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Quasi-stationarity for one-dimensional renormalized Brownian motion

William Oçafrain¹

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Abstract

We are interested in the quasi-stationarity of the time-inhomogeneous Markov process

$$X_t = \frac{B_t}{(t+1)^\kappa}$$

where $(B_t)_{t \geq 0}$ is a one-dimensional Brownian motion and $\kappa \in (0, \infty)$. We first show that the law of X_t conditioned not to go out from $(-1, 1)$ until the time t converges weakly towards the Dirac measure δ_0 when $\kappa > \frac{1}{2}$ as t goes to infinity. Then we show that this conditioned probability converges weakly towards the quasi-stationary distribution of an Ornstein-Uhlenbeck process when $\kappa = \frac{1}{2}$. Finally, when $\kappa < \frac{1}{2}$, it is shown that the conditioned probability converges towards the quasi-stationary distribution of a Brownian motion. We also prove the existence of a Q -process and a quasi-ergodic distribution for $\kappa = \frac{1}{2}$ and $\kappa < \frac{1}{2}$.

Key words : quasi-stationary distribution, Q -process, quasi-limiting distribution, quasi-ergodic distribution, Brownian motion, expanding boundaries

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1 Introduction

1.1 Introduction of the problem and quasi-stationarity

In this paper, we are interested in some notions related to quasi-stationarity for a one-dimensional Brownian motion $(B_t)_{t \geq 0}$ killed when crossing the moving boundaries $t \rightarrow (-(t+1)^\kappa, (t+1)^\kappa)$, with $\kappa \in (0, \infty)$. Quasi-stationarity with moving boundaries was studied in [13] and [14] for periodic or converging boundaries, but expanding boundaries were not yet considered. Actually, instead of considering the process B absorbed at $t \rightarrow (-(t+1)^\kappa, (t+1)^\kappa)$, we will study the quasi-stationarity of the process $X = (X_t)_{t \geq 0}$ absorbed at $(-1, 1)^c$ and defined by

$$X_t := \frac{B_t}{(t+1)^\kappa}, \quad \forall t < \tau_X$$

where $\tau_X := \inf\{t \geq 0 : |X_t| = 1\}$.

The process X is a time-inhomogeneous Markov process. For any $x \in \mathbb{R}$ and $s \geq 0$, denote by $\mathbb{P}_{x,s}$ the probability measure satisfying $\mathbb{P}_{x,s}(X_s = x) = 1$, and denote by $\mathbb{E}_{x,s}$ its corresponding expectation. Also, for any measure μ , for any $s \geq 0$, one denotes by $\mathbb{P}_{\mu,s} := \int_{\mathbb{R}} \mathbb{P}_{x,s} \mu(dx)$ and $\mathbb{E}_{\mu,s} := \int_{\mathbb{R}} \mathbb{E}_{x,s} \mu(dx)$.

A *quasi-stationary distribution of X absorbed at $(-1, 1)^c$* is a probability measure α supported on $(-1, 1)$ such that

$$\mathbb{P}_{\alpha,s}(X_t \in \cdot | \tau_X > t) = \alpha, \quad \forall s \leq t$$

We refer the reader to [10, 12] for more details on the theory. Note however that these references only deal with the time-homogeneous setting and that quasi-stationary distributions for time-inhomogeneous Markov processes do not exist except for particular cases (especially we will see that the existence of one quasi-stationary distribution holds only for $\kappa = \frac{1}{2}$).

Usually, in the literature dealing with quasi-stationarity, mathematicians are interested in showing the weak convergence of marginal laws of Markov processes conditioned not to be absorbed by a cemetery set. The corresponding limit is called *quasi-limiting distribution*. For our purpose, we define a quasi-limiting distribution as follows

Definition 1. α is a quasi-limiting distribution of X if for some initial law μ supported on $(-1, 1)$ and for any $s \geq 0$,

$$\lim_{t \rightarrow \infty} \mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > t) = \alpha$$

where the limit refers to the weak convergence of measures. In [12] it is noted that, in the time-homogeneous setting, quasi-stationary distribution and quasi-limit distribution are equivalent notions. In the time-inhomogeneous setting, this equivalence does not hold anymore. More particularly a time-inhomogeneous Markov process could admit a quasi-limit distribution without admitting a quasi-stationary distribution. However a quasi-stationary distribution is necessarily a quasi-limiting distribution.

Quasi-limiting distribution is not the only point of interest in the theory of quasi-stationarity. Another point is the Q -process, which can be considered as the law of the considered Markov process conditioned not be absorbed. Concerning the process X , we define the Q -process as follows

Definition 2. We say that there is a Q -process for X if there exists a family $(\mathbb{Q}_{x,s})_{x \in (-1,1), s \geq 0}$ of probability measure defined by : for any $x \in (-1, 1)$ and for any $s \leq t$

$$\mathbb{Q}_{x,s}(X_{[s,t]} \in \cdot) := \lim_{T \rightarrow \infty} \mathbb{P}_{x,s}(X_{[s,t]} \in \cdot | T < \tau_X)$$

where, for any $u \leq v$, $X_{[u,v]}$ is the trajectory of X between times u and v . Then the Q -process is defined as the law of X under $(\mathbb{Q}_{x,s})_{x \in (-1,1), s \geq 0}$.

In general, the Q -process is also a Markov process and the theory of quasi-stationarity allows to get some results of ergodicity for the Q -process.

Finally, a third concept to study is the existence of a *quasi-ergodic distribution* defined as follows

Definition 3. β is a quasi-ergodic distribution of X if for some initial law μ supported on $(-1, 1)$ and for any $s \geq 0$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_s^t \mathbb{P}_{\mu,s}(X_u \in \cdot | \tau_X > t) du = \beta$$

In the literature, this notion is also called *mean-ratio quasi-stationary distribution*. The references [10, 12] does not deal with quasi-ergodic distribution. See for example [5, 8] which provide general assumptions implying the existence of quasi-ergodic distribution for time-homogeneous Markov processes.

Some general results on quasi-stationarity for time-inhomogeneous Markov process are established, particularly in [9], where it is shown that criteria based on Doeblin-type condition implies a mixing property (or *merging* or *weak ergodicity*) and the existence of the Q -process. However it will be difficult to apply these results for our purpose. See also [16, 11, 14, 13] for a few

results about quasi-stationarity in the time-inhomogeneous setting, and [1] for ergodic properties for general non-conservative (time-homogeneous and inhomogeneous) semi-group.

Now let us come back to our process X . As we can expect, the existence of quasi-limiting, Q -process and quasi-ergodic distribution will strongly depend on κ . More precisely, three regimes are identified :

- $\kappa > \frac{1}{2}$, we will say that X is *supercritical*
- $\kappa = \frac{1}{2}$, we will say that X is *critical*
- $\kappa < \frac{1}{2}$, we will say that X is *subcritical*

The aim of this paper is therefore to show the existence of quasi-limiting, Q -process and quasi-ergodic distribution for each regime. More precisely, it will be shown in a first step that, for any probability measure μ on $(-1, 1)$ and $s \geq 0$,

$$\lim_{t \rightarrow \infty} \mathbb{P}_{\mu, s}(X_t \in \cdot | \tau_X > t) = \delta_0$$

in the supercritical regime. This regime is of little interest and the existence of a Q -process and a quasi-ergodic distribution will not be shown. In a second step, the existence of quasi-limiting, Q -process and quasi-ergodic distribution will be stated in the critical regime and these probability measures are connected to the quasi-stationarity of an Ornstein-Uhlenbeck process absorbed at $(-1, 1)^c$. Finally, the existence of these three notions will also be established in the subcritical regime where the quasi-stationarity of X is linked with the quasi-stationarity of a Brownian motion on $[-1, 1]$.

1.2 A few notation

For any $E \subset \mathbb{R}$, one denotes by $\mathcal{M}_1(E)$ the set of the probability measures supported on E and, for any measurable bounded function f on $(-1, 1)$ and $\mu \in \mathcal{M}_1((-1, 1))$, denote by

$$\mu(f) := \int_{(-1, 1)} f d\mu$$

For a general Markov process $(\Omega, (\mathcal{F}_t^A)_{t \geq 0}, (A_t)_{t \geq 0}, (\mathbb{P}_{x, s}^A)_{x \in \mathbb{R}, s \geq 0})$, for any probability measure μ on \mathbb{R} and any $s \geq 0$, we define $\mathbb{P}_{\mu, s}^A := \int_{\mathbb{R}} \mathbb{P}_{x, s}^A \mu(dx)$. Then the family of probability measures $(\mathbb{P}_{\mu, s}^A)_{\mu \in \mathcal{M}_1(\mathbb{R}), s \geq 0}$ satisfies

$$\mathbb{P}_{\mu, s}^A(A_s \in \cdot) = \mu$$

If the process A is time-homogeneous, we define, for any $\mu \in \mathcal{M}_1(\mathbb{R})$, $\mathbb{P}_\mu^A := \mathbb{P}_{\mu,0}^A$ and we have, for any $s \leq t$,

$$\mathbb{P}_{\mu,s}^A(X_{[s,t]} \in \cdot) = \mathbb{P}_\mu^A(X_{[0,t-s]} \in \cdot)$$

For $A = X$, we will keep the notation established at the beginning of the introduction.

2 The supercritical regime : $\kappa > \frac{1}{2}$

The following theorem states the existence of a unique quasi-limiting distribution, which is δ_0

Theorem 1. *For any $\mu \in \mathcal{M}_1((-1, 1))$ and $s \geq 0$,*

$$\lim_{t \rightarrow \infty} \mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > t) = \delta_0$$

Proof. By Markov's inequality, for any $\epsilon > 0$ and any probability measure μ ,

$$\begin{aligned} \mathbb{P}_{\mu,s}(|X_t| \geq \epsilon | \tau_X > t) &\leq \frac{\mathbb{E}_{\mu,s}(X_t^2 | \tau_X > t)}{\epsilon^2} \\ &\leq \frac{\mathbb{E}_{\mu,s}(X_t^2)}{\epsilon^2 \mathbb{P}_{\mu,s}(\tau_X > t)} \\ &= \frac{t - s + (s + 1)^{2\kappa} \int_{(-1,1)} x^2 d\mu(x)}{\epsilon^2 (t + 1)^{2\kappa} \mathbb{P}_{\mu,s}(\tau_X > t)} \end{aligned}$$

It is well known that $\mathbb{P}_{\mu,s}$ -almost surely,

$$\limsup_{t \rightarrow \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1$$

and

$$\liminf_{t \rightarrow \infty} \frac{B_t}{\sqrt{2t \log \log t}} = -1$$

Thus, since $\kappa > \frac{1}{2}$, $\mathbb{P}_{\mu,s}$ -almost surely,

$$X_t = \frac{B_t}{(t + 1)^\kappa} \xrightarrow[t \rightarrow \infty]{} 0$$

As a result, for any $s \geq 0$ and any probability measure μ on $(-1, 1)$,

$$\lim_{t \rightarrow \infty} \mathbb{P}_{\mu,s}(\tau_X > t) = \mathbb{P}_{\mu,s}(\tau_X = \infty) > 0$$

Thus, for any $\epsilon > 0$,

$$\mathbb{P}_{\mu,s}(|X_t| \geq \epsilon | \tau_X > t) \xrightarrow[t \rightarrow \infty]{} 0$$

In other words, $(X_t)_{t \geq 0}$ converges in conditional probability towards 0. This concludes the proof. \square

3 The critical case : $\kappa = \frac{1}{2}$

3.1 Existence and uniqueness of a quasi-stationary distribution

We state a first theorem on the existence of the quasi-limiting distribution (and quasi-stationary distribution) in the critical regime.

Theorem 2. *Let α_{OU} be the unique quasi-stationary distribution of the Ornstein-Uhlenbeck process absorbed by $(-1, 1)^c$ whose the generator is*

$$L := \frac{1}{2}\Delta - \frac{1}{2}x\nabla \quad (1)$$

Then α_{OU} is also the unique quasi-stationary distribution of X and there exist $C_{OU}, \gamma_{OU} > 0$ such that for any probability measure μ on $(-1, 1)$ and any $0 \leq s \leq t$,

$$\|\mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > t) - \alpha_{OU}\|_{TV} \leq C_{OU} \left(\frac{s+1}{t+1}\right)^{\gamma_{OU}} \quad (2)$$

In particular, for any $\mu \in \mathcal{M}_1((-1, 1))$ and $s \geq 0$, $t \rightarrow \mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > t)$ converges weakly towards α_{OU} when t goes to infinity.

Remark 1. Using the spectral theory of the Ornstein-Uhlenbeck generator, α_{OU} can be computed and

$$\alpha_{OU}(dx) := K \times (1 - x^2)e^{-\frac{x^2}{2}} dx$$

where K is the renormalization constant.

Remark 2. It is well-known (see [10, 12]) that there exists $\lambda_{OU} > 0$ such that

$$\mathbb{P}_{\alpha_{OU}}^Z(\tau_Z > t) = e^{-\lambda_{OU}t}, \quad \forall t \geq 0 \quad (3)$$

where $\tau_Z := \inf\{t \geq 0 : |Z_t| = 1\}$. Moreover, for any $f \in \{g \in C^2([-1, 1]) : g(-1) = g(1) = 0\}$,

$$\alpha_{OU}(Lf) = -\lambda_{OU}\alpha_{OU}(f)$$

where L is defined in (1). Using the explicit formula of α_{OU} , it is easy to check that

$$\lambda_{OU} = 1 \quad (4)$$

Proof of Theorem 2. Let Z be the Ornstein-Uhlenbeck process whose infinitesimal generator is L . Then, for any probability measure μ on $(-1, 1)$ and any $s \geq 0$,

$$\mathbb{P}_{\mu,s}((X_t)_{t \geq s} \in \cdot) = \mathbb{P}_{\mu}^Z \left(\left(Z_{\log\left(\frac{t+1}{s+1}\right)} \right)_{t \geq s} \in \cdot \right) \quad (5)$$

Hence, using (5), one has for any $s \leq t$,

$$\begin{aligned} \mathbb{P}_{\alpha_{OU},s}(X_t \in \cdot | \tau_X > t) &= \mathbb{P}_{\alpha_{OU}}^Z \left(Z_{\log\left(\frac{t+1}{s+1}\right)} \in \cdot \mid \tau_Z > \log\left(\frac{t+1}{s+1}\right) \right) \\ &= \alpha_{OU} \end{aligned}$$

In other words α_{OU} is also the unique quasi-stationary distribution of the time-inhomogeneous Markov process X . Moreover, since Z satisfies the assumptions (A1) and (A2) of [7] (this is actually shown in [6]), then, by Theorem 2.1. in [7], there exist $C_{OU} > 0$ and $\gamma_{OU} > 0$ such that for any $t \geq 0$ and for any probability measure μ ,

$$\|\mathbb{P}_{\mu}^Z(Z_t \in \cdot | \tau_Z > t) - \alpha_{OU}\|_{TV} \leq C_{OU} e^{-\gamma_{OU} t}$$

Using (5) one deduce that, for any $s \leq t$ and for any probability measure μ on $(-1, 1)$,

$$\|\mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > t) - \alpha_{OU}\|_{TV} \leq C_{OU} \left(\frac{s+1}{t+1} \right)^{\gamma_{OU}}$$

This concludes the proof. \square

3.2 Existence of the Q -process

Before tackling the existence of the Q -process, we need the following proposition:

Proposition 1. *There exists a non-negative function $\eta_{OU} : [-1, 1] \rightarrow \mathbb{R}_+$, positive on $(-1, 1)$ and vanishing on $\{-1, 1\}$ such that for any $x \in (-1, 1)$ and any $s \geq 0$,*

$$\eta_{OU}(x) = \lim_{t \rightarrow \infty} \frac{t+1}{s+1} \mathbb{P}_{x,s}(\tau_X > t)$$

where the convergence holds uniformly on $[-1, 1]$ and $\alpha_{OU}(\eta_{OU}) = 1$. Moreover the function η_{OU} is bounded, belongs to the domain of L defined in (1) and

$$L\eta_{OU} = -\lambda_{OU}\eta_{OU} = -\eta_{OU}$$

Remark 3. More precisely,

$$\eta_{OU}(x) = K' \times (1 - x^2) \quad (6)$$

where K' is the positive constant such that $\alpha_{OU}(\eta_{OU}) = 1$.

An interesting consequence of Proposition 1 can be stated as the following corollary

Corollary 1. *Let B a Brownian motion on \mathbb{R} and denote by*

$$\tau_B^{\sqrt{\cdot}} := \inf\{t \geq 0 : |B_t| \geq \sqrt{t+1}\}$$

Then for any $x \in (-1, 1)$,

$$\mathbb{P}_x^B(\tau_B^{\sqrt{\cdot}} > t) \sim_{t \rightarrow \infty} K' \frac{1 - x^2}{t + 1}$$

Proof of Proposition 1 and Corollary 1. Using Proposition 2.3 in [7] applied to the process Z and (5), one has for any $x \in (-1, 1)$ and $s \geq 0$,

$$\begin{aligned} \eta_{OU}(x) &= \lim_{t \rightarrow \infty} e^{\lambda_{OU} \log\left(\frac{t+1}{s+1}\right)} \mathbb{P}_x^Z\left(\tau_Z > \log\left(\frac{t+1}{s+1}\right)\right) \\ &= \lim_{t \rightarrow \infty} \left(\frac{t+1}{s+1}\right)^{\lambda_{OU}} \mathbb{P}_{x,s}(\tau_X > t) \\ &= \lim_{t \rightarrow \infty} \frac{t+1}{s+1} \mathbb{P}_{x,s}(\tau_X > t) \end{aligned}$$

where we finally used (4). This ends the proof of Proposition 1. Now it is easy to see that, for any $x \in (-1, 1)$ and $t \geq 0$, $\mathbb{P}_x^B(\tau_B^{\sqrt{\cdot}} > t) = \mathbb{P}_{x,0}(\tau_X > t)$. Thus, using Proposition 1 and (6), we conclude the corollary. \square

Remark 4. In [4], Breiman shows a similar result for one-dimensional Brownian motion absorbed by a one-sided square boundary. More precisely, denoting $T_c^* := \inf\{t \geq 0 : B_t \geq c\sqrt{t+1}\}$ for any $c > 0$, he shows that $\mathbb{P}_0^B(T_c^* > t) \sim_{t \rightarrow \infty} at^{-b(c)}$ for $a > 0$ and b such that $b(1) = 1$. In particular, for $c = 1$, $\mathbb{P}_0^B(T_1^* > t)$ and $\mathbb{P}_0^B(\tau_B^{\sqrt{\cdot}} > t)$ decay as $1/t$. The reader can also see [15] for more general boundaries.

We turn to the existence of the Q -process and its ergodicity.

Proposition 2. • *There exists a Q -process and the family of probability measure $(\mathbb{Q}_{x,s})_{x \in (-1,1), s \geq 0}$ defined in Definition 2 is given by : for any $x \in (-1, 1)$ and $s \leq t$,*

$$\begin{aligned} \mathbb{Q}_{x,s}(X_{[s,t]} \in \cdot) &= \mathbb{E}_{x,s} \left(\mathbb{1}_{X_{[s,t]} \in \cdot, \tau_X > t} \left(\frac{t+1}{s+1} \right)^{\lambda_{OU}} \frac{\eta_{OU}(X_t)}{\eta_{OU}(x)} \right) \\ &= \frac{t+1}{s+1} \times \mathbb{E}_{x,s} \left(\mathbb{1}_{X_{[s,t]} \in \cdot, \tau_X > t} \frac{1 - X_t^2}{1 - x^2} \right) \end{aligned}$$

- *The probability measure β_{OU} defined by*

$$\beta_{OU}(dx) := \eta_{OU}(x) \alpha_{OU}(dx) = KK'(1-x^2)^2 e^{-\frac{x^2}{2}} dx$$

is the unique stationary distribution of X under $(\mathbb{Q}_{x,s})_{s \geq 0, x \in (-1,1)}$. Moreover, for any $0 \leq s \leq t$ and any $x \in (-1, 1)$,

$$\|\mathbb{Q}_{x,s}(X_t \in \cdot) - \beta_{OU}\|_{TV} \leq C_{OU} \left(\frac{s+1}{t+1} \right)^{\gamma_{OU}}$$

where C_{OU} and γ_{OU} are the same constant as used in (2).

Proof. Straightforward using (5) and Proposition 3.1. in [7] applied to the Ornstein-Uhlenbeck process Z \square

3.3 Quasi-ergodic distribution

Now let us provide and show the existence and the uniqueness of the quasi-ergodic distribution

Theorem 3. *For any probability measure μ on $(-1, 1)$ and any $s \geq 0$, for any measurable set S*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}_{\mu,s}(X_u \in S | \tau_X > t) du = \int_S \mathbb{E}_x^Z(\tau_Z) \alpha_{OU}(dx)$$

Remark 5. In the time-homogeneous setting, it is usually expected that the quasi-ergodic distribution is the stationary distribution of the Q -process (see [5, 8]). A similar result could even be expected in the time-inhomogeneous case when the Q -process converges weakly at the infinity (see [13]). It is therefore astonishing to see that this is not the case for our process in the critical regime, even though the Q -process admits a stationary measure. In particular, the quasi-ergodic distribution of X is different from the quasi-ergodic distribution of the process Z .

Proof. First, using the variable change $u = s + q(t - s)$, one has, for any $\mu \in \mathcal{M}_1((-1, 1))$, $s < t$ and f bounded measurable,

$$\frac{1}{t-s} \int_s^t \mathbb{E}_{\mu,s}(f(X_u) | \tau_X > t) du = \int_0^1 \mathbb{E}_{\mu,s}(f(X_{s+q(t-s)}) | \tau_X > t) dq$$

As a result it is enough to show the weak convergence of $(\mathbb{P}_{\mu,s}(X_{s+q(t-s)} \in \cdot | \tau_X > t))_{t \geq 0}$ for any $q \in (0, 1)$, then to conclude with the Lebesgue's dominated convergence theorem.

Let $\mu \in \mathcal{M}_1((-1, 1))$, $s \geq 0$, $q \in (0, 1)$ and f bounded measurable. Using Markov property and (5), for any $t \geq s$,

$$\mathbb{E}_{\mu,s}(f(X_{s+q(t-s)}) \mathbb{1}_{\tau_X > t}) = \mathbb{E}_{\mu,s} \left(f(X_{s+q(t-s)}) \mathbb{1}_{\tau_X > s+q(t-s)} \mathbb{P}_{X_{s+q(t-s)}, s+q(t-s)}(\tau_X > t) \right) \quad (7)$$

$$= \mathbb{E}_{\mu,s} (f_t(X_{s+q(t-s)}) \mathbb{1}_{\tau_X > s+q(t-s)}) \quad (8)$$

where we set for any $y \in (-1, 1)$,

$$f_t(y) := f(y) \mathbb{P}_{y, s+q(t-s)}[\tau_X > t]$$

By (5), for any $y \in (-1, 1)$ and $t \geq s$,

$$f_t(y) = f(y) \mathbb{P}_y^Z \left[\tau_Z > \log \left(\frac{t+1}{s+q(t-s)+1} \right) \right]$$

Now define for any $y \in (-1, 1)$,

$$f_\infty(y) := f(y) \mathbb{P}_y^Z [\tau_Z > -\log(q)]$$

It is easy to see that $(f_t)_{t \geq 0}$ converges pointwise towards f_∞ . Moreover, a simple curve sketching entails that the function $t \rightarrow \frac{t+1}{s+q(t-s)+1}$ is increasing, which implies that the sequence $(f_t)_{t \geq 0}$ is a decreasing sequence of continuous functions defined on $[-1, 1]$. Likewise, f_∞ is continuous on $[-1, 1]$. As a result, by Dini's theorem for the decreasing sequences of continuous function, the pointwise convergence is equivalent to uniform convergence on $[-1, 1]$. Thus,

$$\lim_{t \rightarrow \infty} \sup_{y \in (-1, 1)} |f_t(y) - f_\infty(y)| = 0 \quad (9)$$

Now let us show that

$$\lim_{t \rightarrow \infty} \frac{s+q(t-s)+1}{s+1} \mathbb{E}_{\mu,s}(f_\infty(X_{s+q(t-s)}) \mathbb{1}_{\tau_X > s+q(t-s)}) = \mu(\eta_{OU}) \alpha_{OU}(f_\infty) \quad (10)$$

To show this, let us begin with

$$\begin{aligned} & \frac{s+q(t-s)+1}{s+1} \mathbb{E}_{\mu,s} (f_\infty(X_{s+q(t-s)}) \mathbb{1}_{\tau_X > s+q(t-s)}) \\ &= \frac{s+q(t-s)+1}{s+1} \mathbb{P}_{\mu,s}(\tau_X > s+q(t-s)) \times \mathbb{E}_{\mu,s}(f_\infty(X_{s+q(t-s)}) | \tau_X > s+q(t-s)) \end{aligned}$$

On the one hand, by Proposition 1,

$$\lim_{t \rightarrow \infty} \frac{s+q(t-s)+1}{s+1} \mathbb{P}_{\mu,s}(\tau_X > s+q(t-s)) = \mu(\eta_{OU})$$

On the other hand, by (2),

$$\lim_{t \rightarrow \infty} \mathbb{E}_{\mu,s}(f_\infty(X_{s+q(t-s)}) | \tau_X > s+q(t-s)) = \alpha_{OU}(f_\infty)$$

(10) follows from these two convergences. Now, by (10) and (9),

$$\begin{aligned} & \frac{s+q(t-s)+1}{s+1} \mathbb{E}_{\mu,s} (f_t(X_{s+q(t-s)}) \mathbb{1}_{\tau_X > s+q(t-s)}) \\ &= \frac{s+q(t-s)+1}{s+1} \mathbb{E}_{\mu,s} (f_\infty(X_{s+q(t-s)}) \mathbb{1}_{\tau_X > s+q(t-s)}) \\ & \quad + \frac{s+q(t-s)+1}{s+1} \mathbb{E}_{\mu,s} ([f_\infty(X_{s+q(t-s)}) - f_t(X_{s+q(t-s)})] \mathbb{1}_{\tau_X > s+q(t-s)}) \\ & \xrightarrow[t \rightarrow \infty]{} \mu(\eta_{OU}) \alpha_{OU}(f_\infty) \end{aligned}$$

because

$$\begin{aligned} & \left| \frac{s+q(t-s)+1}{s+1} \mathbb{E}_{\mu,s} ([f_\infty(X_{s+q(t-s)}) - f_t(X_{s+q(t-s)})] \mathbb{1}_{\tau_X > s+q(t-s)}) \right| \\ & \leq \frac{s+q(t-s)+1}{s+1} \mathbb{P}_{\mu,s}(\tau_X > s+q(t-s)) \times \sup_{y \in (-1,1)} |f_t(y) - f_\infty(y)| \xrightarrow[t \rightarrow \infty]{} 0 \end{aligned}$$

Hence, using (7), (8),

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{s+q(t-s)+1}{s+1} \mathbb{E}_{\mu,s}(f(X_{s+q(t-s)}) \mathbb{1}_{\tau_X > t}) = \mu(\eta_{OU}) \alpha_{OU}(f_\infty) \\ & = \mu(\eta_{OU}) \int_{(-1,1)} f(x) \mathbb{P}_x^Z(\tau_Z > -\log(q)) \alpha_{OU}(dx) \end{aligned}$$

Moreover, taking $f = \mathbb{1}$, using (3) and (4),

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{s+q(t-s)+1}{s+1} \mathbb{P}_{\mu,s}(\tau_X > t) = \mu(\eta_{OU}) \mathbb{P}_{\alpha_{OU}}^Z(\tau_Z > -\log(q)) \\ & = \mu(\eta_{OU}) q \end{aligned}$$

Thus we deduce that

$$\lim_{t \rightarrow \infty} \mathbb{E}_{\mu, s}(f(X_{s+q(t-s)}) | \tau_X > t) = q^{-1} \int_{(-1,1)} \alpha_{OU}(dx) f(x) \mathbb{P}_x(\tau_Z > -\log(q))$$

Then, by Lebesgue's theorem, for any probability measure μ on $(-1, 1)$ and any bounded measurable function f ,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t-s} \int_s^t \mathbb{E}_{\mu, s}(f(X_u) | \tau_X > t) du &= \lim_{t \rightarrow \infty} \int_0^1 \mathbb{E}_{\mu, s}(f(X_{s+q(t-s)}) | \tau_X > t) dq \\ &= \int_0^1 q^{-1} \int_{(-1,1)} f(x) \mathbb{P}_x^Z(\tau_Z > -\log(q)) \alpha_{OU}(dx) dq \\ &= \int_{(-1,1)} \alpha_{OU}(dx) f(x) \int_0^1 q^{-1} \mathbb{P}_x^Z(\tau_Z > -\log(q)) dq \\ &= \int_{(-1,1)} \alpha_{OU}(dx) f(x) \int_0^\infty \mathbb{P}_x^Z(\tau_Z > s) ds \\ &= \int_{(-1,1)} \alpha_{OU}(dx) f(x) \mathbb{E}_x^Z(\tau_Z) \end{aligned}$$

This concludes the proof. \square

4 The subcritical case : $\kappa < \frac{1}{2}$

In this section, we will show that quasi-limiting distribution, quasi-ergodic distribution and Q -process exist when $\kappa < \frac{1}{2}$. To do this, the strategy will be to compare (in a sense described later) the process X to the process Y defined by

$$Y_t := \int_0^t \frac{1}{(u+1)^\kappa} dB_u, \quad \forall t \geq 0$$

Then the quasi-stationarity of X will be deduced from the one of Y .

4.1 Approximation by Y through asymptotic pseudotrajectories

Denote by $\tau_Y := \inf\{t \geq 0 : |Y_t| = 1\}$. The aim of this subsection is to show the following proposition :

Proposition 3. *There exists a function $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that*

$$\lim_{s \rightarrow \infty} F(s) = 0$$

and such that, for any $0 \leq s \leq t \leq T$, for any probability measure μ on $(-1, 1)$,

$$\|\mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > T) - \mathbb{P}_{\mu,s}^Y(Y_t \in \cdot | \tau_Y > T)\|_{TV} \leq F(s) \quad (11)$$

Remark 6. (11) provides us with a decay towards 0 uniformly in the initial law, t and T . It can be seen as an analogue of the asymptotic pseudotrajectories introduced by Benaïm and Hirsch in [3]. See also [2] for more details about asymptotic pseudotrajectories in general case.

Proof of Proposition 3. By Itô's formula, one has for any $t \geq 0$,

$$X_t = X_0 + Y_t - \langle Y, M \rangle_t$$

where

$$M_t := \int_0^t \kappa(u+1)^{\kappa-1} X_u dB_u = \int_0^t \kappa(u+1)^{2\kappa-1} X_u dX_u + \int_0^t (\kappa(u+1)^{\kappa-1} X_u)^2 du$$

For any $s \leq t$, denote by

$$M_{s,t} := M_t - M_s = \int_s^t \kappa(u+1)^{\kappa-1} X_u dB_u$$

and

$$\langle M, M \rangle_{s,t} := \langle M, M \rangle_t - \langle M, M \rangle_s = \int_s^t (\kappa(u+1)^{\kappa-1} X_u)^2 du$$

and denote by $\mathcal{E}(M)_{s,t}$ the exponential local martingale defined by

$$\begin{aligned} \mathcal{E}(M)_{s,t} &:= \exp\left(M_{s,t} - \frac{1}{2} \langle M, M \rangle_{s,t}\right) \\ &:= \exp\left(\int_s^t \kappa(u+1)^{2\kappa-1} X_u dX_u + \frac{1}{2} \int_s^t (\kappa(u+1)^{\kappa-1} X_u)^2 du\right) \\ &= \exp\left(\frac{1}{2} N_{s,t}\right) \end{aligned}$$

where $N_{s,t} := 2 \int_s^t \kappa(u+1)^{2\kappa-1} X_u dX_u + \int_s^t (\kappa(u+1)^{\kappa-1} X_u)^2 du$. By Itô's formula applied to $t \rightarrow \kappa(t+1)^{2\kappa-1} X_t^2$, for any $s \leq t$,

$$\begin{aligned} N_{s,t} &:= \kappa(t+1)^{2\kappa-1} X_t^2 - \kappa(s+1)^{2\kappa-1} X_s^2 \\ &\quad - \int_s^t [\kappa(u+1)^{2\kappa-1}]' X_u^2 du - \int_s^t \frac{\kappa}{u+1} du + \int_s^t (\kappa(u+1)^{\kappa-1} X_u)^2 du \\ &= \kappa(t+1)^{2\kappa-1} X_t^2 - \kappa(s+1)^{2\kappa-1} X_s^2 \\ &\quad - \int_s^t [\kappa(u+1)^{2\kappa-1}]' X_u^2 du - \kappa \log\left(\frac{t+1}{s+1}\right) + \int_s^t (\kappa(u+1)^{\kappa-1} X_u)^2 du \end{aligned}$$

Note that the process $(N_{s,t \wedge \tau_X})_{s \leq t}$ is almost surely uniformly bounded, thus $\mathcal{E}(M)_{s,t \wedge \tau_X}$ is a martingale. For any $t \geq s \geq 0$ and $\mu \in \mathcal{M}_1((-1, 1))$, define $\mathbb{G}_{\mu,s}$ the probability measure satisfying

$$\mathbb{G}_{\mu,s}(A) = \mathbb{E}_{\mu,s}(\mathcal{E}(M)_{s,t \wedge \tau_X} \mathbb{1}_A), \quad \forall A \in \sigma(X_u, s \leq u \leq t)$$

Then, by Girsanov's theorem, the law of $(X_{t \wedge \tau_X})_{t \geq s}$ under $\mathbb{G}_{\mu,s}$ is the law of $(Y_{t \wedge \tau_Y})_{t \geq s}$ under $\mathbb{P}_{\mu,s}$. In particular, for any S measurable set, probability measure μ on $(-1, 1)$ and $0 \leq s \leq t \leq T$,

$$\begin{aligned} \mathbb{P}_{\mu,s}(Y_t \in S, \tau_Y > T) &= \mathbb{G}_{\mu,s}(X_t \in S, \tau_X > T) \\ &= \mathbb{E}_{\mu,s}(\mathcal{E}(M)_{s,T \wedge \tau_X} \mathbb{1}_{X_t \in S, \tau_X > T}) \\ &= \mathbb{E}_{\mu,s}(\mathcal{E}(M)_{s,T} \mathbb{1}_{X_t \in S, \tau_X > T}) \\ &= \mathbb{E}_{\mu,s} \left(\exp \left(\frac{1}{2} N_{s,T} \right) \mathbb{1}_{X_t \in S, \tau_X > T} \right) \\ &= \left(\frac{s+1}{T+1} \right)^{\frac{\kappa}{2}} \mathbb{E}_{\mu,s} \left(\exp \left(\frac{1}{2} N'_{s,T} \right) \mathbb{1}_{X_t \in S, \tau_X > T} \right) \end{aligned}$$

with $N'_{s,T} = N_{s,T} + \kappa \log \left(\frac{T+1}{s+1} \right)$. Thus,

$$\mathbb{P}_{\mu,s}(Y_t \in S | \tau_Y > T) = \frac{\mathbb{E}_{\mu,s} \left(\exp \left(\frac{1}{2} N'_{s,T} \right) \mathbb{1}_{X_t \in S, \tau_X > T} \right)}{\mathbb{E}_{\mu,s} \left(\exp \left(\frac{1}{2} N'_{s,T} \right) \mathbb{1}_{\tau_X > T} \right)}$$

Thus for any $0 \leq s \leq t \leq T$ and S measurable set,

$$\begin{aligned}
& |\mathbb{P}_{\mu,s}(X_t \in S | \tau_X > T) - \mathbb{P}_{\mu,s}^Y(Y_t \in S | \tau_Y > T)| \\
&= \left| \mathbb{P}_{\mu,s}(X_t \in S | \tau_X > T) - \frac{\mathbb{E}_{\mu,s} \left(\exp \left(\frac{1}{2} N'_{s,T} \right) \mathbb{1}_{X_t \in S, \tau_X > T} \right)}{\mathbb{E}_{\mu,s} \left(\exp \left(\frac{1}{2} N'_{s,T} \right) \mathbb{1}_{\tau_X > T} \right)} \right| \\
&\leq \left| \frac{\mathbb{P}_{\mu,s}(\tau_X > T)}{\mathbb{E}_{\mu,s} \left(\exp \left(\frac{1}{2} N'_{s,T} \right) \mathbb{1}_{\tau_X > T} \right)} \frac{\mathbb{E}_{\mu,s} \left(\exp \left(\frac{1}{2} N'_{s,T} \right) \mathbb{1}_{X_t \in S, \tau_X > T} \right)}{\mathbb{P}_{\mu,s}(\tau_X > T)} - \frac{\mathbb{E}_{\mu,s} \left(\exp \left(\frac{1}{2} N'_{s,T} \right) \mathbb{1}_{X_t \in S, \tau_X > T} \right)}{\mathbb{P}_{\mu,s}(\tau_X > T)} \right| \\
&\quad + \left| \frac{\mathbb{E}_{\mu,s} \left(\exp \left(\frac{1}{2} N'_{s,T} \right) \mathbb{1}_{X_t \in S, \tau_X > T} \right)}{\mathbb{P}_{\mu,s}(\tau_X > T)} - \frac{\mathbb{P}_{\mu,s}(X_t \in S, \tau_X > T)}{\mathbb{P}_{\mu,s}(\tau_X > T)} \right| \\
&\leq \underbrace{\left| \frac{\mathbb{P}_{\mu,s}(\tau_X > T)}{\mathbb{E}_{\mu,s} \left(\exp \left(\frac{1}{2} N'_{s,T} \right) \mathbb{1}_{\tau_X > T} \right)} - 1 \right|}_{=: A_s(\mu, T)} \times \frac{\mathbb{E}_{\mu,s} \left(\exp \left(\frac{1}{2} N'_{s,T} \right) \mathbb{1}_{\tau_X > T} \right)}{\mathbb{P}_{\mu,s}(\tau_X > T)} \\
&\quad + \underbrace{\left| \frac{\mathbb{E}_{\mu,s} \left(\exp \left(\frac{1}{2} N'_{s,T} \right) \mathbb{1}_{X_t \in S, \tau_X > T} \right) - \mathbb{P}_{\mu,s}(X_t \in S, \tau_X > T)}{\mathbb{P}_{\mu,s}(\tau_X > T)} \right|}_{=: C_s(\mu, t, T, S)}
\end{aligned}$$

In order to show (11), we will bound the functions A_s and C_s .

Step 1 : Upper bound for C_s .

For any $0 \leq s \leq t \leq T$, probability measure μ and B measurable set,

$$C_s(\mu, t, T, S) = \left| \underbrace{\mathbb{E}_{\mu,s} \left[\left(\exp \left(\frac{1}{2} N'_{s,T} \right) - 1 \right) \mathbb{1}_{X_t \in S} \middle| \tau_X > T \right]}_{=: f(s, t, T, \mu, S)} \right|$$

On the event $\{\tau_X > T\}$, $X_u^2 < 1$ for any $0 \leq u \leq T$. Hence the function f defined as above is bounded as follows

$$\begin{aligned}
& \exp \left(-\frac{\kappa}{2} (s+1)^{2\kappa-1} - \frac{1}{2} \int_s^T [\kappa(u+1)^{2\kappa-1}]' du \right) - 1 \\
& \leq f(s, t, T, \mu, S) \leq \exp \left(\frac{1}{2} \kappa (T+1)^{2\kappa-1} + \frac{1}{2} \int_s^T (\kappa(u+1)^{\kappa-1})^2 du \right) - 1
\end{aligned}$$

In particular, for any $0 \leq s \leq t \leq T$, for any probability measure μ and S measurable set,

$$\begin{aligned} |f(s, t, T, \mu, S)| &\leq \left(1 - \exp\left(-\frac{1}{2}\kappa(s+1)^{2\kappa-1}\right) \right) \\ &\vee \left(\exp\left(\frac{1}{2}\left(\kappa + \frac{\kappa^2}{1-2\kappa}\right)(s+1)^{2\kappa-1}\right) - 1 \right) \\ &=: \phi(s) \end{aligned}$$

Hence,

$$C_s(\mu, t, T, S) \leq \phi(s)$$

Step 2 : Upper bound for A_s .

Taking $S = (-1, 1)$,

$$\begin{aligned} C_s(\mu, t, T, (-1, 1)) &= \frac{|\mathbb{E}_{\mu, s} \left(\exp\left(\frac{1}{2}N'_{s, T}\right) \mathbb{1}_{X_t \in (-1, 1), \tau_X > T} \right) - \mathbb{P}_{\mu, s}(X_t \in (-1, 1), \tau_X > T)|}{\mathbb{P}_{\mu, s}(\tau_X > T)} \\ &= \left| \frac{\mathbb{E}_{\mu, s} \left(\exp\left(\frac{1}{2}N'_{s, T}\right) \mathbb{1}_{\tau_X > T} \right)}{\mathbb{P}_{\mu, s}(\tau_X > T)} - 1 \right| \end{aligned}$$

According to the previous bound we have shown, for any for any $s \leq T$, for any probability measure μ on $(-1, 1)$,

$$1 - \phi(s) \leq \frac{\mathbb{E}_{\mu, s} \left(\exp\left(\frac{1}{2}N'_{s, t}\right) \mathbb{1}_{\tau_X > t} \right)}{\mathbb{P}_{\mu, s}(\tau_X > t)} \leq 1 + \phi(s) \quad (12)$$

We deduce from this last inequality that

$$A_s(\mu, T) \leq \left(1 - \frac{1}{1 + \phi(s)} \right) \vee \left(\frac{1}{1 - \phi(s)} - 1 \right) =: \psi(s)$$

We set then, for any $s \geq 0$,

$$F(s) = \phi(s) + \psi(s)(1 + \phi(s))$$

which concludes the proof. \square

4.2 Quasi-stationarity of Y

Now we are interested in the quasi-stationarity of the process Y . Note that, by Dubin-Schwartz's theorem, there exists a Brownian motion \tilde{B} such that for any $t \geq 0$

$$Y_t = \tilde{B}_{\frac{(t+1)^{1-2\kappa}-1}{1-2\kappa}} \quad (13)$$

Denote $\tau_{\tilde{B}} := \inf\{t \geq 0 : |\tilde{B}_t| = 1\}$. Then, by (13), for any initial law μ and $s \geq 0$,

$$\mathbb{P}_{\mu,s}^Y(Y_t \in \cdot | \tau_Y > t) = \mathbb{P}_{\mu}^{\tilde{B}} \left(\tilde{B}_{\frac{(t+1)^{1-2\kappa} - (s+1)^{1-2\kappa}}{1-2\kappa}} \in \cdot \mid \tau_{\tilde{B}} > \frac{(t+1)^{1-2\kappa} - (s+1)^{1-2\kappa}}{1-2\kappa} \right)$$

It is well known that a Brownian motion absorbed at $(-1, 1)^c$ admits a unique quasi-stationary distribution α_{Bm} whose explicit form is

$$\alpha_{Bm}(dx) := \frac{1}{2} \cos\left(\frac{\pi}{2}x\right) dx$$

and that there exists $\lambda_{Bm} > 0$ such that

$$\mathbb{P}_{\alpha_{Bm}}^{\tilde{B}}(\tau_Y > t) = e^{-\lambda_{Bm}t}, \quad \forall t \geq 0$$

Remark that λ_{Bm} satisfies also

$$\alpha_{Bm} \left(\frac{1}{2} \Delta f \right) = -\frac{\lambda_{Bm}}{2} \Delta f, \quad \forall f \in \{g \in \mathcal{C}^2([-1, 1]) : g(1) = g(-1) = 0\}$$

The Brownian motion absorbed at $(-1, 1)^c$ satisfies the Champagnat-Villemonais condition (A1) – (A2) in [7], which implies the existence of $C_{Bm}, \gamma_{Bm} > 0$ such that for any probability measure μ and any $t \geq 0$,

$$\|\mathbb{P}_{\mu}^{\tilde{B}}(\tilde{B}_t \in \cdot | \tau_{\tilde{B}} > t) - \alpha_{Bm}\|_{TV} \leq C_{Bm} e^{-\gamma_{Bm}t}$$

Thus, using the Dubins-Schwartz transformation, for any $s \leq t$ and probability measure μ

$$\|\mathbb{P}_{\mu,s}^Y(Y_t \in \cdot | \tau_Y > t) - \alpha_{Bm}\|_{TV} \leq C_{Bm} \exp\left(-\gamma_{Bm} \times \frac{(t+1)^{1-2\kappa} - (s+1)^{1-2\kappa}}{1-2\kappa}\right) \quad (14)$$

Moreover, let η_{Bm} be the function defined by

$$\eta_{Bm}(x) := \lim_{t \rightarrow \infty} e^{\lambda_{Bm}t} \mathbb{P}_x^{\tilde{B}}(\tau_{\tilde{B}} > t)$$

This definition makes sense by Proposition 2.3. in [7]. We recall moreover that η_{Bm} is positive on $(-1, 1)$, vanishing on $\{-1, 1\}$, $\alpha_{Bm}(\eta_{Bm}) = 1$ and the convergence holds uniformly on $[-1, 1]$. Then, in the same way as in the critical case, an analogous version of Propositions 1 and 2 can be stated as follows

Proposition 4. (i) For any $x \in (-1, 1)$ and any $s \geq 0$,

$$\eta_{Bm}(x) = \lim_{t \rightarrow \infty} e^{\lambda_{Bm} \frac{(t+1)^{1-2\kappa} - (s+1)^{1-2\kappa}}{1-2\kappa}} \mathbb{P}_{x,s}^Y(\tau_Y > t)$$

where the convergence holds uniformly on $[-1, 1]$.

(ii) There exists a Q -process for Y in the sense of Definition 2 and the family of probability measure $(\mathbb{Q}_{x,s}^Y)_{x \in (-1,1), s \geq 0}$ defined by $\mathbb{Q}_{s,x}^Y(Y_{[s,t]} \in \cdot) := \lim_{T \rightarrow \infty} \mathbb{P}_{x,s}^Y(Y_{[s,t]} \in \cdot | T < \tau_Y)$ satisfies also

$$\mathbb{Q}_{x,s}^Y(Y_{[s,t]} \in \cdot) = \mathbb{E}_{x,s} \left(\mathbb{1}_{Y_{[s,t]} \in \cdot, t < \tau_Y} e^{\lambda_{Bm} \frac{(t+1)^{1-2\kappa} - (s+1)^{1-2\kappa}}{1-2\kappa}} \frac{\eta_{Bm}(Y_t)}{\eta_{Bm}(x)} \right)$$

for any $x \in (-1, 1)$ and $s \leq t$

(iii) The probability measure β_{Bm} defined by

$$\beta_{Bm}(dx) = \eta_{Bm}(x) \alpha_{Bm}(dx) \quad (15)$$

is the unique stationary distribution of Y under $(\mathbb{Q}_{x,s}^Y)_{x \in (-1,1), s \geq 0}$ and, for any $x \in (-1, 1)$ and $s \geq 0$,

$$\|\mathbb{Q}_{x,s}^Y(Y_t \in \cdot) - \beta_{Bm}\|_{TV} \leq C_{Bm} \exp \left(-\gamma_{Bm} \times \frac{(t+1)^{1-2\kappa} - (s+1)^{1-2\kappa}}{1-2\kappa} \right)$$

where C_{Bm} and γ_{Bm} are the same as in (14).

Proof. The proof is essentially the same as for the proof of Proposition 2. \square

4.3 Quasi-limiting distribution of X

Now we will use Proposition 3 in order to show the existence of a quasi-limiting distribution for the process X .

Theorem 4. For any probability measure μ on $(-1, 1)$ and any $0 \leq s \leq t$,

$$\|\mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > t) - \alpha_{Bm}\|_{TV} \leq F \left(\frac{t}{2} \right) + C_{Bm} \exp \left(-\gamma_{Bm} \times \frac{(t+1)^{1-2\kappa} - (\frac{t}{2} + 1)^{1-2\kappa}}{1-2\kappa} \right) \quad (16)$$

where the function F is defined in Proposition 3. In particular, for any $\mu \in \mathcal{M}_1((-1, 1))$ and any $s \geq 0$,

$$\lim_{t \rightarrow \infty} \mathbb{P}_{\mu, s}(X_t \in \cdot | \tau_X > t) = \alpha_{Bm}$$

Proof. Let $\mu \in \mathcal{M}_1((-1, 1))$. For any $s \leq t$ define

$$\mu_{(s, t)} := \mathbb{P}_{\mu, s}(X_t \in \cdot | \tau_X > t)$$

Then, according to Markov property, for any $s \leq t \leq u$,

$$\mu_{(s, u)} = \mathbb{P}_{\mu_{(s, t)}, t}(X_u \in \cdot | \tau_X > u)$$

Thus, for any $s \leq t$,

$$\begin{aligned} \|\mu_{(s, 2t)} - \alpha_{Bm}\|_{TV} &\leq \|\mu_{(s, 2t)} - \mathbb{P}_{\mu_{(s, t)}, t}^Y(Y_{2t} \in \cdot | \tau_Y > 2t)\|_{TV} \\ &\quad + \|\mathbb{P}_{\mu_{(s, t)}, t}^Y(Y_{2t} \in \cdot | \tau_Y > 2t) - \alpha_{Bm}\|_{TV} \\ &= \|\mathbb{P}_{\mu_{(s, t)}, t}(X_{2t} \in \cdot | \tau_X > 2t) - \mathbb{P}_{\mu_{(s, t)}, t}^Y(Y_{2t} \in \cdot | \tau_Y > 2t)\|_{TV} \\ &\quad + \|\mathbb{P}_{\mu_{(s, t)}, t}^Y(Y_{2t} \in \cdot | \tau_Y > 2t) - \alpha_{Bm}\|_{TV} \\ &\leq F(t) + C_{Bm} \exp\left(-\gamma_{Bm} \times \frac{(2t+1)^{1-2\kappa} - (t+1)^{1-2\kappa}}{1-2\beta}\right) \end{aligned}$$

where we used the inequalities (11) and (14). This shows the inequality (16).

Now, since $\lim_{t \rightarrow \infty} F(t) = 0$ by Proposition 3 and noting that

$$\lim_{t \rightarrow \infty} \exp\left(-\gamma_{Bm} \times \frac{(2t+1)^{1-2\kappa} - (t+1)^{1-2\kappa}}{1-2\kappa}\right) = 0$$

because $\kappa < \frac{1}{2}$, this shows that for any $\mu \in \mathcal{M}_1((-1, 1))$ and $s \geq 0$,

$$\lim_{t \rightarrow \infty} \mathbb{P}_{\mu, s}(X_t \in \cdot | \tau_X > t) = \alpha_{Bm}$$

□

4.4 Quasi-ergodic distribution

The following theorem states the existence and uniqueness of the quasi-ergodic distribution (in the sense of Definition 3) for the process X and that this quasi-ergodic distribution is the probability measure β_{Bm} defined in (15).

Theorem 5. *For any probability measure μ on $(-1, 1)$ and any $s \geq 0$,*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}_{\mu, s}(X_u \in \cdot | \tau_X > t) du = \beta_{Bm}$$

where β_{Bm} is defined in (15) (Proposition 4)

Proof. Let $\mu \in \mathcal{M}_1((-1, 1))$. We recall the notation

$$\mu_{(s, t)} = \mathbb{P}_{\mu, s}(X_t \in \cdot | \tau_X > t), \quad \forall s \leq t$$

For any probability measure μ and $s \leq t$,

$$\begin{aligned} & \left\| \int_0^1 \mathbb{P}_{\mu, s}(X_{s+q(t-s)} \in \cdot | \tau_X > t) dq - \beta_{Bm} \right\|_{TV} \\ & \leq \left\| \int_0^1 \mathbb{P}_{\mu, s}(X_{s+q(t-s)} \in \cdot | \tau_X > t) dq - \int_0^1 \mathbb{P}_{\mu_{(s, s+\frac{q}{2}(t-s)), \frac{q}{2}(t-s)}}^Y(Y_{s+q(t-s)} \in \cdot | \tau_Y > t) dq \right\|_{TV} \\ & \quad + \left\| \int_0^1 \mathbb{P}_{\mu_{(s, s+\frac{q}{2}(t-s)), \frac{q}{2}(t-s)}}^Y(Y_{s+q(t-s)} \in \cdot | \tau_Y > t) dq - \beta_{Bm} \right\|_{TV} \\ & \leq \left\| \int_0^1 \mathbb{P}_{\mu_{(s, s+\frac{q}{2}(t-s)), \frac{q}{2}(t-s)}}(X_{s+q(t-s)} \in \cdot | \tau_X > t) dq - \int_0^1 \mathbb{P}_{\mu_{(s, s+\frac{q}{2}(t-s)), \frac{q}{2}(t-s)}}^Y(Y_{s+q(t-s)} \in \cdot | \tau_Y > t) dq \right\|_{TV} \\ & \quad + \left\| \int_0^1 \mathbb{P}_{\mu_{(s, s+\frac{q}{2}(t-s)), \frac{q}{2}(t-s)}}^Y(Y_{s+q(t-s)} \in \cdot | \tau_Y > t) dq - \beta_{Bm} \right\|_{TV} \\ & \leq \int_0^1 F\left(s + \frac{q}{2}(t-s)\right) dq + \int_0^1 \left\| \mathbb{P}_{\mu_{(s, s+\frac{q}{2}(t-s)), \frac{q}{2}(t-s)}}^Y(Y_{s+q(t-s)} \in \cdot | \tau_Y > t) dq - \beta_{Bm} \right\|_{TV} dq \end{aligned}$$

By Lebesgue's theorem,

$$\lim_{t \rightarrow \infty} \int_0^1 F\left(s + \frac{q}{2}(t-s)\right) dq = 0$$

In order to prove the convergence towards the quasi-ergodic distribution, it remains therefore to show that

$$\lim_{t \rightarrow \infty} \int_0^1 \left\| \mathbb{P}_{\mu_{(s, s+\frac{q}{2}(t-s)), \frac{q}{2}(t-s)}}^Y(Y_{s+q(t-s)} \in \cdot | \tau_Y > t) dq - \beta_{Bm} \right\|_{TV} dq = 0 \quad (17)$$

The idea of the following reasoning is the same as in the critical case. Similarly one has for any $x \in (-1, 1)$, $s \leq t$, $q \in (0, 1)$ and f bounded measurable,

$$\begin{aligned} & e^{\lambda_{Bm} \frac{(t+1)^{1-2\kappa} - (s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}} \mathbb{E}_{\mu_{(s, s+\frac{q}{2}(t-s)), s+\frac{q}{2}(t-s)}}^Y(f(Y_{s+q(t-s)}) \mathbb{1}_{\tau_Y > t}) \\ & = e^{\lambda_{Bm} \frac{(s+q(t-s)+1)^{1-2\kappa} - (s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}} \mathbb{E}_{\mu_{(s, s+\frac{q}{2}(t-s)), s+\frac{q}{2}(t-s)}}^Y(g_t(Y_{s+q(t-s)}) \mathbb{1}_{\tau_Y > s+q(t-s)}) \end{aligned}$$

with for any $y \in (-1, 1)$

$$g_t(y) := e^{\lambda_{Bm} \frac{(t+1)^{1-2\kappa} - (s+q(t-s)+1)^{1-2\kappa}}{1-2\kappa}} f(y) \mathbb{P}_{y, s+q(t-s)+1}^Y [\tau_Y > t+1]$$

Also define for any $y \in (-1, 1)$,

$$g_\infty(y) := f(y) \eta_{Bm}(y)$$

Reminding that

$$\mathbb{P}_{y, s+q(t-s)+1}^Y (\tau_Y > t+1) = \mathbb{P}_y^{\tilde{B}} \left[\tau_{\tilde{B}} > \frac{(t+1)^{1-2\kappa} - (s+q(t-s)+1)^{1-2\kappa}}{1-2\kappa} \right]$$

and using Proposition 2.3. in [7] applied to the process \tilde{B} , $(g_t)_{t \geq 0}$ converges uniformly on $(-1, 1)$ towards g_∞ , which implies that the following quantity

$$\mathbb{E}_{\mu_{(s, s+\frac{q}{2}(t-s)), s+\frac{q}{2}(t-s)}}^Y (|g_t(Y_{s+q(t-s)}) - g_\infty(Y_{s+q(t-s)})| | \tau_Y > s+q(t-s))$$

goes to 0 when t goes to infinity. As a result, if one of the limit in the following equality exists, then the other limit exists also and one has

$$\lim_{t \rightarrow \infty} e^{\lambda_{Bm} \frac{(t+1)^{1-2\kappa} - (s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}} \mathbb{E}_{\mu_{(s, s+\frac{q}{2}(t-s)), s+\frac{q}{2}(t-s)}}^Y (f(Y_{s+q(t-s)}) \mathbb{1}_{\tau_Y > t}) \quad (18)$$

$$= \lim_{t \rightarrow \infty} e^{\lambda_{Bm} \frac{(s+q(t-s)+1)^{1-2\kappa} - (s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}} \mathbb{E}_{\mu_{(s, s+\frac{q}{2}(t-s)), s+\frac{q}{2}(t-s)}}^Y (g_\infty(Y_{s+q(t-s)}) \mathbb{1}_{\tau_Y > s+q(t-s)}) \quad (19)$$

By the definition of conditional expectation, one has

$$\begin{aligned} & \mathbb{E}_{\mu_{(s, s+\frac{q}{2}(t-s)), s+\frac{q}{2}(t-s)}}^Y (g_\infty(Y_{s+q(t-s)}) \mathbb{1}_{\tau_Y > s+q(t-s)}) \\ &= \mathbb{E}_{\mu_{(s, s+\frac{q}{2}(t-s)), s+\frac{q}{2}(t-s)}}^Y (g_\infty(Y_{s+q(t-s)}) | \tau_Y > s+q(t-s)) \mathbb{P}_{\mu_{(s, s+\frac{q}{2}(t-s)), s+\frac{q}{2}(t-s)}}^Y (\tau_Y > s+q(t-s)) \end{aligned}$$

On the one hand, by (14),

$$\lim_{t \rightarrow \infty} \mathbb{E}_{\mu_{(s, s+\frac{q}{2}(t-s)), s+\frac{q}{2}(t-s)}}^Y (g_\infty(Y_{s+q(t-s)}) | \tau_Y > s+q(t-s)) = \alpha_{Bm}(g_\infty) \quad (20)$$

On the other hand, using again Proposition 2.3. in [7] applied to the process \tilde{B} , we deduce that the following quantity

$$\left| e^{\lambda_{Bm} \frac{(s+q(t-s)+1)^{1-2\kappa} - (s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}} \mathbb{P}_{\mu_{(s, s+\frac{q}{2}(t-s)), s+\frac{q}{2}(t-s)}}^Y (\tau_Y > s+q(t-s)) - \mu_{(s, s+\frac{q}{2}(t-s))}(\eta_{Bm}) \right|$$

goes to 0 when t goes to infinity, and again by (14),

$$\lim_{t \rightarrow \infty} \mu_{(s, s + \frac{q}{2}(t-s))}(\eta_{Bm}) = \alpha_{Bm}(\eta_{Bm}) = 1$$

As a result,

$$\lim_{t \rightarrow \infty} e^{\lambda_{Bm} \frac{(s+q(t-s)+1)^{1-2\kappa} - (s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}} \mathbb{P}_{\mu_{(s, s + \frac{q}{2}(t-s)), s + \frac{q}{2}(t-s)}}^Y(\tau_Y > s + q(t-s)) = 1 \quad (21)$$

Hence we deduce from (18), (19), (20) and (21) that, for any bounded measurable function f ,

$$\begin{aligned} & \lim_{t \rightarrow \infty} e^{\lambda_{Bm} \frac{(t+1)^{1-2\kappa} - (s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}} \mathbb{E}_{\mu_{(s, s + \frac{q}{2}(t-s)), s + \frac{q}{2}(t-s)}}^Y (f(Y_{s+q(t-s)}) \mathbb{1}_{\tau_Y > t}) \\ &= \lim_{t \rightarrow \infty} e^{\lambda_{Bm} \frac{(s+q(t-s)+1)^{1-2\kappa} - (s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}} \mathbb{E}_{\mu_{(s, s + \frac{q}{2}(t-s)), s + \frac{q}{2}(t-s)}}^Y (g_\infty(Y_{s+q(t-s)}) \mathbb{1}_{\tau_Y > s+q(t-s)}) \\ &= \alpha_{Bm}(g_\infty) = \int_{(-1,1)} \alpha_{Bm}(dx) f(x) \eta_{Bm}(x) = \beta_{Bm}(f) \end{aligned}$$

Taking $f = \mathbb{1}$,

$$\lim_{t \rightarrow \infty} e^{\lambda_{Bm} \frac{(t+1)^{1-2\kappa} - (s+\frac{q}{2}(t-s)+1)^{1-2\kappa}}{1-2\kappa}} \mathbb{P}_{\mu_{(s, s + \frac{q}{2}(t-s)), s + \frac{q}{2}(t-s)}}^Y(\tau_Y > t) = 1$$

Thus, for any bounded measurable function f ,

$$\lim_{t \rightarrow \infty} \mathbb{E}_{\mu_{(s, s + \frac{q}{2}(t-s)), s + \frac{q}{2}(t-s)}}^Y (f(Y_{s+q(t-s)}) | \tau_Y > t) = \beta_{Bm}(f)$$

We conclude to (17) by Lebesgue's theorem. \square

4.5 Q -process

4.5.1 Existence of the Q -process

Now it remains to prove the existence of the Q -process. More precisely, this subsection is devoted to the proof of the following theorem

Theorem 6. *For any $s \leq t$ and $\mu \in \mathcal{M}_1((-1, 1))$, the family of probability measure $(\mathbb{P}_{\mu, s}(X_{[s, t]} \in \cdot | T < \tau_X))_{T > t}$ converges weakly when T goes to infinity towards*

$$\mathbb{Q}_{\mu, s}(X_{[s, t]} \in \cdot) = \mathbb{E}_{\mu, s} \left(\mathbb{1}_{X_{[s, t]} \in \cdot, \tau_X > t} \frac{\eta_t(X_t)}{\mu(\eta_s)} \right) \quad (22)$$

where $(\eta_t)_{t \geq 0}$ is defined in Proposition 5. Moreover, for any $s \leq t$ and $\mu \in \mathcal{M}_1((-1, 1))$, one has

$$\|\mathbb{Q}_{\mu,s}(X_t \in \cdot) - \mathbb{Q}_{\mu,s}^Y(Y_t \in \cdot)\|_{TV} \leq F(s) \quad (23)$$

where F is the same function as in Proposition 3 and \mathbb{Q}^Y is as defined in Proposition 4.

Before proving this theorem, let us first state the following key proposition.

Proposition 5. *There exist a family of positive bounded functions $(\eta_s)_{s \geq 0}$ satisfying*

$$\mathbb{E}_{x,s}(\mathbb{1}_{\tau_X > t} \eta_t(X_t)) = \eta_s(x), \quad \forall x \in (-1, 1), \forall s \leq t \quad (24)$$

and $H : \mathcal{M}_1((-1, 1)) \times \{s, t \in \mathbb{R}_+ : s \leq t\} \rightarrow (0, \infty)$ such that, for any $\nu \in \mathcal{M}_1((-1, 1))$ and $s \geq 0$,

$$\lim_{t \rightarrow \infty} H(\nu, s, t) = 0$$

and that, for any $s \leq t$ and for any $\mu, \nu \in \mathcal{M}_1((-1, 1))$,

$$\left| \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} - \frac{\mu(\eta_s)}{\nu(\eta_s)} \right| \leq H(\nu, s, t) \quad (25)$$

The proof of this proposition is postponed after the proof of Theorem 6.

Proof of Theorem 6. Let $\mu \in \mathcal{M}_1((-1, 1))$ and $s \leq t$. We define $\mathbb{Q}_{\mu,s}(X_{[s,t]} \in \cdot)$ as the formula (22). Then, for any $T > t$,

$$\begin{aligned} & \|\mathbb{P}_{\mu,s}(X_{[s,t]} \in \cdot | \tau_X > T) - \mathbb{Q}_{\mu,s}(X_{[s,t]} \in \cdot)\|_{TV} \\ &= \left\| \mathbb{E}_{\mu,s} \left[\mathbb{1}_{X_{[s,t]} \in \cdot, \tau_X > t} \left(\frac{\mathbb{P}_{X_t,t}(\tau_X > T)}{\mathbb{P}_{\mu,s}(\tau_X > T)} - \frac{\eta_t(X_t)}{\mu(\eta_s)} \right) \right] \right\|_{TV} \\ &= \left\| \mathbb{E}_{\mu,s} \left[\frac{\mathbb{1}_{X_{[s,t]} \in \cdot, \tau_X > t}}{\mathbb{P}_{\mu,s}(\tau_X > t)} \left(\frac{\mathbb{P}_{X_t,t}(\tau_X > T)}{\mathbb{P}_{\mu(s,t),t}(\tau_X > T)} - \frac{\eta_t(X_t)}{\mu(s,t)(\eta_t)} \right) \right] \right\|_{TV} \end{aligned}$$

where (24) was used. Hence, by (25) in Proposition 5,

$$\begin{aligned} \|\mathbb{P}_{\mu,s}(X_{[s,t]} \in \cdot | \tau_X > T) - \mathbb{Q}_{\mu,s}(X_{[s,t]} \in \cdot)\|_{TV} &\leq H(\mu(s,t), t, T) \left\| \mathbb{E}_{\mu,s} \left[\frac{\mathbb{1}_{X_{[s,t]} \in \cdot, \tau_X > t}}{\mathbb{P}_{\mu,s}(\tau_X > t)} \right] \right\|_{TV} \\ &\leq H(\mu(s,t), t, T) \end{aligned}$$

Since, for $s \leq t$ fixed, $\lim_{T \rightarrow \infty} H(\mu(s,t), t, T) = 0$, this implies the weak convergence of $(\mathbb{P}_{\mu,s}(X_{[s,t]} \in \cdot | T < \tau_X))_{T \geq t}$ towards $\mathbb{Q}_{\mu,s}$ defined in (22). The inequality (23) is a straightforward consequence of (11) in Proposition 3, letting $T \rightarrow \infty$. \square

4.5.2 Proof of Proposition 5

The remaining of the paper is dedicated to prove Proposition 5. In the proof, two important lemmata are used. So we will start by proving these lemmata before tackling the proof of Proposition 5.

Lemma 1. • *For any $s \geq 0$ and $a \in (0, 1)$, there exists $C_{s,a} > 0$ such that*

$$\inf_{x \in [-a, a]} \mathbb{P}_{x,s}(\tau_X > t) \geq C_{s,a} \sup_{x \in (-1, 1)} \mathbb{P}_{x,s}(\tau_X > t), \quad \forall t \geq 0$$

• *For any $a \in (0, 1)$, there exists $C_a > 0$ such that*

$$\inf_{x \in [-a, a]} \mathbb{P}_{x,s}^Y(\tau_Y > t) \geq C_a \sup_{x \in (-1, 1)} \mathbb{P}_{x,s}^Y(\tau_Y > t), \quad \forall s \leq t$$

Proof. • Let $a > 0$. To prove this, note that for any $x \in (-1, 1)$ and $t \geq s \geq 0$,

$$\mathbb{P}_{x,s}(\tau_X > t) = \mathbb{P}_{(s+1)^\kappa x}^B \left[\tau_B^{(\cdot+s+1)^\kappa} > t - s \right]$$

where, for any $s \geq 0$,

$$\tau_B^{(\cdot+s+1)^\kappa} := \inf \{ t \geq 0 : |B_t| = (t + s + 1)^\kappa \}$$

So the Harnack inequality to show becomes : for any $t \geq 0$,

$$\inf_{x \in [-a(s+1)^\kappa, a(s+1)^\kappa]} \mathbb{P}_x(\tau_B^{(\cdot+s+1)^\kappa} > t) \geq C_{s,a} \sup_{x \in (-(s+1)^\kappa, (s+1)^\kappa)} \mathbb{P}_x(\tau_B^{(\cdot+s+1)^\kappa} > t)$$

Actually, for any $t \geq 0$,

$$\inf_{x \in [-a(s+1)^\kappa, a(s+1)^\kappa]} \mathbb{P}_x(\tau_B^{(\cdot+s+1)^\kappa} > t) = \mathbb{P}_{a(s+1)^\kappa}(\tau_B^{(\cdot+s+1)^\kappa} > t)$$

and

$$\sup_{x \in (-(s+1)^\kappa, (s+1)^\kappa)} \mathbb{P}_x(\tau_B^{(\cdot+s+1)^\kappa} > t) = \mathbb{P}_0(\tau_B^{(\cdot+s+1)^\kappa} > t)$$

Then, for any $t \geq 0$,

$$\begin{aligned} \mathbb{P}_{a(s+1)^\kappa}(\tau_B^{(\cdot+s+1)^\kappa} > t) &\geq \mathbb{P}_{a(s+1)^\kappa} \left(\tau_B^0 < \tau_B^{(\cdot+s+1)^\kappa}, \tau_B^{(\cdot+s+1)^\kappa} > t + \tau_B^0 \right) \\ &= \mathbb{E}_{a(s+1)^\kappa} \left(\mathbb{1}_{\tau_B^0 < \tau_B^{(\cdot+s+1)^\kappa}} \mathbb{P}_0(\tau_B^{(\cdot+s+v+1)^\kappa} > t) \Big|_{v=\tau_B^0} \right) \\ &\geq \mathbb{P}_{a(s+1)^\kappa} \left(\tau_B^0 < \tau_B^{(\cdot+s+1)^\kappa} \right) \mathbb{P}_0 \left(\tau_B^{(\cdot+s+1)^\kappa} > t \right) \end{aligned}$$

where

$$\tau_B^0 := \inf\{t \geq 0 : B_t = 0\}$$

Then setting $C_{s,a} := \mathbb{P}_{a(s+1)^\kappa} \left(\tau_B^0 < \tau_B^{(+s+1)^\kappa} \right)$, one has $C_{s,a} > 0$ for any $s \geq 0$ and

$$\inf_{x \in [-a, a]} \mathbb{P}_{x,s}(\tau_X > t) \geq C_{s,a} \sup_{x \in (-1, 1)} \mathbb{P}_{x,s}(\tau_X > t), \quad \forall t \geq 0$$

- This is straightforward using the Harnack inequality for a Brownian motion and using the change of time provided by the Dubin-Schwartz's transformation (13). □

Now let us state and prove Lemma 2.

Lemma 2. *Let $a > 0$. Then there exists a function $\chi_a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that, for any $s \leq t$, for any $\mu \in \mathcal{M}_1((-1, 1))$ and any $\nu \in \mathcal{M}_1((-1, 1))$ such that $\nu([-a, a]) > \frac{1}{2}$,*

$$\left| \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} - \frac{\mathbb{P}_{\mu,s}^Y(\tau_Y > t)}{\mathbb{P}_{\nu,s}^Y(\tau_Y > t)} \right| \leq \chi_a(s)$$

with $\chi_a(s) \rightarrow 0$ when s goes to infinity

Proof. Let $s \leq t$. Remind the equality for any $\mu \in \mathcal{M}_1((-1, 1))$,

$$\mathbb{P}_{\mu,s}(\tau_Y > t) = \left(\frac{s+1}{t+1} \right)^{\frac{\kappa}{2}} \mathbb{E}_{\mu,s} \left[\exp \left(\frac{1}{2} N'_{s,t} \right) \mathbb{1}_{\tau_X > t} \right] \quad (26)$$

where we recall that $N'_{s,t}$ is defined in the proof of Proposition 3, and by (12), for any $\mu \in \mathcal{M}_1((-1, 1))$,

$$1 - \phi(s) \leq \frac{\mathbb{E}_{\mu,s}(\exp(\frac{1}{2} N'_{s,t}) \mathbb{1}_{\tau_X > t})}{\mathbb{P}_{\mu,s}(\tau_X > t)} \leq 1 + \phi(s)$$

where the function ϕ is also defined in the proof of Proposition 3. Thus, by (26), for any $\mu \in \mathcal{M}_1((-1, 1))$,

$$\left(\frac{s+1}{t+1} \right)^{\frac{\kappa}{2}} (1 - \phi(s)) \leq \frac{\mathbb{P}_{\mu,s}(\tau_Y > t)}{\mathbb{P}_{\mu,s}(\tau_X > t)} \leq \left(\frac{s+1}{t+1} \right)^{\frac{\kappa}{2}} (1 + \phi(s))$$

and, since $\phi(s) < 1$ for any $s \geq 0$, one has also,

$$\left(\frac{t+1}{s+1}\right)^{\frac{\kappa}{2}} \frac{1}{1+\phi(s)} \leq \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\mu,s}(\tau_Y > t)} \leq \left(\frac{t+1}{s+1}\right)^{\frac{\kappa}{2}} \frac{1}{1-\phi(s)} \quad (27)$$

Thus, for any $\mu, \nu \in \mathcal{M}_1((-1, 1))$,

$$\frac{1-\phi(s)}{1+\phi(s)} \leq \frac{\mathbb{P}_{\mu,s}(\tau_X > t) \mathbb{P}_{\nu,s}(\tau_Y > t)}{\mathbb{P}_{\mu,s}(\tau_Y > t) \mathbb{P}_{\nu,s}(\tau_X > t)} \leq \frac{1+\phi(s)}{1-\phi(s)} \quad (28)$$

Thus, it is deduced from (28) that, for any $\mu, \nu \in \mathcal{M}_1((-1, 1))$,

$$\begin{aligned} & \left| \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} - \frac{\mathbb{P}_{\mu,s}^Y(\tau_Y > t)}{\mathbb{P}_{\nu,s}^Y(\tau_Y > t)} \right| \\ & \leq \frac{\mathbb{P}_{\mu,s}^Y(\tau_Y > t)}{\mathbb{P}_{\nu,s}^Y(\tau_Y > t)} \left| \frac{\mathbb{P}_{\mu,s}(\tau_X > t) \mathbb{P}_{\nu,s}(\tau_Y > t)}{\mathbb{P}_{\nu,s}(\tau_X > t) \mathbb{P}_{\mu,s}(\tau_Y > t)} - 1 \right| \\ & \leq \frac{\mathbb{P}_{\mu,s}^Y(\tau_Y > t)}{\mathbb{P}_{\nu,s}^Y(\tau_Y > t)} \left[\left(\frac{1+\phi(s)}{1-\phi(s)} - 1 \right) \vee \left(1 - \frac{1-\phi(s)}{1+\phi(s)} \right) \right] \end{aligned}$$

Now, if ν is such that $\nu([-a, a]) > \frac{1}{2}$, by Lemma 1, for any $s \leq t$,

$$\begin{aligned} \mathbb{P}_{\mu,s}^Y(\tau_Y > t) & \leq \sup_{x \in (-1, 1)} \mathbb{P}_{x,s}^Y(\tau_Y > t) \\ & \leq \frac{1}{C_a} \inf_{x \in [-a, a]} \mathbb{P}_{x,s}^Y(\tau_Y > t) \\ & \leq \frac{1}{C_a \nu([-a, a])} \mathbb{P}_{\nu,s}^Y(\tau_Y > t) \\ & \leq \frac{2}{C_a} \mathbb{P}_{\nu,s}^Y(\tau_Y > t) \end{aligned}$$

As a result,

$$\left| \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} - \frac{\mathbb{P}_{\mu,s}^Y(\tau_Y > t)}{\mathbb{P}_{\nu,s}^Y(\tau_Y > t)} \right| \leq \frac{2}{C_a} \left[\left(\frac{1+\phi(s)}{1-\phi(s)} - 1 \right) \vee \left(1 - \frac{1-\phi(s)}{1+\phi(s)} \right) \right]$$

It remains to set $\chi_a(t) := \frac{2}{C_a} \left(\frac{1+\phi(s)}{1-\phi(s)} - 1 \right) \vee \left(1 - \frac{1-\phi(s)}{1+\phi(s)} \right)$. Then, since $\phi(s) \rightarrow 0$ when $s \rightarrow \infty$, χ_a goes also to 0 when s goes to infinity. \square

Remark 7. The inequalities (27) allows us to get that, for any $\mu \in \mathcal{M}_1((-1, 1))$ and $s \geq 0$,

$$\mathbb{P}_{\mu,s}(\tau_X > t) =_{t \rightarrow \infty} O \left(\left(\frac{t+1}{s+1} \right)^{\frac{\kappa}{2}} e^{-\lambda_{Bm} \frac{(t+1)^{1-2\kappa} - (s+1)^{1-2\kappa}}{1-2\kappa}} \right)$$

Hence, defining

$$\tau_B^{(+1)^\kappa} := \inf\{t \geq 0 : |B_t| = (t+1)^\kappa\},$$

then, for $\kappa < \frac{1}{2}$, for any $\mu \in \mathcal{M}_1((-1, 1))$,

$$\mathbb{P}_\mu^B(\tau_B^{(+1)^\kappa} > t) =_{t \rightarrow \infty} O\left((t+1)^{\frac{\kappa}{2}} e^{-\lambda_{Bm} \frac{(t+1)^{1-2\kappa}-1}{1-2\kappa}}\right)$$

However, contrary to the critical regime, it is difficult to provide an asymptotical equivalent for $\mathbb{P}_\mu^B(\tau_B^{(+1)^\kappa} > t)$.

Now we can prove Proposition 5.

Proof of Proposition 5. Let $\nu \in \mathcal{M}_1((-1, 1))$ and $s \geq 0$. We recall then the notation

$$\nu_{(s,t)} := \mathbb{P}_{\nu,s}(X_t \in \cdot | \tau_X > t), \quad \forall s \leq t$$

By Theorem 4, the family $(\nu_{(s,t)})_{s \leq t}$ converges weakly when t goes to infinity towards α_{Bm} . Thus, by Prokhorov's theorem, $(\nu_{(s,t)})_{s \leq t}$ is tight. This implies that there exists $a_s(\nu) \in (0, 1)$ such that, for any $t \geq s$, $\nu_{(s,t)}([-a_s(\nu), a_s(\nu)]) > \frac{1}{2}$.

Let $\mu \in \mathcal{M}_1((-1, 1))$, $s \leq t$ and $T \geq 0$. Then, using Markov's property,

$$\frac{\mathbb{P}_{\mu,s}(\tau_X > t+T) - \mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t+T) - \mathbb{P}_{\nu,s}(\tau_X > t)} = \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} \left(\frac{\mathbb{P}_{\mu_{(s,t),t}}(\tau_X > t+T)}{\mathbb{P}_{\nu_{(s,t),t}}(\tau_X > t+T)} - 1 \right)$$

Using the same argument as in the proof of Lemma 2, by Lemma 1 one has

$$\frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} \leq \frac{2}{C_{s,a_s(\nu)}}$$

Thus,

$$\left| \frac{\mathbb{P}_{\mu,s}(\tau_X > t+T)}{\mathbb{P}_{\nu,s}(\tau_X > t+T)} - \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} \right| \leq \frac{2}{C_{s,a_s(\nu)}} \left| \frac{\mathbb{P}_{\mu_{(s,t),t}}(\tau_X > t+T)}{\mathbb{P}_{\nu_{(s,t),t}}(\tau_X > t+T)} - 1 \right|$$

Using Lemma 2, one has

$$\left| \frac{\mathbb{P}_{\mu,s}(\tau_X > t+T)}{\mathbb{P}_{\nu,s}(\tau_X > t+T)} - \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} \right| \leq \frac{2}{C_{s,a_s(\nu)}} \left(\chi_{a_s(\nu)}(t) + \left| \frac{\mathbb{P}_{\mu_{(s,t),t}}^Y(\tau_Y > t+T)}{\mathbb{P}_{\nu_{(s,t),t}}^Y(\tau_Y > t+T)} - 1 \right| \right)$$

Now,

$$\begin{aligned} \left| \frac{\mathbb{P}_{\mu(s,t),t}^Y(\tau_Y > t+T)}{\mathbb{P}_{\nu(s,t),t}^Y(\tau_Y > t+T)} - 1 \right| &= \frac{|\mathbb{P}_{\mu(s,t),t}^Y(\tau_Y > t+T) - \mathbb{P}_{\nu(s,t),t}^Y(\tau_Y > t+T)|}{\mathbb{P}_{\nu(s,t),t}^Y(\tau_Y > t+T)} \\ &\leq \frac{\sup_{x \in (-1,1)} \mathbb{P}_{x,t}^Y(\tau_Y > t+T)}{\mathbb{P}_{\nu(s,t),t}^Y(\tau_Y > t+T)} \|\mu(s,t) - \nu(s,t)\|_{TV} \\ &\leq \frac{4}{C_{a_s(\nu)}} \times \left(F\left(\frac{t}{2}\right) + C_{Bm} \exp\left(-\gamma_{Bm} \frac{(t+1)^{1-2\kappa} - (\frac{t}{2}+1)^{1-2\kappa}}{1-2\kappa}\right) \right) \end{aligned}$$

where we used Lemma 1 and (16).

We conclude from all these computations that $t \rightarrow \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)}$ is a Cauchy sequence, hence converges as $t \rightarrow \infty$. Denote by $h(s, \mu, \nu)$ the limit and set

$$H(\nu, s, t) := \frac{2}{C_{s,a_s(\nu)}} \left[\chi_{a_s(\nu)}(t) + \frac{4}{C_{a_s(\nu)}} \times \left(F\left(\frac{t}{2}\right) + C_{Bm} \exp\left(-\gamma_{Bm} \frac{(t+1)^{1-2\kappa} - (\frac{t}{2}+1)^{1-2\kappa}}{1-2\kappa}\right) \right) \right]$$

One has therefore, for any $\mu, \nu \in \mathcal{M}_1((-1, 1))$,

$$\left| \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} - h(s, \mu, \nu) \right| \leq H(\nu, s, t)$$

and $\lim_{t \rightarrow \infty} H(\nu, s, t) = 0$.

In order to complete the proof, we will inspire by the proof of Proposition 3.1. in [9]. We define for any $s \geq 0$

$$\eta_s : x \rightarrow h(s, \delta_x, \delta_0)$$

Since, on the one hand,

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{0,s}(\tau_X > t)} = h(s, \mu, \delta_0)$$

and, on the other hand, by Lebesgue's theorem,

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{0,s}(\tau_X > t)} = \mu(\eta_s),$$

then, for any $\mu, \nu \in \mathcal{M}_1((-1, 1))$,

$$h(s, \mu, \nu) = \lim_{t \rightarrow \infty} \frac{\mathbb{P}_{\mu,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)} = \lim_{t \rightarrow \infty} \frac{\mathbb{P}_{\mu,s}(\tau_X > t)/\mathbb{P}_{0,s}(\tau_X > t)}{\mathbb{P}_{\nu,s}(\tau_X > t)/\mathbb{P}_{0,s}(\tau_X > t)} = \frac{\mu(\eta_s)}{\nu(\eta_s)}$$

Moreover, for any $s \leq t \leq u$,

$$\begin{aligned} \mathbb{E}_{x,s} \left(\mathbb{1}_{\tau_X > t} \frac{\mathbb{P}_{X_t,t}(\tau_X > u)}{\mathbb{P}_{0,t}(\tau_X > u)} \right) &= \frac{\mathbb{P}_{x,s}(\tau_X > u)}{\mathbb{P}_{0,t}(\tau_X > u)} \\ &= \frac{\mathbb{P}_{x,s}(\tau_X > u)}{\mathbb{P}_{0,s}(\tau_X > u)} \mathbb{E}_{0,s} \left(\mathbb{1}_{\tau_X > t} \frac{\mathbb{P}_{X_t,t}(\tau_X > u)}{\mathbb{P}_{0,t}(\tau_X > u)} \right) \end{aligned}$$

For any $\mu \in \mathcal{M}_1((-1, 1))$, integrating both sides of the equation with respect to μ , letting $u \rightarrow \infty$ and using Lebesgue's theorem, we deduce that, for any $s \leq t$, there exists a positive constant $c_{s,t}$ which does not depend on μ such that

$$c_{s,t} = \frac{\mathbb{E}_{\mu,s}(\mathbb{1}_{\tau_X > t} \eta_t(X_t))}{\mu(\eta_s)}$$

In addition, for any $s \leq t \leq u$ and for any $\nu \in \mathcal{M}_1((-1, 1))$,

$$c_{s,t} c_{t,u} = \frac{\mathbb{E}_{\mu,s}(\mathbb{1}_{\tau_X > t} \eta_t(X_t))}{\mu(\eta_s)} \frac{\mathbb{E}_{\nu,t}(\mathbb{1}_{\tau_X > u} \eta_u(X_u))}{\nu(\eta_t)}$$

Choosing $\nu = \mu_{(s,t)} = \mathbb{P}_{\mu,s}(X_t \in \cdot | \tau_X > t)$ and using Markov's property, we obtain

$$\begin{aligned} c_{s,t} c_{t,u} &= \frac{\mathbb{E}_{\mu,s}(\mathbb{1}_{\tau_X > t} \eta_t(X_t))}{\mu(\eta_s)} \times \frac{\mathbb{E}_{\mu_{(s,t)},t}(\mathbb{1}_{\tau_X > u} \eta_u(X_u))}{\mu_{(s,t)}(\eta_t)} \\ &= \frac{\mathbb{E}_{\mu,s}(\mathbb{1}_{\tau_X > t} \eta_t(X_t)) \times \mathbb{E}_{\mu,s}(\mathbb{E}_{X_t,t}(\mathbb{1}_{\tau_X > u} \eta_u(X_u)) | \tau_X > t)}{\mu(\eta_s) \times \mathbb{E}_{\mu,s}(\eta_t(X_t) | \tau_X > t)} \\ &= \frac{\mathbb{E}_{\mu,s}(\mathbb{1}_{\tau_X > t} \eta_t(X_t)) \times \mathbb{E}_{\mu,s}(\mathbb{E}_{X_t,t}(\mathbb{1}_{\tau_X > u} \eta_u(X_u)) \mathbb{1}_{\tau_X > t})}{\mu(\eta_s) \times \mathbb{E}_{\mu,s}(\eta_t(X_t) \mathbb{1}_{\tau_X > t})} \\ &= \frac{\mathbb{E}_{\mu,s}(\mathbb{1}_{\tau_X > u} \eta_u(X_u))}{\mu(\eta_s)} \\ &= c_{s,u} \end{aligned}$$

Because of the last equality, replacing for all $s \geq 0$ the function $\eta_s(x)$ by $\eta_s(x)/c_{0,s}$ entails (24). \square

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