' : X' \to Z'$ between two compact Kähler manifolds and satisfies the following commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Z' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\mu} \\ X & \xrightarrow{f} & Z \end{array}$$

such that the morphisms π and μ are bimeromorphic, and moreover, each hypersurface $W \subset X'$ such that $\operatorname{codim}_{Y'} f'(W) \geqslant 2$ is π-contractible, i.e., $\operatorname{codim}_X \pi(W) \geqslant 2$. Following the notations in Proposition 4.3, for m large enough, we can equip $L := \frac{1}{m} f'_\star(m(K_{X'/Z'} + B'))^{\star\star}$ with the Narashimhan-Simha type metric $e^{-\phi}$. Thanks to Corollary 4.5, we can find the "NS-type" Kähler-Einstein metric ω_{KE} on Z' which satisfies

$$\operatorname{Ric}\omega_{\text{KE}} = -\omega_{\text{KE}} + \frac{i}{2\pi}\Theta_\phi(L) \quad \text{on } Z'.$$

We now establish a relation between the Kähler-Einstein metric ω_{KE} and the canonical metric ω_{can}.

Proposition 4.6. — *With the notation above, let ω_{KE} be the Kähler-Einstein metric on Z' solution of (4.5) and let ω_{can} be the canonical metric on Z solution of (4.6). Then, one has $\mu_*\omega_{\text{KE}} = \omega_{\text{can}}$.*

Proof. — Recall that by (4.4), we have

$$m(K_{X'} + B') + E_- = (f')^*(m(K_{Z'} + L)) + E_+, \tag{4.7}$$

and by construction, we have

$$m(K_{X'} + B' - E_{X'}) = (f' \circ \mu)^* mA, \tag{4.8}$$

for some $\mathbb{Q}$-effective divisor $E_{X'}$ such that $\pi_\star(E_{X'}) = E_X$.

We first establish the relation between $\mu^* A$ and $K_{Z'} + L$. Remember that $f'_*\mathcal{O}_{X'}(kE_+) \simeq \mathcal{O}_{Z'}$ for any integer k such that kE_+ has integral coefficients. We deduce that $m(K_{Z'} + L) = \mu^*(mA) \otimes f'_*(mE_{X'} + E_-)$. In particular, we get that $f'_*(mE_{X'} + E_-)$ is a locally trivial sheaf of rank one, hence associated to a divisor $mE_{Z'}$; it is clearly effective and μ-exceptional, as $\mathrm{Supp}(f')_*(mE_{X'}) \subset \mu^{-1}(Z_{\mathrm{sing}})$ and $\mathrm{codim}_{Z'} f'(E_-) \geqslant 2$. Then we have the Zariski decomposition

$$K_{Z'} + L \equiv_{\mathbb{Q}} \mu^* A + E_{Z'} \tag{4.9}$$

By construction, we have $f'^*(mE_{Z'}) + E_+ = mE_{X'} + E_-$.

Now, take $U \subset Z'$ a small coordinate open subset, and let $e_{\mu^* mA} \in H^0(U, \mu^* mA)$ and $e_{mE_{Z'}} \in H^0(U, mE_{Z'})$ be trivializations of $\mu^* mA$ and $mE_{Z'}$ respectively. Let $d\underline{z}$ be a trivialisation of $K_{Z'}$ over U. They induce a trivialization $e \in H^0(U, mL)$ of mL such that

$$d\underline{z}^{\otimes m} \otimes e = e_{\mu^* mA} \otimes e_{mE_{Z'}}. \tag{4.10}$$

Set $e^{-\varphi_{\mu^* A}} := |e_{\mu^* mA}|^{\frac{2}{m}}_{\mu^* mh_A}$, $e^{-\varphi_{E_{Z'}}} := |e_{mE_{Z'}}|^{2/m}_{e^{-\phi_{mE'_Z}}}$ and $e^{-\varphi} := |e|^{2/m}_{e^{-\phi}}$. Let

$$\sigma := (f')^* e_{\mu^* mA} \in H^0(f'^{-1}(U), m(K_{X'} + B' - E_{X'})).$$

Thanks to (4.7) and (4.10), we have

$$\tau := \sigma \otimes (f')^*(e_{mZ'}) \otimes s_{E_+} \in H^0(f'^{-1}(U), m(K_{X'} + B') + E_-),$$

and

$$e^{-\varphi(z)} = \int_{X'_z} |\tau|^{2/m} e^{-\phi_{B'} - \frac{1}{m}\phi_{E_-}} = e^{-\varphi_{E_{Z'}}} \cdot \int_{X'_z} |\sigma|^{2/m} e^{-\phi_{B'} + \phi_{E_{X'}}}. \tag{4.11}$$

The canonical measure ν on Z' has density with respect to the Lebesgue measure $d\lambda = |d\underline{z}|^2$ given by the formula

$$\frac{d\nu}{d\lambda}(z) = \int_{X'_z} \frac{(\sigma \wedge \bar{\sigma})^{1/m} e^{-\phi_{B'} + \phi_{E_{X'}}}}{|\sigma|^{\frac{2}{m}}_{f'^*\mu^* h_A}} = e^{\varphi_{\mu^* A}} \int_{X'_z} |\sigma|^{2/m} e^{-\phi_{B'} + \phi_{E_{X'}}}$$

for $z \in Z'$ generic. Together with (4.11), we get

$$\frac{d\nu}{d\lambda} = e^{\varphi_{\mu^* A} - \varphi + \varphi_{E_{Z'}}}$$

Therefore, $\omega := \mu^* \omega_{\mathrm{can}} + [mE_{Z'}]$ satisfies

$$\mathrm{Ric}\, \omega = -\mu^* \omega_{\mathrm{can}} + dd^c \varphi - [E_{Z'}] = -\omega + \frac{i}{2\pi} \Theta_\phi(L)$$

As $\omega \in c_1(K_{Z'} + L)$ has minimal singularities by the Zariski decomposition (4.9) and satisfies the same Monge-Ampère equation as ω_{KE}, one deduces that $\omega = \omega_{\mathrm{KE}}$, i.e.,

$$\omega_{\mathrm{KE}} = \mu^* \omega_{\mathrm{can}} + [E_{Z'}].$$

As $E_{Z'}$ is μ-exceptional, the proposition is proved. □

Remark 4.7. — *The proof of Proposition 4.6 above shows the more precise identity $\omega_{\mathrm{KE}} = \mu^*\omega_{\mathrm{can}} + E_{Z'}$ for some explicit divisor $E_{Z'}$ on Z'.*

4.3. Relative Kähler-Einstein and canonical metrics, main Theorem. — To finish this section, we now discuss the positivity of the relative Kähler-Einstein or the canonical metrics when the fiber is of intermediate Kodaira dimension.

Theorem 4.8. — *Let $p : X \to Y$ be a projective fibration between two Kähler manifolds of relative dimension n and let B be an effective klt $\mathbb{Q}$-divisor on X. We assume that for a generic fiber X_y, the log Kodaira dimension satisfies $\kappa(K_{X_y} + B_y) > 0$. Let $f : X \dashrightarrow Z$ be the relative Iitaka fibration of $K_{X/Y} + B$, and let $f' : X' \to Z'$ a birational model of f such that X' and Z' are smooth.*

$$\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[d] & & Z' \arrow[d] \\
X \arrow[rr, dashed, "f"] \arrow[dr, "p"'] & & Z \arrow[dl] \\
& Y &
\end{tikzcd}$$

For y generic, let $\omega_{\mathrm{can},y}$ be the canonical metric on Z'_y of the pair (X'_y, B'_y); it induces a current $\omega^\circ_{\mathrm{can}}$ over the smooth locus of $Z' \to Y$.
Assuming that Conjecture 1.4 holds, then the current $\omega^\circ_{\mathrm{can}}$ is positive and extends canonically to a closed positive current on Z'.

The proof of Theorem 4.8 consists mostly in putting together all the constructions explained above. By using [**BCHM10**, **Fuj15**], the canonical ring

$$R(X_y, B_y) = \bigoplus_{m \geqslant 0} H^0(X_y, \lfloor m(K_{X_y} + B_y) \rfloor)$$

is finitely generated. Together with the fact that $h^0(X_y, m(K_{X_y} + B|_{X_y}))$ is constant with respect to $y \in Y^\circ$ for some m large enough, one can construct the relative Iitaka fibration $f : X \dashrightarrow Z := \mathrm{Proj}(p_*(m(K_{X/Y} + B))$. Thanks to the Subsection 4.2, we can find a desingularization $f' : X' \to Z'$ fitting the commutative diagram

$$\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[d, "\pi"'] & & Z' \arrow[d, "\mu"] \arrow[ddl, bend left, "q'"] \\
X \arrow[rr, dashed, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\
& Y &
\end{tikzcd}$$

such that we have a f'-Zariski decomposition over Y_0

$$K_{X'} + B' \equiv_{\mathbb{Q}} (f' \circ \mu)^* A + E \qquad \text{on} \quad (f')^{-1}(Y_0),$$

where A is q-ample.

Let $q' := q \circ \mu : Z' \to Y$ be the projection to the base. Each fiber Z'_y for $y \in Y^\circ$ can be endowed with a canonical metric $\omega_{\mathrm{can},y} \in c_1(\mu^* A_y)$ and a Kähler-Einstein metric $\omega_{\mathrm{KE},y} \in c_1(K_{Z'_y} + L_y)$, where $L_y := L|_{Z'_y}$ is the restriction of the $\mathbb{Q}$-line bundle $L := \frac{1}{m} f'_*(m(K_{X'/Z'} + B'))^{**}$ to Z'_y, endowed with the corresponding restriction of the Narasimhan-Simha metric on L.

In particular, these fiberwise metrics induce singular hermitian metrics $e^{-\phi_{\mathrm{can}}}$ on $\mu^* A|_{q^{-1}(Y^\circ)}$ and $e^{-\phi_{\mathrm{KE}}}$ on $L|_{q'^{-1}(Y^\circ)}$ respectively. As seen in the Subsection 4.2, there exists a μ-exceptional effective divisor $E_{Z'}$ on Z' such that $\phi_{\mathrm{KE}} = \phi_{\mathrm{can}} + [E_{Z'}]$.

Assuming Conjecture 1.4, $e^{-\phi_{\mathrm{KE}}}$ is a positively curved metric on $(K_{Z'/Y} + L)|_{q'^{-1}(Y^\circ)}$ that extends canonically to a positively curved metric on $K_{Z'/Y} + L$ on the whole Z'. As ϕ_{can} comes from Z and $E_{Z'}$ is μ-exceptional, if follows that $e^{-\phi_{\mathrm{can}}}$ is a positively curved metric on $\mu^* A|_{q'^{-1}(Y^\circ)}$ that extends canonically to Z'. This proves Theorem 4.8.

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