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Linear wavelet estimation in regression with additive and multiplicative noise

Christophe Chesneau, Junke Kou and Fabien Navarro

Abstract In this paper, we deal with the estimation of an unknown function from a nonparametric regression model with both additive and multiplicative noises. The case of the uniform multiplicative noise is considered. We develop a projection estimator based on wavelets for this problem. We prove that it attains a fast rate of convergence under the mean integrated square error over Besov spaces. A practical extension to automatically select the truncation parameter of this estimator is discussed. A numerical study illustrates the usefulness of this extension.

Key words: Nonparametric regression, Multiplicative noise, Rates of convergence, Wavelets

1 Introduction

We consider the following unidimensional nonparametric regression model

$$Y_i = U_i f(X_i) + V_i, \quad i \in \{1, \dots, n\}, \quad (1)$$

where $f : [0, 1] \rightarrow \mathbb{R}$ is an unknown regression function, X_1, \dots, X_n are n identically distributed random variables with support on $[0, 1]$, U_1, \dots, U_n are n identically dis-

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tributed random variables having the uniform distribution on a symmetric interval around 0 and V_1, \dots, V_n are n identically distributed random variables. Moreover, it is supposed that X_i and U_i are independent, and U_i and V_i are independent for any $i \in \{1, \dots, n\}$. Additional technical assumptions on the model will be formulated later. We aim to estimate the unknown function $r := f^2$ from $(X_1, Y_1), \dots, (X_n, Y_n)$; the random vectors $(U_1, V_1), \dots, (U_n, V_n)$ form the multiplicative-additive noise. The model (1) can be viewed as a natural extension of the standard nonparametric regression model; the main novelty is the presence of a multiplicative uniform noise that perturbed the unknown function f . Such multiplicative regression model as (1) is very popular in various application areas, particularly in signal processing (e.g. for Global Positioning System (GPS) signal detection in which not only additive noise but also multiplicative noise is encountered [14]), or in econometrics (e.g. for volatility estimation where the source of noise is multiplicative [13], also for deterministic and stochastic frontier estimation where the noise is multiplicative and both multiplicative and additive respectively [22]). On the other hand, let us mention that some connexions exist with the so called heteroscedastic nonparametric regression model. See, for instance, [6], [8] and [3]. In particular, [3] consider the estimation of r in the heteroscedastic nonparametric regression model defined as (1) with X_1 deterministic, V_1 deterministic but unknown (it is an unknown function of X_1) and general assumptions on U_1 . The form of the model is the same but the intrinsic definition is different. In this paper, we propose to estimate r with wavelet methods. Such methods have the advantage to capture the possible complexity of this unknown function. A natural linear wavelet estimator is then developed. With a suitable choice of a tuning parameter inherent of this estimator, we prove that it attains a fast rate of convergence under the mean integrated square error over Besov spaces. One drawback of this estimator is that the theoretical choice for the tuning parameter depends on a supposed unknown smoothness of r . We then provide practical solution to this problem to choose the truncation level of our linear wavelet estimator using an adapted version of the 2-Fold Cross Validation (2FCV) method introduced by Nason [18]. A numerical study is performed to show the applicability of this extension.

The rest of this paper is organized as follows. In Section 2, we briefly present basics on wavelets and Besov balls. Additional assumptions on the model (1), the considered wavelet estimator and the main result are given in Section 3. Section 4 is devoted to the simulation study. The technical details for the proof of our main result are postponed in Section 6.

2 Basics on wavelets and Besov balls

For the purpose of this paper, we use the compactly supported wavelets of the Daubechies family. We present the essential below, all the details can be found in, e.g., [9] and [16]. For any $j \geq 0$, we set $\Lambda_j = \{0, \dots, 2^j - 1\}$ and, for $k \in \Lambda_j$,

$$\phi_{j,k}(x) = 2^{j/2} \phi(2^j x - k), \quad \psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k).$$

Following the methodology of [7], there exists an integer τ such that, for any integer $j_0 \geq \tau$, the collection of functions

$$\mathcal{S} = \{\phi_{j_0,k}, k \in \Lambda_{j_0}; \psi_{j,k}; j \in \mathbb{N} - \{0, \dots, j_0 - 1\}, k \in \Lambda_j\}$$

forms an orthonormal basis of $\mathbb{L}^2([0, 1])$. Therefore, for any integer $j_0 \geq \tau$ and $h \in \mathbb{L}^2([0, 1])$, we have the following wavelet expansion:

$$h(x) = \sum_{k \in \Lambda_{j_0}} \alpha_{j_0,k} \phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_{k \in \Lambda_j} \beta_{j,k} \psi_{j,k}(x), \quad x \in [0, 1],$$

where

$$\alpha_{j_0,k} = \int_0^1 h(x) \phi_{j_0,k}(x) dx, \quad \beta_{j,k} = \int_0^1 h(x) \psi_{j,k}(x) dx,$$

Also, let us mention that $\int_0^1 \phi_{j,k}(x) dx = 2^{-j/2}$, which will be a crucial technical point in the proof. Let P_j be the orthogonal projection operator from $L^2([0, 1])$ onto the space V_j with the orthonormal basis $\{\phi_{j,k}(\cdot) = 2^{j/2} \phi(2^j \cdot - k), k \in \Lambda_j\}$. Then, for any $h \in L^2([0, 1])$, we have

$$P_j h(x) = \sum_{k \in \Lambda_j} \alpha_{j,k} \phi_{j,k}(x), \quad x \in [0, 1].$$

Besov spaces have the feature to capture a wide variety of smoothness properties in a function including spatially inhomogeneous behavior, see [12, 17, 23] for further details. Definitions of those spaces are given below. Suppose that ϕ is m regular (i.e. $\phi \in C^m$ and $|D^\alpha \phi(x)| \leq c(1 + |x|^2)^{-l}$ for each $l \in \mathbb{Z}$, with $\alpha = 0, 1, \dots, m$). Let $h \in L^p([0, 1])$, $p, q \in [1, \infty]$ and $0 < s < m$. Then the following assertions are equivalent:

(1) $h \in B_{p,q}^s([0, 1])$; (2) $\{2^{js} \|P_{j+1} h - P_j h\|_p\} \in l_q$; (3) $\{2^{j(s - \frac{1}{p} + \frac{1}{2})} \|\beta_{j,\cdot}\|_p\} \in l_q$.
The Besov norm of h can be defined by

$$\|h\|_{B_{p,q}^s} := \|(\alpha_{\tau,\cdot})\|_p + \|(2^{j(s - \frac{1}{p} + \frac{1}{2})} \|\beta_{j,\cdot}\|_p)_{j \geq \tau}\|_q, \quad \text{where } \|\beta_{j,\cdot}\|_p^p = \sum_{k \in \Lambda_j} |\beta_{j,k}|^p.$$

3 Assumptions, estimators and main result

Technical assumptions on the model (1) are formulated below.

- A.1** We suppose that $f : [0, 1] \rightarrow \mathbb{R}$ is bounded from above.
- A.2** We suppose that $X_1 \sim \mathcal{U}([0, 1])$.
- A.3** We suppose that $U_1 \sim \mathcal{U}([- \theta, \theta])$ with $\theta > 0$ a fixed real number.
- A.4** We suppose that V_1 has a moment of order 4.

A.5 We suppose that X_i and V_i are independent for any $i \in \{1, \dots, n\}$.

Let us observe that **A.2** specifies that we consider a uniform design and that **A.3** specifies that the uniform multiplicative noise is considered over a symmetric interval around 0. The assumption **A.5** implies that V_i is not a function of X_i a fortiori.

We construct our linear wavelet estimators for r as follows:

$$\hat{r}_{j_0, n}(x) := \sum_{k \in \Lambda_{j_0}} \hat{\alpha}_{j_0, \mathbf{k}} \phi_{j_0, k}(x), \quad x \in [0, 1], \quad (2)$$

where

$$\hat{\alpha}_{j, k} := \frac{3}{\theta^2} \left(\frac{1}{n} \sum_{i=1}^n Y_i^2 \phi_{j, k}(X_i) - \mathbb{E}(V_1^2) 2^{-j/2} \right). \quad (3)$$

The definition of $\hat{\alpha}_{j, k}$ rests on technical consideration which will be presented later. In spite of the simplicity of its construction, its performances strongly depend on the choice of level j_0 . Further details on the linear wavelet estimator in a standard nonparametric regression setting can be found in [12]. Recent developments can be found in [4].

The following result determines the rates of convergence attained by $\hat{r}_{j_0, n}$ via the MISE over Besov spaces.

Proposition 1. *Consider the problem defined by (1) under the assumptions **A.1–A.5**, let $r \in B_{p, q}^s([0, 1])$ with $p, q \in [1, \infty)$, $s > 1/p$. Then the linear wavelet estimator $\hat{r}_{j_0, n}$ with $2^{j_*} \sim n^{\frac{1}{2s'+1}}$ and $s' = s - (1/p - 1/2)_+$ satisfies*

$$\mathbb{E} \left[\int_0^1 (\hat{r}_{j_0, n}(x) - r(x))^2 dx \right] \lesssim n^{-\frac{2s'}{2s'+1}}.$$

The level j_0 as defined in Proposition 1 is chosen to minimize as possible the MISE of $\hat{r}_{j_0, n}$ over Besov spaces. The rate of convergence $n^{-\frac{2s'}{2s'+1}}$ is not a surprise ; it generally corresponds to the one obtained in the standard nonparametric regression estimation. See [12], [24] and [8]. The proof of Proposition 1 is based on a suitable decomposition of the MISE and some intermediary results on the probabilistic properties of the wavelet coefficient estimator (3) (see Lemmas 1 and 2 in Section 6). The rest of this section is devoted to the practical aspect of the estimator (2), with alternatives on the choice of the level j_0 . In particular, we propose a candidate by adapting version of the 2-Fold Cross Validation (2FCV) method originally developed by Nason for choosing the threshold parameter in wavelet shrinkage [18].

4 Simulation study

In order to illustrate the empirical performance of the proposed estimator, a numerical illustration was produced. In order to set in a realistic context, we pro-

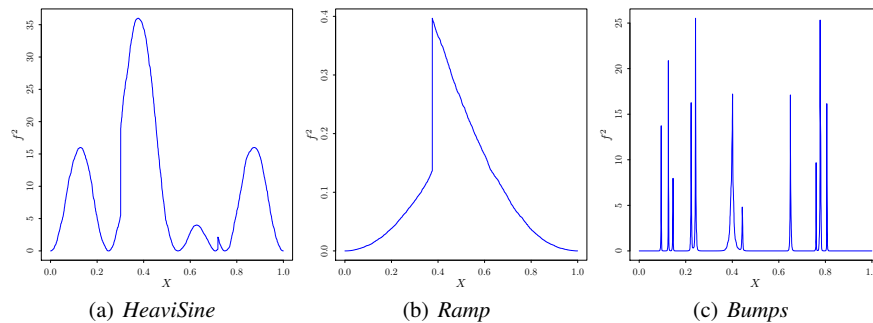


Fig. 1 (a)–(c): The three test (squared) functions to be estimated.

posed to use an automatic selection method of the estimator truncation parameter (not depending on the regularity of the function to be estimated). Simulations were performed using R and in particular the `rwavelet` package [21], available from <https://github.com/fabnavarro/rwavelet>.

The simulated data were generated according to (1), where $n = 4096$, X_i 's are uniformly distributed on $[0, 1]$, U_i 's are $\mathcal{U}([-1, 1])$ (so $\theta = 1$) and V_i are $\mathcal{N}(0, \sigma^2)$ variables and independent of X_i 's with $\sigma^2 = 0.01$. Daubechies' compactly-supported wavelet with 8 vanishing moments were used. We consider three standard test functions for f , commonly used in the wavelet literature (*HeaviSine*, *Ramp* and *Bumps*, see [10]). Recall that we wish to estimate $r = f^2$. The squared version of those functions are plotted in Figure 1.

In the case of fixed design, the calculation of wavelet-based estimators is simple and fast, thanks to Mallat's pyramidal algorithm ([16]). In the case of uniform random design, the implementation requires some changes and several strategies have been developed in the literature (see e.g. [1, 11]). For uniform design regression, [2] proposed to use and studied an approach in which the wavelet coefficients are computed by a simple application of Mallat's algorithm using the ordered Y_i 's as input variables. We have followed this approach because it preserves the simplicity of calculation and the efficiency of the equispaced algorithm. In the context of wavelet regression in random design with heteroscedastic noise, [20] and [15] also adopted this approach. Nason adjusted the usual 2FCV method to choose the threshold parameter in wavelet shrinkage (see [18]). His strategy was used for the selection of linear wavelet estimators by [20]. We have chosen to use this approach to select the truncation parameter j_0 of the linear estimator $\hat{r}_{j_0, n}$. More precisely, we built a collection of linear estimators $\hat{r}_{j_0, n}, j_0 = 0, 1, \dots, \log_2(n) - 1$ (by successively adding whole resolution levels of wavelet coefficients), and select the best among this collection by minimizing a 2FCV criterion denoted by $2FCV(j_0)$. The resulting estimator of the truncation level is denoted by \hat{j}_0 and the corresponding estimator of r by $\hat{r}_{\hat{j}_0, n}$ (see [20, 19] for more details).

For a single experiment, and for each of the three test functions, with a sample size $n = 4096$, we display the observations and the unknown function r in

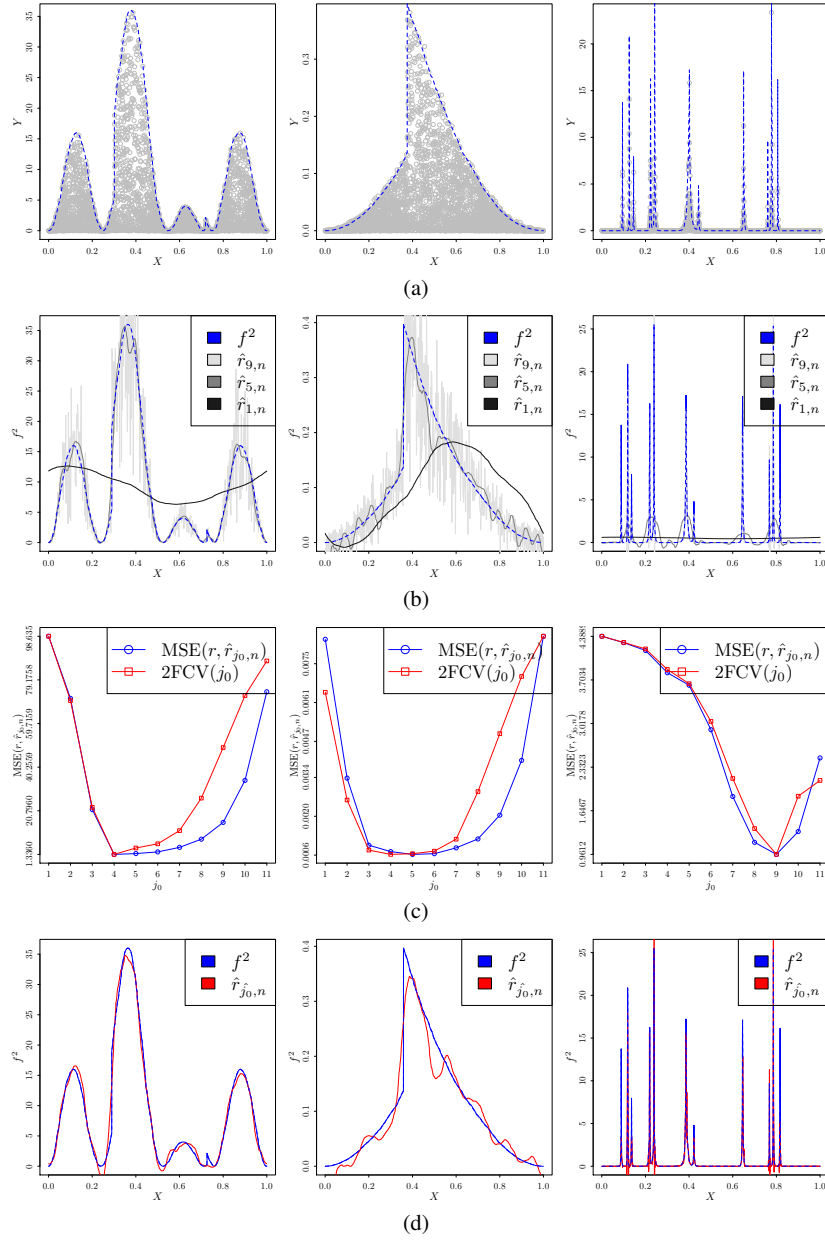


Fig. 2 (a): Noisy observations (X, Y^2) . (b): Sample of the model collection. (c): Graph of the MSE (blue) against j_0 and (re-scaled) 2FCV criterion. (d): Typical estimations from one simulation with $n = 4096$. Blue lines indicate the true functions, red lines correspond to the estimators $\hat{r}_{j_0,n}$.

Figure 2(a). A sample of three estimators from the collection is also shown in the Figure 2(b). Graphs of the curves associated with the selection criterion (i.e. $2FCV(j_0)$) are also displayed in Figure 2(c). In order to be able to evaluate the performance of this criterion, the Mean Square Error curves (i.e. $MSE(\hat{r}_{j_0,n}, r) = \frac{1}{n} \sum_{i=1}^n (r(X_i) - \hat{r}_{j_0,n}(X_i))^2$) are also shown (in blue). We denote by j_0^* , the parameter selected by minimizing this quantity. It can be observed that $2FCV(j_0)$ gives very reliable estimate for the $MSE(\hat{r}_{j_0,n}, r)$, and in turn, also a high-quality estimate of the optimal model. Indeed, in this case, the method allows to find the oracle of the collection (i.e. that obtained by assuming the regularity of the function to be estimated known) for *HeaviSine* and *Bumps* and a model of lower complexity for *Ramp* (i.e. $\hat{j}_0 = 4$ and $j_0^* = 5$).

5 Conclusion

In this paper, we develop a simple wavelet methodology for the problem of estimating an unknown function subject to additive and multiplicative noises. Focusing on a uniform multiplicative noise, we construct a linear wavelet estimator that attains a fast rate of convergence. Then some extensions of the estimator are presented, with a numerical study showing the usefulness of the method.

A possible extension of this work would be to consider a more general model with θ unknown for instance or a more general assumption on the distribution of the multiplicative noise. Another possible extension would be to construct another wavelet estimation procedure involving thresholding of the wavelet coefficient estimators and also dependence on the observations, as in [5] for the additive noise only. These aspects need further investigations that we leave for a future work.

6 Proofs

To prove Proposition 1, we use the following two lemmas.

Lemma 1. *Let $j \geq \tau$, $k \in \Lambda_j$, $\hat{\alpha}_{j,k}$ be (3). Then, under A.1–A.5, we have*

$$\mathbb{E}[\hat{\alpha}_{j,k}] = \alpha_{j,k}.$$

Proof of Lemma 1. Using the independence assumptions on the random variables, A.1–A.5 with $\mathbb{E}[U_1] = 0$, observe that

$$\mathbb{E}[U_1 V_1 f(X_1) \phi_{j,k}(X_1)] = \mathbb{E}[U_1] \mathbb{E}[V_1] \mathbb{E}[f(X_1) \phi_{j,k}(X_1)] = 0$$

and

$$\mathbb{E}[V_1^2 \phi_{j,k}(X_1)] = \mathbb{E}[V_1^2] \mathbb{E}[\phi_{j,k}(X_1)] = \mathbb{E}[V_1^2] \int_0^1 \phi_{j,k}(x) dx = \mathbb{E}[V_1^2] 2^{-j/2}.$$

Therefore, using similar mathematical arguments with $\mathbb{E}[U_1^2] = \frac{\theta^2}{3}$, we have

$$\begin{aligned}
\mathbb{E}[\hat{\alpha}_{j,k}] &= \frac{3}{\theta^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n Y_i^2 \phi_{j,k}(X_i) - \mathbb{E}[V_1^2] 2^{-j/2} \right] \\
&= \frac{3}{\theta^2} \left(\mathbb{E}[Y_1^2 \phi_{j,k}(X_1)] - \mathbb{E}[V_1^2] 2^{-j/2} \right) \\
&= \frac{3}{\theta^2} \left(\mathbb{E}[U_1^2 r(X_1) \phi_{j,k}(X_1)] + 2\mathbb{E}[U_1 V_1 f(X_1) \phi_{j,k}(X_1)] + \mathbb{E}[V_1^2 \phi_{j,k}(X_1)] \right. \\
&\quad \left. - \mathbb{E}[V_1^2 \phi_{j,k}(X_1)] \right) \\
&= \frac{3}{\theta^2} \mathbb{E}[U_1^2] \mathbb{E}[r(X_1) \phi_{j,k}(X_1)] = \int_0^1 r(x) \phi_{j,k}(x) dx = \alpha_{j,k}.
\end{aligned}$$

Lemma 1 is proved. \square

Lemma 2. *Let $j \geq \tau$ such that $2^j \leq n$, $k \in \Lambda_j$, $\hat{\alpha}_{j,k}$ be (3). Then, under (A.A.1)–(A.A.5),*

$$\mathbb{E}[(\hat{\alpha}_{j,k} - \alpha_{j,k})^2] \lesssim \frac{1}{n}.$$

Proof of Lemma 2. Owing to Lemma 1 we have $\mathbb{E}[\hat{\alpha}_{j,k}] = \alpha_{j,k}$. Therefore

$$\begin{aligned}
\mathbb{E}[(\hat{\alpha}_{j,k} - \alpha_{j,k})^2] &= \text{Var}[\hat{\alpha}_{j,k}] = \frac{9}{\theta^4} \text{Var} \left[\frac{1}{n} \sum_{i=1}^n Y_i^2 \phi_{j,k}(X_i) - \mathbb{E}[V_1^2] 2^{-j/2} \right] \\
&= \frac{9}{\theta^4} \text{Var} \left[\frac{1}{n} \sum_{i=1}^n Y_i^2 \phi_{j,k}(X_i) \right] \\
&= \frac{9}{\theta^4} \frac{1}{n} \text{Var}[Y_1^2 \phi_{j,k}(X_1)] \lesssim \frac{1}{n} \mathbb{E}[Y_1^4 \phi_{j,k}^2(X_1)] \\
&\lesssim \frac{1}{n} [\mathbb{E}[U_1^4 f^4(X_1) \phi_{j,k}^2(X_1)] + \mathbb{E}[V_1^4 \phi_{j,k}^2(X_1)]] \\
&= \frac{1}{n} [\mathbb{E}[U_1^4] \mathbb{E}[f^4(X_1) \phi_{j,k}^2(X_1)] + \mathbb{E}[V_1^4 \phi_{j,k}^2(X_1)]]. \quad (4)
\end{aligned}$$

By **A.1** and $\mathbb{E}[\phi_{j,k}^2(X_1)] = \int_0^1 \phi_{j,k}^2(x) dx = 1$, we have $\mathbb{E}[f^4(X_1) \phi_{j,k}^2(X_1)] \lesssim 1$. On the other hand, by **A.4** and **A.5**, we have

$$\mathbb{E}[V_1^4 \phi_{j,k}^2(X_1)] = \mathbb{E}[V_1^4] \mathbb{E}[\phi_{j,k}^2(X_1)] = \mathbb{E}[V_1^4] \lesssim 1$$

Thus all the terms in the brackets of (4) are bounded from above. This ends the proof of Lemma 2. \square

Proof of Proposition 1 from Lemmas 1 and 2. The main lines of the proof use standard arguments (see, for instance, [12]). The key result remains Lemma 2 above

and a suitable choice for j_0 which balance the biases and the rest term of term. More precisely, by the definition of projector, we have

$$\mathbb{E} \left[\int_0^1 |\hat{r}_{j_0,n}(x) - r(x)|^2 dx \right] = \mathbb{E} \left[\|\hat{r}_{j_0,n} - P_{j_*} r\|_2^2 \right] + \|P_{j_*} r - r\|_2^2. \quad (5)$$

The orthonormality of the wavelet basis gives

$$\mathbb{E} \left[\|\hat{r}_{j_0,n} - P_{j_*} r\|_2^2 \right] = \mathbb{E} \left[\left\| \sum_{k \in \Lambda_{j_*}} (\hat{\alpha}_{j_*,k} - \alpha_{j_*,k}) \phi_{j_*,k} \right\|_2^2 \right] = \sum_{k \in \Lambda_{j_*}} \mathbb{E}[(\hat{\alpha}_{j_*,k} - \alpha_{j_*,k})^2].$$

According to Lemma 2, $|\Lambda_{j_*}| \sim 2^{j_*}$ and $2^{j_*} \sim n^{\frac{1}{2s'+1}}$,

$$\mathbb{E} \left[\|\hat{r}_{j_0,n} - P_{j_*} r\|_2^2 \right] \lesssim \frac{2^{j_0}}{n} \lesssim n^{-\frac{2s'}{2s'+1}}. \quad (6)$$

When $p \geq 2$, $s' = s$. By Hölder inequality and $r \in B_{p,q}^s([0, 1])$,

$$\|P_{j_0} r - r\|_2^2 \lesssim \|P_{j_0} r - r\|_p^2 \lesssim 2^{-2j_0 s} \lesssim n^{-\frac{2s}{2s'+1}}.$$

When $1 \leq p < 2$ and $s > 1/p$, $B_{p,q}^s([0, 1]) \subseteq B_{2,\infty}^{s'}([0, 1])$

$$\|P_{j_0} r - r\|_2^2 \lesssim \sum_{j=j_0}^{\infty} 2^{-2js'} \lesssim 2^{-2j_0 s'} \lesssim n^{-\frac{2s'}{2s'+1}}.$$

Therefore, in both cases,

$$\|P_{j_0} r - r\|_2^2 \lesssim n^{-\frac{2s'}{2s'+1}}. \quad (7)$$

By (5), (6) and (7), we obtain

$$\mathbb{E} \left[\int_0^1 |\hat{r}_{j_0,n}(x) - r(x)|^2 dx \right] \lesssim n^{-\frac{2s'}{2s'+1}}.$$

Proposition 1 is proved. \square

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