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# A BRIEF INTRODUCTION TO THE GEOMETRIC ANALYSIS OF DIFFERENTIAL EQUATIONS

LANG XIA

<sup>1</sup>While differential equations (of ordinary or partial) comprise an essential part in the field of scientific studies and assist in formulating science mathematically, they also complicate the understanding of physical phenomena with formidable forms and intricate solutions. The formulation of a differential equation for a physical phenomenon is not easy, and the general solutions to the differential equation are even hard to obtain. However, the solution set of a given differential equation may be treated as an embedded manifold of a submanifold in the Euclidean space. From this point of view, the well-developed geometric theory could be employed to study differential equations. Study differential equations using modern geometries could be called *geometric analysis of differential equations*. In this brief, by sacrificing some mathematical rigorousness, the development of geometric analysis of differential equation is presented gradually. This preliminary idea is then generalized by the employment of the modern jet machinery, which the Lie symmetry methods are then unified and included naturally.

Let's begin with a first order ordinary differential equation (ODE) in the form of

$$(1) \quad \frac{d}{dx}u(x) = u(x) + x$$

which is commonly seen in ODE textbooks. It represents a linear non-homogeneous equation that can be solved by superposing its homogeneous and particular solutions. Actually, we can inspect the behaviors and properties of the solutions without solving the equation by the employment of the following geometrical technique: at each point  $(x_0, u_0)$  assign a vector  $(1, u_0 + x_0)$  associated to the equation (it's nothing but the slope of the solution). The set of the vectors forms a vector field. We then visualize the vector field by drawing each vector  $(1, u + x)$  at the point  $(x, u)$ , of which the final results are shown in Figure 1. The resultant vector field is also called the **direction field**. From the above construction, we may know that the possible solutions  $u(x)$  to Eq.(1) must be the curves (**trajectories**, solid curves in Figure 1) tangent to the vectors. We can readily find how the solution curves evolve in the figure. Thus, the differential equation (1) determines a straight line with a slope equal to  $u(x) + x$  at each point  $(x, y)$ . This is how the solution of a differential equation is visualized without knowing its solution. This very preliminary example demonstrates what the idea of geometric analysis of differential equation is.

The vector field can be expressed in the local coordinate as

$$(2) \quad X = 1 \cdot \frac{\partial}{\partial x} + (u + x) \cdot \frac{\partial}{\partial u}$$

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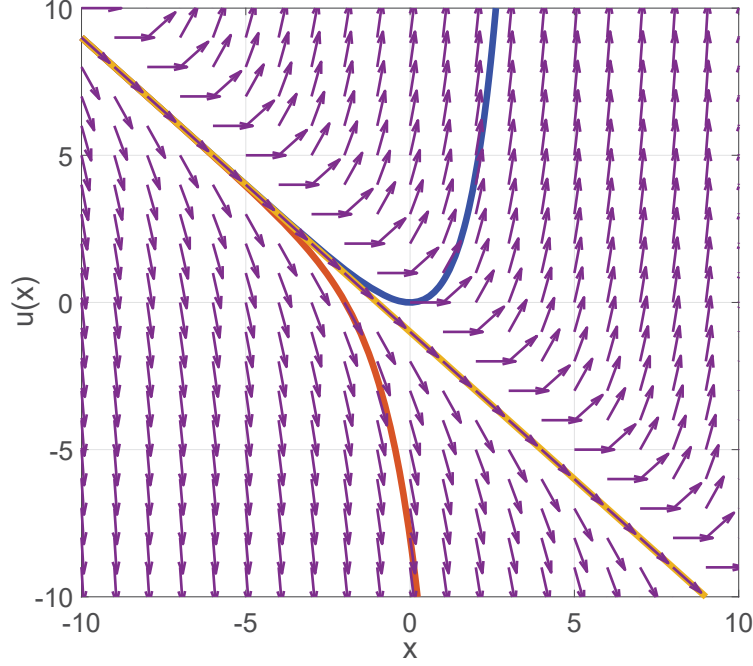


FIGURE 1. The vector field (arrows) of Eq.(1) and its 3 different trajectories (solid curves)

Usually, a differential equation that can be resolved with respect to its highest order, e.g., written as  $\dot{u}(x) = f(x, u)$ , is considered as in the **normal form**. More frequently, most of differential equations are in **the implicit form**, e.g.,  $F(x, u, \dot{u}) = 0$  such that the above method is not convenient in dealing with them. For example, consider the following implicit differential equation

$$(3) \quad F(x, u, \dot{u}) = x^2 + u^2 + \dot{u}^2 - 1 = 0$$

If we want to analyze Eq.(3) by the aforementioned technique, first, the equation should be converted into the normal form, that is

$$(4) \quad \dot{u} = \pm \sqrt{1 - x^2 - u^2}$$

which brings an issue in. Because, at each point  $(x_0, u_0)$  (assume  $x_0^2 + u_0^2 < 1$ ) the vector  $(1, \pm \sqrt{1 - x^2 - u^2})$  has two different values, and the integral curve cannot be identified uniquely. Thus it still does not guarantee the effectiveness of the above geometric approach even by converting the implicit equation to its normal form.

However, in analytical geometry we do visualize implicit curves without using an explicit form. It indicates that we may study the above equation in this way: Let  $p = \dot{u}$  be an independent variable, then the differential equation (3) becomes an algebraic equation  $x^2 + u^2 + p^2 - 1 = 0$ , which actually plots a sphere  $\mathcal{E} = S^2(1)$  being embedded in  $\mathbb{R}^3$ . The solution  $u(x)$  is the set of injective curves on the sphere

parametrized by the equations of

$$(5) \quad \begin{cases} x = x \\ u = u(x) \\ p = \dot{u}(x) \end{cases}$$

with the following restriction

$$(6) \quad p = \frac{d}{dx} u(x)$$

Take the derivative of the curve defined by Eq.(5), we will have

$$(7) \quad \begin{cases} \dot{x} = 1 \\ \dot{u} = p \\ \dot{p} = \ddot{u} \end{cases}$$

then the tangent vectors attached to the curve can be written as

$$(8) \quad X = \frac{\partial}{\partial x} + p \frac{\partial}{\partial u} + \ddot{u} \frac{\partial}{\partial \dot{u}}$$

At a given point  $\theta = (x_0, u_0, p_0)$ , the tangent vector is

$$(9) \quad X(\theta) = 1 \cdot \frac{\partial}{\partial x} + p_0 \frac{\partial}{\partial u} + \ddot{u}(0) \frac{\partial}{\partial \dot{u}}$$

of which the linear combinations span a two dimensional plane  $\mathcal{C}_\theta$  such that  $X \in \mathcal{C}_\theta$ . Since the third component  $\ddot{u}(0)$  is arbitrary (at the point  $\theta$ , it corresponds to the curvature of Eq.(5)), to obtain the equation for the plane  $\mathcal{C}_\theta$ , we can choose the projection of  $X$  onto the  $x - u$  plane, that is in the form of

$$(10) \quad Y = 1 \cdot \frac{\partial}{\partial x} + p_0 \frac{\partial}{\partial u} + 0 \frac{\partial}{\partial \dot{u}}$$

The normal vector  $\mathbf{n}$  of the plane  $\mathcal{C}_\theta$  then can be calculated by

$$(11) \quad \mathbf{n} = X \times Y$$

with being written in the vector form of

$$\mathbf{n} = (p_0, -1, 0)$$

thus the plane  $\mathcal{C}_\theta$  has the following form

$$(12) \quad p_0(x - x_0) - (u - u_0) = 0$$

which is called the **Cartan plane** (see the demonstration in Figure 2). Geometrically, the projection of the Cartan plane onto the  $x - u$  plane is exactly the tangent line of the solution curve (the bottom graph in Figure 2). The set of all Cartan planes is called the **Cartan distribution**. Note that Eq.(12) may be changed into the form of

$$(13) \quad \delta u - p_0 \delta x = 0$$

indicating that the Cartan distribution belongs to the kernel of the following one-form in the local coordinate

$$(14) \quad \omega = du - p dx$$

e.g.,  $\mathcal{C}_\theta \subset \ker\{\omega\}$ . This is because  $\omega(X) = 0$  gives rise to exactly the Eq.(13). The set of  $\mathcal{C} \cap T\mathcal{E}$  forms a direction field on  $\mathcal{E}$ , which is called the **characteristic vector**

**field.** (In this example, a vector in the characteristic field is the intersection of the Cartan plane  $\mathcal{C}_\theta$  and the tangent plane  $T_\theta$  in the top graph of Figure 2).

From the above discussion, we may know that the Cartan distribution is exactly what determines the integral curves. This result can be seen more explicitly by defining the following map

$$\begin{aligned} j^1 f : \quad \mathbb{R} &\rightarrow \mathbb{R} \times \mathbb{R}^2 \\ x &\mapsto (x, u = f(x), p = \dot{u}(x)) \end{aligned}$$

and the corresponding pull-back map

$$\begin{aligned} (j^1 f)_x^* : \quad T_{f(x)}^*(\mathbb{R} \times \mathbb{R}^2) &\rightarrow T_x^* \mathbb{R} \\ \omega &\mapsto (j^1 f)_x^* \omega = \omega(f(x)) \end{aligned}$$

then

$$(15) \quad (j^1 f)_x^* \omega = (j^1 f)_x^* (du - p dx) = df(x) - \frac{df(x)}{dx} dx = 0$$

Therefore, the solutions of the equation are integral curves (Figure 2 just shows one integral curve) on the sphere that annihilate the Cartan distribution  $\mathcal{C}$ . We can easily see that the Cartan distribution is a geometric structure naturally endowed on the sphere. It determines the integral curve on the sphere.

We may abstract the above ideal with considering a more general first order differential equation

$$(16) \quad F(x^1, \dots, x^n, u, u_1, \dots, u_n) = 0$$

where  $u = f(x^1, \dots, x^n)$  and  $u_i = \partial u / \partial x^i$ .

We can always assume that  $(x^1, \dots, x^n, u, u_1, \dots, u_n) \in \mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ , which extends the first  $n+1$  coordinates to  $2n+1$ . Actually, the equation defines a submanifold  $\mathcal{E}$  in the product manifold  $\mathbb{R}^{2n+1}$  written as

$$\mathcal{E} = \{(x^1, \dots, x^n, u, u_1, \dots, u_n) \in \mathbb{R}^{2n+1} \mid F(x^1, \dots, x^n, u, u_1, \dots, u_n) = 0\}$$

The projection of the integral manifolds in  $\mathcal{E}$  onto the hyperplane  $\mathbb{R}^n \times \mathbb{R}$  gives rise the solution  $u(x^1, \dots, x^n)$ . e.g. the hyperplane  $(x^1, \dots, x^n, u)$  (or in Figure 2, the integral curve projected onto  $x-u$  plane).

On the other hand, even more frequently, we can consider the graph of the solution

$$(17) \quad \Gamma(f) := \sigma(x^1, \dots, x^n) = (x^1, \dots, x^n, u(x^1, \dots, x^n)) \in \mathbb{R}^n \times \mathbb{R}$$

as a product manifold, which is similar to the starting example of how we plotted the vector field for the Eq.(1). Due to the graph containing both domain and codomain, it renders a natural way to deal with geometric objects.

The above geometric analysis can be generalized by using the machinery of jet bundles. let  $(x^1, \dots, x^n) \in M$  be a smooth manifold,  $E$  another smooth manifold ( $\dim(E) > \dim(M)$ ), as well as a smooth map  $\pi : E \rightarrow M$ . Thus the triple  $(E, \pi, M)$  forms a **fibered manifold**. If we impose the condition  $E = M \times \mathbb{R}$  here, then  $(E, \pi, M)$  becomes a **fiber bundle**. Meanwhile,  $M$  is called the **base manifold**,  $E$  the **total space** and  $\pi$  the **projection**. For example, to the Eq.(16), we may write that

$$\begin{aligned} M &= \mathbb{R}^n \\ E &= M \times \mathbb{R} \\ \pi : E &\rightarrow M; \text{ with } \pi \circ \sigma = id_M \end{aligned}$$

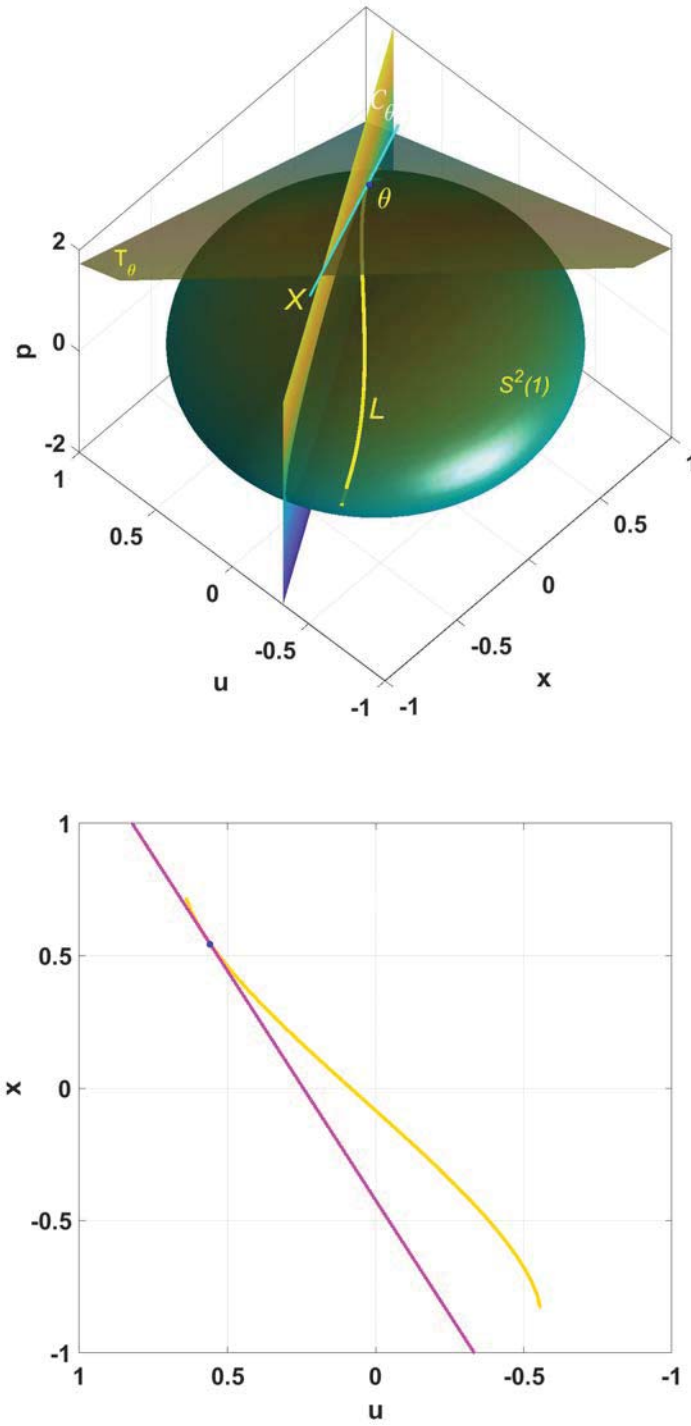


FIGURE 2. Illustrations of Cartan plane ( $C_\theta$ ), tangent plane ( $T_\theta$ ), and tangent vector ( $X$ ) at the point  $\theta$  on the sphere (blue dot).  $L$  is a trajectory (a solution is the projection of  $L$  onto  $x - u$  plane demonstrated by the curve on the bottom graph).

and the **section** of the fiber bundle is defined as a set of graphs (since the functions at each point form a ring, the section is also a  $C^\infty(M)$ -module)

$$\Gamma(M) := \{\sigma : \pi \circ \sigma = id_M\}$$

here  $\sigma = (x, f(x))$  in our example of Eq.(16). If we also count in the derivatives as additional dimensions, the fiber bundle is then extended to the so-called first **jet bundle**, of which the coordinate can be expressed as

$$(x^1, \dots, x^n, u, u_1, \dots, u_n)$$

Remember that the Taylor expansion can also be seen as a jet bundle at a point in the base manifold.

An inverse function of the projection can be written as

$$j_x^1 \sigma : (x^1, \dots, x^n) \mapsto (x^1, \dots, x^n, u, u_1, \dots, u_n)$$

However, there may be many different sections in  $\Gamma(M)$  such that their first derivatives are equal. To avoid this ambiguity, we require that  $j_x^1 \sigma \sim j_x^1 \tau$  in  $T_x(\Gamma(M))$  iff  $\sigma_x = \tau_x$  and  $\frac{\partial \sigma}{\partial x^i}|_x = \frac{\partial \tau}{\partial x^i}|_x$ . Then the **1-jet fiber** is denoted as

$$j_x^1 M := \{j_x^1 \sigma | \sigma \in \Gamma(M)\}$$

and the **1-jet bundle** is the disjoint union of the 1-jet fiber over all points in the base manifold  $M$

$$J^1 M = \bigsqcup_{x \in M} j_x^1 M$$

Thus the first prolonged jet bundle is the triple  $(J^1 M, \pi_1, M)$ , and the 0-jet bundle is  $(J^0 M = E, \pi, M)$ . They are related by

$$\begin{array}{ccc} J^1 M & \xrightarrow{\pi_{1,0}} & J^0 M \\ \pi_1 \downarrow & & \downarrow \pi \\ M & \longrightarrow & M \end{array}$$

Following the jet bundle construction, the k-th prolonged jet bundle is denoted by the triple  $(J^k M, \pi_k, M)$ .

Since  $(u_1, \dots, u_n) \subset T^*M$ , we have  $J^1 M = T^*M \times \mathbb{R}$ .

We may better understand the language of the jet bundles by simply expressing the above relations in the local coordinate.

$$\begin{aligned} J^0 M &= \{(x^1, \dots, x^n, u)\} \\ J^1 M &= \{(x^1, \dots, x^n, u, u_1, \dots, u_n)\} \end{aligned}$$

and the corresponding projections

$$\begin{aligned} \pi_1 : \quad & J^1 M \rightarrow M \\ & (x^1, \dots, x^n, u, u_1, \dots, u_n) \mapsto (x^1, \dots, x^n) \\ \pi_{1,0} : \quad & J^1 M \rightarrow J^0 M \\ & (x^1, \dots, x^n, u, u_1, \dots, u_n) \mapsto (x^1, \dots, x^n, u) \\ (\pi_{1,0})_* : \quad & T(J^1 M) \rightarrow T(J^0 M) \\ & (x^1, \dots, x^n, u_1, \dots, u_n, u_{11}, \dots, u_{1n}, \dots, u_{nn}) \mapsto (x^1, \dots, x^n, u_1, \dots, u_n) \end{aligned}$$

$\forall \alpha = (a, u = f(a)) \in J^0 M$ , we can define a contact element  $\theta = (\alpha, l)$ , where  $l = T_a \Gamma \subset T_\alpha (J^0 M)$ . The base for each space will be

$$\begin{aligned} T_\alpha (M) &= \text{span} \left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}_\alpha \\ T_\alpha^* (M) &= \text{span} \{ dx^1, \dots, dx^n \}_\alpha \\ T_\alpha (J^0 M) &= \text{span} \left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial u} \right\}_\alpha \\ T_\alpha (J^1 M) &= \text{span} \left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial u}, \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_n} \right\}_\alpha \end{aligned}$$

So, the Cartan plane can be defined as

$$\mathcal{C}_\theta = \{ X \in T_\theta (J^1 M) \mid (\pi_{1,0})_* (X) \in l_\theta \}$$

in a sense, the Cartan distribution is the “field” of the Cartan planes in  $T_\theta (J^1 M)$ , of which the projection on  $J^0 M$  is the direction vector field that tangent to the solution  $u = f(x^1, \dots, x^n)$ . In terms of Eq.(14), we can also define the Cartan distribution as

$$\mathcal{C} = \ker \{ \omega \}$$

By far, we have known that the Cartan distribution is determined by the one-form on  $J^1 M$ . We can also derive its expression in the local coordinate.

Let  $X \in T_\theta (J^1 M)$  with the expression in the local coordinate as

$$(18) \quad X = X^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \eta^i \frac{\partial}{\partial u_i}$$

where  $\eta = u_i X^i$  (which is a plane tangent to the graph  $\Gamma(f)$ ). When  $X \in \mathcal{C}$ , we have

$$(19) \quad (\pi_{1,0})_* (X) \in l_\theta$$

and

$$(20) \quad \omega(X) = 0$$

thus from the above definition of Cartan plane, we have

$$(\pi_{1,0})_* (X) = X^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u}$$

remember we also have the relation

$$(21) \quad (\pi_{1,0}^* \omega)_\theta (X_\theta) = \omega_{\pi_{1,0}(\theta)} ((\pi_{1,0})_* (X_\theta))$$

and in terms of Eq.(20)

$$(22) \quad (\pi_{1,0}^* \omega)_\theta (X_\theta) = 0$$

we have

$$(23) \quad (\pi_{1,0}^* \omega)_\theta (X_\theta) = \omega_{\pi_{1,0}(\theta)} ((\pi_{1,0})_* (X_\theta)) = 0$$

substitute the push-forward relation and  $\eta = u_i X^i$  into the above equation, we find

$$(24) \quad \omega_{\pi_{1,0}(\theta)} \left( X^i \frac{\partial}{\partial x^i} + u_i X^i \frac{\partial}{\partial u} \right) = 0$$

$$(25) \quad X^i \left( \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} \right) \lrcorner \omega_{\pi_{1,0}(\theta)} = 0$$



thus if  $X^i \neq 0$ , we finally get

$$(26) \quad \omega = du - u_i dx^i$$

which is also called the **contact form**. To this end, we have known that:

**Definition.** A general solution to a given differential equation is a maximal integral manifold of the Cartan distribution on the differential equation.

Using the language of jet bundles, we can readily define the **point transformation** as a diffeomorphism

$$\phi^0 : J^0 M \rightarrow J^0 M$$

of which the lifting preserves the Cartan distribution  $\mathcal{C}$ , e.g. if there is a one-parameter group  $\phi_t^0$  such that the vector field

$$(27) \quad X = (\phi_t^0)_* \left( \frac{d}{dt} \right) |_{t=0}$$

where

$$(28) \quad X(f) = (\phi_t^0)_* \left( \frac{d}{dt} \right) |_{t=0} f = \frac{df \circ \phi_t^0}{dt} |_{t=0}, \quad \forall f \in C^\infty(J^0(M))$$

and

$$(29) \quad \mathcal{L}_{(\pi_{1,0})_*^{-1} X} \omega = (\pi_{1,0})_*^{-1} X(\omega) = \lambda \omega, \quad \forall \lambda \in C^\infty(J^1(M))$$

then we call  $X$  the **Lie point symmetry**.

Similarly, the **contact transformation** is a diffeomorphism

$$\phi^1 : J^1 M \rightarrow J^1 M$$

which preserves the Cartan distribution  $\mathcal{C}$ , and if there is a one-parameter group  $\phi_t^1$  such that the vector field

$$(30) \quad X = (\phi_t^1)_* \left( \frac{d}{dt} \right) |_{t=0}$$

e.g.

$$(31) \quad X(f) = (\phi_t^1)_* \left( \frac{d}{dt} \right) |_{t=0} f = \frac{df \circ \phi_t^1}{dt} |_{t=0}, \quad \forall f \in C^\infty(J^1(M))$$

and

$$(32) \quad \mathcal{L}_X \omega = X(\omega) = \lambda \omega, \quad \forall \lambda \in C^\infty(J^1(M))$$

then we call  $X$  the **Lie contact symmetry**. Since

$$(33) \quad \pi_{1,0} \circ \phi^1 = \phi^0 \circ \pi_{1,0}$$

the point transformation can be seen as a contact transform “lifted” from the transformation of  $J^0 M$ .

The **Lie field** is just a vector field  $X$  on  $J^k M$  corresponding to the one-parameter group  $\phi_t^k$ , where  $k = 0, 1, \dots, n$ .

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