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Guillaume Chèze

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G. Chèze[a]

[a] Institut de Mathématiques de Toulouse, UMR 5219
Université de Toulouse ; CNRS
UPS IMT, F-31062 Toulouse Cedex 9, France.

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ABSTRACT
It has been shown that the current IAAF decathlon scoring tables involve some bias. In the actual scoring system, the scoring rules depend on some parameters. These parameters are difficult to choose. Indeed, different arguments give different values for the parameters. Then, a natural question arises: Are these parameters mandatory to obtain a fair ranking between the athletes? In this article we show that the answer to this question is no.
Furthermore, we show that if we want a ranking method satisfying four natural fairness axioms then this ranking method is unique. Moreover, this ranking method is easy to perform in practice.

KEYWORDS
Decathlon; rules; fairness; ranking; axioms

Introduction

Combined events have a long history. Indeed pentathlon was already practice during the antiquity. The events are known: there were the long jump, javelin throw, discus throw, the stadion foot race and wrestling. Unfortunately, the used method to choose the winner is not clear and nowadays several hypotheses exist. However, some elements allow us to deduce that a victorious athlete in the first three events was declared the overall winner, see e.g. Young (2004). This implies that the final result of the competition depends of the ranking in the different events and not on the performance in each event.
Nowadays, a different method is used. Since the beginning of the XX-th century, the strategy used to rank the athletes is the following: associate to each performance a number of points, after the ten events sum the points and then the winner is the athlete with the maximum number of points. Thus, if an athlete wins the first six events we cannot declare him or her as the overall winner. Indeed, suppose that during the first six events an athlete A, wins 800 points in each event and an athlete B, wins 790 points in each event. Suppose also that during the four last events A obtains 790 in each event and B obtains 830 in each event. Then the final score of A is: $800 \times 6 + 790 \times 4 = 7960$ points, and the final score of B is: $790 \times 6 + 830 \times 4 = 8060$

CONTACT G. Chèze. Email: guillaume.cheze@math.univ-toulouse.fr

points. Therefore, A is not the winner of the decathlon even if A is the best during the first 6 events.
The approach used today is thus based on the comparison of the performances and not on the comparison of the rankings in the different events. With the modern approach, competitors are more incited to do their best in all events. However, this leads to a serious problem: How can we analyse and compare fairly the performance in different events? Indeed, how can we compare the result of a race expressed in seconds and the the result of a jump expressed in centimeter?

The IAAF decathlon scoring tables used formulas which convert performances into a number of points. Some authors have shown that these formulas are not fair and that some events are favored, see e.g. Tidow (1989, 2000); Westera (2006, 2011). Obviously, the same kind of criticism can be made for heptathlon, see Gassmann et al. (2016).

Westera has proposed a method which allows to consider all events in the same way, see Westera (2006). In this article, we are going to extend Westera's work. We propose a new formula which transforms performances into points. Our formula is a simplification of Westera's one. Then, we are going to show how to aggregate all the scores obtained by each competitor in order to get a fair ranking. Indeed, usually the idea to find the overall winner is to consider the mean of the scores obtained by each competitor. The winner is the athlete with the highest mean. In practice, we compute the sum of the scores which corresponds to the arithmetic mean. But, why don't we use another mean: the geometric mean or the harmonic mean? Indeed, it is not proved that computing the sum of the score is the fairest way to find the overall winner of a decathlon. In this article we are going to show that the geometric mean is better suited for our problem. Roughly speaking, **we prove that it is better to consider the product of the scores rather than the sum**.

We can already explain why it is not unnatural to consider the product instead of the sum of the scores. Indeed, suppose that we consider an imaginary competition with only two events. We also suppose that each event give at most 100 points. Now, consider a competitor A with the following scores (40,100). This means that A wins 40 points in the first event and 100 points in the second. Consider also another competitor B who has reached the scores (65,65). If we consider the sum of the scores then A wins the competition because A has 140 points and B has 130 points. However, it seems that A is a specialist of the second event and he or she is not very good for the first event. The competitor B seems to be more versatile. The product of the scores gives 4225 points for B and 4000 points for A. Thus by considering the product, B becomes the winner of the competition. This shows that the product of the scores allows to avoid that a specialist wins a competition of combined events. This is the goal of combined events. This kind of competitions do not look for a specialist. A decathlon wants to reward the more versatile athlete. We will detail this point in Section 1. We can already remark that the use of a product instead of a sum in order to get a fair ranking is already implemented in combined climbing competitions, see IFSC (2018).

In this article we study separately two problems:

(1) How can we associate fairly a score to each event?

(2) How can we find a fair final ranking thanks to the score of each event?

It seems that the study of decathlon rules has never been divided into these two questions. We think that the second question has never been studied because usually we always add points and the sum gives the ranking. This is the reason why this question was considered as trivial. However, it is not the case. Indeed, as shown in the previous example the product of the score gives another possibility of ranking. The fairness of the scoring rule for each event and the fairness of the final ranking are two different problems.

In this article, we are going to use an axiomatic approach in order to answer to these two questions. More precisely, we are going to define axioms, i.e. basic rules, that we want to satisfy. These basic rules will keep the spirit of the existing scoring rules and their improvements proposed by Westerra. Then **we will show that there exists only one possible method of ranking satisfying these axioms.** We do not claim that the method proposed in this article is the better one. We prove that if we want to have a ranking method satisfying these properties then this method is unique.

The study of ranking rules is not new. Mathematicians and economists have proved a lot of results about this problem. Our result is in the spirit of classical results in social choice theory. In our situation, the proof of our theorem follows easily from our four axioms.

The structure of this article is thus simple: First, we study how to associate fairly a score to a performance. Second, we show how to obtain a fair ranking from scores. At last, we will give some examples in order to show the impact of the proposed ranking method.

1. Fair scoring rules

1.1. State of the art

In the document "IAAF scoring tables for combined events", see IAAF (2001), the history of the tables is presented. Different systems have been used since the XIX-th century. We recall here just some steps in order to introduce our scoring rule.
The first scoring tables were linear. This means that the relation between the performance and the number of points were of this kind:

$$S = A \times (P - B),$$

where S is the score (the number of points), P is the performance of the athlete (for example the height in centimeters for the high jump). The number A and B were chosen in this way: B represents the performance giving 0 points and A was computed in order to give 1000 points to an identified top performance, for example the world record. For the running events the factor $(P - B)$ become $(B - P)$. Indeed, a good performance for these events corresponds to a shorter time.

The problem with this method is the following: a 1 cm improvement in a high jump performance increases in the same way the number of points whatever the performance is. However, if you jump 75 cm and then 76 cm it is not the same performance as if you jump 220 cm and then 221 cm. Thus, new scoring rules have been introduced: progressive scoring rules. These rules are based on formula of this kind:

$$S = A \times (P - B)^C, \text{ with } C > 1.$$

We remark that a new constant $C > 1$ has been introduced. To each event is thus associated three parameters A, B and C.
With this kind of formula, we get more points if we improve our performance from 220 cm to 221 cm than from 75 cm to 76 cm.

The current decathlon scoring tables use these kinds of progressive formulas and they have been used without modification since the 1980s.
When these rules have been established in march 1983, nine points were accepted as basic principles for a new set of tables, see (IAAF, 2001, page 18):

(1) The new set of tables should be used for combined events only.
(2) Results in various events should, as far as possible, yield about the same number of points if the results are comparable as to quality and difficulty.
(3) The new tables should be either:
  (a) a modification of the existing ones
  (b) a straight line in all events
  (c) slightly progressive tables in all events.
(4) It must be possible to use the scoring tables for beginners, juniors and top athletes as well.
(5) There will be a special scoring table for men and another table for women.
(6) All the new versions of the scoring tables should be based on the statistical data for the combined events by paying due regard to the statistical data for performances by single event athletes.
(7) The new tables should be applicable now and for the future.
(8) It is desirable without creating other problems, that the total scores using the new tables for the top world class athletes should remain approximately the same. That is about 8500 points for the decathlon and about 6500 points for the heptathlon.
(9) As far as possible the new tables must insure that a specialist in one event cannot overcome performances in the other events.

However, nowadays the actual system do not respect the second point.
Indeed, if you equal the mens long jump world record (8.95m, Mike Powell, 30 August 1991) you get 1312 points and if you equal the mens 1500 metres world record (3 min 26 s, Hicham El Guerrouj, 14 July 1998) you win 1218 points. This means that the same outstanding performance in different events do not give the same number of points.
Furthermore, this difference $1318 - 1218 = 94$ points is not negligible. For example, during the IAAF world championship in London (2017), the difference of the number of points between the silver medal (Freimuth, 8564 points) and the bronze medal (Kazmirek, 8488 points) is just 76 points.
Moreover, 9000 points is an outstanding performance during a decathlon. Only three men have already done such a performance. Thus 900 points is the average value for an event during an outstanding decathlon. Therefore, if we think about the 94

points as an error of measurement related to the IAAF method, then this error of measurement corresponds to more than 10% of the score obtained during an event. Thus a difference of 94 points is not negligible.

Westera has proposed a method which avoids this problem. Instead of considering a performance P we consider a normalized performance P_N defined in the following way:

$$P_N = \frac{P - P_0}{P_1 - P_0}.$$

where P_1 represents a high level performance (for example the world record) and P_0 a low level performance. Thus if $P = P_1$ then $P_N = 1$ and if $P = P_0$ then $P_N = 0$.
For running events the performance will be the speed, or the inverse of the time. Thus for running events if P is the time in seconds then P_N is defined in the following way:

$$P_N = \frac{\frac{1}{P} - \frac{1}{P_0}}{\frac{1}{P_1} - \frac{1}{P_0}}.$$

The idea of the normalized performance is thus to compare a performance with a standard. If an athlete jumps an height corresponding to 80% of the world record we can consider that this performance is equivalent to a long jump corresponding to 80% of the length of the world record. This leads to the following formula:

$$S = A \times P_N^C.$$

In order to treat all events in the same way, Westera has proposed to take the same values A et C for all events. He has suggested: $A = 863.9$ and $C = 1.479$. These values have been given in order to get scores relatively close to the ones given by the current scoring tables.
We remark that by construction this method satisfies the second point of the nine basic principles accepted by IAAF.

1.2. A new formula

Westera has suggested a method in order to consider events in the same way, but a question is still open: *How do we choose P_0 and P_1 for each event?*

In the previous discussion we have written that we can take P_1 as the world record but we can also take P_1 as the average of the 100 best performances of all time. If we consider the last proposition, do we consider the 100 best performances obtained during a decathlon? As already remarked by Günter Tidow, see Tidow (1989), this raises the question of considering an absolute or a relative basis for scoring rules. If we consider a relative system (this means that we consider uniquely the performance obtained during a decathlon) then we take into account that after two days of competitions it is not fair to compare a performance during a decathlon to performances of single event specialists. Nevertheless, if decathlon is a test of all-round ability than it seems natural to compare the performance of an athlete with the performance of a specialist. The choice of P_1 seems thus to be an unsolvable problem. There does not exist scientific nor philosophic arguments for choosing P_1.

The situation for P_0 seems more easy. For example, for running events we have to choose a reference for low level performances. Nowadays this reference is 18 seconds for the 100 m. Thus 18 seconds gives zero point. Why 18 and not 19? If somebody walks slowly and needs 50 seconds for 100 m he will also get zero points. It is not fair and moreover we can always imagine a worst situation. Here, we thus suggest to take $P_0 = \infty$, then $1/P_0 = 0$. This just means that the worst performance corresponds to a situation with a speed equals to 0. This gives for running events:

$$\tilde{P}_N = \frac{P_1}{P},$$

where P is the time in seconds.

For the other events (jumps and throws), if we define a low level performance we can always imagine a lower performance except if the low level performance P_0 is fixed at 0. This gives:

$$\tilde{P}_N = \frac{P}{P_1}.$$

This leads us to suggest the following kind of scoring rules:

$$S = A \times \tilde{P}_N^C.$$

We can write, for all events except running, this formula as:

$$S = \frac{A \times P^C}{P_1^C}, \qquad \text{(Score 1)}$$

and for running events we get:

$$S = \frac{A \times P_1^C}{P^C}. \qquad \text{(Score 2)}$$

The parameter P_0 has disappeared and we still do not know how to fix A, P_1 and C. However, in the following we will prove that these values are unnecessary to get a ranking between the athletes.

As before, we remark that by construction this scoring rule satisfied the second point of the nine basic principles accepted by IAAF.

2. Fair ranking based on scores

At the end of the competition, when we have all the scores of all competitors, we have to rank the athletes. The problem is thus the following: consider the scores obtained by two athletes $At_{\underline{s}}$ and $At_{\underline{t}}$, respectively $\underline{s} = (s_1, \ldots, s_{10})$ and $\underline{t} = (t_1, \ldots, t_{10})$, how can we rank $At_{\underline{s}}$ and $At_{\underline{t}}$ thanks to $\underline{s}$ and $\underline{t}$?

This means that we are looking for a ranking.

Definition 2.1. A ranking $\succ$ is a relation between athletes. In the following:
$At_{\underline{s}} \succ At_{\underline{t}}$ means $At_{\underline{s}}$ is strictly better than $At_{\underline{t}}$,
$At_{\underline{s}} \succeq At_{\underline{t}}$ means $At_{\underline{s}}$ is better or equal to $At_{\underline{t}}$,
$At_{\underline{s}} \approx At_{\underline{t}}$ means $At_{\underline{s}}$ is equal to $At_{\underline{t}}$, there is a tie.

In the following, if the scores associated to athlete $At_{\underline{s}}$ are $\underline{s} = (s_1, \ldots, s_{10})$ and to athlete $At_{\underline{t}}$ are $\underline{t} = (t_1, \ldots, t_{10})$ then we will write without distinction $\underline{s} \succ \underline{t}$ or $At_{\underline{s}} \succ At_{\underline{t}}$. In the same way, we will write without distinction $\underline{s} \approx \underline{t}$ or $At_{\underline{s}} \approx At_{\underline{t}}$.

2.1. One event is sufficient to give the victory

A natural axiom satisfied by the current rule is **"one event is sufficient to give the victory".** In the following we will denote this axiom by **1EV** (one event victory).

Definition 2.2. A ranking satisfied the rule "one event is sufficient to give the victory" (1EV) if it satisfies the following:
If $s_1 = t_1, s_2 = t_2, \ldots, s_9 = t_9$ and $s_{10} > t_{10}$ then $\underline{s} \succ \underline{t}$.
Reciprocally, if p is a performance and $(p, p, \ldots, p, s_{10}) \succ (p, p, \ldots, p, t_{10})$ then $s_{10} > t_{10}$.

This axiom is natural and means that if two athletes $At_{\underline{s}}$ and $At_{\underline{t}}$ do exactly the same performance during the first nine events and $At_{\underline{s}}$ is better than $At_{\underline{t}}$ in the last event then $At_{\underline{s}}$ is better than $At_{\underline{t}}$ in the overall competition.

2.2. The anonymity axiom

The second natural and basic rule that we want to satisfy for a ranking is called **anonymity**. The idea is the following: it is not important to know the name of the event where competitors win their points, the important information is the number of points.

Definition 2.3. A ranking is *anonymous* if it satisfies the following rule:
if $\underline{t}$ is obtained from $\underline{s}$ by exchanging coordinates then $\underline{t} \approx \underline{s}$

This means that the order of the score is not important, for example:

$$(801, 802, 803, 804, 805, 806, 807, 808, 809, 810) \approx (810, 803, 805, 802, 807, 806, 801, 804, 809, 808).$$

The anonymity axiom is satisfied by the current rule. Indeed, the sum of the scores is independent of the order of the different scores.

2.3. The gradation axiom

Definition 2.4. Let $z_{\underline{s}}$ be the number of scores equal to zero in $\underline{s}$.
A ranking satisfies the *gradation axiom* means that: If $z_{\underline{s}} < z_{\underline{t}}$ then $\underline{s} \succ \underline{t}$.

This axiom means, for example, that an athlete with one score equals to zero cannot be considered better than an athlete with all his or her scores positive. This axiom is not theoretically satisfied by the current rule. However, in practice it is impossible to win a decathlon when one of the score is equal to zero.
A criticism for this axiom can be the following: "An international decathlete failing to

complete one event ends up with a lower rank than a novice who completes ten events to a poor level of performance. This is a problem. Even if an athlete has a score equal to zero in an event then he or she could get a good rank."
However, if we accept the last statement then it implies that an athlete can avoid an event. This leads to the question: how many events an athlete can avoid? Just one, two... In this situation, the competition will not reward the more versatile athlete. It seems that it does not correspond to the spirit of combined events (see in Section 1 the ninth basic principle accepted by IAAF).

2.4. The scale independence axiom

Now, we are going to introduce a new axiom in order to avoid some difficulties raised by the choice of parameters A, and P_1. The problem with these parameters is the following: if we change them then we change the scores and thus probably the ranking. For example, if we use (Score 1) with $P_1 = P_W$ where P_W is the world record of the considered event or with P_a where P_a is the average of the 100 best performances of all time then we get two different scoring rules S_W and S_a where:

$$S_W = \frac{A \times P^C}{P_W^C}; \quad S_a = \frac{A \times P^C}{P_a^C}.$$

It follows

$$S_W = \alpha S_a \text{ where } \alpha = \frac{P_a^C}{P_W^C}.$$

In the same way, if we want to change the parameter A by a new parameter A' then we get two new scoring rules:

$$S = \frac{A \times P^C}{P_1^C} \text{ and } S' = \frac{A' \times P^C}{P_1^C}.$$

As before these rules are related by the formula $S = \alpha' S'$ where $\alpha' = A/A'$.

As it is impossible to objectively decide a good value for P_1 and A, we can look for ranking independent of these parameters. If it is possible to get a ranking satisfying this property then we avoid discussions and arbitrary decisions. More precisely, we want to satisfy the next axiom: **scale independence**.

Definition 2.5. We say that a ranking $\succ$ is *scale independent* if

$$\underline{s} \succeq \underline{t} \iff \underline{u}.\underline{s} \succeq \underline{u}.\underline{t},$$

where $\underline{u}.\underline{s} = (u_1 \times s_1, u_2 \times s_2, \ldots, u_{10} \times s_{10})$, with $\underline{u} = (u_1, \ldots, u_{10})$, $\underline{s} = (s_1, \ldots, s_{10})$ and $u_i > 0$ for all i.

Thus as $S_W = \alpha S_a$ and $S = \alpha' S'$ if we want to have a ranking independent of the choice of the high level performance P_1 and independent of the parameter A then the ranking must be scale independent.
Furthermore, whatever the choice that we make for the parameter P_1, it will change in

the future. Indeed new records will be broken. The scale independence axiom says also that it will be unnecessary to change the rule if the high level performance P_1 changes. Therefore, this last axiom avoids a lot of practical problems. Furthermore, it means that the seventh basic principle accepted by the IAAF, see Section 1, is satisfied.

2.5. The Nash relation

The question now is: *Is there a ranking satisfying the gradation axiom, anonymity, 1EV and scale independence?*

The following ranking $\underset{Nash}{\succ}$ gives an answer to this question. The notation $\prod\limits_{s_i \neq 0} s_i$ corresponds to the product of all positive scores.

Definition 2.6. We set $\underline{s} \underset{Nash}{\succ} \underline{t}$ when $z_{\underline{s}} < z_{\underline{t}}$ or when $z_{\underline{s}} = z_{\underline{t}}$ and $\prod\limits_{s_i \neq 0} s_i > \prod\limits_{t_i \neq 0} t_i$.

We set $\underline{s} \underset{Nash}{\approx} \underline{t}$ when $z_{\underline{s}} = z_{\underline{t}}$ and $\prod_{s_i \neq 0} s_i = \prod_{t_i \neq 0} t_i$.

We call this relation the Nash relation because it is inspired from the work of John Forbes Nash who has studied this kind of relation. John F. Nash was a mathematician who won the Nobel memorial prize in Economic Science in 1994 and the Abel prize in 2015.

The Nash relation uses the geometric mean and not the arithmetic mean in order to compare two series. We recall that the geometric mean of n numbers is $\sqrt[n]{a_1 \times \cdots \times a_n}$. Here the n-th root of the product is not taken into account because we just want to compare two numbers and $x > y \iff \sqrt[n]{x} > \sqrt[n]{y}$.

The Nash relation satisfies the gradation axiom. Indeed, by definition we have $\underline{s} \underset{Nash}{\succ} \underline{t}$ when $z_{\underline{s}} < z_{\underline{t}}$.

The Nash relation satisfies the anonymity axiom. Indeed, a product is independent of the order of the different factors.

The Nash relation satisfies 1EV because if $s_1 = t_1, \ldots, s_9 = t_9$ and $s_{10} > t_{10}$ then $\prod_{s_i \neq 0} s_i > \prod_{t_i \neq 0} t_j$ and $\underline{s} \underset{Nash}{\succ} \underline{t}$.

The Nash relation is also scale independent. For example if we multiply by 2 the parameter A of the first event (100 m) and by 3 the one in the last event (1500m) then this situation corresponds to $\underline{u} = (2, 1, 1, 1, \ldots, 1, 3)$. Suppose that $\underline{s} \underset{Nash}{\succ} \underline{t}$ and in order to simplify the notations $s_i \neq 0$, $t_i \neq 0$, for $i = 1, \ldots, 10$. Then we have

$$\begin{aligned} s_1 \times s_2 \times \cdots \times s_{10} > t_1 \times t_2 \times \cdots \times t_{10} &\iff 2s_1 \times s_2 \times \cdots \times 3s_{10} > 2t_1 \times t_2 \times \cdots \times 3t_{10} \\ &\iff \underline{u}.\underline{s} \underset{Nash}{\succ} \underline{u}.\underline{t}. \end{aligned}$$

Actually, the Nash relation is the only relation satisfying our four axioms. In order to state this characterization we have to define the following notion:

Definition 2.7. We say that two rankings $\succ_1$ and $\succ_2$ coincide when:

$$\underline{s} \succ_1 \underline{t} \iff \underline{s} \succ_2 \underline{t}.$$

Example 2.8. Let f be a strictly increasing function.
We set $\underline{s} \underset{f}{\succ} \underline{t}$ when $z_{\underline{s}} < z_{\underline{t}}$ or when $z_{\underline{s}} = z_{\underline{t}}$ and $f(\prod_{s_i \neq 0} s_i) > f(\prod_{t_i \neq 0} t_i)$.
We set $\underline{s} \underset{f}{\approx} \underline{t}$ when $z_{\underline{s}} = z_{\underline{t}}$ and $f(\prod_{s_i \neq 0} s_i) = f(\prod_{t_i \neq 0} t_i)$.
Then $\underset{Nash}{\succ}$ and $\underset{f}{\succ}$ coincide.
In the following we will use the function $deca(x) = 7670x^{0.23}$.
Then $\underset{Nash}{\succ}$ and $\underset{deca}{\succ}$ coincide. The ranking $\underset{deca}{\succ}$ will be used when we will study some examples. Indeed, the number $7670 \times \left(\prod_{i=1}^{10} s_i\right)^{0.23}$ has approximatively the same order of magnitude than the number of points obtained with the IAAF scoring method. Thus with $\underset{deca}{\succ}$ we get the same ranking than the one obtained with the Nash relation but the number of points given by the formula $7670 \times \left(\prod_{i=1}^{10} s_i\right)^{0.23}$ can be compared with the IAAF score.

We have then the following characterization:

Theorem 2.9.
- *The Nash relation satisfies the gradation axiom, anonymity, 1EV and scale independence.*
- *Conversely, if a relation defined on non-negative numbers satisfied the gradation axiom, anonymity, 1EV and is scale independent then it coincides with the Nash relation.*

Proof. We have already prove the first point. For the proof of the second point, see the appendix. □

Similar characterizations of the Nash relation with other axioms already exist, see Moulin (1988). However, in these characterizations the Nash relation is only used and proved on positive numbers. Here, we consider non-negative numbers because it is possible to have a performance equal to zero.

2.6. Application

We apply the previous theorem to the decathlon when scoring rules defined by formulas (Score 1) and (Score 2) are used.

Consider two athletes $At_{\underline{s}}$ and $At_{\underline{t}}$ with associated scores $(s_1, \ldots, s_{10})$ and $(t_1, \ldots, t_{10})$. The first score corresponds to the first event (100 m) and the last score to the last event (1500 m). We have thus

$$s_1 = \frac{A \times P^C_{1,100}}{P^C_{\underline{s},100}}, s_2 = \frac{A \times P^C_{\underline{s},L}}{P^C_{1,L}}, \ldots, s_{10} = \frac{A \times P^C_{1,1500}}{P^C_{\underline{s},1500}},$$

where $P_{1,100}$ is the high level performance chosen for the first event (100 m), $P_{1,L}$ the high level performance chosen for the second event (long jump),. . . , and $P_{1,1500}$ the high level performance chosen for the last event (1500 m). In the same way $P_{\underline{s},100}$ is the performance of the athlete $At_{\underline{s}}$ in the first event, etc.

Now, suppose that $\succ$ is a ranking defined on the scores which satisfies our four axioms, then by Theorem 2.9 this ranking coincides with the Nash relation. Thus we have the following equivalences:

$$\underline{s} \succ \underline{t} \iff z_{\underline{s}} < z_{\underline{t}} \text{ or } z_{\underline{s}} = z_{\underline{t}} \text{ and } \prod_{s_i \neq 0} s_i > \prod_{t_i \neq 0} t_i.$$

In order to simplify the notation we suppose $z_{\underline{s}} = z_{\underline{t}} = 0$, this gives

$$\underline{s} \succ \underline{t} \text{ and } z_{\underline{s}} = z_{\underline{t}} = 0$$
$$\Updownarrow$$
$$\frac{A \times P_{1,100}^C}{P_{\underline{s},100}^C} \times \frac{A \times P_{\underline{s},L}^C}{P_{1,L}^C} \times \cdots \times \frac{A \times P_{1,1500}^C}{P_{\underline{s},1500}^C} > \frac{A \times P_{1,100}^C}{P_{\underline{t},100}^C} \times \frac{P_{\underline{t},L}^C}{P_{1,L}^C} \times \cdots \times \frac{A \times P_{1,1500}^C}{P_{\underline{t},1500}^C}.$$

We can simplify the right hand side and the left hand side by A, $P_{1,100}^C$, . . . , $P_{1,1500}^C$ this gives:

$$\underline{s} \succ \underline{t} \text{ and } z_{\underline{s}} = z_{\underline{t}} = 0$$
$$\Updownarrow$$
$$\frac{P_{\underline{s},L}^C \times P_{\underline{s},H}^C \times \times P_{\underline{s},PV}^C \times P_{\underline{s},JT}^C \times P_{\underline{s},DT}^C \times P_{\underline{s},SP}^C}{P_{\underline{s},100}^C \times P_{\underline{s},400}^C \times P_{\underline{s},110}^C \times P_{\underline{s},1500}^C} > \frac{P_{\underline{t},L}^C \times P_{\underline{t},H}^C \times \times P_{\underline{t},PV}^C \times P_{\underline{t},JT}^C \times P_{\underline{t},DT} \times P_{\underline{t},SP}^C}{P_{\underline{t},100}^C \times P_{\underline{t},400}^C \times P_{\underline{t},110}^C \times P_{\underline{t},1500}^C}$$

where the indices H means high jump, PV pole vault, JT javelin throw, DT discus throw and SP shot put.

At last, we simplify the power C in the previous inequality (we compute $\sqrt[C]{\cdot}$ of the left hand side and right hand side of the inequality). We get:

$$\underline{s} \succ \underline{t} \text{ and } z_{\underline{s}} = z_{\underline{t}} = 0$$
$$\Updownarrow$$
$$(\mathcal{R}): \frac{P_{\underline{s},L} \times P_{\underline{s},H} \times \times P_{\underline{s},PV} \times P_{\underline{s},JT} \times P_{\underline{s},DT} \times P_{\underline{s},SP}}{P_{\underline{s},100} \times P_{\underline{s},400} \times P_{\underline{s},110} \times P_{\underline{s},1500}} > \frac{P_{\underline{t},L} \times P_{\underline{t},H} \times \times P_{\underline{t},PV} \times P_{\underline{t},JT} \times P_{\underline{t},DT} \times P_{\underline{t},SP}}{P_{\underline{t},100} \times P_{\underline{t},400} \times P_{\underline{t},110} \times P_{\underline{t},1500}}.$$

Remark 1. If we use different coefficients of progressivity C for different events then the simplification is still valid.

We have thus proved the following:

Theorem 2.10. *Suppose that the score of the different events of the decathlon are given by formulas (Score 1) and (Score 2). Consider $\succ$ a relation on scores which satisfies gradation, anonymity, 1EV and scale independence axioms. Then this relation is given by ($\mathcal{R}$). That is to say, the relation follows from the final score given by the* *formula:*

$$S_F = \frac{P_{\underline{s},L} \times P_{\underline{s},H} \times P_{\underline{s},PV} \times P_{\underline{s},JT} \times P_{\underline{s},DT} \times P_{\underline{s},SP}}{P_{\underline{s},100} \times P_{\underline{s},400} \times P_{\underline{s},110} \times P_{\underline{s},1500}}.$$

Thus if we want to use a ranking satisfying some natural axioms then we necessarily must used a ranking which coincides with the one based on the final score computed by S_F. We remark that this final score is independent of the choice of the parameters A, C, $P_{1,100}$, $P_{1,L}$, $\ldots$, $P_{1,1500}$.
Furthermore, we remark that in the numerator we have performances in centimeters (the greater, the better), and in the denominator performances in seconds (the shorter, the better).

3. Some examples

Now, we give some examples in order to show what kind of results are given by this new ranking. We thus compare the ranking obtained with the current score and with the score S_F.

The values obtained with S_F are very different from the current score. However, if we consider the score $S_{deca} = 7670 \times S_F^{0.23}$, then the current score and S_{deca} have the same order of magnitude. This means that in practice we can use the relation $\succ_{deca}$ previously defined, see Exemple 2.8. We have seen that this ranking coincides with the Nash ranking. Thus the ranking constructed from S_F and S_{deca} are the same and satisfy our four axioms.

In S_{deca} we have used the function $f(x) = 7670x^{0.23}$. The value 7670 and 0.23 have been chosen in order to have a simple relation: If $S_F = 2$ then $S_{deca} = 9000$ and if $S_F = 1.2$ then $S_{deca} = 8000$.
The previous choice is arbitrary. We can use other strictly increasing functions f in order to define a new score $S = f(S_F)$. The obtained ranking will coincide with the Nash ranking and $\succ_{deca}$.

Here, the scores S_F and S_{deca} are rounding with a sufficient precision in order to rank the athletes.

In Table B1, we study the results of the 2017 IAAF World Championships. We give the score obtained by the first eight athletes with the current method and their ranking. In the last three columns we give the score obtained by these athletes with the proposed scores S_F, S_{deca} and the associated ranking.

The score S_F is computed with the time in seconds and the length and height in meters.
For example, K. Mayer's results were:
100 m: 10 s 70; Long Jump: 7.52 m; Shot Put: 15.72 m; High Jump: 2.08 m;
400 m: 48 s 26; 110 m hurdles: 13 s 75; Discus throw: 47.14 m; Pole vault: 5.10 m, Javelin throw: 66.10 m; 1 500m: 4 min 36 s 73.

This gives:

$$S_F = \frac{7.52 \times 15.72 \times 2.08 \times 47.14 \times 5.10 \times 66.10}{10.70 \times 48.26 \times 13.75 \times 276.73} = 1.988679948.$$

In Table B1, we remark that the proposed ranking based on the Nash relation with the formula S_F and the current ranking do not coincide.

In Table B2, we do the same comparison with the last five world records. In this situation we remark that T. Dvořák's result was better than Šebrle and Eaton records if we used the Nash relation. However, Mayer's IAAF world record (16/09/2018) gives also the best result with the Nash relation.

4. Conclusion

In this article we have suggested a method for ranking athletes at the end of a decathlon. Our approach has been divided in several steps. First, associate to each event a fair score. Here, fair means satisfied the second basic principle accepted by IAAF: Results in various events should, as far as possible, yield about the same number of points if the results are comparable as to quality and difficulty.
This has led to a new scoring method using formulas (Score 1) and (Score 2). These formulas extend the work done by Westera. Indeed, the score is computed thanks to a normalized performance. In each event, if a performance corresponds to the same ratio of a chosen high level performance than the number of points obtained is the same. With our method for each events three parameters must be chosen. In Westera approach four was needed.
This proposed new rule is in the same spirit of the current rule but allows to have the same construction for each event.
In a second part, we have proved that there exists a unique ranking method based on scores which satisfies four fairness axioms. Our first two axioms 1EV and anonymity are already satisfied by the current rule. The gradation axiom is satisfied in practice by the current rule and allows to satisfy the ninth basic principle accepted by IAAF. The fourth axiom, scale independence, has been introduced in order to have a ranking independent of the choice of the high level performance. Indeed, it seems not possible to define in a fair way what is a high level reference for each event. As a result we obtain a unique possible ranking method given by the Nash relation.
The advantage of this ranking is that it do not depend on arbitrary coefficients. Thus this method does not depend on old or future performances. This means that it satisfies the seventh basic principle accepted by IAAF.
We remark that the proposed method is the same for men and women because the formula S_F is independent of parameters. However, for each event, formulas (Score 1) and (Score 2) use parameters P_1, A and C. Thus, if we want to give explicitly a score after an event we need to chose these parameters and they will be different for men and women. This means that our approach do not contradict the fifth basic principle accepted by IAAF: there will be a special scoring table for men and another for women.
Thus, if we want to give a score after each event then it is necessary to choose parameters and they will be different for men and women. However, these scores and parameters are not necessary to compute the final score S_F which give the ranking.

At last, the proposed ranking is easy to compute and if we want to satisfy our four natural axioms we have no other possible method.

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Appendix A. Proof of Theorem 2.9.

The first point of the theorem has been already proved, see section 2.5.

Here, we consider the second part of the theorem.
We recall that $\mathbb{R}_+ = [0; +\infty[$ and $\mathbb{R}_+^{10}$ is the set of vectors with 10 coordinates in $\mathbb{R}_+$. We consider a relation $\succ$ defined on $\mathbb{R}_+^{10}$ which satisfies gradation, anonymity, 1EV and scale independence axioms, and we are going to show that $\succ$ coincides with $\underset{Nash}{\succ}$.

We consider $(s_1, s_2, \dots, s_{10})$ and $(t_1, t_2, \dots, t_{10})$.
If $z_{\underline{s}} < z_{\underline{t}}$ then $\underline{s} \succ \underline{t}$, by the gradation axiom. Thus the relation $\succ$ coincides with the Nash relation.

Now we suppose $z_{\underline{s}} = z_{\underline{t}}$. In order to simplify the notation we suppose $z_{\underline{s}} = z_{\underline{t}} = 0$, this means that all scores are positive. We set:

$$(s_1, s_2, \dots, s_{10}) \succ (t_1, t_2, \dots, t_{10})$$

and we want to prove $(s_1, s_2, \dots, s_{10}) \underset{Nash}{\succ} (t_1, t_2, \dots, t_{10})$.

By the scale invariance axiom with $\underline{u} = \left(\frac{1}{s_1}, \frac{1}{s_2}, \dots, \frac{1}{s_9}, \frac{1}{t_{10}}\right)$ we have:

$$(s_1, s_2, \dots, s_{10}) \succ (t_1, t_2, \dots, t_{10}) \iff \left(1, 1, \dots, 1, \frac{s_{10}}{t_{10}}\right) \succ \left(\frac{t_1}{s_1}, \frac{t_2}{s_2}, \dots, \frac{t_9}{s_9}, 1\right).$$

Furthermore, the anonymity axiom gives:

$$\left(1, 1, \dots, 1, \frac{s_{10}}{t_{10}}\right) \succ \left(\frac{t_1}{s_1}, \frac{t_2}{s_2}, \dots, \frac{t_9}{s_9}, 1\right) \iff \left(1, 1, \dots, 1, \frac{s_{10}}{t_{10}}\right) \succ \left(1, \frac{t_1}{s_1}, \frac{t_2}{s_2}, \dots, \frac{t_9}{s_9}\right).$$

Now, the strategy is to obtained more coordinates equal to 1 in the vector $(1, t_1/s_1, \dots, t_9/s_9)$.
The scale independence axiom with $\underline{u} = \left(1, 1, \dots, 1, \frac{s_9}{t_9}\right)$ gives:

$$\left(1, 1, \dots, 1, \frac{s_{10}}{t_{10}}\right) \succ \left(1, \frac{t_1}{s_1}, \frac{t_2}{s_2}, \dots, \frac{t_9}{s_9}\right) \iff \left(1, 1, \dots, 1, \frac{s_9 s_{10}}{t_9 t_{10}}\right) \succ \left(1, \frac{t_1}{s_1}, \dots, \frac{t_8}{s_8}, 1\right).$$

The anonymity axiom gives:

$$\left(1, 1, \dots, 1, \frac{s_9 s_{10}}{t_9 t_{10}}\right) \succ \left(1, \frac{t_1}{s_1}, \dots, \frac{t_8}{s_8}, 1\right) \iff \left(1, 1, \dots, 1, \frac{s_9 s_{10}}{t_9 t_{10}}\right) \succ \left(1, 1, \frac{t_1}{s_1}, \dots, \frac{t_8}{s_8}\right).$$

The scale independence axiom with $\underline{u} = \left(1, 1, \dots, 1, \frac{s_8}{t_8}\right)$ gives:

$$\left(1, 1, \dots, 1, \frac{s_9 s_{10}}{t_9 t_{10}}\right) \succ \left(1, 1, \frac{t_1}{s_1}, \dots, \frac{t_8}{s_8}\right) \iff \left(1, 1, \dots, 1, \frac{s_8 s_9 s_{10}}{t_8 t_9 t_{10}}\right) \succ \left(1, 1, \frac{t_1}{s_1}, \dots, \frac{t_7}{s_7}, 1\right).$$

The anonymity axiom gives:

$$\Big(1,1,\dots,1,\frac{s_8s_9s_{10}}{t_8t_9t_{10}}\Big) \succ \Big(1,1,\frac{t_1}{s_1},\dots,\frac{t_7}{s_7},1\Big) \iff \Big(1,1,\dots,1,\frac{s_8s_9s_{10}}{t_8t_9t_{10}}\Big) \succ \Big(1,1,1,\frac{t_1}{s_1},\dots,\frac{t_7}{s_7}\Big).$$

We continue with the same process and we get:

$$\Big(1,\dots,1,\frac{s_2s_3\cdots s_{10}}{t_2t_3\cdots t_{10}}\Big) \succ \Big(1,\dots,1,\frac{t_1}{s_1}\Big).$$

Now, the 1EV axiom gives:

$$\Big(1,\dots,1,\frac{s_2s_3\cdots s_{10}}{t_2t_3\cdots t_{10}}\Big) \succ \Big(1,\dots,1,\frac{t_1}{s_1}\Big) \iff \frac{s_2s_3\cdots s_{10}}{t_2t_3\cdots t_{10}} > \frac{t_1}{s_1} \iff \prod_{i=1}^{10} s_i > \prod_{i=1}^{10} t_i.$$

In conclusion we have:

$$(s_1,s_2,\dots,s_{10}) \succ (t_1,t_2,\dots,t_{10}) \iff \prod_{i=1}^{10} s_i > \prod_{i=1}^{10} t_i \iff \underline{s} \underset{Nash}{\succ} \underline{t}.$$

This proves Theorem 2.9.

Appendix B. Tables

Name	IAAF Points	Ranking with IAAF Points	Points with S_F	Points with S_{deca}	Ranking with S_F and S_{deca}
K. Mayer	8768	1	1.9886799	8983.9078	1
R. Freimuth	8564	2	1.7237276	8693.2696	2
K. Kazmirek	8488	3	1.5189938	8444.0976	4
J. Õiglane	8371	4	1.5677620	8505.6948	3
D. Warner	8309	5	1.2000365	7998.5283	8
O. Kasyanov	8234	6	1.2744814	8110.0227	7
K. Felix	8227	7	1.3922411	8276.5566	6
Adam Helcelet	8222	8	1.4589213	8366.0931	5

Table B1.: Study of the different rankings for the IAAF World Championships London 2017.

Name	World Record with IAAF Points	Points with S_F	Points with S_{deca}
K. Mayer	9126	2.61256	9565.7884
A. Eaton	9045	2.03207	9028.6263
R. Šebrle	9026	2.29382	9283.7678
T. Dvořák	8994	2.39846	9379.5093
D. O'Brien	8891	2.12567	9122.6191

Table B2.: Study of the different rankings for the world record