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p -Laplacians on Directed Graphs

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I. INTRODUCTION AND NOTATIONS

The graph Laplacian plays an important role in describing the structure of a graph signal from weights that measure the similarity between the vertices of the graph. In the literature, three definitions of the graph Laplacian have been considered for undirected graphs: the combinatorial, the normalized and the random-walk Laplacians. Moreover, a nonlinear extension of the Laplacian, called the p -Laplacian, has been put forward for undirected graphs [1], [2]. In this paper, we propose several formulations for p -Laplacians on directed graphs directly inspired from the Laplacians on undirected graphs. Then, we consider the problem of p -Laplacian regularization of graph signals. Finally, we provide experimental results to illustrate the effect of the proposed p -laplacians on two different types of graph signals (images and colored meshes).

A graph represents a set of elements and a set of pairwise relationships between them. The elements are called vertices and the relationships are called edges. Formally, a graph \mathcal{G} is defined by the sets $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in which $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. We denote the i th vertex as $v_i \in \mathcal{V}$. Since each edge is a subset of two vertices, we write $e_{ij} = \{v_i, v_j\}$. A graph is called directed when each edge e_{ij} contains an ordering of the vertices. A directed edge from v_j to v_i will be denoted $v_j \rightarrow v_i$. The edges of a graph can be weighted with a function denoted by $w : \mathcal{E} \rightarrow \mathbb{R}^+$. The out-degree of a node v_i , $d^+(v_i)$, is equal to $d^+(v_i) = \sum_{v_j \rightarrow v_i \in \mathcal{E}} w_{ij}$. The in-degree of a node v_i , $d^-(v_i)$, is equal to $d^-(v_i) = \sum_{v_j \rightarrow v_i \in \mathcal{E}} w_{ji}$. Note that in an undirected graph, $d^+(v_i) = d^-(v_i)$, $\forall v_i \in \mathcal{V}$ and is denoted $d(v_i)$. Let $\mathcal{H}(\mathcal{V})$ be the Hilbert space of real-valued functions defined on the vertices of a graph, a graph signal is a function $f : \mathcal{V} \rightarrow \mathbb{R}^n$ of $\mathcal{H}(\mathcal{V})$ that maps each vertex to a vector $\mathbf{f}(v_i)$. The space $\mathcal{H}(\mathcal{V})$ is endowed with the usual inner product $\langle f, h \rangle_{\mathcal{H}(\mathcal{V})} = \sum_{v_i \in \mathcal{V}} f(v_i)h(v_i)$, where $f, h : \mathcal{V} \rightarrow \mathbb{R}$. Similarly, let $\mathcal{H}(\mathcal{E})$ be the space of real-valued functions defined on the edges of \mathcal{G} . It is endowed with the inner product $\langle F, H \rangle_{\mathcal{H}(\mathcal{E})} = \sum_{e_{ij} \in \mathcal{E}} F(e_{ij})H(e_{ij})$, where $F, H : \mathcal{E} \rightarrow \mathbb{R}$ are two functions of $\mathcal{H}(\mathcal{E})$.

The directed difference operator of a graph signal $f \in \mathcal{H}(\mathcal{V})$, called $d_w : \mathcal{H}(\mathcal{V}) \rightarrow \mathcal{H}(\mathcal{E})$, over a directed edge $v_i \rightarrow v_j$ is denoted by $(d_w f)(v_i, v_j)$. The adjoint operator $d_w^* : \mathcal{H}(\mathcal{E}) \rightarrow \mathcal{H}(\mathcal{V})$, of a function $H \in \mathcal{H}(\mathcal{E})$, can then be expressed at a vertex $v_i \in \mathcal{V}$ by using the definition of the inner products $\langle H, d_w f \rangle_{\mathcal{H}(\mathcal{E})} = \langle d_w^* H, f \rangle_{\mathcal{H}(\mathcal{V})}$. The gradient operator of a function $f \in \mathcal{H}(\mathcal{V})$, at vertex $v_i \in \mathcal{V}$, is the vector of all the weighted directed differences $(d_w f)(v_i, v_j)$, with respect to the set of outgoing edges $v_i \rightarrow v_j \in \mathcal{E}$ and thus its \mathcal{L}_p norm is defined as follows: $\|(\nabla_{\mathbf{w}} \mathbf{f})(\mathbf{v}_i)\|_p = \left[\sum_{v_i \rightarrow v_j \in \mathcal{E}} |(d_w f)(v_i, v_j)|^p \right]^{1/p}$

Then, the p -Laplacian $\Delta_w^p f : \mathcal{H}(\mathcal{V}) \rightarrow \mathcal{H}(\mathcal{V})$ can be formulated as the discrete analogue of the continuous one by: $\Delta_w^p f(v_i) = \frac{1}{2} d_w^* \left(\|\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{v}_i)\|_2^{p-2} (d_w f)(v_i, v_j) \right) = \frac{1}{2} d_w^* \left(\frac{(d_w f)(v_i, v_j)}{\|\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{v}_i)\|_2^{2-p}} \right)$ where $p \in (0, +\infty)$.

II. DIRECTED p -LAPLACIANS

In this paper, we propose three p -laplacians on directed graphs, based on three directed difference operators, similar to the ones that have been considered for the Laplacian on undirected graphs (proofs are not provided due to the lack of space):

- A combinatorial p -Laplacian, denoted by Δ_w^p and obtained from $(d_w f)(v_i, v_j) = w(v_i, v_j)(f(v_j) - f(v_i))$ and its adjoint $(d_w^* H)(v_i) = \sum_{v_j \rightarrow v_i} H(v_j, v_i)w(v_j, v_i) - \sum_{v_i \rightarrow v_j} H(v_i, v_j)w(v_i, v_j)$
- A normalized p -Laplacian, denoted by $\tilde{\Delta}_w^p$ and obtained from $(d_w f)(v_i, v_j) = w(v_i, v_j) \left(\frac{f(v_j)}{\sqrt{d^-(v_j)}} - \frac{f(v_i)}{\sqrt{d^+(v_i)}} \right)$ and its adjoint $(d_w^* H)(v_i) = \sum_{v_j \rightarrow v_i} \frac{H(v_j, v_i)w(v_j, v_i)}{\sqrt{d^-(v_i)}} - \sum_{v_i \rightarrow v_j} \frac{H(v_i, v_j)w(v_i, v_j)}{\sqrt{d^+(v_i)}}$.
- A random-walk laplacian denoted by $\Delta_w^{p, rw}$ and obtained from $(d_w f)(v_i, v_j) = \frac{w(v_i, v_j)}{\sqrt{d^+(v_i)}} (f(v_j) - f(v_i))$ and its adjoint $(d_w^* H)(v_i) = \sum_{v_j \rightarrow v_i} \frac{H(v_j, v_i)w(v_j, v_i)}{\sqrt{d^+(v_i)}} - \sum_{v_i \rightarrow v_j} \frac{H(v_i, v_j)w(v_i, v_j)}{\sqrt{d^+(v_i)}}$

Then the p -Laplacian of each of the aforementioned directed difference operators can be expressed as follows:

$$\begin{aligned} \Delta_w^{p,*} f(v_i) = & \frac{1}{2} \left(f(v_i) \left(\sum_{v_j \rightarrow v_i} \frac{w(v_j, v_i)^2}{\phi(v_j, v_i) \|\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{v}_j)\|_2^{2-p}} + \sum_{v_i \rightarrow v_j} \frac{w(v_i, v_j)^2}{\phi(v_i, v_j) \|\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{v}_i)\|_2^{2-p}} \right) \right. \\ & \left. - \left(\sum_{v_j \rightarrow v_i} \frac{w(v_j, v_i)^2}{\gamma_1(v_j, v_i) \|\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{v}_j)\|_2^{2-p}} f(v_j) + \sum_{v_i \rightarrow v_j} \frac{w(v_i, v_j)^2}{\gamma_2(v_i, v_j) \|\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{v}_i)\|_2^{2-p}} f(v_j) \right) \right) \end{aligned} \quad (1)$$

where ϕ , γ_1 and γ_2 are defined as follows, depending on the chosen directed p -Laplacian $\Delta_w^{p,*}$:

- Δ_w^p : $\phi(v_i, v_j) = \phi(v_j, v_i) = \gamma_1(v_j, v_i) = \gamma_2(v_i, v_j) = 1$,
- $\tilde{\Delta}_w^p$: $\phi(v_i, v_j) = \sqrt{d^-(v_j) d^+(v_i)}$, $\gamma_1(v_j, v_i) = d^-(v_i)$ and $\gamma_2(v_i, v_j) = d^+(v_i)$,
- $\Delta_w^{p,rw}$: $\phi(v_i, v_j) = d^+(v_j)$, $\phi(v_j, v_i) = d^+(v_j)$ and $\gamma_1(v_j, v_i) = \gamma_2(v_i, v_j) = d^+(v_i)$.

With specific weights and $p = 2$, we can recover the classical Laplacians on undirected graphs and some formulations on directed graphs [3].

III. LAPLACIAN REGULARIZATION

We consider the following variational problem of p -Laplacian regularization on directed graphs:

$$g \approx \min_{f: \mathcal{V} \rightarrow \mathbb{R}} \left\{ E_w^{p,*}(f, f^0, \lambda) = \frac{1}{p} R_w^{p,*}(f) + \frac{\lambda}{2} \|f - f^0\|_2^2 \right\}, \quad (2)$$

where the regularization functional $R_w^{p,*}$ can be induced from one of the proposed p -Laplacians on directed graphs, such that $R_w^{p,*}(f) = \langle \Delta_w^{p,*} f, f \rangle_{\mathcal{H}(\mathcal{V})} = \langle d_w f, d_w f \rangle_{\mathcal{H}(\mathcal{E})} = \sum_{\mathbf{v}_i \in \mathcal{V}} \|(\nabla_{\mathbf{w}} \mathbf{f})(\mathbf{v}_i)\|_2^p$. When $p \geq 1$, the energy $E_w^{p,*}$ is a convex functional of functions of $\mathcal{H}(\mathcal{V})$.

For the p -Laplacians we propose (i.e., Δ_w^p , $\tilde{\Delta}_w^p$ and $\Delta_w^{p,rw}$), it can be proven that $\frac{1}{p} \frac{\partial R_w^{p,*}}{\partial f(v_i)} = 2\Delta_w^{p,*} f(v_i)$, and solving Equation (2) then amounts to solve $2\Delta_w^{p,*} f(v_i) + \lambda(f(v_i) - f^0(v_i)) = 0$. By substituting the expression of $\Delta_w^{p,*} f(v_i)$ with one of the proposed p -Laplacians (Δ_w^p , $\tilde{\Delta}_w^p$ or $\Delta_w^{p,rw}$), the system of equations can then be solved using a linearized Gauss-Jacobi iterative method. Let t be an iteration step, and $f^{(t)}$ be the solution at step t , the following iterative algorithm is obtained for each of the proposed p -Laplacians on directed graphs:

$$f^{t+1}(v_i) = \frac{\lambda f^0(v_i) + \left(\sum_{v_j \rightarrow v_i} \frac{w(v_j, v_i)^2 f^t(v_j)}{\phi(v_j, v_i) \|\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{v}_j)\|_2^{2-p}} + \sum_{v_i \rightarrow v_j} \frac{w(v_i, v_j)^2 f^t(v_j)}{\phi(v_i, v_j) \|\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{v}_i)\|_2^{2-p}} \right)}{\lambda + \sum_{v_j \rightarrow v_i} \frac{w(v_j, v_i)^2}{\gamma_1(v_j) \|\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{v}_j)\|_2^{2-p}} + \sum_{v_i \rightarrow v_j} \frac{w(v_i, v_j)^2}{\gamma_2(v_i) \|\nabla_{\mathbf{w}} \mathbf{f}(\mathbf{v}_i)\|_2^{2-p}}} \quad (3)$$

To shortly illustrate the behavior of these p -Laplacians on directed graphs, we show sample results for the filtering of colored graph signals. First an image corrupted by Gaussian noise is filtered on a 8-adjacency directed grid graph. Second a colored mesh is filtered on a triangular directed mesh graph augmented with additional directed edges obtained from a 5-nearest neighbor graph within a 3-hop. We have experimentally observed that the filtering behavior is better with the directed random walk p -Laplacian $\Delta_w^{p,rw}$.



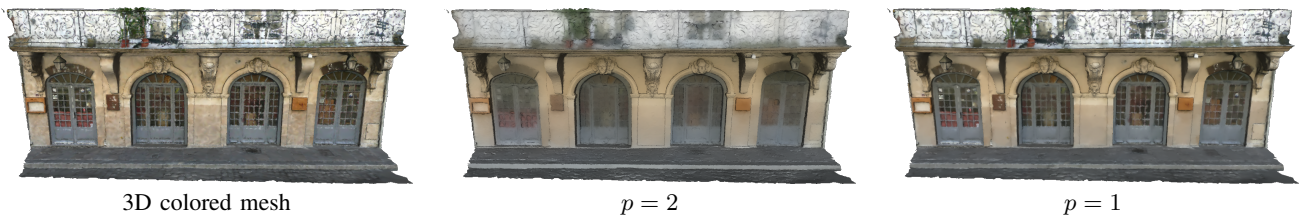
Corrupted image

26.47dB

28.79dB

29.33dB

Fig. 1. p -Laplacian regularization of an image (corrupted by Gaussian noise with $\sigma = 15$) on a 8-adjacency directed grid graph ($\lambda = 0.05$ and $p = 1$) with from left to right: Δ_w^1 , $\tilde{\Delta}_w^1$, $\Delta_w^{1,rw}$.



3D colored mesh

 $p = 2$ $p = 1$

Fig. 2. p -Laplacian regularization of a colored mesh using $\Delta_w^{p,rw}$ with $\lambda = 0.05$.

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