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APPROXIMATE NORMAL FORMS VIA FLOQUET-BLOCH THEORY

PART 1: NEHOROSĚV STABILITY FOR LINEAR WAVES IN QUASIPERIODIC MEDIA

MITIA DUERINCKX, ANTOINE GLORIA, AND CHRISTOPHER SHIRLEY

ABSTRACT. We study the long-time behavior of the Schrödinger flow in a heterogeneous potential λV with small intensity $0 < \lambda \ll 1$. The main new ingredient, which we introduce in the general setting of a stationary ergodic potential, is an approximate stationary Floquet-Bloch theory that is used to put the perturbed Schrödinger operator into approximate normal form. In this first contribution we apply this approach to quasiperiodic potentials and establish a Nehorošev-type stability result. In particular, this ensures asymptotic ballistic transport up to a stretched exponential timescale $\exp(\lambda^{-\frac{1}{s}})$ for some $s > 0$. More precisely, the approximate normal form decomposition leads to long-time effective equations obtained by adding suitable unitary corrections to the free flow. The approach is robust and generically applies to linear waves. For classical waves, for instance, this allows to revisit diffractive geometric optics in quasiperiodically perturbed media. The application to the random setting is postponed to a companion article.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

1.1. **General overview.** We consider the perturbed Schrödinger operator $\mathcal{L}_\lambda := \mathcal{L}_0 + \lambda V$ on $L^2(\mathbb{R}^d)$, where $\mathcal{L}_0 := -\Delta$, where $V : \mathbb{R}^d \rightarrow \mathbb{R}$ denotes a stationary ergodic potential (see definition below), and where the coupling constant $\lambda > 0$ measures the intensity of the perturbation, and we study the long-time behavior of the corresponding Schrödinger flow

$$i\partial_t u_\lambda = \mathcal{L}_\lambda u_\lambda, \quad u_\lambda|_{t=0} = u^\circ, \tag{1.1}$$

in the small disorder regime $\lambda \ll 1$. This well-travelled model describes the motion of an electron in the ambient space \mathbb{R}^d in the presence of a weak disorder. The main difficulty is that the perturbation λV mixes up the eigenspaces of the free operator \mathcal{L}_0 , which can be compared to some extent to the effect of a nonlinear term.

In this contribution, we introduce a general method in form of an approximate stationary Floquet-Bloch theory, which allows to put the perturbed Schrödinger operator \mathcal{L}_λ into an approximate normal form. This provides an accurate description of the flow on long timescales by means of an effective equation where the perturbation λV is replaced by some *constant-coefficient* (typically pseudo-differential) operator. In other words, the flow is approximately described as an effective unitary correction of the free flow. This implies in particular ballistic transport in the considered regimes. In this first contribution, we apply this method to the specific setting of a quasiperiodic disorder V , which leads to several new results. Some of them were announced in [4]. The case of a random disorder is briefly discussed in Section 1.4 and is postponed to the companion article [9].

Let us describe the general strategy. Since the Fourier transform \mathcal{F} diagonalizes the free operator \mathcal{L}_0 and since the perturbation has small intensity $\lambda \ll 1$, we may look for a perturbation \mathcal{F}_λ of the Fourier transform that diagonalizes the perturbed operator \mathcal{L}_λ , turning it into a multiplication operator. If it exists, \mathcal{F}_λ is known as a *Bloch wave transform*. In other words, this amounts to writing \mathcal{L}_λ in *normal form*: decomposing the \mathcal{F}_λ -symbol of \mathcal{L}_λ as $k \mapsto |k|^2 + \kappa_{k,\lambda}$ (that is, as a perturbation of the \mathcal{F} -symbol $k \mapsto |k|^2$ of \mathcal{L}_0), we would find

$$\mathcal{L}_\lambda \mathcal{T}_\lambda = \mathcal{T}_\lambda (\mathcal{L}_0 + \mathcal{K}_\lambda), \quad (1.2)$$

in terms of the transform $\mathcal{T}_\lambda := \mathcal{F}_\lambda^{-1} \mathcal{F}$ and of the pseudo-differential operator $\mathcal{K}_\lambda := \kappa_{-\cdot, \lambda}$ (a multiplication operator in Fourier space). Alternatively, at the level of the Schrödinger flow, this yields

$$u_\lambda^t = e^{-it\mathcal{L}_\lambda} u^\circ = \mathcal{T}_\lambda e^{-it(\mathcal{L}_0 + \mathcal{K}_\lambda)} \mathcal{T}_\lambda^{-1} u^\circ. \quad (1.3)$$

Since \mathcal{F}_λ is a perturbation of \mathcal{F} , the transform \mathcal{T}_λ should be close to identity, in which case the flow u_λ would be close to $U_\lambda := e^{-it(\mathcal{L}_0 + \mathcal{K}_\lambda)} u^\circ$, that is, the solution of the (pseudo-differential) effective equation

$$i\partial_t U_\lambda = (\mathcal{L}_0 + \mathcal{K}_\lambda) U_\lambda, \quad U_\lambda|_{t=0} = u^\circ. \quad (1.4)$$

In this effective evolution, the operator \mathcal{K}_λ is viewed as a constant-coefficient unitary correction of the free flow due to the perturbation λV on long timescales.

In order to implement this general strategy, we proceed in five steps:

- (S1) **Stationary Floquet-Bloch fibration.** Assume that the stationary potential V is constructed on a probability space (Ω, \mathbb{P}) ; see Section 2.1.1 for precise definitions (in the periodic setting the space Ω reduces to the torus \mathbb{T}^d endowed with the Lebesgue measure). We may then decompose the Schrödinger operator \mathcal{L}_λ on $L^2(\mathbb{R}^d \times \Omega)$ as a direct integral of simpler fibered operators $\mathcal{L}_{k,\lambda}$ on the elementary space $L^2(\Omega)$. These (centered) fibered operators take the form

$$\mathcal{L}_{k,\lambda} := e^{-ik \cdot x} (-\Delta + \lambda V) e^{ik \cdot x} - |k|^2 = \mathcal{L}_{k,0} + \lambda V, \quad \mathcal{L}_{k,0} := -\Delta - 2ik \cdot \nabla,$$

and act on $L^2(\Omega)$ viewed as the space of stationary fields. In particular, the Schrödinger flow u_λ can be decomposed as

$$u_\lambda^t(x) = \int_{\mathbb{R}^d} \hat{u}^\circ(k) e^{ik \cdot x - it|k|^2} (e^{-it\mathcal{L}_{k,\lambda}} \mathbf{1})(x, \omega) d^*k, \quad (1.5)$$

in terms of the fibered evolutions $e^{-it\mathcal{L}_{k,\lambda}}1$ on $L^2(\Omega)$. We refer to this decomposition as the stationary Floquet-Bloch fibration since it extends the well-known corresponding construction in the periodic setting [23, 16].

- (S2) **Fibered perturbation problem.** By Step (S1), the Schrödinger operator \mathcal{L}_λ on $L^2(\mathbb{R}^d \times \Omega)$ decomposes into the family of fibered operators $(\mathcal{L}_{k,\lambda})_{k \in \mathbb{R}^d}$ on $L^2(\Omega)$. More precisely, the constant function 1 is an eigenfunction of $\mathcal{L}_{k,0}$ associated with the eigenvalue 0, and the decomposition (1.5) shows that it suffices to study the perturbation of this constant eigenfunction in the spectrum of $\mathcal{L}_{k,\lambda}$ for $\lambda \ll 1$. If the fibered operator $\mathcal{L}_{k,0}$ on $L^2(\Omega)$ had discrete spectrum, the Kato-Rellich perturbation theory [15, 24] would ensure the existence of a (local) branch $\lambda \mapsto (\kappa_{k,\lambda}, \psi_{k,\lambda})$ of eigenvalues and eigenvectors of $\mathcal{L}_{k,\lambda}$ starting at the eigenvalue 0 and the constant eigenfunction. As shown in [10], for periodic V , the operators $\mathcal{L}_{k,0}$ indeed have discrete spectrum and 0 is typically a simple eigenvalue, while for quasiperiodic V the eigenvalue 0 is embedded in dense pure point spectrum and for random V it is embedded in absolutely continuous spectrum. In the latter cases, 0 is thus not isolated in the spectrum of $\mathcal{L}_{k,0}$, hence there is no general perturbation theory at our disposal that would allow to construct branches $\lambda \mapsto (\kappa_{k,\lambda}, \psi_{k,\lambda})$. As briefly discussed in Section 1.4, such branches could actually be not smooth or even not exist.
- (S3) **Approximate Bloch waves.** Rather than investigating the existence of a branch $\lambda \mapsto (\kappa_{k,\lambda}, \psi_{k,\lambda})$ as in Step (S2), we consider the formal Rayleigh-Schrödinger perturbation series that would describe such a branch, should it exist, the terms of which are characterized by PDEs in $L^2(\Omega)$. By truncating this series and regularizing the terms if needed, we are led to an approximate branch $\lambda \mapsto (\kappa_{k,\lambda,\mu}^\ell, \psi_{k,\lambda,\mu}^\ell)$, where ℓ and μ are truncation and regularization parameters. This is referred to as a branch of approximate Bloch waves. By construction, $(\kappa_{k,\lambda,\mu}^\ell, \psi_{k,\lambda,\mu}^\ell)$ satisfies the eigenvalue equation for $\mathcal{L}_{k,\lambda}$ up to an error, called eigendefect. As explained in Section 1.4, regularization is here only needed in the random setting.
- (S4) **Control of the eigendefect.** Using the approximate Bloch waves of Step (S3) instead of exact ones yields errors involving the eigendefect, which thus needs to be controlled. In order to reach optimality in the scaling in ℓ , we proceed to a crucial resummation of the Rayleigh-Schrödinger series using a tree-counting argument. The control of the eigendefect depends in an essential way on the nature of the perturbation V . These estimates are the main technical ingredient required by the approach.
- (S5) **Approximate normal form decomposition.** The above construction naturally leads to defining the following transform,

$$\mathcal{T}_{\lambda,\mu}^\ell v(x, \omega) := \int_{\mathbb{R}^d} \hat{v}(k) e^{ik \cdot x} \psi_{k,\lambda,\mu}^\ell(x, \omega) d^*k, \quad v \in L^2(\mathbb{R}^d).$$

Considering the pseudo-differential operator $\mathcal{K}_{\lambda,\mu}^\ell := \kappa_{-i\nabla,\lambda,\mu}^\ell$, we obtain

$$\begin{aligned} (\mathcal{L}_\lambda \mathcal{T}_{\lambda,\mu}^\ell v)(x, \omega) &= (\mathcal{T}_{\lambda,\mu}^\ell (\mathcal{L}_0 + \mathcal{K}_{\lambda,\mu}^\ell) v)(x, \omega) \\ &\quad + \underbrace{\int_{\mathbb{R}^d} \hat{v}(k) e^{ik \cdot x} (\mathcal{L}_{\lambda,k} - \kappa_{k,\lambda,\mu}^\ell) \psi_{k,\lambda,\mu}^\ell(x, \omega) d^*k}_{=: (\mathcal{R}_{\lambda,\mu}^\ell v)(x, \omega)}, \end{aligned} \quad (1.6)$$

where the residual operator $\mathcal{R}_{\lambda,\mu}^\ell$ involves the eigendefect and is hopefully shown to be small in Step (S4). This yields an approximate normal form decomposition of \mathcal{L}_λ to be compared to (1.2). In addition, since $\psi_{k,\lambda,\mu}^\ell$ is by definition close to the constant function 1, the transform $\mathcal{T}_{\lambda,\mu}^\ell$ is close to identity. A description of the flow on long timescales then follows as in (1.3) in terms of an effective equation: more precisely, if $\mathcal{R}_{\lambda,\mu}^\ell$ is of order $O(g(\lambda))$ in some scaling of μ and ℓ , then on the timescale $t \leq O(g(\lambda)^{-1})$ the flow u_λ remains close to the solution $U_{\lambda,\mu}^\ell$ of the effective equation

$$i\partial_t U_{\lambda,\mu}^\ell = (\mathcal{L}_0 + \mathcal{K}_{\lambda,\mu}^\ell) U_{\lambda,\mu}^\ell, \quad U_{\lambda,\mu}^\ell|_{t=0} = u^\circ. \quad (1.7)$$

Steps (S1)–(S3) (and the crucial resummation argument in Step (S4)) are detailed in Section 2 below, which serves as a basis for the rest of this contribution and for the companion article [9]. The truncation in Step (S3) and the dynamical properties of approximate branches in Step (S5) are inspired by the treatment of the classical wave operator with heterogeneous coefficients at low wave number by Benoit and the second author in [5].

In the rest of this introduction, we describe the application of this general strategy to the quasiperiodic setting, before concluding with the extension to the case of classical (rather than quantum) waves, which illustrates the robustness of the method.

Notation.

- Throughout the article, we denote by $C \geq 1$ any constant that only depends on the dimension and on controlled quantities, the value of which may change from line to line. We use the notation \lesssim (resp. \gtrsim) for $\leq C \times$ (resp. $\geq \frac{1}{C} \times$) up to such a multiplicative constant C . We write \simeq when both \lesssim and \gtrsim hold. We also use the notation $a = O(b)$ for $a \lesssim b$. In an assumption, we write \ll for $\leq \frac{1}{C} \times$ for some large enough C . We add subscripts to $C, \lesssim, \gtrsim, \simeq, O(\cdot), \ll$ in order to indicate dependence on other parameters. The notation $a_\lambda = o(b_\lambda)$ stands for $a_\lambda/b_\lambda \rightarrow 0$ as $\lambda \downarrow 0$.
- We denote by $\hat{f}(k) := \mathcal{F}[f](k) := \int_{\mathbb{R}^d} e^{-ik \cdot x} f(x) dx$ the usual Fourier transform of a smooth function f on \mathbb{R}^d . The inverse Fourier transform is then given by $f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \int_{\mathbb{R}^d} e^{ik \cdot x} \hat{f}(k) d^*k$ in terms of the rescaled Lebesgue measure $d^*k := (2\pi)^{-d} dk$. Likewise, for $M \in \mathbb{N}$, when dealing with periodic functions on the torus $\mathbb{T}^M = [0, 2\pi)^M$, we denote by $\hat{f}(k) := \mathcal{F}[f](k) := \int_{\mathbb{T}^M} e^{-ik \cdot x} f(x) dx$ the associated Fourier series on \mathbb{Z}^M .
- The ball centered at x and of radius r in dimension n is denoted by $B^n(x, r)$, by B_r^n if $x = 0$, and by $B^n(x)$ if $r = 1$. We drop the superscript n whenever $n = d$.
- $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class (we always implicitly consider complex-valued maps when discussing the Schrödinger equation).
- \mathbb{N} denotes the set of natural numbers (including 0). For all $m \in \mathbb{N}$ and $b \in \mathbb{N}^m$, we set $|b| = \sum_i b_i$.
- We write $\langle x \rangle := (1 + |x|^2)^{1/2}$.
- LHS and RHS stand for “left-hand side” and “right-hand side”, respectively.

1.2. Quantum waves in a quasiperiodic potential. Consider the perturbed Schrödinger flow

$$i\partial_t u_\lambda = (-\Delta + \lambda V) u_\lambda, \quad u_\lambda|_{t=0} = u^\circ, \quad (1.8)$$

where the disorder V is quasiperiodic in the following sense.

(QP) *Quasiperiodic setting:*

$$V(x) := \tilde{V}(F^T x),$$

for $M \geq d$, some (winding) matrix $F \in \mathbb{R}^{d \times M}$ (the transpose of which is denoted by F^T), and some lifted map $\tilde{V} \in C(\mathbb{T}^M)$. The natural interpretation as a stationary potential is recalled in Section 2.1.1, cf. Example 2.1.

In this setting, our main result shows that for all initial data the flow remains close to an effective unitary correction of the free flow on very long timescales. More precisely, if the winding matrix F is Diophantine and if \tilde{V} is smooth enough, the conclusion holds up to some stretched exponential timescale.

Theorem 1 (Nehorošev-type stability for quantum waves). *Consider the quasiperiodic setting (QP). Denote by u_λ the Schrödinger flow (1.8) with initial data $u^\circ \in \mathcal{S}(\mathbb{R}^d)$. Assume that the winding matrix $F \in \mathbb{R}^{d \times M}$ satisfies a Diophantine condition, that is, for some $r_0 > 0$,*

$$|F\xi| \geq \frac{1}{C}|\xi|^{-r_0} \quad \text{for all } \xi \in \mathbb{Z}^M \setminus \{0\}, \quad (1.9)$$

that the lifted map \tilde{V} is Gevrey-regular, that is, for some $\alpha > 0$,

$$\|\mathbb{1}_{|\cdot| > K} \mathcal{F}\tilde{V}\|_{L^1} \leq \exp(-K^\alpha) \quad \text{for all } K \geq 0, \quad (1.10)$$

and let $s_0 > M + r_0$. There exist a dense open set $\mathcal{O} \subset \mathbb{R}^d$ and a sequence $(k \mapsto \nu_k^n)_n \subset C(\mathcal{O})$ (cf. Definition 2.4) with the following property: Given $\ell \in \mathbb{N}$, setting $\kappa_{k,\lambda}^\ell := \lambda \sum_{n=0}^{\ell} \lambda^n \nu_k^n$, the flow

$$U_\lambda^{\ell;t}(x) := \int_{\mathbb{R}^d} e^{-it(|k|^2 + \mathbb{1}_{\mathcal{O}}(k)\kappa_{k,\lambda}^\ell)} e^{ik \cdot x} \hat{u}^\circ(k) d^*k, \quad (1.11)$$

which solves the effective (pseudo-differential) equation

$$(i\partial_t + \Delta)U_\lambda^\ell = \kappa_{-i\nabla,\lambda}^\ell \Pi_{\mathcal{O}} U_\lambda^\ell, \quad U_\lambda^\ell|_{t=0} := u^\circ, \quad (1.12)$$

where $\Pi_{\mathcal{O}} := \mathbb{1}_{\mathcal{O}}(-i\nabla)$ denotes the projection on \mathcal{O} in Fourier space, satisfies for all $\gamma < \frac{1}{2d+1}$ and $\lambda \ll 1$,

$$\sup_{0 \leq \lambda' \leq \lambda} \sup_{0 \leq t \leq T(\lambda)} \|u_{\lambda'}^t - U_{\lambda'}^{\ell(\lambda);t}\|_{L^2} \lesssim_{u^\circ} \lambda^\gamma, \quad (1.13)$$

with the choice

$$T(\lambda) := \exp(\lambda^{-\frac{1}{s}}), \quad \ell(\lambda) := \lambda^{-\frac{1}{s}},$$

and $s := \frac{s_0 + M + 1}{1 - (2d+1)\gamma} + \frac{1}{\alpha} \frac{s_0 + 2M + 1}{1 - (2d+1)\gamma}$. If in addition \tilde{V} is a trigonometric polynomial (that is, $\mathcal{F}\tilde{V}$ compactly supported), the same holds with $s := \frac{s_0 + M + 1}{1 - (2d+1)\gamma}$. \diamond

The existence of an (exact) Bloch wave transform \mathcal{F}_λ in the quasiperiodic setting is a difficult and open question. Under some strong assumptions on the quasiperiodic structure, it was recently shown by Karpeshina and coauthors [12, 14, 13] that an (exact) normal form decomposition (1.2) holds with \mathcal{T}_λ replaced by some non-bijective map. More precisely, there is a large set of initial data for which (1.3) holds, which in particular implies that ballistic transport holds for all times for such initial data. This can be viewed as a KAM-type result in an infinite-dimensional setting. Pursuing this analogy with nearly

integrable Hamiltonian systems, the so-called Arnol'd diffusion phenomenon [3] would suggest that (1.3) should indeed only hold for some (typically strict) subspace of initial data and under strong assumptions on the quasiperiodic structure. However, in the small disorder regime $\lambda \ll 1$, recent results in 1D [26, 27] rather advocate that the normal form decomposition (1.2) might generically hold with bijective \mathcal{T}_λ . The main difficulty for such a result is related to the dense crossings of eigenvalues as explained in the spectral interpretation in Section 1.4. In contrast, in the present contribution, we focus on approximate versions (1.6) of the normal form decomposition (1.2). The validity up to a stretched exponential timescale is obtained by optimizing wrt the truncation parameter ℓ in (1.6). Rather than a KAM result, this corresponds to a Nehorošev-type stability result [18, 19, 20], which holds for all initial data and for (essentially) any quasiperiodic structure. As such, this nicely completes the recent KAM-type results [12, 14, 13, 26, 27].

In particular, the above stability result leads to ballistic transport up to stretched exponential timescales. Although ballistic transport is expected to hold on all timescales, this result is new and stands out by its generality as it is established for all initial data and under a mere Diophantine condition. More precisely, for $m \geq 1$, we define the rescaled moments of the flow,

$$M_m^t(u_\lambda) := \left\| \left(\frac{\cdot}{t}\right)^m u_\lambda^t \right\|_{L^2}, \quad (1.14)$$

which can be viewed as measuring the asymptotic ballistic velocity of the wavefunction, and we show that the velocity $M_m^t(u_\lambda)$ remains close to $M_m^t(u_0)$, the velocity of the free flow u_0 , which can be explicitly computed and obviously remains of order 1.

Corollary 1 (Ballistic transport for quantum waves). *Under the assumptions of Theorem 1, for all $u^\circ \in \mathcal{S}(\mathbb{R}^d)$ and $m \geq 1$, we have*

$$\lim_{t, \lambda} |M_m^t(u_\lambda) - M_m^t(u_0)| = 0$$

as $t \uparrow \infty$ and $\lambda \downarrow 0$ in the regime $t \leq \exp(\lambda^{-\frac{1}{s}})$ with $s > \frac{\alpha+1}{\alpha}(m+1)(2(s_0+1)+M)$. \diamond

Remarks 1.1.

- *Regularity and algebraic assumptions:*

As clear in the proof, the largest timescale allowed by the approach depends both on the regularity of \tilde{V} and on the algebraic properties of F , which is not surprising in view of the known results in 1D (e.g. [6]). Stretched exponential timescales are obtained only for Gevrey-regular \tilde{V} and Diophantine F . On the one hand, if F is Diophantine but if we decrease the regularity of \tilde{V} , the allowed timescale is shortened: for algebraically decaying $\mathcal{F}\tilde{V}$ the same results hold up to some corresponding algebraic timescale. On the other hand, if \tilde{V} is a trigonometric polynomial (that is, $\mathcal{F}\tilde{V}$ compactly supported) but if F is only irrational (that is, $F\xi \neq 0$ for all $\xi \in \mathbb{Z}^M \setminus \{0\}$), then the results hold up to any algebraic timescale (cf. Remark 3.4).

- *Peaked initial data in Fourier space:*

Provided u° has a fixed compact support in Fourier space, the error estimates in Theorem 1 depend on u° via its $L^2(\mathbb{R}^d)$ -norm only, so that the result holds uniformly for initial data that are peaked in Fourier space, e.g. of the form $u_\varepsilon^\circ(x) = \varepsilon^{d/2} e^{ik_0 \cdot x} g(\varepsilon x)$ with $0 < \varepsilon \ll 1$, $\hat{g} \in C_c^\infty(\mathbb{R}^d)$, and $k_0 \in \mathcal{O}$. Choosing a scaling relation $\varepsilon = \lambda^\beta$ for some $\beta > 0$, injecting this in the formula for U_λ^ℓ , and Taylor expanding $k \mapsto \kappa_{k, \lambda}^\ell$ around k_0 , we are lead to an effective PDE that involves a localization at wavenumber k_0 of the

pseudo-differential operator appearing in (1.12). We do not dwell on this here and refer to Corollary 2 below for the corresponding result in the case of classical waves.

- *Periodic setting:*

A corresponding result also holds in the case of a periodic disorder V . In that case, perturbation series actually become summable, so the error estimates are the best possible: we may take $\ell \uparrow \infty$ in (1.13) and the results hold for all $t \geq 0$ and $\lambda \ll 1$ without any restriction. The proof follows from the additional periodic estimates of Remarks 3.2 and 3.4. \diamond

The proof of Theorem 1 is the object of Section 4 based on the estimates of the eigen-defect in Section 3. Section 4 is written in such a way that it can be easily adapted to situations where eigendefects satisfy different estimates, which will be used in [9].

1.3. Classical waves in quasiperiodic media. We turn to the corresponding perturbed classical wave flow

$$\partial_{tt}^2 u_\lambda = \nabla \cdot (\text{Id} + \lambda a) \nabla u_\lambda, \quad u_\lambda|_{t=0} = u^\circ, \quad \partial_t u_\lambda|_{t=0} = v^\circ, \quad (1.15)$$

where the matrix field a is quasiperiodic in the following sense.

(QP') *Quasiperiodic setting:*

$$a(x) := \tilde{a}(F^T x),$$

for $M \geq d$, some (winding) matrix $F \in \mathbb{R}^{d \times M}$ (the transpose of which is denoted by F^T), and some lifted map $\tilde{a} \in C(\mathbb{T}^M)$. The natural interpretation as a stationary matrix field is recalled in Section 2.1.1, cf. Example 2.1.

For classical (as opposed to quantum) waves, a large body of literature is devoted to the identification of effective equations to describe the wave flow on long timescales in perturbed media, e.g. [1, 2] where diffractive corrections to geometric optics are established in the case of a periodic perturbation of a periodic medium. In the following, we extend such results to quasiperiodic perturbations. We start with a result for fixed initial data; the proof follows that of Theorem 1 and the main modifications are indicated in Section 5.

Theorem 2 (Nehorošev-type stability for classical waves). *Consider the quasiperiodic setting (QP'). Denote by u_λ the classical flow (1.15) with initial data $(u^\circ, v^\circ) \in \mathcal{S}(\mathbb{R}^d)^2$ and assume that the Fourier transform \hat{v}° is supported away from 0. Assume that the winding matrix $F \in \mathbb{R}^{d \times M}$ satisfies a Diophantine condition (1.9) for some $r_0 > 0$, that the lifted map \tilde{a} is Gevrey-regular in the sense of (1.10) for some $\alpha > 0$, and let $s_0 > M + r_0$. There exist a dense open set $\mathcal{O} \subset \mathbb{R}^d$ and a sequence $(k \mapsto \nu_k^n)_n \subset C(\mathcal{O})$ with the following property: Given $\ell \in \mathbb{N}$, setting $\kappa_{k,\lambda}^\ell := \lambda \sum_{n=0}^\ell \lambda^n \nu_k^n$, defining $\mathcal{O}_\lambda^\ell := \{k \in \mathcal{O} : |k|^2 - \lambda \sum_{n=0}^\ell \lambda^n |\nu_k^n| \geq 0\}$ (note that $\mathcal{O}_\lambda^\ell \uparrow \mathcal{O}$ as $\lambda \downarrow 0$), and recalling the notation $\text{sinc } x = \frac{\sin x}{x}$, the flow*

$$U_\lambda^{\ell;t}(x) := \int_{\mathbb{R}^d} \left(\cos \left(t \sqrt{|k|^2 + \mathbf{1}_{\mathcal{O}_\lambda^\ell}(k) \kappa_{k,\lambda}^\ell} \right) \hat{u}^\circ(k) + t \text{sinc} \left(t \sqrt{|k|^2 + \mathbf{1}_{\mathcal{O}_\lambda^\ell}(k) \kappa_{k,\lambda}^\ell} \right) \hat{v}^\circ(k) \right) e^{ik \cdot x} d^* k,$$

which solves the effective (pseudo-differential) equation

$$(\partial_{tt}^2 - \Delta + \kappa_{-i\nabla, \lambda}^\ell \Pi_{\mathcal{O}_\lambda^\ell}) U_\lambda^\ell = 0, \quad U_\lambda^\ell|_{t=0} = u^\circ, \quad \partial_t U_\lambda^\ell|_{t=0} = v^\circ, \quad (1.16)$$

where $\Pi_{\mathcal{O}^\lambda} := \mathbb{1}_{\mathcal{O}^\lambda}(-i\nabla)$ denotes the projection on \mathcal{O}^λ in Fourier space, satisfies the estimate (1.13) of Theorem 1. \diamond

Next, we specialize this result to the case when the initial data are peaked in Fourier space, thus replacing equation (1.15) by

$$\begin{aligned} \partial_{tt}^2 u_{\lambda,\varepsilon} &= \nabla \cdot (\text{Id} + \lambda a) \nabla u_{\lambda,\varepsilon}, \\ u_{\lambda,\varepsilon}(x)|_{t=0} &= \varepsilon^{\frac{d}{2}} e^{ik_0 \cdot x} u^\circ(\varepsilon x), \quad \partial_t u_{\lambda,\varepsilon}(x)|_{t=0} = \varepsilon^{\frac{d}{2}} e^{ik_0 \cdot x} v^\circ(\varepsilon x), \end{aligned} \quad (1.17)$$

for some $k_0 \in \mathbb{R}^d \setminus \{0\}$. (Note that the case $k_0 = 0$ is of a very different nature as it is about zooming at the bottom of the spectrum, which will be considered in a future work.) In this setting, for $\lambda, \varepsilon \downarrow 0$, the pseudo-differential equation (1.16) is replaced by a local PDE. As in [1, 2], the result is naturally expressed by factorizing out the free flow and the group velocity $\pm \frac{k_0}{|k_0|}$.

Corollary 2. *Under the assumptions of Theorem 2, we denote by u_λ the classical flow (1.3) with $k_0 \in \mathcal{O} \setminus \{0\}$ and the scaling relation $\varepsilon = \lambda^\beta$ for some $\beta > 0$. For simplicity, assume that $(u^\circ, v^\circ) \in \mathcal{S}(\mathbb{R}^d)^2$ has compactly supported Fourier transform and that \tilde{a} is a trigonometric polynomial. For all $m, p \geq 0$ there exist p -th order tensors $C_{m,p}(k_0)$ with the following property: Given $\ell \in \mathbb{N}$, the flow $\tilde{U}_\lambda^{\ell;t} := \frac{1}{2}(\tilde{U}_{\lambda,+}^{\ell;t} + \tilde{U}_{\lambda,-}^{\ell;t})$ with*

$$\tilde{U}_{\lambda,\pm}^{\ell;t}(x) := \frac{\varepsilon^{\frac{d}{2}}}{2} e^{ik_0 \cdot (x \mp t \frac{k_0}{|k_0|})} A_{\lambda,\pm}^{\ell;t}(\varepsilon(x \mp t \frac{k_0}{|k_0|})),$$

where the amplitudes $A_{\lambda,\pm}^\ell$ solve the effective diffractive PDEs

$$\begin{aligned} \left(i\partial_t \pm \varepsilon^2 \frac{|k_0|^2 \Delta - (k_0 \cdot \nabla)^2}{2|k_0|^3} \mp \lambda \frac{\nu_{k_0}^0}{2|k_0|} \right) A_{\lambda,\pm}^\ell \\ = \pm \sum_{\substack{0 \leq m \leq \ell \\ 0 \leq p \leq \lfloor \ell/\alpha \rfloor}} \mathbb{1}_{\substack{m+p \geq 2 \\ (m,p) \neq (0,2)}} \lambda^m \varepsilon^p C_{m,p}(k_0) \odot (-i\nabla)^{\otimes p} A_{\lambda,\pm}^\ell, \end{aligned} \quad (1.18)$$

with initial data $A_{\lambda,\varepsilon,\pm}^\ell|_{t=0} = u^\circ \pm \frac{iv^\circ}{|k_0|}$ and with \odot denoting the total contraction of tensors, satisfies for some $0 < \gamma = \gamma(\beta, \ell) \leq 1$ and for all $t \geq 0$ and $\lambda \ll 1$,

$$\|u_\lambda^t - \tilde{U}_\lambda^{\ell;t}\|_{L^2} \lesssim_{k_0, \ell, u^\circ, v^\circ, \tilde{a}} \lambda^\gamma (1 + \lambda^\ell t). \quad \diamond$$

For the choice $\lambda = \varepsilon^2$, this result extends [1, 2] to the case of quasiperiodic perturbations (up to changing variables $(\varepsilon x, \varepsilon t) \rightsquigarrow (x', t')$). In terms of diffractive geometric optics, it is read as follows:

- For times $t \ll \varepsilon^{-2} \wedge \lambda^{-1}$, the equation (1.18) for the amplitudes $A_{\lambda,\pm}^\ell$ reduces to $i\partial_t A_{\lambda,\pm}^\ell = 0$ up to negligible terms, hence the flow $u_\lambda^t(x)$ simply remains close to

$$\frac{\varepsilon^{\frac{d}{2}}}{2} \sum_{\pm} e^{ik_0 \cdot (x \mp t \frac{k_0}{|k_0|})} \left(u^\circ \pm \frac{v^\circ}{i|k_0|} \right) (\lambda^\beta (x \mp \frac{k_0}{|k_0|} t)),$$

which is the geometric optic approximation with group velocity $\pm \frac{k_0}{|k_0|}$.

- For times $t \gtrsim \varepsilon^{-2} \wedge \lambda^{-1}$, the transported profiles $u^\circ \pm \frac{v^\circ}{i|k_0|}$ are further deformed and their spread is described by the Schrödinger equation (1.18). This is known as diffractive geometric optics. The first diffractive correction due to the background appears on times

$t \gtrsim \varepsilon^{-2}$, while the first diffractive correction due to the disorder a appears on times $t \gtrsim \lambda^{-1}$ and is nonzero whenever $\nu_{k_0}^0 := k_0 \cdot (\int_{\mathbb{T}^M} \tilde{a})k_0 \neq 0$.

1.4. Other settings and general comments. The general strategy that we introduce in this contribution is robust and can be applied to different types of disorder and of linear wave operators — as already illustrated by treating both quantum and classical waves (in addition, classical wave equations with higher-order elliptic operators or other types of linear waves like Maxwell’s equations could be treated similarly). It also extends to the discrete setting. Below, we announce the corresponding results of the companion article [9] for a random disorder V , we include a spectral discussion of the approach depending on the nature of the spectrum of the fibered operators, and we conclude with emphasizing that normal form decompositions are much stronger results than ballistic transport.

Random disorder. For a random perturbation λV , the construction of approximate Bloch waves becomes particularly difficult: the coefficients of the Rayleigh-Schrödinger perturbation series are no longer defined in $L^2(\Omega)$, which compels us to further use a regularization procedure (cf. parameter μ in Step (S3)). This is related to the expected emergence of complex resonances in the spectrum of the fibered operators, cf. below. In the companion article [9], our results are as follows: If V has fast decaying correlations, we prove the validity for all $\ell \geq 1$ of an approximate normal form decomposition (1.6) with residual operator \mathcal{R}_λ^ℓ of order $O_\ell(\lambda^{2-1/\ell})$. Optimizing wrt ℓ , we deduce that the Schrödinger flow remains close to the free flow for $t = o(\lambda^{-2}|\log \lambda|^{-1})$. This seemingly simple result crucially uses stochastic cancellations: closeness to the free flow is obtained for free only on the timescale $t \ll \lambda^{-1}$ and is more demanding beyond that. Such a result is however not new (and is essentially already contained in [25]), but the norm in probability in which results are established is much stronger. The approach also allows to treat the case of a strongly correlated random disorder V , in which case closeness to the free flow is valid only up to some timescale that explicitly depends on the decay of correlations. In addition, we consider the corresponding problem of the random perturbation of a periodic operator $-\Delta + V_0$, which requires to first establish fine properties of the periodic Bloch variety (which reduces to a paraboloid for $V_0 \equiv 0$).

Spectral interpretation. The general approach reduces the analysis of the Schrödinger flow u_λ to the perturbation of the eigenvalue 0 in the spectrum of each of the fibered operators $\mathcal{L}_{k,\lambda}$, $k \in \mathbb{R}^d$, in the regime $\lambda \ll 1$. The goal is to construct a branch $\lambda \mapsto \kappa_{k,\lambda}$ of eigenvalues of $\mathcal{L}_{k,\lambda}$ starting at 0 for each k . This perturbation problem however fundamentally depends on the nature of the spectrum.

- If the spectrum of $\mathcal{L}_{k,0}$ is discrete (and if the perturbation λV is $\mathcal{L}_{k,0}$ -compact), as is the case for a periodic disorder V , the Kato-Rellich perturbation theory ensures the existence and analyticity of (most of) the branches $\lambda \mapsto \kappa_{k,\lambda}$, which are then given by their (convergent) Rayleigh-Schrödinger perturbation series. Note that, as discussed in Remark 3.2, the analyticity of the branches cannot hold uniformly when k gets close to the so-called diffraction hyperplanes, that is, to the values such that the eigenvalue 0 is not simple in the spectrum of $\mathcal{L}_{k,0}$. Regardless of these subtleties, in the periodic case, a Bloch transform \mathcal{F}_λ actually exists, is bijective, and is strongly close to \mathcal{F} [16].
- If the spectrum of $\mathcal{L}_{k,0}$ is dense pure point with a (typically simple) eigenvalue at 0, as is the case for a quasiperiodic disorder V , and if a branch of eigenvalues $\lambda \mapsto \kappa_{k,\lambda}$ exists, then it is certainly not analytic due to dense crossings with other eigenvalues. In

particular, the formal Rayleigh-Schrödinger perturbation series should not converge in that setting. Our approach then amounts to using the Rayleigh-Schrödinger series as an asymptotic series describing a likely existing branch. Such asymptotic information are of course not strong enough to obtain conclusions on all timescales.

- If the spectrum of $\mathcal{L}_{k,0}$ consists of a simple eigenvalue at 0 embedded in an absolutely continuous part, as is typically the case for random disorder V (cf. [10]), we expect the eigenvalue to disappear whenever $\lambda > 0$ in view of Fermi's Golden Rule. Given that no branch of eigenvalues should thus exist, we may wonder about the meaning of the approximate branch $\lambda \mapsto \kappa_{k,\lambda,\mu}^\ell$ that we construct. A key observation in [10] shows that $\kappa_{k,\lambda,\mu}^\ell$ admits a limit as $\mu \downarrow 0$ and that this limit is complex-valued with an imaginary part is of order $\simeq \lambda^2$. A natural interpretation is that the eigenvalue at 0 gets perturbed into a branch of complex resonances and that $\kappa_{k,\lambda,\mu}^\ell$ is an approximate resonance. The need to regularize the coefficients in the Rayleigh-Schrödinger series in this setting is precisely related to the fact that the corresponding resonant modes cannot belong to the space $L^2(\Omega)$: the approximate Bloch waves $\psi_{k,\lambda,\mu}^\ell$ are viewed as approximate resonant modes and diverge in $L^2(\Omega)$ as $\mu \downarrow 0$. This explains the limitation in the companion article [9]: since our approach requires to stick to the strong space $L^2(\Omega)$, we are limited to timescales such that resonances are not visible, that is, to timescales $t = o(\lambda^{-2})$. We refer to [10] for a discussion of the Floquet-Bloch approach beyond that timescale.

Normal forms and ballistic transport: a counterexample. Whereas the approximate normal form decomposition (1.6) implies ballistic transport in some asymptotic regime, ballistic transport is itself a weaker property and may occur even when the operator cannot be put in any approximate normal form. This can be seen on the following instructive example: define the operator $\mathcal{L}'_0 := i\nabla_1$ on $L^2(\mathbb{R}^d)$ and its perturbed version $\mathcal{L}'_\lambda := \mathcal{L}'_0 + \lambda V$ where V is a centered stationary Gaussian random field on \mathbb{R}^d with covariance function $\mathcal{C} \in C_c^\infty(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}} \mathcal{C} \neq 0$. A direct computation shows that the corresponding flow $i\partial_t v_\lambda = \mathcal{L}'_\lambda v_\lambda$, $v_\lambda|_{t=0} = v^\circ$ is explicitly given by

$$v_\lambda^t(x) = v^\circ(x_1 + t, x') e^{-i\lambda \int_0^t V(x_1+t-s, x') ds}, \quad x := (x_1, x'), \quad x' := (x_2, \dots, x_d).$$

Hence, it satisfies ballistic transport on all timescales $t \geq 0$ in the form

$$\mathbb{E} \left[\int_{\mathbb{R}^d} |x|^2 |v_\lambda^t(x)|^2 dx \right] = \mathbb{E} \left[\int_{\mathbb{R}^d} |x|^2 |v^\circ(x_1 + t, x')|^2 dx \right] \simeq (1+t)^2.$$

On the one hand, (exact) Bloch waves for \mathcal{L}'_λ do not exist. On the other hand, we claim that this operator cannot even be put in approximate normal form (1.6) with a residual operator of order $o(\lambda^2)$. If it were the case, it would imply that v_λ remains close to the solution V_λ of a corresponding effective equation up to the timescale $t \lesssim \lambda^{-2}$. Since V_λ is deterministic, this entails that the averaged wavefunction $\mathbb{E}[v_\lambda]$ also remains close to V_λ (hence to v_λ) up to that timescale. Now an explicit computation yields

$$\mathbb{E} [v_\lambda^t(x)] = v^\circ(x_1 + t, x') e^{-\lambda^2 \int_0^t s \mathcal{C}(t-s, 0) ds},$$

so that $\|\mathbb{E}[v_\lambda^t] - v_\lambda^t\|_{L^2} \gtrsim \|v^\circ\|_{L^2} (1 - e^{-\lambda^2 \int_0^t s \mathcal{C}(t-s, 0) ds})$, which is not small in the regime $t \simeq \lambda^{-2}$ (since $\int_0^t s \mathcal{C}(t-s, 0) ds \sim \frac{t}{2} \int_{\mathbb{R}} \mathcal{C}$ as $t \uparrow \infty$). In conclusion, this example displays ballistic transport on all timescales but can be accurately described as a unitary correction

of the free flow only up to the timescale $t = o(\lambda^{-2})$. This illustrates that approximate normal form decompositions are indeed much stronger than ballistic transport properties.

2. APPROXIMATE STATIONARY FLOQUET-BLOCH THEORY

In this section, we adapt the standard periodic Floquet-Bloch theory (e.g. [16]) to the general stationary setting, we show how the behavior of the Schrödinger flow is reduced to a fibered perturbation problem, and we approximately solve this perturbation problem in terms of suitable approximate Bloch waves. This covers Steps (S1)–(S3) of our general approach.

2.1. Stationary Floquet-Bloch theory. We start by adapting the standard periodic Bloch-Floquet theory (e.g. [16]) to the general stationary setting.

2.1.1. Preliminary: notion of stationarity. Assume that the probability space (Ω, \mathbb{P}) is endowed with a measurable action $\tau := (\tau_x)_{x \in \mathbb{R}^d}$ of the group $(\mathbb{R}^d, +)$ on Ω , that is, the maps $\tau_x : \Omega \rightarrow \Omega$ are measurable for all x and they satisfy

- $\tau_x \circ \tau_y = \tau_{x+y}$ for all $x, y \in \mathbb{R}^d$;
- $\mathbb{P}[\tau_x A] = \mathbb{P}[A]$ for all $x \in \mathbb{R}^d$ and all measurable $A \subset \Omega$;
- the map $\mathbb{R}^d \times \Omega \rightarrow \Omega : (x, \omega) \mapsto \tau_x \omega$ is jointly measurable.

We then assume that the potential V is given by $V(x, \omega) := \tilde{V}(\tau_{-x}\omega)$ for some random variable $\tilde{V} : \Omega \rightarrow \mathbb{R}$. More generally, a measurable function $f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ is said to be τ -stationary (or simply stationary) if it satisfies $f(x, \omega) = f(0, \tau_{-x}\omega)$ for all x, ω . In particular, this implies $\mathbb{E}[f(x, \cdot)] = \mathbb{E}[f(0, \cdot)]$ for all x , and it ensures that f is jointly measurable and that the map $\omega \mapsto f(x, \omega)$ is measurable for all x . Setting $\tilde{f}(\omega) := f(0, \omega)$, stationarity obviously yields a bijection between random variables $\tilde{f} : \Omega \rightarrow \mathbb{R}$ and stationary measurable functions $f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$. The function f is then called the *stationary extension* of the random variable \tilde{f} . In particular, the subspace of stationary functions $f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ in $L^2(\Omega, L^2_{\text{loc}}(\mathbb{R}^d))$ is identified with the Hilbert space $L^2(\Omega)$, and the weak gradient ∇ on locally square integrable functions then turns by stationarity into a linear operator on $L^2(\Omega)$. For all $l \geq 0$, we may further define the (Hilbert) space $H^l(\Omega)$ as the space of all random variables $\tilde{f} \in L^2(\Omega)$ the stationary extension f of which belongs to $L^2(\Omega; H^l_{\text{loc}}(\mathbb{R}^d))$. Also note that, by a stochastic version of Lusin's theorem, the joint measurability condition above implies that τ -stationary functions are necessarily stochastically continuous; in particular, for all $\tilde{f} \in L^2(\Omega)$, the map $\mathbb{R}^d \rightarrow L^2(\Omega) : y \mapsto \tilde{f}(\tau_{-y}\cdot)$ is continuous. We refer to e.g. [8, Appendix A.2] for details.

Examples 2.1.

(a) It is well-known that periodic and quasi-periodic (as well as almost periodic [22]) potentials V can be viewed as random stationary potentials:

- For periodic V , we set $\Omega := \mathbb{T}^d$ endowed with the Lebesgue measure, we define $\tau_{-x}\omega = \omega + x \pmod{\mathbb{T}^d}$, and we set $W(x, \omega) := V(\omega + x)$, which defines a stationary field W . The stationary gradient on $L^2(\Omega)$ then coincides with the usual weak gradient on $L^2(\mathbb{T}^d)$
- For quasiperiodic V as in (QP), we set $\Omega := \mathbb{T}^M$ endowed with the Lebesgue measure, we define $\tau_{-x}\omega = \omega + F^T x \pmod{\mathbb{T}^M}$, and we set $W(x, \omega) := \tilde{V}(\omega + F^T x)$, which

defines a stationary field W . The stationary gradient on $L^2(\Omega)$ then coincides with $F\nabla_{\mathbb{T}^M}$ in terms of the weak gradient $\nabla_{\mathbb{T}^M}$ on $L^2(\mathbb{T}^M)$.

In the sequel, we consider the Schrödinger flow with V replaced by $W(\cdot, \omega)$ and we exploit averaging wrt the translation ω . Since our main results are stated for $\omega = 0$, we however need to get rid of averaging wrt ω a posteriori and establish estimates that are uniform in ω . For simplicity, we make no difference between V and W in the notation.

- (b) Any \mathbb{Z}^d -stationary random potential (that is, satisfying the stationarity assumption (H) for an action τ of $(\mathbb{Z}^d, +)$ on Ω) can also be seen as a stationary random potential as above. Indeed, assume that $\tau' := (\tau'_z)_{z \in \mathbb{Z}^d}$ is a measurable action of $(\mathbb{Z}^d, +)$ on a probability space (Ω', \mathbb{P}') , and that V is τ' -stationary, that is, $V(x+z, \omega) = V(x, \tau'_z \omega)$ for all $x \in \mathbb{R}^d$, $z \in \mathbb{Z}^d$, and $\omega \in \Omega'$. Let us endow $\Omega := \Omega' \times [0, 1]^d$ with the product probability measure $\mathbb{P} := \mathbb{P}' \times \mathcal{L}eb$, where $\mathcal{L}eb$ here denotes the Lebesgue measure restricted to the torus $[0, 1]^d$, and let us define the action $\tau := (\tau_x)_{x \in \mathbb{R}^d}$ of $(\mathbb{R}^d, +)$ on $\Omega = \Omega' \times [0, 1]^d$ by

$$\tau_x(\omega, y) := (\tau'_{\lfloor x \rfloor} \omega, y + x - \lfloor y + x \rfloor),$$

where $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \dots, \lfloor x_d \rfloor)$ for $x \in \mathbb{R}^d$. The map $\bar{V}(x, (\omega, y)) := V(x - y, \omega)$ then defines a τ -stationary random function on $\mathbb{R}^d \times \Omega$. \diamond

2.1.2. *Stationary Floquet transform.* For $f \in L^2(\mathbb{R}^d \times \Omega)$, we first define the (non-stationary) Floquet transform $\mathcal{U}f : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ by

$$\mathcal{U}f(k, x, \omega) = \mathcal{F}[\mathcal{O}_x f(\cdot, \omega)](k), \quad \mathcal{O}_x f(y, \omega) = f(x + y, \tau_y \omega). \quad (2.1)$$

The following properties directly follow from this definition.

Lemma 2.2. *Writing $e_k(x) := e^{ik \cdot x}$,*

- (i) *the map \mathcal{O}_x (hence also the map $f \mapsto \mathcal{U}f(\cdot, x, \cdot)$) is unitary on $L^2(\mathbb{R}^d \times \Omega)$ for all x ;*
- (ii) *$\mathcal{U}f(k, \cdot, \cdot)$ is e_k -stationary in the sense that $\mathcal{U}f(k, x + z, \omega) = e_k(z) \mathcal{U}f(k, x, \tau_{-z} \omega)$;*
- (iii) *$f(x, \omega) = \mathcal{F}^{-1}[\mathcal{U}f(\cdot, x, \omega)](0)$, where the RHS is well-defined in $L^2(\mathbb{R}^d \times \Omega)$.* \diamond

(The expression $\mathcal{F}^{-1}[\mathcal{U}f(\cdot, x, \omega)](0)$ is indeed well-defined in $L^2(\mathbb{R}^d \times \Omega)$ since the stochastic continuity that follows from the measurability of the action τ ensures that the map $\mathbb{R}^d \rightarrow L^2(\mathbb{R}^d \times \Omega) : y \mapsto ((x, \omega) \mapsto \mathcal{F}^{-1}[\mathcal{U}f(\cdot, x, \omega)](y) = f(x + y, \tau_y \omega))$ is continuous, cf. Section 2.1.1.)

For $f \in L^2(\mathbb{R}^d \times \Omega)$, it is then natural to define

$$\mathcal{V}f(k, x, \omega) := e^{-ik \cdot x} \mathcal{U}f(k, x, \omega), \quad (2.2)$$

which, for any fixed $k \in \mathbb{R}^d$, is stationary by the above properties. Also, for all $x \in \mathbb{R}^d$, the map $f \mapsto \mathcal{V}f(\cdot, x, \cdot)$ is unitary on $L^2(\mathbb{R}^d \times \Omega)$. With the usual identification of $\mathcal{V}f$ with its restriction $\mathcal{V}f(\cdot, 0, \cdot)$, we may thus view \mathcal{V} as a unitary operator on $L^2(\mathbb{R}^d \times \Omega)$, which we refer to as the *stationary Floquet transform*.

Lemma 2.3. *The stationary Floquet transform \mathcal{V} satisfies*

- (i) *$f(x, \omega) = \mathcal{F}^{-1}[k \mapsto e_k(x) \mathcal{V}f(k, x, \omega)](0)$, where the RHS is defined in $L^2(\mathbb{R}^d \times \Omega)$;*
- (ii) *denoting by $\iota : L^2(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d \times \Omega)$ the canonical injection, we have $\mathcal{V} \circ \iota = \iota \circ \mathcal{F}$ on $L^2(\mathbb{R}^d)$;*
- (iii) *for all $f \in L^2(\mathbb{R}^d \times \Omega)$ and $g \in L^2(\Omega)$ with $gf \in L^2(\mathbb{R}^d \times \Omega)$, we have $\mathcal{V}(gf) = g \mathcal{V}f$.* \diamond

2.1.3. *Stationary Floquet-Bloch fibration.* The stationary Floquet transform \mathcal{V} decomposes differential operators on $L^2(\mathbb{R}^d \times \Omega)$ into direct integrals of elementary fibered operators on the simpler space $L^2(\Omega)$ of stationary functions.

On the one hand, the Laplacian $-\Delta$ on $L^2(\mathbb{R}^d \times \Omega)$ is transformed as follows by the stationary Floquet transform \mathcal{V} , for all $f \in L^2(\Omega; H^2(\mathbb{R}^d))$,

$$\mathcal{V}[(-\Delta)f](k, x, \omega) = -(\nabla + ik) \cdot (\nabla + ik) \mathcal{V}f(k, x, \omega) = (|k|^2 - \Delta_k) \mathcal{V}f(k, x, \omega) \quad (2.3)$$

in terms of the (centered) fibered Laplacian

$$-\Delta_k := e^{-ik \cdot x} (-\Delta) e^{ik \cdot x} - |k|^2 = -(\nabla + ik) \cdot (\nabla + ik) - |k|^2 = -\Delta - 2ik \cdot \nabla,$$

where all the derivatives are taken in the weak sense wrt the x -variable. The action of the operator $-\Delta_k$ is considered in (2.3) on stationary functions, hence equivalently on $L^2(\Omega)$. Its domain is then clearly $D(-\Delta_k) = H^2(\Omega)$, and the centering ensures that the constant function 1 belongs to its kernel.

On the other hand, since the potential $V \in L^2(\Omega)$ is stationary, it (densely) defines a multiplicative operator on $L^2(\mathbb{R}^d \times \Omega)$. In the periodic and quasiperiodic settings in the present article, V turns out to be uniformly bounded, hence it defines a bounded self-adjoint operator on $L^2(\mathbb{R}^d \times \Omega)$. In view of the random case in the companion article [9], we emphasize that the boundedness of V is not needed for our purposes: if V satisfies a lower bound $V(x, \omega) \geq -K(\omega)(1 + |x|^2)$ for some random variable K with $\mathbb{E}[|K|^2] < \infty$ (which is a mild requirement), the Faris-Lavine argument [11] ensures that the corresponding Schrödinger operator $\mathcal{L}_\lambda = -\Delta + \lambda V$ on $L^2(\mathbb{R}^d \times \Omega)$ is essentially self-adjoint on $L^\infty(\Omega; H^2(\mathbb{R}^d, |x|^2 dx))$. As in (2.3), we then find for all $f \in D(\mathcal{L}_\lambda)$, using Lemma 2.3(iii),

$$\mathcal{V}[\mathcal{L}_\lambda f](k, \omega) = (|k|^2 + \mathcal{L}_{k, \lambda}) \mathcal{V}f(k, \omega), \quad (2.4)$$

in terms of the fibered Schrödinger operator

$$\mathcal{L}_{k, \lambda} := e^{-ik \cdot x} (-\Delta + \lambda V) e^{ik \cdot x} - |k|^2 = -\Delta_k + \lambda V.$$

For fixed k , we view the fibered operator $\mathcal{L}_{k, \lambda}$ as an essentially self-adjoint operator on $L^2(\Omega)$. Using direct integral representation (see e.g. [23, p.280]), we may reformulate Lemma 2.3(i) as

$$\begin{aligned} L^2(\mathbb{R}^d \times \Omega) &= \int_{\oplus} e_k L^2(\Omega) d^*k, \\ -\Delta &= \int_{\oplus} e_k (|k|^2 - \Delta_k) d^*k, \quad \mathcal{L}_\lambda = \int_{\oplus} e_k (|k|^2 + \mathcal{L}_{k, \lambda}) d^*k. \end{aligned}$$

This fibration leads to the following useful decomposition of the Schrödinger flow (1.1): for an initial condition $u^\circ \in L^2(\mathbb{R}^d)$, denoting as before by $\iota : L^2(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d \times \Omega)$ the canonical injection and using Lemma 2.3(i)–(ii) and (2.4), we find

$$\begin{aligned} u_\lambda^t(x, \omega) &= (e^{-it\mathcal{L}_\lambda} \iota u^\circ)(x, \omega) \stackrel{(i)}{=} \mathcal{F}^{-1} \left[k \mapsto e^{ik \cdot x} \mathcal{V}(e^{-it\mathcal{L}_\lambda} \iota u^\circ)(k, x, \omega) \right] (0) \\ &\stackrel{(2.4)}{=} \mathcal{F}^{-1} \left[k \mapsto e^{ik \cdot x} e^{-it|k|^2} (e^{-it\mathcal{L}_{k, \lambda}} \mathcal{V} \iota u^\circ)(k, x, \omega) \right] (0) \\ &\stackrel{(ii)}{=} \mathcal{F}^{-1} \left[k \mapsto \hat{u}^\circ(k) e^{ik \cdot x} e^{-it|k|^2} (e^{-it\mathcal{L}_{k, \lambda}} 1)(k, x, \omega) \right] (0) \\ &= \mathcal{F}^{-1} \left[k \mapsto \hat{u}^\circ(k) e^{ik \cdot x} e^{-it|k|^2} \int_{\mathbb{R}} e^{-it\kappa} d\mu_{k, \lambda}^1(\kappa)(x, \omega) \right] (0), \end{aligned}$$

in terms of the $L^2(\Omega)$ -valued spectral measure $\mu_{k,\lambda}^1$ of $\mathcal{L}_{k,\lambda}$ associated with the constant function 1. Provided we have enough integrability with respect to the k -variable, this takes the simpler form

$$\begin{aligned} u_\lambda^t(x, \omega) &= \int_{\mathbb{R}^d} \hat{u}^\circ(k) e^{-it|k|^2} e^{ik \cdot x} (e^{-it\mathcal{L}_{k,\lambda}})(x, \omega) d^*k \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}} \hat{u}^\circ(k) e^{-it(|k|^2 + \kappa)} e^{ik \cdot x} d\mu_{k,\lambda}^1(\kappa)(x, \omega) d^*k. \end{aligned} \quad (2.5)$$

For $\lambda = 0$, we simply have $d\mu_{k,0}^1(\kappa) = d\delta_0(\kappa)$, while for $\lambda > 0$ the planar wave e_k is corrected into a (potentially non-atomic) *Bloch measure* $e_k d\mu_{k,\lambda}^1(\kappa)$, which is adapted to the potential V . If $\mu_{k,\lambda}^1$ admits an atom at κ_* , the function $e_k \mu_{k,\lambda}^1(\{\kappa_*\}) \in L^2(\Omega; L_{\text{loc}}^2(\mathbb{R}^d))$ is called a *Bloch wave*, which is in particular a “generalized eigenfunction” of \mathcal{L}_λ associated with the “generalized eigenvalue” $|k|^2 + \kappa_*$. In the periodic case, the measures $\mu_{k,\lambda}^1$ are all discrete and the situation is thus much simplified [16].

2.1.4. Fibered perturbation problem. We have seen that the Schrödinger operator \mathcal{L}_λ on $L^2(\mathbb{R}^d \times \Omega)$ is equivalent to the collection of all the fibered operators $\mathcal{L}_{k,\lambda}$ on $L^2(\Omega)$, for $k \in \mathbb{R}^d$. More precisely, the decomposition (2.5) implies that the behavior of the Schrödinger flow u_λ is equivalent to that of all the spectral measures $\mu_{k,\lambda}^1$ associated with the constant function 1, for $k \in \mathbb{R}^d$. Since 1 is an eigenfunction of $\mathcal{L}_{k,0}$ associated with the eigenvalue 0, we are reduced to study the perturbation problem for this eigenvalue in the spectrum of $\mathcal{L}_{k,\lambda}$ in the regime $\lambda \ll 1$. A naïve approach consists in postulating that for $\lambda > 0$ the eigenvalue 0 (resp. the eigenfunction 1) is perturbed into an eigenvalue $\kappa_{k,\lambda}$ (resp. an eigenfunction $\psi_{k,\lambda}$), and in trying to construct them via their Taylor series, that is, as the sum of the so-called Rayleigh-Schrödinger perturbation series

$$\kappa_{k,\lambda} = \lambda \sum_{n=0}^{\infty} \lambda^n \nu_k^n, \quad \psi_{k,\lambda} = 1 + \sum_{n=1}^{\infty} \lambda^n \phi_k^n. \quad (2.6)$$

This can indeed be done in the periodic setting (for most k). The eigenvalue equation

$$\mathcal{L}_{k,\lambda} \psi_{k,\lambda} = \kappa_{k,\lambda} \psi_{k,\lambda}, \quad \kappa_{k,\lambda} \in \mathbb{R}, \quad \psi_{k,\lambda} \in L^2(\Omega), \quad (2.7)$$

then splits into a hierarchy of Rayleigh-Schrödinger equations for the coefficients $\nu_k^n \in \mathbb{R}$ and $\phi_k^n \in L^2(\Omega)$. In line with the wording in [5] related to the homogenization theory, we refer to the ϕ_k^n 's as the correctors. This approach however quickly fails: for quasiperiodic V the coefficients (ν_k^n, ϕ_k^n) can be constructed from the Rayleigh-Schrödinger equations but the series (2.6) is not summable, while for random V the correctors ϕ_k^n cannot even be defined in $L^2(\Omega)$. This is related to the spectral interpretation in Section 1.4: for quasiperiodic V dense crossings of eigenvalues are expected to destroy analyticity of the branch $\lambda \mapsto (\kappa_{k,\lambda}, \psi_{k,\lambda})$, while for random V no such branch should even exist.

2.2. Approximate Bloch waves. While solving the eigenvalue problem (2.7) beyond the periodic setting is very difficult or even impossible, we may at least construct approximate solutions of (2.7), that is, approximate Bloch waves, in the small disorder regime $\lambda \ll 1$. In the quasiperiodic setting, since the Rayleigh-Schrödinger coefficients (ν_k^n, ϕ_k^n) can all be constructed, we view (2.6) as asymptotic series describing a likely existing branch, hence

we define approximate Bloch waves as the partial sums of this formal series,

$$\kappa_{k,\lambda}^\ell := \lambda \sum_{n=0}^{\ell} \lambda^n \nu_{k,\mu}^n, \quad \psi_{k,\lambda}^\ell := \sum_{n=0}^{\ell} \lambda^n \phi_{k,\mu}^n.$$

Such truncated Rayleigh-Schrödinger series are referred to in the sequel as *Taylor-Bloch waves*. The Bloch decomposition (2.5) then naturally urges to compare the Schrödinger flow u_λ with the approximate solution

$$V_\lambda^{\ell;t}(x, \omega) := \int_{\mathbb{R}^d} \hat{u}^\circ(k) e^{-it(|k|^2 + \kappa_{k,\lambda}^\ell)} e^{ik \cdot x} \psi_{k,\lambda}^\ell(x, \omega) d^*k.$$

In the random setting, since the correctors ϕ_k^n are not defined in $L^2(\Omega)$, we further need to suitably regularize the Rayleigh-Schrödinger equations to ensure the existence of solutions in the desired space; details are postponed to [9].

We focus here on the situation when the Rayleigh-Schrödinger coefficients (ν_k^n, ϕ_k^n) can be constructed, thus defining the jet of a formal branch $\lambda \mapsto (\kappa_{k,\lambda}, \psi_{k,\lambda})$ at $\lambda = 0$. We further require continuity wrt the parameter k .

Definition 2.4. Given $1 \leq \ell < \infty$ and a nonempty open set $O \subset \mathbb{R}^d$, a family $(\nu_k^n, \phi_k^n : k \in O, 0 \leq n \leq \ell) \subset L^2(\Omega) \times \mathbb{R}$ is called a *field of ℓ -jets of Bloch waves* if

- (i) for all n , the map $O \rightarrow L^2(\Omega) \times \mathbb{R} : k \mapsto (\nu_k^n, \phi_k^n)$ is continuous;
- (ii) $\nu_k^n := \mathbb{E}[V\phi_k^n]$ for all $n \geq 0$;
- (iii) for all $k \in O$, we have $\phi_k^0 \equiv 1$, and for all n the function ϕ_k^{n+1} satisfies $\mathbb{E}[\phi_k^{n+1}] = 0$ and

$$-\Delta_k \phi_k^{n+1} = -\Pi V \phi_k^n + \sum_{l=0}^{n-1} \mathbb{E}[V \phi_k^l] \phi_k^{n-l}, \quad (2.8)$$

where Π denotes the orthogonal projection onto $\{1\}^\perp$, that is $\Pi f := f - \mathbb{E}[f]$.

The corresponding family $(\kappa_{k,\lambda}^\ell, \psi_{k,\lambda}^\ell : k \in O, \lambda \geq 0)$ of partial sums,

$$\kappa_{k,\lambda}^\ell := \lambda \mathbb{E}[V \psi_{k,\lambda}^\ell] = \lambda \sum_{n=0}^{\ell} \lambda^n \nu_k^n, \quad \psi_{k,\lambda}^\ell := \sum_{n=0}^{\ell} \lambda^n \phi_k^n, \quad (2.9)$$

is then called the *sheet of Taylor-Bloch waves* of order ℓ . Note that $\nu_k^0 = \mathbb{E}[V]$. \diamond

As the following shows, for small λ , these Taylor-Bloch waves almost satisfy the eigenvalue equation (2.7).

Lemma 2.5. *Let $\ell \geq 1$, let $(\nu_k^n, \phi_k^n)_{k,n}$ be a field of ℓ -jets of Bloch waves, and let $(\kappa_{k,\lambda}^\ell, \psi_{k,\lambda}^\ell)_{k,\lambda}$ be the corresponding sheet of Taylor-Bloch waves. Then we have*

$$(-\Delta_k + \lambda V) \psi_{k,\lambda}^\ell = \kappa_{k,\lambda}^\ell \psi_{k,\lambda}^\ell + \lambda^{\ell+1} \mathfrak{d}_{k,\lambda}^\ell,$$

in terms of the Taylor-Bloch eigendefect

$$\mathfrak{d}_{k,\lambda}^\ell := \left(\Pi V \phi_k^\ell - \sum_{l=0}^{\ell-1} \nu_k^l \phi_k^{\ell-l} \right) - \lambda \sum_{n=1}^{\ell} \sum_{l=\ell-n}^{\ell-1} \lambda^{n+l-\ell} \nu_k^{l+1} \phi_k^n. \quad \diamond$$

Proof. The proof is elementary and follows from several resummations:

$$\begin{aligned}
\mathcal{L}_{k,\lambda}\psi_{k,\lambda}^\ell &= \mathcal{L}_{k,\lambda} \sum_{n=0}^{\ell} \lambda^n \phi_k^n \\
&= - \sum_{n=1}^{\ell} \lambda^n (V\phi_k^{n-1} - \mathbb{E}[V\phi_k^{n-1}]) + \sum_{n=2}^{\ell} \lambda^n \sum_{l=0}^{n-2} \phi_k^{n-l-1} \mathbb{E}[V\phi_k^l] + V \sum_{n=0}^{\ell} \lambda^{n+1} \phi_k^n \\
&= \lambda \sum_{n=0}^{\ell-1} \lambda^n \sum_{l=0}^n \nu_k^l \phi_k^{n-l} + \lambda^{\ell+1} V \phi_k^\ell \\
&= \kappa_{k,\lambda}^\ell \psi_{k,\lambda}^\ell + \lambda^{\ell+1} \left(V \phi_k^\ell - \mathbb{E}[V \phi_k^\ell] - \sum_{l=0}^{\ell-1} \nu_k^l \phi_k^{\ell-l} \right) - \lambda^{\ell+2} \sum_{n=1}^{\ell} \sum_{l=\ell-n}^{\ell-1} \lambda^{n+l-\ell} \nu_k^{l+1} \phi_k^n,
\end{aligned}$$

as claimed. \square

2.3. Tree formulas for Rayleigh-Schrödinger coefficients. For later purposes, we explicitly solve the nonlinear recurrence equations (2.8) for the Rayleigh-Schrödinger coefficients (ν_k^n, ϕ_k^n) . Indeed, while solving (2.8) would naïvely lead to a sum of C^{n^2} terms, the formula below only involves a sum of C^n terms (cf. exponential number of trees). This is crucial to prove sharp corrector estimates in Proposition 3.3, which are key to the stretched exponential timescale in the main results. Although we believe that the formulas below could be obtained by a careful counting and recombination of the many terms occurring when solving the nonlinear recurrence (2.8), we rather display a shorter indirect argument based on a Lagrange series expansion.

Proposition 2.6. *Let V be stationary on (Ω, \mathbb{P}) and assume that $V \in L^p(\Omega)$ for all $p < \infty$. For $m \geq 1$ let $\mathcal{T}_m \subset \mathbb{N}^m$ denote the set of rooted m -trees, which we define as the following set of indices (cf. Figure 1),*

$$\mathcal{T}_m := \{a = (a_1, \dots, a_m) \in \mathbb{N}^m : a_j + \dots + a_m \leq m - j, \forall 1 \leq j \leq m\}. \quad (2.10)$$

Note that $\#\mathcal{T}_m \leq 4^m$. If $(\nu_k^n, \phi_k^n)_{k,n}$ is a field of jets of Bloch waves as in Definition 2.4, then we have for all $n \geq 0$,

$$\begin{aligned}
\nu_k^n &= \sum_{m=1}^{n+1} (-1)^{n+1-m} \sum_{a \in \mathcal{T}_m} \sum_{\substack{c \in \mathbb{N}^m \\ |c|=n+1-m}} \sum_{\substack{b^1 \in \mathbb{N}^{c_1} \\ |b^1|=a_1}} \dots \sum_{\substack{b^m \in \mathbb{N}^{c_m} \\ |b^m|=a_m}} \\
&\quad \times \mathbb{E}[V(-\Delta_k)^{-b_1^1-1} \Pi V \dots (-\Delta_k)^{-b_{c_1}^1-1} \Pi V] \\
&\quad \times \dots \mathbb{E}[V(-\Delta_k)^{-b_1^m-1} \Pi V \dots (-\Delta_k)^{-b_{c_m}^m-1} \Pi V],
\end{aligned}$$

and for all $n \geq 1$,

$$\phi_k^n = \sum_{m=1}^n (-1)^m \sum_{\ell=0}^{n-m} \sum_{\substack{a \in \mathbb{N}^\ell \\ |a|=n-m-\ell}} \sum_{\substack{b \in \mathbb{N}^m \\ |b|=\ell}} \nu_k^{a_1} \dots \nu_k^{a_\ell} (-\Delta_k)^{-b_1-1} \Pi V \dots (-\Delta_k)^{-b_m-1} \Pi V,$$

assuming that all the terms make sense in $L^2(\Omega)$ (in fact, whenever $k \in \mathcal{O}$ with the notation of Section 3.1). \diamond

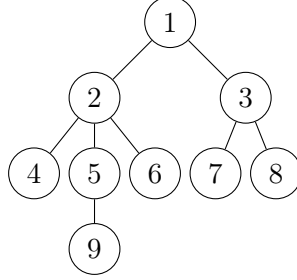


FIGURE 1. Given a rooted plane tree, we let the vertices be labelled from the left to the right, from the top to the bottom. A tree of order m then uniquely defines an element $a \in \mathcal{T}_m$ by defining a_j as the number of children of the j th vertex. For instance, the above plotted tree corresponds to the element $a = (2, 3, 2, 0, 1, 0, 0, 0, 0) \in \mathcal{T}_9$.

Proof. The cardinality $\#\mathcal{T}_m$ is obviously bounded by the number of ways to put $m - 1$ unlabelled balls in m labelled boxes, that is, by $\binom{2(m-1)}{m-1} \leq 4^m$. By a density argument, we may assume that the potential V belongs to $L^\infty(\Omega)$. By an approximation argument, it further suffices to establish the result for *regularized* Rayleigh-Schrödinger coefficients (as used in the random setting in [9]): for $\mu > 0$ we define $(\nu_{k,\mu}^n, \phi_{k,\mu}^n)$ as follows,

- (i) $\nu_{k,\mu}^n := \mathbb{E}[V\phi_{k,\mu}^n]$ for all $n \geq 0$;
- (ii) $\phi_{k,\mu}^0 \equiv 1$, and for all $n \geq 0$ the function $\phi_{k,\mu}^{n+1}$ satisfies $\mathbb{E}[\phi_{k,\mu}^{n+1}] = 0$ and

$$(-i\mu - \Delta_k)\phi_{k,\mu}^{n+1} = -\Pi V\phi_{k,\mu}^n + \sum_{l=0}^{n-1} \mathbb{E}[V\phi_{k,\mu}^l]\phi_{k,\mu}^{n-l}. \quad (2.11)$$

We then define the regularized Bloch waves

$$\kappa_{k,\lambda,\mu} := \lambda \mathbb{E}[V\psi_{k,\lambda,\mu}] = \lambda \sum_{n=0}^{\infty} \lambda^n \nu_{k,\mu}^n, \quad \psi_{k,\lambda,\mu} := \sum_{n=0}^{\infty} \lambda^n \phi_{k,\mu}^n, \quad (2.12)$$

where the series indeed converge for λ small enough (for fixed $\mu > 0$) and satisfy

$$(-i\mu - \Delta_k + \lambda V)\psi_{k,\lambda,\mu} = \kappa_{k,\lambda,\mu}\psi_{k,\lambda,\mu} - i\mu, \quad \mathbb{E}[\psi_{k,\lambda,\mu}] = 1. \quad (2.13)$$

Note that the Rayleigh-Schrödinger coefficients (ν_k^n, ϕ_k^n) are clearly retrieved in the limit $\mu \downarrow 0$, so that it suffices to establish the statement of the proposition for $(\nu_{k,\mu}^n, \phi_{k,\mu}^n)$ and $-\Delta_k$ replaced by $(\nu_{k,\mu}^n, \phi_{k,\mu}^n)$ and $-i\mu - \Delta_k$. The strategy of the proof is as follows: we establish an explicit series expansion for $\psi_{k,\lambda,\mu}$ in the restricted regime $\lambda \ll \mu^2 \wedge 1$ using a fixed point argument together with Lagrange and Neumann expansions, and then we identify the coefficients $(\nu_{k,\mu}^n, \phi_{k,\mu}^n)$ in (2.12), which leads to formulas for these coefficients that hold independently of λ .

Applying the projectors $1 - \Pi = \mathbb{E}[\cdot]$ and Π , we deduce that the equation (2.13) is equivalent to

$$\kappa_{k,\lambda,\mu} = \lambda \mathbb{E}[V\psi_{k,\lambda,\mu}], \quad (-i\mu - \kappa_{k,\lambda,\mu} - \Delta_k + \lambda \Pi V)(\psi_{k,\lambda,\mu} - 1) = -\lambda \Pi V. \quad (2.14)$$

Provided that $\Im(i\mu + \kappa_{k,\lambda,\mu}) \neq 0$, the μ -regularized eigenvalue $\kappa_{k,\lambda,\mu}$ is therefore a solution of the following fixed-point problem,

$$\begin{aligned} \kappa_{k,\lambda,\mu} &= G_{k,\lambda,\mu}(\kappa_{k,\lambda,\mu}), \\ G_{k,\lambda,\mu}(\kappa) &:= \lambda \mathbb{E}[V] - \lambda^2 \mathbb{E}[V(-i\mu - \kappa - \Delta_k + \lambda\Pi V)^{-1}\Pi V]. \end{aligned}$$

Provided that $\Im(i\mu + \kappa) \neq 0$, the quantity $G_{k,\lambda,\mu}(\kappa)$ can be rewritten in form of a Neumann series for all $\lambda > 0$ small enough,

$$\begin{aligned} G_{k,\lambda,\mu}(\kappa) &= \lambda \mathbb{E}[V] - \lambda^2 \mathbb{E}\left[V(1 + \lambda(-i\mu - \kappa - \Delta_k)^{-1}\Pi V)^{-1}(-i\mu - \kappa - \Delta_k)^{-1}\Pi V\right] \\ &= \lambda \mathbb{E}[V] - \lambda^2 \sum_{n=0}^{\infty} (-\lambda)^n \mathbb{E}\left[V(\Gamma_{k,\mu}(\kappa)\Pi V)^{n+1}1\right], \end{aligned}$$

where we use the shorthand notation $\Gamma_{k,\mu}(\kappa) := (-i\mu - \kappa - \Delta_k)^{-1}$. For $|\kappa| < \mu$, we may further use the Neumann series

$$\Gamma_{k,\mu}(\kappa) = (1 - \kappa\Gamma_{k,\mu}(0))^{-1}\Gamma_{k,\mu}(0) = \sum_{n=0}^{\infty} \kappa^n \Gamma_{k,\mu}(0)^{n+1}.$$

Injecting this into the above then leads to the following series expansion for $G_{k,\lambda,\mu}$: given $|\kappa| < \mu$, we obtain for all $\lambda > 0$ small enough,

$$\begin{aligned} G_{k,\lambda,\mu}(\kappa) &= \lambda \mathbb{E}[V] - \sum_{n=1}^{\infty} (-\lambda)^{n+1} \sum_{a_1, \dots, a_n=0}^{\infty} \kappa^{a_1 + \dots + a_n} \mathbb{E}\left[V\Gamma_{k,\mu}(0)^{a_1+1}\Pi V \dots \Gamma_{k,\mu}(0)^{a_n+1}\Pi V\right] \\ &= \sum_{\ell=0}^{\infty} \kappa^\ell G_{k,\lambda,\mu;\ell}, \end{aligned}$$

where we have set for all $\ell \in \mathbb{N}$,

$$G_{k,\lambda,\mu;\ell} := - \sum_{n=0}^{\infty} (-\lambda)^{n+1} \sum_{\substack{a \in \mathbb{N}^n \\ |a|=\ell}} \mathbb{E}\left[V\Gamma_{k,\mu}(0)^{a_1+1}\Pi V \dots \Gamma_{k,\mu}(0)^{a_n+1}\Pi V\right].$$

(Note that we use here the natural convention for empty sums and products: for $\ell = 0$ the sum takes the form

$$- \sum_{n=0}^{\infty} (-\lambda)^{n+1} \mathbb{E}\left[V(\Gamma_{k,\mu}(0)\Pi V)^n 1\right],$$

while for $\ell \geq 1$ the sum is reduced to $n \geq 1$.) We are now in position to solve the fixed-point equation $\kappa = G_{k,\lambda,\mu}(\kappa)$ in the form of a Lagrange series expansion as e.g. in [7]. By a simple tree-counting argument, the unique solution $\kappa_{k,\lambda,\mu}$ for λ small enough can be expressed as the following sum on all possible trees of all sizes,

$$\kappa_{k,\lambda,\mu} = \sum_{m=1}^{\infty} \sum_{a \in \mathcal{T}_m} G_{k,\lambda,\mu;a_1} \dots G_{k,\lambda,\mu;a_m}, \quad (2.15)$$

or alternatively,

$$\kappa_{k,\lambda,\mu} = \sum_{n=1}^{\infty} \lambda^n \kappa_{k,\mu;n},$$

where we have defined

$$\begin{aligned} \kappa_{k,\mu;n} := & \sum_{m=1}^n (-1)^{n+m} \sum_{a \in \mathcal{T}_m} \sum_{\substack{c \in \mathbb{N}^m \\ |c|=n-m}} \sum_{\substack{b^1 \in \mathbb{N}^{c_1} \\ |b^1|=a_1}} \cdots \sum_{\substack{b^m \in \mathbb{N}^{c_m} \\ |b^m|=a_m}} \\ & \times \mathbb{E}[V\Gamma_{k,\mu}(0)^{b_1^1+1}\Pi V \dots \Gamma_{k,\mu}(0)^{b_{c_1}^1+1}\Pi V] \\ & \times \dots \mathbb{E}[V\Gamma_{k,\mu}(0)^{b_1^m+1}\Pi V \dots \Gamma_{k,\mu}(0)^{b_{c_m}^m+1}\Pi V], \quad (2.16) \end{aligned}$$

provided that the above power series is absolutely convergent and satisfies $|\kappa_{k,\lambda,\mu}| < \mu$. Let us quickly check that these two conditions are indeed satisfied for $\lambda > 0$ small enough. For that purpose, we make use of the following rough estimate,

$$|\mathbb{E}[V\Gamma_{k,\mu}(0)^{b_1+1}\Pi V \dots \Gamma_{k,\mu}(0)^{b_m+1}\Pi V]| \leq \|V\|_{L^\infty}^{m+1} \mu^{-(b_1+\dots+b_m)-m},$$

which indeed yields for all $n \geq 1$,

$$\begin{aligned} |\kappa_{k,\mu;n}| & \leq \sum_{m=1}^n \sum_{a \in \mathcal{T}_m} \sum_{c_1, \dots, c_m=0}^{\infty} \mathbb{1}_{c_1+\dots+c_m=n-m} \\ & \quad \times \sum_{\substack{b^1 \in \mathbb{N}^{c_1} \\ |b^1|=a_1}} \cdots \sum_{\substack{b^m \in \mathbb{N}^{c_m} \\ |b^m|=a_m}} \|V\|_{L^\infty}^{m+c_1+\dots+c_m} \mu^{-(|b^1|+\dots+|b^m|)-(c_1+\dots+c_m)} \\ & \leq \|V\|_{L^\infty}^n \sum_{m=1}^n \sum_{a \in \mathcal{T}_m} \sum_{c_1, \dots, c_m=0}^{\infty} \mathbb{1}_{c_1+\dots+c_m=n-m} \\ & \quad \times \binom{a_1+c_1-1}{c_1-1} \cdots \binom{a_m+c_m-1}{c_m-1} \mu^{m-n-(a_1+\dots+a_m)} \\ & \leq \|V\|_{L^\infty}^n \sum_{m=1}^n \sum_{j=0}^{m-1} \mu^{m-n-j} \mathcal{I}_{m,n,j}, \end{aligned}$$

where we have set

$$\mathcal{I}_{m,n,j} := \sum_{a_1, \dots, a_m=0}^{\infty} \sum_{c_1, \dots, c_m=0}^{\infty} \mathbb{1}_{\substack{a_1+\dots+a_m=j \\ c_1+\dots+c_m=n-m}} \binom{a_1+c_1-1}{c_1-1} \cdots \binom{a_m+c_m-1}{c_m-1}.$$

(Here we take the convention $\binom{-1}{-1} = 1$ and $\binom{s}{-1} = 0$ for $s \geq 0$.) Using the rough bound $\mathcal{I}_{m,n,j} \leq 2^{2(n+j-1)-m}$, we deduce

$$|\kappa_{k,\mu;n}| \leq \|V\|_{L^\infty}^n 2^{3n} (\mu \wedge 1)^{1-n},$$

which leads to

$$|\kappa_{k,\lambda,\mu}| \leq \sum_{n=1}^{\infty} \lambda^n |\kappa_{k,\mu;n}| \leq \sum_{n=1}^{\infty} \left(\frac{8\lambda}{\mu \wedge 1} \|V\|_{L^\infty} \right)^n < \mu$$

for λ small enough (say $\lambda \ll \mu^2 \wedge 1$).

We turn to the series representation of $\psi_{k,\lambda,\mu}$, and insert the expression (2.15) into (2.14). Since $|\kappa_{k,\lambda,\mu}| < \mu$, one may invert the equation for $\psi_{k,\lambda,\mu}$. Proceeding as above using

Neumann series, we obtain the following absolutely convergent expansion in $L^2(\Omega)$ for λ small enough,

$$\begin{aligned}\psi_{k,\lambda,\mu} &= 1 + \sum_{n=1}^{\infty} (-\lambda)^n \sum_{\ell=0}^{\infty} \kappa_{k,\lambda,\mu}^{\ell} \sum_{\substack{a \in \mathbb{N}^n \\ |a|=\ell}} (-i\mu - \Delta_k)^{-a_1-1} \Pi V \dots (-i\mu - \Delta_k)^{-a_n-1} \Pi V \\ &= 1 + \sum_{n=1}^{\infty} \lambda^n \psi_{k,\mu;n},\end{aligned}$$

where we have set for all $n \geq 1$

$$\begin{aligned}\psi_{k,\mu;n} &:= \sum_{m=1}^n (-1)^m \sum_{\ell=0}^{n-m} \sum_{\substack{a \in \mathbb{N}^{\ell} \\ |a|=n-m-\ell}} \sum_{\substack{b \in \mathbb{N}^m \\ |b|=\ell}} \\ &\quad \times \kappa_{k,\mu;1+a_1} \dots \kappa_{k,\mu;1+a_{\ell}} (-i\mu - \Delta_k)^{-b_1-1} \Pi V \dots (-i\mu - \Delta_k)^{-b_m-1} \Pi V.\end{aligned}\quad (2.17)$$

We are in position to conclude. Comparing the above series representations with the Rayleigh-Schrödinger series (2.12) satisfying the same regularized eigenvalue equation (2.13), and identifying the powers of λ in these (locally convergent) series, we deduce for all $n \geq 1$,

$$\psi_{k,\mu;n} = \phi_{k,\mu}^n, \quad \kappa_{k,\mu;n} = \nu_{k,\mu}^{n-1}.$$

Passing to the limit $\mu \downarrow 0$ in these formulas, the conclusion follows. \square

3. APPROXIMATE BLOCH WAVES IN THE QUASIPERIODIC SETTING

In this section, we establish fine bounds on the Rayleigh-Schrödinger coefficients (ν_k^n, ϕ_k^n) in the quasiperiodic setting (QP). As emphasized in Remark 1.1, the quality of such bounds depends both on the regularity of the lifted map \tilde{V} and on the algebraic properties of the winding matrix F . We start by introducing a suitable form of a Diophantine condition for F , adapted to the fibered structure.

3.1. Diophantine condition. The key ingredient to estimate the Rayleigh-Schrödinger coefficients in the quasiperiodic setting is the inversion of the fibered Laplacians $-\Delta_k$, $k \in \mathbb{R}^d$. The Fourier symbol of $-\Delta_k$ is given by $\xi \mapsto |F\xi + k|^2 - |k|^2$ on \mathbb{Z}^M and can in general vanish for $\xi \neq 0$, which prohibits any invertibility. In order to ensure invertibility, we must obviously assume that F is irrational (that is, $F\xi \neq 0$ for all $\xi \in \mathbb{Z}^M \setminus \{0\}$) and we must also restrict to values of k away from the so-called diffraction hyperplanes $P_{\xi} := \{k' \in \mathbb{R}^d : |F\xi + k'| = |k'|\}$, $\xi \in \mathbb{Z}^M \setminus \{0\}$. The complement of the union of all those hyperplanes constitutes the so-called *non-resonant set*,

$$\mathcal{O} := \mathbb{R}^d \setminus \bigcup_{\xi \in \mathbb{Z}^M \setminus \{0\}} P_{\xi},$$

which typically has a Cantor-like structure. To obtain precise bounds on the Rayleigh-Schrödinger coefficients, fine estimates on the inverse symbol $\xi \mapsto (|F\xi + k|^2 - |k|^2)^{-1}$ are further required for non-resonant k . This is provided by the following when F is Diophantine. To ensure uniform bounds, we must naturally restrict to values of k outside some *fattened* resonant set \mathcal{R}_R , $R \geq 1$, with $\mathcal{O}_R := \mathbb{R}^d \setminus \overline{\mathcal{R}_R} \uparrow \mathcal{O}$ as $R \uparrow \infty$.

Lemma 3.1 (Diophantine condition). *Assume that the winding matrix $F \in \mathbb{R}^{d \times M}$ satisfies for some $r_0 > 0$ the following Diophantine condition,*

$$|F\xi| \geq \frac{1}{C_0} |\xi|^{-r_0} \quad \text{for all } \xi \in \mathbb{Z}^M \setminus \{0\}, \quad (3.1)$$

and let $s_0 > M + r_0$ be fixed. Then there exists a decreasing collection $(\mathcal{R}_R)_{R \geq 1}$ of open (resonant) subsets $\mathcal{R}_R \subset \mathbb{R}^d$ and there exists a constant $C > 0$ (depending on C_0, F, M, s_0) such that the following hold for all $R \geq 1$:

(i) For all $k \in \mathbb{R}^d \setminus \mathcal{R}_R$ and all $\xi \in \mathbb{Z}^M \setminus \{0\}$, we have

$$||F\xi + k|^2 - |k|^2| \geq R^{-1} |\xi|^{-s_0}. \quad (3.2)$$

(ii) For all $\kappa > 0$, we have $|\mathcal{R}_R \cap B_\kappa| \leq CR^{-1} |\partial B_\kappa|$.

(iii) We can decompose $\mathcal{R}_R = \bigcup_{n=1}^{\infty} \mathcal{R}_R^n$, where $(\mathcal{R}_R^n)_n$ is an increasing sequence of open subsets of \mathbb{R}^d , such that for all n ,

– \mathcal{R}_R^n is a finite union of regular open sets;

– the condition (3.2) holds for all $k \in \mathbb{R}^d \setminus \mathcal{R}_R^n$ and $\xi \in \mathbb{Z}^M \setminus \{0\}$ with $|\xi| \leq n$. \diamond

(In the sequel, we conveniently write $\mathcal{R}_R^t := \mathcal{R}_R^{\lceil t \rceil}$ for non-integer $t \geq 0$.)

Remarks 3.2.

- *Diophantine condition (3.1)*: Given $r_0 > d - 1$, the standard theory of Diophantine conditions ensures that for almost every $F \in \mathbb{R}^{d \times M}$ there exists $C_0 > 0$ (depending on F, M, r_0) such that (3.1) holds.
- *Property (ii)*: Two dual behaviors are included in property (ii): on the one hand the density of the resonant set \mathcal{R}_R decreases to 0 as $R \uparrow \infty$, and on the other hand for fixed R the set $\mathbb{R}^d \setminus \mathcal{R}_R$ is extensive in the sense that the density of \mathcal{R}_R in a ball B_κ decreases to 0 as $\kappa \uparrow \infty$.
- *Periodic setting*: Let us argue that in the periodic setting (that is, (QP) with $M = d$ and $F = \text{Id}$) the above lemma holds with $s_0 = 0$. More precisely, there exists a decreasing collection $(\mathcal{R}_R)_{R \geq 1}$ of regular open subsets $\mathcal{R}_R \subset \mathbb{R}^d$ and there exists a constant $C > 0$ such that the following hold for all $R \geq 1$:

(i') For all $k \in \mathbb{R}^d \setminus \mathcal{R}_R$ and all $\xi \in \mathbb{Z}^M \setminus \{0\}$, we have

$$||\xi + k|^2 - |k|^2| \geq R^{-1}. \quad (3.3)$$

(ii') For all $\kappa > 0$, we have $|\mathcal{R}_R \cap B_\kappa| \leq C_\kappa R^{-1}$.¹

This is obtained as a direct adaptation of the proof below, noting that the diffraction hyperplane $P_\xi = \{k' \in \mathbb{R}^d : |F\xi + k'| = |k'|\}$ does not intersect the ball $B_{\frac{1}{2}|\xi|}$. \diamond

Proof of Lemma 3.1. For $R \geq 1$ and $\xi \in \mathbb{Z}^M \setminus \{0\}$, we consider the fattened diffraction hyperplane

$$\mathcal{R}_R(\xi) := \{k \in \mathbb{R}^d : ||F\xi + k|^2 - |k|^2| < R^{-1} |\xi|^{-s_0}\},$$

and the fattened resonance set

$$\mathcal{R}_R := \bigcup_{\xi \in \mathbb{Z}^M \setminus \{0\}} \mathcal{R}_R(\xi).$$

1. Similarly as in Lemma 3.1, the factor C_κ could be improved into $C|\partial B_\kappa|$ if we replace the RHS R^{-1} in (3.3) by $R^{-1} |\xi|^{-s_0}$ with $s_0 > d - 1$.

For $\xi \in \mathbb{Z}^M \setminus \{0\}$, we note that the distance of a point $k \in \mathbb{R}^d$ to the diffraction hyperplane $P_\xi := \{k' \in \mathbb{R}^d : |F\xi + k'| = |k'|\}$ is given by $|k \cdot \frac{F\xi}{|F\xi|} - \frac{1}{2}|F\xi||$, hence a point $k \in \mathbb{R}^d$ satisfies $||F\xi + k|^2 - |k|^2| < R^{-1}|\xi|^{-s_0}$ if and only if it is at distance $< \frac{1}{2}R^{-1}|F\xi|^{-1}|\xi|^{-s_0}$ from P_ξ . This implies

$$\mathcal{R}_R(\xi) = P_\xi + B_{\frac{1}{2}R^{-1}|F\xi|^{-1}|\xi|^{-s_0}}, \quad (3.4)$$

and the Diophantine condition (3.1) then allows to estimate, for all $\kappa > 0$,

$$|\mathcal{R}_R(\xi) \cap B_\kappa| \leq CR^{-1}\kappa^{d-1}|F\xi|^{-1}|\xi|^{-s_0} \leq CR^{-1}\kappa^{d-1}|\xi|^{r_0-s_0}.$$

By a rough union bound and the choice $s_0 > M + r_0$, this yields

$$|\mathcal{R}_R \cap B_\kappa| \leq \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} CR^{-1}\kappa^{d-1}|\xi|^{r_0-s_0} \leq CR^{-1}\kappa^{d-1}.$$

Items (i)–(ii) follow. It remains to prove item (iii). For that purpose, for all $n \geq 1$, we consider

$$\mathcal{R}_R^n := \bigcup_{\substack{\xi \in \mathbb{Z}^M \setminus \{0\} \\ |\xi| \leq n}} \mathcal{R}_R(\xi).$$

By definition, the sequence $(\mathcal{R}_R^n)_n$ is increasing and satisfies $\mathcal{R}_R = \bigcup_n \mathcal{R}_R^n$, and (3.2) holds for all $k \in \mathbb{R}^d \setminus \mathcal{R}_R^n$ and $|\xi| \leq n$. \square

3.2. Bounds on the Rayleigh-Schrödinger coefficients. For $k \in \mathcal{O}$, the Fourier symbol of $-\Delta_k$ does not vanish outside the origin, which ensures that all the terms of the form $(-\Delta_k)^{-b_1-1}\Pi V \dots (-\Delta_k)^{-b_m-1}\Pi V$ are well-defined in $L^2(\Omega)$. In view of Proposition 2.6, this entails that the Rayleigh-Schrödinger coefficients (ν_k^n, ϕ_k^n) are well-defined for all n whenever $k \in \mathcal{O}$. It remains to establish fine estimates on these coefficients, for which we exploit the Diophantine condition (3.1).

Proposition 3.3. *Consider the quasiperiodic setting (QP), assume that the winding matrix F satisfies the Diophantine condition (3.1) with $r_0 > 0$, let $s_0 > M + r_0$, assume that the lifted map \tilde{V} has compactly supported Fourier transform, and set $K := \sup\{1 \vee |\xi| : \xi \in \text{supp } \mathcal{F}\tilde{V}\}$. Then there exists a constant C (depending on F, M, s_0) and for all $\ell, R \geq 1$ there exists a field of ℓ -jets of Bloch waves $(\nu_k^n, \phi_k^n : k \in \mathbb{R}^d \setminus \mathcal{R}_R^{K\ell}, 0 \leq n \leq \ell)$ in the sense of Definition 2.4, which satisfy the following estimates for all $n \geq 1, k \in \mathbb{R}^d \setminus \mathcal{R}_R^{Kn}$, and $s, j \geq 0$,*

$$|\nabla_k^j \nu_k^n| \leq (CRjK^{s_0+1}n^{s_0+1})^j (CRK^{s_0+M}n^{s_0})^n \|\mathcal{F}\tilde{V}\|_{L^\infty}^{n+1}, \quad (3.5)$$

$$\|\nabla_k^j \phi_k^n\|_{H^s(\Omega)} \leq (CKn)^s (CRjK^{s_0+1}n^{s_0+1})^j (CRK^{s_0+M}n^{s_0})^n \|\mathcal{F}\tilde{V}\|_{L^\infty}^n. \quad (3.6)$$

In particular, for all $\hat{u} \in C_c^\infty(\mathbb{R}^d)$ supported in $\mathbb{R}^d \setminus \mathcal{R}_R^{Kn}$, we deduce

$$\begin{aligned} & \sup_{\omega \in \Omega} \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{ik \cdot x} \nabla_k^j \nabla^s \phi_k^n(x, \omega) \hat{u}(k) d^*k \right|^2 dx \right)^{\frac{1}{2}} \\ & \leq (CKn)^{s+M+1} (CRjK^{s_0+1}n^{s_0+1})^j (CRK^{s_0+M}n^{s_0})^n \|\mathcal{F}\tilde{V}\|_{L^\infty}^n \|\hat{u}\|_{L^2}, \end{aligned} \quad (3.7)$$

which holds uniformly wrt translations on $\Omega = \mathbb{T}^M$. \diamond

Remarks 3.4.

- *Dependence on n :* We believe that the $(Cn^{s_0})^n$ -growth of the above bounds on (ν_k^n, ϕ_k^n) is optimal, which implies in particular that the Rayleigh-Schrödinger series (2.6) cannot be absolutely convergent, so that if a branch of Bloch waves exists it cannot be analytic in λ , cf. the discussion in Section 1.4.
- *Irrational winding matrix:* If we merely assume $F\xi \neq 0$ for all $\xi \in \mathbb{Z}^M \setminus \{0\}$, then (3.6), (3.5), and (3.7) hold in a modified form where the dependence on K and n is replaced by some constant $C_{K,n}$ that is no longer explicit (and can grow much faster, for instance for Liouville frequencies). The easy adaptation of the proof is left to the reader.
- *Periodic setting:* In the periodic setting (that is, (QP) with $M = d$ and $F = \text{Id}$), the bounds of Proposition 3.3 above hold in the following improved form: for all $R, n \geq 1$, $s, j \geq 0$, and $k \in \mathbb{R}^d \setminus \mathcal{R}_R$ (with \mathcal{R}_R defined as in the last item of Remarks 3.2),

$$\|\nabla_k^j \phi_k^n\|_{H^s(\Omega)} \leq n^s (CRjn)^j (CR)^n.$$

This ensures the convergence of the Rayleigh-Schrödinger perturbation series (2.6) for small λ , which also follows from the Kato-Rellich theorem. \diamond

Proof of Proposition 3.3. We use Fourier series \mathcal{F} in the (high-dimensional) torus $\Omega = \mathbb{T}^M$. We make a slight abuse of notation in this proof and write $\hat{V} = \mathcal{F}\tilde{V} : \mathbb{Z}^M \rightarrow \mathbb{C}$. We split the proof into two steps.

Step 1. Proof of (3.5) and (3.6) for $j = 0$.

Lemma 3.1 allows to invert the Fourier symbol of $-\Delta_k$. By Proposition 2.6, it suffices to control for all $m \geq 1$ and $b \in \mathbb{N}^m$ the functions

$$\chi_k^{m,b} := (-\Delta_k)^{-b_1-1} \Pi \tilde{V} \dots (-\Delta_k)^{-b_m-1} \Pi \tilde{V} : \mathbb{T}^M \rightarrow \mathbb{C}.$$

Since $\mathcal{F}[\Pi f](\xi) = \hat{f}(\xi) \mathbf{1}_{\xi \neq 0}$ for all $f \in L^2(\mathbb{T}^M)$, the Fourier transform of $\chi_k^{m,b}$ takes the form

$$\hat{\chi}_k^{m,b}(\xi) = \mathbf{1}_{\xi \neq 0} \sum_{\xi_2, \dots, \xi_m \in \mathbb{Z}^M \setminus \{0\}} \frac{\hat{V}(\xi - \xi_2) \dots \hat{V}(\xi_{m-1} - \xi_m) \hat{V}(\xi_m)}{(|F\xi + k|^2 - |k|^2)^{b_1+1} \dots (|F\xi_m + k|^2 - |k|^2)^{b_m+1}}, \quad (3.8)$$

so that Parseval's formula yields

$$\mathbb{E}[V\chi_k^{m,b}] = \sum_{\xi_1, \dots, \xi_m \in \mathbb{Z}^M \setminus \{0\}} \frac{\hat{V}(-\xi_1) \hat{V}(\xi_m)}{(|F\xi_m + k|^2 - |k|^2)^{b_m+1}} \prod_{j=1}^{m-1} \frac{\hat{V}(\xi_j - \xi_{j+1})}{(|F\xi_j + k|^2 - |k|^2)^{b_j+1}}.$$

Recall that $K := \max\{1 \vee |\xi| : \xi \in \text{supp } \hat{V}\} < \infty$, so that the above sum can be restricted to

$$\{(\xi_1, \dots, \xi_m) \in (\mathbb{Z}^M \setminus \{0\})^m : |\xi_\ell|, |\xi_{m-\ell+1}| \leq \ell K, \forall 1 \leq \ell \leq \lceil \frac{m}{2} \rceil\}.$$

Combining this observation with Lemma 3.1 then yields the following for all $m \geq 1$ and $b \in \mathbb{N}^m$: if $m \in 2\mathbb{N}$, we have for all $k \in \mathbb{R}^d \setminus \mathcal{R}_R^{K\frac{m}{2}}$,

$$\begin{aligned} |\mathbb{E}[V\chi_k^{m,b}]| &\leq (RK^{s_0})^{m+|b|} \|\hat{V}\|_{L^\infty}^{m+1} \sum_{\xi_1, \dots, \xi_m \in \mathbb{Z}^M} \mathbf{1}_{|\xi_1|, |\xi_m| \leq K} \\ &\quad \times \prod_{j=2}^{m/2} \left(j^{s_0(b_j + b_{m-j+1} + 2)} \mathbf{1}_{|\xi_j - \xi_{j-1}| \leq K} \mathbf{1}_{|\xi_{m-j+1} - \xi_{m-j+2}| \leq K} \right) \\ &\leq (CK)^{mM} (RK^{s_0} m^{s_0})^{m+|b|} \|\hat{V}\|_{L^\infty}^{m+1}. \end{aligned}$$

Likewise, if $m \in 2\mathbb{N} + 1$, the same estimate holds for all $k \in \mathbb{R}^d \setminus \mathcal{R}_R^{K\frac{m+1}{2}}$. Injecting this into the formula of Proposition 2.6 for the ν_k^n 's (recall that \mathcal{T}_m denotes the set of rooted m -trees (2.10)), we obtain for all $n \geq 1$ and $k \in \mathbb{R}^d \setminus \mathcal{R}_R^{K\lceil \frac{m}{2} \rceil}$,

$$\begin{aligned} |\nu_k^n| &\leq \sum_{m=1}^{n+1} \sum_{a \in \mathcal{T}_m} \sum_{\substack{c \in \mathbb{N}^m \\ |c| = n+1-m}} \sum_{\substack{b^1 \in \mathbb{N}^{c_1} \\ |b^1| = a_1}} \dots \sum_{\substack{b^m \in \mathbb{N}^{c_m} \\ |b^m| = a_m}} (RK^{s_0} c_1^{s_0})^{c_1+a_1} (CK)^{c_1 M} \|\hat{V}\|_{L^\infty}^{c_1+1} \\ &\quad \times \dots (RK^{s_0} c_m^{s_0})^{c_m+a_m} (CK)^{c_m M} \|\hat{V}\|_{L^\infty}^{c_m+1}, \end{aligned}$$

for some C depending on M . Since $\#\mathcal{T}_m \leq 4^m$, this directly leads to

$$|\nu_k^n| \leq (CRK^{s_0+M} n^{s_0})^n \|\hat{V}\|_{L^\infty}^{n+1}. \quad (3.9)$$

Since $\nu_k^0 = \mathbb{E}[V]$, the same estimate obviously holds for $n = 0$, and the conclusion (3.5) for $j = 0$ follows. We turn to the bounds on the correctors. For that purpose, for all $s \geq 0$, $m \geq 1$, and $b \in \mathbb{N}^m$, we combine (3.8) with Parseval's identity in the form

$$\begin{aligned} \|\chi_k^{m,b}\|_{H^s(\Omega)}^2 &\lesssim \sum_{\xi_1 \in \mathbb{Z}^M \setminus \{0\}} \langle \xi_1 \rangle^{2s} \\ &\quad \times \left| \sum_{\xi_2, \dots, \xi_m \in \mathbb{Z}^M \setminus \{0\}} \frac{\hat{V}(\xi_1 - \xi_2) \dots \hat{V}(\xi_{m-1} - \xi_m) \hat{V}(\xi_m)}{(|F\xi_1 + k|^2 - |k|^2)^{b_1+1} \dots (|F\xi_m + k|^2 - |k|^2)^{b_m+1}} \right|^2. \end{aligned}$$

By the Cauchy-Schwarz inequality together with compactness of the support of \hat{V} ,

$$\begin{aligned} \|\chi_k^{m,b}\|_{H^s(\Omega)}^2 &\leq \|\hat{V}\|_{L^2}^{2m} \\ &\quad \times \sup_{\xi_1 \in \mathbb{Z}^M \setminus \{0\}} \sum_{\xi_2, \dots, \xi_m \in \mathbb{Z}^M \setminus \{0\}} \frac{\langle \xi_1 \rangle^{2s} \mathbf{1}_{|\xi_1 - \xi_2| \leq K} \dots \mathbf{1}_{|\xi_{m-1} - \xi_m| \leq K} \mathbf{1}_{|\xi_m| \leq K}}{(|F\xi_1 + k|^2 - |k|^2)^{2(b_1+1)} \dots (|F\xi_m + k|^2 - |k|^2)^{2(b_m+1)}}. \end{aligned}$$

Hence, using Lemma 3.1, for all $k \in \mathbb{R}^d \setminus \mathcal{R}_R^{Km}$,

$$\begin{aligned} \|\chi_k^{m,b}\|_{H^s(\Omega)}^2 &\leq (CK)^{2mM} (Km + 1)^{2s} \|\hat{V}\|_{L^\infty}^{2m} \prod_{j=1}^m (RK^{s_0} (m - j + 1)^{s_0})^{2(b_j+1)} \\ &\leq (CK)^{2mM} (Km + 1)^{2s} (RK^{s_0} m^{s_0})^{2(m+|b|)} \|\hat{V}\|_{L^\infty}^{2m}, \end{aligned}$$

for some C depending on M . Injecting this estimate into the formula of Proposition 2.6 for the ϕ_k^n 's, we obtain for all $n \geq 1$ and $k \in \mathbb{R}^d \setminus \mathcal{R}_R^{Kn}$,

$$\|\phi_k^n\|_{H^s(\Omega)} \leq (CKn)^s \sum_{m=1}^n \sum_{\ell=0}^{n-m} (CK)^{mM} (CRK^{s_0} m^{s_0})^{m+\ell} \|\hat{V}\|_{L^\infty}^m \sum_{\substack{a \in \mathbb{N}^\ell \\ |a|=n-m-\ell}} |\nu_k^{a_1}| \dots |\nu_k^{a_\ell}|.$$

Combined with the bound (3.9) on the ν_k^n 's, this yields

$$\|\phi_k^n\|_{H^s(\Omega)} \leq (CKn)^s (CRK^{s_0+M} n^{s_0})^n \|\hat{V}\|_{L^\infty}^n,$$

that is, (3.6) for $j = 0$.

Step 2. Conclusion.

We start with the proof of (3.6) for all $j \geq 1$. For that purpose, we note that, for $n \geq 1$, $|\xi| \leq n$, and $k \in \mathbb{R}^d \setminus \mathcal{R}_R^n$, Lemma 3.1 yields

$$\left| \nabla_k^j \frac{1}{|F\xi + k|^2 - |k|^2} \right| = \frac{j! |2F\xi|^j}{(|F\xi + k|^2 - |k|^2)^{j+1}} \leq \frac{(Cjn)^j (Rn^{s_0})^j}{||F\xi + k|^2 - |k|^2|}.$$

Taking the derivative ∇_k in both sides of the formulas of Proposition 2.6 for (ν_k^n, ϕ_k^n) , this bound allows to repeat the argument of Step 1, and the conclusion easily follows.

We turn to the proof of (3.7) and show that it follows from (3.6). In the rest of this proof, we distinguish between the corrector ϕ_k^n defined on $\mathbb{R}^d \times \mathbb{T}^M$ and its folded version $\tilde{\phi}_k^n$ defined on \mathbb{T}^M , which are related via $\phi_k^n(x, \omega) = \tilde{\phi}_k^n(F^T x + \omega)$. By the Sobolev embedding, for $a > M$, the space $H^a(\mathbb{T}^M)$ is embedded into $L^\infty(\mathbb{T}^M)$. Denote by $\nabla_{\mathbb{T}^M}$ the weak gradient on \mathbb{T}^M . For all $\hat{u} \in C_c^\infty(\mathbb{R}^d)$, we have by definition of ϕ_k^n , the Sobolev embedding, and Fubini's theorem,

$$\begin{aligned} & \sup_{\omega \in \mathbb{T}^M} \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{ik \cdot x} \nabla_k^j \nabla^s \phi_k^n(x, \omega) \hat{u}(k) d^* k \right|^2 dx \right)^{\frac{1}{2}} \\ &= \sup_{\omega \in \mathbb{T}^M} \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{ik \cdot x} \hat{u}(k) \nabla_k^j (F \nabla_{\mathbb{T}^M})^s \tilde{\phi}_k^n(F^T x + \omega) d^* k \right|^2 dx \right)^{\frac{1}{2}} \\ &\leq \left(\int_{\mathbb{R}^d} \left\| \int_{\mathbb{R}^d} e^{ik \cdot x} \hat{u}(k) \nabla_k^j \nabla_{\mathbb{T}^M}^s \tilde{\phi}_k^n d^* k \right\|_{L^\infty(\mathbb{T}^M)}^2 dx \right)^{\frac{1}{2}} \\ &\leq C \left(\int_{\mathbb{R}^d} \left\| \int_{\mathbb{R}^d} e^{ik \cdot x} \hat{u}(k) \nabla_k^j \nabla_{\mathbb{T}^M}^{s+a} \tilde{\phi}_k^n d^* k \right\|_{L^2(\mathbb{T}^M)}^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Hence, by Parseval's identity,

$$\begin{aligned} & \sup_{\omega \in \mathbb{T}^M} \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{ik \cdot x} \nabla_k^j \nabla^s \phi_k^n(x, \omega) \hat{u}(k) d^* k \right|^2 dx \right)^{\frac{1}{2}} \\ &\leq C \|u\|_{L^2(\mathbb{R}^d)} \|\nabla_k^j \nabla_{\mathbb{T}^M}^{s+a} \tilde{\phi}_k^n\|_{L^2(\mathbb{T}^M)} \leq C \|u\|_{L^2(\mathbb{R}^d)} \|\nabla_k^j \phi_k^n\|_{H^{s+a}(\mathbb{T}^M)}. \end{aligned}$$

The conclusion (3.7) then follows from (3.6). \square

4. SCHRÖDINGER FLOW

4.1. Main result and structure of the proof. The following proposition asserts that bounds on the Rayleigh-Schrödinger coefficients imply a description of the Schrödinger flow on long timescales in terms of an effective equation.

Proposition 4.1 (Approximate normal form from approximate Bloch waves). *Given $s_0 > 0$ and $\ell, R, K, M \geq 1$, assume that the Rayleigh-Schrödinger coefficients $(\nu_k^n, \phi_k^n)_{k,n}$ satisfy the conclusions of Proposition 3.3 for all $1 \leq n \leq \ell$ and $k \in \mathbb{R}^d \setminus \mathbb{R}_R^{K^n}$, and consider an initial data $u^\circ \in \mathcal{S}(\mathbb{R}^d)$ with \hat{u}° compactly supported in $\mathbb{R}^d \setminus \mathcal{R}_R^{K^\ell}$. Denote by $u_\lambda \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d))$ the unique solution of the Schrödinger equation*

$$i\partial_t u_\lambda = (-\Delta + \lambda V)u_\lambda, \quad u_\lambda|_{t=0} = u^\circ, \quad (4.1)$$

and consider the following approximate flow

$$U_\lambda^{\ell;t}(x) := \int_{\mathbb{R}^d} e^{-it(|k|^2 + \kappa_{k,\lambda}^\ell)} e^{ik \cdot x} \hat{u}^\circ(k) d^*k. \quad (4.2)$$

Then, for all $\lambda \leq \frac{1}{2}(CRK^{s_0+M}\ell^{s_0}\|\mathcal{F}\tilde{V}\|_{L^\infty})^{-1}$ and $T \geq 0$,

$$\sup_{\Omega} \sup_{0 \leq \lambda' \leq \lambda} \sup_{0 \leq t \leq T} \|u_{\lambda'}^t - U_{\lambda'}^{\ell;t}\|_{L^2} \leq \lambda CRK^{s_0+2M+1}\ell^{s_0+M+1}\|\mathcal{F}\tilde{V}\|_{L^\infty}\|u^\circ\|_{L^2} + \lambda^{\ell+1}T(K\ell)^{M+1}(CRK^{s_0+M}\ell^{s_0})^\ell\|\mathcal{F}\tilde{V}\|_{L^\infty}^{\ell+1}\|u^\circ\|_{L^2}. \quad \diamond$$

The following result shows that such L^2 approximation results lead to ballistic transport properties. More precisely, the moments of u_λ are shown to be close to those of the free flow u_0 (although u_λ^t is not close to u_0^t in an L^2 sense for $t \gtrsim \lambda^{-2}$ if $\nu_k^1 \neq 0$, say).

Proposition 4.2 (Asymptotic ballistic transport). *Given $m \geq 1$, $u^\circ \in \mathcal{S}(\mathbb{R}^d)$, and $V \in W^{m,\infty}(\mathbb{R}^d)$, denote by $u_\lambda \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d))$ the Schrödinger flow (4.1) and by u_0 the corresponding free flow, and for $\ell \geq 1$ consider the approximate flow U_λ^ℓ defined in (4.2). Then, for all $t \geq 1$,*

$$|M_m^t(u_\lambda) - M_m^t(u_0)| \lesssim_{m,u^\circ} \|u_\lambda^t - U_\lambda^{\ell;t}\|_{L^2}^{\frac{1}{m+1}} + \|u_\lambda^t - U_\lambda^{\ell;t}\|_{L^2} + \sum_{j=1}^{m+1} \sum_{l=1}^j \left(\int_{\mathbb{R}^d} |\nabla_k^l \kappa_{k,\lambda}^\ell|^{\frac{2j}{l}} |\langle \frac{1}{t} \nabla \rangle^{m+1} \hat{u}_0^t(k)|^2 d^*k \right)^{\frac{1}{2}},$$

where the multiplicative constant only depends on $d, m, \|V\|_{W^{m,\infty}}, \|\langle \nabla \rangle^{m+1} u^\circ\|_{L^2}$, and $\|\langle \cdot \rangle^{m+1} u^\circ\|_{L^2}$. \diamond

Theorem 1 will follow from Proposition 4.1 and Proposition 3.3 together with an optimization in the truncation parameter ℓ and an approximation argument for the initial data. Corollary 1 will follow from Proposition 4.2 together with further approximation arguments. In the following subsection we split the proof of Proposition 4.1 into a string of lemmas, which are then proved in the subsequent subsections. Next, we turn to the proof of Proposition 4.2 and we conclude the section with the proofs of Theorem 1 and Corollary 1.

4.2. Structure of the proof of Proposition 4.1. As motivated in Section 2, we start by considering the following approximate Bloch expansion of the initial data u° ,

$$W_\lambda^{\ell;\circ}(x) := \int_{\mathbb{R}^d} e^{ik \cdot x} \psi_{k,\lambda}^\ell(x) \hat{u}^\circ(k) d^*k. \quad (4.3)$$

For small λ , we can indeed formally replace the Fourier modes $x \mapsto e^{ik \cdot x}$ by the corresponding approximate Bloch waves $x \mapsto e^{ik \cdot x} \psi_{k,\lambda}^\ell(x)$. The following lemma quantifies the resulting error and its time propagation.

Lemma 4.3 (Preparation of initial data). *In the setting of Proposition 4.1, we define $W_\lambda^{\ell;\circ}$ as in (4.3) and we denote by $W_\lambda^\ell \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d))$ the solution of*

$$i\partial_t W_\lambda^\ell = (-\Delta + \lambda V) W_\lambda^\ell, \quad W_\lambda^\ell|_{t=0} = W_\lambda^{\ell;\circ}. \quad (4.4)$$

Then, for all $\lambda \leq \frac{1}{2}(CRK^{s_0+M} \ell^{s_0} \|\mathcal{F}\tilde{V}\|_{L^\infty})^{-1}$ and $t \geq 0$,

$$\sup_{\Omega} \sup_{0 \leq \lambda' \leq \lambda} \|u_{\lambda'}^t - W_{\lambda'}^{\ell;t}\|_{L^2} \leq \lambda CRK^{s_0+2M+1} \ell^{s_0+M+1} \|\mathcal{F}\tilde{V}\|_{L^\infty} \|u^\circ\|_{L^2}. \quad \diamond$$

Next, starting with the approximate Bloch expansion $W_\lambda^{\ell;\circ}$ of the initial data and using that approximate Bloch waves approximately diagonalize the Schrödinger operator (cf. Lemma 2.5), we arrive at an approximate Bloch expansion of the Schrödinger flow.

Lemma 4.4 (Approximate diagonalization). *In the setting of Proposition 4.1, denote by $W_\lambda^\ell \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d))$ the solution of (4.4) and let $V_\lambda^\ell \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d))$ be given by*

$$V_\lambda^{\ell;t}(x) := \int_{\mathbb{R}^d} e^{-it(|k|^2 + \kappa_{k,\lambda}^\ell)} e^{ik \cdot x} \psi_{k,\lambda}^\ell(x) \hat{u}^\circ(k) d^*k. \quad (4.5)$$

Then, for all $\lambda \leq \frac{1}{2}(CRK^{s_0+M} \ell^{s_0} \|\mathcal{F}\tilde{V}\|_{L^\infty})^{-1}$ and $t \geq 0$,

$$\sup_{\Omega} \sup_{0 \leq \lambda' \leq \lambda} \|W_{\lambda'}^{\ell;t} - V_{\lambda'}^{\ell;t}\|_{L^2} \leq \lambda^{\ell+1} t (K\ell)^{M+1} (CRK^{s_0+M} \ell^{s_0})^\ell \|\mathcal{F}\tilde{V}\|_{L^\infty}^{\ell+1} \|u^\circ\|_{L^2}. \quad \diamond$$

Finally, the approximate Bloch waves $x \mapsto e^{ik \cdot x} \psi_{k,\lambda}^\ell(x)$ can be replaced back by the Fourier modes $x \mapsto e^{ik \cdot x}$, which leads to the expected effective flow U_λ^ℓ .

Lemma 4.5. *In the setting of Proposition 4.1, let $V_\lambda^\ell \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d))$ be defined as in (4.5) and let $U_\lambda^\ell \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d))$ be given by*

$$U_\lambda^{\ell;t}(x) := \int_{\mathbb{R}^d} e^{-it(|k|^2 + \kappa_{k,\lambda}^\ell)} e^{ik \cdot x} \hat{u}^\circ(k) d^*k.$$

Then, for all $\lambda \leq \frac{1}{2}(CRK^{s_0+M} \ell^{s_0} \|\mathcal{F}\tilde{V}\|_{L^\infty})^{-1}$ and $t \geq 0$,

$$\sup_{\Omega} \sup_{0 \leq \lambda' \leq \lambda} \|V_{\lambda'}^{\ell;t} - U_{\lambda'}^{\ell;t}\|_{L^2} \leq \lambda CRK^{s_0+2M+1} \ell^{s_0+M+1} \|\mathcal{F}\tilde{V}\|_{L^\infty} \|u^\circ\|_{L^2}. \quad \diamond$$

Proposition 4.1 follows from the decomposition $u_\lambda - U_\lambda^\ell = (u_\lambda - W_\lambda^\ell) + (W_\lambda^\ell - V_\lambda^\ell) + (V_\lambda^\ell - U_\lambda^\ell)$, the triangle inequality, and the combination of Lemmas 4.3, 4.4, and 4.5.

4.3. Proof of Lemma 4.3: Preparation of initial data. Since the difference $u_\lambda - W_\lambda^\ell$ satisfies

$$i\partial_t(u_\lambda - W_\lambda^\ell) = (-\Delta + \lambda V)(u_\lambda - W_\lambda^\ell), \quad (u_\lambda - W_\lambda^\ell)|_{t=0} = u^\circ - W_\lambda^{\ell;\circ},$$

the unitarity of the Schrödinger flow yields for all $t \geq 0$,

$$\|u_\lambda^t - W_\lambda^{\ell;t}\|_{L^2} = \|u^\circ - W_\lambda^{\ell;\circ}\|_{L^2},$$

so that it suffices to prove that for all $\lambda \leq \frac{1}{2}(CRK^{s_0+2M}\ell^{s_0}\|\mathcal{F}\tilde{V}\|_{L^\infty})^{-1}$,

$$\sup_{\Omega} \sup_{0 \leq \lambda' \leq \lambda} \|u^\circ - W_{\lambda'}^{\ell;\circ}\|_{L^2} \leq \lambda CRK^{s_0+2M+1}\ell^{s_0+M+1}\|\mathcal{F}\tilde{V}\|_{L^\infty}\|u^\circ\|_{L^2}. \quad (4.6)$$

By definition (4.3) of $W_\lambda^{\ell;\circ}$ and by Definition 2.4,

$$(u^\circ - W_\lambda^{\ell;\circ})(x) = -\sum_{n=1}^{\ell} \lambda^n \int_{\mathbb{R}^d} e^{ik \cdot x} \phi_k^n(x) \hat{u}^\circ(k) d^*k.$$

Hence, by assumption (3.7),

$$\begin{aligned} \sup_{\Omega} \sup_{0 \leq \lambda' \leq \lambda} \|u^\circ - W_{\lambda'}^{\ell;\circ}\|_{L^2} &\leq \sum_{n=1}^{\ell} \lambda^n \sup_{\omega \in \Omega} \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{ik \cdot x} \phi_k^n(x, \omega) \hat{u}^\circ(k) d^*k \right|^2 dx \right)^{\frac{1}{2}} \\ &\lesssim K^{M+1} \|u^\circ\|_{L^2} \sum_{n=1}^{\ell} \lambda^n n^{M+1} (CRK^{s_0+M} n^{s_0})^n \|\mathcal{F}\tilde{V}\|_{L^\infty}^n, \end{aligned} \quad (4.7)$$

and the claim (4.6) follows. \square

4.4. Proof of Lemma 4.4: Approximate diagonalization. We first claim that

$$\sup_{\Omega} \sup_{0 \leq \lambda' \leq \lambda} \sup_{0 \leq t \leq T} \|W_{\lambda'}^{\ell;t} - V_{\lambda'}^{\ell;t}\|_{L^2} \lesssim \int_0^T \lambda^{\ell+1} \sup_{\Omega} \sup_{0 \leq \lambda' \leq \lambda} \|F_{\lambda'}^{\ell;t}\|_{L^2} dt, \quad (4.8)$$

in terms of

$$F_\lambda^{\ell;t}(x) := \int_{\mathbb{R}^d} e^{-it(|k|^2 + \kappa_{k,\lambda}^\ell)} e^{ik \cdot x} \mathfrak{d}_{k,\lambda}^\ell(x) \hat{u}^\circ(k) d^*k, \quad (4.9)$$

where $\mathfrak{d}_{k,\lambda}^\ell$ denotes the eigendefect (cf. Lemma 2.5). Indeed, by definition of V_λ^ℓ and by Lemma 2.5, we find

$$\begin{aligned} &(i\partial_t + \Delta - \lambda V(x)) V_\lambda^{\ell;t}(x) \\ &= \int_{\mathbb{R}^d} e^{-it(|k|^2 + \kappa_{k,\lambda}^\ell)} e^{ik \cdot x} ((\Delta_k - \lambda V + \kappa_{k,\lambda}^\ell) \psi_{k,\lambda}^\ell)(x) \hat{u}^\circ(k) d^*k \\ &= -\lambda^{\ell+1} F_\lambda^{\ell;t}(x), \end{aligned}$$

so that the difference $W_\lambda^\ell - V_\lambda^\ell$ satisfies

$$(i\partial_t + \Delta - \lambda V)(W_\lambda^\ell - V_\lambda^\ell) = \lambda^{\ell+1} F_\lambda^\ell, \quad (W_\lambda^\ell - V_\lambda^\ell)|_{t=0} = 0.$$

Duhamel's formula together with the unitarity of the Schrödinger flow then yields

$$\sup_{0 \leq t \leq T} \|W_\lambda^{\ell;t} - V_\lambda^{\ell;t}\|_{L^2} \leq \int_0^T \lambda^{\ell+1} \|F_\lambda^{\ell;t}\|_{L^2} dt,$$

and (4.8) follows. By (4.8), it now suffices to prove that for all $\lambda \leq \frac{1}{2}(CRK^{s_0+M}\ell^{s_0}\|\mathcal{F}\tilde{V}\|_{L^\infty})^{-1}$ and $0 \leq t \leq T$,

$$\sup_{\Omega} \sup_{0 \leq \lambda' \leq \lambda} \|F_{\lambda'}^{\ell;t}\|_{L^2} \lesssim (K\ell)^{M+1}(CRK^{s_0+M}\ell^{s_0})^\ell \|\mathcal{F}\tilde{V}\|_{L^\infty}^{\ell+1} \|u^\circ\|_{L^2}. \quad (4.10)$$

By definition (4.9) of F_λ^ℓ and by definition of $\mathfrak{d}_{k,\lambda}^\ell$ (cf. Lemma 2.5), the assumptions (3.5) and (3.7) yield

$$\begin{aligned} \|F_\lambda^{\ell;t}\|_{L^2} &= \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{-it(|k|^2 + \kappa_{k,\lambda}^\ell)} e^{ik \cdot x} \mathfrak{d}_{k,\lambda}^\ell(x) \hat{u}^\circ(k) d^*k \right|^2 dx \right)^{\frac{1}{2}} \\ &\leq (K\ell)^{M+1}(CRK^{s_0+M}\ell^{s_0})^\ell \|\mathcal{F}\tilde{V}\|_{L^\infty}^{\ell+1} \|u^\circ\|_{L^2} \\ &\quad + (K\ell)^{M+1} \|u^\circ\|_{L^2} \sum_{n=1}^{\ell} \lambda^n (CRK^{s_0+M}\ell^{s_0})^{n+\ell} \|\mathcal{F}\tilde{V}\|_{L^\infty}^{n+\ell+1}, \end{aligned}$$

and the claim (4.10) follows. \square

4.5. Proof of Lemma 4.5. Since

$$(V_\lambda^{\ell;t} - U_\lambda^{\ell;t})(x) = \sum_{n=1}^{\ell} \lambda^n \int_{\mathbb{R}^d} e^{-it(|k|^2 + \kappa_{k,\lambda}^\ell)} e^{ik \cdot x} \phi_k^n(x) \hat{u}^\circ(k) d^*k,$$

the desired estimate directly follows from assumption (3.7) as in (4.7). \square

4.6. Proof of Proposition 4.2: Asymptotic ballistic transport. Before proceeding to the proof, we recall the following a priori estimate for the Schrödinger flow in weighted norms. This result is due to Ozawa [21, Theorem 1] (see also [9] for the extension to mildly unbounded potentials).

Lemma 4.6 ([21]). *Given $z^\circ \in \mathcal{S}(\mathbb{R}^d)$, $F \in L_{\text{loc}}^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d))$, and a (real-valued) potential $V \in W^{m-1,\infty}(\mathbb{R}^d)$ with $m \geq 1$, denote by $z \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d))$ the solution of the Schrödinger equation*

$$(i\partial_t + \Delta - V)z^t = F^t, \quad z^t|_{t=0} = z^\circ. \quad (4.11)$$

Then, for all $t \geq 0$ and $i, j \geq 0$ with $i + j = m$, we have

$$\begin{aligned} \langle t \rangle^{-i} \|\langle \cdot \rangle^i \langle \nabla \rangle^j z^t\|_{L^2} &\lesssim_m \|\langle \nabla \rangle^m z^\circ\|_{L^2} + \langle t \rangle^{-m} \|\langle \cdot \rangle^m z^\circ\|_{L^2} \\ &\quad + \int_0^t \left(\|\langle \nabla \rangle^m F^s\|_{L^2} + \langle t \rangle^{-m} \|\langle \cdot \rangle^m F^s\|_{L^2} \right) ds, \end{aligned}$$

where the multiplicative constant depends only on $d, m, \|V\|_{W^{m-1,\infty}}$. \diamond

With this estimate at hand, we show that the rescaled moments $M_m^t(u_\lambda)$ of the Schrödinger flow (cf. (1.14)) can be truncated in a ballistic scaling. For $C_0 \geq 1$, we define the ballistically truncated moment via

$$\tilde{M}_m^t(u_\lambda; C_0) := \left\| \left(\frac{\cdot}{t} \right)^m e^{-\frac{1}{2} \left(\frac{\cdot}{C_0 t} \right)^2} u_\lambda^t \right\|_{L^2}.$$

This is the starting point for the proof of Proposition 4.2.

Corollary 4.7 (Ballistic truncation). *Given $u^\circ \in \mathcal{S}(\mathbb{R}^d)$ and a (real-valued) potential $V \in W^{m,\infty}(\mathbb{R}^d)$ with $m \geq 1$, denote by $u \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d))$ the solution of the Schrödinger flow*

$$i\partial_t u = (-\Delta + V)u, \quad u|_{t=0} = u^\circ.$$

Then, for all $C_0 \geq 1$ and $t \geq 0$, we have

$$|M_m^t(u) - \tilde{M}_m^t(u; C_0)| \lesssim_m C_0^{-1} (\|\langle \nabla \rangle^{m+1} u^\circ\|_{L^2} + \|\langle \cdot \rangle^{m+1} u^\circ\|_{L^2}),$$

where the multiplicative constant depends only on $d, m, \|V\|_{W^{m,\infty}}$. \diamond

Proof. We claim that it suffices to establish the following estimate,

$$|M_m^t(u) - \tilde{M}_m^t(u; C_0)| \lesssim C_0^{-1} M_{m+1}^t(u), \quad (4.12)$$

since then the conclusion follows from Lemma 4.6 in the form $M_{m+1}^t(u) \lesssim_m \|\langle \nabla \rangle^{m+1} u^\circ\|_{L^2} + \|\langle \cdot \rangle^{m+1} u^\circ\|_{L^2}$. In order to prove (4.12), it suffices to show that for all $R > 0$,

$$\| |\cdot|^m (1 - \gamma_R) u \|_{L^2} \lesssim R^{-1} \| |\cdot|^{m+1} u \|_{L^2}, \quad (4.13)$$

in terms of the Gaussian cut-off $\gamma_R(x) := e^{-\frac{1}{2}(\frac{|x|}{R})^2}$. For that purpose, we write in Fourier space

$$\| |\cdot|^m (1 - \gamma_R) u \|_{L^2}^2 = \int_{\mathbb{R}^d} |\nabla^m \hat{u}(k) - \hat{\gamma}_R * \nabla^m \hat{u}(k)|^2 d^*k,$$

where $\hat{\gamma}_R(k) := (\sqrt{2\pi}R)^d e^{-\frac{1}{2}(R|k|)^2}$. Since $\int_{\mathbb{R}^d} \hat{\gamma}_R(k) d^*k = 1$ and $\int_{\mathbb{R}^d} |k|^2 \hat{\gamma}_R(k) d^*k \lesssim R^{-2}$, the Cauchy-Schwarz inequality yields

$$\begin{aligned} \| |\cdot|^m (1 - \gamma_R) u \|_{L^2}^2 &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \hat{\gamma}_R(k') (\nabla^m \hat{u}(k) - \nabla^m \hat{u}(k+k')) d^*k' \right|^2 d^*k \\ &\lesssim R^{-2} \int_{\mathbb{R}^d} |\nabla^{m+1} \hat{u}(k)|^2 d^*k, \end{aligned}$$

that is, (4.13). \square

We may now turn to the proof of Proposition 4.2.

Proof of Proposition 4.2. Let $m \geq 0$ be fixed. In this proof, we use the notation \lesssim_{m,u° for \leq up to a multiplicative constant that only depends on $d, m, \|V\|_{W^{m,\infty}}, \|\langle \nabla \rangle^{m+1} u^\circ\|_{L^2}$, and $\|\langle \cdot \rangle^{m+1} u^\circ\|_{L^2}$. The starting point is the triangle inequality in the following form, for all $C_0 \geq 1$,

$$\begin{aligned} |M_m^t(u_\lambda) - M_m^t(u_0)| &\leq |M_m^t(u_\lambda) - \tilde{M}_m^t(u_\lambda; C_0)| + |\tilde{M}_m^t(u_\lambda; C_0) - \tilde{M}_m^t(U_\lambda^\ell; C_0)| \\ &\quad + |M_m^t(U_\lambda^\ell) - \tilde{M}_m^t(U_\lambda^\ell; C_0)| + |M_m^t(U_\lambda^\ell) - M_m^t(u_0)|. \end{aligned}$$

Using Corollary 4.7 to estimate the first RHS term and using (4.12) to estimate the third one, this yields

$$\begin{aligned} |M_m^t(u_\lambda) - M_m^t(u_0)| &\lesssim_{m,u^\circ} C_0^{-1} + C_0^m \|u_\lambda^t - U_\lambda^{\ell;t}\|_{L^2} \\ &\quad + C_0^{-1} M_{m+1}^t(U_\lambda^\ell) + |M_m^t(U_\lambda^\ell) - M_m^t(u_0)|. \quad (4.14) \end{aligned}$$

It remains to prove the following estimates for all $t \geq 1$:

$$M_{m+1}^t(U_\lambda^\ell) \lesssim_{m,u^\circ} 1 + \sum_{j=1}^{m+1} \sum_{l=1}^j \left(\int_{\mathbb{R}^d} |\nabla_k^l \kappa_{k,\lambda}^\ell|^{\frac{2j}{l}} |(\frac{1}{t}\nabla)^{m+1-j} \hat{u}_0^t(k)|^2 d^*k \right)^{\frac{1}{2}}, \quad (4.15)$$

$$|M_m^t(U_\lambda^\ell) - M_m^t(u_0)| \lesssim_{m,u^\circ} \sum_{j=1}^m \sum_{l=1}^j \left(\int_{\mathbb{R}^d} |\nabla_k^l \kappa_{k,\lambda}^\ell|^{\frac{2j}{l}} |(\frac{1}{t}\nabla)^{m-j} \hat{u}_0^t(k)|^2 d^*k \right)^{\frac{1}{2}}, \quad (4.16)$$

Injecting these estimates into (4.14) and optimizing wrt $C_0 \geq 1$, the conclusion follows.

We start with the proof of (4.16), for which we argue in Fourier space. By definition (4.2) of U_λ^ℓ , we have

$$\hat{U}_\lambda^{\ell;t}(k) = e^{-it(|k|^2 + \kappa_{k,\lambda}^\ell)} \hat{u}^\circ(k) = e^{-it\kappa_{k,\lambda}^\ell} \hat{u}_0^t(k),$$

so that the Leibniz rule leads to

$$|\nabla^m \hat{U}_\lambda^{\ell;t}(k) - e^{-it\kappa_{k,\lambda}^\ell} \nabla^m \hat{u}_0^t(k)| \leq \sum_{j=1}^m \binom{m}{j} |\nabla_k^j e^{-it\kappa_{k,\lambda}^\ell}| |\nabla^{m-j} \hat{u}_0^t(k)|.$$

Integrating wrt k and using the triangle inequality, we deduce

$$|M_m^t(U_\lambda^\ell) - M_m^t(u_0)| \lesssim_m t^{-m} \sum_{j=1}^m \left(\int_{\mathbb{R}^d} |\nabla_k^j e^{-it\kappa_{k,\lambda}^\ell}|^2 |\nabla^{m-j} \hat{u}_0^t(k)|^2 d^*k \right)^{\frac{1}{2}}.$$

For $j \geq 1$, we compute

$$|\nabla_k^j e^{-it\kappa_{k,\lambda}^\ell}| \lesssim_j \sum_{l=1}^j t^{\frac{j}{l}} |\nabla_k^l \kappa_{k,\lambda}^\ell|^{\frac{j}{l}} \lesssim_j \langle t \rangle^j \sum_{l=1}^j |\nabla_k^l \kappa_{k,\lambda}^\ell|^{\frac{j}{l}},$$

and the claim (4.16) follows for all $t \geq 1$. Likewise, this argument for m replaced by $m+1$ and combined with the a priori estimate of Lemma 4.6 in the form $M_m^t(u_0) \lesssim_{m,u^\circ} 1$ yields (4.15). \square

4.7. Proof of Theorem 1. In order to apply Proposition 4.1 with finite parameters $\ell, R, K \geq 1$, we first need to proceed to two truncation procedures:

- cut frequencies higher than K in the potential V ;
- project the initial data u° onto the restricted non-resonant set $\mathbb{R}^d \setminus \mathcal{R}_R^{K\ell}$ in Fourier space.

We start with the frequency cut-off: for all $K \geq 1$ we define $\tilde{V}_K := \mathcal{F}^{-1}(\mathcal{F}\tilde{V} \mathbf{1}_{|\cdot| \leq K})$, and we consider the corresponding Schrödinger flow,

$$i\partial_t u_{K,\lambda} = (-\Delta + \lambda V_K) u_{K,\lambda}, \quad u_{K,\lambda}|_{t=0} = u^\circ.$$

The Gevrey regularity assumption on V implies $\|V_K - V\|_{L^\infty} \leq \|\mathcal{F}(\tilde{V}_K - \tilde{V})\|_{L^1} \leq e^{-K^\alpha}$, so that Duhamel's formula and unitarity yield for all $t \geq 0$,

$$\|u_\lambda^t - u_{K,\lambda}^t\|_{L^2} \lesssim \lambda t e^{-K^\alpha} \|u^\circ\|_{L^2}. \quad (4.17)$$

We turn to the projection of the initial data: for $R \geq 1$ we define

$$u_R^\circ := \mathcal{F}^{-1}[\mathbf{1}_{B_{R^{1/d}}} \mathbf{1}_{\mathbb{R}^d \setminus \mathcal{R}_R^{K\ell}} \hat{u}^\circ],$$

and we consider the corresponding Schrödinger flow,

$$i\partial_t u_{K,R,\lambda} = (-\Delta + \lambda V_K) u_{K,R,\lambda}, \quad u_{K,R,\lambda}|_{t=0} = u_R^\circ. \quad (4.18)$$

By unitarity and by Lemma 3.1(ii), we find for all $t \geq 0$,

$$\begin{aligned} \|u_{K,\lambda}^t - u_{K,R,\lambda}^t\|_{L^2} &= \|u^\circ - u_R^\circ\|_{L^2} \leq \|\mathbb{1}_{\mathbb{R}^d \setminus B_{R^{1/d}}}\hat{u}^\circ\|_{L^2} + \|\mathbb{1}_{B_{R^{1/d}} \cap \mathcal{R}_R^{K\ell}}\hat{u}^\circ\|_{L^2} \\ &\leq R^{-\frac{1}{2d}}\|\langle \cdot \rangle \hat{u}^\circ\|_{L^2} + |B_{R^{1/d}} \cap \mathcal{R}_R^{K\ell}|^{\frac{1}{2}}\|\hat{u}^\circ\|_{L^\infty} \\ &\lesssim R^{-\frac{1}{2d}}(\|\langle \cdot \rangle \hat{u}^\circ\|_{L^2} + \|\hat{u}^\circ\|_{L^\infty}). \end{aligned} \quad (4.19)$$

Combining this with (4.17), we deduce

$$\|u_\lambda^t - u_{K,R,\lambda}^t\|_{L^2} \lesssim (R^{-\frac{1}{2d}} + \lambda t e^{-K\alpha})(\|\langle \cdot \rangle \hat{u}^\circ\|_{L^2} + \|\hat{u}^\circ\|_{L^\infty}). \quad (4.20)$$

We may now apply Proposition 4.1 to the truncated Schrödinger flow $u_{K,R,\lambda}$. Denote by $\kappa_{K,k,\lambda}^\ell$ the Bloch eigenvalues associated with V replaced by V_K (and by $\nu_{K,k}^n$ the corresponding Rayleigh-Schrödinger coefficients), and define the corresponding truncated approximate flow,

$$U_{K,R,\lambda}^{\ell;t}(x) := \int_{\mathbb{R}^d} e^{-it(|k|^2 + \kappa_{K,k,\lambda}^\ell)} e^{ik \cdot x} \hat{u}_R^\circ(k) d^*k. \quad (4.21)$$

Proposition 4.1 together with (4.20) and the assumption $\|\mathcal{F}\tilde{V}\|_{L^\infty} \leq 1$ then yields for all $\lambda \leq \frac{1}{2}(CRK^{s_0+M}\ell^{s_0})^{-1}$ and $T \geq 0$,

$$\begin{aligned} \sup_{\Omega} \sup_{0 \leq \lambda' \leq \lambda} \sup_{0 \leq t \leq T} \|u_{\lambda'}^t - U_{K,R,\lambda'}^{\ell;t}\|_{L^2} &\lesssim (\|\langle \cdot \rangle \hat{u}^\circ\|_{L^2} + \|\hat{u}^\circ\|_{L^\infty}) \\ &\times (R^{-\frac{1}{2d}} + \lambda T e^{-K\alpha} + \lambda R K^{-s_0+2M+1} \ell^{s_0+M+1} + \lambda^{\ell+1} T (K\ell)^{M+1} (CRK^{s_0+M}\ell^{s_0})^\ell). \end{aligned} \quad (4.22)$$

Finally, we show how to replace $U_{K,R,\lambda}^\ell$ by the approximate flow U_λ^ℓ without cut-off as defined in (1.11). Since \hat{u}_R° is supported in $\mathbb{R}^d \setminus \overline{\mathcal{R}_R^{K\ell}} \subset \mathcal{O}$, we can write

$$U_{K,R,\lambda}^{\ell;t}(x) = \int_{\mathbb{R}^d} e^{-it(|k|^2 + \mathbb{1}_{\mathcal{O}(k)}\kappa_{K,k,\lambda}^\ell)} e^{ik \cdot x} \hat{u}_R^\circ(k) d^*k,$$

so that comparing with (1.11) yields

$$\|U_\lambda^{\ell;t} - U_{K,R,\lambda}^{\ell;t}\|_{L^2} \lesssim \|u^\circ - u_R^\circ\|_{L^2} + t \left(\int_{\mathbb{R}^d \setminus \overline{\mathcal{R}_R^{K\ell}}} |\kappa_{K,k,\lambda}^\ell - \kappa_{k,\lambda}^\ell|^2 |\hat{u}^\circ(k)|^2 d^*k \right)^{\frac{1}{2}}.$$

It remains to estimate the second RHS term. Since $\|\mathcal{F}(\tilde{V} - \tilde{V}_K)\|_{L^\infty} \leq \|\mathcal{F}(\tilde{V} - \tilde{V}_K)\|_{L^1} \leq e^{-K\alpha}$, the argument in Step 1 of the proof of Proposition 3.3 easily yields for all $n \geq 1$ and $k \in \mathbb{R}^d \setminus \mathcal{R}_R^{K\ell}$,

$$|\nu_{K,k}^n - \nu_{2K,k}^n| \lesssim e^{-K\alpha} (CRK^{s_0+M}n^{s_0})^n,$$

and hence for all $\lambda \leq \frac{1}{2}(CRK^{s_0+M}\ell^{s_0})^{-1}$ and $k \in \mathbb{R}^d \setminus \mathcal{R}_R^{K\ell}$,

$$|\kappa_{k,\lambda}^\ell - \kappa_{K,k,\lambda}^\ell| \lesssim \lambda e^{-K\alpha}.$$

Injecting this together with (4.19) into the above, we find

$$\|U_\lambda^{\ell;t} - U_{K,R,\lambda}^{\ell;t}\|_{L^2} \lesssim (R^{-\frac{1}{2d}} + \lambda t e^{-K\alpha})(\|\langle \cdot \rangle \hat{u}^\circ\|_{L^2} + \|\hat{u}^\circ\|_{L^\infty}),$$

and (4.22) becomes

$$\begin{aligned} \sup_{\Omega} \sup_{0 \leq \lambda' \leq \lambda} \sup_{0 \leq t \leq T} \|u_{\lambda'}^t - U_{\lambda'}^{\ell;t}\|_{L^2} &\lesssim (\|\langle \cdot \rangle \hat{u}^\circ\|_{L^2} + \|\hat{u}^\circ\|_{L^\infty}) \\ &\times (R^{-\frac{1}{2d}} + \lambda T e^{-K\alpha} + \lambda R K^{-s_0+2M+1} \ell^{s_0+M+1} + \lambda^{\ell+1} T (K\ell)^{M+1} (CRK^{s_0+M}\ell^{s_0})^\ell). \end{aligned} \quad (4.23)$$

For $\gamma < \frac{1}{2d+1}$, choosing

$$\begin{aligned} R &= R(\lambda) := \lambda^{-2d\gamma}, \\ K &= K(\lambda) := \ell(\lambda)^{\frac{1}{\alpha}}, \\ T &= T(\lambda) := e^{\ell(\lambda)}, \\ \ell &= \ell(\lambda) := \lambda^{-\frac{\alpha(1-(2d+1)\gamma)}{s_0+2M+1+\alpha(s_0+M+1)}}, \end{aligned}$$

the conclusion follows after straightforward simplifications. \square

4.8. Proof of Corollary 1. Let $m \geq 0$ be fixed. In this proof, we use the notation \lesssim_{m,u° for \leq up to a multiplicative constant that only depends on $d, m, \|\langle \nabla \rangle^{m+1} u^\circ\|_{L^2}$, $\|\langle \cdot \rangle^{m+1} u^\circ\|_{L^2}$, and $\|\langle \cdot \rangle^m \hat{u}^\circ\|_{L^\infty}$. Again, we may not directly apply Proposition 4.2 since the approximate Bloch eigenvalues $\kappa_{k,\lambda}^\ell$ can only be estimated for k away from the resonant set. Therefore, we first need to proceed to similar truncations of the initial data u° as in the proof of Theorem 1. More care is however needed here as all truncations need to be chosen smooth. We thus start with the construction of suitable smooth truncations. Given $R \geq 1$, for each $\xi \in \mathbb{Z}^M \setminus \{0\}$, recalling the definition (3.4) of $\mathcal{R}_R(\xi)$ in the proof of Lemma 3.1, we choose a cut-off function χ_R^ξ with $\chi_R^\xi = 0$ in $\mathcal{R}_R(\xi)$, $\chi_R^\xi = 1$ outside $\mathcal{R}_{2R}(\xi)$, and $\|\nabla^j \chi_R^\xi\|_{L^\infty} \lesssim_j (R|\xi|^{s_0+1})^j$ for all $0 \leq j \leq m$. We also choose a cut-off function χ_R with $\chi_R = 1$ in $B_{R^{1/d}}$, $\chi_R = 0$ outside $B_{2R^{1/d}}$, and $\|\nabla^j \chi_R\|_{L^\infty} \lesssim_j 1$. We then define the product cut-off function

$$\zeta_R := \chi_R \prod_{\substack{\xi \in \mathbb{Z}^M \setminus \{0\} \\ |\xi| \leq K^\ell}} \chi_R^\xi.$$

By construction, for all $0 \leq j \leq m$, we find

$$\|\nabla^j \zeta_R\|_{L^\infty} \lesssim (RK^{s_0+M+1}\ell^{s_0+M+1})^j, \quad (4.24)$$

and

$$\mathbb{1}_{B_{R^{1/d}}} \mathbb{1}_{\mathbb{R}^d \setminus \overline{\mathcal{R}_{2R}^{K^\ell}}} \leq \zeta_R \leq \mathbb{1}_{B_{2R^{1/d}}} \mathbb{1}_{\mathbb{R}^d \setminus \overline{\mathcal{R}_R^{K^\ell}}}.$$

We now define the truncated initial data $u_R^\circ := \mathcal{F}^{-1}[\zeta_R \hat{u}^\circ]$, the truncated Schrödinger flow $u_{K,R,\lambda}$ as the solution of the corresponding equation (4.18), and we let $u_{R,0}$ denote the corresponding truncation of the free flow u_0 . We further define the truncated approximate flow $U_{K,R,\lambda}^\ell$ as in (4.21).

With these definitions at hand, we now turn to the proof of Corollary 1. This is about repeating the proof of Proposition 4.2 with suitable truncation arguments. The starting point is the triangle inequality in the form

$$\begin{aligned} |M_m^t(u_\lambda) - M_m^t(u_0)| &\leq |M_m^t(u_\lambda) - \tilde{M}_m^t(u_\lambda; C_0)| + |\tilde{M}_m^t(u_\lambda; C_0) - \tilde{M}_m^t(u_{K,R,\lambda}; C_0)| \\ &\quad + |\tilde{M}_m^t(u_{K,R,\lambda}; C_0) - \tilde{M}_m^t(U_{K,R,\lambda}^\ell; C_0)| + |M_m^t(U_{K,R,\lambda}^\ell) - \tilde{M}_m^t(U_{K,R,\lambda}^\ell; C_0)| \\ &\quad + |M_m^t(U_{K,R,\lambda}^\ell) - M_m^t(u_{R,0})| + |M_m^t(u_0) - M_m^t(u_{R,0})|. \end{aligned}$$

Using Corollary 4.7 to estimate the first RHS term and using (4.12) to estimate the fourth one, this takes the form

$$\begin{aligned} |M_m^t(u_\lambda) - M_m^t(u_0)| &\lesssim_{m,u^\circ} C_0^{-1} + C_0^m \|u_\lambda - u_{K,R,\lambda}\|_{L^2} + C_0^m \|u_{K,R,\lambda} - U_{K,R,\lambda}^\ell\|_{L^2} \\ &\quad + C_0^{-1} M_{m+1}^t(U_{K,R,\lambda}^\ell) + |M_m^t(U_{K,R,\lambda}^\ell) - M_m^t(u_{R,0})| + |M_m^t(u_0) - M_m^t(u_{R,0})|. \quad (4.25) \end{aligned}$$

We separately estimate the last five RHS terms and we start with the first one, which is a truncation error. Arguing as for (4.20) (now with smooth truncations), we find

$$\|u_\lambda^t - u_{K,R,\lambda}^t\|_{L^2} \lesssim_{u^\circ} R^{-\frac{1}{2d}} + \lambda t e^{-K^\alpha}.$$

Applying Proposition 4.1 to estimate the third RHS term in (4.25), we obtain for all $\lambda \leq \frac{1}{2}(CRK^{s_0+M}\ell^{s_0})^{-1}$,

$$\|u_{K,R,\lambda}^t - U_{K,R,\lambda}^{\ell;t}\|_{L^2} \lesssim_{u^\circ} \lambda CRK^{s_0+2M+1}\ell^{s_0+M+1} + \lambda^{\ell+1}t(K\ell)^{M+1}(CRK^{s_0+M}\ell^{s_0})^\ell.$$

Using (4.15) to estimate the fourth RHS term in (4.25) yields

$$M_{m+1}^t(U_{K,R,\lambda}^\ell) \lesssim_{m,u^\circ} 1 + \sum_{j=1}^{m+1} \sum_{l=1}^j \left(\int_{\mathbb{R}^d} |\nabla_k^l \kappa_{K,k,\lambda}^\ell|^{\frac{2j}{l}} |(\frac{1}{t}\nabla)^{m+1-j} \hat{u}_{R,0}^t(k)|^2 d^*k \right)^{\frac{1}{2}},$$

and hence, by the definition (2.9) of $\kappa_{K,k,\lambda}^\ell$ and the bounds of Proposition 3.3, we deduce for all $\lambda \leq \frac{1}{2}(CRK^{s_0+M}\ell^{s_0})^{-1}$,

$$M_{m+1}^t(U_{K,R,\lambda}^\ell) \lesssim_{m,u^\circ} 1 + \lambda(RK^{s_0+1}\ell^{s_0+1})^{m+1} \|\langle \frac{1}{t}\nabla \rangle^{m+1} \hat{u}_{R,0}^t\|_{L^2}.$$

Since Lemma 4.6 and (4.24) yield

$$\begin{aligned} \|\langle \frac{1}{t}\nabla \rangle^{m+1} \hat{u}_{R,0}^t\|_{L^2} &\lesssim_m \|\langle \nabla \rangle^{m+1} (\zeta_R \hat{u}^\circ)\|_{L^2} + \|\langle \cdot \rangle^{m+1} \zeta_R \hat{u}^\circ\|_{L^2} \\ &\lesssim_{m,u^\circ} (RK^{s_0+M+1}\ell^{s_0+M+1})^{m+1}, \end{aligned}$$

the above turns into

$$M_{m+1}^t(U_{K,R,\lambda}^\ell) \lesssim_{m,u^\circ} 1 + \lambda(R^2 K^{2(s_0+1)+M} \ell^{2(s_0+1)+M})^{m+1}.$$

Likewise, combining (4.16) with the bounds of Proposition 3.3, we may estimate the fifth RHS term in (4.25) as follows: for all $\lambda \leq \frac{1}{2}(CRK^{s_0+M}\ell^{s_0})^{-1}$,

$$\begin{aligned} |M_m^t(U_{K,R,\lambda}^\ell) - M_m^t(u_{R,0})| &\lesssim \sum_{j=1}^m \sum_{l=1}^j \left(\int_{\mathbb{R}^d} |\nabla_k^l \kappa_{K,k,\lambda}^\ell|^{\frac{2j}{l}} |(\frac{1}{t}\nabla)^{m-j} \hat{u}_{R,0}^t(k)|^2 d^*k \right)^{\frac{1}{2}} \\ &\lesssim_m \lambda(RK^{s_0+1}\ell^{s_0+1})^m \|\langle \frac{1}{t}\nabla \rangle^m \hat{u}_{R,0}^t\|_{L^2} \lesssim_{m,u^\circ} \lambda(R^2 K^{2(s_0+1)+M} \ell^{2(s_0+1)+M})^m. \end{aligned}$$

It remains to estimate the last RHS term in (4.25), which is a truncation error. We write

$$\begin{aligned} |\nabla^m(\hat{u}_0^t - \hat{u}_{R,0}^t)(k)| &= |\nabla_k^m(e^{-it|k|^2}(1 - \zeta_R(k))\hat{u}^\circ(k))| \\ &\lesssim \sum_{j=0}^m (t^j |k|^j + t^{\frac{j}{2}}) |\nabla^{m-j}((1 - \zeta_R(k))\hat{u}^\circ(k))|, \end{aligned}$$

so that for $t \geq 1$,

$$t^{-m} \|\nabla^m(\hat{u}_0^t - \hat{u}_{R,0}^t)\|_{L^2} \lesssim_{m,u^\circ} \|\langle \cdot \rangle^m (1 - \zeta_R) \hat{u}^\circ\|_{L^2} + \sum_{j=0}^{m-1} t^{j-m} \|\zeta_R\|_{W^{m-j,\infty}}.$$

Applying (4.19) in the form $\|\langle \cdot \rangle^m (1 - \zeta_R) \hat{u}^\circ\|_{L^2} \lesssim_{m, u^\circ} R^{-\frac{1}{2d}}$, and using (4.24), we deduce for all $t \geq RK^{s_0+M+1} \ell^{s_0+M+1}$,

$$\begin{aligned} t^{-m} \|\nabla^m (\hat{u}_0^t - \hat{u}_{R,0}^t)\|_{L^2} &\lesssim_{m, u^\circ} R^{-\frac{1}{2d}} + \sum_{j=0}^{m-1} (t^{-1} RK^{s_0+M+1} \ell^{s_0+M+1})^{m-j} \\ &\lesssim_m R^{-\frac{1}{2d}} + t^{-1} RK^{s_0+M+1} \ell^{s_0+M+1}. \end{aligned}$$

Injecting all the above estimates into (4.25) and optimizing wrt $C_0 \geq 1$, we find for all $\lambda \leq \frac{1}{2}(CRK^{s_0+M} \ell^{s_0})^{-1}$,

$$\begin{aligned} &|M_m^t(u_\lambda) - M_m^t(u_0)| \\ &\lesssim_{m, u^\circ} \left(R^{-\frac{1}{2d}} + \lambda t e^{-K^\alpha} + \lambda CRK^{s_0+2M+1} \ell^{s_0+M+1} + \lambda^{\ell+1} t (K\ell)^{M+1} (CRK^{s_0+M} \ell^{s_0})^\ell \right)^{\frac{1}{m+1}} \\ &\quad + \lambda (R^2 K^{2(s_0+1)+M} \ell^{2(s_0+1)+M})^{m+1} + t^{-1} RK^{s_0+M+1} \ell^{s_0+M+1}. \end{aligned}$$

For $\gamma < \frac{1}{4d(m+1)+1}$, choosing

$$\begin{aligned} R &= R(\lambda) := \lambda^{-2d\gamma}, \\ K &= K(\lambda) := \ell(\lambda)^{\frac{1}{\alpha}}, \\ \ell &= \ell(\lambda) := \lambda^{-\frac{\alpha(1-(4d(m+1)+1)\gamma)}{(m+1)(\alpha+1)(2(s_0+1)+M)}}, \end{aligned}$$

the conclusion follows in the regime $\lambda^{-\frac{1}{m+1}} \leq t \leq e^{\ell(\lambda)}$ after straightforward simplifications. For shorter timescales, the conclusion is easier. \square

5. CLASSICAL FLOW

This section is devoted to the corresponding main results for the wave flow. It is organized as follows. We start by adapting the approximate stationary Floquet-Bloch theory to the operator $-\nabla \cdot (\text{Id} + \lambda a) \nabla$, defining a notion of approximate Bloch waves in the present setting via the corresponding Rayleigh-Schrödinger perturbation series. We then turn to the proof of Theorem 2 and we mainly focus on the differences with respect to the proofs of the corresponding results for the Schrödinger flow, to which we refer the reader for most of the arguments. Finally, we study the case of peaked initial data in Fourier space and prove Corollary 2.

5.1. Approximate Bloch waves. Consider the operator $-\nabla \cdot (\text{Id} + \lambda a) \nabla$ where λa is a quasiperiodic perturbation in the sense of (QP'). We construct a branch of approximate Bloch waves in terms of the corresponding Rayleigh-Schrödinger series for the perturbed fibered operator $-(\nabla + ik) \cdot (\text{Id} + \lambda a) (\nabla + ik) = -\Delta_k + |k|^2 - \lambda (\nabla + ik) \cdot a (\nabla + ik)$. The corresponding Rayleigh-Schrödinger coefficients $(\nu_k^n, \phi_k^n)_{k,n}$ (compare with Definition 2.4 in the Schrödinger case) then takes on the following guise:

- $\nu_k^n := -ik \cdot \mathbb{E} [a(\nabla + ik) \phi_k^n]$ for all $n \geq 0$;
- for all $k \in \mathbb{R}^d$, we have $\phi_k^0 \equiv 1$, and for all n the function ϕ_k^{n+1} satisfies $\mathbb{E} [\phi_k^{n+1}] = 0$ and

$$-\Delta_k \phi_k^{n+1} = \Pi(\nabla + ik) \cdot a(\nabla + ik) \phi_k^n + \sum_{l=0}^{n-1} \nu_k^l \phi_k^{n-l}.$$

We are exactly in the same situation as in the previous sections with the multiplication by V replaced by the operator $-(\nabla + ik) \cdot a(\nabla + ik)$. A direct adaptation of the proof of Proposition 3.3 yields the following corrector estimates.

Proposition 5.1. *Consider the quasiperiodic setting (QP'), assume that the winding matrix F satisfies the Diophantine condition (3.1) with $r_0 > 0$, let $s_0 > M + r_0$, assume that the lifted map \tilde{a} has compactly supported Fourier transform, and set $K := \sup\{1 \vee |\xi| : \xi \in \text{supp } \mathcal{F}\tilde{a}\}$. Then there exists a constant C (depending on F, M, s_0) and for all $\ell, R \geq 1$ there exists a field of ℓ -jets of Bloch waves $(\nu_k^n, \phi_k^n : k \in \mathbb{R}^d \setminus \mathcal{R}_R^{K\ell}, 0 \leq n \leq \ell)$ in the above sense, which satisfy the following estimates for all $n \geq 1, k \in \mathbb{R}^d \setminus \mathcal{R}_R^{Kn}$, and $s, j \geq 0$,*

$$\begin{aligned} |\nabla_k^j \nu_k^n| &\leq \langle k \rangle^{2n} (CRjK^{s_0+1}n^{s_0+1})^j (CRK^{s_0+M}n^{s_0})^n \|\mathcal{F}\tilde{a}\|_{L^\infty}^{n+1}, \\ \|\nabla_k^j \phi_k^n\|_{H^s(\Omega)} &\leq \langle k \rangle^{2n} (CKn)^s (CRjK^{s_0+1}n^{s_0+1})^j (CRK^{s_0+M}n^{s_0})^n \|\mathcal{F}\tilde{a}\|_{L^\infty}^n. \end{aligned} \quad (5.1)$$

and for all $\hat{u} \in C_c^\infty(\mathbb{R}^d)$ supported in $\mathbb{R}^d \setminus \mathcal{R}_R^{Cn}$,

$$\begin{aligned} \sup_{\omega \in \Omega} \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{ik \cdot x} \nabla_k^j \nabla^s \phi_k^n(x, \omega) \hat{u}(k) d^*k \right|^2 dx \right)^{\frac{1}{2}} \\ \leq (CKn)^{s+M+1} (CRjK^{s_0+1}n^{s_0+1})^j (CRK^{s_0+M}n^{s_0})^n \|\mathcal{F}\tilde{a}\|_{L^\infty}^n \|\langle \cdot \rangle^{2n} \hat{u}\|_{L^2}, \end{aligned} \quad (5.2)$$

which holds uniformly wrt translations on $\Omega = \mathbb{T}^M$. \diamond

For later purposes, we further show that the corrector ϕ_k^n can be written as $\phi_k^n = (\nabla + ik) \cdot \Phi_k^n$ for some Φ_k^n that satisfies the same bounds as ϕ_k^n itself.

Proposition 5.2. *Under the assumptions of Proposition 5.1, for all $n \geq 0$ and $k \in \mathbb{R}^d \setminus \mathcal{R}_R^{Kn}$, there exists $\Phi_k^n \in L^2(\Omega)^d$ such that $\phi_k^n = (\nabla + ik) \cdot \Phi_k^n$ and such that Φ_k^n satisfies the same bounds (5.1)–(5.2) multiplied by a factor $\langle k \rangle^{-1} (1 + |k|^{-1})^{2+2(n-1)\vee 0}$. \diamond*

Proof. As in Proposition 2.6, the correctors can be expressed via the following tree formulas: for all $n \geq 1$,

$$\begin{aligned} \phi_k^n &= \sum_{m=1}^n \sum_{\ell=0}^{n-m} \sum_{\substack{c \in \mathbb{N}^\ell \\ |c|=n-m-\ell}} \sum_{\substack{b \in \mathbb{N}^m \\ |b|=\ell}} \nu_k^{c_1} \dots \nu_k^{c_\ell} \\ &\quad \times (-\Delta_k)^{-b_1-1} \Pi(\nabla + ik) \cdot a(\nabla + ik) \\ &\quad \dots (-\Delta_k)^{-b_m-1} \Pi(\nabla + ik) \cdot a(\nabla + ik) 1. \end{aligned} \quad (5.3)$$

On the one hand, we claim that for all $G \in C_c^\infty(\mathbb{R}^d)^d$ there holds

$$(-\Delta_k)^{-b-1} (\nabla + ik) \cdot G = \frac{1}{(-|k|^2)^{b+1}} (\nabla + ik) \cdot \left(\text{Id} + (\nabla + ik)(-\Delta_k)^{-1}(\nabla + ik) \cdot \right)^{b+1} G.$$

Indeed, the relation

$$-\Delta_k f = (\nabla + ik) \cdot G$$

implies $f = (\nabla + ik) \cdot F$ with

$$F = \frac{1}{-|k|^2} (G + (\nabla + ik)f),$$

and the claim follows. Injecting this identity into (5.3) and decomposing the first projection $\Pi = \text{Id} - \mathbb{E}[\cdot]$, we deduce $\phi_k^n = (\nabla + ik) \cdot \Phi_k^n$ where Φ_k^n is given by

$$\begin{aligned} \Phi_k^n &= \sum_{m=1}^n \sum_{\ell=0}^{n-m} \sum_{\substack{c \in \mathbb{N}^\ell \\ |c|=n-m-\ell}} \sum_{\substack{b \in \mathbb{N}^m \\ |b|=\ell}} \frac{\nu_k^{c_1} \dots \nu_k^{c_\ell}}{(-|k|^2)^{b_1+1}} \\ &\quad \times \left(\text{Id} + (\nabla + ik)(-\Delta_k)^{-1}(\nabla + ik) \cdot \right)^{b_1+1} a(\nabla + ik) \\ &\quad \times (-\Delta_k)^{-b_2-1} \Pi(\nabla + ik) \cdot a(\nabla + ik) \\ &\quad \dots (-\Delta_k)^{-b_m-1} \Pi(\nabla + ik) \cdot aik. \end{aligned}$$

We may now proceed to the estimate of Φ_k^n exactly as for ϕ_k^n . The details are left to the reader. \square

5.2. Proof of Theorem 2. The proof of Theorem 2 follows the general strategy of Section 4: the initial data (u°, v°) is first replaced by an approximate Bloch expansion and the corresponding flow then admits an explicit approximate formula using approximate Bloch waves (since the latter approximately diagonalize the wave operator). In order to control the errors, we exploit the corrector estimates of Proposition 5.1. We recall that L^2 -estimates for the wave equation are easily seen to take on the following guise.

Lemma 5.3 (L^2 -estimates for classical waves). *Given $z^\circ, w_1^\circ \in L^2(\mathbb{R}^d)$, $w_2^\circ \in L^2(\mathbb{R}^d)^d$, $F \in L^1_{\text{loc}}(\mathbb{R}^+; L^2(\mathbb{R}^d))$, and given a uniformly elliptic matrix field a , denote by z the solution of the classical wave equation*

$$(\partial_{tt}^2 - \nabla \cdot a \nabla)z = F, \quad z|_{t=0} = z^\circ, \quad \partial_t z|_{t=0} = w_1^\circ + \nabla \cdot w_2^\circ.$$

Then, for all $t \geq 0$, we have

$$\|z\|_{L_t^\infty L^2} \lesssim \|z^\circ\|_{L^2} + t\|w_1^\circ\|_{L^2} + \|w_2^\circ\|_{L^2} + \int_0^t \left\| \int_0^s F \right\|_{L^2} ds. \quad \diamond$$

The rest of the proof is similar to that of Theorem 1 in Section 4 with however three main differences:

- (I) In view of Lemma 5.3, general initial velocities v° in (1.15) lead the L^2 -norm of the flow to grow linearly in time. The initial data in Theorem 2 are chosen so that this does not happen, and we need to ensure that the various errors made in the approximation procedure do not grow in time either.
- (II) As opposed to quantum waves, the approximate solution is not necessarily well-defined here. This requires to make some further assumptions on the initial data, which can a posteriori be dropped by an approximation argument.
- (III) Comparing the L^2 -estimates for quantum and classical waves with a nontrivial source term F in the equation, we note that an additional time integral appears in the case of classical waves (cf. Lemma 5.3). If treated naively, this would lead to an additional time factor in the error estimates and would reduce the maximal allowed timescale. However, we only have to deal with source terms F displaying a particular structure that indeed ensures that $\int_0^t F$ remains bounded uniformly in time.

We now briefly comment on these three differences.

Argument for (I). As the only possible difficulty here comes from the initial velocity, we assume $u^\circ \equiv 0$, $v^\circ \neq 0$, so that (1.15) reads

$$\partial_{tt}^2 u_\lambda = \nabla \cdot (\text{Id} + \lambda a) \nabla u_\lambda, \quad u_\lambda|_{t=0} = 0, \quad \partial_t u_\lambda|_{t=0} = v^\circ,$$

where v° has Fourier transform \hat{v}° compactly supported in $\mathbb{R}^d \setminus \{0\}$. In particular, this entails that $v^\circ = \nabla \cdot g^\circ$ for some $g^\circ \in H^1(\mathbb{R}^d)^d$. The first step in the proof of Theorem 2 is to replace v° by its approximate Bloch wave expansion

$$Z_\lambda^{\ell;\circ}(x) := \int_{\mathbb{R}^d} e^{ik \cdot x} \psi_{k,\lambda}^\ell(x) \hat{v}^\circ(k) d^*k, \quad (5.4)$$

to consider the corresponding flow,

$$\partial_{tt}^2 W_\lambda^\ell = \nabla \cdot (\text{Id} + \lambda a) \nabla W_\lambda^\ell = 0, \quad W_\lambda^\ell|_{t=0} = 0, \quad \partial_t W_\lambda^\ell|_{t=0} = Z_\lambda^{\ell;\circ},$$

and to estimate $\|u_\lambda^t - W_\lambda^{\ell;t}\|_{L^2}$. In order to avoid a linear time growth, it is crucial to check that $Z_\lambda^{\ell;\circ}$ can be written in divergence form. This indeed follows from Proposition 5.2, which yields $Z_\lambda^{\ell;\circ} = \nabla \cdot g_\lambda^{\ell;\circ}$ with

$$g_\lambda^{\ell;\circ} := \int_{\mathbb{R}^d} e^{ik \cdot x} \Psi_{k,\lambda}^\ell(x) \hat{v}^\circ(k) d^*k, \quad \Psi_{k,\lambda}^\ell := \sum_{n=0}^{\ell} \lambda^n \Phi_k^n.$$

By Lemma 5.3, we then obtain for all $t \geq 0$,

$$\|u_\lambda^t - W_\lambda^{\ell;t}\|_{L_t^\infty L^2} \lesssim \|g^\circ - g_\lambda^{\ell;\circ}\|_{L^2} \leq \sum_{n=1}^{\ell} \lambda^n \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{ik \cdot x} \Phi_k^n(x) \hat{v}^\circ(k) d^*k \right|^2 dx \right)^{\frac{1}{2}},$$

and it remains to use the corrector estimates of Proposition 5.2 for the Φ_k^n 's to conclude.

Argument for (II). The second step of the proof of Theorem 2 consists in writing an approximate representation formula for the solution W_λ^ℓ of

$$\partial_{tt}^2 W_\lambda^\ell = \nabla \cdot (\text{Id} + \lambda a) \nabla W_\lambda^\ell, \quad W_\lambda^\ell|_{t=0} = W_\lambda^{\ell;\circ}, \quad \partial_t W_\lambda^\ell|_{t=0} = Z_\lambda^{\ell;\circ},$$

where $W_\lambda^{\ell;\circ}$ and $Z_\lambda^{\ell;\circ}$ are the approximate Bloch expansions of u° and v° , cf. (5.4). We naturally define

$$V_\lambda^{\ell;t}(x) := \int_{\mathbb{R}^d} \left(\cos \left(t \sqrt{|k|^2 + \kappa_{k,\lambda}^\ell} \right) \hat{u}^\circ(k) + t \text{sinc} \left(t \sqrt{|k|^2 + \kappa_{k,\lambda}^\ell} \right) \hat{v}^\circ(k) \right) e^{ik \cdot x} \psi_{k,\lambda}^\ell(x) d^*k.$$

This is however a priori not well-defined since the approximate Bloch eigenvalues $|k|^2 + \kappa_{k,\lambda}^\ell$ may be non-positive. To avoid that, given $\ell, R \geq 1$ and $\lambda_0 > 0$, we must further assume that the Fourier transform of the initial data (u°, v°) is compactly supported in

$$\mathcal{O}_{R,\lambda_0}^\ell := \{k \in \mathbb{R}^d \setminus \mathcal{R}_R^\ell : |k|^2 - \lambda_0 \sum_{n=0}^{\ell} \lambda_0^n |\nu_k^n| \geq 0\}.$$

Under this assumption, for $0 < \lambda \leq \lambda_0$, the formula for V_λ^ℓ makes perfect sense. Since $\mathcal{O}_{R,\lambda_0}^\ell \uparrow \mathbb{R}^d \setminus \mathcal{R}_R^\ell$ as $\lambda_0 \downarrow 0$, we deduce that this assumption on (u°, v°) can a posteriori be dropped by an approximation argument as in the proof of Theorem 1.

Argument for (III). The error $W_\lambda^\ell - V_\lambda^\ell$ satisfies

$$\begin{aligned} (\partial_{tt}^2 - \nabla \cdot (\text{Id} + \lambda a) \nabla)(W_\lambda^\ell - V_\lambda^\ell) &= F_\lambda^\ell, \\ (W_\lambda^\ell - V_\lambda^\ell)|_{t=0} &= 0, \quad \partial_t(W_\lambda^\ell - V_\lambda^\ell)|_{t=0} = 0, \end{aligned}$$

in terms of

$$\begin{aligned} F_\lambda^{\ell;t}(x) &:= - \int_{\mathbb{R}^d} \left(\cos \left(t \sqrt{|k|^2 + \kappa_{k,\lambda}^\ell} \right) \hat{u}^\circ(k) + t \operatorname{sinc} \left(t \sqrt{|k|^2 + \kappa_{k,\lambda}^\ell} \right) \hat{v}^\circ(k) \right) e^{ik \cdot x} \\ &\quad \times \left(-\Delta_k - \lambda(\nabla + ik) \cdot a(\nabla + ik) - \kappa_{k,\lambda}^\ell \right) \psi_{k,\lambda}^\ell(x) d^*k. \end{aligned}$$

We then apply Lemma 5.3 and we must estimate the contribution $\int_0^t \| \int_0^s F_\lambda^\ell \|_{L^2} ds$ due to the source term. Note that this contribution displays two time integrals in contrast to the case of quantum waves. However, the explicit time integration of F_λ^ℓ yields

$$\begin{aligned} \int_0^t F_\lambda^{\ell;s}(x) ds &= - \int_{\mathbb{R}^d} \left(t \operatorname{sinc} \left(t \sqrt{|k|^2 + \kappa_{k,\lambda}^\ell} \right) \hat{u}^\circ(k) - \frac{1 - \cos \left(t \sqrt{|k|^2 + \kappa_{k,\lambda}^\ell} \right)}{|k|^2 + \kappa_{k,\lambda}^\ell} \hat{v}^\circ(k) \right) e^{ik \cdot x} \\ &\quad \times \left(-\Delta_k - \lambda(\nabla + ik) \cdot a(\nabla + ik) - \kappa_{k,\lambda}^\ell \right) \psi_{k,\lambda}^\ell(x) d^*k, \end{aligned}$$

which can then be estimated as in the case of quantum waves (without loosing any time factor). A similar argument was used in [5, Proof of Proposition 3, Substep 3.2].

We may then conclude that W_λ^ℓ remains close to V_λ^ℓ in a suitable regime, and therefore close to U_λ^ℓ defined by

$$U_\lambda^{\ell;t}(x) := \int_{\mathbb{R}^d} \left(\cos \left(t \sqrt{|k|^2 + \kappa_{k,\lambda}^\ell} \right) \hat{u}^\circ(k) + t \operatorname{sinc} \left(t \sqrt{|k|^2 + \kappa_{k,\lambda}^\ell} \right) \hat{v}^\circ(k) \right) e^{ik \cdot x} d^*k, \quad (5.5)$$

and solution of

$$(\partial_{tt}^2 - \Delta + \kappa_{\lambda, -i\nabla}^\ell) U_\lambda^\ell = 0, \quad U_\lambda^\ell|_{t=0} = u^\circ, \quad \partial_t U_\lambda^\ell|_{t=0} = v^\circ. \quad \square$$

5.3. Proof of Corollary 2. Let $k_0 \in \mathcal{O} \setminus \{0\}$ and consider the case when the initial data in (1.15) is a slow modulation of the plane wave $x \mapsto e^{ik_0 \cdot x}$,

$$\begin{aligned} \partial_{tt}^2 u_{\lambda,\varepsilon} &= \nabla \cdot (\text{Id} + \lambda a) \nabla u_{\lambda,\varepsilon}, \\ u_{\lambda,\varepsilon}(x)|_{t=0} &= \varepsilon^{\frac{d}{2}} e^{ik_0 \cdot x} u^\circ(\varepsilon x), \quad \partial_t u_{\lambda,\varepsilon}(x)|_{t=0} = \varepsilon^{\frac{d}{2}} e^{ik_0 \cdot x} v^\circ(\varepsilon x), \end{aligned}$$

where ε satisfies a scaling relation $\varepsilon := \lambda^\beta$ for some $\beta > 0$. Let $\ell \geq 1$ be fixed. We assume for simplicity that $(\hat{u}^\circ, \hat{v}^\circ)$ is compactly supported in the unit ball B (say) and that \tilde{a} is a trigonometric polynomial with $K := \sup\{1 \vee |\xi| : \xi \in \operatorname{supp} \mathcal{F}\tilde{a}\} < \infty$. There exist $R = R(k_0, \ell, K)$ and $\varepsilon_0 = \varepsilon_0(k_0, \ell, K)$ such that $B_\varepsilon(k_0) \subset \mathbb{R}^d \setminus (\mathcal{R}_R^{K,\ell} \cup \{0\})$. Since $\hat{u}_{\lambda,\varepsilon}(k)|_{t=0} = \varepsilon^{-\frac{d}{2}} \hat{u}^\circ(\frac{1}{\varepsilon}(k - k_0))$ and $\partial_t \hat{u}_{\lambda,\varepsilon}(k)|_{t=0} = \varepsilon^{-\frac{d}{2}} \hat{v}^\circ(\frac{1}{\varepsilon}(k - k_0))$, the proof of Proposition 4.1 yields for all $t \geq 0$, $\varepsilon \leq \varepsilon_0$, and $\lambda \ll_{k_0, \ell, K} 1$,

$$\|u_{\lambda,\varepsilon}^t - U_{\lambda,\varepsilon}^{\ell;t}\|_{L^2} \lesssim_{k_0, \ell, K, u^\circ, v^\circ} \lambda(1 + \lambda^\ell T), \quad (5.6)$$

in terms of

$$\begin{aligned} U_{\lambda,\varepsilon}^{\ell;t}(x) &:= \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} \left(\cos \left(t \sqrt{|k|^2 + \kappa_{k,\lambda}^\ell} \right) \hat{u}^\circ\left(\frac{1}{\varepsilon}(k - k_0)\right) \right. \\ &\quad \left. + t \operatorname{sinc} \left(t \sqrt{|k|^2 + \kappa_{k,\lambda}^\ell} \right) \hat{v}^\circ\left(\frac{1}{\varepsilon}(k - k_0)\right) \right) e^{ik \cdot x} d^*k, \end{aligned}$$

where the smallness constraint on λ ensures that $|k|^2 + \kappa_{k,\lambda}^\ell \geq 0$ on $B_\varepsilon(k_0)$. We now simplify the formula for $U_{\lambda,\varepsilon}^\ell$ in the limit $\lambda, \varepsilon \downarrow 0$. We split the approximate flow $U_{\lambda,\varepsilon}^\ell$ into two contributions $U_{\lambda,\varepsilon}^\ell = \frac{1}{2}(U_{\lambda,\varepsilon,+}^\ell + U_{\lambda,\varepsilon,-}^\ell)$, where

$$U_{\lambda,\varepsilon,\pm}^{\ell;t}(x) := \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\pm it\sqrt{|k|^2 + \kappa_{k,\lambda}^\ell}} \left(\hat{u}^\circ\left(\frac{1}{\varepsilon}(k - k_0)\right) \mp \frac{i\hat{v}^\circ\left(\frac{1}{\varepsilon}(k - k_0)\right)}{\sqrt{|k|^2 + \kappa_{k,\lambda}^\ell}} \right) e^{ik \cdot x} d^*k.$$

Changing variables yields $U_{\lambda,\varepsilon,\pm}^{\ell;t}(x) = \varepsilon^{\frac{d}{2}} e^{ik_0 \cdot x} R_{\lambda,\varepsilon,\pm}^{\ell;t}(\varepsilon x)$ with

$$R_{\lambda,\varepsilon,\pm}^{\ell;t}(x) := \int_{\mathbb{R}^d} e^{\pm it\sqrt{|k_0 + \varepsilon k|^2 + \kappa_{k_0 + \varepsilon k,\lambda}^\ell}} \left(\hat{u}^\circ(k) \mp \frac{i\hat{v}^\circ(k)}{\sqrt{|k_0 + \varepsilon k|^2 + \kappa_{k_0 + \varepsilon k,\lambda}^\ell}} \right) e^{ik \cdot x} d^*k.$$

The boundedness of $k \mapsto \kappa_{k,\lambda}^\ell$ on $\mathbb{R}^d \setminus \mathcal{R}_R^{K\ell}$ (cf. Proposition 5.1) yields for all $\varepsilon \leq \varepsilon_0$ and $\lambda \ll_{k_0,\ell,K} 1$,

$$R_{\lambda,\varepsilon,\pm}^{\ell;t}(x) = \int_{\mathbb{R}^d} e^{\pm it\sqrt{|k_0 + \varepsilon k|^2 + \kappa_{k_0 + \varepsilon k,\lambda}^\ell}} \left(\hat{u}^\circ(k) \mp \frac{i\hat{v}^\circ(k)}{|k_0|} \right) e^{ik \cdot x} d^*k + O_{\ell,v^\circ}(\lambda + \varepsilon), \quad (5.7)$$

where the approximation holds in $L^2(\mathbb{R}^d)$. It remains to make a Taylor expansion of the square-root appearing in the time exponential. We start by expanding the approximate Bloch eigenvalue: for $L \geq \ell$ and $\lambda \ll_{k_0,\ell,K} 1$, using the boundedness of $k \mapsto \kappa_{k,\lambda}^\ell$ on $\mathbb{R}^d \setminus \mathcal{R}_R^{K\ell}$ (cf. Proposition 5.1),

$$\begin{aligned} \sqrt{|k|^2 + \kappa_{k,\lambda}^\ell} &= |k| \sum_{n=0}^L \frac{(-1)^n (2n)!}{(1-2n)4^n (n!)^2} \left(\frac{\kappa_{k,\lambda}^\ell}{|k|^2} \right)^n + O_{k,L}((\kappa_{k,\lambda}^\ell)^{L+1}) \\ &= \sum_{n=0}^L \lambda^n \frac{(-1)^n (2n)!}{(1-2n)4^n (n!)^2} \left(\sum_{m=0}^\ell \lambda^m \nu_k^m \right)^n |k|^{1-2n} + O_{k,\ell,L,K}(\lambda^{L+1}) \\ &= \sum_{n=0}^L \sum_{m=0}^\ell \lambda^{n+m} \left(\frac{(-1)^n (2n)!}{(1-2n)4^n (n!)^2} \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha|=m}} \nu_k^{\alpha_1} \dots \nu_k^{\alpha_n} \right) |k|^{1-2n} + O_{k,\ell,L,K}(\lambda^{\ell+1}). \end{aligned}$$

Given $P \geq 0$, we then replace k by $k_0 + \varepsilon k$ and Taylor expand the corresponding symbol up to order P , using the smoothness of $k \mapsto \nu_k^n$ on $\mathbb{R}^d \setminus \mathcal{R}_R^{K\ell}$ for $n \leq \ell$ (cf. Proposition 5.1),

$$\begin{aligned} \sqrt{|k_0 + \varepsilon k|^2 + \kappa_{k_0 + \varepsilon k,\lambda}^\ell} &= \sum_{n=0}^L \sum_{m=0}^\ell \sum_{p=0}^P \lambda^{n+m} \varepsilon^p |k_0 + \varepsilon k|^{1-2n} k^{\otimes p} \odot C_{n,m,p}^L(k_0) \\ &\quad + O_{k_0,\ell,L,P,K}(\lambda^{\ell+1} + \varepsilon^{P+1}), \end{aligned}$$

where we have set

$$C_{n,m,p}^L(k_0) := \frac{(-1)^n (2n)!}{(1-2n)4^n (n!)^2} \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha|=m}} \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta|=p}} \frac{1}{\beta!} \nabla^{\beta_1} \nu_{k_0}^{\alpha_1} \otimes \dots \otimes \nabla^{\beta_n} \nu_{k_0}^{\alpha_n}$$

(with the convention that the double sum equals 1 if $n = 0$). Finally, noting that

$$\begin{aligned}
|k_0 + \varepsilon k|^{1-2n} &= (|k_0|^2 + 2\varepsilon k \cdot k_0 + \varepsilon^2 |k|^2)^{\frac{1}{2}-n} \\
&= |k_0|^{1-2n} \sum_{s=0}^{\infty} (-1)^s \frac{\prod_{j=0}^{s-1} (2n-1+2j)}{2^s s!} \left(2\varepsilon k \cdot \frac{k_0}{|k_0|^2} + \varepsilon^2 \frac{|k|^2}{|k_0|^2} \right)^s \\
&= |k_0|^{1-2n} \sum_{s=0}^{\infty} (-1)^s \frac{\prod_{j=0}^{s-1} (2n-1+2j)}{2^s s!} \sum_{l=0}^s \binom{s}{l} \varepsilon^{2s-l} |k_0|^{l-2s} |k|^{2(s-l)} \left(2k \cdot \frac{k_0}{|k_0|} \right)^l \\
&= \sum_{r=0}^P \varepsilon^r |k_0|^{1-2n-r} \sum_{\substack{0 \leq l \leq s \\ 2s-l=r}} (-1)^s \frac{\prod_{j=0}^{s-1} (2n-1+2j)}{2^{s-l} l! (s-l)!} \left(\frac{k \cdot k_0}{|k_0|} \right)^l |k|^{r-l} + O_{k_0, P}(\varepsilon^{P+1}),
\end{aligned}$$

and using the boundedness of $k \mapsto \nu_k^n$ on $\mathbb{R}^d \setminus \mathcal{R}_R^{K\ell}$ for $n \leq \ell$ (cf. Proposition 5.1), we conclude for all $\lambda, \varepsilon \ll_{k_0, \ell, P, K} 1$,

$$\sqrt{|k_0 + \varepsilon k|^2 + \kappa_{k_0 + \varepsilon k, \lambda}^\ell} = \sum_{m=0}^{\ell} \sum_{p=0}^P \lambda^m \varepsilon^p k^{\otimes p} \odot C_{m,p}(k_0) + O_{k_0, \ell, P, K}(\lambda^{\ell+1} + \varepsilon^{P+1}),$$

where we have set

$$\begin{aligned}
C_{m,p}(k_0) &:= \sum_{r=0}^p \sum_{n=0}^m |k_0|^{1-2n-r} \left(\sum_{\substack{0 \leq l \leq s \\ 2s-l=r}} \frac{(-1)^s \prod_{j=0}^{s-1} (2n-1+2j)}{2^{s-l} l! (s-l)!} \left(\frac{k_0}{|k_0|} \right)^{\otimes l} \otimes \text{Id}^{\otimes (s-l)} \right) \\
&\quad \otimes \left(\frac{(-1)^n (2n)!}{(1-2n)4^n (n!)^2} \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha|=m-n}} \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta|=p-r}} \frac{1}{\beta!} \nabla^{\beta_1} \nu_{k_0}^{\alpha_1} \otimes \dots \otimes \nabla^{\beta_n} \nu_{k_0}^{\alpha_n} \right).
\end{aligned}$$

Note that the first terms in this series can be explicitly computed,

$$C_{0,0} = |k_0|, \quad C_{1,0} = \frac{\nu_{k_0}^0}{2|k_0|}, \quad C_{0,1} = \frac{k_0}{|k_0|}, \quad C_{0,2} = \frac{|k_0|^2 \text{Id} - k_0 \otimes k_0}{2|k_0|^3}.$$

Injecting this expansion into (5.7), we are lead to

$$\begin{aligned}
R_{\lambda, \varepsilon, \pm}^{\ell; t}(x) &= \int_{\mathbb{R}^d} e^{\pm it \sum_{m=0}^{\ell} \sum_{p=0}^P \lambda^m \varepsilon^p k^{\otimes p} \odot C_{m,p}(k_0)} \left(\hat{u}^\circ(k) \mp \frac{i\hat{v}^\circ(k)}{|k_0|} \right) e^{ik \cdot x} d^* k \\
&\quad + O_{k_0, \ell, P, K, u^\circ, v^\circ}(\lambda + \varepsilon + t(\lambda^{\ell+1} + \varepsilon^{P+1})),
\end{aligned}$$

where the approximation holds in $L^2(\mathbb{R}^d)$. In view of the scaling relation $\varepsilon = \lambda^\beta$, we naturally choose $P = \lfloor \ell/\beta \rfloor$. Injecting the explicit form of $C_{0,0}, C_{1,0}, C_{0,1}$ and combining with (5.6), the conclusion follows. \square

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