



HAL
open science

Robustness of the pathwise structure of fluctuations in stochastic homogenization

Mitia Duerinckx, Antoine Gloria, Felix Otto

► **To cite this version:**

Mitia Duerinckx, Antoine Gloria, Felix Otto. Robustness of the pathwise structure of fluctuations in stochastic homogenization. 2018. hal-01876842v1

HAL Id: hal-01876842

<https://hal.science/hal-01876842v1>

Preprint submitted on 18 Sep 2018 (v1), last revised 29 Nov 2021 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ROBUSTNESS OF THE PATHWISE STRUCTURE OF FLUCTUATIONS IN STOCHASTIC HOMOGENIZATION

MITIA DUERINCKX, ANTOINE GLORIA, AND FELIX OTTO

ABSTRACT. We consider a linear elliptic system in divergence form with random coefficients and study the random fluctuations of large-scale averages of the field and the flux of the solution. In a previous contribution, for discrete elliptic equations with independent and identically distributed conductances, we developed a theory of fluctuations based on the notion of homogenization commutator, defined as the flux minus the homogenized coefficients times the field of the solution: we proved that the two-scale expansion of this special quantity is accurate at leading order when averaged on large scales (as opposed to the two-scale expansion of the field and flux taken separately) and that the large-scale fluctuations of both the field and the flux can be recovered from those of the commutator. This implies that the large-scale fluctuations of the commutator of the corrector drive all other large-scale fluctuations to leading order, which we refer to as the *pathwise structure* of fluctuations in stochastic homogenization. In the present contribution we extend this result in two directions: we treat *continuum* elliptic (possibly non-symmetric) systems and with strongly *correlated* coefficient fields (Gaussian-like with a covariance function that displays an arbitrarily slow algebraic decay at infinity). Our main result shows that the two-scale expansion of the homogenization commutator is still accurate to leading order when averaged on large scales, which illustrates the robustness of the pathwise structure of fluctuations.

CONTENTS

1. Introduction	1
2. Main results and structure of the proof	5
2.1. Notation and statement of the main results	5
2.2. Structure of the proof	8
3. Proofs	12
3.1. Proof of Lemma 2.4: Representation formulas	12
3.2. Dyadic partitions and reformulation of weighted norms	14
3.3. Proof of Proposition 2.5: Main estimates	15
3.4. Proof of Proposition 2.6: Main estimates (cont'd)	19
3.5. Proof of the main results	24
Acknowledgements	28
References	28

1. INTRODUCTION

This article constitutes the second part of a series that develops a theory of fluctuations in stochastic homogenization of linear elliptic (non-necessarily symmetric) systems in divergence form. In the first part [5], we provided a complete picture of our theory (with

optimal error estimates and convergence rates) in the simplified framework of discrete elliptic equations with independent and identically distributed (iid) conductances. We proved three main results: the pathwise structure of fluctuations, the Gaussian character of fluctuations, and the identification of the limiting covariance structure. In the present contribution, we focus on the fundamental *pathwise* aspect of the theory, based on the accuracy of the two-scale expansion for fluctuations of the homogenization commutator of the solution operator. The aim is to extend the validity of this pathwise structure of fluctuations to continuum (non-symmetric) systems with strongly correlated coefficient fields. We take this as a sign of the *robustness* of the pathwise structure. We mainly consider the class of Gaussian coefficient fields (with covariance function the decay of which can be arbitrarily slow at infinity). Questions regarding the convergence in law of the rescaled standard homogenization commutator are addressed in the forthcoming contribution [2] in the case of correlated Gaussian fields. In [6], we further explain how this whole pathwise theory of fluctuations naturally extends to higher orders. For a thorough discussion of the literature, we refer the reader to the introduction of the companion article [5].

Let \mathbf{a} be a stationary and ergodic random coefficient field that is bounded,

$$|\mathbf{a}(x)\xi| \leq |\xi|, \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } x \in \mathbb{R}^d, \quad (1.1)$$

and satisfies the ellipticity property

$$\int \nabla u \cdot \mathbf{a} \nabla u \geq \lambda \int |\nabla u|^2, \quad \text{for all smooth and compactly supported } u, \quad (1.2)$$

for some $\lambda > 0$ (this notion of functional coercivity is weaker than pointwise ellipticity for systems). Throughout the article we use scalar notation, but no iota in the proofs would change for systems under assumptions (1.1) and (1.2). For all $\varepsilon > 0$ we set $\mathbf{a}_\varepsilon := \mathbf{a}(\frac{\cdot}{\varepsilon})$, and for all deterministic vector fields $f \in C_c^\infty(\mathbb{R}^d)^d$ we consider the random family $(u_\varepsilon)_{\varepsilon > 0}$ of unique Lax-Milgram solutions in \mathbb{R}^d (which henceforth means the unique weak solutions in $\dot{H}^1(\mathbb{R}^d)$) of the rescaled problems

$$-\nabla \cdot \mathbf{a}_\varepsilon \nabla u_\varepsilon = \nabla \cdot f. \quad (1.3)$$

(The choice of considering an equation on the whole space rather than on a bounded set allows us to focus on fluctuations in the bulk, and avoid effects of boundary layers. The choice of taking a right-hand side (RHS) in divergence form allows us to treat all dimensions at once.) It is known since the pioneering work of Papanicolaou and Varadhan [14] and Kozlov [12] that, almost surely, u_ε converges weakly (in $\dot{H}^1(\mathbb{R}^d)$) as $\varepsilon \downarrow 0$ to the unique Lax-Milgram solution \bar{u} in \mathbb{R}^d of

$$-\nabla \cdot \bar{\mathbf{a}} \nabla \bar{u} = \nabla \cdot f, \quad (1.4)$$

where $\bar{\mathbf{a}}$ is a deterministic and constant matrix that only depends on the law of \mathbf{a} . More precisely, for any direction $e \in \mathbb{R}^d$, the projection $\bar{\mathbf{a}}e$ is the expectation of the flux of the corrector in the direction e ,

$$\bar{\mathbf{a}}e = \mathbb{E}[\mathbf{a}(\nabla \phi_e + e)], \quad (1.5)$$

where the corrector ϕ_e is the unique (up to a random additive constant) almost-sure solution of the corrector equation in \mathbb{R}^d ,

$$-\nabla \cdot \mathbf{a}(\nabla \phi_e + e) = 0,$$

in the class of functions the gradient of which is stationary, has finite second moment, and has zero expectation. We denote by $\phi = (\phi_i)_{i=1}^d$ the vector field the entries of which are the correctors ϕ_i in the canonical directions e_i of \mathbb{R}^d .

In [5], we developed a complete theory of fluctuations in stochastic homogenization of discrete elliptic equations with iid conductances (see also [11] for heuristic arguments). The key in our theory consists in focusing on the *homogenization commutator of the solution* $\mathbf{a}_\varepsilon \nabla u_\varepsilon - \bar{\mathbf{a}} \nabla u_\varepsilon$ and in studying its relation to the (*standard*) *homogenization commutator* $\Xi := (\Xi_i)_{i=1}^d$, where the harmonic coordinates $x \mapsto x_i + \phi_i(x)$ play the role of u_ε ,

$$\Xi_i := \mathbf{a}(\nabla \phi_i + e_i) - \bar{\mathbf{a}}(\nabla \phi_i + e_i), \quad \Xi_{ij} := (\Xi_i)_j. \quad (1.6)$$

In the framework of [5], we showed that this stationary random (non-symmetric) 2-tensor field Ξ , which by (1.5) has vanishing expectation, enjoys the following three crucial properties (which we reformulate below in the nonsymmetric continuum setting of the present contribution):

- (I) First and most importantly, the two-scale expansion of the *homogenization commutator* of the solution

$$\mathbf{a}_\varepsilon \nabla u_\varepsilon - \bar{\mathbf{a}} \nabla u_\varepsilon - \mathbb{E}[\mathbf{a}_\varepsilon \nabla u_\varepsilon - \bar{\mathbf{a}} \nabla u_\varepsilon] \approx \Xi_i(\frac{\cdot}{\varepsilon}) \nabla_i \bar{u} \quad (1.7)$$

is accurate in the fluctuation scaling in the sense that for all $g \in C_c^\infty(\mathbb{R}^d)^d$ and $q < \infty$,

$$\begin{aligned} \mathbb{E} \left[\left| \int_{\mathbb{R}^d} g \cdot (\mathbf{a}_\varepsilon \nabla u_\varepsilon - \bar{\mathbf{a}} \nabla u_\varepsilon - \mathbb{E}[\mathbf{a}_\varepsilon \nabla u_\varepsilon - \bar{\mathbf{a}} \nabla u_\varepsilon]) - \int_{\mathbb{R}^d} g \cdot \Xi_i(\frac{\cdot}{\varepsilon}) \nabla_i \bar{u} \right|^q \right]^{\frac{1}{q}} \\ \lesssim_{f,g,q} \varepsilon \mathbb{E} \left[\left| \int_{\mathbb{R}^d} g \cdot \Xi_i(\frac{\cdot}{\varepsilon}) \nabla_i \bar{u} \right|^q \right]^{\frac{1}{q}}, \quad (1.8) \end{aligned}$$

up to a $|\log \varepsilon|$ factor in the critical dimension $d = 2$.

- (II) Second, both the fluctuations of the field ∇u_ε and of the flux $\mathbf{a}_\varepsilon \nabla u_\varepsilon$ can be recovered through *deterministic* projections from those of the homogenization commutator $\mathbf{a}_\varepsilon \nabla u_\varepsilon - \bar{\mathbf{a}} \nabla u_\varepsilon$, which shows that no information is lost by passing to the homogenization commutator. More precisely,

$$\begin{aligned} \int_{\mathbb{R}^d} g \cdot \nabla(u_\varepsilon - \mathbb{E}[u_\varepsilon]) &= - \int_{\mathbb{R}^d} (\bar{\mathcal{P}}_H^* g) \cdot (\mathbf{a}_\varepsilon \nabla u_\varepsilon - \bar{\mathbf{a}} \nabla u_\varepsilon - \mathbb{E}[\mathbf{a}_\varepsilon \nabla u_\varepsilon - \bar{\mathbf{a}} \nabla u_\varepsilon]), \quad (1.9) \\ \int_{\mathbb{R}^d} g \cdot (\mathbf{a}_\varepsilon \nabla u_\varepsilon - \mathbb{E}[\mathbf{a}_\varepsilon \nabla u_\varepsilon]) &= \int_{\mathbb{R}^d} (\bar{\mathcal{P}}_L^* g) \cdot (\mathbf{a}_\varepsilon \nabla u_\varepsilon - \bar{\mathbf{a}} \nabla u_\varepsilon - \mathbb{E}[\mathbf{a}_\varepsilon \nabla u_\varepsilon - \bar{\mathbf{a}} \nabla u_\varepsilon]), \end{aligned}$$

in terms of the Helmholtz and Leray projections in $L^2(\mathbb{R}^d)^d$,

$$\begin{aligned} \bar{\mathcal{P}}_H &:= \nabla(\nabla \cdot \bar{\mathbf{a}} \nabla)^{-1} \nabla \cdot, & \bar{\mathcal{P}}_L &:= \text{Id} - \bar{\mathcal{P}}_H \bar{\mathbf{a}}, \\ \bar{\mathcal{P}}_H^* &:= \nabla(\nabla \cdot \bar{\mathbf{a}}^* \nabla)^{-1} \nabla \cdot, & \bar{\mathcal{P}}_L^* &:= \text{Id} - \bar{\mathcal{P}}_H \bar{\mathbf{a}}^*, \end{aligned} \quad (1.10)$$

where $\bar{\mathbf{a}}^*$ denotes the transpose of $\bar{\mathbf{a}}$. In addition, the fluctuations of the field $\nabla \phi(\frac{\cdot}{\varepsilon})$ and of the flux $\mathbf{a}_\varepsilon \nabla \phi(\frac{\cdot}{\varepsilon})$ of the corrector are also determined by those of the standard commutator $\Xi(\frac{\cdot}{\varepsilon})$: the definition of Ξ leads to $-\nabla \cdot \bar{\mathbf{a}} \nabla \phi_i = \nabla \cdot \Xi_i$ and $\mathbf{a}(\nabla \phi_i + e_i) - \bar{\mathbf{a}} e_i = \Xi_i + \bar{\mathbf{a}} \nabla \phi_i$, to the effect of $\nabla \phi_i = -\bar{\mathcal{P}}_H \Xi_i$ and $\mathbf{a}(\nabla \phi_i + e_i) - \bar{\mathbf{a}} e_i =$

$(\text{Id} - \bar{\mathbf{a}}\bar{\mathcal{P}}_H)\Xi_i$ in the stationary sense, and hence, formally,

$$\begin{aligned} \int_{\mathbb{R}^d} F : \nabla\phi(\tfrac{\cdot}{\varepsilon}) &= - \int_{\mathbb{R}^d} \bar{\mathcal{P}}_H^* F : \Xi(\tfrac{\cdot}{\varepsilon}), \\ \int_{\mathbb{R}^d} F : (\mathbf{a}_\varepsilon(\nabla\phi(\tfrac{\cdot}{\varepsilon}) + \text{Id}) - \bar{\mathbf{a}}) &= \int_{\mathbb{R}^d} \mathcal{P}_L^* F : \Xi(\tfrac{\cdot}{\varepsilon}), \end{aligned} \quad (1.11)$$

where $\bar{\mathcal{P}}_H^*$ and $\bar{\mathcal{P}}_L^*$ act on the second index of the tensor field $F \in C_c^\infty(\mathbb{R}^d)^{d \times d}$. A suitable sense to these identities is given in Corollary 1.

(III) Third, the standard homogenization commutator Ξ is an approximately local function of the coefficients \mathbf{a} , which allows to infer the large-scale behavior of Ξ from the large-scale behavior of \mathbf{a} itself. This locality is best seen when formally computing partial derivatives of Ξ with respect to \mathbf{a} , cf. [5, equation (1.10)].

Let us comment on the *pathwise structure* of fluctuations revealed in (I)–(II). Combined with (1.8), identities (1.9) and (1.11) imply that the fluctuations of ∇u_ε , $\mathbf{a}_\varepsilon \nabla u_\varepsilon$, $\nabla\phi(\tfrac{\cdot}{\varepsilon})$, and $\mathbf{a}_\varepsilon \nabla\phi(\tfrac{\cdot}{\varepsilon})$ are determined to leading order by those of $\Xi(\tfrac{\cdot}{\varepsilon})$ in a strong norm in probability. This almost sure (“pathwise” in the language of SPDE) relation thus reduces the leading-order fluctuations of all quantities of interest to those of the sole homogenization commutator Ξ in a pathwise sense. As we emphasized in [5], besides its theoretical importance, this *pathwise structure* is bound to affect multi-scale computing and uncertainty quantification in an essential way. For weakly correlated coefficient field \mathbf{a} we expect from property (III) that $\Xi(\tfrac{\cdot}{\varepsilon})$ displays the CLT scaling and that the rescaling $\varepsilon^{-d/2}\Xi(\tfrac{\cdot}{\varepsilon})$ converges in law to a Gaussian white noise, so that the pathwise structure (I)–(II) then leads to the known scaling limit results for the different quantities of interest in stochastic homogenization (which we indeed proved in the setting of [5]).

In the present contribution, we focus on the pathwise structure (I)–(II). More precisely, we consider the class of Gaussian coefficient fields, the covariance function of which decays at infinity at some fixed (yet arbitrary) algebraic rate $(1+|x|)^{-\beta}$ parametrized by $\beta > 0$, and we show that properties (I)–(II) still hold for this whole Gaussian class, which illustrates the surprising robustness of the pathwise structure with respect to the large-scale behavior of the homogenization commutator. Indeed, in dimension $d = 1$ (in which case the quantities under investigation are simpler and explicit), two typical behaviors have been identified in terms of the scaling limit of the homogenization commutator, depending on the parameter β (cf. [1]),

- For $\beta > d = 1$: The commutator Ξ displays the CLT scaling and $\varepsilon^{-\frac{d}{2}}\Xi(\tfrac{\cdot}{\varepsilon})$ converges to a white noise (Gaussian fluctuations, local limiting covariance structure).
- For $0 < \beta < d = 1$: The suitable rescaling $\varepsilon^{-\frac{\beta}{2}}\Xi(\tfrac{\cdot}{\varepsilon})$ converges up to a subsequence to a fractional Gaussian field (Gaussian fluctuations, nonlocal limiting covariance structure, potentially no uniqueness of the limit).

(Note that a different, non-Gaussian behavior may also occur in degenerate cases, cf. [10, 13]; see also the second item in Remark 2.1.) In particular, the pathwise result holds in these two examples whereas the rescaled homogenization commutator Ξ does not necessarily converge to white noise or may even not converge at all. The identification of the scaling limit of the homogenization commutator in higher dimensions is a separate question and is addressed in [2] for the whole range of values of $\beta > 0$, where we investigate the consequences of the locality of Ξ with respect to the coefficient field (cf. (III) above),

combining techniques developed in [8] with Malliavin calculus. This work will in particular extend [1] to dimensions $d > 1$.

Although we focus on the model case of Gaussian coefficient fields, the arguments that we give in this contribution are robust enough to cover the general setting of weighted functional inequalities introduced and studied in [3, 4], and therefore to treat all the models of coefficient fields considered in [16], a reference textbook on heterogenous materials. This makes the results of this contribution not only of theoretical but also of practical interest.

2. MAIN RESULTS AND STRUCTURE OF THE PROOF

2.1. Notation and statement of the main results. For some $k \geq 1$ let a be an \mathbb{R}^k -valued Gaussian random field, constructed on some probability space (Ω, \mathbb{P}) (with expectation \mathbb{E}), which is stationary and centered, and thus characterized by its covariance function

$$c(x) := \mathbb{E}[a(x) \otimes a(0)], \quad c : \mathbb{R}^d \rightarrow \mathbb{R}^{k \times k}.$$

We assume that the covariance function decays algebraically at infinity in the sense that there exist $\beta, C_0 > 0$ such that for all $x \in \mathbb{R}^d$

$$\frac{1}{C_0}(1 + |x|)^{-\beta} \leq |c(x)| \leq C_0(1 + |x|)^{-\beta}. \quad (2.1)$$

Given a map $h \in C_b^1(\mathbb{R}^k)^{d \times d}$, we define $\mathbf{a} : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ by $\mathbf{a}(x) = h(a(x))$, and assume that it satisfies the conditions (1.1) and (1.2) almost surely. We then (abusively) call the coefficient field \mathbf{a} *Gaussian with parameter* $\beta > 0$. From a technical point of view, we shall rely on (and frequently refer to) results and methods developed in [3, 8, 7]. If \mathbf{a} is Gaussian with parameter β , then \mathbf{a} is ergodic, and we have existence and uniqueness of correctors ϕ and of the homogenized coefficients $\bar{\mathbf{a}}$ (cf. Lemma 2.2 below).

Throughout the article, we use the notation $\lesssim_{(\dots)}$ (resp. $\gtrsim_{(\dots)}$) for $\leq C \times$ (resp. $\geq C \times$), where the multiplicative constant C depends on $d, \lambda, \beta, \|\nabla h\|_{L^\infty}$, on the constant C_0 in (2.1), and on the additional parameters “ \dots ” if any. We write $\simeq_{(\dots)}$ when both $\lesssim_{(\dots)}$ and $\gtrsim_{(\dots)}$ hold. In an assumption, we use the notation $\ll_{(\dots)}$ for $\leq \frac{1}{C} \times$ for some (large enough) constant $C \simeq_{(\dots)} 1$.

In this setting, we consider the fluctuations of the field ∇u_ε and of the flux $\mathbf{a}_\varepsilon \nabla u_\varepsilon$ of the solution of (1.3), as encoded by the random bilinear functionals $I_1^\varepsilon : (f, g) \mapsto I_1^\varepsilon(f, g)$ and $I_2^\varepsilon : (f, g) \mapsto I_2^\varepsilon(f, g)$ defined for all $f, g \in C_c^\infty(\mathbb{R}^d)^d$ by

$$\begin{aligned} I_1^\varepsilon(f, g) &:= \int_{\mathbb{R}^d} g \cdot \nabla(u_\varepsilon - \mathbb{E}[u_\varepsilon]), \\ I_2^\varepsilon(f, g) &:= \int_{\mathbb{R}^d} g \cdot (\mathbf{a}_\varepsilon \nabla u_\varepsilon - \mathbb{E}[\mathbf{a}_\varepsilon \nabla u_\varepsilon]). \end{aligned}$$

We further encode the fluctuations of the corrector field $\nabla \phi$ and flux $\mathbf{a}(\nabla \phi + \text{Id})$ by the (centered) random linear functionals $J_1^\varepsilon : F \mapsto J_1^\varepsilon(F)$ and $J_2^\varepsilon : F \mapsto J_2^\varepsilon(F)$ defined for all $F \in C_c^\infty(\mathbb{R}^d)^{d \times d}$ by

$$\begin{aligned} J_1^\varepsilon(F) &:= \int_{\mathbb{R}^d} F(x) : \nabla \phi\left(\frac{x}{\varepsilon}\right) dx, \\ J_2^\varepsilon(F) &:= \int_{\mathbb{R}^d} F(x) : (\mathbf{a}_\varepsilon(x)(\nabla \phi\left(\frac{x}{\varepsilon}\right) + \text{Id}) - \bar{\mathbf{a}}) dx. \end{aligned}$$

As explained above, a crucial role is played by the (standard) homogenization commutator Ξ defined in (1.6), and by the error in the two-scale expansion of the homogenization commutator of the solution. These quantities are encoded by the random linear functionals $J_0^\varepsilon : F \mapsto J_0^\varepsilon(F)$ and $E^\varepsilon : (f, g) \mapsto E^\varepsilon(f, g)$ defined for all $F \in C_c^\infty(\mathbb{R}^d)^{d \times d}$ and all $f, g \in C_c^\infty(\mathbb{R}^d)^d$ by

$$\begin{aligned} J_0^\varepsilon(F) &:= \int_{\mathbb{R}^d} F(x) : \Xi\left(\frac{x}{\varepsilon}\right) dx, \\ E^\varepsilon(f, g) &:= \int_{\mathbb{R}^d} g \cdot (\mathbf{a}_\varepsilon \nabla u_\varepsilon - \bar{\mathbf{a}} \nabla u_\varepsilon - \mathbb{E}[\mathbf{a}_\varepsilon \nabla u_\varepsilon - \bar{\mathbf{a}} \nabla u_\varepsilon]) - \int_{\mathbb{R}^d} g \cdot \Xi_i\left(\frac{\cdot}{\varepsilon}\right) \nabla_i \bar{u}. \end{aligned}$$

Since the case $d = 1$ is much simpler and well-understood [9], we only focus in the sequel on dimensions $d > 1$.

We first establish the following boundedness result for J_0^ε , establishing the suitable β -dependent scaling for the fluctuations of the homogenization commutator (see also [7, Theorem 1]).

Proposition 1 (Fluctuation scaling). *Let $d \geq 2$, assume that the coefficient field \mathbf{a} is Gaussian with parameter $\beta > 0$, define $\pi_* : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by*

$$\pi_*(t) := \begin{cases} (1+t)^\beta & : \beta < d, \\ (1+t)^d \log^{-1}(2+t) & : \beta = d, \\ (1+t)^d & : \beta > d, \end{cases} \quad (2.2)$$

and define the rescaled functional

$$\widehat{J}_0^\varepsilon := \pi_*\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2}} J_0^\varepsilon.$$

For all $0 < \varepsilon \leq 1$, $F \in C_c^\infty(\mathbb{R}^d)^{d \times d}$, $0 < p - 1 \ll 1$, and $\alpha > \frac{d - (\beta \wedge d)}{2} + d \frac{p-1}{2p}$, we have

$$|\widehat{J}_0^\varepsilon(F)| \leq C_{\alpha, p}^{\varepsilon, F} (\|w_1^\alpha F\|_{L^{2p}} + \mathbf{1}_{\beta \leq d} \| [F]_2 \|_{L^p}), \quad (2.3)$$

where $w_1(z) := 1 + |z|$, $[F]_2(x) := (\int_{B(x)} |F|^2)^{\frac{1}{2}}$, and where $C_{\alpha, p}^{\varepsilon, F}$ is a random variable with stretched exponential moments: there exists $\gamma_1 \simeq 1$ such that

$$\sup_{0 < \varepsilon < 1} \mathbb{E} \left[\exp \left(\frac{1}{C_{\alpha, p}} (C_{\alpha, p}^{\varepsilon, F})^{\gamma_1} \right) \right] \leq 2$$

for some (deterministic) constant $C_{\alpha, p} \simeq_{\alpha, p} 1$. \diamond

Our next main result establishes the smallness of the error E^ε in the two-scale expansion of the homogenization commutator in the fluctuation scaling, which is the key to the pathwise theory of fluctuations.

Theorem 1 (Pathwise structure of fluctuations). *Let $d \geq 2$, assume that the coefficient field \mathbf{a} is Gaussian with parameter $\beta > 0$, let π_* be defined in (2.2), define $\mu_* : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by*

$$\mu_*(r) := \begin{cases} 1 & : \beta > 2, d > 2, \\ \log(2+r)^{\frac{1}{2}} & : \beta > 2, d = 2, \\ \log(2+r) & : \beta = 2, d \geq 2, \\ (1+r)^{1-\frac{\beta}{2}} & : \beta < 2, d \geq 2, \end{cases} \quad (2.4)$$

set $\mu_*(z) := \mu_*(|z|)$, recall $w_1(z) := 1 + |z|$, and consider the rescaled functionals

$$\widehat{E}^\varepsilon := \pi_*(\tfrac{1}{\varepsilon})^{\frac{1}{2}} E^\varepsilon, \quad \widehat{J}_0^\varepsilon := \pi_*(\tfrac{1}{\varepsilon})^{\frac{1}{2}} J_0^\varepsilon, \quad \widehat{I}_1^\varepsilon := \pi_*(\tfrac{1}{\varepsilon})^{\frac{1}{2}} I_1^\varepsilon.$$

For all $0 < \varepsilon \leq 1$, $f, g \in C_c^\infty(\mathbb{R}^d)^d$, $0 < p - 1 \ll 1$, and $\alpha > \frac{d - (\beta \wedge d)}{2} + d \frac{p-1}{4p}$, we have

$$\begin{aligned} |\widehat{E}^\varepsilon(f, g)| &\leq \varepsilon \mu_*(\tfrac{1}{\varepsilon}) C_{\alpha, p}^{\varepsilon, f, g} \left(\|\mu_* \nabla f\|_{L^4} \|w_1^\alpha g\|_{L^{4p}} + \|\mu_* \nabla g\|_{L^4} \|w_1^\alpha f\|_{L^{4p}} \right. \\ &\quad \left. + \mathbb{1}_{\beta \leq d} (\|\mu_* \nabla f\|_{L^2} \|g\|_{L^2 \cap L^{2p}} + \|\mu_* \nabla g\|_{L^2} \|f\|_{L^2 \cap L^{2p}}) \right), \end{aligned} \quad (2.5)$$

where $C_{\alpha, p}^{\varepsilon, f, g}$ is a random variable with stretched exponential moments: there exists $\gamma_2 \simeq 1$ such that

$$\sup_{0 < \varepsilon < 1} \mathbb{E} \left[\exp \left(\frac{1}{C_{\alpha, p}} (C_{\alpha, p}^{\varepsilon, f, g})^{\gamma_2} \right) \right] \leq 2$$

for some (deterministic) constant $C_{\alpha, p} \simeq_{\alpha, p} 1$. \diamond

Remark 2.1.

- The exponents γ_1 and γ_2 in the above results are made explicit in the proofs: we obtain

$$\gamma_1 = \frac{2(\beta \wedge d)}{\beta \wedge d + d} - \eta \quad \text{and} \quad \gamma_2 = \frac{2(\beta \wedge d)}{2(\beta \wedge d) + 3d} - \eta,$$

where $\eta > 0$ can be chosen arbitrarily; these values are not expected to be optimal.

- The ε -scaling in the above results is believed to be optimal. The rescaling in the definition of $\widehat{J}_0^\varepsilon$ and $\widehat{I}_1^\varepsilon$ is natural since it precisely leads to a variance of order 1 for averages of the coefficient field \mathbf{a} itself. For some non-generic examples, (2.3) may however overestimate the variance. In dimension $d = 1$, one may indeed construct explicit Gaussian coefficient fields \mathbf{a} such that fluctuations of the homogenization commutator J_0^ε are of smaller order than what (2.3) predicts [15, 10, 13], in which case the suitable rescaling of J_0^ε has a non-Gaussian limit. In such situations, the pathwise property (2.5) implies that the variance of $\widehat{I}_1^\varepsilon$ is overestimated as well, and the validity of a higher-order pathwise relation is an open question. General necessary and sufficient conditions for the sharpness of (2.3) are studied in [2].
- The proofs of the above results are robust enough to cover the general setting of weighted functional inequalities introduced and studied in [3, 4]. In the case of functional inequalities with oscillation, we may indeed use Cauchy-Schwarz' inequality and an energy estimate to replace the perturbed functions $\tilde{\phi}$ and $\nabla \tilde{u}$ appearing in the representation formula (3.3) below by their unperturbed versions ϕ and ∇u . This allows to conclude whenever the weight has a superalgebraic decay (see indeed [7, proof of Theorem 4]). If one is only interested in Gaussian coefficient fields, one may replace the use of functional inequalities by a direct use of the Brascamp-Lieb inequality, which would allow to slightly shorten some of the proofs (and potentially improve the norms of the test functions F, f, g). \diamond

In view of the identities (1.9) and (1.11), the above results imply that the large-scale fluctuations of $\widehat{I}_1^\varepsilon$, $\widehat{I}_2^\varepsilon$, $\widehat{J}_1^\varepsilon$, and $\widehat{J}_2^\varepsilon$ are driven by those of J_0^ε in a pathwise sense (see [5, Corollary 2.4] for details).

Corollary 1 ([5]). *Let $d \geq 2$, assume that the coefficient field \mathbf{a} is Gaussian with parameter $\beta > 0$, let π_* and μ_* be defined by (2.2) and (2.4), let $\bar{\mathcal{P}}_H$, $\bar{\mathcal{P}}_H^*$, and $\bar{\mathcal{P}}_L^*$ be as in (1.10), and recall the rescaled functionals*

$$\widehat{J}_0^\varepsilon := \pi_*\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2}} J_0^\varepsilon, \quad \widehat{I}_i^\varepsilon := \pi_*\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2}} I_i^\varepsilon, \quad \widehat{J}_i^\varepsilon := \pi_*\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2}} J_i^\varepsilon, \quad i = 1, 2.$$

For all $\varepsilon > 0$ and $f, g \in C_c^\infty(\mathbb{R}^d)^d$, we have for all $0 < p - 1 \ll 1$ and $\alpha > \frac{d - (\beta \wedge d)}{2} + d\frac{p-1}{4p}$,

$$\begin{aligned} & |\widehat{I}_1^\varepsilon(f, g) - \widehat{J}_0^\varepsilon(\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_H^* g)| + |\widehat{I}_2^\varepsilon(f, g) + \widehat{J}_0^\varepsilon(\bar{\mathcal{P}}_H f \otimes \bar{\mathcal{P}}_L^* g)| \\ & \leq \varepsilon \mu_*\left(\frac{1}{\varepsilon}\right) \mathcal{C}_{\alpha, p}^{\varepsilon, f, g} \left(\|\mu_* \nabla f\|_{L^4} \|w_1^\alpha g\|_{L^{4p}} + \|\mu_* \nabla g\|_{L^4} \|w_1^\alpha f\|_{L^{4p}} \right. \\ & \quad \left. + \mathbf{1}_{\beta \leq d} (\|\mu_* \nabla f\|_{L^2} \|g\|_{L^2 \cap L^{2p}} + \|\mu_* \nabla g\|_{L^2} \|f\|_{L^2 \cap L^{2p}}) \right), \end{aligned}$$

where $\mathcal{C}_{\alpha, p}^{\varepsilon, f, g}$ is a random variable with stretched exponential moments independent of ε as in the statement of Theorem 1. In addition, for all $\varepsilon > 0$ and $F \in C_c^\infty(\mathbb{R}^d)^{d \times d}$, we have almost surely

$$J_1^\varepsilon(F) = -J_0^\varepsilon(\bar{\mathcal{P}}_H^* F), \quad J_2^\varepsilon(F) = J_0^\varepsilon(\bar{\mathcal{P}}_L^* F),$$

where in particular we may give an almost sure meaning to $J_0^\varepsilon(\bar{\mathcal{P}}_H^* F)$ and $J_0^\varepsilon(\bar{\mathcal{P}}_L^* F)$ for all $F \in C_c^\infty(\mathbb{R}^d)^{d \times d}$, even when $\bar{\mathcal{P}}_H^* F$ and $\bar{\mathcal{P}}_L^* F$ do not have integrable decay. \diamond

2.2. Structure of the proof. We describe the string of arguments that leads to Proposition 1 and Theorem 1. Next to the corrector ϕ , we first need to recall the notion of flux corrector σ , which was recently introduced in the stochastic setting in [8, Lemma 1] and allows to put the equation for the two-scale homogenization error in divergence form (cf. (3.26) below). The extended corrector (ϕ, σ) is only defined up to an additive constant, and we henceforth choose the anchoring $f_B(\phi, \sigma) = 0$ on the centered unit ball B .

Lemma 2.2 ([8]). *Let the coefficient field \mathbf{a} be stationary and ergodic. Then there exist two random tensor fields $(\phi_i)_{1 \leq i \leq d}$ and $(\sigma_{ijk})_{1 \leq i, j, k \leq d}$ with the following properties: The gradient fields $\nabla \phi_i$ and $\nabla \sigma_{ijk}$ are stationary¹ and have finite second moments and vanishing expectations:*

$$\mathbb{E} [|\nabla \phi_i|^2] \leq \frac{1}{\lambda^2}, \quad \sum_{j, k=1}^d \mathbb{E} [|\nabla \sigma_{ijk}|^2] \leq 4d \left(\frac{1}{\lambda^2} + 1 \right), \quad \mathbb{E} [\nabla \phi_i] = \mathbb{E} [\nabla \sigma_{ijk}] = 0. \quad (2.6)$$

Moreover, for all i , the field $\sigma_i := (\sigma_{ijk})_{1 \leq j, k \leq d}$ is skew-symmetric, that is,

$$\sigma_{ijk} = -\sigma_{ikj}. \quad (2.7)$$

Finally, the following equations are satisfied a.s. in the distributional sense on \mathbb{R}^d ,

$$-\nabla \cdot \mathbf{a}(\nabla \phi_i + e_i) = 0, \quad (2.8)$$

$$\nabla \cdot \sigma_i = q_i - \mathbb{E} [q_i], \quad (2.9)$$

$$-\Delta \sigma_{ijk} = \partial_j q_{ik} - \partial_k q_{ij},$$

¹That is, $\nabla \phi_i(\mathbf{a}; \cdot + z) = \nabla \phi_i(\mathbf{a}(\cdot + z); \cdot)$ and $\nabla \sigma_{ijk}(\mathbf{a}; \cdot + z) = \nabla \sigma_{ijk}(\mathbf{a}(\cdot + z); \cdot)$ a.e. in \mathbb{R}^d , for all shift vectors $z \in \mathbb{R}^d$.

where $q_i =: (q_{ij})_{1 \leq j \leq d}$ is given by $q_i := \mathbf{a}(\nabla \phi_i + e_i)$, and where the (distributional) divergence of a tensor field is defined as

$$(\nabla \cdot \sigma_i)_j := \sum_{k=1}^d \nabla_k \sigma_{ijk}. \quad \diamond$$

The proofs of Proposition 1 and Theorem 1 are based on the combination of two main ingredients. The first ingredient is a sensitivity calculus combined with functional inequalities for Gaussian ensembles [3, 4]. The second ingredient is a duality argument combined with the large-scale (weighted) Calderón-Zygmund estimates of [8].

The sensitivity calculus measures the influence of changes of the coefficient field \mathbf{a} on random variables $X = X(\mathbf{a})$ via the functional (or Malliavin-type) derivative $\partial^{\text{fct}} X(x) = \frac{\partial X}{\partial \mathbf{a}}(\mathbf{a}, x)$, that is, the $L^2(\mathbb{R}^d)^{d \times d}$ -gradient of X wrt \mathbf{a} . We recall that this functional derivative is characterized as follows, for any compactly supported perturbation $\mathbf{b} \in L^\infty(\mathbb{R}^d)^{d \times d}$,

$$\int_{\mathbb{R}^d} \partial^{\text{fct}} X(\mathbf{a}, x) : \mathbf{b}(x) dx := \lim_{t \downarrow 0} \frac{1}{t} (X(\mathbf{a} + t\mathbf{b}) - X(\mathbf{a})). \quad (2.10)$$

In fact, rather than in this derivative itself, we are interested in the local L^1 -norms of $\partial^{\text{fct}} F$ on measurable sets $D \subset \mathbb{R}^d$, which in view of (2.10) can be characterized by duality as follows,

$$\int_D |\partial^{\text{fct}} X| = \sup \left\{ \limsup_{t \downarrow 0} \frac{1}{t} (X(\mathbf{a} + t\mathbf{b}) - X(\mathbf{a})) : \sup |\mathbf{b}| \leq 1, \mathbf{b}|_{\mathbb{R}^d \setminus D} = 0 \right\}. \quad (2.11)$$

This quantity measures the sensitivity of the random variable $X = X(\mathbf{a})$ wrt changes in the coefficient field localized in D . This sensitivity calculus is a building block to control the variance and the entropy of X via functional inequalities in the probability space [3]. To this aim, we define a weighted norm $\|\cdot\|_\beta^2$ on random fields G depending on $\beta > 0$:

$$\|G\|_\beta^2 := \int_0^\infty \int_{\mathbb{R}^d} \left(\int_{B_{\ell+1}(z)} |G| \right)^2 dz (\ell + 1)^{-d-\beta-1} d\ell. \quad (2.12)$$

As shown in [3, Proposition 2.4], in the integrable case $\beta > d$, we can drop the integral over ℓ , in which case

$$\|G\|_\beta^2 \simeq \|G\|^2 := \int_{\mathbb{R}^d} \left(\int_{B(z)} |G| \right)^2 dz. \quad (2.13)$$

In these terms, we may formulate the following weighted logarithmic Sobolev inequality (WLSI) for the Gaussian coefficient field \mathbf{a} . In view of (2.13), for $\beta > d$, this reduces to the standard (unweighted) logarithmic Sobolev inequality (LSI). The proof is based on a corresponding Brascamp-Lieb inequality (cf. [4, Theorem 3.1]).

Lemma 2.3 ([4]). *Assume that the coefficient field \mathbf{a} is Gaussian with parameter $\beta > 0$. Then for all random variables $X = X(\mathbf{a})$,*

$$\text{Ent}[X^2] := \mathbb{E}[X^2 \log X^2] - \mathbb{E}[X^2] \mathbb{E}[\log X^2] \lesssim \mathbb{E} \left[\|\partial^{\text{fct}} X\|_\beta^2 \right]. \quad \diamond$$

Our general strategy for the proof of Proposition 1 and Theorem 1 consists in estimating the weighted norm of the functional derivatives of $J_0^\varepsilon(F)$ and of $E^\varepsilon(f, g)$. The following lemma provides a useful representation formula for these functional derivatives. This is a

continuum version of [5, Lemma 3.2]. By scaling, it is enough to consider $\varepsilon = 1$, and we write for simplicity $J_0 := J_0^1$ and $E := E^1$.

Lemma 2.4 (Representation formulas). *For all $f \in C_c^\infty(\mathbb{R}^d)^d$, let $u := u_1$ denote the solution of (1.3) (with $\varepsilon = 1$), let \bar{u} denote the solution of (1.4), and define the two-scale expansion error $w_f := u - (1 + \phi_i \nabla_i) \bar{u}$. Then, for all $F \in C_c^\infty(\mathbb{R}^d)^{d \times d}$,*

$$\partial^{\text{fct}} J_0(F) = (F_{ij} e_j + \nabla S_i) \otimes (\nabla \phi_i + e_i), \quad (2.14)$$

and for all $g \in C_c^\infty(\mathbb{R}^d)^d$,

$$\begin{aligned} \partial^{\text{fct}} E(f, g) &= g_j (\nabla \phi_j^* + e_j) \otimes (\nabla w_f + \phi_i \nabla \nabla_i \bar{u}) + (\phi_j^* \nabla g_j + \nabla r) \otimes \nabla u \\ &\quad - (\phi_j^* \nabla (g_j \nabla_i \bar{u}) + \nabla R_i) \otimes (\nabla \phi_i + e_i), \end{aligned} \quad (2.15)$$

where the auxiliary fields $S = (S_i)_{i=1}^d$, r , and $R = (R_i)_{i=1}^d$ are the Lax-Milgram solutions in \mathbb{R}^d of

$$-\nabla \cdot \mathbf{a}^* \nabla S_i = \nabla \cdot (F_{ij} (\mathbf{a}^* - \bar{\mathbf{a}}^*) e_j), \quad (2.16)$$

$$-\nabla \cdot \mathbf{a}^* \nabla r = \nabla \cdot ((\phi_j^* \mathbf{a}^* - \sigma_j^*) \nabla g_j), \quad (2.17)$$

$$-\nabla \cdot \mathbf{a}^* \nabla R_i = \nabla \cdot ((\phi_j^* \mathbf{a}^* - \sigma_j^*) \nabla (g_j \nabla_i \bar{u})), \quad (2.18)$$

and \mathbf{a}^* denotes the pointwise transpose coefficient field of \mathbf{a} , and (ϕ^*, σ^*) denotes the corresponding extended corrector (recall that $\bar{\mathbf{a}}^* = \bar{\mathbf{a}}^*$). \diamond

It remains to estimate $\|\partial^{\text{fct}} J_0(F)\|_\beta$ and $\|\partial^{\text{fct}} E(f, g)\|_\beta$, which is achieved in the following two (technical) propositions.

Proposition 2.5 (Main estimates). *Let the coefficient field \mathbf{a} be Gaussian with parameter $\beta > 0$. Let π_* and μ_* be defined by (2.2) and (2.4), respectively, and let r_* be the minimal radius (a random field) of Lemma 2.7 below. For $F \in C_c^\infty(\mathbb{R}^d)$ we denote by $[F]_2 : \mathbb{R}^d \rightarrow \mathbb{R}_+$ the moving local quadratic average of F on unit balls, that is,*

$$[F]_2(x) := \left(\int_{B(x)} |F|^2 \right)^{\frac{1}{2}},$$

and for all $R \geq 1$ we set $w_R(x) := \frac{|x|}{R} + 1$. Then the following hold:

(i) If $\beta > d$, we have for all $R \geq 1$, $0 < \alpha - d \ll 1$, and $0 < p - 1 \ll_\alpha 1$,

$$\|\partial^{\text{fct}} J_0(F)\|_\beta^2 \lesssim_{\alpha, p} r_*(0)^{\alpha \frac{p-1}{p}} \left(\int_{\mathbb{R}^d} r_*^{d \frac{p-1}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} |F|^{2p} \right)^{\frac{1}{p}}. \quad (2.19)$$

(ii) If $\beta \leq d$, we have for all $R \geq 1$, $0 < \gamma < \beta$, $0 < \alpha - d \ll 1$, and $0 < p - 1 \ll_{\gamma, \alpha} 1$,

$$\begin{aligned} \|\partial^{\text{fct}} J_0(F)\|_\beta^2 &\lesssim_{\alpha, p} R^d \pi_*(R)^{-1} [\text{RHS}(2.19)] + R^{2d - \beta - \frac{2d}{p}} r_*(0)^{2d \frac{p-1}{p}} \left(\int_{\mathbb{R}^d} r_*^{d(1 - \frac{p}{2})} [F]_2^p \right)^{\frac{2}{p}} \\ &\quad + R^{d - \beta} r_*(0)^{d - \gamma + \alpha \frac{p-1}{p}} \left(\int_{\mathbb{R}^d} r_*^{\gamma \frac{p-1}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} w_R^{(d - \gamma)p + \alpha(p-1)} |F|^{2p} \right)^{\frac{1}{p}}, \end{aligned} \quad (2.20)$$

where we use the short-hand notation [RHS(2.19)] for the RHS of (2.19). \diamond

Proposition 2.6 (Main estimates — cont'd). *Let the coefficient field \mathbf{a} be Gaussian with parameter $\beta > 0$. Let π_* and μ_* be defined by (2.2) and (2.4), respectively, and let r_* and \mathcal{C} be the random fields defined in Lemmas 2.7 and 2.8 below. For $F \in C_c^\infty(\mathbb{R}^d)^d$ we denote by $[F]_\infty : \mathbb{R}^d \rightarrow \mathbb{R}_+$ the moving local supremum of F on unit balls, that is,*

$$[F]_\infty(x) := \sup_{B(x)} |F|,$$

and recall $w_R(x) = \frac{|x|}{R} + 1$. Then the following hold:

(i) *If $\beta > d$, we have for all $R \geq 1$, $0 < \alpha - d \ll 1$, and $0 < p - 1 \ll_\alpha 1$,*

$$\begin{aligned} \|\partial^{\text{fct}} E(f, g)\|^2 &\lesssim_{\alpha, p} r_*(0)^{\alpha \frac{p-1}{2p}} \left(\int_{\mathbb{R}^d} r_*^{2d \frac{2p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{2p}} \\ &\times \left(\left(\int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 [\nabla^2 \bar{u}]_\infty^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} [g]_\infty^{4p} \right)^{\frac{1}{2p}} \right. \\ &\quad \left. + \left(\int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 [\nabla g]_\infty^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} (|f| + [\nabla \bar{u}]_\infty)^{4p} \right)^{\frac{1}{2p}} \right). \end{aligned} \quad (2.21)$$

(ii) *If $\beta \leq d$, we have for all $R \geq 1$, $0 \leq \gamma < \beta$, $0 < \alpha - d \ll 1$, and $0 < p - 1 \ll_\alpha 1$,*

$$\begin{aligned} \|\partial^{\text{fct}} E(f, g)\|_\beta^2 &\lesssim_{\alpha, p} R^d \pi_*(R)^{-1} [\text{RHS}(2.21)] \\ &+ R^{-\beta} \left(\int_{\mathbb{R}^d} \mathcal{C}^2 \mu_*^2 [\nabla g]_\infty^2 \right) \left(\left(\int_{\mathbb{R}^d} |f|^2 \right) + (Rr_*(0))^{d \frac{p-1}{p}} \left(\int_{\mathbb{R}^d} [\nabla \bar{u}]_\infty^{2p} \right)^{\frac{1}{p}} \right) \\ &+ R^{-\beta} \left(\int_{\mathbb{R}^d} \mathcal{C}^2 \mu_*^2 [\nabla^2 \bar{u}]_\infty^2 \right) \left(\left(\int_{\mathbb{R}^d} r_*^d [g]_\infty^2 \right) + (Rr_*(0))^{d \frac{p-1}{p}} \left(\int_{\mathbb{R}^d} [g]_\infty^{2p} \right)^{\frac{1}{p}} \right) \\ &+ R^{d-\beta} r_*(0)^{d-\gamma+\alpha \frac{p-1}{2p}} \left(\int_{\mathbb{R}^d} r_*^{\gamma \frac{2p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{2p}} \\ &\times \left(\left(\int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 [\nabla g]_\infty^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{2p(d-\gamma)+\alpha(p-1)} (|f| + [\nabla \bar{u}]_\infty)^{4p} \right)^{\frac{1}{2p}} \right. \\ &\quad \left. + \left(\int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 [\nabla^2 \bar{u}]_\infty^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} r_*^{2pd} w_R^{2p(d-\gamma)+\alpha(p-1)} [g]_\infty^{4p} \right)^{\frac{1}{2p}} \right), \end{aligned} \quad (2.22)$$

where we use the short-hand notation $[\text{RHS}(2.21)]$ for the RHS of (2.21). \diamond

The proofs of Propositions 2.5 and 2.6 rely on two further ingredients: large-scale weighted Calderón-Zygmund estimates, and moment bounds on the extended corrector (ϕ, σ) (which are at the origin of the scaling μ_* in the estimates). We start by recalling the former, which follows from [8, Theorem 1, Corollary 4, and Corollary 5].

Lemma 2.7 ([8]). *Assume that the coefficient field \mathbf{a} is Gaussian with parameter $\beta > 0$, and let π_* be as in (2.2). There exists a stationary, 1-Lipschitz continuous random field $r_* \geq 1$ (the minimal radius), satisfying for some (deterministic) constant $C \simeq 1$,*

$$\mathbb{E} \left[\exp \left(\frac{1}{C} \pi_*(r_*) \right) \right] \leq 2, \quad (2.23)$$

and there exists a random partition $\mathcal{P} \subset \{2^k(z + [-\frac{1}{2}, \frac{1}{2}]^d) : z \in \mathbb{Z}^d, k \in \mathbb{N}\}$ of \mathbb{R}^d into dyadic cubes, satisfying a.s. for some (deterministic) constant $C \simeq 1$,

$$\frac{1}{C} \sup_Q r_* \leq \text{diam } Q \leq C \inf_Q r_*, \quad \text{for all } Q \in \mathcal{P}, \quad (2.24)$$

such that the following properties hold a.s.,

- (a) Mean-value property: For any \mathbf{a} -harmonic function u in B_R (that is, $-\nabla \cdot \mathbf{a} \nabla u = 0$ in B_R), we have for all radii $r_*(0) \leq r \leq R$,

$$\int_{B_r} |\nabla u|^2 \lesssim \int_{B_R} |\nabla u|^2. \quad (2.25)$$

- (b) Large-scale Calderón-Zygmund estimates: For $1 < p < \infty$, for any (sufficiently fast) decaying scalar field u and vector field g related in \mathbb{R}^d by

$$-\nabla \cdot \mathbf{a} \nabla u = \nabla \cdot g,$$

we have

$$\sum_{Q \in \mathcal{P}} |Q| \left(\int_Q |\nabla u|^2 \right)^{\frac{p}{2}} \lesssim_p \sum_{Q \in \mathcal{P}} |Q| \left(\int_Q |g|^2 \right)^{\frac{p}{2}}. \quad (2.26)$$

- (c) Large-scale weighted Calderón-Zygmund estimates: For $2 \leq p < \infty$, $0 \leq \gamma < d(p-1)$, and for any non-decreasing radial weight $w \geq 1$ satisfying

$$w(r) \leq w(r') \leq \left(\frac{r'}{r} \right)^\gamma w(r), \quad \text{for all } 0 \leq r \leq r',$$

we have for u and g as in (b) above,

$$\sum_{Q \in \mathcal{P}} |Q| \left(\sup_Q w \right) \left(\int_Q |\nabla u|^2 \right)^{\frac{p}{2}} \lesssim_{p,\gamma} \sum_{Q \in \mathcal{P}} |Q| \left(\sup_Q w \right) \left(\int_Q |g|^2 \right)^{\frac{p}{2}}. \quad (2.27) \quad \diamond$$

Whereas the minimal radius r_* quantifies the sublinearity of the extended corrector at infinity [8], the precise growth of the latter is estimated as follows (cf. [7, Theorem 2]).

Lemma 2.8 ([8, 7]). *Assume that the coefficient field \mathbf{a} is Gaussian with parameter $\beta > 0$, let μ_* be as in (2.4), and let r_* be as in Lemma 2.7. Then the extended corrector (ϕ, σ) defined in Lemma 2.2 satisfies for all $x \in \mathbb{R}^d$,*

$$\left(\int_{B(x)} |(\nabla \phi, \nabla \sigma)|^2 \right)^{\frac{1}{2}} \lesssim r_*(x)^{\frac{d}{2}},$$

and

$$\left(\int_{B(x)} |(\phi, \sigma)|^2 \right)^{\frac{1}{2}} \leq \mathcal{C}(x) \mu_*(x), \quad (2.28)$$

where $\mathcal{C} \geq 1$ is a 1-Lipschitz continuous random field with the following stretched exponential moments: for all $\eta > 0$, there exists a (deterministic) constant $C_\eta \simeq_\eta 1$ such that

$$\mathbb{E} \left[\exp \left(\frac{1}{C_\eta} \mathcal{C}^{2 \frac{\beta \wedge d}{\beta \wedge d + d} - \eta} \right) \right] \leq 2. \quad (2.29) \quad \diamond$$

3. PROOFS

3.1. Proof of Lemma 2.4: Representation formulas. We first introduce some notation. Let \mathbf{a} and $\tilde{\mathbf{a}}$ be two admissible coefficient fields, and set $\delta \mathbf{a} := \tilde{\mathbf{a}} - \mathbf{a}$. For all random variables (or fields) $F = F(\mathbf{a})$, we set $\tilde{F} := F(\tilde{\mathbf{a}})$ and $\delta F := \tilde{F} - F$. We then denote by (ϕ, σ) , (ϕ^*, σ^*) , $(\tilde{\phi}, \tilde{\sigma})$, and $(\tilde{\phi}^*, \tilde{\sigma}^*)$ the extended correctors associated with \mathbf{a} , \mathbf{a}^* , $\tilde{\mathbf{a}}$, and $\tilde{\mathbf{a}}^*$, respectively.

Step 1. Proof of identity (2.14).

The definition (1.6) of Ξ_{ij} yields

$$\delta J_0(F) = \int_{\mathbb{R}^d} F_{ij} \delta \Xi_{ij} = \int_{\mathbb{R}^d} F_{ij} e_j \cdot (\mathbf{a} - \bar{\mathbf{a}}) \nabla \delta \phi_i + \int_{\mathbb{R}^d} F_{ij} e_j \cdot \delta \mathbf{a} (\nabla \tilde{\phi}_i + e_i).$$

Using the definition (2.16) of the auxiliary field S as well as the corrector equation (2.8) for $\tilde{\phi}_i$ and $\tilde{\phi}_i$ in the form

$$\nabla \cdot \mathbf{a} \nabla \delta \phi_i = -\nabla \cdot \delta \mathbf{a} (\nabla \tilde{\phi}_i + e_i), \quad (3.1)$$

the first RHS term above can be rewritten as

$$\int_{\mathbb{R}^d} F_{ij} e_j \cdot (\mathbf{a} - \bar{\mathbf{a}}) \nabla \delta \phi_i = - \int_{\mathbb{R}^d} \nabla S_i \cdot \mathbf{a} \nabla \delta \phi_i = \int_{\mathbb{R}^d} \nabla S_i \cdot \delta \mathbf{a} (\nabla \tilde{\phi}_i + e_i),$$

and the conclusion (2.14) follows from the definition (2.10) of the functional derivative.

Step 2. Proof of identity (2.15).

We start by giving a suitable representation formula for the functional derivative of the homogenization commutator of the solution $(\mathbf{a} - \bar{\mathbf{a}}) \nabla u$. By the property (2.9) of the flux corrector σ_j^* in the form $(\mathbf{a}^* - \bar{\mathbf{a}}^*) e_j = -\mathbf{a}^* \nabla \phi_j^* + \nabla \cdot \sigma_j^*$, and by its skew-symmetry (2.7) in the form $(\nabla \cdot \sigma_j^*) \cdot \nabla \delta u = -\nabla \cdot (\sigma_j^* \nabla \delta u)$, we find

$$\begin{aligned} \delta(e_j \cdot (\mathbf{a} - \bar{\mathbf{a}}) \nabla u) &= e_j \cdot \delta \mathbf{a} \nabla \tilde{u} + e_j \cdot (\mathbf{a} - \bar{\mathbf{a}}) \nabla \delta u \\ &= e_j \cdot \delta \mathbf{a} \nabla \tilde{u} - \nabla \cdot (\sigma_j^* \nabla \delta u) - \nabla \phi_j^* \cdot \mathbf{a} \nabla \delta u. \end{aligned}$$

Equation (1.3) for u and \tilde{u} in the form

$$-\nabla \cdot \mathbf{a} \nabla \delta u = \nabla \cdot \delta \mathbf{a} \nabla \tilde{u} \quad (3.2)$$

allows us to rewrite the last RHS term as

$$\begin{aligned} -\nabla \phi_j^* \cdot \mathbf{a} \nabla \delta u &= -\nabla \cdot (\phi_j^* \mathbf{a} \nabla \delta u) + \phi_j^* \nabla \cdot \mathbf{a} \nabla \delta u \\ &= -\nabla \cdot (\phi_j^* \mathbf{a} \nabla \delta u) - \phi_j^* \nabla \cdot \delta \mathbf{a} \nabla \tilde{u} \\ &= -\nabla \cdot (\phi_j^* \mathbf{a} \nabla \delta u) - \nabla \cdot (\phi_j^* \delta \mathbf{a} \nabla \tilde{u}) + \nabla \phi_j^* \cdot \delta \mathbf{a} \nabla \tilde{u}. \end{aligned}$$

Hence, we conclude

$$\delta(e_j \cdot (\mathbf{a} - \bar{\mathbf{a}}) \nabla u) = (\nabla \phi_j^* + e_j) \cdot \delta \mathbf{a} \nabla \tilde{u} - \nabla \cdot ((\phi_j^* \mathbf{a} + \sigma_j^*) \nabla \delta u) - \nabla \cdot (\phi_j^* \delta \mathbf{a} \nabla \tilde{u}),$$

and similarly, replacing $x \mapsto u(x)$ by $x \mapsto \phi_i(x) + x_i$,

$$\delta \Xi_{ij} = (\nabla \phi_j^* + e_j) \cdot \delta \mathbf{a} (\nabla \tilde{\phi}_i + e_i) - \nabla \cdot ((\phi_j^* \mathbf{a} + \sigma_j^*) \nabla \delta \phi_i) - \nabla \cdot (\phi_j^* \delta \mathbf{a} (\nabla \tilde{\phi}_i + e_i)).$$

Considering $\delta(e_j \cdot (\mathbf{a} - \bar{\mathbf{a}}) \nabla u) - \nabla_i \bar{u} \delta \Xi_{ij}$ and multiplying by g_j , we are led to

$$\begin{aligned} \delta E(f, g) &= \int_{\mathbb{R}^d} g_j (\nabla \phi_j^* + e_j) \cdot \delta \mathbf{a} (\nabla \tilde{u} - (\nabla \tilde{\phi}_i + e_i) \nabla_i \bar{u}) \\ &\quad + \int_{\mathbb{R}^d} \phi_j^* \nabla g_j \cdot \delta \mathbf{a} \nabla \tilde{u} - \int_{\mathbb{R}^d} \phi_j^* \nabla (g_j \nabla_i \bar{u}) \cdot \delta \mathbf{a} (\nabla \tilde{\phi}_i + e_i) \\ &\quad + \int_{\mathbb{R}^d} \nabla g_j \cdot (\phi_j^* \mathbf{a} + \sigma_j^*) \nabla \delta u - \int_{\mathbb{R}^d} \nabla (g_j \nabla_i \bar{u}) \cdot (\phi_j^* \mathbf{a} + \sigma_j^*) \nabla \delta \phi_i. \end{aligned}$$

For the first RHS term we use the definition of w_f in form of $\nabla u - (\nabla\phi_i + e_i)\nabla_i\bar{u} = \nabla w_f + \phi_i\nabla\nabla_i\bar{u}$, whereas for the last two RHS terms we use the definitions (2.17) and (2.18) of the auxiliary fields r and R , combined with equations (3.1) and (3.2), so that

$$\begin{aligned} \delta E(f, g) &= \int_{\mathbb{R}^d} g_j(\nabla\phi_j^* + e_j) \cdot \delta\mathbf{a}(\nabla\tilde{w}_f + \tilde{\phi}_i\nabla\nabla_i\bar{u}) \\ &\quad + \int_{\mathbb{R}^d} \phi_j^*\nabla g_j \cdot \delta\mathbf{a}\nabla\tilde{u} - \int_{\mathbb{R}^d} \phi_j^*\nabla(g_j\nabla_i\bar{u}) \cdot \delta\mathbf{a}(\nabla\tilde{\phi}_i + e_i) \\ &\quad + \int_{\mathbb{R}^d} \nabla r \cdot \delta\mathbf{a}\nabla\tilde{u} - \int_{\mathbb{R}^d} \nabla R_i \cdot \delta\mathbf{a}(\nabla\tilde{\phi}_i + e_i), \end{aligned} \quad (3.3)$$

and the conclusion (2.15) follows from the definition (2.10) of the functional derivative.

3.2. Dyadic partitions and reformulation of weighted norms. Following [7, proof of Proposition 1], it is convenient to rewrite the weighted norm (2.12) as a sum over coarsened dyadic partitions. We start with the partition \mathcal{P} of \mathbb{R}^d into dyadic cubes given by Lemma 2.7. For all dyadic $\ell + 1$ (that is, $\ell = 2^k - 1$, $k \in \mathbb{N}$), we then denote by $\mathcal{P}_\ell \subset \{2^j(z + [-\frac{1}{2}, \frac{1}{2}]^d) : z \in \mathbb{Z}^d, j \in \mathbb{N}\}$ the finest dyadic partition of \mathbb{R}^d that is coarser than both \mathcal{P} and $\{(\ell + 1)(z + [-\frac{1}{2}, \frac{1}{2}]^d) : z \in \mathbb{Z}^d\}$. By definition, $\mathcal{P}_0 = \mathcal{P}$. Occasionally, we will have to single out the cube $Q_\ell^0 \in \mathcal{P}_\ell$ that contains the origin, and we set $\mathcal{P}_\ell^* := \mathcal{P}_\ell \setminus \{Q_\ell^0\}$. We recall the following elementary properties of these partitions :

- For any $Q_\ell \in \mathcal{P}_\ell$, we have for all $x \in Q_\ell$,

$$\text{diam } Q_\ell \simeq \ell + r_*(x), \quad (3.4)$$

so that in particular

$$|Q_\ell| \simeq \left(\ell + \inf_{Q_\ell} r_*\right)^d \lesssim (\ell + 1)^d \left(\inf_{Q_\ell} r_*\right)^d. \quad (3.5)$$

Indeed, given $x \in Q_\ell \in \mathcal{P}_\ell$, the definition of \mathcal{P}_ℓ implies $\text{diam } Q_\ell \geq \ell + 1$, as well as the existence of some $Q' \in \mathcal{P}$ such that $x \in Q' \subset Q_\ell$. The latter property together with (2.24) yields $\text{diam } Q_\ell \gtrsim r_*(x)$, so that we may conclude $\text{diam } Q_\ell \gtrsim \ell + r_*(x)$. For the converse inequality, we distinguish between two cases:

- If $Q_\ell \in \mathcal{P}$, then (2.24) yields $\text{diam } Q_\ell \lesssim r_*(x) \leq \ell + r_*(x)$.
- If $Q_\ell \notin \mathcal{P}$, then by definition $\text{diam } Q_\ell = \ell + 1 \leq \ell + r_*(x)$.

- For any $Q \in \mathcal{P}$, we have

$$\sup_{x \in Q} |x| \lesssim \inf_{x \in Q} |x| + r_*(0). \quad (3.6)$$

Indeed, for all $x, y \in Q$, by definition of the partition \mathcal{P} , we have $|x| \leq |y| + \text{diam } Q \lesssim |y| + r_*(y)$, and the conclusion follows from the Lipschitz continuity of r_* .

- The mean-value property (2.25) applied to the corrector ϕ , together with the bound (2.6), yields

$$\int_{Q_\ell} |\nabla\phi + \text{Id}|^2 \lesssim |Q_\ell|. \quad (3.7)$$

- Since by definition \mathcal{P}_ℓ is coarser than the partition of \mathbb{R}^d into cubes of sidelength $\ell + 1$, we find for all fields G ,

$$(\ell + 1)^{-d} \int_{\mathbb{R}^d} \left(\int_{B_{\ell+1}(z)} |G| \right)^2 dz \lesssim \sum_{Q_\ell \in \mathcal{P}_\ell} \left(\int_{Q_\ell} |G| \right)^2 =: \|G\|_{\ell+1}^2. \quad (3.8)$$

We now turn to the reformulation of the weighted norm $\|\cdot\|_\beta$ in terms of this family of coarsened partitions $(\mathcal{P}_\ell)_\ell$:

- For $\beta > d$, combining (2.13) and (3.8) for $\ell = 0$ yields

$$\|G\|_\beta^2 \simeq \|G\|^2 \lesssim \sum_{Q \in \mathcal{P}} \left(\int_Q |G| \right)^2 = \|G\|_1^2. \quad (3.9)$$

- For $\beta \leq d$, using (3.8) to reformulate the definition (2.12),

$$\begin{aligned} \|G\|_\beta^2 &\leq \sum_{k=0}^{\infty} 2^k \int_{\mathbb{R}^d} \left(\int_{B_{2^k}(z)} |G| \right)^2 (2^{k-1})^{-d-1-\beta} dz \\ &\lesssim \sum_{k=0}^{\infty} 2^{-\beta k} \sum_{Q \in \mathcal{P}_{2^{k-1}}} \left(\int_Q |G| \right)^2 = \sum_{k=0}^{\infty} 2^{-\beta k} \|G\|_{2^k}^2. \end{aligned} \quad (3.10)$$

3.3. Proof of Proposition 2.5: Main estimates. We split the proof into two main steps, first addressing the case of the standard LSI ($\beta > d$), and then turning to the general weighted case ($\beta \leq d$). Let $R \geq 1$ be arbitrary.

Step 1. Proof of (2.19) for standard LSI ($\beta > d$).

Since for standard LSI ($\beta > d$) we have $\|\partial^{\text{fct}} J_0(F)\|_\beta \lesssim \|\partial^{\text{fct}} J_0(F)\|_1$ (cf. (3.9)), it suffices to prove the following estimate: for all dyadic $\ell + 1$, for all $\alpha > d$ and $p > 1$ with $\alpha(p-1) < d(2p-1)$,

$$\|\partial^{\text{fct}} J_0(F)\|_{\ell+1}^2 \lesssim_{\alpha,p} (\ell+1)^d r_*(0)^\alpha \frac{p-1}{p} \left(\int_{\mathbb{R}^d} r_*^{\frac{d-p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} |F|^{2p} \right)^{\frac{1}{p}}. \quad (3.11)$$

We split the proof into two further substeps.

Substep 1.1. Preliminary: If v is the Lax-Milgram solution of $-\nabla \cdot \mathbf{a} \nabla v = \nabla \cdot h$ with $h \in C_c^\infty(\mathbb{R}^d)$, then for all $s \geq 0$, $q \geq 1$, $\alpha > d$, and $p > 1$ with $\alpha(p-1) < d(2pq-1)$, we claim

$$\begin{aligned} \sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell|^{1+s} \left(\int_{Q_\ell} (|h| + |\nabla v|)^2 \right)^q \\ \lesssim_{\alpha,p,q,s} (\ell+1)^{ds} r_*(0)^\alpha \frac{p-1}{p} \left(\int_{\mathbb{R}^d} r_*^{\frac{ds-p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} [h]_2^{2pq} \right)^{\frac{1}{p}}, \end{aligned} \quad (3.12)$$

where for $s = 0$ we may even choose $p = 1$, in which case (3.12) is replaced by

$$\sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell| \left(\int_{Q_\ell} (|h| + |\nabla v|)^2 \right)^q \lesssim_q \int_{\mathbb{R}^d} [h]_2^{2q}, \quad (3.13)$$

which we state and prove for future reference only.

By property (3.5) of Q_ℓ and Jensen's inequality (which we may use since \mathcal{P}_ℓ is coarser than \mathcal{P}), we have

$$\begin{aligned} & \sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell|^{1+s} \left(\fint_{Q_\ell} (|h| + |\nabla v|)^2 \right)^q \\ & \lesssim_s (\ell + 1)^{ds} \sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell| \left(\inf_{Q_\ell} r_*^{ds} \right) \left(\fint_{Q_\ell} (|h| + |\nabla v|)^2 \right)^q \\ & \leq (\ell + 1)^{ds} \sum_{Q \in \mathcal{P}} |Q| \left(\inf_Q r_*^{ds} \right) \left(\fint_Q (|h| + |\nabla v|)^2 \right)^q. \end{aligned} \quad (3.14)$$

For all $\alpha > d$ and $p > 1$, we smuggle in the weight $Q \mapsto \sup_Q w_R$ to the power $\alpha \frac{p-1}{p}$, and use Hölder's inequality with exponents $(\frac{p}{p-1}, p)$, so that

$$\begin{aligned} \sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell|^{1+s} \left(\fint_{Q_\ell} (|h| + |\nabla v|)^2 \right)^q & \lesssim_s (\ell + 1)^{ds} \left(\sum_{Q \in \mathcal{P}} |Q| \left(\inf_Q r_*^{ds \frac{p}{p-1}} \right) \left(\sup_Q w_R \right)^{-\alpha} \right)^{\frac{p-1}{p}} \\ & \quad \times \left(\sum_{Q \in \mathcal{P}} |Q| \left(\sup_Q w_R \right)^{\alpha(p-1)} \left(\fint_Q (|h| + |\nabla v|)^2 \right)^{pq} \right)^{\frac{1}{p}}, \end{aligned}$$

where the first RHS sum is bounded by $\int_{\mathbb{R}^d} r_*^{ds \frac{p}{p-1}} w_R^{-\alpha}$. Provided $\alpha(p-1) < d(2pq-1)$, we may apply the large-scale weighted Calderón-Zygmund estimate (2.27) to the equation for v , to the effect of

$$\begin{aligned} \sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell|^{1+s} \left(\fint_{Q_\ell} (|h| + |\nabla v|)^2 \right)^q & \lesssim_{\alpha, p, q, s} (\ell + 1)^{ds} \left(\int_{\mathbb{R}^d} r_*^{ds \frac{p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{p}} \\ & \quad \times \left(\sum_{Q \in \mathcal{P}} |Q| \left(\sup_Q w_R \right)^{\alpha(p-1)} \left(\fint_Q |h|^2 \right)^{pq} \right)^{\frac{1}{p}}. \end{aligned}$$

Using (3.6) in the form $\sup_Q w_R \lesssim r_*(0) + \inf_Q w_R \leq r_*(0) \inf_Q w_R$, Jensen's inequality, and the fact that \mathcal{P} is coarser than the partition of \mathbb{R}^d into unit cubes (to take local averages), the conclusion (3.12) follows. For $s = 0$, we appeal to the large-scale (unweighted) Calderón-Zygmund estimate (2.26), which amounts to choosing $p = 1$ in the above.

Substep 1.2. Proof of (3.11).

Starting point is formula (2.14), which, by definition (3.8) of $\|\cdot\|_{\ell+1}$, Cauchy-Schwarz' inequality, and the mean-value property (3.7), implies

$$\begin{aligned} \|\partial^{\text{fct}} J_0(F)\|_{\ell+1}^2 & \leq \sum_{Q_\ell \in \mathcal{P}_\ell} \left(\int_{Q_\ell} (|F| + |\nabla S|)^2 \right) \left(\int_{Q_\ell} |\nabla \phi + \text{Id}|^2 \right) \\ & \lesssim \sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell|^2 \left(\fint_{Q_\ell} (|F| + |\nabla S|)^2 \right). \end{aligned}$$

Applying (3.12) with $s = q = 1$ to v replaced by the solution S of (2.16), the conclusion (3.11) follows.

Step 2. Proof of (2.20) in the general weighted case ($\beta \leq d$).

The combination of (3.10) with (3.11) is not enough to prove (2.20), and we have to refine (3.11) in the regime $\ell \geq R$. We split this step into three substeps.

Substep 2.1. Preliminary: If v is the Lax-Milgram solution of $-\nabla \cdot \mathbf{a} \nabla v = \nabla \cdot h$ with $h \in C_c^\infty(\mathbb{R}^d)$, then for all $s \geq 0$, $q \geq 1$, $0 \leq \gamma \leq ds$, $\alpha > d$, and $p > 1$ with $(ds - \gamma)p + \alpha(p - 1) < d(2pq - 1)$, we claim

$$\begin{aligned} \sum_{Q_\ell \in \mathcal{P}_\ell^*} |Q_\ell|^{1+s} \left(\int_{Q_\ell} (|h| + |\nabla v|)^2 \right)^q \\ \lesssim_{\gamma, \alpha, p, q, s} (\ell + 1)^\gamma R^{ds - \gamma} r_*(0)^{ds - \gamma + \alpha \frac{p-1}{p}} \left(\int_{\mathbb{R}^d} r_*^{\gamma \frac{p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{p}} \\ \times \left(\int_{\mathbb{R}^d} w_R^{(ds - \gamma)p + \alpha(p-1)} [h]_2^{2pq} \right)^{\frac{1}{p}}. \end{aligned} \quad (3.15)$$

Note that the sum on the left-hand side (LHS) of (3.15) runs over $Q_\ell \in \mathcal{P}_\ell^*$ (which does not contain the cube centered at the origin). The same argument as for (3.14) yields

$$\begin{aligned} \sum_{Q_\ell \in \mathcal{P}_\ell^*} |Q_\ell|^{1+s} \left(\int_{Q_\ell} (|h| + |\nabla v|)^2 \right)^q &\lesssim_s \sum_{Q_\ell \in \mathcal{P}_\ell^*} |Q_\ell| \left(\ell + \inf_{Q_\ell} r_* \right)^{ds} \left(\int_{Q_\ell} (|h| + |\nabla v|)^2 \right)^q \\ &\lesssim_s \sum_{\substack{Q \in \mathcal{P} \\ Q \cap Q_\ell^0 = \emptyset}} |Q| \left(\ell + \inf_Q r_* \right)^{ds} \left(\int_Q (|h| + |\nabla v|)^2 \right)^q. \end{aligned}$$

For $0 \leq \gamma \leq ds$, $\alpha > d$, and $p > 1$, we smuggle in the weight $Q \mapsto \sup_Q w_R$ to the power $\alpha \frac{p-1}{p}$ and the weight $Q \mapsto \sup_Q w_1$ to the power $ds - \gamma$, and use Hölder's inequality with exponents $(\frac{p}{p-1}, p)$, and obtain

$$\begin{aligned} \sum_{Q_\ell \in \mathcal{P}_\ell^*} |Q_\ell|^{1+s} \left(\int_{Q_\ell} (|h| + |\nabla v|)^2 \right)^q \\ \lesssim_s \left(\sum_{\substack{Q \in \mathcal{P} \\ Q \cap Q_\ell^0 = \emptyset}} |Q| \left(\sup_Q w_R \right)^{-\alpha} \left(\sup_Q w_1 \right)^{-(ds - \gamma) \frac{p}{p-1}} \left(\ell + \inf_Q r_* \right)^{ds \frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ \times \left(\sum_{Q \in \mathcal{P}} |Q| \left(\sup_Q w_R \right)^{\alpha(p-1)} \left(\sup_Q w_1 \right)^{(ds - \gamma)p} \left(\int_Q (|h| + |\nabla v|)^2 \right)^{pq} \right)^{\frac{1}{p}}. \end{aligned}$$

In the first RHS factor, for $Q \in \mathcal{P}$ with $Q \cap Q_\ell^0 = \emptyset$, we use the bound $\sup_Q w_1 \gtrsim \ell + \inf_Q r_*$, while in the second RHS factor we use $w_1 \leq R w_R$. The above then leads to

$$\begin{aligned} \sum_{Q_\ell \in \mathcal{P}_\ell^*} |Q_\ell|^{1+s} \left(\int_{Q_\ell} (|h| + |\nabla v|)^2 \right)^q &\lesssim_{\gamma, s} R^{ds - \gamma} \left(\sum_{Q \in \mathcal{P}} |Q| \left(\sup_Q w_R \right)^{-\alpha} \left(\ell + \inf_Q r_* \right)^{\gamma \frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ &\times \left(\sum_{Q \in \mathcal{P}} |Q| \left(\sup_Q w_R \right)^{(ds - \gamma)p + \alpha(p-1)} \left(\int_Q (|h| + |\nabla v|)^2 \right)^{pq} \right)^{\frac{1}{p}}. \end{aligned}$$

Provided $(ds - \gamma)p + \alpha(p - 1) < d(2pq - 1)$, we may apply the large-scale weighted Calderón-Zygmund estimate (2.27) to equation (2.16) for S . Further using the estimate (3.6) in the form $\sup_Q w_R \lesssim r_*(0) \inf_Q w_R$ and using Jensen's inequality, the conclusion (3.15) follows.

Substep 2.2. Improvement of (3.11) in the range $\ell \geq R$: for all dyadic $\ell + 1$, all $0 \leq \gamma \leq d$, $\alpha > d$, and $p > 1$ with $(d - \gamma)p + \alpha(p - 1) < d(2p - 1)$ and $p \leq 2$,

$$\begin{aligned} \|\partial^{\text{fct}} J_0(F)\|_{\ell+1}^2 &\lesssim_{\gamma, \alpha, p} (\ell + 1)^{2d\frac{p-1}{p}} r_*(0)^{2d\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} r_*^{d(1-\frac{p}{2})} [F]_2^p \right)^{\frac{2}{p}} \\ &+ (\ell + 1)^\gamma R^{d-\gamma} r_*(0)^{d-\gamma+\alpha\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} r_*^{\gamma\frac{p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} w_R^{(d-\gamma)p+\alpha(p-1)} |F|^{2p} \right)^{\frac{1}{p}}. \end{aligned} \quad (3.16)$$

By formula (2.14) and the definition (3.8) of $\|\cdot\|_{\ell+1}$, after singling out the cube Q_ℓ^0 centered at the origin, and using Cauchy-Schwarz' inequality on the other cubes, we obtain

$$\begin{aligned} \|\partial^{\text{fct}} J_0(F)\|_{\ell+1}^2 &\leq \left(\int_{Q_\ell^0} (|F| + |\nabla S|) |\nabla \phi + \text{Id}| \right)^2 \\ &+ \sum_{Q_\ell \in \mathcal{P}_\ell^*} \left(\int_{Q_\ell} (|F| + |\nabla S|)^2 \right) \left(\int_{Q_\ell} |\nabla \phi + \text{Id}|^2 \right). \end{aligned} \quad (3.17)$$

We start by treating the first RHS term. For $p > 1$, we decompose the cube Q_ℓ^0 according to the partition \mathcal{P} , use Cauchy-Schwarz' inequality, the mean-value property (3.7), and Hölder's inequality with exponents $(\frac{p}{p-1}, p)$, so that

$$\begin{aligned} \int_{Q_\ell^0} (|F| + |\nabla S|) |\nabla \phi + \text{Id}| &\leq \sum_{\substack{Q \in \mathcal{P} \\ Q \subset Q_\ell^0}} \left(\int_Q (|F| + |\nabla S|)^2 \right)^{\frac{1}{2}} \left(\int_Q |\nabla \phi + \text{Id}|^2 \right)^{\frac{1}{2}} \\ &\lesssim \sum_{\substack{Q \in \mathcal{P} \\ Q \subset Q_\ell^0}} |Q| \left(\int_Q (|F| + |\nabla S|)^2 \right)^{\frac{1}{2}} \\ &\lesssim |Q_\ell^0|^{\frac{p-1}{p}} \left(\sum_{Q \in \mathcal{P}} |Q| \left(\int_Q (|F| + |\nabla S|)^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}. \end{aligned} \quad (3.18)$$

By (3.5) in the form $|Q_\ell^0| \lesssim (\ell + 1)^d r_*(0)^d$ and the large-scale Calderón-Zygmund estimate (2.26) with exponent $1 < p \leq 2$ applied to the solution S of (2.16), this turns into

$$\int_{Q_\ell^0} (|F| + |\nabla S|) |\nabla \phi + \text{Id}| \lesssim_p (\ell + 1)^{d\frac{p-1}{p}} r_*(0)^{d\frac{p-1}{p}} \left(\sum_{Q \in \mathcal{P}} |Q| \left(\int_Q |F|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}, \quad (3.19)$$

and hence, by the discrete $\ell^p - \ell^2$ estimate (with \mathcal{P} being coarser than the partition of \mathbb{R}^d into unit cubes) and by the property (2.24) of \mathcal{P} ,

$$\begin{aligned} \int_{Q_\ell^0} (|F| + |\nabla S|) |\nabla \phi + \text{Id}| &\lesssim_p (\ell + 1)^{d\frac{p-1}{p}} r_*(0)^{d\frac{p-1}{p}} \left(\sum_{Q \in \mathcal{P}} |Q|^{1-\frac{p}{2}} \left(\int_Q |F|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\ &\lesssim_p (\ell + 1)^{d\frac{p-1}{p}} r_*(0)^{d\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} r_*^{d(1-\frac{p}{2})} [F]_2^p \right)^{\frac{1}{p}}. \end{aligned} \quad (3.20)$$

We turn to the second RHS term in (3.17). The mean-value property (3.7) yields

$$\sum_{Q_\ell \in \mathcal{P}_\ell^*} \left(\int_{Q_\ell} (|F| + |\nabla S|)^2 \right) \left(\int_{Q_\ell} |\nabla \phi + \text{Id}|^2 \right) \lesssim \sum_{Q_\ell \in \mathcal{P}_\ell^*} |Q_\ell|^2 \left(\int_{Q_\ell} (|F| + |\nabla S|)^2 \right).$$

Applying (3.15) with $s = q = 1$ and with v replaced by the solution S of (2.16), and then combining with (3.17) and (3.20), the conclusion (3.16) follows.

Substep 2.3. Conclusion.

We appeal to (3.10), which we combine to (3.11) for $\ell = 2^k - 1 \leq R$ and to (3.16) for $\ell = 2^k - 1 > R$. Denote by K the smallest integer k such that $2^k \geq R$. Provided that $0 \leq \gamma < \beta$, we then have

$$\begin{aligned} \sum_{k=0}^K 2^{-\beta k} 2^{dk} &\lesssim \left\{ \begin{array}{l} 2^{(d-\beta)K} \quad : \quad \beta < d \\ \log(2^K) \quad : \quad \beta = d \end{array} \right\} \simeq R^d \pi_*(R)^{-1}, \\ \sum_{k=K}^{\infty} 2^{-\beta k} &\lesssim 2^{-\beta K} \simeq R^{-\beta}, \\ R^{d-\gamma} \sum_{k=K}^{\infty} 2^{-\beta k} 2^{\gamma k} &\lesssim R^{d-\gamma} 2^{-(\beta-\gamma)K} \simeq R^{d-\beta}, \end{aligned}$$

and the conclusion (2.20) follows.

3.4. Proof of Proposition 2.6: Main estimates (cont'd). By (2.15) in Lemma 2.4, we have $\partial^{\text{fct}} E(f, g) = G_1 + G_2 + G_3$ with

$$\begin{aligned} G_1 &:= g_j (\nabla \phi_j^* + e_j) \otimes (\nabla w_f + \phi_i \nabla \nabla_i \bar{u}), \quad G_2 := (\phi_j^* \nabla g_j + \nabla r) \otimes \nabla u, \\ G_3 &:= -(\phi_j^* \nabla (g_j \nabla_i \bar{u}) + \nabla R_i) \otimes (\nabla \phi_i + e_i), \end{aligned}$$

so that it suffices to estimate the norms of each of the G_i 's separately. We split the proof into two main steps, first address the case of the standard LSI ($\beta > d$), and then turn to the general weighted case ($\beta \leq d$). Let $R \geq 1$ be arbitrary.

Step 1. Proof of (2.21) for standard LSI ($\beta > d$).

Since for standard LSI ($\beta > d$) property (3.9) yields $\|\partial^{\text{fct}} J_0(F)\|_\beta \lesssim \|\partial^{\text{fct}} J_0(F)\|_1$, it suffices to establish the following estimates: for all dyadic $\ell + 1$, for all $0 < \alpha - d \ll 1$, and $0 < p - 1 \ll_\alpha 1$,

$$\|G_1\|_{\ell+1}^2 \lesssim_{\alpha, p} (\ell + 1)^d \left(\int_{\mathbb{R}^d} r_*^{4d} [g]_\infty^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 [\nabla^2 \bar{u}]_\infty^4 \right)^{\frac{1}{2}}, \quad (3.21)$$

$$\begin{aligned} \|G_2\|_{\ell+1}^2 &\lesssim_{\alpha, p} (\ell + 1)^d r_*(0)^{\alpha \frac{p-1}{2p}} \left(\int_{\mathbb{R}^d} r_*^{d \frac{2p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{2p}} \\ &\quad \times \left(\int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 [\nabla g]_\infty^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} |f|^{4p} \right)^{\frac{1}{2p}}, \end{aligned} \quad (3.22)$$

$$\begin{aligned} \|G_3\|_{\ell+1}^2 &\lesssim_{\alpha, p} (\ell + 1)^d r_*(0)^{\alpha \frac{p-1}{p}} \left(\int_{\mathbb{R}^d} r_*^{d \frac{p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{p}} \\ &\quad \times \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} \mathcal{C}^{2p} \mu_*^{2p} [\nabla (g \nabla \bar{u})]_\infty^{2p} \right)^{\frac{1}{p}}. \end{aligned} \quad (3.23)$$

Indeed, replacing p by $\frac{2p}{p+1}$, estimating $[\nabla(g\nabla\bar{u})]_\infty \lesssim [\nabla g]_\infty[\nabla\bar{u}]_\infty + [g]_\infty[\nabla^2\bar{u}]_\infty$, and using Hölder's inequality, the estimate (3.23) easily leads to

$$\begin{aligned} \|G_3\|_{\ell+1}^2 &\lesssim_{\alpha,p} (\ell+1)^d r_*(0)^{\alpha\frac{p-1}{2p}} \left(\int_{\mathbb{R}^d} r_*^{\frac{d}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{2p}} \\ &\quad \times \left(\left(\int_{\mathbb{R}^d} C^4 \mu_*^4 [\nabla^2\bar{u}]_\infty^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} [g]_\infty^{4p} \right)^{\frac{1}{2p}} \right. \\ &\quad \left. + \left(\int_{\mathbb{R}^d} C^4 \mu_*^4 [\nabla g]_\infty^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} [\nabla\bar{u}]_\infty^{4p} \right)^{\frac{1}{2p}} \right), \end{aligned}$$

so that (2.21) follows in combination with (3.21) and (3.22). We address the estimates (3.21)–(3.23) separately, and split the proof into three substeps.

Substep 1.1. Proof of (3.21).

The definition (3.8) of $\|\cdot\|_{\ell+1}$, followed by Cauchy-Schwarz' inequality, yields

$$\begin{aligned} \|G_1\|_{\ell+1}^2 &\leq \left(\sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell|^3 \left(\int_{Q_\ell} |g|^2 |\nabla\phi + \text{Id}|^2 \right)^2 \right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell| \left(\int_{Q_\ell} (|\nabla w_f| + |\phi| |\nabla^2\bar{u}|)^2 \right)^2 \right)^{\frac{1}{2}}. \quad (3.24) \end{aligned}$$

We start by treating the first RHS factor. Using (3.5), taking the local supremum of g , applying Lemma 2.8, and using Jensen's inequality, we find

$$\begin{aligned} \sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell|^3 \left(\int_{Q_\ell} |g|^2 |\nabla\phi + \text{Id}|^2 \right)^2 &\lesssim (\ell+1)^{2d} \sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell| \left(\inf_{Q_\ell} r_*^{2d} \right) \left(\int_{Q_\ell} r_*^d [g]_\infty^2 \right)^2 \\ &\leq (\ell+1)^{2d} \int_{\mathbb{R}^d} r_*^{4d} [g]_\infty^4. \quad (3.25) \end{aligned}$$

We turn to the second RHS factor of (3.24). Note that the two-scale expansion error w_f satisfies the following equation (cf. [7, proof of Theorem 3])

$$-\nabla \cdot \mathbf{a} \nabla w_f = \nabla \cdot ((\mathbf{a}\phi_j + \sigma_j) \nabla \nabla_j \bar{u}). \quad (3.26)$$

By (3.13) with $q = 2$ applied to w_f , we obtain after taking local suprema of $\nabla^2\bar{u}$, and controlling correctors by Lemma 2.8,

$$\sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell| \left(\int_{Q_\ell} (|\nabla w_f| + |\phi| |\nabla^2\bar{u}|)^2 \right)^2 \lesssim \int_{\mathbb{R}^d} [(|\phi| + |\sigma|) \nabla^2\bar{u}]_2^4 \lesssim \int_{\mathbb{R}^d} C^4 \mu_*^4 [\nabla^2\bar{u}]_\infty^4. \quad (3.27)$$

Combined with (3.24) and (3.25), it yields the conclusion (3.21).

Substep 1.2. Proof of (3.22).

The definition (3.8) of $\|\cdot\|_{\ell+1}$, followed by Cauchy-Schwarz' inequality, yields

$$\|G_2\|_{\ell+1}^2 \leq \left(\sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell| \left(\int_{Q_\ell} (|\phi| |\nabla g| + |\nabla r|)^2 \right)^2 \right)^{\frac{1}{2}} \left(\sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell|^3 \left(\int_{Q_\ell} |\nabla u|^2 \right)^2 \right)^{\frac{1}{2}}. \quad (3.28)$$

We start with the first RHS factor. By (3.13) with $q = 2$ applied to the solution r of (2.17), we obtain after taking local suprema of ∇g and controlling correctors by Lemma 2.8,

$$\sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell| \left(\int_{Q_\ell} (|\phi| |\nabla g| + |\nabla r|)^2 \right) \lesssim \int_{\mathbb{R}^d} [(|\phi| + |\sigma|) |\nabla g|_2^4] \lesssim \int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 [\nabla g]_\infty^4. \quad (3.29)$$

We turn to the second RHS factor of (3.28). By (3.12) with $s = q = 2$ applied to the solution u of (1.3), we obtain for all $\alpha > d$ and $p > 1$ with $\alpha(p-1) < d(4p-1)$,

$$\begin{aligned} \sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell|^3 \left(\int_{Q_\ell} |\nabla u|^2 \right)^2 \\ \lesssim_{\alpha,p} (\ell+1)^{2d} r_*(0)^{\alpha \frac{p-1}{p}} \left(\int_{\mathbb{R}^d} r_*^{d \frac{2p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} |f|^{4p} \right)^{\frac{1}{p}}. \end{aligned} \quad (3.30)$$

Combined with (3.28) and (3.29), it yields (3.22).

Substep 1.3. Proof of (3.23).

The definition (3.8) of $\|\cdot\|_{\ell+1}$, followed by Cauchy-Schwarz' inequality and the mean-value property (3.7), yields

$$\begin{aligned} \|G_3\|_{\ell+1}^2 &\leq \sum_{Q_\ell \in \mathcal{P}_\ell} \left(\int_{Q_\ell} (|\phi^*| |\nabla(g\nabla\bar{u})| + |\nabla R|)^2 \right) \left(\int_{Q_\ell} |\nabla\phi + \text{Id}|^2 \right) \\ &\lesssim \sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell|^2 \left(\int_{Q_\ell} (|\phi^*| |\nabla(g\nabla\bar{u})| + |\nabla R|)^2 \right). \end{aligned}$$

By (3.12) with $s = q = 1$ applied to the solution R of (2.18), we obtain for all $\alpha > d$ and $p > 1$ with $\alpha(p-1) < d(2p-1)$,

$$\|G_3\|_{\ell+1}^2 \lesssim (\ell+1)^d r_*(0)^{\alpha \frac{p-1}{p}} \left(\int_{\mathbb{R}^d} r_*^{d \frac{p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} [|\phi^*| |\nabla(g\nabla\bar{u})|_2]^{2p} \right)^{\frac{1}{p}}.$$

Taking local suprema of $\nabla(g\nabla\bar{u})$ and using Lemma 2.8 to control correctors, (3.23) follows.

Step 2. Proof of (2.22) in the general weighted case ($\beta \leq d$).

As in Step 2 of the proof of Proposition 2.5, we need to refine (3.21)–(3.23) in the range $\ell > R$. More precisely, we shall establish that for all dyadic $\ell+1$, for all $0 \leq \gamma \leq d$, $0 < \alpha - d \ll 1$, and $0 < p-1 \ll_\alpha 1$,

$$\begin{aligned} \|G_1\|_{\ell+1}^2 &\lesssim_{\gamma,\alpha,p} \left(\int_{\mathbb{R}^d} \mathcal{C}^2 \mu_*^2 [\nabla^2 \bar{u}]_\infty^2 \right) \left(\int_{\mathbb{R}^d} r_*^d [g]_\infty^2 \right) \\ &\quad + (\ell+1)^\gamma R^{d-\gamma} r_*(0)^{d-\gamma+\alpha \frac{p-1}{2p}} \left(\int_{\mathbb{R}^d} r_*^{\gamma \frac{2p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{2p}} \\ &\quad \times \left(\int_{\mathbb{R}^d} r_*^{2pd} w_R^{2p(d-\gamma)+\alpha(p-1)} [g]_{4p}^2 \right)^{\frac{1}{2p}} \left(\int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 [\nabla^2 \bar{u}]_\infty^4 \right)^{\frac{1}{2}}, \end{aligned} \quad (3.31)$$

$$\begin{aligned} \|G_2\|_{\ell+1}^2 &\lesssim_{\gamma,\alpha,p} \left(\int_{\mathbb{R}^d} \mathcal{C}^2 \mu_*^2 [\nabla g]_\infty^2 \right) \left(\int_{\mathbb{R}^d} |f|^2 \right) \\ &\quad + (\ell+1)^\gamma R^{d-\gamma} r_*(0)^{d-\gamma+\alpha \frac{p-1}{2p}} \left(\int_{\mathbb{R}^d} r_*^{\gamma \frac{2p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{2p}} \\ &\quad \times \left(\int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 [\nabla g]_\infty^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{2p(d-\gamma)+\alpha(p-1)} |f|^{4p} \right)^{\frac{1}{2p}}, \end{aligned} \quad (3.32)$$

$$\begin{aligned}
\|G_3\|_{\ell+1}^2 &\lesssim_{\gamma,\alpha,p} (\ell+1)^{2d\frac{p-1}{p}} r_*(0)^{2d\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} \mathcal{C}^p \mu_*^p [\nabla(g\nabla\bar{u})]_\infty^p \right)^{\frac{2}{p}} \\
&\quad + (\ell+1)^\gamma R^{d-\gamma} r_*(0)^{d-\gamma+\alpha\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} r_*^{\gamma\frac{p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{p}} \\
&\quad \times \left(\int_{\mathbb{R}^d} w_R^{p(d-\gamma)+\alpha(p-1)} \mathcal{C}^{2p} \mu_*^{2p} [\nabla(g\nabla\bar{u})]_\infty^{2p} \right)^{\frac{1}{p}}. \tag{3.33}
\end{aligned}$$

Replacing p by $\frac{2p}{p+1}$, estimating $[\nabla(g\nabla\bar{u})]_\infty \lesssim [\nabla g]_\infty [\nabla\bar{u}]_\infty + [g]_\infty [\nabla^2\bar{u}]_\infty$, and using Hölder's inequality, the estimate (3.33) easily leads to

$$\begin{aligned}
\|G_3\|_{\ell+1}^2 &\lesssim_{\gamma,\alpha,p} (\ell+1)^{d\frac{p-1}{p}} r_*(0)^{d\frac{p-1}{p}} \left(\left(\int_{\mathbb{R}^d} \mathcal{C}^2 \mu_*^2 [\nabla g]_\infty^2 \right) \left(\int_{\mathbb{R}^d} [\nabla\bar{u}]_\infty^{2p} \right)^{\frac{1}{p}} \right. \\
&\quad \left. + \left(\int_{\mathbb{R}^d} \mathcal{C}^2 \mu_*^2 [\nabla^2\bar{u}]_\infty^2 \right) \left(\int_{\mathbb{R}^d} [g]_\infty^{2p} \right)^{\frac{1}{p}} \right) \\
&\quad + (\ell+1)^\gamma R^{d-\gamma} r_*(0)^{d-\gamma+\alpha\frac{p-1}{2p}} \left(\int_{\mathbb{R}^d} r_*^{\gamma\frac{2p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{2p}} \\
&\quad \times \left(\left(\int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 [\nabla g]_\infty^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{2p(d-\gamma)+\alpha(p-1)} [\nabla\bar{u}]_\infty^{4p} \right)^{\frac{1}{2p}} \right. \\
&\quad \left. + \left(\int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 [\nabla^2\bar{u}]_\infty^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{2p(d-\gamma)+\alpha(p-1)} [g]_\infty^{4p} \right)^{\frac{1}{2p}} \right).
\end{aligned}$$

By (3.10), $\|\partial^{\text{fct}} E(F)\|_\beta^2 \lesssim \sum_{k=0}^\infty 2^{-\beta k} \|\partial^{\text{fct}} E(F)\|_{2^k}^2$. We then apply (3.21)–(3.23) for $\ell = 2^k - 1 \leq R$ and (3.31)–(3.33) for $\ell = 2^k - 1 > R$, which yields the desired estimate (2.22) after arguing as in Substep 2.3 of the proof of Proposition 2.5. The rest of this step, which we split into three parts, is dedicated to the proof of (3.31)–(3.33).

Substep 2.1. Proof of (3.31).

By the definition (3.8) of $\|\cdot\|_{\ell+1}$, we obtain after singling out the cube Q_ℓ^0 centered at the origin, and using Cauchy-Schwarz' inequality,

$$\begin{aligned}
\|G_1\|_{\ell+1}^2 &\leq \left(\int_{Q_\ell^0} |g|^2 |\nabla\phi + \text{Id}|^2 \right) \left(\int_{Q_\ell^0} (|\nabla w_f| + |\phi| |\nabla^2\bar{u}|)^2 \right) \\
&\quad + \left(\sum_{Q_\ell \in \mathcal{P}_\ell^*} |Q_\ell|^3 \left(\int_{Q_\ell} |g|^2 |\nabla\phi + \text{Id}|^2 \right)^2 \right)^{\frac{1}{2}} \left(\sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell| \left(\int_{Q_\ell} (|\nabla w_f| + |\phi| |\nabla^2\bar{u}|)^2 \right)^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

We first take local suprema of g and apply Lemma 2.8 to control correctors, so that

$$\int_{Q_\ell^0} |g|^2 |\nabla\phi + \text{Id}|^2 \lesssim \int_{\mathbb{R}^d} r_*^d [g]_\infty^2.$$

Second, the energy estimate associated with equation (3.26) for w_f yields after taking local suprema of $\nabla^2\bar{u}$, and using Lemma 2.8,

$$\int_{Q_\ell^0} (|\nabla w_f| + |\phi| |\nabla^2\bar{u}|)^2 \lesssim \int_{\mathbb{R}^d} (|\phi| + |\sigma|)^2 |\nabla^2\bar{u}|^2 \lesssim \int_{\mathbb{R}^d} \mathcal{C}^2 \mu_*^2 [\nabla^2\bar{u}]_\infty^2.$$

Third, by (3.15) with $s = 2$, $q = 2$ applied to $|h| = |g| |\nabla\phi + \text{Id}|$ (and $v = 0$) and by Lemma 2.8, we obtain for all $0 \leq 2\gamma \leq 2d$, $\alpha > d$, and $p > 1$ with $2p(d-\gamma) + \alpha(p-1) <$

$d(4p - 1)$,

$$\begin{aligned} \sum_{Q_\ell \in \mathcal{P}_\ell^*} |Q_\ell|^3 \left(\int_{Q_\ell} |g|^2 |\nabla \phi + \text{Id}|^2 \right)^2 &\lesssim_{\gamma, \alpha, p} (\ell + 1)^{2\gamma} R^{2(d-\gamma)} r_*(0)^{2(d-\gamma) + \alpha \frac{p-1}{p}} \\ &\times \left(\int_{\mathbb{R}^d} r_*^{2\gamma \frac{p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} w_R^{2p(d-\gamma) + \alpha(p-1)} \underbrace{[g(\nabla \phi + \text{Id})]_2^{4p}}_{\lesssim r_*^{2dp} [g]_\infty^{4p}} \right)^{\frac{1}{p}}. \end{aligned}$$

The conclusion (3.31) then follows from the combination of the above estimates with (3.27).

Substep 2.2. Proof of (3.32).

We start from the definition (3.8) of $\|\cdot\|_{\ell+1}$, single out the cube Q_ℓ^0 , and use Cauchy-Schwarz' inequality on the other cubes, obtaining

$$\begin{aligned} \|G_2\|_{\ell+1}^2 &\leq \left(\int_{Q_\ell^0} (|\phi| |\nabla g| + |\nabla r|)^2 \right) \left(\int_{Q_\ell^0} |\nabla u|^2 \right) \\ &+ \left(\sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell| \left(\int_{Q_\ell} (|\phi| |\nabla g| + |\nabla r|)^2 \right)^2 \right)^{\frac{1}{2}} \left(\sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell|^3 \left(\int_{Q_\ell} |\nabla u|^2 \right)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

We first use an energy estimate for equations (2.17) and (1.3) for r and u , and obtain after taking local suprema of g and applying Lemma 2.8,

$$\begin{aligned} \left(\int_{Q_\ell^0} |\phi|^2 |\nabla g|^2 + |\nabla r|^2 \right) \left(\int_{Q_\ell^0} |\nabla u|^2 \right) &\lesssim \left(\int_{\mathbb{R}^d} (|\phi| + |\sigma|)^2 |\nabla g|^2 \right) \left(\int_{\mathbb{R}^d} |f|^2 \right) \\ &\lesssim \left(\int_{\mathbb{R}^d} \mathcal{C}^2 \mu_*^2 [\nabla g]_\infty^2 \right) \left(\int_{\mathbb{R}^d} |f|^2 \right). \end{aligned}$$

Second, we apply (3.13) with $q = 2$ to the solution r of (2.17), and obtain after taking local suprema of g and applying Lemma 2.8,

$$\sum_{Q_\ell \in \mathcal{P}_\ell} |Q_\ell| \left(\int_{Q_\ell} (|\phi| |\nabla g| + |\nabla r|)^2 \right)^2 \lesssim \int_{\mathbb{R}^d} [(|\phi| + |\sigma|) \nabla g]_2^4 \lesssim \int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 [\nabla g]_\infty^4.$$

Finally, we apply (3.15) with $s = q = 2$ to the solution u of (1.3), and obtain for all $0 \leq 2\gamma \leq 2d$, $\alpha > d$, and $p > 1$ with $2p(d - \gamma) + \alpha(p - 1) < d(4p - 1)$,

$$\begin{aligned} \sum_{Q_\ell \in \mathcal{P}_\ell^*} |Q_\ell|^3 \left(\int_{Q_\ell} |\nabla u|^2 \right)^2 &\lesssim_{\gamma, \alpha, p} (\ell + 1)^{2\gamma} R^{2(d-\gamma)} r_*(0)^{2(d-\gamma) + \alpha \frac{p-1}{p}} \left(\int_{\mathbb{R}^d} r_*^{\gamma \frac{2p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{p}} \\ &\times \left(\int_{\mathbb{R}^d} w_R^{2p(d-\gamma) + \alpha(p-1)} |f|^{4p} \right)^{\frac{1}{p}}. \quad (3.34) \end{aligned}$$

The combination of these last four estimates yields (3.32).

Substep 2.3. Proof of (3.33).

We start from the definition (3.8) of $\|\cdot\|_{\ell+1}$, single out the cube Q_ℓ^0 , and use Cauchy-Schwarz' inequality on the other cubes and the mean-value property (3.7). This yields

$$\begin{aligned} \|G_3\|_{\ell+1}^2 &\lesssim \left(\int_{Q_\ell^0} (|\phi^*|\nabla(g\nabla\bar{u})| + |\nabla R|)|\nabla\phi + \text{Id}| \right)^2 \\ &\quad + \sum_{Q_\ell \in \mathcal{P}_\ell^*} |Q_\ell|^2 \left(\int_{Q_\ell} (|\phi^*|\nabla(g\nabla\bar{u})| + |\nabla R|)^2 \right). \end{aligned} \quad (3.35)$$

We first treat the first RHS term. For $p > 1$, we argue as in (3.18), and obtain

$$\begin{aligned} \int_{Q_\ell^0} (|\phi^*|\nabla(g\nabla\bar{u})| + |\nabla R|)|\nabla\phi + \text{Id}| \\ \lesssim |Q_\ell^0|^{\frac{p-1}{p}} \left(\sum_{Q \in \mathcal{P}} |Q| \left(\int_Q (|\phi^*|\nabla(g\nabla\bar{u})| + |\nabla R|)^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}. \end{aligned}$$

Using (3.5) in the form $|Q_\ell^0| \lesssim (\ell+1)^d r_*(0)^d$, the large-scale Caderón-Zygmund estimate (2.26) with exponent $1 < p \leq 2$ applied to the solution R of (2.18) yields after taking local suprema of $\nabla(g\nabla\bar{u})$ and using Lemma 2.8,

$$\begin{aligned} \int_{Q_\ell^0} (|\phi^*|\nabla(g\nabla\bar{u})| + |\nabla R|)|\nabla\phi + \text{Id}| \\ \lesssim (\ell+1)^{d\frac{p-1}{p}} r_*(0)^{d\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} [(|\phi^*| + |\sigma^*|)\nabla(g\nabla\bar{u})]_2^p \right)^{\frac{1}{p}} \\ \lesssim (\ell+1)^{d\frac{p-1}{p}} r_*(0)^{d\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} \mathcal{C}^p \mu_*^p [\nabla(g\nabla\bar{u})]_\infty^p \right)^{\frac{1}{p}}. \end{aligned}$$

We then turn to the second RHS term in (3.35), apply (3.15) with $s = q = 1$ to the solution R of (2.18), and take local suprema of $\nabla(g\nabla\bar{u})$ and use Lemma 2.8 to control correctors. The conclusion (3.33) follows.

3.5. Proof of the main results. We mainly focus on the proof of the statements for the standard LSI ($\beta > d$), and quickly argue how to adapt the argument to general WLSI ($\beta \leq d$) in the last step.

Step 1. Proof of Proposition 1 for standard LSI ($\beta > d$).

Let $F \in C_c^\infty(\mathbb{R}^d)^{d \times d}$. Starting point is (2.19) in Proposition 2.5. By Hölder's inequality, the triangle inequality in probability, and the stationarity of r_* , we obtain for all $R \geq 1$, $0 < \alpha - d \ll 1$, $0 < p - 1 \ll_\alpha 1$, and $q \gg \frac{1}{p-1}$,

$$\mathbb{E} \left[\|\partial^{\text{fct}} J_0(F)\|^{2q} \right]^{\frac{1}{q}} \lesssim_{\alpha,p} \mathbb{E} \left[\left(r_*^{d+\alpha\frac{p-1}{p}} \right)^q \right]^{\frac{1}{q}} R^{d\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} |F|^{2p} \right)^{\frac{1}{p}}.$$

Replacing F by $\varepsilon^{\frac{d}{2}} F(\varepsilon \cdot)$ and choosing $R = \frac{1}{\varepsilon}$, this yields

$$\mathbb{E} \left[\|\partial^{\text{fct}} \widehat{J}_0^\varepsilon(F)\|^{2q} \right]^{\frac{1}{q}} \lesssim_{\alpha,p} \mathbb{E} \left[\left(r_*^{d+\alpha\frac{p-1}{p}} \right)^q \right]^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} w_1^{\alpha(p-1)} |F|^{2p} \right)^{\frac{1}{p}}. \quad (3.36)$$

We now recall the following implication (which follows from WLSI in form of the moment bounds in [3, Proposition 3.1(i)]; see also [7, Step 1 of the proof of Theorem 1]): for all random variables Y_1, Y_2 , given $q_0 \geq 1$ and $\kappa > 0$,

$$\begin{aligned} \mathbb{E} \left[\left\| \partial^{\text{fct}} Y_1 \right\|_{\beta}^{2q} \right]^{\frac{1}{q}} &\leq \mathbb{E} [Y_2^q]^{\frac{1}{q}} \quad \text{for all } q \geq q_0, \quad \text{and} \quad \mathbb{E} [\exp(Y_2^{\kappa})] \leq 2 \\ &\implies \exists C \simeq_{q_0, \kappa} 1 : \mathbb{E} \left[\exp \left(\frac{1}{C} Y_1^{2 \frac{\kappa}{1+\kappa}} \right) \right] \leq 2. \end{aligned} \quad (3.37)$$

Using this property and the moment bound of Lemma 2.7 for r_* , the estimate (3.36) leads to the conclusion (2.3).

Step 2. Proof of Theorem 1 for standard LSI ($\beta > d$).

Let $f, g \in C_c^\infty(\mathbb{R}^d)$. We split the proof into two substeps, first improving (2.21) to avoid local suprema in the estimate, and then turning to (2.5) itself.

Substep 2.1. Improvement of (2.21): for all $R \geq 1$, $0 < \alpha - d \ll 1$, and $0 < p - 1 \ll_{\alpha} 1$,

$$\begin{aligned} \left\| \partial^{\text{fct}} E(f, g) \right\|^2 &\lesssim_{\alpha, p} r_*(0)^{\alpha \frac{p-1}{2p}} \left(\int_{\mathbb{R}^d} r_*^{2d \frac{2p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{2p}} \\ &\quad \times \left(\left(\int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 (|\nabla f| + |\nabla^2 \bar{u}|)^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} |g|^{4p} \right)^{\frac{1}{2p}} \right. \\ &\quad \left. + \left(\int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 |\nabla g|^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} (|f| + |\nabla \bar{u}|)^{4p} \right)^{\frac{1}{2p}} \right). \end{aligned} \quad (3.38)$$

We first apply (2.21) to the averaged functions f_1 and g_1 defined by $f_1(x) := \mathcal{f}_{B(x)} f$ and $g_1(x) := \mathcal{f}_{B(x)} g$. Noting that $[f_1]_{\infty} \lesssim \mathcal{f}_{B_2(x)} f$ and that the solution \bar{u}_1 of the homogenized equation (1.4) with averaged RHS f_1 is given by $\bar{u}_1(x) = \mathcal{f}_{B(x)} \bar{u}$, and using the Lipschitz continuity of \mathcal{C} , we obtain for all $0 < \alpha - d \ll 1$ and $0 < p - 1 \ll_{\alpha} 1$,

$$\begin{aligned} \left\| \partial^{\text{fct}} E(f, g) \right\|^2 &\lesssim_{\alpha, p} \left\| \partial^{\text{fct}} E(f_1 - f, g_1) \right\|^2 + \left\| \partial^{\text{fct}} E(f, g_1 - g) \right\|^2 \\ &\quad + r_*(0)^{\alpha \frac{p-1}{2p}} \left(\int_{\mathbb{R}^d} r_*^{2d \frac{2p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{2p}} \left(\left(\int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 |\nabla^2 \bar{u}|^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} |g|^{4p} \right)^{\frac{1}{2p}} \right. \\ &\quad \left. + \left(\int_{\mathbb{R}^d} \mathcal{C}^4 \mu_*^4 |\nabla g|^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} (|f| + |\nabla \bar{u}|)^{4p} \right)^{\frac{1}{2p}} \right). \end{aligned} \quad (3.39)$$

It remains to estimate the first two RHS terms of (3.39), which we will prove to be small not because the two-scale expansion is accurate, but because $f_1 - f$ and $g_1 - g$ are small themselves. Arguing as in the proof of (2.15), we have the alternative formula

$$\partial^{\text{fct}} E(f, g) = g \otimes \nabla u - g \otimes \nabla \bar{u} - g \nabla_i \bar{u} \otimes (\nabla \phi_i + e_i) + \nabla t \otimes \nabla u - \nabla T_i \otimes (\nabla \phi_i + e_i),$$

where the auxiliary fields t and T are the Lax-Milgram solutions in \mathbb{R}^d of

$$\begin{aligned} -\nabla \cdot \mathbf{a}^* \nabla t &= \nabla \cdot ((\mathbf{a}^* - \bar{\mathbf{a}}^*)g), \\ -\nabla \cdot \mathbf{a}^* \nabla T_i &= \nabla \cdot ((\mathbf{a}^* - \bar{\mathbf{a}}^*)g \nabla_i \bar{u}). \end{aligned}$$

Using this decomposition and arguing as in the proof of Proposition 2.5, we obtain for all $R \geq 1$, $0 < \alpha - d \ll 1$, and $0 < p - 1 \ll_\alpha 1$,

$$\begin{aligned} \|\partial^{\text{fct}} E(f, g)\|^2 &\lesssim r_*(0)^{\alpha \frac{p-1}{2p}} \left(\int_{\mathbb{R}^d} r_*^{\frac{d-2p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{2p}} \\ &\quad \times \min \left\{ \left(\int_{\mathbb{R}^d} (|f| + |\nabla \bar{u}|)^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} |g|^{4p} \right)^{\frac{1}{2p}} ; \right. \\ &\quad \left. \left(\int_{\mathbb{R}^d} |g|^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} (|f| + |\nabla \bar{u}|)^{4p} \right)^{\frac{1}{2p}} \right\}. \end{aligned}$$

Using in addition that $|f - f_1| \leq \int_0^1 \int_{tB} |\nabla f(\cdot + y)| dy dt$, this turns into

$$\begin{aligned} &\|\partial^{\text{fct}} E(f_1 - f, g_1)\| + \|\partial^{\text{fct}} E(f, g_1 - g)\| \\ &\lesssim r_*(0)^{\alpha \frac{p-1}{2p}} \left(\int_{\mathbb{R}^d} r_*^{\frac{d-2p}{p-1}} w_R^{-\alpha} \right)^{\frac{p-1}{2p}} \left(\left(\int_{\mathbb{R}^d} (|\nabla f| + |\nabla^2 \bar{u}|)^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} |g|^{4p} \right)^{\frac{1}{2p}} \right. \\ &\quad \left. + \left(\int_{\mathbb{R}^d} |\nabla g|^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} (|f| + |\nabla \bar{u}|)^{4p} \right)^{\frac{1}{2p}} \right). \end{aligned}$$

Combining this with (3.39) leads to the conclusion (3.38).

Substep 2.2. Conclusion.

By Hölder's inequality, the triangle inequality in probability, and the stationarity of r_* , the estimate (3.38) leads to the following: for all $R \geq 1$, $0 < \alpha - d \ll 1$, $0 < p - 1 \ll_\alpha 1$, and $q \gg \frac{1}{p-1}$,

$$\begin{aligned} \mathbb{E} \left[\|\partial^{\text{fct}} E(f, g)\|^{2q} \right]^{\frac{1}{q}} &\lesssim_{\alpha, p} \mathbb{E} \left[\left(r_*^{2d + \alpha \frac{p-1}{2p}} \mathcal{C}^2 \right)^q \right]^{\frac{1}{q}} \\ &\quad \times R^{\frac{d}{2}(1 - \frac{1}{p})} \left(\left(\int_{\mathbb{R}^d} \mu_*^4 (|\nabla f| + |\nabla^2 \bar{u}|)^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} |g|^{4p} \right)^{\frac{1}{2p}} \right. \\ &\quad \left. + \left(\int_{\mathbb{R}^d} \mu_*^4 |\nabla g|^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{\alpha(p-1)} (|f| + |\nabla \bar{u}|)^{4p} \right)^{\frac{1}{2p}} \right). \end{aligned}$$

We then apply the standard weighted Calderón-Zygmund theory to the constant-coefficient equation (1.4) for \bar{u} , and replace f and g by $\varepsilon^{\frac{d}{4}} f(\varepsilon \cdot)$ and $\varepsilon^{\frac{d}{4}} g(\varepsilon \cdot)$. For the choice $R = \frac{1}{\varepsilon}$, and by the bound $\mu_*(\frac{\cdot}{\varepsilon}) \lesssim \mu_*(\frac{1}{\varepsilon}) \mu_*(\cdot)$, this implies

$$\begin{aligned} \mathbb{E} \left[\|\partial^{\text{fct}} \widehat{E}^\varepsilon(f, g)\|^{2q} \right]^{\frac{1}{q}} &\lesssim_{\alpha, p} \mathbb{E} \left[\left(r_*^{2d + \alpha \frac{p-1}{2p}} \mathcal{C}^2 \right)^q \right]^{\frac{1}{q}} \\ &\quad \times \varepsilon^2 \mu_*(\frac{1}{\varepsilon})^2 \left(\left(\int_{\mathbb{R}^d} \mu_*^4 |\nabla f|^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_1^{\alpha(p-1)} |g|^{4p} \right)^{\frac{1}{2p}} \right. \\ &\quad \left. + \left(\int_{\mathbb{R}^d} \mu_*^4 |\nabla g|^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_1^{\alpha(p-1)} |f|^{4p} \right)^{\frac{1}{2p}} \right). \quad (3.40) \end{aligned}$$

We now recall the following version of Hölder's inequality: for all random variables Y_1, Y_2 , given $\kappa_1, \kappa_2 > 0$,

$$\begin{aligned} \mathbb{E} [\exp(Y_1^{\kappa_1})] \leq 2 \quad \text{and} \quad \mathbb{E} [\exp(Y_2^{\kappa_2})] \leq 2 \\ \implies \exists C \simeq_{\kappa_1, \kappa_2} 1 : \mathbb{E} \left[\exp \left(\frac{1}{C} (Y_1 Y_2)^{\frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}} \right) \right] < \infty. \end{aligned} \quad (3.41)$$

Using this property, the moment bounds of Lemmas 2.7 and 2.8 for r_* and \mathcal{C} yield, for all $\eta > 0$, $\mathbb{E} [\exp(\frac{1}{C_\eta} (r_*^{2d} \mathcal{C}^2)^{\frac{1}{4} - \eta})] \leq 2$ for some $C_\eta \simeq_\eta 1$. Combining this with (3.40), property (3.37) yields the conclusion (2.5).

Step 3. General weighted LSI ($\beta \leq d$).

We start with Proposition 1. By Hölder's inequality, the triangle inequality in probability, and the stationarity of r_* , the estimate (2.20) in Proposition 2.5 leads to the following: for all $R \geq 1$, $0 < \gamma < \beta$, $0 < \alpha - d \ll 1$, $0 < p - 1 \ll_{\gamma, \alpha} 1$, and $q \gg \frac{1}{p-1}$,

$$\begin{aligned} \mathbb{E} \left[\|\partial^{\text{fct}} J_0(F)\|_\beta^{2q} \right]^{\frac{1}{q}} \lesssim_{\gamma, \alpha, p} \mathbb{E} \left[\left(r_*^{d + \alpha \frac{p-1}{p}} \right)^q \right]^{\frac{1}{q}} R^{2d} \pi_*(R)^{-1} \\ \times \left(R^{-\frac{d}{p}} \left(\int_{\mathbb{R}^d} w_R^{(d-\gamma)p + \alpha(p-1)} |F|^{2p} \right)^{\frac{1}{p}} + R^{-\frac{2d}{p}} \left(\int_{\mathbb{R}^d} [F]_2^p \right)^{\frac{2}{p}} \right). \end{aligned}$$

Replacing F by $\varepsilon^d \pi_*(\frac{1}{\varepsilon})^{\frac{1}{2}} F(\varepsilon \cdot)$ and choosing $R = \frac{1}{\varepsilon}$, this yields

$$\begin{aligned} \mathbb{E} \left[\|\partial^{\text{fct}} \widehat{J}_0^\varepsilon(F)\|_\beta^{2q} \right]^{\frac{1}{q}} \lesssim_{\gamma, \alpha, p} \mathbb{E} \left[\left(r_*^{d + \alpha \frac{p-1}{p}} \right)^q \right]^{\frac{1}{q}} \\ \times \left(\left(\int_{\mathbb{R}^d} w_1^{(d-\gamma)p + \alpha(p-1)} |F|^{2p} \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^d} [F]_2^p \right)^{\frac{2}{p}} \right). \end{aligned}$$

The combination of this estimate with property (3.37) and with the moment bound of Lemma 2.7 for r_* implies the desired estimate (2.3).

We finally turn to Theorem 1. Arguing as in Substep 2.1 above, we may get rid of local suprema in the estimate (2.22) in Proposition 2.6. Using then Hölder's inequality, the triangle inequality in probability, and the stationarity of r_* , we obtain the following: for all $R \geq 1$, $0 \leq \gamma < \beta$, $0 < \alpha - d \ll 1$, $0 < p - 1 \ll_\alpha 1$, and $q \gg \frac{1}{p-1}$,

$$\begin{aligned} \mathbb{E} \left[\|\partial^{\text{fct}} E(f, g)\|^{2q} \right]^{\frac{1}{q}} \lesssim_{\gamma, \alpha, p} \mathbb{E} \left[\left(r_*^{2d + \alpha \frac{p-1}{2p}} \mathcal{C}^2 \right)^q \right]^{\frac{1}{q}} \\ \times \left(R^{d + \frac{d}{2}(1 - \frac{1}{p})} \pi_*(R)^{-1} \left(\left(\int_{\mathbb{R}^d} \mu_*^4 (|\nabla f| + |\nabla^2 \bar{u}|)^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{2p(d-\gamma) + \alpha(p-1)} |g|^{4p} \right)^{\frac{1}{2p}} \right. \right. \\ \left. \left. + \left(\int_{\mathbb{R}^d} \mu_*^4 |\nabla g|^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} w_R^{2p(d-\gamma) + \alpha(p-1)} (|f| + |\nabla \bar{u}|)^{4p} \right)^{\frac{1}{2p}} \right) \right. \\ \left. + R^{-\beta} \left(\int_{\mathbb{R}^d} \mu_*^2 (|\nabla f| + |\nabla^2 \bar{u}|)^2 \right) \left(\left(\int_{\mathbb{R}^d} |g|^2 \right) + R^{d - \frac{d}{p}} \left(\int_{\mathbb{R}^d} |g|^{2p} \right)^{\frac{1}{p}} \right) \right. \\ \left. + R^{-\beta} \left(\int_{\mathbb{R}^d} \mu_*^2 |\nabla g|^2 \right) \left(\left(\int_{\mathbb{R}^d} (|f| + |\nabla \bar{u}|)^2 \right) + R^{d - \frac{d}{p}} \left(\int_{\mathbb{R}^d} (|f| + |\nabla \bar{u}|)^{2p} \right)^{\frac{1}{p}} \right) \right). \end{aligned}$$

Since in dimension $d \geq 2$ the weights μ_*^2 and μ_*^4 always belong to the Muckenhoupt classes A_2 and A_4 , respectively, we may apply the standard weighted Calderón-Zygmund theory to the constant-coefficient equation (1.4) for \bar{u} in order to simplify the above RHS. Replacing then f and g by $\pi_*(\frac{1}{\varepsilon})^{\frac{1}{4}}\varepsilon^{\frac{d}{2}}f(\varepsilon\cdot)$ and $\pi_*(\frac{1}{\varepsilon})^{\frac{1}{4}}\varepsilon^{\frac{d}{2}}g(\varepsilon\cdot)$, choosing $R = \frac{1}{\varepsilon}$, and using the bound $\mu_*(\frac{\cdot}{\varepsilon}) \lesssim \mu_*(\frac{1}{\varepsilon})\mu_*(\cdot)$, the conclusion (2.5) follows as in Substep 2.2.

ACKNOWLEDGEMENTS

The work of MD is supported by F.R.S.-FNRS (Belgian National Fund for Scientific Research) through a Research Fellowship. MD and AG acknowledge financial support from the European Research Council under the European Community's Seventh Framework Programme (FP7/2014-2019 Grant Agreement QUANTHOM 335410).

REFERENCES

- [1] G. Bal, J. Garnier, S. Motsch, and V. Perrier. Random integrals and correctors in homogenization. *Asymptot. Anal.*, 59(1-2):1–26, 2008.
- [2] M. Duerinckx, J. Fischer, and A. Gloria. Scaling limit of the homogenization commutator for Gaussian coefficient fields. In preparation.
- [3] M. Duerinckx and A. Gloria. Weighted functional inequalities: Concentration properties. Preprint, arXiv:1711.03148, 2017.
- [4] M. Duerinckx and A. Gloria. Weighted functional inequalities: Constructive approach. Preprint, arXiv:1711.03152, 2017.
- [5] M. Duerinckx, A. Gloria, and F. Otto. The structure of fluctuations in stochastic homogenization. Preprint, arXiv:1602.01717v3.
- [6] M. Duerinckx and F. Otto. Higher-order pathwise theory of fluctuations in stochastic homogenization. In preparation.
- [7] A. Gloria, S. Neukamm, and F. Otto. Quantitative estimates in stochastic homogenization for correlated fields. Preprint, arXiv:1409.2678v3.
- [8] A. Gloria, S. Neukamm, and F. Otto. A regularity theory for random elliptic operators. Preprint, arXiv:1409.2678v3.
- [9] Y. Gu. A central limit theorem for fluctuations in 1D stochastic homogenization. *Stoch. Partial Differ. Equ. Anal. Comput.*, 4(4):713–745, 2016.
- [10] Y. Gu and G. Bal. Random homogenization and convergence to integrals with respect to the Rosenblatt process. *J. Differential Equations*, 253(4):1069–1087, 2012.
- [11] Y. Gu and J.-C. Mourrat. Scaling limit of fluctuations in stochastic homogenization. *Multiscale Model. Simul.*, 14(1):452–481, 2016.
- [12] S. M. Kozlov. The averaging of random operators. *Mat. Sb. (N.S.)*, 109(151)(2):188–202, 327, 1979.
- [13] A. Lechiheb, I. Nourdin, G. Zheng, and E. Haouala. Convergence of random oscillatory integrals in the presence of long-range dependence and application to homogenization. *Probab. Math. Statist.*, in press.
- [14] G. C. Papanicolaou and S. R. S. Varadhan. Boundary value problems with rapidly oscillating random coefficients. In *Random fields, Vol. I, II (Esztergom, 1979)*, volume 27 of *Colloq. Math. Soc. János Bolyai*, pages 835–873. North-Holland, Amsterdam, 1981.
- [15] M. S. Taqqu. Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrsch. Verw. Gebiete*, 50(1):53–83, 1979.
- [16] S. Torquato. *Random heterogeneous materials*, volume 16 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, 2002. Microstructure and macroscopic properties.

(Antoine Gloria) SORBONNE UNIVERSITÉ, UMR 7598, LABORATOIRE JACQUES-LOUIS LIONS, PARIS,
FRANCE & UNIVERSITÉ LIBRE DE BRUXELLES, BRUSSELS, BELGIUM

E-mail address: `antoine.gloria@upmc.fr`

(Felix Otto) MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, LEIPZIG, GERMANY

E-mail address: `otto@mis.mpg.de`