# Middle multiplicative convolution and hypergeometric equations

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# MIDDLE MULTIPLICATIVE CONVOLUTION AND HYPERGEOMETRIC EQUATIONS

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Abstract. — Using a relation due to Katz linking up additive and multiplicative convolutions, we make explicit the behaviour of some Hodge invariants by middle multiplicative convolution, following [DS13] and [Mar18a] in the additive case. Moreover, the main theorem gives a new proof of a result of Fedorov computing the Hodge invariants of hypergeometric equations.

The starting point of this article is a work of Dettweiler and Sabbah [DS13] consisting in making explicit the behaviour of Hodge invariants by middle additive convolution by a Kummer module, motivated by the Katz algorithm [Kat96]. In [Mar18a], we developed this work without doing the assumption of scalar monodromy at infinity assumed in the Katz algorithm and in [DS13], and more precisely we made precise the behaviour of nearby cycle local Hodge numerical data.

There exists a tricky link between middle additive convolution with a Kummer module and middle multiplicative convolution with a particular hypergeometric module, due to Katz [Kat96] and detailed in Proposition 2.1. It allows us in §2 to transpose the general results of [Mar18a] to the multiplicative context, after having recalled in §1 the necessary definitions for understanding it.

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An application of these results is another way to prove a theorem due to Fedorov computing the Hodge invariants of hypergeometric equations [Fed17, Th. 3], very different but more direct, insofar as it uses the explicit behaviour of the Hodge invariants at infinity and 0.

## 1. Numerical Hodge data

Let us begin by recalling the definition of local Hodge invariants introduced in [DS13, §2.2]. Let $\Delta$ be a disc centered in 0 with coordinate $t$ and $(V, F^\bullet V, \nabla)$ be a complex polarizable variation of Hodge structure on $\Delta^*$. We denote by $M$ the corresponding $\mathscr{D}_\Delta$-module minimal extension at 0.

**Nearby cycles.** For $a \in (-1, 0]$ and $\lambda = e^{-2i\pi a}$, the nearby cycle space at the origin $\psi_\lambda(M)$ is equipped with the nilpotent endomorphism $\mathrm{N} = -2i\pi(t\partial_t - a)$ and the Hodge filtration is such that $\mathrm{N}F^p\psi_\lambda(M) \subset F^{p-1}\psi_\lambda(M)$. The monodromy filtration induced by N enables us to define the spaces $\mathrm{P}_\ell\psi_\lambda(M)$ of primitive vectors, equipped with a polarizable Hodge structure. The nearby cycle local Hodge numerical data are defined by

$$\nu^p_{\lambda,\ell}(M) := h^p(\mathrm{P}_\ell\psi_\lambda(M)) = \dim \mathrm{gr}^p_F \mathrm{P}_\ell\psi_\lambda(M),$$

with the relation $\nu^p_\lambda(M) := h^p\psi_\lambda(M) = \sum_{\ell\geq 0}\sum_{k=0}^{\ell} \nu^{p+k}_{\lambda,\ell}(M)$. We set

$$\nu^p_{\lambda,\mathrm{prim}}(M) := \sum_{\ell\geq 0} \nu^p_{\lambda,\ell}(M) \quad \text{and} \quad \nu^p_{\lambda,\mathrm{coprim}}(M) := \sum_{\ell\geq 0} \nu^{p+\ell}_{\lambda,\ell}(M).$$

**Vanishing cycles.** For $\lambda \neq 1$, the vanishing cycle space at the origin is given by $\phi_\lambda(M) = \psi_\lambda(M)$ and comes with N and $F^p$ as before. For $\lambda = 1$, the Hodge filtration on $\phi_1(M)$ is such that $F^p\mathrm{P}_\ell\phi_1(M) = \mathrm{N}(F^p\mathrm{P}_{\ell+1}\psi_1(M))$. Similarly to nearby cycles, the vanishing cycle local Hodge numerical data is defined by $\mu^p_{\lambda,\ell}(M) := h^p(\mathrm{P}_\ell\phi_\lambda(M)) = \dim \mathrm{gr}^p_F\mathrm{P}_\ell\phi_\lambda(M)$.

Now let us leave the local point of view, and let $\boldsymbol{x} = \{x_1, ..., x_r\}$ denote a set of points of $\mathbb{G}_m$, $x_0 = 0$, $\mathscr{D} = \mathscr{D}_{\mathbb{G}_m} = \mathbb{C}[t, t^{-1}]\langle\partial_t\rangle$ and $j$ the inclusion $\mathbb{G}_m\backslash\boldsymbol{x} \hookrightarrow \mathbb{P}^1$. Let $(V, F^\bullet V, \nabla)$ be a complex polarizable variation of Hodge structure on $\mathbb{G}_m\backslash\boldsymbol{x}$ and $M$ be the $\mathscr{D}$-module minimal extension on points of $\boldsymbol{x}$. We set $\mathscr{M}^{\min}$ the $\mathscr{D}_{\mathbb{P}^1}$-module minimal extension of $M$ at 0 and infinity.

**Degrees $\delta^p$.** The Deligne extension $V^0$ of $(V, \nabla)$ on $\mathbb{P}^1$ is contained in $M$, and endowed with the filtration $j_*F^pV \cap V^0$. We set

$$\delta^p(M) = \deg \mathrm{gr}^p_F V^0 = \deg \frac{j_*F^pV \cap V^0}{j_*F^{p+1}V \cap V^0}\,.$$

## 2. Middle multiplicative convolution with $H_{0,\gamma_0}$

Let us fix $\gamma \in (0,1]$ and set $\lambda = \exp(-2i\pi\gamma)$. The Kummer module $\mathscr{L}_\lambda$ is defined by $\mathscr{L}_\lambda = \mathscr{D}/\mathscr{D}\cdot(t\partial_t - \gamma)$ and the middle additive convolution functor with $\mathscr{L}_\lambda$ is denoted by $\mathrm{MC}_\lambda$. The next proposition links up additive and multiplicative convolution and is due to Katz [Kat96, Lemma 2.13.1], and adapted here to the point of view of $\mathscr{D}$-modules:

PROPOSITION 2.1. — *Let us denote by $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ the inclusion and by $H_{0,\gamma}$ the hypergeometric module $\mathscr{D}/\mathscr{D}\cdot(t\partial_t - t(t\partial_t - \gamma))$. We have the following formula for every holonomic $\mathscr{D}$-module $M$:*

$$M *_{\mathrm{mid}\times} H_{0,\gamma} = j^+(\mathrm{MC}_\lambda(j_{\dagger+}(M \otimes \mathscr{L}_{\overline{\lambda}}))).$$

ASSUMPTION 2.2. — In everything that follows, we fix $\gamma_0 \in (0,1)$ and set $\lambda_0 = \exp(-2i\pi\gamma_0)$. If we assume that $M$ is an irreducible regular holonomic $\mathscr{D}$-module, not isomorphic to $\mathscr{L}_{\lambda_0}$ and not supported on a point, then $j_{\dagger+}(M \otimes \mathscr{L}_{\overline{\lambda_0}})$ satifies Assumption 1.2.2 of [DS13] and we can apply to it the results of [DS13] and [Mar18a]. Therefore, we do this assumption in what follows.

The following proposition gives the behaviour of vanishing cycle local Hodge numerical data by middle convolution with $H_{0,\gamma_0}$:

PROPOSITION 2.3. — *For all $i \in \{1,...,n\}$, we have:*

$$\mu^p_{x_i,\lambda,\ell}(M *_{\mathrm{mid}\times} H_{0,\gamma_0}) = \begin{cases} \mu^p_{x_i,\lambda/\lambda_0,\ell}(M) & \text{if } \gamma \in (0,\gamma_0] \\ \mu^{p-1}_{x_i,\lambda/\lambda_0,\ell}(M) & \text{if } \gamma \in (\gamma_0,1]. \end{cases}$$

*Proof.* — For $i \in \{1,...,n\}$, Proposition 2.1 gives

$$\mu^p_{x_i,\lambda,\ell}(M *_{\mathrm{mid}\times} H_{0,\gamma_0}) = \mu^p_{x_i,\lambda,\ell}(\mathrm{MC}_{\lambda_0}(j_{\dagger+}(M \otimes \mathscr{L}_{\overline{\lambda_0}}))).$$

According to Assumption 2.2, we know that $j_{\dagger+}(M \otimes \mathscr{L}_{\overline{\lambda_0}})$ satisfies Assumption 1.2.2 of [DS13], then we can apply [DS13, Th. 3.1.2(2)] and get

$$\mu^p_{x_i,\lambda,\ell}(M *_{\mathrm{mid}\times} H_{0,\gamma_0}) = \begin{cases} \mu^p_{x_i,\lambda/\lambda_0,\ell}(M \otimes \mathscr{L}_{\overline{\lambda_0}}) & \text{if } \gamma \in (0,\gamma_0] \\ \mu^{p-1}_{x_i,\lambda/\lambda_0,\ell}(M \otimes \mathscr{L}_{\overline{\lambda_0}}) & \text{if } \gamma \in (\gamma_0,1]. \end{cases}$$

As $\mathscr{L}_{\overline{\lambda_0}}$ has trivial monodromy around $x_i \neq 0$, we have $\mu^p_{x_i,\lambda/\lambda_0,\ell}(M \otimes \mathscr{L}_{\overline{\lambda_0}}) = \mu^p_{x_i,\lambda/\lambda_0,\ell}(M)$ and it is possible to conclude the proof. □

Concerning nearby cycle local Hodge numerical data at infinity, Proposition 2.1 combined with Theorem 1 of [Mar18a] and [DS13, 2.2.13] directly gives the following proposition:

PROPOSITION 2.4. — *We have the following data:*

$$\nu^p_{\infty,\lambda,\ell}(M *_{\mathrm{mid}\times} H_{0,\gamma_0}) = \begin{cases} \nu^{p-1}_{\infty,\lambda,\ell}(M) & \text{if } \gamma \in (0, 1-\gamma_0) \\ \nu^{p}_{\infty,\lambda,\ell}(M) & \text{if } \gamma \in (1-\gamma_0, 1) \\ \nu^{p}_{\infty,1,\ell+1}(M) & \text{if } \lambda = 1 \\ \nu^{p-1}_{\infty,\overline{\lambda_0},\ell-1}(M) & \text{if } \lambda = \overline{\lambda_0},\ \ell \geq 1. \end{cases}$$

REMARK 2.5. — We also have an explicit but more complicated formula for $\nu^p_{\infty,\overline{\lambda_0},0}(M *_{\mathrm{mid}\times} H_{0,\gamma_0})$, given and proved in [Mar18b, Prop 6.4.3].

The nearby cycle local Hodge numerical data at 0 are given by the following proposition:

PROPOSITION 2.6. — *We have the following data:*

$$\nu^p_{0,\lambda,\ell}(M *_{\mathrm{mid}\times} H_{0,\gamma_0}) = \begin{cases} \nu^{p}_{0,\lambda,\ell}(M) & \text{if } \gamma \in (0, \gamma_0) \\ \nu^{p-1}_{0,\lambda,\ell}(M) & \text{if } \gamma \in (\gamma_0, 1) \\ \nu^{p}_{0,\lambda_0,\ell+1}(M) & \text{if } \lambda = \lambda_0 \\ \nu^{p-1}_{0,1,\ell-1}(M) & \text{if } \lambda = 1,\ \ell \geq 1 \\ h^pH^1(\mathbb{P}^1, \mathrm{DR}\mathscr{M}^{\min}) & \text{if } \lambda = 1,\ \ell = 0. \end{cases}$$

*Proof.* — Similarly to Proposition 2.4, Proposition 2.1 combined with [DS13, Th. 3.1.2(2)] and [DS13, 2.2.14] directly gives the result, except if $\lambda = 1$ and $\ell = 0$. This last case is treated in [Mar18b, Prop 6.4.5]. □

REMARK 2.7. — Summing the nearby cycle local Hodge numerical data, we deduce an explicit formula for Hodge numbers:

$$\begin{aligned} h^p(M *_{\mathrm{mid}\times} H_{0,\gamma_0}) = h^p(M) &+ \nu^{p-1}_{0,1,\mathrm{prim}}(M) - \nu^{p-1}_{0,\lambda_0,\mathrm{prim}}(M) \\ &+ h^pH^1(\mathbb{P}^1, \mathrm{DR}\mathscr{M}^{\min}) + \sum_{\gamma\in[\gamma_0,1)} (\nu^{p-1}_{0,\lambda}(M) - \nu^{p}_{0,\lambda}(M)). \end{aligned}$$

To finish this study of the behaviour of Hodge invariants by middle multiplicative convolution with $H_{0,\gamma_0}$, let us make explicit the degrees $\delta^p$ defined in §1:

PROPOSITION 2.8. — *The degrees $\delta^p$ are given by:*

$$\delta^p(M *_{\mathrm{mid}\times} H_{0,\gamma_0}) = \delta^p(M) + \sum_{\gamma\in[\gamma_0,1)} (\nu^p_{0,\lambda}(M) - \nu^{p-1}_{0,\lambda}(M)) + \nu^{p-1}_{0,\lambda_0,\mathrm{prim}}(M) - \sum_{i=1}^{r} \left( \mu^p_{x_i,1}(M) + \sum_{\gamma\in(0,1-\gamma_0)} \mu^{p-1}_{x_i,\lambda}(M) \right).$$

*Proof.* — According to Proposition 2.1 and Theorem 3 of [Mar18a], we have

$$\delta^p(M *_{\mathrm{mid}\times} H_{0,\gamma_0}) = \delta^p(M \otimes \mathscr{L}_{\overline{\lambda_0}}) + \sum_{\gamma\in[\gamma_0,1)} \nu^p_{\infty,\lambda}(M \otimes \mathscr{L}_{\overline{\lambda_0}}) - \sum_{i=0}^{r} \left( \mu^p_{x_i,1}(M \otimes \mathscr{L}_{\overline{\lambda_0}}) + \sum_{\gamma\in(0,1-\gamma_0)} \mu^{p-1}_{x_i,\lambda}(M \otimes \mathscr{L}_{\overline{\lambda_0}}) \right).$$

Let us make precise each of these terms. Applying [DS13, Prop. 2.3.2], we get

$$\delta^p(M \otimes \mathscr{L}_{\overline{\lambda_0}}) = \delta^p(M) - h^p(M) + \sum_{\gamma\in[\gamma_0,1)} \nu^p_{0,\lambda}(M) + \sum_{\gamma\in[1-\gamma_0,1)} \nu^p_{\infty,\lambda}(M).$$

Applying [DS13, 2.2.13], we have

$$\sum_{\gamma\in[\gamma_0,1)} \nu^p_{\infty,\lambda}(M \otimes \mathscr{L}_{\overline{\lambda_0}}) = \sum_{\gamma\in[\gamma_0,1)} \nu^p_{\infty,\lambda\overline{\lambda_0}}(M) = \sum_{\gamma\in[0,1-\gamma_0)} \nu^p_{\infty,\lambda}(M) \tag{2.9}$$

$$\sum_{\gamma\in(0,1-\gamma_0)} \mu^{p-1}_{0,\lambda}(M \otimes \mathscr{L}_{\overline{\lambda_0}}) = \sum_{\gamma\in(\gamma_0,1)} \nu^{p-1}_{0,\lambda}(M) \tag{2.10}$$

$$\begin{aligned} \mu^p_{0,1}(M \otimes \mathscr{L}_{\overline{\lambda_0}}) &= \nu^{p-1}_{0,1}(M \otimes \mathscr{L}_{\overline{\lambda_0}}) - \nu^{p-1}_{0,1,\mathrm{prim}}(M \otimes \mathscr{L}_{\overline{\lambda_0}}) \\ &= \nu^{p-1}_{0,\lambda_0}(M) - \nu^{p-1}_{0,\lambda_0,\mathrm{prim}}(M), \end{aligned} \tag{2.11}$$

and we get the expected formula. □

## 3. Fedorov's formula

For any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in [0,1)^n$, the hypergeometric differential operator $\mathrm{Hyp}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is defined by

$$\mathrm{Hyp}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \prod_{i=1}^{n}(t\partial_t - \alpha_i) - t\prod_{j=1}^{n}(t\partial_t - \beta_j),$$

and the corresponding hypergeometric module by $H_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \mathscr{D}/\mathscr{D} \cdot \mathrm{Hyp}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. These $\mathscr{D}$-modules are irreducibles if and only if $\alpha_i \neq \beta_j$ for all $i, j \in \{1, ..., n\}$ [Kat90, Cor. 3.2.1]. We assume in what follows that this condition is satisfied.

The leading term of the operator is $t^n(1-t)\partial_t^n$, then we have a connection on the trivial holomorphic bundle of rank $n$ on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The three singularities are regular, and Theorem 3.5.4 of [Kat90] shows that the corresponding local system on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is physically rigid. In other words, and Riemann already remarked it in 1857, the hypergeometric equation can be reconstructed, up to isomorphism, with the knowledge of its monodromies at 0, 1 and $\infty$. By [Sim90, Cor. 8.1], the restriction of $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ to $\mathbb{G}_m \setminus \{1\}$ underlies a complex polarizable variation of Hodge structure, unique up to a shift of the Hodge filtration [Del87, Prop. 1.13(i)]. Let us make precise the three monodromies and what that implies on the calculation of local Hodge invariants.

**At $\infty$ :** For $m \in \{1, ..., n\}$, we set $\mathrm{mult}(\beta_m) = \#\{j \in \{1, ..., n\} \mid \beta_j = \beta_m\}$, $\ell_m(\boldsymbol{\beta}) = \mathrm{mult}(\beta_m) - 1$ and $\lambda_m = \exp(2i\pi\beta_m)$. The monodromy matrix at infinity if composed for each eigenvalue $\lambda_m$ with a unique Jordan bloc of size $\mathrm{mult}(\beta_m)$. We deduce that $\dim \mathrm{P}_\ell\psi_{\infty,\lambda_m}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = 0$ except for $\ell = \ell_m(\boldsymbol{\beta})$ for which this quantity is equal to 1. The computation of $\nu^p_{\infty,\lambda_m,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ is reduced to finding the value of $p \in \mathbb{Z}$ for which this quantity for $\ell = \ell_m(\boldsymbol{\beta})$ is non zero (and equal to 1).

**At 0 :** For $m \in \{1, ..., n\}$, we set $\mathrm{mult}(\alpha_m) = \#\{j \in \{1, ..., n\} \mid \alpha_j = \alpha_m\}$, $\ell_m(\boldsymbol{\alpha}) = \mathrm{mult}(\alpha_m) - 1$ and $\mu_m = \exp(-2i\pi\alpha_m)$. The monodromy matrix at 0 if composed for each eigenvalue $\mu_m$ with a unique Jordan bloc of size $\mathrm{mult}(\alpha_m)$. We deduce that $\dim \mathrm{P}_\ell\psi_{0,\mu_m}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = 0$ except for $\ell = \ell_m(\boldsymbol{\alpha})$ for which this quantity is equal to 1. The computation of $\nu^p_{0,\mu_m,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ is reduced to finding the value of $p \in \mathbb{Z}$ for which this quantity for $\ell = \ell_m(\boldsymbol{\alpha})$ is non zero (and equal to 1).

**At 1 :** The monodromy at 1 is a pseudoreflection, sum of the identity and a matrix of rank 1. We know by a Pochhammer's result that there are $n-1$ independant eigenvectors associated to the eigenvalue 1 (see [BH89, Prop. 2.8] and [Beu08, Th. 1.1]). If we set $\gamma_s \in (0, 1]$ such that $\gamma_s = \sum_{k=1}^n(\beta_k - \alpha_k) \bmod \mathbb{Z}$, we deduce that $\lambda_s = \exp(-2i\pi\gamma_s)$ is also an eigenvalue of the monodromy, called the *special eigenvalue*.

• If $\lambda_s \neq 1$, then the monodromy is diagonalizable. We have $\mu_{1,\lambda_s}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \nu_{1,\lambda_s}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = 1$, $\nu_{1,1}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = n-1$ and $\mu_{1,1}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = 0$. The only thing left to be determined is the value of $p \in \mathbb{Z}$ for which $\mu^p_{1,\lambda_s,0}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ is non zero (and equal to 1).

• If $\lambda_s = 1$, then the monodromy is a transvection. We have $\nu_{1,1}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = n$ and $\mu_{1,1}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = 1$. More precisely, $\mu_{1,1,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = 0$ except for $\ell = 0$ for which this quantity is equal to 1. The only thing left to be determined is the value of $p \in \mathbb{Z}$ for which $\mu^p_{1,1,0}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ is non zero (and equal to 1).

Definition 3.1. — Let us set $\alpha, \beta, \gamma \in [0,1)$. We say that the pair $(\alpha, \beta)$ *is separated by* $\gamma$ if $\exp(2i\pi\gamma)$ is in the open interval $(\exp(2i\pi\alpha), \exp(2i\pi\beta))$ of the oriented circle, a property that we denote by $\alpha \to \gamma \to \beta$. It means that either $0 \leq \alpha < \gamma < \beta < 1$, or $0 \leq \gamma < \beta < \alpha < 1$, or $0 \leq \beta < \alpha < \gamma < 1$.

Remark 3.2. — It is the same notation as in the beginning of Chapter 4 of [Fed17], with the difference that $\alpha$, $\beta$ and $\gamma$ are not necessarily distinct (but in this last case, the property $\alpha \to \gamma \to \beta$ is not satisfied).

Definition 3.3. — For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in [0,1)^n$ and $\gamma \in [0,1)$, we set

$$p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma) := \#\{k \mid \neg(\alpha_k \to \gamma \to \beta_k)\} = \#\{k \mid \alpha_k \to \gamma \to \beta_k\}^{\complement}.$$

Note that this quantity does not depend on the numbering of the $n$-tuple of pairs $((\alpha_1, \beta_1), ..., (\alpha_n, \beta_n))$.

We denote by $\{\cdot\}$ the fractional part.

Theorem 3.4. — *Given a decomposition $H_{\boldsymbol{\alpha},\boldsymbol{\beta}} = H_{\alpha_1,\beta_1} * \cdots * H_{\alpha_n,\beta_n}$ into convolutions of hypergeometric modules of rank 1, then $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is equipped of a natural polarizable variation of Hodge structure satisfying:*

(a) $$\nu^p_{0,\mu_m,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \begin{cases} 1 & \text{if } p = p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \alpha_m) \text{ and } \ell = \ell_m(\boldsymbol{\alpha}) \\ 0 & \text{otherwise} \end{cases}$$

(b) $$\nu^p_{\infty,\lambda_m,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \begin{cases} 1 & \text{if } p = p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \beta_m) \text{ and } \ell = \ell_m(\boldsymbol{\beta}) \\ 0 & \text{otherwise} \end{cases}$$

(c) $$\mu^p_{1,\lambda_s,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \begin{cases} 1 & \text{if } p = \#\left\{i \in \{1,...,n\} \,\middle|\, \left\{\sum_{k=1}^{i}(\beta_k - \alpha_k)\right\} < \gamma_s\right\} \\ & \text{and } \ell = 0 \\ 0 & \text{otherwise.} \end{cases}$$

REMARK 3.5. — 1) The order in which the convolutions are done does not matter when we consider the Hodge filtration, because $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is defined as $\pi_+(H_{\alpha_1,\beta_1}\boxtimes\cdots\boxtimes H_{\alpha_n,\beta_n})$, where $\pi:(\mathbb{G}_m)^n\to\mathbb{G}_m$ is the product map. Renumbering the $n$-tuple $((\alpha_1,\beta_1),...,(\alpha_n,\beta_n))$ has no influence on Hodge invariants.
2) Given a decomposition into convolutions of hypergeometric modules of rank one, there exists a unique associated Hodge filtration, if we started from the trivial Hodge filtration for rank one. This means that the filtration is natural only if we give such a decomposition.
3) By uniqueness of the Hodge filtration up to a shift, we deduce that changing the decomposition will induce a shift in the filtration.

*Proof.* — By induction on $n\in\mathbb{N}^*$, length of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The theorem is satisfied for $n=1$. Let us set $n\geq 1$, $(\boldsymbol{\alpha},\boldsymbol{\beta})=((\alpha_0,...,\alpha_n),(\beta_0,...,\beta_n))$ two $(n+1)$-tuples such that $\alpha_i\neq\beta_j$ for all $i,j\in\{0,...,n\}$, and $m\in\{0,...,n\}$.

**Formula (b).** Let us suppose that (b) is satisfied for all tuples of length $n$.
(Case 1) Let us suppose that $\beta_m\neq\beta_0$. According to [DS13, 2.2.13], we have

$$\nu^p_{\infty,\lambda_m,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}})=\nu^p_{\infty,\lambda_m\exp(-2i\pi\alpha_0),\ell}(H_{\{\boldsymbol{\alpha}-\alpha_0\},\{\boldsymbol{\beta}-\alpha_0\}}).$$

We know that

$$H_{\{\boldsymbol{\alpha}-\alpha_0\},\{\boldsymbol{\beta}-\alpha_0\}}=H_{\{\widehat{\boldsymbol{\alpha}_0}-\alpha_0\},\{\widehat{\boldsymbol{\beta}_0}-\alpha_0\}}*H_{0,\{\beta_0-\alpha_0\}},$$

where $\widehat{\boldsymbol{\alpha}_0}$ is the tuple $\boldsymbol{\alpha}$ where we have removed $\alpha_0$, and similarly for $\widehat{\boldsymbol{\beta}_0}$. Applying Proposition 2.4, we get

$$\nu^p_{\infty,\lambda_m\exp(-2i\pi\alpha_0),\ell}(H_{\{\boldsymbol{\alpha}-\alpha_0\},\{\boldsymbol{\beta}-\alpha_0\}})=\begin{cases}\nu^{p-1}_{\infty,\lambda_m\exp(-2i\pi\alpha_0),\ell}\left(H_{\{\widehat{\boldsymbol{\alpha}_0}-\alpha_0\},\{\widehat{\boldsymbol{\beta}_0}-\alpha_0\}}\right)\\ \qquad\text{if } \{\beta_m-\alpha_0\}>\{\beta_0-\alpha_0\}\\ \nu^p_{\infty,\lambda_m\exp(-2i\pi\alpha_0),\ell}\left(H_{\{\widehat{\boldsymbol{\alpha}_0}-\alpha_0\},\{\widehat{\boldsymbol{\beta}_0}-\alpha_0\}}\right)\\ \qquad\text{if } \{\beta_m-\alpha_0\}<\{\beta_0-\alpha_0\}.\end{cases}$$

Applying [DS13, 2.2.13] once again, we have

$$\nu^p_{\infty,\lambda_m,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}})=\begin{cases}\nu^p_{\infty,\lambda_m,\ell}\left(H_{\widehat{\boldsymbol{\alpha}_0},\widehat{\boldsymbol{\beta}_0}}\right) & \text{if } \alpha_0\to\beta_m\to\beta_0\\ \nu^{p-1}_{\infty,\lambda_m,\ell}\left(H_{\widehat{\boldsymbol{\alpha}_0},\widehat{\boldsymbol{\beta}_0}}\right) & \text{otherwise.}\end{cases}$$

By the induction hypothesis, the left quantity is non zero if and only if $p=p(\boldsymbol{\alpha},\boldsymbol{\beta},\beta_m)$ and $\ell=\ell_m(\boldsymbol{\beta})=\ell_m(\widehat{\boldsymbol{\beta}_0})$.

(Case 2) Let us suppose that $\beta_m=\beta_0$ and $\ell_0(\boldsymbol{\beta})\geq 1$. Applying the same reasoning as before and using Proposition 2.4 (case $\lambda=\overline{\lambda_0}$, $\ell\geq 1$), we get

$$\nu^p_{\infty,\lambda_0,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}})=\nu^{p-1}_{\infty,\lambda_0,\ell-1}\left(H_{\widehat{\boldsymbol{\alpha}_0},\widehat{\boldsymbol{\beta}_0}}\right),$$

non zero if and only if $\ell = \ell_0(\boldsymbol{\beta}) = \ell_0(\widehat{\boldsymbol{\beta}_0})+1$. In this case, we have $p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \beta_0) = p(\widehat{\boldsymbol{\alpha}_0}, \widehat{\boldsymbol{\beta}_0}, \beta_0) + 1$ because we do not have $\alpha_0 \to \beta_0 \to \beta_0$.

(Case 3) Let us suppose that $\beta_m = \beta_0$ et $\ell_0(\boldsymbol{\beta}) = 0$, so we have $\beta_1 \neq \beta_0$. Applying the same reasoning as in Case 1, we get

$$\nu^p_{\infty,\lambda_0,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \begin{cases} \nu^p_{\infty,\lambda_0,\ell}\left(H_{\widehat{\boldsymbol{\alpha}_1},\widehat{\boldsymbol{\beta}_1}}\right) & \text{if } \{\beta_0 - \alpha_1\} < \{\beta_1 - \alpha_1\} \\ \nu^{p-1}_{\infty,\lambda_0,\ell}\left(H_{\widehat{\boldsymbol{\alpha}_1},\widehat{\boldsymbol{\beta}_1}}\right) & \text{if } \{\beta_0 - \alpha_1\} > \{\beta_1 - \alpha_1\} \end{cases}$$

$$= \begin{cases} \nu^p_{\infty,\lambda_0,\ell}\left(H_{\widehat{\boldsymbol{\alpha}_1},\widehat{\boldsymbol{\beta}_1}}\right) & \text{if } \alpha_1 \to \beta_0 \to \beta_1 \\ \nu^{p-1}_{\infty,\lambda_0,\ell}\left(H_{\widehat{\boldsymbol{\alpha}_1},\widehat{\boldsymbol{\beta}_1}}\right) & \text{otherwise.} \end{cases}$$

By the induction hypothesis, and as the order in which the convolutions are done does not matter, the left quantity is non zero if and only if $p = p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \beta_0)$ and $\ell = \ell_0(\boldsymbol{\beta}) = \ell_0(\widehat{\boldsymbol{\beta}_1}) = 0$.

To conclude, Formula (b) is satisfied for the couple $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

**Formula (a).** Let us suppose that (a) is satisfied for all tuples of length $n$.
(Case 1) Let us suppose that $\alpha_m \neq \alpha_0$. According to Proposition 2.6 and [DS13, 2.2.13], and applying the same reasoning as in Case 1 of the proof of Formula (b), we have

$$\nu^p_{0,\mu_m,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \begin{cases} \nu^p_{0,\mu_m,\ell}\left(H_{\widehat{\boldsymbol{\alpha}_0},\widehat{\boldsymbol{\beta}_0}}\right) & \text{if } \alpha_0 \to \alpha_m \to \beta_0 \\ \nu^{p-1}_{0,\mu_m,\ell}\left(H_{\widehat{\boldsymbol{\alpha}_0},\widehat{\boldsymbol{\beta}_0}}\right) & \text{otherwise.} \end{cases}$$

By the induction hypothesis, the left quantity is non zero if and only if $p = p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \alpha_m)$ and $\ell = \ell_m(\boldsymbol{\alpha}) = \ell_m(\widehat{\boldsymbol{\alpha}_0})$.

(Case 2) Let us suppose that $\alpha_m = \alpha_0$ and $\ell_0(\boldsymbol{\alpha}) \geq 1$. Applying the same reasoning as before and using Proposition 2.6 (case $\lambda = 1$, $\ell \geq 1$), we get

$$\nu^p_{0,\mu_0,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \nu^{p-1}_{0,\mu_0,\ell-1}\left(H_{\widehat{\boldsymbol{\alpha}_0},\widehat{\boldsymbol{\beta}_0}}\right),$$

non zero if and only if $\ell = \ell_0(\boldsymbol{\alpha}) = \ell_0(\widehat{\boldsymbol{\alpha}_0}) + 1$. In this case, we have $p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \alpha_0) = p(\widehat{\boldsymbol{\alpha}_0}, \widehat{\boldsymbol{\beta}_0}, \alpha_0) + 1$ because we do not have $\alpha_0 \to \alpha_0 \to \beta_0$.

(Case 3) Let us suppose that $\alpha_m = \alpha_0$ et $\ell_0(\boldsymbol{\alpha}) = 0$, so we have $\alpha_1 \neq \alpha_0$. Applying the same reasoning as in Case 1, we get

$$\nu^p_{0,\mu_0,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \begin{cases} \nu^p_{0,\mu_0,\ell}\left(H_{\widehat{\boldsymbol{\alpha}_1},\widehat{\boldsymbol{\beta}_1}}\right) & \text{if } \alpha_1 \to \alpha_0 \to \beta_1 \\ \nu^{p-1}_{0,\mu_0,\ell}\left(H_{\widehat{\boldsymbol{\alpha}_1},\widehat{\boldsymbol{\beta}_1}}\right) & \text{otherwise.} \end{cases}$$

By the induction hypothesis, and as the order in which the convolutions are done does not matter, the left quantity is non zero if and only if $p = p(\boldsymbol{\alpha}, \boldsymbol{\beta}, \alpha_0)$ and $\ell = \ell_0(\boldsymbol{\alpha}) = \ell_0(\widehat{\boldsymbol{\alpha}_1}) = 0$.

To conclude, Formula (a) is satisfied for the couple $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

**Formula (c).** Let us suppose that Formula (c) is satisfied for all tuples of length $n$. We set $\lambda_s$ the special eigenvalue of $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, $\lambda'_s$ the special eigenvalue of $H_{\widehat{\boldsymbol{\alpha}_0},\widehat{\boldsymbol{\beta}_0}}$ and $\gamma_0 = \{\beta_0 - \alpha_0\}$. Reals $\gamma_s$ and $\gamma'_s$ in $(0,1]$ verifying $\lambda_s = \exp(-2i\pi\gamma_s)$ and $\lambda'_s = \exp(-2i\pi\gamma'_s)$ are linked by the relation $\gamma_s = \gamma'_s + \gamma_0 \bmod \mathbb{Z}$.

According to Proposition 2.3 and [DS13, 2.2.14], and applying the same reasoning as in the proof of Case 1 of Formula (b), we have

$$\mu^p_{1,\lambda_s,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \begin{cases} \mu^p_{1,\lambda'_s,\ell}\left(H_{\widehat{\boldsymbol{\alpha}_0},\widehat{\boldsymbol{\beta}_0}}\right) & \text{if } \gamma_s \in (0,\gamma_0] \\ \mu^{p-1}_{1,\lambda'_s,\ell}\left(H_{\widehat{\boldsymbol{\alpha}_0},\widehat{\boldsymbol{\beta}_0}}\right) & \text{if } \gamma_s \in (\gamma_0,1]. \end{cases}$$

By induction hypothesis, we have

$$\mu^p_{1,\lambda'_s,\ell}\Big(H_{\widehat{\boldsymbol{\alpha}_0},\widehat{\boldsymbol{\beta}_0}}\Big) = \begin{cases} 1 & \text{if } p = \#\left\{i \geq 1 \,\middle|\, \left\{\sum_{k=1}^i (\beta_k - \alpha_k)\right\} < \gamma'_s\right\} \text{ and } \ell = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and we can remark that

$$\#\left\{i \geq 0 \,\middle|\, \left\{\sum_{k=0}^{i}(\beta_k - \alpha_k)\right\} < \gamma_s\right\} = \begin{cases} \#\left\{i \geq 1 \,\middle|\, \left\{\sum\limits_{k=1}^{i}(\beta_k - \alpha_k)\right\} < \gamma'_s\right\} \\ \qquad \text{if } \gamma_s \in (0,\gamma_0] \\ \#\left\{i \geq 1 \,\middle|\, \left\{\sum\limits_{k=1}^{i}(\beta_k - \alpha_k)\right\} < \gamma'_s\right\} + 1 \\ \qquad \text{if } \gamma_s \in (\gamma_0,1]. \end{cases}$$

To conclude, Formula (c) is satisfied for the couple $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. □

**Link between Theorem 3.4 and Fedorov's formulas.** Formulas (a) and (b) of the previous theorem corresponds to Formulas (a) et (b) of Theorem 3 in [Fed17]. However, this is not fully obvious in the sense that Fedorov considers in his article the space of solutions of the connection associated with the hypergeometric equation, while we consider the space of horizontal sections of the connection. Let us begin by transposing Fedorov's formulas in terms of horizontal sections with the following lemma. Note that we do not necessarily assume that the tuples are ordered.

LEMMA 3.6. — *Parts* (a) *and* (b) *of* [Fed17, Th. 3] *are equivalent to the following statement:*
*The hypergeometric module $H_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is equipped with a variation of polarized Hodge structure verifying, up to a shift, the following identities:*

$$\text{(a)}\qquad \nu^p_{0,\mu_m,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \begin{cases} 1 & \text{if } p = \#\{j \mid \beta_j < \alpha_m\} - \#\{i \mid \alpha_i < \alpha_m\} \\ & \text{and } \ell = \ell_m(\boldsymbol{\alpha}) \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{(b)}\qquad \nu^p_{\infty,\lambda_m,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \begin{cases} 1 & \text{if } p = \#\{j \mid \beta_j \le \beta_m\} - \#\{i \mid \alpha_i < \beta_m\} \\ & \text{and } \ell = \ell_m(\boldsymbol{\beta}) \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* — The space of solutions and the space of horizontal sections are dual (see for example [Pha79, Cor. 7.1.1]). If we denote by $*$ the dual, we have the relation $(\mathrm{P}_\ell H)^* \simeq \mathrm{N}^\ell \mathrm{P}_\ell(H^*)$ as Hodge structures and then

$$(\mathrm{gr}^p_F \mathrm{P}_\ell H)^* \simeq \mathrm{gr}_F^{-p}(\mathrm{P}_\ell H)^* \simeq \mathrm{gr}_F^{-p}\mathrm{N}^\ell \mathrm{P}_\ell(H^*) \simeq \mathrm{gr}_F^{-p+\ell}\mathrm{P}_\ell(H^*).$$

Consequently, duality translates as the transformation $(p,\ell) \mapsto (-p+\ell,\ell)$. Applying this rule, we deduce that [Fed17, Th. 3(a)] is equivalent to

$$\nu^p_{0,\mu_m,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \begin{cases} 1 & \text{if } p = -(\#\{i \mid \alpha_i \le \alpha_m\} - \#\{j \mid \beta_j < \alpha_m\}) + \ell_m(\boldsymbol{\alpha}) \\ & \text{and } \ell = \ell_m(\boldsymbol{\alpha}) \\ 0 & \text{otherwise,} \end{cases}$$

in other words

$$\nu^p_{0,\mu_m,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \begin{cases} 1 & \text{if } p = \#\{j \mid \beta_j < \alpha_m\} - \#\{i \mid \alpha_i < \alpha_m\} \\ & \text{and } \ell = \ell_m(\boldsymbol{\alpha}) \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, [Fed17, Th. 3(b)] is equivalent to

$$\nu^p_{\infty,\lambda_m,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \begin{cases} 1 & \text{if } p = -(\#\{i \mid \alpha_i < \beta_m\} - \#\{j \mid \beta_j < \beta_m\}) + \ell_m(\boldsymbol{\beta}) \\ & \text{and } \ell = \ell_m(\boldsymbol{\beta}) \\ 0 & \text{otherwise,} \end{cases}$$

in other words

$$\nu^p_{\infty,\lambda_m,\ell}(H_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = \begin{cases} 1 & \text{if } p = \#\{j \mid \beta_j \leq \beta_m\} - \#\{i \mid \alpha_i < \beta_m\} \\ & \text{and } \ell = \ell_m(\boldsymbol{\beta}) \\ 0 & \text{otherwise,} \end{cases}$$

that concludes the proof. □

It remains to show that the formulas of the previous lemma correspond to the formulas of Theorem 3.4, up to a shift. This is a consequence of the following combinatorial lemma, insofar as $\#\{k \mid \alpha_k < \beta_k\}$ only depends on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

LEMMA 3.7. — *We have the following relations:*
*(i)* $p(\boldsymbol{\alpha},\boldsymbol{\beta},\alpha_m) - (\#\{j \mid \beta_j < \alpha_m\} - \#\{i \mid \alpha_i < \alpha_m\}) = \#\{k \mid \alpha_k < \beta_k\}$
*(ii)* $p(\boldsymbol{\alpha},\boldsymbol{\beta},\beta_m) - (\#\{j \mid \beta_j \leq \beta_m\} - \#\{i \mid \alpha_i < \beta_m\}) = \#\{k \mid \alpha_k < \beta_k\}$.

*Proof.* — (i) Let us sum up in the following table the contributions of $k \in \{1,...,n\}$ to $p(\boldsymbol{\alpha},\boldsymbol{\beta},\alpha_m)$ and $\#\{j \mid \beta_j < \alpha_m\} - \#\{i \mid \alpha_i < \alpha_m\}$ according to the relative positions of $\alpha_k$, $\beta_k$ and $\alpha_m$.

| relative positions | | contribution of $k$ to $p(\boldsymbol{\alpha},\boldsymbol{\beta},\alpha_m)$ | contribution of $k$ to $\#\{j \mid \beta_j < \alpha_m\} - \#\{i \mid \alpha_i < \alpha_m\}$ |
|---|---|---|---|
| $\alpha_k < \beta_k$ | $0 \leq \alpha_m < \alpha_k < \beta_k < 1$ | 1 | 0 |
| | $0 \leq \alpha_k = \alpha_m < \beta_k < 1$ | 1 | 0 |
| | $0 \leq \alpha_k < \alpha_m < \beta_k < 1$ | 0 | −1 |
| | $0 \leq \alpha_k < \beta_k < \alpha_m < 1$ | 1 | 0 |
| $\alpha_k > \beta_k$ | $0 \leq \alpha_m < \beta_k < \alpha_k < 1$ | 0 | 0 |
| | $0 \leq \beta_k < \alpha_m < \alpha_k < 1$ | 1 | 1 |
| | $0 \leq \beta_k < \alpha_k = \alpha_m < 1$ | 1 | 1 |
| | $0 \leq \beta_k < \alpha_k < \alpha_m < 1$ | 0 | 0 |

This table proves that $p(\boldsymbol{\alpha},\boldsymbol{\beta},\alpha_m)$ and $\#\{j \mid \beta_j < \alpha_m\} - \#\{i \mid \alpha_i < \alpha_m\}$ differ by $\#\{k \mid \alpha_k < \beta_k\}$, showing Formula (i).

(ii) Let us now sum up in the following table the contributions of the integer $k$ to $p(\boldsymbol{\alpha},\boldsymbol{\beta},\beta_m)$ and $\#\{j \mid \beta_j \leq \beta_m\} - \#\{i \mid \alpha_i < \beta_m\}$ according to the relative positions of $\alpha_k$, $\beta_k$ and $\beta_m$.

| relative positions | | contribution of $k$ to $p(\boldsymbol{\alpha},\boldsymbol{\beta},\beta_m)$ | contribution of $k$ to $\#\{j \mid \beta_j \le \beta_m\} - \#\{i \mid \alpha_i < \beta_m\}$ |
|---|---|---|---|
| $\alpha_k < \beta_k$ | $0 \le \beta_m < \alpha_k < \beta_k < 1$ | 1 | 0 |
| | $0 \le \alpha_k < \beta_m < \beta_k < 1$ | 0 | $-1$ |
| | $0 \le \alpha_k < \beta_k = \beta_m < 1$ | 1 | 0 |
| | $0 \le \alpha_k < \beta_k < \beta_m < 1$ | 1 | 0 |
| $\alpha_k > \beta_k$ | $0 \le \beta_m < \beta_k < \alpha_k < 1$ | 0 | 0 |
| | $0 \le \beta_k = \beta_m < \alpha_k < 1$ | 1 | 1 |
| | $0 \le \beta_k < \beta_m < \alpha_k < 1$ | 0 | 0 |
| | $0 \le \beta_k < \alpha_k < \beta_m < 1$ | 1 | 1 |

This table proves that $p(\boldsymbol{\alpha},\boldsymbol{\beta},\beta_m)$ and $\#\{j \mid \beta_j \le \beta_m\} - \#\{i \mid \alpha_i < \beta_m\}$ differ by $\#\{k \mid \alpha_k < \beta_k\}$, showing Formula (ii). □

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