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**ERRATUM TO ‘AN INVARIANCE PRINCIPLE FOR  
STATIONARY RANDOM FIELDS UNDER HANNAN’S  
CONDITION’**

DALIBOR VOLNÝ AND YIZAO WANG

In this erratum, we point out a mistake in a statement in Volný and Wang [4, Section 6], published in *Stochastic Processes and their Applications*, 124(12):4012–4029, on the relation between Wu’s physical dependence measure [1, 5] and Hannan’s condition [2, 4] on stationary random fields. These conditions have been introduced in the past as easy-to-verify conditions for stationary random fields that lead to the central limit theorem and invariance principle. It had been known before that Wu’s condition is strictly stronger than Hannan’s condition in dimension one, as shown in Wu [5, Theorem 1]. However, it was stated in [4, p. 4026] that the argument for dimension one ‘can be easily adapted to high dimension and the details are omitted’. This statement is not true.

We are grateful to Davide Giraud for having read carefully [4] and pointed out the mistake to us. It turned out that Wu and Hannan’s condition are no longer comparable in dimension  $d \geq 2$ . We have provided an example of stationary random field in [4, Proposition 6.1] that satisfies Hannan’s condition, but not Wu’s. Here, in Proposition 1 below, we provide another example in dimension  $d \geq 2$  that satisfies Wu’s condition, but not Hannan’s.

Similar statements as the aforementioned one on the relationship between Wu and Hannan’s conditions in [4] can also be found in [3]. The mistake does not affect other results in [3, 4], which are on limit theorems under Hannan’s condition for stationary random fields.

Recall that we consider stationary random fields in the form of

$$(1) \quad X_i = f \circ T_i(\epsilon), i \in \mathbb{Z}^d$$

where  $\epsilon = \{\epsilon_i\}_{i \in \mathbb{Z}^d}$  is a collection of i.i.d. random variables,  $f$  is a measurable function from  $\mathbb{R}^{\mathbb{Z}^d}$  to  $\mathbb{R}$ , and  $T_i$  given by  $[T_i(\epsilon)]_j = \epsilon_{i+j}$ ,  $i, j \in \mathbb{Z}^d$ , are the canonical shift operators on  $\mathbb{R}^{\mathbb{Z}^d}$ . For the sake of simplicity, we only recall the two conditions for  $d = 2$  in (2) below, and refer to [4] for the case when  $d \geq 3$ .

**Proposition 1.** *For every  $d \geq 2$ , there exists a random field in the form of (1), such that Wu’s condition holds, but Hannan’s condition does not.*

*Proof.* We first recall the two conditions of interest for  $d = 2$ . The Hannan’s condition is based on the so-called projection operators  $P_i, i \in \mathbb{Z}^2$  defined as, for  $\mathcal{F}_i = \sigma(\epsilon_j : j \leq i, j \in \mathbb{Z}^2)$  with  $j \leq i$  in the coordinate-wise sense,

$$P_i g(\epsilon) = \mathbb{E}(g(\epsilon) | \mathcal{F}_i) - \mathbb{E}(g(\epsilon) | \mathcal{F}_{i-(1,0)}) - \mathbb{E}(g(\epsilon) | \mathcal{F}_{i-(0,1)}) + \mathbb{E}(g(\epsilon) | \mathcal{F}_{i-(1,1)}),$$

for any integrable function  $g$ . On the other hand, to introduce Wu's condition we set  $\epsilon^* = \{\epsilon_i^*\}_{i \in \mathbb{Z}^2}$  with

$$\epsilon_i^* = \begin{cases} \epsilon_i & \text{if } i \neq (0, 0) \\ \epsilon_{(0,0)}^* & \text{otherwise,} \end{cases}$$

where  $\epsilon_{(0,0)}^*$  is an independent copy of  $\epsilon_{(0,0)}$  also independent of  $\{\epsilon_i\}_{i \in \mathbb{Z}^2}$ , and set

$$f_i^*(\epsilon) = f \circ T_i(\epsilon^*).$$

Hannan's condition and Wu's condition for the random field  $\{f \circ T_i(\epsilon)\}_{i \in \mathbb{Z}^2}$  correspond to the following inequalities,

$$(2) \quad \sum_{i \in \mathbb{Z}^2} \|P_i f\|_2 < \infty \quad \text{and} \quad \sum_{i \in \mathbb{Z}^2} \|f - f_i^*\|_2 < \infty,$$

respectively, where here and below we write  $\|h\|_2 = (\mathbb{E}h^2(\epsilon))^{1/2}$ .

Now, we provide a concrete example of  $f$ . Let  $\epsilon = \{\epsilon_i\}_{i \in \mathbb{Z}^2}$  be a family of i.i.d. random variables with  $\mathbb{P}(\epsilon_i = \pm 1) = 1/2$ . Introduce, for a sequence of strictly positive integers  $\{n_k\}_{k \in \mathbb{N}}$  to be determined later,

$$g^{(k)}(\epsilon) = \sum_{i \in \{1, \dots, n_k\}^2} \epsilon_{(i_1, 0)} \epsilon_{(0, i_2)}, k \in \mathbb{N},$$

and here and in the sequel we use the notation  $i = (i_1, i_2) \in \mathbb{Z}^2$ . Next, for a sequence of non-negative real numbers  $\{a_k\}_{k \in \mathbb{N}}$  to be determined later, consider

$$f(\epsilon) = \sum_{k=1}^{\infty} a_k g^{(k)} \circ T_{(m_k, 0)}(\epsilon),$$

with  $m_1 = 0, m_k = \sum_{i=1}^{k-1} n_i, k \geq 2$ .

For the so-constructed  $f$ , we can compute explicitly the two series in (2). For the series corresponding to Hannan's condition, observe that by independence,

$$(3) \quad P_i g^{(k)}(\epsilon) = \begin{cases} \epsilon_{(i_1, 0)} \epsilon_{(0, i_2)} & \text{if } i \in \{1, \dots, n_k\}^2 \\ 0 & \text{otherwise.} \end{cases}$$

So we have

$$\sum_{i \in \mathbb{Z}^2} \|P_i g^{(k)}\|_2 = n_k^2.$$

Furthermore for each  $i \in \mathbb{Z}^2$ ,  $P_i[g^{(k)} \circ T_{(m_k, 0)}](\epsilon) = \epsilon_{(i_1, 0)} \epsilon_{(0, i_2)}$  if  $i - (m_k, 0) \in \{1, \dots, n_k\}^2$  and zero otherwise. By the choice of  $n_k$  and  $m_k$ , it also follows that  $P_i[g^{(k)} \circ T_{(m_k, 0)}](\epsilon) \neq 0$  for at most one  $k$ . So we can write

$$\|P_i f\|_2 = \sum_{k=1}^{\infty} a_k \|P_i [g^{(k)} \circ T_{(m_k, 0)}]\|_2 = \sum_{k=1}^{\infty} a_k \|P_{i - (m_k, 0)} g^{(k)}\|_2.$$

Hence,

$$(4) \quad \sum_{i \in \mathbb{Z}^2} \|P_i f\|_2 = \sum_{k=1}^{\infty} a_k \sum_{i \in \mathbb{Z}^2} \|P_{i - (m_k, 0)} g^{(k)}\|_2 = \sum_{k=1}^{\infty} a_k \sum_{i \in \mathbb{Z}^2} \|P_i g^{(k)}\|_2 = \sum_{k=1}^{\infty} a_k n_k^2.$$

For the series corresponding to Wu's condition, observe first that

$$f_i^*(\epsilon) = \sum_{k=1}^{\infty} a_k g_i^{(k),*}(\epsilon) \quad \text{with} \quad g_i^{(k),*}(\epsilon) = g^{(k)} \circ T_i(\epsilon^*),$$

and

$$\left\| g^{(k)} - g_i^{(k),*} \right\|_2 = \begin{cases} \sqrt{2}n_k^{1/2} & \text{if } i \in (\{1, \dots, n_k\} \times \{0\}) \cup (\{0\} \times \{1, \dots, n_k\}) \\ 0 & \text{otherwise.} \end{cases}$$

So

$$\sum_{i \in \mathbb{Z}^2} \left\| g^{(k)} - g_i^{(k),*} \right\|_2 = 2\sqrt{2}n_k^{3/2}.$$

Since for every  $i \in \mathbb{Z}^2$  there exists at most one  $k \in \mathbb{N}$  such that  $g^{(k)}(\epsilon) - g_i^{(k),*}(\epsilon) \neq 0$ , we arrive at

$$(5) \quad \sum_{i \in \mathbb{Z}^2} \|f - f_i^*\|_2 = \sum_{i \in \mathbb{Z}^2} \sum_{k=1}^{\infty} a_k \left\| g^{(k)} - g_i^{(k),*} \right\|_2 = 2\sqrt{2} \sum_{k=1}^{\infty} a_k n_k^{3/2}.$$

Thus, in order to have an example as desired, in view of (4) and (5) it suffices to choose  $\{a_k\}_{k \in \mathbb{N}}$  and  $\{n_k\}_{k \in \mathbb{N}}$  such that  $\sum_{k=1}^{\infty} a_k n_k^2 = \infty$  but  $\sum_{k=1}^{\infty} a_k n_k^{3/2} < \infty$ . This can be achieved by, for example, taking  $n_k = 2^k$  and  $a_k = n_k^{-2}$ .

The idea can be adapted to  $d \geq 3$ , and we only sketch the key steps. For each  $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$ , let  $i^{(q)}$  be the vector in  $\mathbb{Z}^d$  such that the  $q$ -th coordinate equals  $i_q$ , and all other ones equal zero (e.g.  $i^{(1)} = (i_1, 0, \dots, 0) \in \mathbb{Z}^d$ ). Consider

$$g^{(k)}(\epsilon) = \sum_{i \in \{1, \dots, n_k\}^d} \prod_{q=1}^d \epsilon_{i^{(q)}}.$$

An extension of the projection operator  $P_i$  is introduced for  $i \in \mathbb{Z}^d$  in [4, (2.3) and (2.4)]. The key step is to verify that  $P_i g^{(k)}(\epsilon) = \prod_{q=1}^d \epsilon_{i^{(q)}}$  for  $i \in \{1, \dots, n_k\}^d$  and 0 otherwise, a generalization of (3). This is a consequence of [4, (2.3)] and the independence assumption on  $\epsilon$ . The rest of the calculation is similar to  $d = 2$ , and in particular one can obtain, with similar notation for  $g_i^{(k),*}$ ,

$$\sum_{i \in \mathbb{Z}^d} \left\| P_i g^{(k)} \right\|_2 = n_k^d \quad \text{and} \quad \sum_{i \in \mathbb{Z}^d} \left\| g^{(k)} - g_i^{(k),*} \right\|_2 = d\sqrt{2}n_k^{(d+1)/2}.$$

The details are omitted. □

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