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# A LIMIT THEOREM FOR THE SURVIVAL PROBABILITY OF A SIMPLE RANDOM WALK AMONG POWER-LAW RENEWAL TRAPS

JULIEN POISAT AND FRANÇOIS SIMENHAUS

ABSTRACT. We consider a one-dimensional simple random walk surviving among a field of static soft traps : each time it meets a trap the walk is killed with probability  $1 - e^{-\beta}$ , where  $\beta$  is a positive and fixed parameter. The positions of the traps are sampled independently from the walk and according to a renewal process. The increments between consecutive traps, or gaps, are assumed to have a power-law decaying tail with exponent  $\gamma > 0$ . We prove convergence in law for the properly rescaled logarithm of the quenched survival probability as time goes to infinity. The normalization exponent is  $\gamma/(\gamma + 2)$ , while the limiting law writes as a variational formula with both universal and non-universal features. The latter involves (i) a Poisson point process that emerges as the universal scaling limit of the properly rescaled gaps and (ii) a function of the parameter  $\beta$  that we call *asymptotic cost of crossing per trap* and that may, in principle, depend on the details of the gap distribution. Our proof suggests a confinement strategy of the walk in a single large gap. This model may also be seen as a  $(1 + 1)$ -directed polymer among many repulsive interfaces, in which case  $\beta$  corresponds to the strength of repulsion, the survival probability to the partition function and its logarithm to the finite-volume free energy. Along the way we prove a stochastic monotonicity property for the hitting time of the killed random walk with respect to the non-killed one, that could be of interest in other contexts, see Proposition 3.5.

*Disclaimer.* In this paper we denote by  $\mathbb{N}$  the set of positive integers and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . The letter  $C$  is used for the constants whose values are irrelevant and may change from line to line.

## 1. INTRODUCTION AND MODEL

We consider a one-dimensional simple random walk surviving among a random field of static soft traps : each time it meets a trap the walk is killed with probability  $1 - e^{-\beta}$  or survives with probability  $e^{-\beta}$ , where  $\beta$  is a positive and fixed parameter, see Section 1.1 for a precise definition. The increments between consecutive traps, or gaps, are assumed to be mutually independent and independent from the walk, with a power-law decaying tail. This is what we refer to as *power-law renewal traps* in the title, see Section 1.3 for a precise definition. To be more precise, we deal with the *quenched* version of the model, meaning that the positions of the traps are frozen and the survival probability is computed only with respect to the law of the random walk. Our main result, Theorem 2.8, states a convergence in law for the properly rescaled logarithm of the quenched survival probability, seen as a random variable with respect to the field of traps, as time goes to infinity. The limiting law writes as a variational formula involving (i) a Poisson point process that emerges as the

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*Key words and phrases.* Random walks in random traps, polymers in random environments, parabolic Anderson model, survival probability, FKG inequalities, Ray-Knight theorems.

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universal scaling limit of the properly rescaled gaps and (ii) a function of the parameter  $\beta$  that we call *asymptotic cost of crossing per trap* and that may, in principle, depend on the details of the gap distribution, see the definition of  $\lambda(\beta)$  in Proposition 2.2. Even if we offer no path statement for the walk conditioned on surviving, our proof strongly suggests a *confinement strategy* according to which the walk remains in a large gap with an appropriate scale. Path localization could be considered as future work.

As we will see in Section 1.2, our model may also be seen as a  $(1+1)$ -directed polymer among many repulsive interfaces, in which case  $\beta$  corresponds to the strength of repulsion. We will take advantage of this connection by using and revisiting some estimates obtained by Caravenna and Pétrélis [3, 4] in the case of periodic traps. We point out that the logarithm of the survival probability is the finite-volume free energy of the corresponding polymer model.

*Outline.* Section 1 contains the mathematical definition of the model and a discussion on the relation with other models, such as the directed polymer among multiple interfaces. Section 2 contains the statement of Theorem 2.8, which is our main result. At the beginning of the section we introduce several mathematical objects which are necessary for the understanding of the theorem. Comments and related open questions are listed at the end of Section 2. Sections 3 to 6 constitute the proof of the theorem. Key tools are gathered in Section 3. Note that the decoupling inequality stated as Proposition 3.5 therein is quite general and may be useful in other contexts. The rest of the proof is split into a lower bound part (Section 4), an upper bound part (Section 5) and a conclusion (Section 6). The more technical proofs are deferred to an appendix.

**1.1. A random walk in  $\mathbb{Z}$  among soft traps.** We consider  $S = (S_n)_{n \geq 0}$  a simple random walk on  $\mathbb{Z}$  in presence of traps. We recall that the increments  $(S_n - S_{n-1})_{n \geq 1}$  are independent and identically distributed (i.i.d.) random variables which are uniformly distributed on  $\{-1, 1\}$ , and we shall write  $P_x$  for the law of the walk started at  $S_0 = x$ , for  $x \in \mathbb{Z}$ , with the notational simplification  $P_0 = P$ . The positions of the traps are integers and they will be denoted by  $\tau = \{\tau_n\}_{n \geq 0}$  with  $\tau_0 = 0$ . The increments  $(\tau_{n+1} - \tau_n)_{n \geq 0}$  may be referred to as *gaps*.

Let  $\beta > 0$  be a parameter of the model. Informally, each time the walk meets a trap, it is killed with probability  $1 - e^{-\beta}$  or survives with probability  $e^{-\beta}$  independently from the past. The traps are called *soft*, by opposition to *hard* traps, because the walk has a positive chance to survive when sitting on a trap. For a precise mathematical definition let us first introduce  $(\theta_n)_{n \geq 1}$ , the clock process recording the times when the random walk visits  $\tau$ , that is

$$(1.1) \quad \theta_0 = 0, \quad \theta_{n+1} = \inf\{k > \theta_n : S_k \in \tau\}, \quad n \geq 0.$$

We enlarge the probability space so as to include a  $\mathbb{N}$ -valued geometric random variable  $\mathcal{N}$  with success parameter  $1 - e^{-\beta}$ . This plays the role of the clock that kills the walk after its  $\mathcal{N}$ -th meeting with the set of traps. We now define  $\sigma$  the death time of the walk by

$$(1.2) \quad \sigma = \theta_{\mathcal{N}}.$$

Note that our probability law now depends on the parameter  $\beta$ . We shall write  $P_x^\beta$  when we want to stress this dependence or omit the superscript when no confusion is possible. Again we may write  $P^\beta$  instead of  $P_0^\beta$ . We also point out that  $\sigma$  depends on  $\tau$  through  $\theta$  even if it is not explicit in the notations.

The *hitting times* of the walk are defined by

$$(1.3) \quad H_x = \inf\{n \geq 1 : S_n = x\}, \quad x \in \mathbb{Z},$$

and

$$(1.4) \quad H_{\mathbb{Z}^-} = \inf\{n \geq 1 : S_n \leq 0\}.$$

In this paper we study the limiting behaviour of the probability that the walk survives up to time  $n \in \mathbb{N}$ , as  $n$  gets large. For convenience, we consider walks that do not visit  $\mathbb{Z}^- = -\mathbb{N}_0$ . This extra condition allows to consider traps that are indexed by  $\mathbb{N}_0$  instead of  $\mathbb{Z}$  and does not hide anything deep nor change the main idea of the paper. Thus, our *survival probability* writes

$$(1.5) \quad Z_n = \mathbb{P}^\beta(\sigma \wedge H_{\mathbb{Z}^-} > n), \quad n \in \mathbb{N}.$$

We stress again that  $Z_n$  is a function of the environment of traps  $(\tau_n)_{n \geq 0}$ .

**1.2. (De)pinning of a  $(1 + 1)$ -directed polymer by multiple interfaces.** By integrating on  $\mathcal{N}$  in (1.5), we obtain

$$(1.6) \quad Z_n = \mathbb{E} \left[ \exp \left( -\beta \sum_{k=1}^n \mathbf{1}_{\{S_k \in \tau\}} \right) \mathbf{1}_{\{H_{\mathbb{Z}^-} > n\}} \right].$$

This expression links the survival probability to a certain polymer model from statistical mechanics. More precisely, the expression above is the partition function of a  $(1 + 1)$ -directed polymer above an impenetrable wall and among many repulsive interfaces. Here,  $(k, S_k)_{0 \leq k \leq n}$  plays the role of a polymer with  $n$  monomers and the parameter  $n$ , which is initially a time parameter, becomes the size of the polymer, see Figure 1. Whenever the polymer touches one of the interfaces, located at the levels  $\tau = \{\tau_n\}_{n \geq 0}$ , it is penalized by a factor  $e^{-\beta}$ . Finally, the event  $\{H_{\mathbb{Z}^-} > n\}$  reflects the presence of a hard wall at level 0.

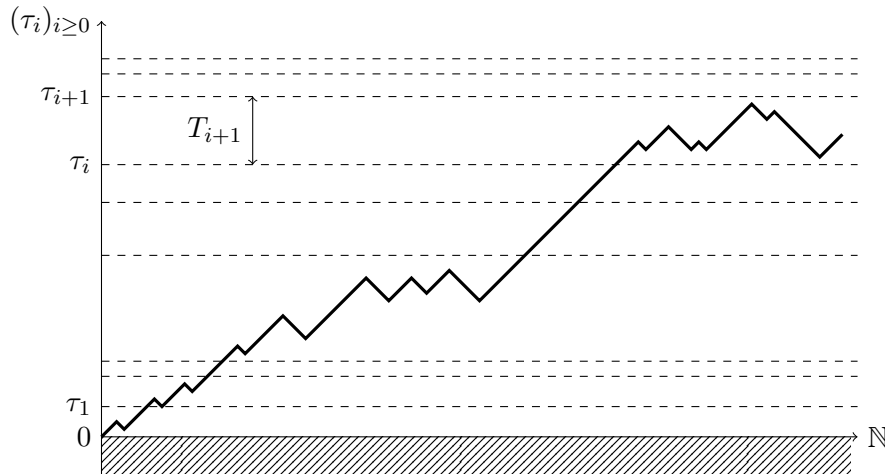


FIGURE 1. Example of a polymer among repulsive interfaces. The dashed lines correspond to the interfaces, the thick one to the polymer and the shaded area to the hard wall.

There is a rich literature about the pinning phenomenon in polymer models, for which we refer to [5, 7, 8]. In general, the parameter  $\beta$  can have any sign (thus  $\beta < 0$  corresponds to the attractive case with our notations) and in the simplest case of a unique interface at

level zero without wall, one observes a localization transition at  $\beta = 0$ . The case  $\beta < 0$  is known as the *repulsive* or *delocalized* phase as the polymer typically touches the interface only finitely many times. The case  $\beta > 0$  is called instead the *attractive* or *localized* phase as the polymer typically touches the interface a positive fraction of time. Caravenna and P  tr  lis studied the case of multiple interfaces periodically located at levels  $t_n\mathbb{Z}$ , where the period  $t_n \geq 1$  depends on the size of the polymer  $n$ , both in the attractive [4] and repulsive case [3]. In contrast, the positions of our interfaces are random, but do not vary in  $n$ . However, the size of the *relevant gaps* does depend on  $n$ , which explains why we use (and extend) some estimates from [3]. Note also the difference between our results, since Caravenna and P  tr  lis obtained results at the level of paths, which means information on the (de)localization of the path of the random walk under the polymer measure

$$(1.7) \quad \frac{d\mathbb{P}_n^\beta}{d\mathbb{P}} \propto \exp\left(-\beta \sum_{k=1}^n \mathbf{1}_{\{S_k \in \tau\}}\right),$$

(where  $\propto$  means ‘‘proportional to’’) while this is not the purpose of the present paper, see Comment 8 in Section 2.4.

**1.3. Assumption on the traps.** We now put a probability measure  $\mathbb{P}$  on the environment of traps. We denote by  $T_k = \tau_k - \tau_{k-1}$ , for  $k \in \mathbb{N}$ , the increments, that is the size of the intervals between two consecutive traps, which we call *gaps*. We assume that, under  $\mathbb{P}$ ,  $\tau$  is a discrete renewal process, that is the  $(T_k)$ ’s are i.i.d.  $\mathbb{N}$ -valued random variables. We further assume that  $\tau_0 = 0$  and that the increments have a power-tail distribution:

$$(1.8) \quad \mathbb{P}(T_1 = n) \sim c_\tau n^{-(1+\gamma)}, \quad \gamma > 0, \quad n \rightarrow \infty,$$

where  $c_\tau$  is a positive constant. We recall the following standard limit theorem, see e.g. Petrov [11, Theorem 14, p. 91].

**Proposition 1.1.** *If  $\gamma \in (0, 1]$ , the sequence  $(\tau_n/n^{1/\gamma})_{n \geq 1}$  converges in law to a (totally skewed to the right)  $\gamma$ -stable random variable with scale parameter  $c_\tau$ . If  $\gamma > 1$  then  $(\tau_n/n)_{n \geq 1}$  converges almost-surely to  $\mathbb{E}(T_1)$ .*

**1.4. Parabolic Anderson model with a correlated potential.** Our model is also connected to a discrete analogue of the parabolic Anderson model (PAM) with potential  $V(x) = -\beta \mathbf{1}_{\{x \in \tau\}} - \infty \mathbf{1}_{\{x \leq 0\}}$ , that is the heat equation with random potential  $V$ ,

$$(1.9) \quad \partial_t u = \Delta u + Vu.$$

There is a rich literature and intense activity around the PAM. We refer to K  nig [10] for a recent survey on this topic. Note that the potential is usually chosen as a sequence of random variables that are independent in the space parameter. In contrast, our potential exhibits long-range spatial correlations, that is one of the research direction suggested in [10, Section 7.2]. For a review of potentials in discrete and continuous space, we refer to [10, Section 1.5].

Let us end this section with a reference to the classical monograph by Sznitman [15] on random motions in random media. Chapter 6 is of particular interest to us as it highlights the link with directed polymers in the presence of columnar defects and introduces the concept of pinning effect of quenched path measures, see again Comment 8 in Section 2.4.

## 2. RESULTS

In this section we first introduce the various tools needed to state our main theorem. We first prove the existence of the asymptotic exponential cost for the walker to cross a new trap without being killed (Section 2.1). Then, in Section 2.2, we consider the environment as a point process and investigate some properties of its asymptotic limit that is a Poisson point process with explicit intensity measure. We finally state our result in Section 2.3 and conclude with a few comments in Section 2.4.

**2.1. Asymptotic cost of crossing traps.** We first recall without proof the following standard proposition:

**Proposition 2.1.** *For all  $t \in \mathbb{N}$ ,*

$$(2.1) \quad \mathbb{P}_1(H_t < H_0) = 1/t.$$

Let us define, for  $\ell \geq 1$ , the random variable (with respect to  $\tau$ )

$$(2.2) \quad \lambda(\ell, \beta) = -\frac{1}{\ell} \log \mathbb{P}^\beta(H_{\tau_\ell} < H_0 \wedge \sigma).$$

Note that in the above definition  $H_0$  could be safely replaced by  $H_{\mathbb{Z}^-}$ . The next proposition gives the existence of an *asymptotic cost of crossing per trap*, which will play a crucial role in our theorem.

**Proposition 2.2.** *For all  $\beta > 0$  there exists a positive constant  $\lambda(\beta) = \lambda(\beta, \mathbb{P})$  such that,  $\mathbb{P}$ -a.s. and in  $L_1(\mathbb{P})$ ,*

$$(2.3) \quad \lim_{\ell \rightarrow \infty} \lambda(\ell, \beta) = \lambda(\beta),$$

with

$$(2.4) \quad 0 \leq \lambda(\beta) - \beta \leq \mathbb{E}(\log T_1) + \log 2.$$

Note that  $\log T_1$  is integrable because of (1.8).

*Proof of Proposition 2.2.* Let us define a collection of random variables indexed by:

$$(2.5) \quad \mathcal{Z}(i, j) = -\log \mathbb{P}_{\tau_i}(H_{\tau_j} < H_{\tau_i} \wedge \sigma), \quad 0 \leq i < j.$$

Thus, we are interested in the limit of  $(\mathcal{Z}(0, \ell)/\ell)_{\ell \geq 1}$ . Let  $1 \leq i < j < k$ . To go from  $\tau_i$  to  $\tau_k$  without being killed, one strategy is to go first from  $\tau_i$  to  $\tau_j$  and survive, then from  $\tau_j$  to  $\tau_k$  and survive without coming back to  $\tau_j$ , which, by the Markov property, leads to the inequality

$$(2.6) \quad \mathcal{Z}(i, k) \leq \mathcal{Z}(i, j) + \mathcal{Z}(j, k).$$

By stationarity of the sequence  $(T_k)_{k \geq 1}$ ,  $\mathcal{Z}(i, j)$  has the same law as  $\mathcal{Z}(0, j - i)$ . Moreover, by Proposition 2.1,

$$(2.7) \quad \mathcal{Z}(i - 1, i) \leq \beta + \log T_i + \log 2, \quad i \in \mathbb{N},$$

and one gets by iterating (2.6)

$$(2.8) \quad \beta \ell \leq \mathcal{Z}(0, \ell) \leq (\beta + \log 2)\ell + \sum_{1 \leq i \leq \ell} \log T_i$$

(the lower bound follows simply from the fact that surviving to a newly visited trap costs at least  $e^{-\beta}$ ). We may now conclude with Kingman's sub-additive ergodic theorem (see Theorem 7.4.1 in [6]) and the law of large numbers.  $\blacksquare$

**Proposition 2.3.** *The function  $\beta \rightarrow \lambda(\beta)$  is continuous on  $\mathbb{R}_+^*$ .*

*Proof of Proposition 2.3.* We prove that the function is concave on  $(0, +\infty)$ : as it is finite, it implies continuity. We observe by integrating over  $\mathcal{N}$  that

$$(2.9) \quad \lambda(\ell, \beta) = -\frac{1}{\ell} \log \mathbf{E} \left( \exp \left( -\beta \sum_{k=1}^{H_{\tau_\ell}} 1_{\{S_k \in \tau\}} \right) 1_{\{H_{\tau_\ell} < H_0\}} \right).$$

A basic interchange theorem allows us to write

$$(2.10) \quad \partial_\beta^2 \lambda(\ell, \beta) = -\frac{1}{\ell} \text{Var}_{\tilde{\mathbb{P}}_\beta} \left( \sum_{k=1}^{H_{\tau_\ell}} 1_{\{S_k \in \tau\}} \right) \leq 0,$$

where  $\tilde{\mathbb{P}}_\beta$  is absolutely continuous with respect to  $\mathbb{P}$ , with Radon-Nikodym derivative:

$$(2.11) \quad \frac{d\tilde{\mathbb{P}}_\beta}{d\mathbb{P}} = \exp \left( -\beta \sum_{k=1}^{H_{\tau_\ell}} 1_{\{S_k \in \tau\}} \right) 1_{\{H_{\tau_\ell} < H_0\}}.$$

We deduce thereby that  $\beta \mapsto \lambda(\ell, \beta)$  is concave. Therefore  $\lambda$  is concave as the almost-sure limit of concave functions.  $\blacksquare$

**2.2. Convergence of the environment.** In this section we recall a few elements of point processes, see [13] for more background on this theory.

Define the quadrant  $E := [0, +\infty) \times (0, +\infty)$  and consider  $\mathcal{E}$  the Borel  $\sigma$ -algebra on  $E$ . We say a measure  $\mu$  on  $(E, \mathcal{E})$  to be a point measure if  $\mu$  can be written

$$(2.12) \quad \mu := \sum_{i=1}^{+\infty} \delta_{z_i}$$

where, for any  $a$  in  $E$ ,  $\delta_a$  denotes the Dirac measure in  $a$  and  $(z_i)_{i \geq 1}$  is a family of points in  $E$ . If  $\mu$  can be written as in (2.12) we say that  $z_i \in \mu$  ( $i \geq 1$ ) even if we should say that  $z_i$  is in the support of  $\mu$ .

We call  $M_p(E)$  the set of all Radon point measures on  $E$ , that are the point measures  $\mu$  such that  $\mu(K) < +\infty$  for all compact sets  $K \subset E$ . We endow  $M_p(E)$  with the  $\sigma$ -algebra  $\mathcal{M}_p(E)$  defined as the smallest  $\sigma$ -algebra that makes applications  $\mu \mapsto \mu(F)$  measurable for all  $F \in \mathcal{E}$ . Let  $C_K^+(E)$  be the set of continuous non-negative functions on  $E$  with compact support. A sequence  $(\mu_n)_{n \geq 1}$  in  $M_p(E)$  is said to converge vaguely to  $\mu$ , which we note  $\mu_n \xrightarrow{v} \mu$ , if for any  $f$  in  $C_K^+(E)$

$$(2.13) \quad \int f \, d\mu_n \rightarrow \int f \, d\mu, \quad n \rightarrow +\infty.$$

This provides a topology on  $M_p(E)$  that turns out to be metrisable, separable, and complete. In this context, a sequence of probability measures  $(P_n)_{n \geq 1}$  on  $\mathcal{M}_p(E)$  converges weakly to  $P$ , which we note  $P_n \xrightarrow{w} P$ , if for every  $\Theta$  vaguely continuous and bounded on  $M_p(E)$ ,

$$(2.14) \quad \int \Theta \, dP_n \rightarrow \int \Theta \, dP, \quad n \rightarrow +\infty.$$

We now come back to our context. For  $n \geq 1$  we define

$$(2.15) \quad \begin{aligned} (X_i^n, Y_i^n) &:= \left( \frac{i-1}{n}, \frac{T_i}{n^{1/\gamma}} \right) \quad \text{for all } i \geq 1, \\ \text{and} \quad \Pi_n &= \sum_{i=1}^{+\infty} \delta_{(X_i^n, Y_i^n)}. \end{aligned}$$

We observe that  $\Pi_n$  is a random variable that takes values in  $M_p(E)$ . Recall the definition of  $c_\tau$  in (1.8) and define  $\Pi$  to be a Poisson point process on  $E$  with intensity

$$(2.16) \quad p := dx \otimes \frac{c_\tau \gamma}{y^{\gamma+1}} dy,$$

that is, for all finite families of disjoint events  $(A_i)_{1 \leq i \leq n} \in \mathcal{E}^n$ ,  $(\Pi(A_i))_{1 \leq i \leq n}$  are independent Poisson random variables with respective parameters  $(p(A_i))_{1 \leq i \leq n}$ .

**Proposition 2.4.** *It holds that  $\Pi_n \xrightarrow{w} \Pi$ .*

*Proof.* The proof follows from [13, Proposition 3.21 p. 154] once noticed that for all  $y > 0$ , the sequence  $(n\mathbb{P}(T_1 > yn^{1/\gamma}))_{n \geq 1}$  converges to  $c_\tau/y^\gamma$  when  $n$  goes to infinity.  $\blacksquare$

We define for any  $\lambda > 0$ ,

$$(2.17) \quad \begin{aligned} \psi^\lambda &: E \rightarrow \mathbb{R}^+ \\ (x, y) &\mapsto \lambda x + \frac{\pi^2}{2y^2}, \end{aligned}$$

and for any  $\mu$  in  $M_p(E)$ ,

$$(2.18) \quad \Psi^\lambda(\mu) := \inf_{(x,y) \in \mu} \psi^\lambda(x, y).$$

We can now define the variable that appears as the limit law in Theorem 2.8,

$$(2.19) \quad F := \Psi^{\lambda(\beta)}(\Pi) = \inf_{(x,y) \in \Pi} \left\{ \lambda(\beta)x + \frac{\pi^2}{2y^2} \right\},$$

where we remind that  $(x, y) \in \Pi$  means that  $(x, y)$  is in the support of  $\Pi$ . We shall write  $F^\beta$  instead of  $F$  when we want to stress the dependence on this parameter.

The rest of the section is devoted to various results relative to the environment  $\Pi$ .

**Proposition 2.5.** *For all  $\lambda > 0$ ,  $\Psi^\lambda(\Pi)$  is almost surely well defined and positive. Moreover the infimum in the definition of  $\Psi^\lambda$  is almost surely achieved at a unique point  $(x^*, y^*)$  in  $E$ .*

*Proof of Proposition 2.5.* As there is almost surely at least one point in  $\Pi$ , the infimum is well defined. Moreover, there are almost surely finitely many points in  $[0, 1] \times [1, +\infty)$ , which implies that  $\bar{Y}$ , the maximum of the second coordinates among these points, is almost surely finite. Thus,  $\psi^\lambda(x, y) \geq \min\{\pi^2/(2\bar{Y}^2), \lambda\}$  for any  $(x, y) \in \Pi$ , which proves that  $\Psi^\lambda(\Pi)$  is almost surely positive.

For  $\lambda > 0$  and  $u > 0$ , we introduce the set

$$(2.20) \quad A_u^\lambda := \{(x, y) \in E \text{ such that } \psi^\lambda(x, y) \leq u\},$$

and note that  $\Pi(A_u^\lambda) < +\infty$  almost surely.

Let us denote by  $C$  the event that the minimum in the definition of  $\Psi$  is achieved. We observe that for any  $u > 0$ ,

$$(2.21) \quad P(\Psi(\Pi) < u) = P(\Psi(\Pi) < u, \Pi(A_u^\lambda) < +\infty) \leq P(\Psi(\Pi) < u, \Pi \in C).$$



As  $\lim_{u \rightarrow +\infty} P(\Psi(\Pi) < u) = 1$  and  $\lim_{u \rightarrow +\infty} P(\Psi(\Pi) < u, \Pi \in C) = P(\Pi \in C)$ , we obtain that  $P(\Pi \in C) = 1$ .

It remains to prove that the infimum is almost surely achieved at only one point. For  $u > 0$ , let  $D_u$  be the event that two points of  $\Pi \cap A_u^\lambda$  have the same image by  $\psi^\lambda$  and  $D$  be the event that two points of  $\Pi$  have the same image by  $\psi^\lambda$ . For  $n \in \mathbb{N}$  and conditional on  $\{\Pi(A_u^\lambda) = n\}$ , the restriction of  $\Pi$  to  $A_u^\lambda$  has the same law as  $\sum_{i=1}^n \delta_{X_i}$ , where the  $(X_i)_{1 \leq i \leq n}$ 's are i.i.d. with continuous law  $p$  restricted to  $A_u^\lambda$  and renormalised to a probability measure. This implies that  $P(\Pi \in D_u | \Pi(A_u^\lambda)) = 0$  and thus  $P(\Pi \in D_u) = 0$ . We obtain

$$(2.22) \quad P(\Psi(\Pi) < u) = P(\Psi(\Pi) < u, \Pi \in D_u^c).$$

The first term converges to 1 while the second one converges to  $P(\Pi \in D)$  when  $u$  goes to infinity. This proves  $P(\Pi \in D) = 1$  and concludes the proof.  $\blacksquare$

Unfortunately, it is difficult to work directly with  $\Psi^\lambda$  as it appears not to be a vaguely continuous function on  $M_p(E)$ . For this reason, we introduce the function  $\Psi_K^\lambda$ , for any compact set  $K \subset E$ , defined by

$$\Psi_K^\lambda(\mu) := \inf_{(x,y) \in \mu \cap K} \psi^\lambda(x, y), \quad \mu \in M_p(E).$$

The advantage of restricting the infimum to a compact set lies in the following lemma.

**Lemma 2.6.** *For any compact set  $K \subset E$  and  $\lambda > 0$ , the function  $\Psi_K^\lambda$  is vaguely continuous on  $M_p(E)$ .*

*Proof.* Consider a sequence  $(\mu_n)_{n \geq 1}$  in  $M_p(E)$  that converges vaguely to  $\mu$ . Due to Proposition 3.14 in [13],  $\mu \in M_p(E)$ . Suppose  $\mu(K) = k$ , that is  $\mu(\cdot \cap K)$  writes  $\sum_{i=1}^k \delta_{(x_i, y_i)}$  where  $(x_i, y_i)_{1 \leq i \leq k}$  is a family in  $K$ . By Proposition 3.13 in [13] there exists for all  $n$  larger than some  $n(K)$  a family  $(x_i^n, y_i^n)_{1 \leq i \leq k}$  such that  $\mu_n(\cdot \cap K) = \sum_{i=1}^k \delta_{(x_i^n, y_i^n)}$ . Moreover, for all  $1 \leq i \leq k$ , the sequence  $(x_i^n, y_i^n)_{n \geq 1}$  converges to  $(x_i, y_i)$  as  $n$  goes to infinity. This implies that  $\Psi_K^\lambda(\mu_n)$  converges to  $\Psi_K^\lambda(\mu)$ .  $\blacksquare$

We conclude this section with the following technical lemma:

**Lemma 2.7.** *The family  $(F^{\beta-\varepsilon})_{\varepsilon \geq 0}$  (seen as functions of the random measure  $\Pi$ ) converges non-decreasingly to  $F^\beta$  when  $\varepsilon \rightarrow 0$ , almost surely.*

*Proof.* Using Proposition 2.5 we can define almost surely a random point  $(X^*(\beta), Y^*(\beta))$  such that

$$(2.23) \quad \psi^{\lambda(\beta)}(X^*(\beta), Y^*(\beta)) = \Psi^{\lambda(\beta)}(\Pi).$$

We first prove that  $(X^*(\beta - \varepsilon), Y^*(\beta - \varepsilon)) = (X^*(\beta), Y^*(\beta))$  almost surely if  $\varepsilon > 0$  is chosen small enough. Let  $\varepsilon_0 \in (0, \beta)$ . Consider some  $x \geq x_0 := \frac{2\Psi^{\lambda(\beta)}(\Pi)}{\lambda(\beta - \varepsilon_0)}$  and any  $y > 0$ . As  $\lambda(\cdot)$  and  $\Psi^{\lambda(\cdot)}$  are non-decreasing functions of  $\beta$  it holds that

$$(2.24) \quad \psi^{\lambda(\beta - \varepsilon)}(x, y) \geq \lambda(\beta - \varepsilon)x \geq 2\Psi^{\lambda(\beta - \varepsilon)}(\Pi)$$

for all  $\varepsilon < \varepsilon_0$ , and we conclude that  $X^*(\beta - \varepsilon) < x_0$  for any  $\varepsilon < \varepsilon_0$ .

Consider now some  $y \leq y_0 := \frac{\pi}{2\sqrt{\Psi^{\lambda(\beta)}(\Pi)}}$  and any  $x > 0$ . We thus obtain

$$(2.25) \quad \psi^{\lambda(\beta - \varepsilon)}(x, y) \geq \frac{\pi^2}{2y^2} \geq 2\Psi^{\lambda(\beta - \varepsilon)}(\Pi),$$

and we conclude that  $Y^*(\beta - \varepsilon) > y_0$  for any  $\varepsilon < \varepsilon_0$ . We deduce thereof that almost surely

$$(2.26) \quad F^{\beta-\varepsilon} = \inf_{(x,y) \in \Pi \cap \{x < x_0, y > y_0\}} \psi^{\lambda(\beta-\varepsilon)}(x, y).$$

Finally, observe that  $\Pi(x < x_0, y > y_0) < +\infty$  almost surely, so that there are only finitely many candidates for  $(X^*(\beta - \varepsilon), Y^*(\beta - \varepsilon))$ . Pick  $(X, Y) \in \Pi \cap \{x < x_0, y > y_0\}$  that is not  $(X^*(\beta), Y^*(\beta))$  (if there is no such point there is nothing more to prove as  $(X^*(\beta), Y^*(\beta))$  is then the only candidate). The function  $\varepsilon \mapsto \lambda(\beta - \varepsilon)X + \frac{\pi^2}{2Y^2}$  is (i) strictly larger than  $\lambda(\beta)X^*(\beta) + \frac{\pi^2}{2(Y^*(\beta))^2}$  at  $\varepsilon = 0$  due to Proposition 2.5 and (ii) continuous due to Proposition 2.3. Therefore,

$$(2.27) \quad \lambda(\beta - \varepsilon)X + \frac{\pi^2}{2Y^2} > \lambda(\beta)X^*(\beta) + \frac{\pi^2}{2(Y^*(\beta))^2}$$

for  $\varepsilon > 0$  small enough. As  $\Pi(\{x < x_0, y > y_0\}) < +\infty$  almost surely, we can choose  $\varepsilon$  small enough so that the last inequality holds for all points in  $\Pi \cap \{x < x_0, y > y_0\}$ . Therefore, we may from now on consider  $\varepsilon > 0$  small enough so that  $(X^*(\beta - \varepsilon), Y^*(\beta - \varepsilon)) = (X^*(\beta), Y^*(\beta))$ .

From what precedes, we get

$$(2.28) \quad |\Psi^{\lambda(\beta-\varepsilon)}(\Pi) - \Psi^{\lambda(\beta)}(\Pi)| \leq |\lambda(\beta - \varepsilon) - \lambda(\beta)|X^*(\beta).$$

Again, as  $\lambda$  is continuous,  $\Psi^{\lambda(\beta-\varepsilon)}(\Pi)$  converges almost surely to  $\Psi^{\lambda(\beta)}(\Pi)$  when  $\varepsilon$  goes to 0. Finally,

$$(2.29) \quad F^{\beta-\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} F^\beta \quad a.s.$$

Moreover, as  $\lambda$  is non-decreasing with  $\beta$ , the convergence is monotone:

$$(2.30) \quad F^{\beta-\varepsilon} \nearrow_{\varepsilon \rightarrow 0} F^\beta \quad a.s.$$

■

**2.3. Statement of the result.** From now on we set

$$(2.31) \quad N = N(n) = n^{\frac{\gamma}{\gamma+2}}, \quad F_n = -\frac{1}{N} \log Z_n, \quad n \geq 1.$$

Let us first explain at a heuristic level the choice of this normalization. The argument is of the type one uses to find volume exponents in some polymer models and is sometimes referred to as a *Flory argument*. We assume that at a large time  $n$  the walk has visited at most  $N$  traps and has remained confined in the largest visible gap, and we find the value of  $N$  with the best energy-entropy balance. By basic extreme-value theory, the size of that gap is of order  $N^{1/\gamma}$ , and by a standard small-ball estimate (see Proposition 3.1 below for a precise statement) the entropic cost of being confined in that gap during time  $n$  is of order  $nN^{-2/\gamma}$ . Also, the cost of crossing  $N$  traps should be roughly of order  $N$ , see Proposition 2.2 below for a rigorous proposition. Finally, by equating these two costs, one finds the optimal choice  $N = n^{\frac{\gamma}{\gamma+2}}$ . As a consequence, the walk has travelled a distance of order  $n^{\frac{1+\gamma}{\gamma+2}}$  from the origin (see Proposition 1.1). This *confinement strategy* will be justified during the proof of our theorem, which we now state:

**Theorem 2.8.** *The sequence of random variables  $(F_n)_{n \geq 1}$  converges in  $\mathbb{P}$ -distribution to the random variable*

$$(2.32) \quad F := \Psi^{\lambda(\beta)}(\Pi) = \inf_{(x,y) \in \Pi} \left\{ \lambda(\beta)x + \frac{\pi^2}{2y^2} \right\},$$

where  $\Pi$  is a Poisson point process on  $\mathbb{R}^+ \times \mathbb{R}_*^+$  with intensity  $p = dx \otimes c_\tau \gamma y^{-(1+\gamma)} dy$ .

The rest of the article is devoted to the proof of this result. The general strategy is the following. In Section 3 we introduce the notion of *good environments*, that are environments for which we can provide suitable bounds on the survival probability  $Z_n$ , see (3.60). In Sections 4 and 5 we successively prove lower and upper bounds, assuming only that the environment is good. We complete the proof in Section 6: first we show that we can adjust parameters so that environments are asymptotically good, with a probability arbitrarily close to one; then we use the two bounds obtained in the previous sections to relate the quantity of interest to an explicit functional  $\Psi$  of the properly renormalized environment, see (2.18). Finally we let  $n$  tend to infinity to prove that  $F$  has the same law as  $\Psi$  applied to the limiting environment, that is a Poisson point process (see Proposition 2.4).

**2.4. Comments.** We collect here a number of comments and remarks about our result.

**1.** We do not consider the annealed survival probability, which decreases at most polynomially fast since

$$(2.33) \quad \mathbb{E}\mathbb{P}(\sigma \wedge H_{\mathbb{Z}^-} > n) \geq \frac{1}{2} \mathbb{P}_1(H_0 \geq n) \mathbb{P}(\tau_1 > n) \sim \frac{1}{2} c_\tau n^{-\gamma-\frac{1}{2}},$$

as  $n \rightarrow \infty$ , and thus has a completely different behaviour than the quenched probability.

**2.** Note that we cannot hope for better than weak convergence. Indeed, if  $F_n$  would converge to  $F$  almost-surely, then  $F$  would be measurable with respect to the tail  $\sigma$ -algebra of the family  $(T_i)_{i \geq 1}$ . As the increments are independent, the latter is trivial due to the 0 – 1 law, and  $F$  would be deterministic.

**3.** In the case  $\gamma \leq 1$ , the variational formula in (2.32) admits an alternative representation in terms of a subordinator, which reads

$$(2.34) \quad F = \inf_{t \geq 0} \left\{ \lambda(\beta)t + \frac{\pi^2}{2(\Delta \mathcal{S}_t)^2} \right\},$$

where  $(\mathcal{S}_t)_{t \geq 0}$  is a  $\gamma$ -stable subordinator and  $\Delta \mathcal{S}_t = \mathcal{S}_t - \mathcal{S}_t^- = \mathcal{S}_t - \lim_{u \rightarrow t^-} \mathcal{S}_u$ .

**4.** We can compute explicitly the tail distribution function of the limiting law  $F = \Psi^{\lambda(\beta)}(\Pi)$  in Theorem 2.8. Recall (2.16), (2.17) and (2.20). For any  $u \geq 0$  (we write  $\lambda$  instead of  $\lambda(\beta)$  to lighten notations),

$$(2.35) \quad \mathbb{P}(F \geq u) = \mathbb{P}(\Pi(A_u^\lambda) = 0) = \exp(-p(A_u^\lambda)).$$

Since  $A_u^\lambda = \left\{ (x, y) : 0 \leq x < u/\lambda, y \geq \frac{\pi}{\sqrt{2(u-\lambda x)}} \right\}$ , we get by a straightforward computation that

$$(2.36) \quad \mathbb{P}(F \geq u) = \exp\left(-\frac{c_\tau}{\lambda(\beta)\pi^\gamma(\gamma+2)}(2u)^{\frac{\gamma}{2}+1}\right), \quad u \geq 0.$$

**5.** The case  $\gamma = 0$  is left open. In this case, a gap distribution of the form (1.8) is no longer appropriate and one should instead assume that  $\mathbb{P}(T_1 = n) \sim L(n)/n$ , where  $L$  is a non-negative slowly varying function such that  $\sum L(n)/n$  is finite. Complications may arise at two levels : (i) the normalization of  $\max_{1 \leq i \leq n} T_i$ , that we use to guess the value of  $N$ , and (ii) the integrability of  $\log T_1$ , that we use in Proposition 2.2. For instance, if  $L(n) = (\log n)^{-2}$  then  $\mathbb{E}(\log T_1) = \infty$  and  $\max_{1 \leq i \leq n} T_i$  has a completely different type

of renormalization since, as one can readily show,  $(1/\sqrt{n}) \log \max_{1 \leq i \leq n} T_i$  converges to a non-trivial probability law with cumulative distribution function  $x \mapsto \exp(-x^{-2}) \mathbf{1}_{\{x > 0\}}$ , as  $n \rightarrow \infty$ .

**6.** We state without proof an alternative expression for  $\lambda(\beta)$  based on ergodic theory considerations. To this end, let  $\tilde{\tau}$  be an independent copy of  $\tau$ , as defined in Section 1.3. Suppose that the random walk is now free to visit  $\mathbb{Z}^-$  but is killed by the set  $-\tilde{\tau}$  (note the minus sign), with the same probability  $1 - \exp(-\beta)$ , and denote by  $\tilde{\sigma}$  the corresponding killing time. Then,

$$(2.37) \quad \lambda(\beta) = -\mathbb{E}\tilde{\mathbb{E}} \log P^\beta(H_{\tau_1} < \tilde{\sigma}).$$

Assuming this last equality, we could readily prove using the dominated convergence theorem that  $\lambda$  is also continuous at 0.

**7.** Equation (2.4) does not give much information about the behaviour of  $\lambda(\beta)$  at 0, that remains an open question. We expect however that  $\beta = o(\lambda(\beta))$  as  $\beta \rightarrow 0$  and we now explain why. To this end, recall (2.37) and the related notations above. By integrating over  $\mathcal{N}$  and differentiating in  $\beta$  we obtain

$$(2.38) \quad \lim_{\beta \rightarrow 0} \lambda'(\beta) = \tilde{\mathbb{E}}\mathbb{E}\mathbb{E} \left( \sum_{k=1}^{H_{\tau_1}} \mathbf{1}_{\{S_k \in -\tilde{\tau}\}} \right),$$

that we expect to be infinite. Indeed, by first restricting the walk to make its first step to the left and then using the symmetry of the random walk,

$$(2.39) \quad \tilde{\mathbb{E}}\mathbb{E}\mathbb{E} \left( \sum_{k=1}^{H_{\tau_1}} \mathbf{1}_{\{S_k \in -\tilde{\tau}\}} \right) \geq \frac{1}{2} \tilde{\mathbb{E}}\mathbb{E}_{-1} \left( \sum_{k=1}^{H_0} \mathbf{1}_{\{S_k \in -\tilde{\tau}\}} \right) = \frac{1}{2} \tilde{\mathbb{E}}\mathbb{E}_1 \left( \sum_{k=1}^{H_0} \mathbf{1}_{\{S_k \in \tilde{\tau}\}} \right).$$

We now interchange integrals and use the renewal theorem to obtain, at least for  $\gamma \neq 1$ ,

$$(2.40) \quad \lim_{\beta \rightarrow 0} \lambda'(\beta) \geq \frac{1}{2} \mathbb{E}_1 \left( \sum_{k=1}^{H_0} \tilde{\mathbb{P}}(S_k \in \tilde{\tau}) \right) \geq \frac{C}{2} \mathbb{E}_1 \left( \sum_{k=1}^{H_0} (1 + S_k)^{(\gamma-1) \wedge 0} \right).$$

Since, by Proposition B.2, the mean number of visits to  $x \in \mathbb{N}_0$  between time 1 and  $H_0$  equals 1 under  $\mathbb{P}_1$ , we get

$$(2.41) \quad \lim_{\beta \rightarrow 0} \lambda'(\beta) \geq C \sum_{x \geq 0} (1+x)^{(\gamma-1) \wedge 0} = \infty.$$

**8.** Note that we offer no path statement. In other words, we do not prove anything about the behaviour of the walk conditioned to survive for a long time  $n$ . However, as it is often the case with this type of model, our result and the method of proof suggest a path strategy, which in our context corresponds to a confinement (or localization) strategy. To be more precise, we roughly expect that as  $n$  is large, the walk reaches the trap labelled  $X^*(\beta)N$  and then remains in the corresponding gap, of size  $Y^*(\beta)N^{1/\gamma}$ , where  $(X^*(\beta), Y^*(\beta))$  is distributed as the unique minimizer of the random variational problem in Theorem 2.8. In other words, the path (or polymer) gets stuck in a slab which is *large enough* and *not too far from the origin*. Surprisingly, the repulsive interaction between the path and the traps leads to a pinning effect on the quenched path measures, as explained by Sznitman [15, Chapter 6]. Proving such a result should demand substantial additional work, as one would most likely need sharper bounds on the survival probability (partition function) and fine

controls on the ratio of survival probabilities restricted to suitable events. Nevertheless, this can be considered as an interesting direction of research.

**9.** Let us stress that the scaling  $t^{\gamma/(\gamma+2)}$  that appears in our Theorem 2.8 is different from the scaling of the PAM in a bounded i.i.d. potential. In this case [10, Example 5.10] states that the correct scaling is  $t$  up to a logarithmic correction. Hence we are in a case where the correlations of the potential have a drastic effect on the asymptotic behaviour of the survival probability.

### 3. KEY TOOLS

In this section we introduce several tools which we will use to prove our result. For convenience, various notations are gathered together in Section 3.1, to which the reader may refer. In Section 3.2 we remind the reader of the so-called small ball estimate and establish a rough upper bound on the probability that the walker stays confined in a fixed gap until time  $n$ , see Proposition 3.4. Section 3.3 contains Proposition 3.5, which states that a walk conditioned to hit a given point  $x$  before coming back to 0 does it faster when it is also conditioned on survival until its first visit to  $x$ . In Section 3.4 we state the two technical Lemmas 3.6 and 3.7 that we will use in Section 5 while proving the upper bound on  $Z_n$ . Finally we introduce the key notion of *good environment* in Section 3.5. Informally, *good environments* are those for which we are able to efficiently bound  $Z_n$ . We thus give a list of events, see (3.60), that are essentially the technical conditions we will need in Proposition 5.1 and 4.1.

**3.1. Notations.** Let us introduce notations that will be necessary in what comes next.

*Killing clock.* We recall the definition of the  $\mathbb{N}$ -valued geometric random variable  $\mathcal{N}$  with success parameter  $1 - e^{-\beta}$  that plays the role of the killing clock.

*Records.* As we already hinted in the heuristics, only the *largest* gaps matter. To be more precise, a particular attention is given to record values of the sequence  $(T_\ell)$ . Therefore, we let

$$(3.1) \quad i(0) = 0, \quad i(k) = \inf\{i > i(k-1) : T_{i+1} > T_{i(k-1)+1}\}, \quad k \geq 1,$$

be the sequence of record indexes, while

$$(3.2) \quad \tau_k^* = \tau_{i(k)} \quad \text{and} \quad T_k^* = T_{i(k)+1}, \quad k \geq 0.$$

We also define

$$(3.3) \quad R(a, b) = \{k \geq 1 : a \leq i(k) \leq b\}, \quad \mathcal{R}(a, b) = i(R(a, b)), \quad a, b \in \mathbb{N}, \quad a < b,$$

and

$$(3.4) \quad R_\varepsilon(n) = R(\varepsilon N, \varepsilon^{-1} N), \quad \mathcal{R}_\varepsilon(n) = \mathcal{R}(\varepsilon N, \varepsilon^{-1} N), \quad n \in \mathbb{N}, \quad \varepsilon > 0.$$

Finally we write

$$(3.5) \quad \mathcal{R} = \mathcal{R}(1, +\infty),$$

for the set of all records.

*Auxiliary random walk.* We remind that the clock process  $(\theta_n)_{n \geq 1}$  is defined by

$$(3.6) \quad \theta_0 = 0, \quad \theta_{n+1} = \inf\{k > \theta_n : S_k \in \tau\}, \quad n \geq 0.$$

The process that chronologically keeps track of the traps visited by the walk will be denoted by  $X = (X_n)_{n \geq 0}$  and is uniquely determined by  $\tau_{X_n} = S_{\theta_n}$ . It is not difficult to see that  $X$  is a Markov chain on  $\mathbb{N}_0$ , the hitting times of which are denoted by

$$(3.7) \quad \zeta_x = \inf\{n \geq 1: X_n = x\}, \quad x \in \mathbb{N}_0,$$

and

$$(3.8) \quad \zeta_k^* = \inf\{n \geq 1: X_n = i(k)\}, \quad x \in \mathbb{N}_0.$$

*Transition matrices and their moment-generating functions.* Let us define

$$(3.9) \quad q_{ij}(n) = \mathbb{P}_{\tau_i}(S_k \notin \tau, 1 \leq k < n, S_n = \tau_j), \quad i, j \in \mathbb{N}_0, \quad n \geq 1,$$

and the associated family of matrices  $\{Q(\phi)\}_{\phi \geq 0}$  by

$$(3.10) \quad Q_{ij}(\phi) = \sum_{n \geq 1} e^{\phi n} q_{ij}(n) = \mathbb{E}_{\tau_i} \left( e^{\phi \theta_1} \mathbf{1}_{\{S_{\theta_1} = \tau_j\}} \right), \quad i, j \in \mathbb{N}_0, \quad \phi \geq 0.$$

Note that the matrix  $\{Q_{ij}(0)\}_{i, j \geq 0}$  is nothing but the transition matrix of the Markov chain  $X$  defined above. These quantities will appear in Lemma 3.6 below and are zero as soon as  $|j - i| > 1$ . Finally, we will also use the following notations for the gap associated to a non-oriented edge  $\{i, j\}$  with  $|j - i| \leq 1$ :

$$(3.11) \quad t_{ij} = \begin{cases} t_{i+1} & \text{if } j = i + 1, \\ t_i & \text{if } j = i - 1, \\ t_{i+1} \vee t_i & \text{if } i = j, \end{cases}$$

where  $(t_i)$  is a sequence of integers.

**3.2. Confinement estimates.** One of the key standard estimates in our arguments are the so-called *small-ball estimates*, that control the probability that a simple random walk stays confined in an interval:

**Proposition 3.1.** *There exist  $t_0, c_1, c_2, c_3, c_4 > 0$  such that for all  $t > t_0$ , the following inequalities hold for all  $n \geq 1$  such that  $n \in 2\mathbb{N}$  or  $n - t \in 2\mathbb{N}$ :*

$$(3.12) \quad \frac{c_1}{t \wedge n^{1/2}} e^{-g(t)n} \leq \mathbb{P}(H_t \wedge H_0 \wedge H_{-t} > n) \leq \frac{c_2}{t \wedge n^{1/2}} e^{-g(t)n}$$

$$(3.13) \quad \frac{c_3}{t^3 \wedge n^{3/2}} e^{-g(t)n} \leq \mathbb{P}(H_t \wedge H_0 \wedge H_{-t} = n) \leq \frac{c_4}{t^3 \wedge n^{3/2}} e^{-g(t)n},$$

where

$$(3.14) \quad g(t) = -\log \cos\left(\frac{\pi}{t}\right) = \frac{\pi^2}{2t^2} + O\left(\frac{1}{t^4}\right), \quad t \rightarrow +\infty.$$

This proposition is taken from Lemma 2.1 in Caravenna and P  tr  lis [3]. A crucial point here is the uniformity of the constants, which gives the uniformity of the constant  $C$  in Proposition 3.4.

Caravenna and P  tr  lis [3] treated the case of equally spaced traps, which we refer to as the *homogeneous* case, in the sense that increments of  $\tau$  are all equal. We summarize their results here.

**Proposition 3.2** (Homogeneous case, see Eq. (2.1)-(2.3) in [3]). *Let  $t \in \mathbb{N}$  and  $\tau = t\mathbb{Z}$ . There exists a constant  $\phi(\beta, t)$  such that*

$$(3.15) \quad \phi(\beta, t) = - \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\sigma > n),$$

with

$$(3.16) \quad \phi(\beta, t) = \frac{\pi^2}{2t^2} \left( 1 - \frac{4}{e^\beta - 1} \frac{1}{t} + o\left(\frac{1}{t}\right) \right).$$

Moreover, it is the only solution of the equation:

$$(3.17) \quad \mathbb{E}(\exp(\phi \inf\{n \geq 1: S_n \in \tau\})) = \exp(\beta), \quad \beta \geq 0.$$

Note that the first order term in the expansion of  $\phi$  does not depend on  $\beta$ . It turns out that we are able to extend this proposition, at the price of additional technical work, to deal with the *weakly-inhomogeneous* case, that is when increments of  $\tau$  follow a periodic pattern. We obtain the following:

**Proposition 3.3** (Weakly-inhomogeneous case). *Let  $p \geq 2$ ,  $t_1, \dots, t_p$  positive integers and  $\tau$  be the periodic set  $\{\tau_i: 0 \leq i < p\} + \tau_p\mathbb{Z}$ , where  $\tau_0 = 0$  and  $\tau_i = \sum_{1 \leq j \leq i} t_j$  for all  $0 < i < p$ . There exists a constant  $\phi = \phi(\beta; t_1, \dots, t_p)$  such that*

$$(3.18) \quad \phi(\beta; t_1, \dots, t_p) = - \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\sigma > n).$$

Moreover,

$$(3.19) \quad \mathbb{P}(\sigma > n) \leq Cn^2 p \exp(-\phi(\beta; t_1, \dots, t_p)n), \quad n \geq 1,$$

and

$$(3.20) \quad \phi(\beta, t_{\max}) \leq \phi(\beta; t_1, \dots, t_p) < g(t_{\max}), \quad t_{\max} = \max_{1 \leq i \leq p} t_i.$$

The proof is deferred to Appendix A. Remark that both inequalities in (3.20) are intuitive: the first one asserts that it is easier to survive in a homogeneous environment with gap  $t_{\max}$  than in the original environment. The second one states that one of the strategy to survive is to stay confined in the largest gap. With this estimate in hand, we get our first building block, that is an upper bound on the probability to survive in-between two traps, for a general environment  $\tau$ .

**Proposition 3.4.** *There exists a constant  $C > 0$  such that for all  $0 \leq k < r < \ell$ , one has*

$$(3.21) \quad \mathbb{P}_{\tau_r}(\sigma \wedge H_{\tau_k} \wedge H_{\tau_\ell} > n) \leq Cn^2(\ell - k) \exp(-\phi(\beta; \max\{t_i: k < i \leq \ell\})n),$$

where  $\phi(\beta; \cdot)$  is defined in Proposition 3.2.

*Proof of Proposition 3.4.* The proof relies on periodization. Since the random walk does not leave the interval  $(\tau_k, \tau_\ell)$  on the event considered, we may as well replace the renewal  $\tau$  by a periodized version, and by translation invariance, consider that the random walk starts at zero. Finally, everything is as if the walk evolves in a new environment  $\tilde{\tau}$ , with periodic increments, defined by

$$(3.22) \quad \tilde{\tau} = \{\tau_i: k \leq i \leq \ell\} - \tau_r + (\tau_\ell - \tau_k)\mathbb{Z},$$

and we now have to bound from above  $\mathbb{P}^{\tilde{\tau}}(\sigma \wedge H_{\tilde{\tau}_{k-r}} \wedge H_{\tilde{\tau}_{\ell-r}} > n)$ , where we put a superscript on  $\mathbb{P}$  to stress that the walk evolves among  $\tilde{\tau}$ . This probability is certainly smaller than  $\mathbb{P}_0^{\tilde{\tau}}(\sigma > n)$ , and we may now conclude thanks to Proposition 3.3.  $\blacksquare$



**3.3. A decoupling inequality.** The next building block is a control on the probability that the walk reaches a given point before a certain fixed time, conditional on survival and not coming back to 0. In essence, the following proposition tells us that the walk reaches this point *stochastically faster* in the presence of traps:

**Proposition 3.5.** *For all  $\beta > 0$ ,  $x \in \mathbb{N}$  and  $n \in \mathbb{N}$ ,*

$$(3.23) \quad \mathbb{P}^\beta(H_x \leq n \mid \sigma \wedge H_0 > H_x) \geq \mathbb{P}(H_x \leq n \mid H_0 > H_x).$$

Let us stress that this proposition is general, as it does not depend on the position of the traps. We defer its proof to Appendix B.

**3.4. Two technical lemmas.** Recall the notations in (3.10) and (3.11).

**Lemma 3.6.** *Define the function  $f: z \in (0, \pi) \mapsto z/\sin(z)$ . Let  $n \in \mathbb{N}$ . For  $\varepsilon > 0$  small enough, there exist  $\alpha = \alpha(\varepsilon) > 0$ ,  $C > 0$  and  $T_0(\varepsilon) \in \mathbb{N}$  such that for  $T > T_0(\varepsilon) \vee \max_{0 \leq x \leq n} T_x$  and  $\phi = \frac{\pi^2}{2T^2}$ ,*

$$(3.24) \quad \frac{Q_{x,y}(\phi)}{Q_{x,y}(0)} \leq \begin{cases} \exp(\varepsilon) & \text{if } x \neq y \text{ and } T_{xy} \leq \alpha T, \text{ or } x = y \\ 2f\left(\pi \frac{\max_{0 \leq x \leq n} T_x}{T} \left(1 + \frac{C}{T^2}\right)\right) & \text{else.} \end{cases}$$

The ratio in (3.24) is the positive Laplace transform of the hitting time of  $\tau$  for the walk conditioned to go from  $\tau_x$  to  $\tau_y$ . As we will consider large values of  $T$  and thus small values of  $\phi$ , Lemma 3.6 can be understood as a control of this transform near 0.

*Proof of Lemma 3.6.* We consider  $\varepsilon > 0$  small enough (it will appear in the following how small it has to be). Let us start with the case  $x \neq y$ . From the explicit expressions of the Laplace transforms, see e.g. (A.5) in Caravenna and P  tr  lis [4], we get

$$(3.25) \quad Q_{x,y}(\phi) = \frac{\tan \Delta}{2 \sin(T_{xy} \Delta)}, \quad \text{where } \Delta = \Delta(\phi) = \arctan(\sqrt{e^{2\phi} - 1}),$$

and we note that

$$(3.26) \quad Q_{x,y}(0) = \frac{1}{2T_{xy}}.$$

Let us notice that (3.25) is well-defined if  $T_{xy} \Delta < \pi$ , which occurs as soon as  $T$  is large enough. Indeed, by expanding  $\Delta$  we see that there exists a constant  $C > 0$  and  $T_1 \in \mathbb{N}$  such that for  $T \geq T_1$ ,

$$(3.27) \quad \Delta \leq \frac{\pi}{T} \left(1 + \frac{C}{T^2}\right).$$

If we assume moreover that  $T > T_{xy}$ , we obtain, as  $T$  and  $T_{xy}$  are integers,

$$(3.28) \quad \frac{T_{xy} \Delta}{\pi} \leq \frac{T-1}{T} \left(1 + \frac{C}{T^2}\right) = 1 - \frac{1+o(1)}{T} < \frac{1}{1+\varepsilon},$$

provided  $T$  is larger than some  $T_1(\varepsilon) \in \mathbb{N}$ . For the rest of the proof, we assume that  $T > T_1 \vee T_1(\varepsilon) \vee \max\{T_x : 0 \leq x \leq n\}$ .

By combining (3.25) and (3.26), we now obtain

$$(3.29) \quad \frac{Q_{x,y}(\phi)}{Q_{x,y}(0)} = \frac{T_{xy} \tan(\Delta)}{\sin(T_{xy} \Delta)}.$$

By using (3.27) and expanding  $\tan$  to first order, there exists  $T_2(\varepsilon)$  such that for  $T \geq T_2(\varepsilon)$ ,

$$(3.30) \quad \tan(\Delta) \leq (1 + \varepsilon)\Delta.$$



By adding this latter condition on  $T$ , we get, since  $f$  is increasing,

$$(3.31) \quad \begin{aligned} \frac{Q_{x,y}(\phi)}{Q_{x,y}(0)} &\leq (1 + \varepsilon) \frac{T_{xy}\Delta}{\sin(T_{xy}\Delta)} \\ &= (1 + \varepsilon) f(T_{xy}\Delta) \\ &\leq (1 + \varepsilon) f\left(\pi \frac{T_{xy}}{T} \left(1 + \frac{C}{T^2}\right)\right). \end{aligned}$$

As  $\varepsilon < 1$ , that concludes the proof of the second inequality in our statement. To get the first inequality when  $x \neq y$ , notice first that, as  $f(z) \rightarrow 1$  when  $z \rightarrow 0$ , there exists  $z_\varepsilon$  such that  $(1 + \varepsilon)f(z) \leq \exp(\varepsilon)$  for  $z \leq z_\varepsilon$ . Therefore, it is enough to define

$$(3.32) \quad \alpha(\varepsilon) = \frac{z_\varepsilon}{\pi(1 + \varepsilon)},$$

assume that  $T > T_3(\varepsilon) := (C/\varepsilon)^{1/2}$  and use (3.31) to get what we need.

We are left with the case  $x = y$ . Again, with the help of (A.5) in Caravenna and P  tr  lis [4],

$$(3.33) \quad Q_{xx}(\phi) = 1 - \frac{1}{2} \frac{\tan(\Delta)}{\tan(T_{x-1,x}\Delta)} - \frac{1}{2} \frac{\tan(\Delta)}{\tan(T_{x,x+1}\Delta)},$$

where  $\Delta$  is defined as in (3.25). We thereby retrieve the standard formula:

$$(3.34) \quad Q_{xx}(0) = 1 - \frac{1}{2T_{x-1,x}} - \frac{1}{2T_{x,x+1}}.$$

Note that it is enough to treat the case  $T_{x,x+1} = T_{x-1,x}$  since

$$(3.35) \quad Q_{xx}(\phi) = \frac{1}{2} \left(1 - \frac{\tan(\Delta)}{\tan(T_{x-1,x}\Delta)}\right) + \frac{1}{2} \left(1 - \frac{\tan(\Delta)}{\tan(T_{x,x+1}\Delta)}\right).$$

We may now consider the ratio

$$(3.36) \quad \frac{Q_{xx}(\phi)}{Q_{xx}(0)} = \frac{1 - \frac{\tan(\Delta)}{\tan(T_{x,x+1}\Delta)}}{1 - \frac{1}{T_{x,x+1}}}.$$

By choosing  $T \geq T_2(\varepsilon)$  and expanding  $\tan$  to first order, we obtain

$$(3.37) \quad 1 - \frac{\tan(\Delta)}{\tan(T_{x,x+1}\Delta)} \leq \begin{cases} 1 - \frac{\Delta}{\tan(T_{x,x+1}\Delta)} & \text{if } T_{x,x+1}\Delta \leq \frac{\pi}{2}, \\ 1 - (1 + \varepsilon) \frac{\Delta}{\tan(T_{x,x+1}\Delta)} & \text{if } \frac{\pi}{2} < T_{x,x+1}\Delta < \pi. \end{cases}$$

We remind that our conditions on  $T$  guarantee that  $T_{x,x+1}\Delta < \pi$ . The reason why we split cases above is that  $\tan$  changes sign at the value  $\pi$ . We further make a dichotomy : (i)  $T$  is large and  $T_{x+1}$  is at least  $\varepsilon T$  and (ii)  $T$  is large and  $T_{x+1}$  less than  $\varepsilon T$ . Let us start with (i). If actually  $T_{x,x+1}\Delta < \pi/2$ , we may simply bound the denominator in (3.36) by 1. Otherwise, we note that  $z \mapsto -z/\tan(z)$  is increasing on  $(\pi/2, \pi)$ , so we may write, as  $T_{x,x+1}\Delta < \pi/(1 + \varepsilon)$  by (3.28),

$$(3.38) \quad 1 - (1 + \varepsilon) \frac{\Delta}{\tan(T_{x,x+1}\Delta)} \leq 1 - \frac{1}{T_{x,x+1}} \frac{\pi}{\tan(\pi/(1 + \varepsilon))}.$$

Thus, if we define

$$(3.39) \quad T_4(\varepsilon) = \frac{3}{\varepsilon^2} \left( \frac{\pi}{|\tan(\pi/(1 + \varepsilon))|} \vee 1 \right)$$

and assume that  $T > T_4(\varepsilon)$ , we obtain

$$(3.40) \quad \frac{Q_{xx}(\phi)}{Q_{xx}(0)} \leq \frac{1 - \frac{1}{T_{x,x+1}} \frac{\pi}{\tan(\pi/(1+\varepsilon))}}{1 - \frac{1}{T_{x,x+1}}} \leq \frac{1 + \varepsilon/3}{1 - \varepsilon/3} = 1 + \frac{2}{3}\varepsilon + o(\varepsilon), \quad \varepsilon \rightarrow 0,$$

which is smaller than  $\exp(\varepsilon)$  when  $\varepsilon$  is small enough. We now deal with (ii) and to this end we assume  $T \geq T_2(\varepsilon)$  and  $T_{x,x+1} \leq \varepsilon T$ , in which case we expand  $\tan(T_{x,x+1}\Delta)$ . By using (3.27) and assuming that  $T > T_3(\varepsilon)$  we get  $T_{x,x+1}\Delta \leq \varepsilon(1+\varepsilon)\pi$ . Thus, there exists a constant  $C = C(\varepsilon) > 0$  such that

$$(3.41) \quad \tan(T_{x,x+1}\Delta) \leq T_{x,x+1}\Delta + C(T_{x,x+1}\Delta)^3, \quad \text{so } 1 - \frac{\Delta}{\tan(T_{x,x+1}\Delta)} \leq 1 - \frac{1}{T_{x,x+1}}(1 - CT_{x,x+1}^2\Delta^2).$$

Finally, since  $T_{x,x+1} \geq 2$  necessarily,

$$(3.42) \quad \frac{Q_{xx}(\phi)}{Q_{xx}(0)} \leq 1 + 2CT_{x,x+1}\Delta^2 \leq 1 + 2C\varepsilon(1+\varepsilon)\Delta.$$

Now we pick  $T \geq T_5(\varepsilon)$  such that  $\Delta \leq [2C(1+\varepsilon)]^{-1}$  and we get the claim.

We conclude the proof by setting  $T_0(\varepsilon) = \max(T_1, T_1(\varepsilon), T_2(\varepsilon), T_3(\varepsilon), T_4(\varepsilon), T_5(\varepsilon))$ .  $\blacksquare$

Recall the notations in (3.2) and (3.11). Given  $\alpha > 0$  and  $k \in \mathbb{N}$ , we define a set of *bad* edges as

$$(3.43) \quad \mathcal{B}_{k,\alpha} = \{1 \leq x, y \leq i(k) : x \neq y, T_{x,y} > \alpha T_k^*\}.$$

and its cardinal

$$(3.44) \quad L_{k,\alpha} = |\mathcal{B}_{k,\alpha}|.$$

These bad edges correspond to the second case in Lemma 3.6. Recall also (3.7) and (3.8). The following lemma controls the visits to the bad edges:

**Lemma 3.7.** *There exists a function  $h$  such that, for any  $A > 0$ ,  $k \geq 0$ , and  $\alpha > 0$ , if  $T_k^* > h(A, L_{k,\alpha}, \alpha)$  then*

$$(3.45) \quad \mathbb{E}^\beta \left( A^{\#\{i \leq \zeta_k^* : \{X_{i-1}, X_i\} \in \mathcal{B}_{k,\alpha}\}} \mathbf{1}_{\{\zeta_k^* < \zeta_0 \wedge \mathcal{N}\}} \right) \leq 2A^{L_{k,\alpha}} \tau_k^* \mathbb{P}^\beta(\zeta_k^* < \zeta_0 \wedge \mathcal{N}).$$

*Proof of Lemma 3.7.* We start with the case  $L_{k,\alpha} = 1$  and denote by  $(s, s+1)$  the bad edge. By using the geometric nature of  $\mathcal{N}$  and applying the Markov property at  $\zeta_{s+1}$ , we get

$$(3.46) \quad \begin{aligned} \mathbb{E}^\beta \left( A^{\#\{i \leq \zeta_k^* : \{X_{i-1}, X_i\} \in \mathcal{B}_{k,\alpha}\}} \mathbf{1}_{\{\zeta_k^* < \zeta_0 \wedge \mathcal{N}\}} \right) &= \mathbb{E}^\beta \left( A^{\#\{i \leq \zeta_{s+1} : \{X_{i-1}, X_i\} = \{s, s+1\}\}} \mathbf{1}_{\{\zeta_k^* < \zeta_0 \wedge \mathcal{N}\}} \right) \\ &\leq \mathbb{P}^\beta(\zeta_{s+1} < \zeta_0 \wedge \mathcal{N}) A \mathbb{E}_{\tau_{s+1}}^\beta \left( A^{\#\{i \leq \zeta_{s+1} : \{X_{i-1}, X_i\} = \{s, s+1\}\}} \mathbf{1}_{\{\zeta_k^* < \zeta_0 \wedge \mathcal{N}\}} \right), \end{aligned}$$

and we now focus on the last factor in the line above. By considering the consecutive visits of  $X$  to  $s+1$ , we may write

$$(3.47) \quad \mathbb{E}_{\tau_{s+1}}^\beta \left( A^{\#\{i \leq \zeta_{s+1} : \{X_{i-1}, X_i\} = \{s, s+1\}\}} \mathbf{1}_{\{\zeta_k^* < \zeta_0 \wedge \mathcal{N}\}} \right) = \mathbb{E}(v^G) \mathbb{P}_{\tau_{s+1}}(\zeta_k^* < \mathcal{N} | \zeta_k^* < \zeta_{s+1}),$$

where  $G$  is a  $\mathbb{N}_0$ -valued geometric random variable with parameter  $\mathbb{P}_{\tau_{s+1}}(\zeta_k^* < \zeta_{s+1})$  and

$$(3.48) \quad v = \mathbb{E}_{\tau_{s+1}} \left( A^{\#\{i \leq \zeta_{s+1} : \{X_{i-1}, X_i\} = \{s, s+1\}\}} \mathbf{1}_{\{\zeta_{s+1} < \mathcal{N}\}} | \zeta_{s+1} < \zeta_k^* \right).$$

We are going to show now that  $v \leq 1$  when  $T_k^* \geq h_0(A, \alpha)$  where

$$(3.49) \quad h_0(A, \alpha) = \frac{A^2}{2\alpha e^\beta (e^\beta - 1)}.$$

To this end, note that

$$(3.50) \quad v \leq \frac{1}{2}e^{-\beta} + \frac{1}{2}\left(1 - \frac{1}{T_{s,s+1}}\right)e^{-\beta} + \frac{1}{2T_{s,s+1}}A^2e^{-2\beta}.$$

Indeed, the first term is given by walks which make their first step to the right. The second term comes from those who make their first step to the left but come back to  $\tau_{s+1}$  before hitting  $\tau_s$ , whereas the third term comes from the walks who hit  $\tau_s$  before coming back to  $\tau_{s+1}$ . Then, as  $T_{s,s+1} \geq \alpha T_k^*$ ,

$$(3.51) \quad v \leq e^{-\beta} + \frac{A^2 e^{-2\beta}}{2\alpha T_k^*},$$

which, by (3.49), proves that  $v \leq 1$ . To complete the proof in this case, we write

$$(3.52) \quad \begin{aligned} \text{r.h.s(3.46)} &\leq A \mathbb{P}^\beta(\zeta_{s+1} < \zeta_0 \wedge \mathcal{N}) \mathbb{P}_{\tau_{s+1}}(\zeta_k^* < \mathcal{N} | \zeta_k^* < \zeta_{s+1}) \\ &\leq A \mathbb{P}^\beta(\zeta_{s+1} < \zeta_0 \wedge \mathcal{N}) \frac{\mathbb{P}_{\tau_{s+1}}^\beta(\zeta_k^* < \mathcal{N} \wedge \zeta_{s+1})}{\mathbb{P}_{\tau_{s+1}}(\zeta_k^* < \zeta_{s+1})} \\ &\leq 2A(\tau_k^* - \tau_{s+1}) \mathbb{P}^\beta(\zeta_{s+1} < \zeta_0 \wedge \mathcal{N}) \mathbb{P}_{\tau_{s+1}}^\beta(\zeta_k^* < \mathcal{N} \wedge \zeta_{s+1}) \\ &\leq 2A \tau_k^* \mathbb{P}^\beta(\zeta_k^* < \zeta_0 \wedge \mathcal{N}). \end{aligned}$$

Let us now conclude the proof in the general case  $L_{k,\alpha} \geq 1$ . Our strategy is to decouple the contribution of each bad set by Hölder's inequality and reduce the problem to the case  $L_{k,\alpha} = 1$  with  $A$  replaced by  $A^L$ . Indeed, if we note

$$(3.53) \quad \mathcal{B}_{k,\alpha} = \{(s_\ell, s_{\ell+1}) : 1 \leq \ell \leq L, 1 \leq s_\ell < i(k)\},$$

and suppose that  $T_k^* \geq h(A, L_{k,\alpha}, \alpha)$ , where

$$(3.54) \quad h(A, L, \alpha) := h_0(A^L, \alpha) = \frac{A^{2L}}{2\alpha e^\beta (e^\beta - 1)},$$

we get

$$(3.55) \quad \begin{aligned} &\mathbb{E}^\beta \left( A^{\#\{i \leq \zeta_k^* : \{X_{i-1}, X_i\} \in \mathcal{B}_{k,\alpha}\}} \mathbf{1}_{\{\zeta_k^* < \zeta_0 \wedge \mathcal{N}\}} \right) \\ &\leq \prod_{i=1}^{L_{k,\alpha}} \mathbb{E}^\beta \left( (A^{L_{k,\alpha}})^{\#\{i \leq \zeta_k^* : \{X_{i-1}, X_i\} = \{s_\ell, s_{\ell+1}\}\}} \mathbf{1}_{\{\zeta_k^* < \zeta_0 \wedge \mathcal{N}\}} \right)^{1/L_{k,\alpha}} \\ &\leq 2A^{L_{k,\alpha}} \tau_k^* \mathbb{P}^\beta(\zeta_k^* < \zeta_0 \wedge \mathcal{N}). \end{aligned}$$

This concludes the proof. ■

**3.5. Good environments.** We define here a notion of *good environments*, that are environments where it is possible to give a good control on the survival probability. We will show in Section 6.1 that these environments are typical, meaning that by tuning some parameters and considering  $n$  large enough, their probability of occurrence can be made arbitrarily close to one.

*Additional notations.* Beforehand, we remind of the functions  $f$  and  $h$  introduced in Lemma 3.6 and Lemma 3.7. We define

$$(3.56) \quad f_k := 2f \left( \pi \frac{T_{k-1}^*}{T_k^*} \left[ 1 + \frac{C}{(T_k^*)^2} \right] \right),$$

that appears in the events  $A_n^{(6)}$  and  $A_n^{(7)}$  below. The constant  $C$  above is the same as the one in (3.24) in Lemma 3.6. From (3.14) and (3.16), there exists (for a fixed  $\beta$ ) a constant  $C_1 > 0$  so that

$$(3.57) \quad 1/(C_1 t^2) \leq g(t) \wedge \phi(\beta, t) \leq g(t) \vee \phi(\beta, t) \leq C_1/t^2, \quad t \geq 1.$$

This constant appears in the event  $A_n^{(9)}$ . Finally, we define the exponent

$$(3.58) \quad \kappa = \begin{cases} \frac{\gamma}{4} & \text{if } \gamma \leq 1 \\ \frac{1}{2\gamma} - \frac{1}{4} & \text{if } 1 < \gamma < 2 \\ \frac{1}{2\gamma} & \text{if } \gamma \geq 2, \end{cases}$$

which appears in the event  $A_n^{(1)}$ .

*Definition.* Let  $\delta, \varepsilon_0, \varepsilon, \eta > 0$ . The set of good environments, denoted by  $\Omega_n(\delta, \varepsilon_0, \varepsilon, \eta)$ , is defined as the intersection of the events defined below (we stress that  $\alpha(\varepsilon)$  and  $T_0(\varepsilon)$  appearing here are the same as in Lemma 3.6):

$$(3.59) \quad \Omega_n(\delta, \varepsilon_0, \varepsilon, \eta) = \bigcap_{i=1}^{11} A_n^{(i)}(\delta, \varepsilon_0, \varepsilon, \eta),$$

with

$$(3.60) \quad \begin{aligned} A_n^{(1)} &= \begin{cases} \{\tau_{N^{1+\kappa}}^2 < n^{1-\frac{\gamma \wedge (2-\gamma)}{4(\gamma+2)}}\} & \text{if } \gamma < 2 \\ \{\tau_{N^{1+\kappa}}^2 < n^{1+\frac{2\gamma-1}{2(\gamma+2)}}\} & \text{if } \gamma \geq 2, \end{cases} \\ A_n^{(2)}(\varepsilon_0) &:= \{T_k \leq \varepsilon_0^{\frac{1}{2\gamma}} N^{\frac{1}{\gamma}}, \quad \forall k \leq \varepsilon_0 N\} \\ A_n^{(3)}(\varepsilon_0) &:= \{\tau_{N/\varepsilon_0} < n\} \\ A_n^{(4)}(\delta, \varepsilon) &:= \{\exists \ell \in \{N, \dots, 2N\}: T_\ell \geq T_0 \vee \delta N^{\frac{1}{\gamma}}\} \\ A_n^{(5)}(\varepsilon_0, \varepsilon) &:= \{\forall k \in R_{\varepsilon_0}(n), T_k^* > T_0 \vee \varepsilon_0^{\frac{3}{2\gamma}} N^{\frac{1}{\gamma}}\} \\ A_n^{(6)}(\varepsilon_0, \varepsilon) &:= \{\forall k \in R_{\varepsilon_0}(n), f_k^{L_k} \leq \exp(n^{\frac{\gamma}{2(\gamma+2)}})\} \\ A_n^{(7)}(\varepsilon_0, \varepsilon) &:= \{\forall k \in R_{\varepsilon_0}(n), T_k^* > h(f_k, L_k, \alpha, \alpha)\} \\ A_n^{(8)}(\varepsilon_0) &:= \{|R(1, N/\varepsilon_0)| \leq [\log(N/\varepsilon_0)]^2\} \\ A_n^{(9)}(\delta) &:= \{|\lambda(2N, \beta) - \lambda(\beta)| \leq \frac{C_1}{2\delta^2}\} \\ A_n^{(10)}(\varepsilon_0, \varepsilon, \eta) &:= \{|\lambda(\ell-1, b) - \lambda(b)| \leq \frac{\varepsilon_0 \eta}{2}, \quad \forall \ell \geq \varepsilon_0 N, b \in \{\beta, \beta - \varepsilon\}\} \\ A_n^{(11)}(\varepsilon_0) &:= \{\Pi_N(\{y > 1/\varepsilon_0\}) = 0\}. \end{aligned}$$

We might omit some or all parameters of  $\Omega_n$  when no confusion is possible to alight notations. Event  $A_n^{(1)}$  is used to provide a lower bound on  $Z_n$ , see Proposition 4.1. Events from  $A_n^{(2)}$  to  $A_n^{(9)}$  are used to establish the upper bound, see Proposition 5.1. Finally, the

events  $A_n^{(10)}$  and  $A_n^{(11)}$  are used in the conclusion of the proof, see (6.58) and (6.56) in Section 6.2.

#### 4. LOWER BOUND

In this section we prove a lower bound on  $Z_n$  that is an upper bound on  $F_n$ . Let us set for  $\beta \geq 0$ ,  $n \in \mathbb{N}$  and  $\ell > 1$ ,

$$(4.1) \quad G_n^\beta(\ell) = -\frac{1}{N} \log \mathbb{P}^\beta(\sigma \wedge H_0 > H_{\tau_{\ell-1}}) + g(T_\ell) \frac{n}{N} = \lambda(\ell - 1, \beta) \frac{\ell - 1}{N} + g(T_\ell) \frac{n}{N},$$

where  $\lambda(\ell, \beta)$  has been defined in (2.2). Recall the definition of  $\kappa$  in (3.58). Our goal in this section is to prove

**Proposition 4.1.** *On  $\Omega_n$ ,*

$$(4.2) \quad F_n \leq \min_{1 < \ell \leq N^{1+\kappa}} G_n(\ell) + o(1)$$

where the  $o(1)$  holds when  $n$  goes to  $+\infty$ .

Actually, only  $A_n^{(1)}$  is necessary in Lemma 4.2 and this event does not depend on any parameter. That is why we omit the parameters in  $\Omega_n$  above.

In order to prove this proposition we need the following lemma, that states that the exponential cost of reaching level  $N^{1+\kappa}$  (and so any level  $\ell \leq N^{1+\kappa}$ ) before time  $n$  is negligible in front of  $N$ .

**Lemma 4.2.** *There exists a function  $k(N)$  such that  $k(N) = o(N)$  as  $N \rightarrow \infty$  and, on  $\Omega_n$ , for all  $1 < \ell < N^{1+\kappa}$ ,*

$$(4.3) \quad \mathbb{P}(H_{\tau_\ell} \leq n | H_{\tau_\ell} < H_0) \geq \exp(-k(N)).$$

*Proof of Lemma 4.2.* Observe that

$$(4.4) \quad \begin{aligned} \mathbb{P}(H_{\tau_\ell} \leq n | H_{\tau_\ell} < H_0) &\geq \mathbb{P}(S_n \geq \tau_\ell ; S_k > 0, 0 < k \leq n) \\ &\geq \mathbb{P}(S_n \geq \tau_\ell) \mathbb{P}(S_k > 0, 0 < k \leq n) \\ &\geq \mathbb{P}(S_n \geq \tau_{N^{1+\kappa}}) \mathbb{P}(S_k > 0, 0 < k \leq n). \end{aligned}$$

To go from the first to the second line we use the FKG inequality, since both events are non-decreasing coordinate-wise in the family of i.i.d. increments  $(S_i - S_{i-1})_{1 \leq i \leq n}$ . As there exists some constant  $C > 0$  such that  $\mathbb{P}(S_k > 0, 0 < k \leq n) \geq C/\sqrt{n}$  we only focus on  $\mathbb{P}(S_n \geq \tau_{N^{1+\kappa}})$ .

Assume first that  $\gamma < 2$ . Then, we notice that, on  $A_n^{(1)}$ ,  $\tau_{N^{1+\kappa}} = o(\sqrt{n})$  when  $n$  goes to infinity so that

$$(4.5) \quad \mathbb{P}(S_n \geq \tau_{N^{1+\kappa}}) = \mathbb{P}(S_n \geq 0) - \mathbb{P}(0 \leq S_n < \tau_{N^{1+\kappa}}) = 1/2 + o(1).$$

Assume now that  $\gamma \geq 2$ . In this case,  $\tau_{N^{1+\kappa}}$  is not anymore negligible in front of  $\sqrt{n}$ . However, on  $A_n^{(1)}$ , a standard large deviation estimate for the simple random walk yields

$$(4.6) \quad \mathbb{P}(S_n \geq \tau_{N^{1+\kappa}}) \geq \exp\left(-Cn^{\frac{2\gamma-1}{2(\gamma+2)}}\right).$$

We may conclude by setting  $k(N) = CN^{1-\frac{1}{2\gamma}}$ . ■

*Proof of Proposition 4.1.* We provide a lower bound on  $Z_n$  by computing the cost of various strategies. Here is an informal description of the tested strategies: for  $1 < \ell \leq N^{1+\kappa}$ , the walk reaches  $\tau_{\ell-1}$  before time  $n$  and before its death, which has a probability of order  $e^{-\lambda\ell}$ . Then, it remains confined in the gap  $(\tau_{\ell-1}, \tau_\ell)$  until time  $n$ , with a probability of order  $e^{-g(T_\ell)n}$ . We finally optimise on  $1 < \ell \leq N^{1+\kappa}$ .

We may thus write for all  $\ell > 1$ ,

$$(4.7) \quad Z_n \geq Z_n^{[\ell]} := \mathbb{P}(H_{\tau_\ell} \wedge \sigma \wedge H_0 > n \geq H_{\tau_{\ell-1}}),$$

and then optimize over  $1 < \ell \leq N^{1+\kappa}$ . By decomposing on the value of  $H_{\tau_{\ell-1}}$ , we have on  $\Omega_n$  and for  $n$  large enough,

$$(4.8) \quad \begin{aligned} Z_n^{[\ell]} &\geq \sum_{0 \leq k \leq n} \mathbb{P}(\sigma \wedge H_0 > H_{\tau_{\ell-1}} = k) \mathbb{P}_{\tau_{\ell-1}}(H_{\tau_\ell} \wedge H_{\tau_{\ell-1}} > n - k) \\ &\geq \sum_{0 \leq k \leq n} \mathbb{P}(\sigma \wedge H_0 > H_{\tau_{\ell-1}} = k) \mathbb{P}_{\tau_{\ell-1}}(H_{\tau_\ell} \wedge H_{\tau_{\ell-1}} > n) \\ &\geq \mathbb{P}(\sigma \wedge H_0 > H_{\tau_{\ell-1}}, H_{\tau_{\ell-1}} \leq n) \times \frac{c_1}{2\sqrt{n}} e^{-g(T_\ell)n} \\ &\geq \mathbb{P}(\sigma \wedge H_0 > H_{\tau_{\ell-1}}) \mathbb{P}(H_{\tau_{\ell-1}} \leq n | H_0 > H_{\tau_{\ell-1}}) \times \frac{c_1}{2\sqrt{n}} e^{-g(T_\ell)n} \\ &\geq \frac{c_1}{2\sqrt{n}} \mathbb{P}(\sigma \wedge H_0 > H_{\tau_{\ell-1}}) e^{-g(T_\ell)n - k(N)}. \end{aligned}$$

Note that we have used Proposition 3.1 to go from the second to the third line and Proposition 3.5 to go from the third to the fourth line. Finally, to go from the fourth to the fifth line we use Lemma 4.2 and the fact that the environment is in  $\Omega_n$ . Therefore,

$$(4.9) \quad F_n \leq -\frac{1}{N} \log \left( \frac{c_1}{2\sqrt{n}} \right) + \frac{k(N)}{N} + \inf_{1 < \ell \leq N^{1+\kappa}} G_n^\beta(\ell),$$

where  $G_n^\beta$  is defined in (4.1). Since

$$(4.10) \quad -\frac{1}{N} \log \left( \frac{c_1}{2\sqrt{n}} \right) + \frac{k(N)}{N} = o(1),$$

as  $N \rightarrow \infty$ , this concludes the proof.  $\blacksquare$

## 5. UPPER BOUND

In this section we prove an upper bound on  $Z_n$  or, equivalently, a lower bound on  $F_n$ . Recall the definitions in (3.4).

**Proposition 5.1.** *Let  $\varepsilon, \delta > 0$ . There exists  $\varepsilon_0 > 0$  such that, on  $\Omega_n(\delta, \varepsilon_0, \varepsilon)$ ,*

$$(5.1) \quad F_n \geq \min_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} G_n^{\beta-\varepsilon}(\ell) + o(1).$$

where the  $o(1)$  holds as  $n$  goes to infinity and  $G_n^\beta$  is defined in (4.1).

Before starting the proof of this proposition we need additional notations. Recall (3.1). We define the hitting time of the  $k$ -th record

$$(5.2) \quad H_k^* = H_{\tau_{i(k)}}, \quad k \geq 0.$$

We also define

$$(5.3) \quad \tilde{H}_0 = 0, \quad \tilde{H}_i = \inf\{n > \theta_{i-1} : S_n \in \tau\} - \tilde{H}_{i-1}, \quad i \geq 1.$$

For all  $n \geq 1$  and  $k \geq 1$ , we define

$$(5.4) \quad Z_n^{(k)} = Z_n(H_k^* \leq n < H_{k+1}^*)$$

where  $Z_n(A) = \mathbb{P}^\beta(\sigma \wedge H_{Z^-} > n, A)$  for any event  $A$ . The quantity above corresponds to the survival probability restricted to the event that the walker has reached the  $k$ -th record but not the  $(k+1)$ -th one. These events are clearly incompatible for different values of  $k$ .

*Proof of Proposition 5.1.* Let  $\varepsilon, \delta > 0$ . We choose  $\varepsilon_0$  small enough so that

$$(5.5) \quad \begin{aligned} \frac{\beta}{\varepsilon_0} &> 2(C_1\delta^{-2} + \lambda(\beta)), \\ \varepsilon_0^{-1/\gamma} &> 4C_1(C_1\delta^{-2} + \lambda(\beta)), \end{aligned}$$

(these technical conditions will become clear soon).

We have to prove that (5.1) is satisfied on  $\Omega_n$ . We thus consider until the end of this proof an environment that is in  $\Omega_n$ .

**Step 1.** We first prove that we can actually consider only the records in  $R_{\varepsilon_0}(n)$ , that are the ones lying in the window  $\{\varepsilon_0 N, \dots, N/\varepsilon_0\}$ , see (5.10) for a precise formulation. As the environment is in  $A_n^{(1)} \cap A_n^{(4)}(\delta, \varepsilon)$ , using (4.8) and (3.57), we obtain for  $n$  large enough, a rough lower bound on  $Z_n$ :

$$(5.6) \quad \begin{aligned} Z_n &\geq \max_{N \leq \ell \leq 2N} Z_n^{[\ell]} \\ &\geq \max_{N \leq \ell \leq 2N} \frac{c_1}{2\sqrt{n}} \mathbb{P}(\sigma \wedge H_0 > H_{\tau_{\ell-1}}) e^{-g(T_\ell)n-k(N)} \\ &\geq \frac{c_1}{2\sqrt{n}} \exp\left(-\left\{C_1\delta^{-2}N + k(N) + 2N\lambda(2N, \beta)\right\}\right). \end{aligned}$$

As the environment is in  $A_n^{(9)}(\delta)$ , we finally get

$$(5.7) \quad Z_n \geq \frac{c_1}{2\sqrt{n}} \exp\left\{-N\left(2C_1\delta^{-2} + 2\lambda(\beta) + o(1)\right)\right\},$$

where the  $o(1)$  holds as  $n \rightarrow \infty$ . Observe that

$$(5.8) \quad \sum_{k \in R(N/\varepsilon_0, +\infty)} Z_n^{(k)} \leq e^{-\beta N/\varepsilon_0},$$

while due to Proposition 3.4 and the fact that the environment is in  $A_n^{(2)}(\varepsilon_0)$ , we have for  $n$  large enough,

$$(5.9) \quad \begin{aligned} \sum_{k \in R(0, \varepsilon_0 N)} Z_n^{(k)} &\leq Z_n(H_{\tau_{\varepsilon_0 N}} > n) \leq Cn^2(\varepsilon_0 N) \exp(-\phi(\beta, \max\{T_i, i \leq \varepsilon_0 N\})n) \\ &\leq Cn^2(\varepsilon_0 N) \exp\left(-\frac{n}{C_1(\varepsilon_0^{1/2\gamma} N^{1/\gamma})^2}\right) \\ &\leq \exp\left(-\frac{\varepsilon_0^{-1/\gamma} N}{2C_1}\right). \end{aligned}$$

Combining (5.7), (5.8) and (5.9) and due to the choice of  $\varepsilon_0$  in (5.5), we easily get that for  $n$  large enough

$$(5.10) \quad Z_n \leq 2 \sum_{k \in R_{\varepsilon_0}(n)} Z_n^{(k)}.$$

**Step 2.** The main work is thus to derive an upper bound on  $Z_n^{(k)}$  for  $k \in R_{\varepsilon_0}(n)$  (see (5.21)).

Using the Markov property at time  $H_k^*$  we observe that for  $k \in R_{\varepsilon_0}(n)$

$$(5.11) \quad \begin{aligned} Z_n^{(k)} &= \mathbb{P}^\beta (\sigma \wedge H_{\mathbb{Z}^-} > n, H_k^* \leq n < H_{k+1}^*) \\ &= \sum_{m=0}^n \mathbb{P}^\beta (\sigma \wedge H_{\mathbb{Z}^-} > m, H_k^* = m) \mathbb{P}_{\tau_i(k)}^\beta (\sigma \wedge H_{k+1}^* \wedge H_{\mathbb{Z}^-} > n - m). \end{aligned}$$

For all  $m \geq 0$ , by Proposition 3.4, we have on  $A_n^{(3)}(\varepsilon_0)$ , for  $n$  large enough,

$$(5.12) \quad \begin{aligned} \mathbb{P}_{\tau_i(k)}^\beta (\sigma \wedge H_{k+1}^* \wedge H_{\mathbb{Z}^-} > n - m) &\leq C(n - m)^2 i(k + 1) e^{-\phi(\beta; \max\{T_i : 0 \leq i < i(k+1)\})(n-m)} \\ &\leq C n^3 e^{-\phi(\beta, T_k^*)(n-m)}. \end{aligned}$$

It remains to bound  $\mathbb{P}^\beta (\sigma \wedge H_{\mathbb{Z}^-} > m, H_k^* = m)$  for  $0 \leq m \leq n$ . Recall (3.7) and (3.8). By using Tchebychev's inequality in the first line and then conditioning on  $X$ , we obtain for  $\phi := \frac{\pi^2}{2(T_k^*)^2}$ ,

$$(5.13) \quad \begin{aligned} \mathbb{P}^\beta (\sigma \wedge H_{\mathbb{Z}^-} > m, H_k^* = m) &= \mathbb{P}^\beta (\zeta_k^* < \zeta_0 \wedge \mathcal{N}, H_k^* = m) \\ &\leq e^{-\phi m} \mathbb{E}^\beta (e^{\phi H_k^*} \mathbf{1}_{\{\zeta_k^* < \zeta_0 \wedge \mathcal{N}\}}) \\ &\leq e^{-\phi m} \mathbb{E}^\beta (\mathbb{E}(e^{\phi H_k^*} | X) \mathbf{1}_{\{\zeta_k^* < \zeta_0 \wedge \mathcal{N}\}}) \\ &\leq e^{-\phi m} \mathbb{E}^\beta \left( \prod_{1 \leq i \leq \zeta_k^*} \mathbb{E}(e^{\phi \tilde{H}_i} | X_{i-1}, X_i) \mathbf{1}_{\{\zeta_k^* < \zeta_0 \wedge \mathcal{N}\}} \right). \end{aligned}$$

Next, by integrating on  $\mathcal{N}$  we obtain

$$(5.14) \quad \mathbb{E}^\beta \left( \prod_{1 \leq i \leq \zeta_k^*} \mathbb{E}(e^{\phi \tilde{H}_i} | X_{i-1}, X_i) \mathbf{1}_{\{\zeta_k^* < \zeta_0 \wedge \mathcal{N}\}} \right) = \mathbb{E} \left( \prod_{1 \leq i \leq \zeta_k^*} e^{-\beta} \frac{Q_{X_{i-1}, X_i}(\phi)}{Q_{X_{i-1}, X_i}(0)} \mathbf{1}_{\{\zeta_k^* < \zeta_0\}} \right)$$

with notations similar to (3.10).

On  $A_n^{(5)}(\varepsilon_0, \varepsilon)$  the assumptions of Lemma 3.6 are valid (with  $T_k^*$  playing the role of  $T$ ), which provides  $\alpha > 0$ . Recall the definition of  $\mathcal{B}_{k, \alpha}$  in (3.43). We obtain

$$(5.15) \quad \begin{aligned} &\mathbb{E} \left( \prod_{1 \leq i \leq \zeta_k^*} e^{-\beta} \frac{Q_{X_{i-1}, X_i}(\phi)}{Q_{X_{i-1}, X_i}(0)} \mathbf{1}_{\{\zeta_k^* < \zeta_0\}} \right) \\ &\leq \mathbb{E}^{\beta - \varepsilon} \left( 2f \left( \pi \frac{T_{k-1}^*}{T_k^*} \left( 1 + \frac{C}{(T_k^*)^2} \right) \right)^{\#\{i \leq \zeta_k^* : \{X_{i-1}, X_i\} \in \mathcal{B}_{k, \alpha}\}} \mathbf{1}_{\{\zeta_k^* < \zeta_0 \wedge \mathcal{N}\}} \right). \end{aligned}$$



Recall the definition of  $f_k$  in (3.56). On  $A_n^{(7)}(\varepsilon_0, \varepsilon)$  the assumptions of Lemma 3.7 (with  $f_k$  playing the role of  $A$ ) are satisfied and from (5.15) we obtain that on  $A_n^{(3)}(\varepsilon_0)$

$$(5.16) \quad \mathbb{E} \left( \prod_{1 \leq i \leq \zeta_k^*} e^{-\beta \frac{Q_{X_{i-1}, X_i}(\phi)}{Q_{X_{i-1}, X_i}(0)}} \mathbf{1}_{\{\zeta_k^* < \zeta_0\}} \right) \leq \mathbb{E}^{\beta-\varepsilon} \left( f_k^{\#\{i \leq \zeta_k^* : \{X_{i-1}, X_i\} \in \mathcal{B}_{k, \alpha}\}} \mathbf{1}_{\{\zeta_k^* < \zeta_0 \wedge \mathcal{N}\}} \right) \\ \leq 2f_k^{L_{k, \alpha}} n \mathbb{P}^{\beta-\varepsilon}(\zeta_k^* < \zeta_0 \wedge \mathcal{N}).$$

Finally, combining (5.11), (5.12), (5.13) and (5.16) we obtain for  $k \in R_{\varepsilon_0}(n)$

$$(5.17) \quad Z_n^{(k)} \leq 2C n^4 f_k^{L_{k, \alpha}} e^{-\phi(\beta, T_k^*)n} \mathbb{P}^{\beta-\varepsilon}(\zeta_k^* < \zeta_0 \wedge \mathcal{N}) \sum_{m=0}^n e^{-\left(\frac{\pi^2}{2(T_k^*)^2} - \phi(\beta, T_k^*)\right)m} \\ \leq 2C n^4 f_k^{L_{k, \alpha}} e^{-G_n^{\beta-\varepsilon}(i(k))N} e^{-(\phi(\beta, T_k^*) - g(T_k^*))n} \sum_{m=0}^n e^{-\left(\frac{\pi^2}{2(T_k^*)^2} - \phi(\beta, T_k^*)\right)m}.$$

On  $A_n^{(5)}(\varepsilon_0, \varepsilon)$  we control both the errors  $\phi(\beta; T_k^*) - g(T_k^*)$  and  $\phi(\beta; T_k^*) - \frac{\pi^2}{2(T_k^*)^2}$ . Indeed due to (3.14) and (3.16) there exists some constant  $C(\beta)$  such that for  $t$  large enough (depending on  $\beta$ ),

$$(5.18) \quad |\phi(\beta; t) - \frac{\pi^2}{2t^2}| \vee |\phi(\beta; t) - g(t)| < \frac{C(\beta)}{t^3},$$

and we obtain that on  $A_n^{(5)}(\varepsilon_0, \varepsilon)$  and for  $n$  large enough

$$(5.19) \quad |\phi(\beta; T_k^*) - \frac{\pi^2}{2(T_k^*)^2}| \vee |\phi(\beta; T_k^*) - g(T_k^*)| \leq \frac{C(\beta)}{\varepsilon_0^{9/2\gamma} N^{3/\gamma}}.$$

From (5.17) we thus obtain for  $n$  large enough and  $k \in R_{\varepsilon_0}(n)$

$$(5.20) \quad Z_n^{(k)} \leq 2C n^5 f_k^{L_{k, \alpha}} \exp \left\{ -N \min_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} G_n^{\beta-\varepsilon}(\ell) + C(\beta) \varepsilon_0^{-9/2\gamma} n^{\frac{\gamma-1}{\gamma+2}} \right\}.$$

We also remind that on  $A_n^{(6)}(\varepsilon_0, \varepsilon)$ ,  $f_k^{L_{k, \alpha}} \leq e^{n \frac{\gamma}{2(\gamma+2)}}$  so that finally for  $k \in R_{\varepsilon_0}(n)$ , and for  $n$  large enough,

$$(5.21) \quad Z_n^{(k)} \leq 2C n^5 \exp \left( -N \min_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} G_n^{\beta-\varepsilon}(\ell) + o(N) \right).$$

**Step 3.** It remains to sum this last equation for  $k \in R_{\varepsilon_0}(n)$ . As the environment is in  $A_n^{(8)}(\varepsilon_0)$ , we finally obtain that for  $n$  large enough

$$(5.22) \quad Z_n \leq \varepsilon_0^{-1} 2C n^6 \exp \left( -N \min_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} G_n^{\beta-\varepsilon}(\ell) + o(N) \right)$$

and

$$(5.23) \quad F_n \geq \min_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} G_n^{\beta-\varepsilon}(\ell) + o(1),$$

where the  $o(1)$  holds as  $n \rightarrow \infty$ . ■

## 6. PROOF OF THEOREM 2.8

This section is divided in two parts. In the first one we establish that, for  $n$  large enough, environments are good with probability arbitrary close from 1. The second one is devoted to the proof our main result Theorem 2.8. Due to the control on the environment we can actually restrict our analysis to the event of good environments so that results of Proposition 4.1 and 5.1 are in force and provide a precise control on  $Z_n$ .

**6.1. Environments are good with high probability.** The aim of this part is to prove the following proposition that assures a control of the probability that an environment is good:

**Proposition 6.1.** *For all  $\theta > 0$  there exists  $\delta$  and  $\varepsilon_1(\delta)$  small enough such that for all  $\varepsilon_0 < \varepsilon_1(\delta)$ , for all  $\varepsilon, \eta > 0$*

$$(6.1) \quad \liminf_{n \rightarrow +\infty} \mathbb{P}(\Omega_n(\delta, \varepsilon_0, \varepsilon, \eta)) \geq 1 - \theta.$$

We first establish various results on the *records* of the sequence  $(T_i)_{i \geq 1}$ .

*Preliminaries on records.* We say that  $n \in \mathbb{N}$  is a record if  $n = 1$  or  $n \geq 2$  and  $T_n > \max\{T_1, \dots, T_{n-1}\}$ . Let us define

$$(6.2) \quad I_n = \begin{cases} 1 & \text{if } n \text{ is a record} \\ 0 & \text{otherwise,} \end{cases}$$

the indicator of a record, and

$$(6.3) \quad \mathfrak{R}_n = \sum_{k=1}^n I_k = |R(1, n)|$$

the number of records between 1 and  $n$ . It was noticed (see Rényi [12]) that when  $T_1$  is a continuous random variable, the  $I_n$ 's are independent Bernoulli random variables with mean  $\mathbb{E}(I_n) = 1/n$ . However, we deal here with the discrete case, where this simple structure breaks down because of the possibility of ties. Actually, this case seems to have attracted relatively little attention (see however [9] and references therein). In this section, we provide some results in the discrete case (moments and deviations from the mean for the number of records) that will be useful later, though we do not aim for optimality. We start with:

**Proposition 6.2.** *For all  $p \in \mathbb{N}$  and  $1 \leq n_1 < n_2 < \dots < n_p$ ,*

$$(6.4) \quad \mathbb{E}(I_{n_1} \dots I_{n_p}) \leq (1/n_1) \dots (1/n_p).$$

As a consequence we obtain:

**Proposition 6.3.** *For  $b > 1$  there exists a positive constant  $c = c(b)$  (which we may choose smaller but arbitrarily close to  $\sup_{\lambda > 0} \{b\lambda + 1 - e^\lambda\} = 1 + b(\ln b - 1) > 0$ ) such that for  $n$  large enough,*

$$(6.5) \quad \mathbb{P}(\mathfrak{R}_n \geq b \log n) \leq n^{-c}.$$

*Proof of Proposition 6.2.* We prove it by iteration on  $p \geq 1$  and thus start with the case  $p = 1$ . Let  $n \geq 2$  (the statement is trivial for  $n = 1$ ). Let  $J_n$  be the indicator of the event that there is a strict maximum among the  $n$  first variables, meaning that

$$(6.6) \quad J_n = \mathbf{1}_{\{\exists 1 \leq i \leq n: \forall 1 \leq j \leq n, j \neq i, T_i > T_j\}}.$$

By exchangeability of the  $n$  first variables we get that

$$(6.7) \quad \mathbb{E}(I_n) = \frac{1}{n} \mathbb{E}(J_n) \leq \frac{1}{n}.$$

Suppose now that the statement is true for  $p \geq 1$  and let  $2 \leq n_1 < \dots < n_{p+1}$  (if  $n_1 = 1$ , we can safely remove it). Again, by exchangeability of  $(T_1, \dots, T_{n_1})$  we have

$$(6.8) \quad \mathbb{E}(I_{n_1} \dots I_{n_{p+1}}) = \frac{1}{n_1} \mathbb{E}(J_{n_1} I_{n_2} \dots I_{n_{p+1}}) \leq \frac{1}{n_1} \mathbb{E}(I_{n_2} \dots I_{n_{p+1}}),$$

and the result follows by using the induction hypothesis.  $\blacksquare$

*Proof of Proposition 6.3.* Let  $\lambda > 0$  to be specified later. By Chernoff's bound,

$$(6.9) \quad \mathbb{P}(\mathfrak{R}_n \geq b \log n) \leq e^{-b\lambda \log n} \mathbb{E}[e^{\lambda \mathfrak{R}_n}].$$

Since  $\mathfrak{R}_n = \sum_{k=1}^n I_k$  and the  $I_k$ 's are  $\{0, 1\}$ -valued random variables, we get

$$(6.10) \quad e^{\lambda \mathfrak{R}_n} = \prod_{k=1}^n (1 + [e^\lambda - 1] I_k) = 1 + \sum_{\substack{J \subseteq \{1, \dots, n\} \\ J \neq \emptyset}} [e^\lambda - 1]^{|J|} \prod_{j \in J} I_j.$$

By taking the expectation and using Proposition 6.2, we get

$$(6.11) \quad \mathbb{E}[e^{\lambda \mathfrak{R}_n}] \leq \prod_{k=1}^n \left(1 + [e^\lambda - 1] \frac{1}{k}\right) \leq \exp([e^\lambda - 1][1 + o(1)] \log n),$$

where the  $o(1)$  holds as  $n \rightarrow \infty$ . Finally, we obtain

$$(6.12) \quad \mathbb{P}(\mathfrak{R}_n \geq b \log n) \leq \exp(\{[e^\lambda - 1][1 + o(1)] - b\lambda\} \log n),$$

which concludes the proof.  $\blacksquare$

*Proof of Proposition 6.1.* We will use notations from the previous subsection during the proof, and sometimes write  $\mathfrak{R}(n)$  instead of  $\mathfrak{R}_n$  for the sake of readability. We consider the events  $A_n^{(i)}$  for  $1 \leq i \leq 11$  and conclude by combining the results obtained in each cases. Along the proof we will use that, by (1.8), there exists  $c_1, c_2 > 0$  such that for all  $m \in \mathbb{N}$ ,

$$(6.13) \quad 1 - c_1 m^{-\gamma} \leq \mathbb{P}(T_1 \leq m) \leq 1 - c_2 m^{-\gamma} \leq \exp(-c_2 m^{-\gamma}).$$

*Case  $i = 1$ .* Assume first that  $\gamma \leq 1$ , so that  $\kappa = \frac{\gamma}{4}$ . Since

$$(6.14) \quad N^{(1+\frac{\gamma}{4})\frac{2}{\gamma}} = o(n^{1-\frac{\gamma}{4(\gamma+2)}}),$$

one has  $\lim_{n \rightarrow \infty} \mathbb{P}(A_n^{(1)}) = 0$  from Proposition 1.1. Assume now that  $\gamma > 1$ . Then,

$$(6.15) \quad N^{2(1+\kappa)} = \begin{cases} o(n^{1-\frac{2-\gamma}{4(2+\gamma)}}) & \text{if } \gamma \in (1, 2) \\ o(n^{1+\frac{2\gamma-1}{2(\gamma+2)}}) & \text{if } \gamma \geq 2, \end{cases}$$

and again  $\lim_{n \rightarrow \infty} \mathbb{P}(A_n^{(1)}) = 0$  from Proposition 1.1.

*Case  $i = 2$ .* Note that, by (6.13),

$$(6.16) \quad \mathbb{P}\left(A_n^{(2)}(\varepsilon_0)\right) = \mathbb{P}\left(T_1 \leq \varepsilon_0^{\frac{1}{2\gamma}} N^{1/\gamma}\right)^{\varepsilon_0 N} \geq \left(1 - \frac{c_1}{\varepsilon_0^{\frac{1}{2}} N}\right)^{\varepsilon_0 N} = e^{-c_1 \varepsilon_0^{1/2}} + o(1) \quad n \rightarrow +\infty.$$

We obtain  $\lim_{\varepsilon_0 \rightarrow 0} \liminf_{n \rightarrow +\infty} \mathbb{P}(A_n^{(2)}(\varepsilon_0)) = 1$ .

*Case  $i = 3$ .* Here we note that  $N^{1/\gamma} = o(n)$  when  $\gamma \leq 1$  and  $N = o(n)$  when  $\gamma > 1$ , and we conclude using Proposition 1.1 that for all  $\varepsilon_0 > 0$

$$(6.17) \quad \lim_{n \rightarrow \infty} \mathbb{P}(A_n^{(3)}(\varepsilon_0)) = 0.$$

*Case  $i = 4$ .* By independence of the  $T_\ell$ 's, one has

$$(6.18) \quad \mathbb{P}(A_n^{(4)}(\delta, \varepsilon)^c) \leq \mathbb{P}(T_1 < T_0(\varepsilon) \vee \delta N^{\frac{1}{\gamma}})^N.$$

Therefore,

$$(6.19) \quad \mathbb{P}(A_n^{(4)}(\delta, \varepsilon)^c) \leq \exp(-c(NT_0(\varepsilon))^{-\gamma} \wedge \delta^{-\gamma}),$$

from which we deduce that for all  $\varepsilon > 0$

$$(6.20) \quad \lim_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}(A_n^{(4)}(\delta, \varepsilon)) = 1.$$

*Case  $i = 5$ .* Recall the definitions of  $R_{\varepsilon_0}(n)$  and  $T_k^*$  in (3.2) and (3.4). We get

$$(6.21) \quad \mathbb{P}(A_n^{(5)}(\varepsilon_0, \varepsilon)^c) \leq \mathbb{P}(\forall \ell \leq \varepsilon_0 N, T_\ell \leq T_0(\varepsilon) \vee \varepsilon_0^{\frac{3}{2\gamma}} N^{\frac{1}{\gamma}}) \leq \mathbb{P}(T_1 \leq T_0(\varepsilon) \vee \varepsilon_0^{\frac{3}{2\gamma}} N^{\frac{1}{\gamma}})^{\varepsilon_0 N}.$$

By using again (6.13), we obtain

$$(6.22) \quad \mathbb{P}(A_n^{(5)}(\varepsilon_0, \varepsilon)^c) \leq \exp(-c_2(\varepsilon_0 T_0(\varepsilon))^{-\gamma} N \wedge \varepsilon_0^{-1/2}),$$

from which we get for all  $\varepsilon > 0$

$$(6.23) \quad \lim_{\varepsilon_0 \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}(A_n^{(5)}(\varepsilon_0, \varepsilon)) = 1.$$

*Case  $6 \leq i \leq 8$ .* Let us first prove that for  $\varepsilon_0 > 0$ ,

$$(6.24) \quad \lim_{n \rightarrow \infty} \mathbb{P}(A_n^{(8)}(\varepsilon_0)) = 1.$$

Indeed, since  $|R(1, N/\varepsilon_0)| = \mathfrak{R}(N/\varepsilon_0)$ , it is enough to prove that

$$(6.25) \quad \mathbb{P}(\mathfrak{R}_n \leq (\log n)^2) \rightarrow 1, \quad n \rightarrow \infty,$$

which can be easily deduced from Proposition 6.3. We now deal with the event corresponding to  $i = 7$ . Fix  $\varepsilon_0, \varepsilon > 0$ , and note that

$$(6.26) \quad \begin{aligned} \mathbb{P}(A_n^{(7)}(\varepsilon_0, \varepsilon)^c) &\leq \mathbb{P}(A_n^{(8)}(\varepsilon_0)^c) + \mathbb{P}(A_n^{(5)}(\varepsilon_0, \varepsilon)^c) \\ &\quad + \mathbb{P}\left(\exists k \leq [\log(N/\varepsilon_0)]^2: \varepsilon_0^{\frac{3}{2\gamma}} N^{\frac{1}{\gamma}} \leq T_k^* \leq h(f_k, L_k, \alpha(\varepsilon)), i(k) \leq N/\varepsilon_0\right). \end{aligned}$$

By a union bound we are left to prove that

$$(6.27) \quad p_k(n, \varepsilon_0) = o([\log n]^{-2}),$$

where

$$(6.28) \quad p_k(n, \varepsilon_0, \varepsilon) := \mathbb{P}(\varepsilon_0^{\frac{3}{2\gamma}} N^{\frac{1}{\gamma}} \leq T_k^* \leq h(f_k, L_k, \alpha(\varepsilon)), i(k) \leq N/\varepsilon_0).$$

From (3.54) in the proof of Lemma 3.7, we have for  $n$  large enough,

$$(6.29) \quad p_k(n, \varepsilon_0, \varepsilon) \leq \mathbb{P}(f_k^{L_k} \geq C(\varepsilon_0, \varepsilon) N^{\frac{1}{2\gamma}}, T_k^* \geq \varepsilon_0^{\frac{3}{2\gamma}} N^{\frac{1}{\gamma}}, i(k) \leq N/\varepsilon_0).$$

Let us first show that

$$(6.30) \quad \mathbb{P}\left(f_k > A, T_k^* \geq \varepsilon_0^{\frac{3}{2\gamma}} N^{\frac{1}{\gamma}}\right) \leq C\left[A \wedge \varepsilon_0^{\frac{3}{2\gamma}} N^{\frac{2}{\gamma}}\right]^{-1}.$$

To this end, our first ingredient is the following upper bound, which holds for  $u \in (0, 1)$  and  $k \in \mathbb{N}$ :

$$(6.31) \quad \mathbb{P}\left(\frac{T_k^*}{T_{k+1}^*} \geq 1 - u\right) \leq Cu.$$

We show this inequality at the end of the proof, in order not to disrupt the main line of ideas. We will also use the following elementary bound:

$$(6.32) \quad f(u) = \frac{u}{\sin(u)} \leq \frac{C}{\pi - u}, \quad u \in (0, \pi).$$

Coming back to (6.30), we have by using (6.32),

$$(6.33) \quad \begin{aligned} \mathbb{P}\left(f_k > A, T_k^* \geq \varepsilon_0^{\frac{3}{2\gamma}} N^{\frac{1}{\gamma}}\right) &\leq \mathbb{P}\left(\frac{T_{k-1}^*}{T_k^*} \left[1 + \frac{C}{(T_k^*)^2}\right] \geq 1 - \frac{C}{A}, T_k^* \geq \varepsilon_0^{\frac{3}{2\gamma}} N^{\frac{1}{\gamma}}\right) \\ &\leq \mathbb{P}\left(\frac{T_{k-1}^*}{T_k^*} \geq 1 - \frac{C}{A} - \frac{C}{\varepsilon_0^{\frac{3}{\gamma}} N^{\frac{2}{\gamma}}}\right), \end{aligned}$$

and we obtain (6.30) thanks to (6.31). In view of (6.27), we choose  $A = A_n = [\log n]^3$ . For  $k \geq 1$ , from (6.29),

$$(6.34) \quad p_k(n, \varepsilon_0, \varepsilon) \leq \mathbb{P}\left(f_k \geq A_n, T_k^* \geq \varepsilon_0^{\frac{3}{2\gamma}} N^{\frac{1}{\gamma}}\right) + \mathbb{P}\left(L_k \geq C \frac{\log n}{\log \log n}, T_k^* \geq \varepsilon_0^{\frac{3}{2\gamma}} N^{\frac{1}{\gamma}}, i(k) \leq N/\varepsilon_0\right).$$

Using (6.30), the first term in the sum above is  $o[(\log n)^{-2}]$ . We now deal with the second one. On the corresponding event, we have

$$(6.35) \quad \text{card}\left\{1 \leq j \leq N/\varepsilon_0 : T_j \geq \alpha \varepsilon_0^{\frac{3}{2\gamma}} N^{\frac{1}{\gamma}}\right\} \geq C \frac{\log n}{\log \log n}.$$

Furthermore, the random variable in the l.h.s. of the inequality follows a binomial law with parameters  $N/\varepsilon_0$  and  $\mathbb{P}(T_1 \geq \alpha \varepsilon_0^{\frac{3}{2\gamma}} N^{\frac{1}{\gamma}}) \leq \alpha^{-\gamma} \varepsilon_0^{-3/2} N^{-1}$  (up to a harmless constant). By using a standard binomial-to-Poisson approximation (cf. end of the proof)

$$(6.36) \quad \mathbb{P}(\text{Bin}(\ell, q) \geq m) \leq \left(\frac{q\ell}{m}\right)^m e^{m-ql}, \quad \ell \in \mathbb{N}, q \in (0, 1), m \in \mathbb{N}: ql < m,$$

with  $q = \varepsilon_0^{-3/2} N^{-1}$ ,  $\ell = N/\varepsilon_0$  and  $m = C \log n / \log \log n$ , we get that

$$(6.37) \quad \mathbb{P}((6.35)) = o(n^{-C/2}),$$

which is enough to conclude. From what precedes we finally obtain that for all  $\varepsilon_0, \varepsilon > 0$ ,

$$(6.38) \quad \lim_{n \rightarrow \infty} \mathbb{P}(A_n^{(7)}(\varepsilon_0, \varepsilon)^c) = 0.$$

The event corresponding to  $i = 6$  can be readily treated with the same idea, since the  $N^{\frac{1}{2\gamma}}$  in (6.29) is less than  $\exp(n^{\frac{\gamma}{2(\gamma+2)}})$ . Finally,

$$(6.39) \quad \lim_{n \rightarrow \infty} \mathbb{P}(A_n^{(6)}(\varepsilon_0, \varepsilon)^c \cup A_n^{(7)}(\varepsilon_0, \varepsilon)^c) = 0.$$

*Case  $9 \leq i \leq 10$ .* From the almost-sure convergence in Proposition 2.2, we readily get that for any choice of  $\delta, \varepsilon_0, \varepsilon, \eta$ ,

$$(6.40) \quad \lim_{n \rightarrow \infty} \mathbb{P}(A_n^{(9)}(\delta) \cap A_n^{(10)}(\varepsilon_0, \varepsilon, \eta)) = 1.$$

Case  $i = 11$ . Note that by (6.13),

$$(6.41) \quad \mathbb{P}\left(A_n^{(11)}(\varepsilon_0)\right) = \mathbb{P}\left(T_1 \leq \frac{N^{1/\gamma}}{\varepsilon_0}\right)^N \geq e^{-c_1 \varepsilon_0^\gamma} + o(1) \quad n \rightarrow +\infty.$$

We obtain  $\lim_{\varepsilon_0 \rightarrow 0} \liminf_{n \rightarrow +\infty} \mathbb{P}(A_n^{(11)}(\varepsilon_0)) = 1$ .

*Proof of (6.31).* By writing  $1 + v = (1 - u)^{-1}$  for convenience, one has

$$(6.42) \quad \mathbb{P}\left(T_{k+1}^* \leq (1 + v)T_k^* \mid T_k^* = \ell\right) = \begin{cases} 0 & \text{if } v < 1/\ell \\ \mathbb{P}(T_1 \leq \lfloor (1 + v)\ell \rfloor \mid T_1 > \ell) & \text{otherwise.} \end{cases}$$

From our assumption on the tail of  $T_1$  and with the help of a standard comparison between series and integrals, the probability in the second case is bounded from above by

$$(6.43) \quad C\ell^\gamma \sum_{n=\ell+1}^{\lfloor (1+v)\ell \rfloor} n^{-(1+\gamma)} \leq C\ell^\gamma \int_{\ell}^{\lfloor (1+v)\ell \rfloor - 1} t^{-(1+\gamma)} dt.$$

In turn, the integral above is controlled by

$$(6.44) \quad \int_{\ell}^{\lfloor (1+v)\ell \rfloor} t^{-(1+\gamma)} dt = C[1 - (1 + v)^{-\gamma}]\ell^{-\gamma} \leq C u \ell^{-\gamma},$$

which completes the proof of (6.31).

*Proof of (6.36).* By Chernoff's bound, one has for  $x > 0$ ,

$$(6.45) \quad \mathbb{P}(\text{Bin}(\ell, q) \geq m) \leq e^{-xm} \mathbb{E}(e^{x \text{Ber}(q)})^\ell \leq \exp(-xm + q(e^x - 1)\ell).$$

Since  $m > q\ell$ , we may choose  $x = \log(m/(q\ell))$  to minimize the r.h.s. in the line above and get the result.

**6.2. Conclusion : proof of Theorem 2.8.** To prove Theorem 2.8, we establish that for all  $u \in \mathbb{R}$  which is a continuity point of the distribution function of  $F$ ,

$$(6.46) \quad \lim_{n \rightarrow \infty} \mathbb{P}(F_n \leq u) = \mathbb{P}(F \leq u).$$

By (2.36) all real numbers are continuity points of  $F$ . Moreover since  $F_n$  is positive, we only have to prove (6.46) for  $u > 0$ .

We start with the **upper bound** in (6.46)

$$(6.47) \quad \limsup_{n \rightarrow \infty} \mathbb{P}(F_n \leq u) \leq \mathbb{P}(F \leq u).$$

Fix  $\theta > 0$  that we will let go to 0 only at the very end of the proof. Fix also  $\varepsilon, \eta > 0$ . From Proposition 6.1, there exists  $\delta > 0$  and  $\varepsilon_1(\delta)$  so that for  $\varepsilon_0 < \varepsilon_1(\delta)$

$$(6.48) \quad \liminf_n \mathbb{P}(\Omega_n(\delta, \varepsilon_0, \varepsilon, \eta)) > 1 - \theta.$$

Fix  $\varepsilon_0 < \varepsilon_1(\delta)$  small enough so that the conclusion of Proposition 5.1 is satisfied. Thus we obtain that for  $n$  large enough

$$(6.49) \quad \mathbb{P}(F_n \leq u) \leq \mathbb{P}\left(\min_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} G_n^{\beta-\varepsilon}(\ell) \leq u + \eta, \Omega_n\right) + \mathbb{P}(\Omega_n^c).$$

From Proposition 6.1 and the choices of  $\delta$  and  $\varepsilon_0$ ,  $\mathbb{P}(\Omega_n^c) < \theta$  for  $n$  large enough. Thus we just have to focus on the first term in the last equation. We introduce for  $n, \ell \geq 1$  and  $\beta > 0$ , the random variable

$$(6.50) \quad \tilde{G}_n^\beta(\ell) = \frac{\lambda(\beta)(\ell-1)}{N} + \frac{\pi^2 n}{2T_\ell^2 N} = \psi^{\lambda(\beta)}\left(\frac{\ell}{N}, \frac{T_\ell}{N^{1/\gamma}}\right).$$

We replace  $G$  by  $\tilde{G}$  in (6.49) and control the probability that both processes are not close

$$(6.51) \quad \mathbb{P}\left(\min_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} G_n^{\beta-\varepsilon}(\ell) \leq u + \eta, \Omega_n\right) \\ \leq \mathbb{P}\left(\min_{\ell \in \mathcal{R}} \tilde{G}_n^{\beta-\varepsilon}(\ell) < u + 2\eta, \Omega_n\right) + \mathbb{P}\left(\max_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} |\tilde{G}_n^{\beta-\varepsilon}(\ell) - G_n^{\beta-\varepsilon}(\ell)| \geq \eta, \Omega_n\right).$$

The first term in the sum gives the main contribution. Let us first prove that the second one is zero for  $n$  large enough. For  $\ell \geq 1$  we define

$$(6.52) \quad \Delta_1(n, \ell) := \frac{\ell-1}{N} |\lambda(\beta-\varepsilon) - \lambda(\ell-1, \beta-\varepsilon)|$$

and

$$(6.53) \quad \Delta_2(n, \ell) := \frac{n}{N} \left| g(T_\ell) - \frac{\pi^2}{2T_\ell^2} \right|$$

so that

$$(6.54) \quad \max_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} |\tilde{G}_n^{\beta-\varepsilon}(\ell) - G_n^{\beta-\varepsilon}(\ell)| \leq \max_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} \Delta_1(n, \ell) + \max_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} \Delta_2(n, \ell).$$

We first deal with  $\Delta_2$ . According to (3.14) there exists some  $C > 0$  such that for all  $\ell$ ,  $|g(T_\ell) - \frac{\pi^2}{2T_\ell^2}| \leq \frac{C}{T_\ell^4}$ . We can thus deduce that

$$(6.55) \quad \mathbb{P}\left(\max_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} \Delta_2(n, \ell) \geq \eta, \Omega_n\right) \leq \mathbb{P}\left(\frac{C n}{N(\varepsilon_0^{\frac{2}{\gamma}} N^{1/\gamma})^4} \geq \eta, A_n^{(5)}(\varepsilon_0, \varepsilon)\right),$$

and this last term is 0 for  $n$  large enough. We turn to the control of  $\Delta_1$ . For  $n \geq 1$ ,

$$(6.56) \quad \mathbb{P}\left(\max_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} \Delta_1(n, \ell) \geq \eta, \Omega_n\right) \leq \mathbb{P}\left(\max_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} \Delta_1(n, \ell) \geq \eta, A_n^{(10)}(\varepsilon_0, \varepsilon, \eta)\right),$$

and again the last term is 0 for  $n$  large enough. Let us come back to the first term in (6.51),  $\mathbb{P}(\min_{\ell \in \mathcal{R}} \tilde{G}_n^{\beta-\varepsilon}(\ell) < u + 2\eta, \Omega_n)$ . As  $\ell$  ranges  $\mathcal{R}$  we may write

$$(6.57) \quad \min_{\ell \in \mathcal{R}} \tilde{G}_n^{\beta-\varepsilon}(\ell) = \Psi^{\lambda(\beta-\varepsilon)}(\Pi_N).$$

We thus obtain

$$(6.58) \quad \mathbb{P}\left(\min_{\ell \in \mathcal{R}} \tilde{G}_n^{\beta-\varepsilon}(\ell) < u + 2\eta, \Omega_n\right) \leq \mathbb{P}\left(\Psi^{\lambda(\beta-\varepsilon)}(\Pi_N) < u + 2\eta, A_n^{(11)}(\varepsilon_0)\right) \\ \leq \mathbb{P}\left(\Psi_K^{\lambda(\beta-\varepsilon)}(\Pi_N) < u + 2\eta\right),$$

where  $K := A_{u+2\eta}^{\lambda(\beta-\varepsilon)} \cap \{y \leq 1/\varepsilon_0\}$  with the set  $A$  defined in (2.20). As  $K$  is compact, Proposition 2.4 and Lemma 2.6 assure that

$$(6.59) \quad \mathbb{P}\left(\Psi_K^{\lambda(\beta-\varepsilon)}(\Pi_N) < u + 2\eta\right) \rightarrow \mathbb{P}\left(\Psi_K^{\lambda(\beta-\varepsilon)}(\Pi) < u + 2\eta\right)$$

when  $n$  goes to infinity (we recall that  $\Psi_K^{\lambda(\beta-\varepsilon)}(\Pi)$  is continuous). Using that  $\Psi^{\lambda(\beta-\varepsilon)} \leq \Psi_K^{\lambda(\beta-\varepsilon)}$  we obtain

$$(6.60) \quad \begin{aligned} \mathbb{P}(\Psi_K^{\lambda(\beta-\varepsilon)}(\Pi) < u + 2\eta) &\leq \mathbb{P}(\Psi^{\lambda(\beta-\varepsilon)}(\Pi) < u + 2\eta) \\ &= \mathbb{P}(F^{\beta-\varepsilon} < u + 2\eta). \end{aligned}$$

Finally, we have proven

$$(6.61) \quad \limsup_{n \rightarrow +\infty} \mathbb{P}(F_n \leq u) \leq \mathbb{P}(F^{\beta-\varepsilon} < u + 2\eta) + \theta.$$

As  $u \mapsto \mathbb{P}(F^{\beta-\varepsilon} \leq u)$  is right-continuous,

$$(6.62) \quad \lim_{\eta \rightarrow 0} \mathbb{P}(F^{\beta-\varepsilon} < u + 2\eta) = \mathbb{P}(F^{\beta-\varepsilon} \leq u).$$

From Lemma 2.7,  $F^{\beta-\varepsilon} \nearrow F^\beta$  almost surely when  $\varepsilon$  goes to 0 so that

$$(6.63) \quad \mathbb{P}(F^{\beta-\varepsilon} \leq u) \rightarrow \mathbb{P}(F^\beta \leq u) \quad \varepsilon \rightarrow 0.$$

Finally,

$$(6.64) \quad \limsup_{n \rightarrow +\infty} \mathbb{P}(F_n \leq u) \leq \mathbb{P}(F^\beta \leq u) + \theta,$$

and, as  $\theta$  can be chosen arbitrarily small, we obtain the upper bound

$$(6.65) \quad \limsup_{n \rightarrow +\infty} \mathbb{P}(F_n \leq u) \leq \mathbb{P}(F^\beta \leq u).$$

We turn now to the **lower bound** in (6.46):

$$(6.66) \quad \liminf_{n \rightarrow \infty} \mathbb{P}(F_n \leq u) \geq \mathbb{P}(F \leq u),$$

or, equivalently,

$$(6.67) \quad \limsup_{n \rightarrow \infty} \mathbb{P}(F_n > u) \leq \mathbb{P}(F > u).$$

The proof works essentially in the same way as for the upper bound. Again fix  $\theta, \varepsilon, \eta > 0$  and, using Proposition 6.1,  $\delta > 0$  and  $\varepsilon_1(\delta)$  so that for  $\varepsilon_0 < \varepsilon_1(\delta)$

$$(6.68) \quad \liminf_n \mathbb{P}(\Omega_n(\delta, \varepsilon_0, \varepsilon, \eta)) > 1 - \theta.$$

We choose  $\varepsilon_0 < \varepsilon_1(\delta)$  small enough so that

- (1) the conclusion of Proposition 4.1 is satisfied;
- (2) the following inequality holds

$$(6.69) \quad \frac{\lambda(\beta)}{\varepsilon_0} > 2\lambda(\beta) + \frac{\pi^2}{2\delta^2}.$$

Using Proposition 4.1, for  $n$  large enough,

$$(6.70) \quad \begin{aligned} \mathbb{P}(F_n > u) &\leq \mathbb{P}\left(\min_{1 < \ell \leq N^{1+\kappa}} G_n^\beta(\ell) > u - \eta, \Omega_n\right) + \mathbb{P}(\Omega_n^c) \\ &\leq \mathbb{P}\left(\min_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} G_n^\beta(\ell) > u - \eta, \Omega_n\right) + \mathbb{P}(\Omega_n^c) \\ &\leq \mathbb{P}\left(\min_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} \tilde{G}_n^\beta(\ell) > u - 2\eta, \Omega_n\right) + \mathbb{P}\left(\max_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} |\tilde{G}_n^\beta(\ell) - G_n^\beta(\ell)| \geq \eta, \Omega_n\right) + \mathbb{P}(\Omega_n^c). \end{aligned}$$



The second term in this last equation is treated exactly in the same way as the second term in (6.51) and is thus zero for  $n$  large enough. The third one is smaller than  $\theta$  by (6.68) for  $n$  large enough. We thus focus on the first one. The choice of  $\varepsilon_0$  in (6.69) implies that

$$(6.71) \quad \Omega_n \subset \{\operatorname{argmin} \tilde{G}_n < N/\varepsilon_0\}.$$

Indeed, as  $\Omega_n \subset A_n^{(4)}(\delta, \varepsilon)$ , it holds that, on  $\Omega_n$ ,

$$(6.72) \quad \min_{N \leq \ell \leq 2N} \tilde{G}_n^\beta < 2\lambda(\beta) + \frac{\pi^2}{2\delta^2} < \frac{\lambda(\beta)}{\varepsilon_0} < \min_{\ell > N/\varepsilon_0} \tilde{G}_n^\beta(\ell).$$

Therefore, for any compact set  $K$  in  $E$ ,

$$(6.73) \quad \begin{aligned} \mathbb{P}\left(\min_{\ell \in \mathcal{R}_{\varepsilon_0}(n)} \tilde{G}_n^\beta(\ell) > u - 2\eta, \Omega_n\right) &= \mathbb{P}\left(\min_{\ell \in \mathcal{R}} \tilde{G}_n^\beta(\ell) > u - 2\eta, \Omega_n\right) \\ &= \mathbb{P}\left(\Psi^{\lambda(\beta)}(\Pi_N) > u - 2\eta, \Omega_n\right) \\ &\leq \mathbb{P}\left(\Psi_K^{\lambda(\beta)}(\Pi_N) > u - 2\eta\right). \end{aligned}$$

By Lemma 2.6,  $\mathbb{P}\left(\Psi_K^{\lambda(\beta)}(\Pi_N) > u - 2\eta\right)$  converges to  $\mathbb{P}\left(\Psi_K^{\lambda(\beta)}(\Pi) > u - 2\eta\right)$  when  $N$  goes to infinity. Finally,

$$(6.74) \quad \limsup_{n \rightarrow \infty} \mathbb{P}(F_n > u) \leq \mathbb{P}\left(\Psi_K^{\lambda(\beta)}(\Pi) > u - 2\eta\right) + \theta.$$

By letting  $K$  increase to  $E$ , we obtain

$$(6.75) \quad \limsup_{n \rightarrow \infty} \mathbb{P}(F_n > u) \leq \mathbb{P}\left(\Psi^{\lambda(\beta)}(\Pi) > u - 2\eta\right) + \theta,$$

and we conclude as for the upper bound by letting  $\eta$  and  $\theta$  go to 0.

### APPENDIX A. PROOF OF PROPOSITION 3.3

The proof is divided into several steps. In the following, we partition  $\tau$  into  $p$  disjoint subsets  $\tau^{(i)} = \tau_i + \tau_p \mathbb{Z}$ , for  $0 \leq i < p$ .

**Step 1. Decomposition of the probability.** We first consider the event  $\{\sigma > n, S_n \in \tau\}$  instead of  $\{\sigma > n\}$  and will come back to the original event at the final step. By decomposing according to the visits to  $\tau$  and by using the Markov property, we obtain

$$(A.1) \quad \mathbb{P}(\sigma > n, S_n \in \tau) = \sum_{m=1}^n \sum_{0 < u_1 < \dots < u_m = n} \sum_{x_1, \dots, x_m} \prod_{i=1}^m \left( q_{x_{i-1}, x_i}(u_i - u_{i-1}) e^{-\beta} \right),$$

where  $u_0 = 0, x_0 = 0, x_1, \dots, x_m \in \mathbb{Z}/p\mathbb{Z}$ , and the  $q_{ij}(n)$ 's are a slight modification of the ones defined in (3.9), namely

$$(A.2) \quad q_{ij}(n) = \mathbb{P}_{\tau_i}(S_k \notin \tau, 1 \leq k < n, S_n \in \tau^{(j)}), \quad i, j \in \mathbb{Z}/p\mathbb{Z}, \quad n \geq 1.$$

It will be helpful later in the proof to know the asymptotic behaviour of  $q_{ij}(n)$ , as  $n \rightarrow \infty$ :

$$(A.3) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log q_{ij}(n) = -g(t_{ij}),$$

where  $g$  has been defined in (3.14) and  $t_{ij}$  in (3.11).

**Step 2. Definition of  $\phi(\beta; t_1, \dots, t_p)$ .** Recall the definition of  $Q_{ij}(\phi)$  in (3.10) (with  $q_{ij}$  defined in (A.2)), which is now restricted to  $i, j \in \mathbb{Z}/p\mathbb{Z}$ . For all  $i, j$ ,  $Q_{ij}(\phi)$  is finite and increasing on  $[0, g(t_{ij})]$ , infinite on  $[g(t_{ij}), \infty)$  and its limit at  $g(t_{ij})$  is infinite, by

Proposition 3.1. Let  $\Lambda(\phi)$  be the Perron-Frobenius eigenvalue of  $Q(\phi)$ , defined as infinity when one of the entry is infinite, that is for  $\phi \geq \min g(t_{ij}) = g(t_{\max})$ . We recall that

$$(A.4) \quad \Lambda(\phi) = \sup_{v \neq 0} \min_i \frac{(Q(\phi)v)_i}{v_i}.$$

From what precedes,  $\Lambda$  is increasing on  $[0, g(t_{\max}))$  and tends to  $\infty$  as  $\phi \nearrow g(t_{\max})$ . Therefore, the equation

$$(A.5) \quad \Lambda(\phi) = \exp(\beta)$$

has a unique positive solution on this interval, that we denote by  $\phi(\beta; t_1, \dots, t_p)$ . As  $\phi(\beta; t_1, \dots, t_p) \in [0, g(t_{\max}))$ , this proves the second inequality in (3.20). In the sequel of the proof, for the sake of conciseness, we use the notation  $\phi(\beta) = \phi(\beta; t_1, \dots, t_p)$ . Coming back to (A.1), we get

$$(A.6) \quad \text{r.h.s(A.1)} = e^{-\phi(\beta)n} \sum_{m=1}^n \sum_{0 < u_1 < \dots < u_m = n} \sum_{x_1, \dots, x_m} \prod_{i=1}^m \left( q_{x_{i-1}, x_i}(u_i - u_{i-1}) e^{-\beta + \phi(\beta)(u_i - u_{i-1})} \right).$$

**Step 3. Spectral decomposition and a first upper bound.** The key idea in this step is a spectral decomposition, which is a technique used also in the context of the parabolic Anderson model, see [10, Section 2.2.1]. Let us define a matrix  $Q^\beta$  by

$$(A.7) \quad Q_{ij}^\beta = Q_{ij}(\phi(\beta)) e^{-\beta}, \quad i, j \in \mathbb{Z}/p\mathbb{Z},$$

which is symmetric (by symmetry of the simple random walk) with positive entries. From what precedes, its top eigenvalue is one. Therefore, we may denote by  $1 = \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{p-1}$  its eigenvalues by decreasing order (with possible repetitions), with  $|\lambda_k| < 1$  for  $k > 1$  (Theorem 1.1 in Seneta [14]) and  $(\nu_i)_{0 \leq i < p}$  an associated basis of orthonormal left eigenvectors. Note that for all  $0 \leq x < p$ , we have  $\delta_x = \sum_{i=0}^{p-1} \nu_i(x) \nu_i$ , that is the element of  $\mathbb{R}^p$  which is 1 at coordinate  $x$  and 0 elsewhere. Let us now define, for  $0 \leq a < p$ ,

$$(A.8) \quad \mathcal{Z}_n(a) = \sum_{m=1}^n \sum_{x_0=a, x_1, \dots, x_m} \prod_{i=1}^m Q_{x_{i-1}, x_i}^\beta.$$

By removing the condition  $\{u_m = n\}$  in (A.6), we get the upper bound

$$(A.9) \quad \text{r.h.s(A.6)} \leq e^{-\phi(\beta)n} \mathcal{Z}_n(0).$$

Moreover,

$$(A.10) \quad \mathcal{Z}_n(0) = \langle \delta_0, \mathcal{Z}_n(\cdot) \rangle = \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \nu_i(0) \nu_i(j) \mathcal{Z}_n(j),$$

(with the usual scalar product) which yields

$$(A.11) \quad \mathcal{Z}_n(0) = \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \sum_{m=1}^n \nu_i(0) \nu_i(j) \sum_{x_0=j, x_1, \dots, x_m} \prod_{k=1}^m Q_{x_{k-1}, x_k}^\beta.$$

By definition of the  $\nu_i$ 's we get for  $0 \leq i < p$ ,

$$(A.12) \quad \sum_{j=0}^{p-1} \nu_i(j) \sum_{x_0=j, x_1, \dots, x_m} \prod_{k=1}^m Q_{x_{k-1}, x_k}^\beta = \nu_i(Q^\beta)^m \mathbf{1} = \lambda_i^m \sum_{j=0}^{p-1} \nu_i(j),$$

where  $\mathbf{1}$  is the vector with all one. Therefore,

$$(A.13) \quad \mathcal{Z}_n(0) = \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \sum_{m=1}^n \nu_i(0) \nu_i(j) \lambda_i^m \leq n \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} |\nu_i(0)| |\nu_i(j)| \leq np,$$

where in the first inequality we use that  $|\lambda_i| \leq 1$  and the triangular inequality, while in the second inequality we use the Cauchy-Schwarz inequality and the fact that

$$(A.14) \quad \sum_{j=0}^{p-1} \nu_i(j)^2 = \|\nu_i\|_2^2 = 1, \quad \sum_{i=0}^{p-1} \nu_i(0)^2 = \|\delta_0\|_2^2 = 1.$$

Finally, we have obtained

$$(A.15) \quad \mathbb{P}(\sigma > n, S_n \in \tau) \leq np e^{-\phi(\beta)n}.$$

**Step 4. Lower bound in (3.15).** The components of a Perron-Frobenius eigenvector being all positive (or all negative), we may consider the matrix  $\{Q_{ij}^\beta \frac{\nu_0(j)}{\nu_0(i)}\}_{ij}$ , which turns out to be stochastic. This actually defines a Markov renewal process  $\rho$  on  $\mathbb{Z}/p\mathbb{Z}$  (see e.g. Section VII.4 in Asmussen [1]) with law  $\mathcal{P}_\beta$  determined by the kernel

$$(A.16) \quad q_{ij}^\beta(n) = \exp(\phi(\beta)n - \beta) q_{ij}(n) \frac{\nu_0(j)}{\nu_0(i)}, \quad n \geq 1, \quad i, j \in \mathbb{Z}/p\mathbb{Z},$$

and starting from state 0. Therefore, we may write

$$(A.17) \quad \begin{aligned} \mathbb{P}(\sigma > n, S_n \in \tau) &\geq \mathbb{P}(\sigma > n, S_n \in \tau^{(0)}) \\ &= e^{-\phi(\beta)n} \sum_{m=1}^n \sum_{0 < u_1 < \dots < u_m = n} \sum_{\substack{x_1, \dots, x_{m-1} \\ x_0 = x_m = 0}} \prod_{i=1}^m \left( q_{x_{i-1}, x_i}(u_i - u_{i-1}) e^{-\beta + \phi(\beta)(u_i - u_{i-1})} \right) \\ &= e^{-\phi(\beta)n} \sum_{m=1}^n \sum_{0 < u_1 < \dots < u_m = n} \sum_{\substack{x_1, \dots, x_{m-1} \\ x_0 = x_m = 0}} \prod_{i=1}^m q_{x_{i-1}, x_i}^\beta(u_i - u_{i-1}) \\ &= e^{-\phi(\beta)n} \mathcal{P}_\beta(n \in \rho_0), \end{aligned}$$

where  $\rho_0$  is the subset of  $\rho$  formed by the Markov renewal points with state 0. It turns out that it is a renewal process. By the second inequality in (3.20) (that we have already proven in Step 2)  $q_{ij}^\beta$  decays exponentially in  $n$  for all  $i, j \in \mathbb{Z}/p\mathbb{Z}$ . This implies (as the modulating Markov chain has finite state space) that the inter-arrival law of  $\rho_0$  also decays exponentially in  $n$ , which implies integrability. Therefore, by the renewal theorem,  $\mathcal{P}_\beta(n \in \rho_0)$  converges to some constant (that is the inverse of the mean inter-arrival time). This concludes this step.

**Step 5. Proof of (3.20).** The second inequality has already been established in Step 2, so let us prove the first inequality. A standard coupling argument yields

$$(A.18) \quad \sum_j Q_{ij}(\phi) = \mathbb{E}_{\tau_i}(e^{\phi\theta_1}) \leq \mathbb{E}(e^{\phi\theta_1^{\max}}), \quad i \in \mathbb{Z}/p\mathbb{Z},$$

where  $\theta_1 = \inf\{n \geq 1 : S_n \in \tau\}$  and  $\theta_1^{\max} = \inf\{n \geq 1 : S_n \in t_{\max}\mathbb{Z}\}$ . By Proposition 3.2, we get that  $\sum_j Q_{ij}(\phi(\beta, t_{\max})) = e^\beta$ . Thanks to Lemma A.1 below, it means that  $\Lambda(\phi(\beta, t_{\max})) \leq e^\beta$  and we get the desired bound, as  $\Lambda$  is non-decreasing.

**Step 6. Final upper bound.** We now conclude by removing the condition  $\{S_n \in \tau\}$  in the upper bound. To this end, we decompose according to the last visit to  $\tau$  before  $n$ :

$$(A.19) \quad \mathbb{P}(\sigma > n, S_n \notin \tau) = \sum_{m=0}^{n-1} \sum_{j=0}^{p-1} \mathbb{P}(\sigma > m, S_m \in \tau^{(j)}) \mathbb{P}_{\tau_j}(S_k \notin \tau, k \leq n-m).$$

By using Proposition 3.1, we get that there exists  $C$  such that for all  $0 \leq j < p$  and  $n \geq 1$ ,

$$(A.20) \quad \mathbb{P}_{\tau_j}(S_k \notin \tau, k \leq n) \leq C e^{-\min g(t_{ij})n} = C e^{-g(t_{\max})n}.$$

By using (3.20), we get

$$(A.21) \quad \begin{aligned} \mathbb{P}(\sigma > n, S_n \notin \tau) &\leq C \sum_{m=0}^n \sum_{j=0}^{p-1} \mathbb{P}(\sigma > m, S_m \in \tau^{(j)}) e^{-\phi(\beta)(n-m)} \\ &= C \sum_{m=0}^n \mathbb{P}(\sigma > m, S_m \in \tau) e^{-\phi(\beta)(n-m)} \\ &\leq C n^2 p \exp(-\phi(\beta)n), \end{aligned}$$

where we have used (A.15) to go from the second to the last line.

**Lemma A.1.** *If the sums over lines of a non-negative matrix  $A$  are less than one, then its Perron-Frobenius eigenvalue is less than one.*

*Proof of Lemma A.1.* Let  $\lambda$  be an eigenvalue of  $A$  and  $v$  an associated eigenvector such that  $v_{i_*} = \max_i v_i > 0$ . Then

$$(A.22) \quad \lambda v_{i_*} = (Av)_{i_*} \leq \sum_j A_{i_*,j} v_{i_*} \leq v_{i_*},$$

and that is enough to conclude as  $v_{i_*} > 0$ . ■

## APPENDIX B. PROOF OF PROPOSITION 3.5

The main tool is an inequality of the FKG-type. The strategy is to write the events in Proposition 3.5 in terms of the local times of the walk. Then, we link these local times to a certain Galton-Watson process, by using a version of Ray-Knight's theorem. Finally we prove positive association of the relevant variables, thanks to the Galton-Watson representation.

*Local times.* Let  $m \in \mathbb{N}$ . We introduce the oriented edge local times

$$(B.1) \quad \ell_i^{(m)} = \sum_{1 \leq k \leq m} \mathbf{1}_{\{S_{k-1}=i-1, S_k=i\}}, \quad i \in \mathbb{N},$$

as well as the vertex local times

$$(B.2) \quad L_i^{(m)} = \sum_{1 \leq k \leq m} \mathbf{1}_{\{S_k=i\}}, \quad i \in \mathbb{N}_0.$$

It will be convenient for what comes next to also define

$$(B.3) \quad Z_i^{(m)} = \ell_i^{(m)} - 1, \quad i \in \mathbb{N}.$$

Let  $x \in \mathbb{N}$ . Notice that on the event  $\{S_m \geq x\}$ ,

$$(B.4) \quad L_i^{(m)} = Z_i^{(m)} + Z_{i+1}^{(m)} + 1, \quad 1 \leq i < x.$$

*A Galton-Watson process with immigration.* Let us fix  $x \in \mathbb{N}$  and introduce the random variables  $(\mathcal{Z}_i)_{1 \leq i \leq x}$  with law  $\mathcal{P}_{0,x}$  (and associated expectation  $\mathcal{E}_{0,x}$ ), that are defined by

- $\mathcal{Z}_1 = 0$ ;
- for  $1 < i < x$  and conditionally on  $\sigma(\mathcal{Z}_1, \dots, \mathcal{Z}_i)$ , the law of  $\mathcal{Z}_{i+1}$  is that of the sum of  $\mathcal{Z}_i + 1$  (the additional term 1 is interpreted as an immigrant) i.i.d. random variables with law  $\text{Geo}_0(1/2)$ , where  $\mathbb{P}(\text{Geo}_0(1/2) = k) = 2^{-(k+1)}$  for  $k \in \mathbb{N}_0$ .

In other terms,  $\mathcal{P}_{0,x}$  is the law of a Galton-Watson process with zero ancestor, one immigrant at each generation up to generation  $x$  (included) and  $\text{Geo}_0(1/2)$  as reproduction law. The random variable  $\mathcal{Z}_i$  stands for the number of individuals at generation  $i$  (except for the immigrant). The total population between generations 1 and  $x$  included will be denoted by

$$(B.5) \quad \Xi = \sum_{1 \leq i \leq x} \mathcal{Z}_i.$$

*Ray-Knight's theorem on  $\mathbb{Z}$ .* The link between the local times of the stopped random walk and the Galton-Watson process introduced above is given by the following proposition:

**Proposition B.1.** *For every non-negative or bounded function  $\varphi: \mathbb{R}^x \rightarrow \mathbb{R}$ ,*

$$(B.6) \quad \mathbb{E}[\varphi(Z_i^{(m)}, 1 \leq i \leq x) \mathbf{1}_{\{H_x < H_0, H_x = m\}}] = \mathcal{E}_{0,x}[\varphi(\mathcal{Z}_i, 1 \leq i \leq x) \mathbf{1}_{\{\mathcal{Z}_x = 0, \Xi = (m-x)/2\}}].$$

*Proof of Proposition B.1.* Note that

$$(B.7) \quad \{H_x < H_0, H_x = m\} = \{Z_x^{(m)} = 0, L_0^{(m)} = 0, S_m = x\}.$$

The result follows by applying the version of Ray-Knight's theorem as stated in Theorem 2.3 in Caravenna, den Hollander, Pétrelis and Poisat [2].  $\blacksquare$

Observe that from this last proposition one can recover the classical version of Ray-Knight's theorem:

**Proposition B.2.** *With equal probability the sequence  $(\ell_x^{(H_0)})_{x \in \mathbb{N}}$  is either identically 0 or a critical Galton-Watson process started at 1 with  $\text{Geo}_0(1/2)$  as offspring distribution.*

This classical representation allows us to use the following proposition that provides a useful decoupling inequality.

**Proposition B.3.** *Let  $x \geq 1$ . Under  $\mathcal{P}_{0,x}(\cdot | \mathcal{Z}_x = 0)$ , the components of  $\mathcal{Z} = (\mathcal{Z}_i)_{1 \leq i \leq x}$  are positively associated. This means that for every pair of non-decreasing (coordinate-wise) and bounded functions  $\varphi_1, \varphi_2: \mathbb{R}^x \mapsto \mathbb{R}$ ,*

$$(B.8) \quad \mathcal{E}_{0,x}[\varphi_1(\mathcal{Z})\varphi_2(\mathcal{Z}) | \mathcal{Z}_x = 0] \geq \mathcal{E}_{0,x}[\varphi_1(\mathcal{Z}) | \mathcal{Z}_x = 0] \mathcal{E}_{0,x}[\varphi_2(\mathcal{Z}) | \mathcal{Z}_x = 0].$$

*Proof of Proposition B.3.* We proceed in two steps.

- (i) Let us first establish the same inequality without conditioning. In this case the components of  $\mathcal{Z}$  are that of a Markov chain and, by a suitable conditioning and the standard one-dimensional FKG inequality, the proof reduces to showing the following property: for all  $j \geq 1$  and  $\varphi: \mathbb{R}^{j+1} \mapsto \mathbb{R}$  non-decreasing coordinate-wise it holds that  $(z_1, \dots, z_j) \mapsto \mathcal{E}_{0,x}[\varphi(z_1, \dots, z_j, \mathcal{Z}_{j+1}) | \mathcal{Z}_j = z_j]$  is also non-decreasing coordinate-wise. The latter is true as the law of  $\mathcal{Z}_{i+1}$  conditioned on  $\mathcal{Z}_i$  is, by a standard coupling, stochastically non-decreasing in  $\mathcal{Z}_i$ .
- (ii) Inequality (B.8) then follows with the same arguments once we notice that the law of a Galton-Watson tree conditioned on extinction is that of a Galton-Watson tree with inhomogeneous reproduction law.

■

*Conclusion: proof of Proposition 3.5.* We will prove the following decoupling inequality, that is equivalent to the statement of the proposition:

$$(B.9) \quad \mathbb{P}^\beta(H_x \leq n, \sigma \wedge H_0 > H_x) \geq \mathbb{P}^\beta(\sigma \wedge H_0 > H_x) \mathbb{P}(H_x \leq n \mid H_0 > H_x).$$

Let us denote the time spent on traps up to  $H_x$  by

$$(B.10) \quad \Theta(\tau) = \sum_{j \in \tau \cap [1, x]} L_j^{(H_x)}.$$

By (B.4), observe that

$$(B.11) \quad \Theta(\tau) = \sum_{j \in \tau \cap [1, x]} L_j^{(H_x)} = 1 + \sum_{j \in \tau \cap [1, x]} \left( Z_j^{(H_x)} + Z_{j+1}^{(H_x)} + 1 \right).$$

With these notations in hand, we get by integrating over  $\mathcal{N}$  that

$$(B.12) \quad \text{l.h.s (B.9)} = \mathbb{E} \left( \exp(-\beta \Theta(\tau)) \mathbf{1}_{\{H_x \leq n, H_x < H_0\}} \right).$$

Thanks to Proposition B.1, this is equal to

$$(B.13) \quad \mathcal{E}_{0,x} \left[ \exp \left( -\beta \left( 1 + \sum_{j \in \tau \cap [1, x]} [Z_j + Z_{j+1} + 1] \right) \right) \mathbf{1}_{\{\Xi \leq (n-x)/2, Z_x = 0\}} \right].$$

Since the random variables  $\exp \left( -\beta \left( 1 + \sum_{j \in \tau \cap [1, x]} [Z_j + Z_{j+1} + 1] \right) \right)$  and  $\mathbf{1}_{\{\Xi \leq (n-x)/2\}}$  are both non-increasing in  $(Z_1, \dots, Z_x)$  (coordinate-wise), we get from Proposition B.3, after conditioning on  $\{Z_x = 0\}$ , that

$$(B.14) \quad \text{l.h.s (B.9)} \geq \mathcal{E}_{0,x} \left[ \exp \left( -\beta \left( 1 + \sum_{j \in \tau \cap [1, x]} [Z_j + Z_{j+1} + 1] \right) \right) \mathbf{1}_{\{Z_x = 0\}} \right] \mathcal{P}_{0,x}(\Xi \leq (n-x)/2 \mid Z_x = 0).$$

Proposition B.1 allows us to proceed backwards, meaning that we interpret these two factors in terms of the random walk, and we get the result. ■

## REFERENCES

- [1] S. Asmussen. *Applied probability and queues*, volume 51 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 2003. Stochastic Modelling and Applied Probability.
- [2] F. Caravenna, F. den Hollander, N. P  tr  lis, and J. Poisat. Annealed scaling for a charged polymer. *Math. Phys. Anal. Geom.*, 19(1):Art. 2, 87, 2016.
- [3] F. Caravenna and N. P  tr  lis. Depinning of a polymer in a multi-interface medium. *Electron. J. Probab.*, 14:no. 70, 2038–2067, 2009.
- [4] F. Caravenna and N. P  tr  lis. A polymer in a multi-interface medium. *Ann. Appl. Probab.*, 19(5):1803–1839, 2009.
- [5] F. den Hollander. *Random polymers*, volume 1974 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. Lectures from the 37th Probability Summer School held in Saint-Flour, 2007.
- [6] R. Durrett. *Probability: theory and examples*, volume 31 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, fourth edition, 2010.
- [7] G. Giacomin. *Random polymer models*. Imperial College Press, London, 2007.
- [8] G. Giacomin. *Disorder and critical phenomena through basic probability models*, volume 2025 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011. Lecture notes from the 40th Probability Summer School held in Saint-Flour, 2010,   cole d’  t   de Probabilit  s de Saint-Flour. [Saint-Flour Probability Summer School].
- [9] R. Gouet, F. J. L  pez, and G. Sanz. Central limit theorems for the number of records in discrete models. *Adv. in Appl. Probab.*, 37(3):781–800, 2005.

- [10] W. König. *The parabolic Anderson model*. Pathways in Mathematics. Birkhäuser/Springer, 2016. Random walk in random potential.
- [11] V. V. Petrov. *Limit theorems of probability theory*, volume 4 of *Oxford Studies in Probability*. The Clarendon Press, Oxford University Press, New York, 1995. Sequences of independent random variables, Oxford Science Publications.
- [12] A. Rényi. Théorie des éléments saillants d'une suite d'observations. *Ann. Fac. Sci. Univ. Clermont-Ferrand No.*, 8:7–13, 1962.
- [13] S. I. Resnick. *Extreme values, regular variation and point processes*. Springer Series in Operations Research and Financial Engineering. Springer, New York, 2008. Reprint of the 1987 original.
- [14] E. Seneta. *Non-negative matrices and Markov chains*. Springer Series in Statistics. Springer, New York, 2006. Revised reprint of the second (1981) edition [Springer-Verlag, New York; MR0719544].
- [15] A.-S. Sznitman. *Brownian motion, obstacles and random media*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

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