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Advances on the controllability of bilinear quantum systems on compact graphs

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Abstract

In this work, we consider the bilinear Schrödinger equation (*BSE*) $i\partial_t\psi = -\Delta\psi + u(t)B\psi$ in the Hilbert space $L^2(\mathcal{G}, \mathbb{C})$ with \mathcal{G} a compact graph. The Laplacian $-\Delta$ is equipped with self-adjoint boundary conditions, B is a bounded symmetric operator and $u \in L^2((0, T), \mathbb{R})$ with $T > 0$. We study new hypotheses leading to the global exact controllability of the (*BSE*) in $D(|\Delta|^{s/2})$ with $s \geq 3$. Afterwards, we introduce the “energetic controllability”, a weaker notion of controllability useful when the global exact controllability fails. We provide examples of the main results involving for instance star graphs.

1 Introduction

In this work, we consider $\mathcal{H} := L^2(\mathcal{G}, \mathbb{C})$ with \mathcal{G} a compact graph.

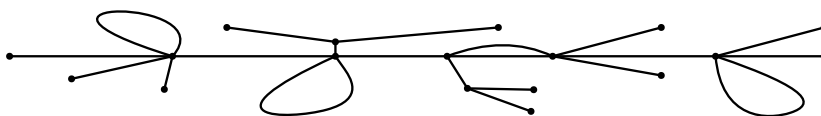


Figure 1: A compact graph is an one-dimensional domain composed by finite vertices (points) connected by edges (segments) of finite lengths.

We study the bilinear Schrödinger equation in the Hilbert space \mathcal{H}

$$(BSE) \quad \begin{cases} i\partial_t\psi(t) = A\psi(t) + u(t)B\psi(t), & t \in (0, T), \\ \psi(0) = \psi_0, & T > 0. \end{cases}$$

The Laplacian $A = -\Delta$ is equipped with self-adjoint boundary conditions, B is a bounded symmetric operator and $u \in L^2((0, T), \mathbb{R})$. When the (*BSE*) is well-posed, we call Γ_t^u the unitary propagator generated by $A + u(t)B$.

A possible use of the (*BSE*) is to model the dynamics of a particle constrained in a one-dimensional structure shaped as \mathcal{G} and subjected to an external field represented by $u(t)B$. The operator B describes the action of the field and u its intensity. The term $u(t)B$ plays the role of a control and it is natural to wonder whether the (*BSE*) is *exactly controllable*, i.e. given any states ψ^1 and ψ^2 , there exists u steering the system from ψ^1 to ψ^2 .

Even though control problems involving graphs have been very popular in the last decades, the bilinear Schrödinger equation on compact graphs has only been studied in [Duc18a]. In the mentioned work, the *well-posedness* and the *global exact controllability* of the (*BSE*) are provided under suitable assumptions in some spaces $D(|A|^{s/2})$ with $s \geq 3$.

In the current manuscript, we develop a new technique leading to the global exact controllability of the (*BSE*) in some $D(|A|^{s/2})$ with $s \geq 3$. In order to study those systems where this outcome is not guaranteed, we introduce a weaker notion of controllability denoted *energetic controllability*.

Proving the exact controllability in subspaces of $D(A)$ is classical for this type of result and it is due to the work [BMS82] on bilinear systems by Ball, Mardsen and Slemrod. Nevertheless [BMS82] guarantees that the bilinear Schrödinger equation is well-posed in \mathcal{H} , it also ensure that the (*BSE*) is not exactly controllable in \mathcal{H} and in $D(A)$ when $B : D(A) \rightarrow D(A)$ (see [BMS82, *Theorem 3.6*]). Because of this negative result, we proceed as in [Duc18a] and we address the problem in suitable subspaces of $D(A)$.

This approach has been widely used in order to study the (*BSE*) on the bounded interval $\mathcal{G} = (0, 1)$ (after being popularized in [BL10]). Let

$$D(A_D) = H^2((0, 1), \mathbb{C}) \cap H_0^1((0, 1), \mathbb{C}), \quad A_D \psi := -\Delta \psi, \quad \forall \psi \in D(A_D).$$

In [BL10], Beauchard and Laurent prove the *well-posedness* and the *local exact controllability* of the bilinear Schrödinger equation in $H_{(0)}^s := D(A_D^{s/2})$ for $s = 3$, when B is a multiplication operator for suitable $\mu \in H^3((0, 1), \mathbb{R})$. In [Mor14], Morancey studies the *simultaneous local exact controllability* of two or three (*BSE*) in $H_{(0)}^3$ for suitable $B = \mu \in H^3((0, 1), \mathbb{R})$.

In [MN15], Morancey and Nersesyan extend the previous result. They achieve the *simultaneous global exact controllability* of finitely many bilinear Schrödinger equations in $H_{(0)}^4$ for suitable $B = \mu \in H^4((0, 1), \mathbb{R})$.

In [Duc18c], the author ensures the *simultaneous global exact controllability in projection* of infinite (*BSE*) in $H_{(0)}^3$ for suitable bounded symmetric B . Under similar assumptions, he exhibits the *global exact controllability* of the (*BSE*) between eigenstates via explicit controls and times in [Duc18b].

1.1 Framework

We adopt the notation introduced in [Duc18a]. Let \mathcal{G} be a compact graph composed by $N \in \mathbb{N}$ edges $\{e_j\}_{j \leq N}$ of lengths $\{L_j\}_{j \leq N}$ and $M \in \mathbb{N}$ vertices

$\{v_j\}_{j \leq M}$. We call V_e and V_i the external and the internal vertices of \mathcal{G} , *i.e.*

$$V_e := \{v \in \{v_j\}_{j \leq M} \mid \exists! e \in \{e_j\}_{j \leq N} : v \in e\}, \quad V_i := \{v_j\}_{j \leq M} \setminus V_e.$$

We study graphs equipped with a metric, which parametrizes each edge e_j with a coordinate going from 0 to its length L_j . A graph is compact when it is composed by a finite number of vertices and edges of finite length.

We consider functions $f := (f^1, \dots, f^N) : \mathcal{G} \rightarrow \mathbb{C}$ with domain a compact metric graph \mathcal{G} so that $f^j : e_j \rightarrow \mathbb{C}$ with $j \leq N$. For $s > 0$, we denote

$$\mathcal{H} = L^2(\mathcal{G}, \mathbb{C}) = \prod_{j \leq N} L^2(e_j, \mathbb{C}), \quad H^s := H^s(\mathcal{G}, \mathbb{C}) = \prod_{j \leq N} H^s(e_j, \mathbb{C}).$$

The Hilbert space \mathcal{H} is equipped with the norm $\|\cdot\|$ and the scalar product

$$\langle \psi, \varphi \rangle := \langle \psi, \varphi \rangle_{\mathcal{H}} = \sum_{j \leq N} \langle \psi^j, \varphi^j \rangle_{L^2(e_j, \mathbb{C})} = \sum_{j \leq N} \int_{e_j} \overline{\psi^j(x)} \varphi^j(x) dx, \quad \forall \psi, \varphi \in \mathcal{H}.$$

In the *(BSE)*, the operator A is a self-adjoint Laplacian such that the functions in $D(A)$ satisfy the following boundary conditions. Each $v \in V_i$ is equipped with *Neumann-Kirchhoff* boundary conditions when

$$\begin{cases} f \text{ is continuous in } v, \\ \sum_{e \ni v} \frac{\partial f}{\partial x_e}(v) = 0, \end{cases} \quad \forall f \in D(A).$$

The derivatives are assumed to be taken in the directions entering to the vertex (ingoing directions). In addition, the external vertices V_e are equipped with *Dirichlet* or *Neumann* type boundary conditions.

The operator A admits purely discrete spectrum (see [Kuc04, *Theorem 18*]). We call $\{\lambda_k\}_{k \in \mathbb{N}}$ the non-decreasing sequence of eigenvalues of A and $\{\phi_k\}_{k \in \mathbb{N}}$ a Hilbert basis of \mathcal{H} composed by corresponding eigenfunctions.

1.2 Novelties of the work

Global exact controllability. In the works [BL10], [Duc18c], [Duc18b] and [Mor14], local and global exact controllability results for the *(BSE)* are proved when $\mathcal{G} = (0, 1)$ thanks to the hypothesis $\inf_{k \in \mathbb{N}} |\lambda_{k+1} - \lambda_k| \geq 0$, which is not guaranteed when \mathcal{G} is a generic compact graphs. However,

$$(1) \quad \exists \mathcal{M} \in \mathbb{N}, \delta' > 0 \quad : \quad \inf_{k \in \mathbb{N}} |\lambda_{k+\mathcal{M}} - \lambda_k| > \delta' \mathcal{M}$$

as showed in [Duc18a, *relation (2)*] (when \mathcal{G} is not an interval $\mathcal{M} \neq 1$). In [Duc18a], the global exact controllability of the *(BSE)* is provided when an additional condition is verified. Indeed, the author assumes the existence of $C > 0$ and a suitable $\tilde{d} \geq 0$ such that $|\lambda_{k+1} - \lambda_k| \geq Ck^{-\tilde{d}}$ for every $k \in \mathbb{N}$.

In the current work, we study an alternative set of assumptions leading to the global exact controllability. We hypothesize the existence of an entire function G such that $G \in L^\infty(\mathbb{R})$ and there exist $J, I > 0$ such that $|G(z)| \leq J e^{I|z|}$ for every $z \in \mathbb{C}$. We also assume that $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^+$ are pairwise distinct numbers, $\{\pm\sqrt{\lambda_k}\}_{k \in \mathbb{N}}$ are simple zeros of G and

$$\exists \tilde{d} \geq 1, C > 0 \quad : \quad |G'(\pm\sqrt{\lambda_k})| \geq C k^{-\tilde{d}}, \quad \forall k \in \mathbb{N}.$$

If these assumptions are verified for suitable $\tilde{d} \geq 0$, then the global exact controllability of the (BSE) can be provided in $H_{\mathcal{G}}^s := D(|A|^{s/2})$ with $s \geq 3$ depending on the parameter \tilde{d} . We resume the result in Theorem 3.2.

Let \mathcal{G} be a *star graph* composed by $N \in \mathbb{N}$ edges $\{e_j\}_{j \leq N}$. Each edge e_j is parametrized with a coordinate going from 0 to the length of the edge L_j . We set the 0 in the external vertex belonging to e_j .

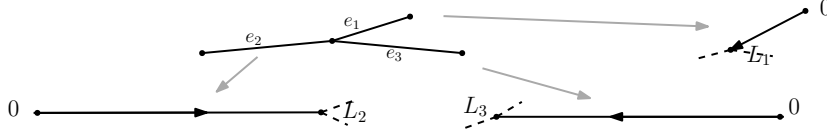


Figure 2: Parametrization of a star graph with $N = 3$ edges.

In Theorem 3.3 and Corollary 3.5, thanks to diophantine techniques, we ensure the global exact controllability of the (BSE) when \mathcal{G} is a suitable star graph. Despite similar results are stated in [Duc18a, Theorem 2.4] and [Duc18a, Remark 2.5] when $N \leq 4$, we prove it for every $N \in \mathbb{N}$.

Definition 1.1. For every $N \in \mathbb{N}$, we define $\mathcal{AL}(N) \subset (\mathbb{R}^+)^N$ as follows. For every $\{L_j\}_{j \leq N} \in \mathcal{AL}(N)$, the numbers $\{1, \{L_j\}_{j \leq N}\}$ are linearly independent over \mathbb{Q} and all the ratios L_k/L_j are algebraic irrational numbers.

Example 1.2. Let \mathcal{G} be a star graph equipped with Dirichlet boundary conditions in V_e and Neumann-Kirchhoff boundary conditions in V_i . Let

$$B : \psi = (\psi^1, \dots, \psi^N) \mapsto ((x - L_1)^4 \psi^1, 0, \dots, 0), \quad \forall \psi \in \mathcal{H}.$$

There exists $\mathcal{C} \subset (\mathbb{R}^+)^N$ countable such that, for every $\{L_j\}_{j \leq N} \in \mathcal{AL}(N) \setminus \mathcal{C}$, the (BSE) is globally exactly controllable in $H_{\mathcal{G}}^{4+\epsilon}$ for every $\epsilon \in (0, 1/2)$, i.e.

$$\forall \psi^1, \psi^2 \in H_{\mathcal{G}}^{4+\epsilon} : \|\psi^1\| = \|\psi^2\|, \exists T > 0, u \in L^2((0, T), \mathbb{R}) : \Gamma_T^u \psi^1 = \psi^2.$$

Proof. See Remark 4.2. □

In Example 1.2, we notice that the controllability holds even though the control field only acts on one edge of \mathcal{G} since $\{L_j\}_{j \leq N} \in \mathcal{AL}(N)$.

Energetic controllability. Let $\{\varphi_k\}_{k \in \mathbb{N}}$ be an orthonormal system of \mathcal{H} composed by eigenfunction of A and let $\{\mu_k\}_{k \in \mathbb{N}}$ be the relative eigenvalues. The bilinear Schrödinger equation is energetically controllable in $\{\mu_k\}_{k \in \mathbb{N}}$ if,

$$\forall m, n \in \mathbb{N}, \quad \exists T > 0, \quad u \in L^2((0, T), \mathbb{R}) \quad : \quad \Gamma_T^u \varphi_m = \varphi_n.$$

Physically speaking, the result guarantees that the energy of the quantum system $i\partial_t \psi = A\psi$ in $L^2(\mathcal{G}, \mathbb{C})$ can be controlled in specific energy levels through the external field $u(t)B$. We resume the outcome in Theorem 3.7.

Example 1.3. Let \mathcal{G} be a star graph equipped with Dirichlet boundary conditions in V_e and Neumann-Kirchhoff boundary conditions in V_i . Let \mathcal{G} be composed by $N \in \mathbb{N}$ edges of equal length L and let $B\psi = ((x-L)^2\psi^1, 0, \dots, 0)$ for every $\psi \in \mathcal{H}$. The (BSE) is energetically controllable in $\{\frac{k^2\pi^2}{4L^2}\}_{k \in \mathbb{N}}$.

Proof. See Section 5. □

The peculiarity of Example 1.3 is that the result is valid even though the spectrum of A presents multiple eigenvalues and Theorem 3.2 is not satisfied (also [Duc18a, Theorem 2.3] is not guaranteed). Moreover, the energetic controllability is ensured with respect to all the energy levels of the quantum system $i\partial_t \psi = A\psi$ in $L^2(\mathcal{G}, \mathbb{C})$. Indeed, $\{\frac{k^2\pi^2}{4L^2}\}_{k \in \mathbb{N}}$ are the eigenvalues of A non-repeated with their multiplicity (see the proof of Example 1.3).

The energetic controllability is also useful when \mathcal{G} is too difficult to analyze. By watching the structure of the graph, it is possible to explicit some eigenvalues $\{\mu_k\}_{k \in \mathbb{N}}$ and test if the energetic controllability in $\{\mu_k\}_{k \in \mathbb{N}}$ is satisfied. In Section 5, we discuss some examples where the result is verified, *e.g.* graphs containing self-closing edges.



Figure 3: Example of graph containing more self-closing edges.

1.3 Scheme of the work

In Section 2, we provide some preliminaries on the (BSE) and we introduce the assumptions adopted in the work. Moreover, Proposition 2.1 presents the well-posedness of the (BSE) in the spaces $H_{\mathcal{G}}^s$ with suitable $s > 0$.

In Section 3.1, we provide the main outcomes of the manuscript. The global exact controllability is presented for generic graphs in Theorem 3.2, while for suitable star graphs in Theorem 3.3 and Corollary 3.5. The abstract result regarding the energetic controllability is ensured in Theorem 3.7.

In Section 4 and Section 5, we discuss some examples of the main results (as Example 1.2 and Example 1.3). In particular, the global exact controllability is treated in Section 4 and the energetic controllability in Section 5.

In Appendix A, we present some spectral results adopted in the work, while we study the solvability of the so-called *moments problems* in Appendix B.

2 Preliminaries

In the current work, we denote a graph \mathcal{G} as *quantum graph* when a self-adjoint Laplacian A is defined on \mathcal{G} . As in [Duc18a], we respectively call (\mathcal{NK}) , (\mathcal{D}) and (\mathcal{N}) the *Neumann-Kirchhoff*, *Dirichlet* and *Neumann* boundary conditions characterizing $D(A)$. We say that \mathcal{G} is equipped with one of the previous boundaries in a vertex v , when each $f \in D(A)$ satisfies it in v .

By simplifying the notation of [Duc18a], we say that \mathcal{G} is equipped with (\mathcal{D}) (or (\mathcal{N})) when, for every $f \in D(A)$, the function f satisfies (\mathcal{D}) (or (\mathcal{N})) in every $v \in V_e$ and verifies (\mathcal{NK}) in every $v \in V_i$. In addition, the graph \mathcal{G} is equipped with $(\mathcal{D}/\mathcal{N})$ when, for every $f \in D(A)$ and $v \in V_e$, the function f satisfies (\mathcal{D}) (or (\mathcal{N})) in v and f verifies (\mathcal{NK}) in every $v \in V_i$.

Let $\Phi := \{\phi_k\}_{k \in \mathbb{N}}$ be a complete orthonormal system of \mathcal{H} composed by eigenfunctions of A and let $\{\lambda_k\}_{k \in \mathbb{N}}$ be the corresponding eigenvalues. Let $[r]$ be the entire part of $r \in \mathbb{R}$. For $s > 0$, we define the spaces

$$H_{\mathcal{NK}}^s := \left\{ \psi \in H^s \mid \partial_x^{2n} \psi \text{ is continuous in } v, \quad \forall n \in \mathbb{N} \cup \{0\}, \quad n < [(s+1)/2]; \right. \\ \left. \sum_{e \in \mathcal{N}(v)} \partial_{x_e}^{2n+1} \psi(v) = 0, \quad \forall n \in \mathbb{N} \cup \{0\}, \quad n < [s/2], \quad \forall v \in V_i \right\},$$

$$H_{\mathcal{G}}^s = H_{\mathcal{G}}^s(\mathcal{G}, \mathbb{C}) := D(A^{s/2}), \quad \|\cdot\|_{(s)} := \|\cdot\|_{H_{\mathcal{G}}^s} = \left(\sum_{k \in \mathbb{N}} |k^s \langle \cdot, \phi_k \rangle|^2 \right)^{\frac{1}{2}},$$

$$h^s(\mathbb{C}) := \left\{ \{a_k\}_{k \in \mathbb{N}} \subset \mathbb{C} \mid \sum_{k \in \mathbb{N}} |k^s a_k|^2 < \infty \right\}, \quad \|\cdot\|_{(s)} := \left(\sum_{k \in \mathbb{N}} |k^s \cdot|^2 \right)^{\frac{1}{2}}$$

(we refer to [Duc18a, Proposition 3.2] for details on $H_{\mathcal{G}}^s$ with $s \in \mathbb{R}^+ \setminus \mathbb{N}$).

Remark. [Duc18a, Remark 2.3] *There exists $c \in \mathbb{R}$ such that $\|\cdot\|_{(s)} \asymp \| |A + c|^{\frac{s}{2}} \cdot \|$. If $0 \notin \sigma(A)$ (the spectrum of A), then we can assume $c = 0$. We underline that the only eigenvalue of A that can be 0 is λ_1 .*

Let $\varphi := \{\varphi_k\}_{k \in \mathbb{N}} \subseteq \{\phi_k\}_{k \in \mathbb{N}}$ be an orthonormal system of \mathcal{H} and let $\{\mu_k\}_{k \in \mathbb{N}} \subseteq \{\lambda_k\}_{k \in \mathbb{N}}$ be the corresponding eigenvalues. We introduce

$$\eta > 0, \quad a \geq 0, \quad I := \{(j, k) \in \mathbb{N}^2 : j \neq k\}, \quad \widetilde{\mathcal{H}} := \overline{\text{span}\{\varphi_k \mid k \in \mathbb{N}\}}^{L^2}.$$

Assumptions (I(φ, η)). The bounded symmetric operator B is such that $\text{Ran}(B|_{H_{\mathcal{G}}^2 \cap \widetilde{\mathcal{H}}}) \subseteq H_{\mathcal{G}}^2 \cap \widetilde{\mathcal{H}}$ and the following conditions are satisfied.

1. There exists $C > 0$ such that $|\langle \varphi_k, B\varphi_1 \rangle| \geq \frac{C}{k^{2+\eta}}$ for every $k \in \mathbb{N}$.
2. For every $(j, k), (l, m) \in I$ such that $(j, k) \neq (l, m)$ and $\mu_j - \mu_k = \mu_l - \mu_m$, it holds $\langle \varphi_j, B\varphi_j \rangle - \langle \varphi_k, B\varphi_k \rangle - \langle \varphi_l, B\varphi_l \rangle + \langle \varphi_m, B\varphi_m \rangle \neq 0$.

Assumptions (I(η)). The couple (A, B) satisfies Assumptions I(Φ, η).

Assumptions (II(φ, η, a)). Let one of the following points be satisfied.

1. When \mathcal{G} is equipped with $(\mathcal{D}/\mathcal{N})$ and $a + \eta \in (0, 3/2)$, there exists $d \in [\max\{a + \eta, 1\}, 3/2)$ such that $\text{Ran}(B|_{H_{\mathcal{G}}^{2+d} \cap \widetilde{\mathcal{H}}}) \subseteq H^{2+d} \cap H_{\mathcal{G}}^2 \cap \widetilde{\mathcal{H}}$.
2. When \mathcal{G} is equipped with (\mathcal{N}) and $a + \eta \in (0, 7/2)$, there exist $d \in [\max\{a + \eta, 2\}, 7/2)$ and $d_1 \in (d, 7/2)$ such that $\text{Ran}(B|_{H_{\mathcal{G}}^{2+d} \cap \widetilde{\mathcal{H}}}) \subseteq H^{2+d} \cap H_{\mathcal{N}\mathcal{K}}^{1+d} \cap H_{\mathcal{G}}^2 \cap \widetilde{\mathcal{H}}$ and $\text{Ran}(B|_{H_{\mathcal{N}\mathcal{K}}^{d_1} \cap \widetilde{\mathcal{H}}}) \subseteq H_{\mathcal{N}\mathcal{K}}^{d_1} \cap \widetilde{\mathcal{H}}$.
3. When \mathcal{G} is equipped with (\mathcal{D}) and $a + \eta \in (0, 5/2)$, there exists $d \in [\max\{a + \eta, 1\}, 5/2)$ such that $\text{Ran}(B|_{H_{\mathcal{G}}^{2+d} \cap \widetilde{\mathcal{H}}}) \subseteq H^{2+d} \cap H_{\mathcal{N}\mathcal{K}}^{1+d} \cap H_{\mathcal{G}}^2 \cap \widetilde{\mathcal{H}}$. If $a + \eta \geq 2$, then there exists $d_1 \in (d, 5/2)$ such that $\text{Ran}(B|_{H^{d_1} \cap \widetilde{\mathcal{H}}}) \subseteq H^{d_1} \cap \widetilde{\mathcal{H}}$.

Assumptions (II(η, a)). The couple (A, B) satisfies Assumptions II(Φ, η, a).

From now on, we omit the terms φ, η and a from the notations of Assumptions I and Assumptions II when their are not relevant.

Proposition 2.1. [Duc18a, Proposition 3.1] *Let \mathcal{G} be a compact quantum graph. Let the couple (A, B) satisfy Assumptions II(η, \tilde{d}) with $\eta > 0$ and $\tilde{d} \geq 0$. Let $\psi^0 \in H_{\mathcal{G}}^{2+d}$ with d introduced in Assumptions II and $u \in L^2((0, T), \mathbb{R})$. There exists a unique mild solution of (BSE) in $H_{\mathcal{G}}^{2+d}$, i.e. a function $\psi \in C_0([0, T], H_{\mathcal{G}}^{2+d})$ such that for every $t \in [0, T]$,*

$$(2) \quad \psi(t, x) = e^{-iAt}\psi^0(x) - i \int_0^t e^{-iA(t-s)}u(s)B\psi(s, x)ds.$$

Moreover, there exists $C = C(T, B, u) > 0$ so that

$$\|\psi\|_{C^0([0, T], H_{\mathcal{G}}^{2+d})} \leq C\|\psi^0\|_{H_{\mathcal{G}}^{2+d}}, \quad \|\psi(t)\| = \|\psi^0\|, \quad \forall t \in [0, T], \psi_0 \in H_{\mathcal{G}}^{2+d}.$$

Remark 2.2. *Let $\varphi := \{\varphi_k\}_{k \in \mathbb{N}} \subseteq \{\phi_k\}_{k \in \mathbb{N}}$ be an orthonormal system of \mathcal{H} made by eigenfunctions of A and $\widetilde{\mathcal{H}} := \overline{\text{span}\{\varphi_k \mid k \in \mathbb{N}\}}^{L^2}$. We notice that, if (A, B) satisfies Assumptions II(φ, η, \tilde{d}) with $\eta > 0$ and $\tilde{d} > 0$, then the following claim is valid. For every $\psi^0 \in H_{\mathcal{G}}^{2+d} \cup \widetilde{\mathcal{H}}$ with d introduced in Assumptions II and $u \in L^2((0, T), \mathbb{R})$, there exists a unique mild solution of (BSE) in $H_{\mathcal{G}}^{2+d} \cup \widetilde{\mathcal{H}}$. The statement follows equivalently to Proposition 2.1 as Γ_t^u preserves $H_{\mathcal{G}}^2 \cap \widetilde{\mathcal{H}}$ since $B : H_{\mathcal{G}}^2 \cap \widetilde{\mathcal{H}} \rightarrow H_{\mathcal{G}}^2 \cap \widetilde{\mathcal{H}}$.*

3 Main results

3.1 Global exact controllability

Definition 3.1. The (BSE) is said to be globally exactly controllable in the space $H_{\mathcal{G}}^s$ with $s \geq 3$ if, for every $\psi^1, \psi^2 \in H_{\mathcal{G}}^s$ such that $\|\psi^1\| = \|\psi^2\|$, there exist a time $T > 0$ and a control $u \in L^2((0, T), \mathbb{R})$ such that $\Gamma_T^u \psi^1 = \psi^2$.

Theorem 3.2. Let \mathcal{G} be a compact quantum graph and $\{\lambda_k\}_{k \in \mathbb{N}}$ be pairwise distinct real numbers satisfying (1) with $\mathcal{M} \in \mathbb{N} \setminus \{1\}$. Let $G \in L^\infty(\mathbb{R})$ be an entire function such that there exist $J, I > 0$ such that $|G(z)| \leq J e^{I|z|}$ for every $z \in \mathbb{C}$. The numbers $\{\pm\sqrt{\lambda_k}\}_{k \in \mathbb{N}}$ are simple zeros of G and

$$\exists \tilde{d} \geq 1, C > 0 \quad : \quad |G'(\pm\sqrt{\lambda_k})| \geq C k^{-\tilde{d}}, \quad \forall k \in \mathbb{N}.$$

If the couple (A, B) satisfies Assumptions I(η) and Assumptions II($\eta, \tilde{d} - 1$) for $\eta > 0$, then the (BSE) is globally exactly controllable in $H_{\mathcal{G}}^s$ for $s = 2 + d$ and d from Assumptions II.

Proof. The result follows from the proof of [Duc18a, Theorem 2.3], which can be found in [Duc18a, Section 4]. In fact, it is obtained by gathering in $H_{\mathcal{G}}^s$ the local exact controllability and the global approximate controllability. The only difference is on the proof of the local exact controllability and in how the solvability of the moments problem [Duc18a, relation (17)] is provided, *i.e.* in how we prove that, for every $\{x_k\}_{k \in \mathbb{N}} \in \{\{x_k\}_{k \in \mathbb{N}} \in h^s(\mathbb{C}) \mid ix_1 \in \mathbb{R}\} \subset h^s$, there exists $u \in L^2((0, T), \mathbb{R})$ with $T > 0$ such that

$$(3) \quad ix_k / \langle \phi_k, B\phi_1 \rangle = \int_0^T u(\tau) e^{i(\lambda_k - \lambda_1)\tau} d\tau, \quad \forall k \in \mathbb{N}.$$

As B is symmetric, $\langle \phi_1, B\phi_1 \rangle \in \mathbb{R}$ and $ix_1 / \langle \phi_1, B\phi_1 \rangle \in \mathbb{R}$. After, we know $\{x_k (\langle \phi_k, B\phi_1 \rangle)^{-1}\}_{k \in \mathbb{N}} \in h^{d-\eta} \subseteq h^{\tilde{d}-1}$ thanks to the point **1.** of Assumptions I. Thus, the solvability of (3) is valid for $\{x_k (\langle \phi_k, B\phi_1 \rangle)^{-1}\}_{k \in \mathbb{N}} \in \{\{c_k\}_{k \in \mathbb{N}} \in h^{\tilde{d}-1}(\mathbb{C}) \mid c_1 \in \mathbb{R}\}$ thanks to Proposition B.7 and the local exact controllability holds in $H_{\mathcal{G}}^s$ with $s = 2 + d$ and d from Assumptions II.

In conclusion, the (BSE) is globally approximately controllable in $H_{\mathcal{G}}^s$ as proved in [Duc18a, Section 4.2]. The statement is achieved by gathering the two results as for the proof of [Duc18a, Theorem 2.3]. \square

In the following theorem, we ensure the global exact controllability when \mathcal{G} is a star graph. The result leads to Example 1.2.

Theorem 3.3. Let \mathcal{G} be a star graph equipped with $(\mathcal{D}/\mathcal{N})$ made by edges long $\{L_j\}_{j \leq N} \in \mathcal{AL}(N)$. If the couple (A, B) satisfies Assumptions I(η) and Assumptions II(η, ϵ) for $\eta, \epsilon > 0$, then the (BSE) is globally exactly controllable in $H_{\mathcal{G}}^s$ for $s = 2 + d$ and d from Assumptions II.

Proof. 1) Star graph equipped with (D). The conditions (D) on V_e imply that $\phi_k = (a_k^1 \sin(\sqrt{\lambda_k}x), \dots, a_k^n \sin(\sqrt{\lambda_k}x))$ with $\{a_k^l\}_{l \leq N} \subset \mathbb{C}$ and $k \in \mathbb{N}$. The conditions (NK) in the internal vertex v_0 ensure that there hold $a_k^1 \sin(\sqrt{\lambda_k}L_1) = \dots = a_k^N \sin(\sqrt{\lambda_k}L_N)$ and $\sum_{l \leq N} a_k^l \cos(\sqrt{\lambda_k}L_l) = 0$ for every $k \in \mathbb{N}$. Then, $\sum_{l=1}^N \cot(\sqrt{\lambda_k}L_l) = 0$ for every $k \in \mathbb{N}$. Let

$$G(x) := \prod_{l \leq N} \sin(xL_l) \sum_{l \leq N} \cot(xL_l) \quad \tilde{G}(x) := \prod_{l \leq N} \sin(xL_l) \sum_{l \leq N} \frac{L_l}{\sin^2(xL_l)}.$$

Now, $G(\lambda_k) = 0$ for every $k \in \mathbb{N}$, $G \in L^\infty(\mathbb{R})$ and G is an entire function such that $|G(z)| \leq N e^{|z| \sum_{l=1}^N L_l}$ for every $z \in \mathbb{C}$ since $|\cos(zL_l)| \leq e^{L_l|z|}$ and $|\sin(zL_l)| \leq e^{L_l|z|}$ for every $l \leq N$. For $L^* := \min_{l \leq N} L_l$ and $x \in \mathbb{R}$,

$$|\tilde{G}(x)| = \frac{\prod_{l \leq N} |\sin(xL_l)| \sum_{l \leq N} L_l \prod_{k \neq l} \sin^2(xL_k)}{\prod_{l \leq N} \sin^2(xL_l)} \geq L^* \sum_{l \leq N} \prod_{k \neq l} |\sin(xL_k)|.$$

After, $G'(\sqrt{\lambda_k}) = -\tilde{G}'(\sqrt{\lambda_k})$ for each $k \in \mathbb{N}$ since $G'(x) = -\tilde{G}'(x) + H(x)$,

$$H(x) := \frac{d}{dx} \left(\prod_{l \leq N} \cos(xL_l) \right) \sum_{l \leq N} \cot(xL_l), \quad H(\sqrt{\lambda_k}) = 0.$$

We refer to [DZ06, Corollary A.10; (2)], which contains a misprint as it is valid for every $\lambda > \frac{\pi}{2} \max\{1/L_j : j \leq N\}$. For every $\epsilon > 0$, there exists $C_1 > 0$ such that $|G'(\pm\sqrt{\lambda_k})| \geq L^* \sum_{l=1}^N \prod_{j \neq l} |\sin(\sqrt{\lambda_k}L_j)| \geq \frac{C_1}{(\sqrt{\lambda_k})^{1+\epsilon}}$ for every $k \geq \hat{k}$ with $\hat{k} \in \mathbb{N}$ such that $\lambda_{\hat{k}} > \frac{\pi}{2} \max\{1/L_j : j \leq N\}$.

Remark 3.4. For every $k \in \mathbb{N}$ and $j \leq N$, we have $|\phi_k^j(L_j)| \neq 0$, otherwise the (NK) conditions would ensure the existence of $l, m \leq N$ so that

$$\phi_k^l(L_l) = a_k^l \sin(L_l \sqrt{\lambda_k}) = a_k^m \sin(L_m \sqrt{\lambda_k}) = \phi_k^m(L_m) = 0, \quad \phi_k^l, \phi_k^m \neq 0$$

with $a_k^l, a_k^m \neq 0$, which is absurd as $\{L_j\}_{j \leq N} \in \mathcal{AL}(N)$.

Remark 3.4 implies $|G'(\pm\sqrt{\lambda_k})| \neq 0$ for $k < \hat{k}$ and, from [Duc18a, Remark A.4],

$$\forall \epsilon > 0 \quad \exists C_2 > 0 : \quad |G'(\pm\sqrt{\lambda_k})| \geq C_2 k^{-1-\epsilon}, \quad \forall k \in \mathbb{N}.$$

Now, the spectrum of A is simple, otherwise it would be possible to construct eigenfunctions of A violating the statement of Remark 3.4. In conclusion, the claim is achieved thanks to Theorem 3.2 for $\tilde{d} = 1 + \epsilon$.

2) Generic star graph. Let $I_1 \subseteq \{1, \dots, N\}$ be the set of indices of those edges containing an external vertex equipped with (N) and $I_2 := \{1, \dots, N\} \setminus I_1$. The proof follows from the techniques adopted in 1) by using Proposition A.2 instead of [DZ06, Corollary A.10; (2)] with

$$G(x) := \prod_{l \in I_2} \sin(xL_l) \prod_{l \in I_1} \cos(xL_l) \left(\sum_{l \in I_2} \cot(xL_l) + \sum_{l \in I_1} \tan(xL_l) \right). \quad \square$$

Corollary 3.5. Let \mathcal{G} be a star graph equipped with (\mathcal{D}/N) . Let \mathcal{G} satisfy the following conditions with $\tilde{N} \in 2\mathbb{N}$ such that $\tilde{N} \leq N$.

- For every $j \leq \tilde{N}/2$, the external vertices of \mathcal{G} belonging to e_{2j-1} and e_{2j} are equipped with the same type of boundary conditions.
- The couples of edges $\{e_{2j-1}, e_{2j}\}_{j \leq \tilde{N}/2}$ are long $\{L_j\}_{j \leq \tilde{N}/2}$, while the edges $\{e_j\}_{\tilde{N} < j \leq N}$ measure $\{L_j\}_{\tilde{N} < j \leq N}$.

If $\{L_j\}_{j \leq \tilde{N}/2} \cup \{L_j\}_{\tilde{N} < j \leq N} \in \mathcal{AL}(\frac{\tilde{N}}{2} + N - \tilde{N})$, while the couple (A, B) satisfies Assumptions I(η) and Assumptions II(η, ϵ) for $\eta, \epsilon > 0$, then the (BSE) is globally exactly controllable in $H_{\mathcal{G}}^s$ for $s = 2 + d$ and d from Assumptions II.

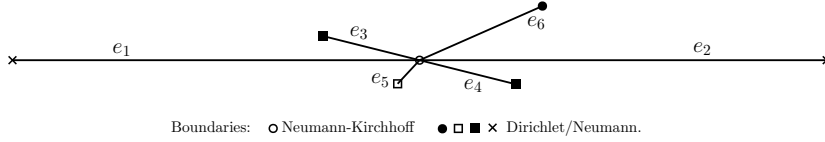


Figure 4: Example of \mathcal{G} described in Corollary 3.5 with $\tilde{N} = 4$ and $N = 6$.

Proof. Let $I_1 \subseteq \{1, \dots, \tilde{N}/2\}$ be the set of indices j such that e_{2j-1} and e_{2j} contain two external vertices of \mathcal{G} equipped with (\mathcal{N}) and $I_2 := \{1, \dots, \tilde{N}/2\} \setminus I_1$. Let $I_3 \subseteq \{\tilde{N} + 1, \dots, N\}$ be the set of indices of those edges in $\{e_j\}_{\tilde{N} < j \leq N}$ containing an external vertex of \mathcal{G} equipped with (\mathcal{N}) and $I_4 := \{\tilde{N} + 1, \dots, N\} \setminus I_3$. We notice that $\left\{ \frac{(2k-1)^2 \pi^2}{4L_j^2} \right\}_{j \in I_1, k \in \mathbb{N}} \cup \left\{ \frac{k^2 \pi^2}{L_j^2} \right\}_{j \in I_2, k \in \mathbb{N}}$ are simple eigenvalues of A corresponding to some eigenfunctions $\{f_{k;j}\}_{j \in I_1 \cup I_2, k \in \mathbb{N}} \subseteq \{\phi_k\}_{k \in \mathbb{N}}$. Afterwards, every $f \in \{\phi_k\}_{k \in \mathbb{N}} \setminus \{f_{k;j}\}_{j \in I_1 \cup I_2, k \in \mathbb{N}}$ is such that $f(v) \neq 0$ with $v \in V_e$ (similarly to Remark 3.4) and it corresponds to a simple eigenvalue. The statement is proved as Theorem 3.3 by considering the function

$$G(x) := \prod_{l \in I_2 \cup I_4} \sin(xL_l) \prod_{l \in I_1 \cup I_3} \cos(xL_l) \left(2 \sum_{l \in I_2} \cot(xL_l) + 2 \sum_{l \in I_1} \tan(xL_l) + \sum_{l \in I_4} \cot(xL_l) + \sum_{l \in I_3} \tan(xL_l) \right). \quad \square$$

3.2 Energetic controllability

Let $\{\varphi_k\}_{k \in \mathbb{N}} \subseteq \{\phi_k\}_{k \in \mathbb{N}}$ be an orthonormal system of \mathcal{H} made by eigenfunctions of A and let $\{\mu_k\}_{k \in \mathbb{N}}$ be the corresponding eigenvalues.

Definition 3.6. The (BSE) is energetically controllable in $\{\mu_k\}_{k \in \mathbb{N}}$ if, for every $m, n \in \mathbb{N}$, there exist $T > 0$ and $u \in L^2((0, T), \mathbb{R})$ so that $\Gamma_T^u \varphi_m = \varphi_n$.

Theorem 3.7. *Let \mathcal{G} be a compact quantum graph and one of the following set of assumptions be satisfied.*

1. *There exists an entire function G such that $G \in L^\infty(\mathbb{R})$ and there exist $J, I > 0$ so that $|G(z)| \leq J e^{I|z|}$ for every $z \in \mathbb{C}$. The numbers $\{\mu_k\}_{k \in \mathbb{N}}$ satisfy (1) with $\mathcal{M} \in \mathbb{N} \setminus \{1\}$, $\{\pm\sqrt{\mu_k}\}_{k \in \mathbb{N}}$ are simple zeros of G and there exist $\tilde{d} \geq 1$ and $C > 0$ so that $|G'(\pm\sqrt{\mu_k})| \geq \frac{C}{k^{\tilde{d}}}$ for every $k \in \mathbb{N}$.*
2. *For every $\epsilon > 0$, there exist $C > 0$ and $\tilde{d} \geq 1$ so that $|\mu_{k+1} - \mu_k| \geq C k^{-\tilde{d}-1}$ for each $k \in \mathbb{N}$.*

If (A, B) satisfies Assumptions I(φ, η) and Assumptions II($\varphi, \eta, \tilde{d} - 1$) for $\eta > 0$, then the (BSE) is globally exactly controllable in $H_{\mathcal{G}}^s \cap \widetilde{\mathcal{H}}$ for $s = 2 + d$ with d from Assumptions I and energetically controllable in $\{\mu_k\}_{k \in \mathbb{N}}$.

Proof. From Remark 2.2, the (BSE) is well-posed in $H_{\mathcal{G}}^s \cap \widetilde{\mathcal{H}}$ with $s = 2 + d$ and d from Assumptions II. The statement of Theorem 3.2 holds in $\widetilde{\mathcal{H}}$ when the point **1.** is valid, while the validity of [Duc18a, Theorem 2.3] in $\widetilde{\mathcal{H}}$ is guaranteed by **2.** . The global exact controllability is provided in $H_{\mathcal{G}}^s \cap \widetilde{\mathcal{H}}$ and the energetic controllability follows as $\varphi_k \in H_{\mathcal{G}}^s \cap \widetilde{\mathcal{H}}$ for every $k \in \mathbb{N}$. \square

Let \mathcal{G} be a generic compact quantum graph. By watching the structure of the graph and the boundary conditions of $D(A)$, it is possible to construct some eigenfunctions $\{\varphi_k\}_{k \in \mathbb{N}}$ of A corresponding to some eigenvalues $\{\mu_k\}_{k \in \mathbb{N}}$. For instance, we consider \mathcal{G} containing a self-closing edge e_1 of length 1.

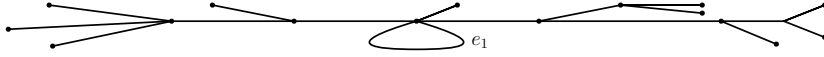


Figure 5: Example of graph containing a self-closing edge.

We construct $\varphi := \{\varphi_k\}_{k \in \mathbb{N}}$ some eigenfunctions of A such that

$$\varphi_k = (\sqrt{2} \sin((2k - 1)\pi x), 0, \dots, 0),$$

and corresponding to $\{\mu_k\}_{k \in \mathbb{N}} = \{(2k - 1)^2 \pi^2\}_{k \in \mathbb{N}} \subseteq \{\lambda_k\}_{k \in \mathbb{N}}$. If Assumptions I(φ, η) and Assumptions II($\varphi, \eta, 0$) are satisfied for $\eta > 0$, then Theorem 3.7 implies to the energetic controllability in $\{\mu_k\}_{k \in \mathbb{N}}$. The same holds when \mathcal{G} contains more self-closing edges as Figure 3 (*i.e.* Example 5.3).

Remark. *The idea described above can be adopted when \mathcal{G} contains suitable sub-graphs denoted “uniform chains”. A uniform chain is a sequence of edges of equal length L connecting $M \in \mathbb{N}$ vertices $\{v_j\}_{j \leq M}$ such that $v_2, \dots, v_{M-1} \in V_i$. We also assume that either $v_1, v_M \in V_e$ are equipped with (\mathcal{D}) , $v_1 = v_M \in V_i$, or $M = 3$ and $v_1, v_3 \in V_e$ are equipped with (\mathcal{N}) .*

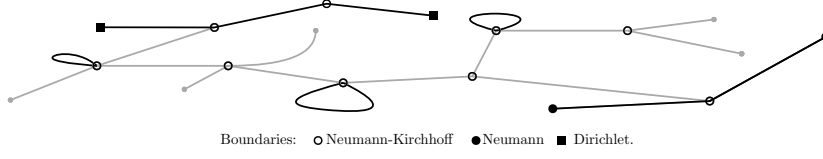


Figure 6: The figure underlines the uniform chains in a generic graph.

Let \mathcal{G} contain $\tilde{N} \in \mathbb{N}$ uniform chains $\{\tilde{\mathcal{G}}_j\}_{j \leq \tilde{N}}$, composed by edges of lengths $\{L_j\}_{j \leq \tilde{N}} \in \mathcal{AL}(\tilde{N})$. Let $I_1 \subseteq \{1, \dots, \tilde{N}\}$ and $I_2 \subseteq \{1, \dots, \tilde{N}\} \setminus I_1$ be respectively the sets of indices j such that the external vertices of $\tilde{\mathcal{G}}_j$ are equipped with (\mathcal{N}) and (\mathcal{D}) , while $I_3 := \{1, \dots, \tilde{N}\} \setminus (I_1 \cup I_2)$. If $\{L_j\}_{j \leq \tilde{N}} \in \mathcal{AL}(\tilde{N})$, then the energetic controllability can be guaranteed in

$$\left\{ \frac{(2k-1)^2 \pi^2}{4L_j^2} \right\}_{\substack{k, j \in \mathbb{N} \\ j \in I_1}} \cup \left\{ \frac{k^2 \pi^2}{L_j^2} \right\}_{\substack{k, j \in \mathbb{N} \\ j \in I_2}} \cup \left\{ \frac{(2k-1)^2 \pi^2}{L_j^2} \right\}_{\substack{k, j \in \mathbb{N} \\ j \in I_3}}.$$

4 Examples: Global exact controllability

Example 4.1. Let \mathcal{G} be a star graph equipped with (\mathcal{N}) . For $\psi \in \mathcal{H}$, let $B(\psi^1, \dots, \psi^N) = ((5x^6 - 24x^5 L_1 + 45x^4 L_1^2 - 40x^3 L_1^3 + 15x^2 L_1^4 - L_1^6)\psi^1, 0, \dots, 0)$.

There exists $\mathcal{C} \subset (\mathbb{R}^+)^N$ countable so that, for every $\{L_j\}_{j \leq N} \in \mathcal{AL}(N) \setminus \mathcal{C}$, the problem (BSE) is globally exactly controllable in $H_{\mathcal{G}}^{5+\epsilon}$ with $\epsilon \in (0, 1/2)$.

Proof. The conditions (\mathcal{N}) in V_i imply $\phi_k = (a_k^1 \cos(x\sqrt{\lambda_k}), \dots, a_k^N \cos(x\sqrt{\lambda_k}))$ with $\{a_k^l\}_{l \leq N} \subset \mathbb{R}$ and $k \in \mathbb{N}$ so that $\{\phi_k\}_{k \in \mathbb{N}}$ forms a Hilbert basis of \mathcal{H} i.e. $1 = \sum_{l \leq N} \int_0^{L_l} |a_k^l|^2 \cos^2(x\sqrt{\lambda_k}) dx = \sum_{l \leq 4} |a_k^l|^2 \left(\frac{L_l}{2} + \frac{\cos(L_l \sqrt{\lambda_k}) \sin(L_l \sqrt{\lambda_k})}{2\sqrt{\lambda_k}} \right)$. For every $k \in \mathbb{N}$, the (\mathcal{NK}) conditions in V_i ensure $a_k^1 \cos(\sqrt{\lambda_k} L_1) = \dots = a_k^N \cos(\sqrt{\lambda_k} L_N)$ and $\sum_{l \leq N} a_k^l \sin(\sqrt{\lambda_k} L_l) = 0$. Thus, we have

$$(4) \quad \sum_{l \leq N} \tan(\sqrt{\lambda_k} L_l) = 0, \quad \sum_{l \leq N} |a_k^l|^2 \sin(L_l \sqrt{\lambda_k}) \cos(L_l \sqrt{\lambda_k}) = 0.$$

Now, $1 = \sum_{l=1}^N |a_k^l|^2 L_l / 2$ and the continuity implies $a_k^l = a_k^1 \frac{\cos(\sqrt{\lambda_k} L_1)}{\cos(\sqrt{\lambda_k} L_l)}$ for $l \neq 1$ and $k \in \mathbb{N}$, which ensures $|a_k^1|^2 \left(L_1 + \sum_{l=2}^4 L_l \frac{\cos^2(\sqrt{\lambda_k} L_1)}{\cos^2(\sqrt{\lambda_k} L_l)} \right) = 2$. Thus,

$$(5) \quad |a_k^1|^2 = 2 \prod_{m=2}^4 \cos^2(\sqrt{\lambda_k} L_m) \left(\sum_{j=1}^4 L_j \prod_{m \neq j} \cos^2(\sqrt{\lambda_k} L_m) \right)^{-1}, \quad \forall k \in \mathbb{N}.$$

From (4) and (5), there holds $\sum_{l=1}^N \sin(\sqrt{\lambda_k} L_l) \prod_{m \neq l} \cos(\sqrt{\lambda_k} L_m) = 0$. Thanks to [Duc18a, Remark A.4] and [Duc18a, Proposition A.5], for every $\epsilon > 0$, there exist $C_1, C_2 > 0$ such that, for every $k \in \mathbb{N}$,

$$(6) \quad |a_k^1| \geq \sqrt{\frac{2}{\sum_{l=1}^N L_l \cos^{-2}(\sqrt{\lambda_k} L_l)}} \geq \sqrt{\frac{2}{\sum_{l=1}^N L_l C_1^{-2} \lambda_k^{1+\epsilon}}} \geq \frac{C_2}{k^{1+\epsilon}}.$$

In addition, $\langle \phi_1^l, B\phi_k^l \rangle_{L^2(e_l, \mathbb{C})} = 0$ for $2 \leq l \leq N$ and, for every $k \in \mathbb{N}$,

$$(7) \quad \langle \phi_1, B\phi_k \rangle = -\frac{120a_k^1 a_1^1 L_1^6}{(\sqrt{\lambda_k} + \sqrt{\lambda_1})^4} - \frac{120a_k^1 a_1^1 L_1^6}{(\sqrt{\lambda_k} - \sqrt{\lambda_1})^4} + o(\sqrt{\lambda_k}^{-5}).$$

As in [Duc18a, *Example 1.2*], the couples (A, B) satisfies Assumptions I($3 + \epsilon$) with $\epsilon > 0$. Indeed, one can define $a_k(\cdot)$ and $B_k(\cdot)$ with $k \in \mathbb{N}$, analytic functions in \mathbb{R}^+ , so that $a_k(L_1) = (a_k^1)^2$, $\sqrt{a_1(L_1)a_k(L_1)}B_k(L_1) = \langle \phi_1, B\phi_k \rangle$ and each $\sqrt{a_1(\cdot)a_k(\cdot)}B_k(\cdot)$ is non-constant and analytic. The set of positive zeros $\tilde{V}_k \subset \mathbb{R}^+$ of $\sqrt{a_1(\cdot)a_k(\cdot)}B_k(\cdot)$ is discrete and $\tilde{V} = \bigcup_{k \in \mathbb{N}} \tilde{V}_k$ is countable. For every $\{L_l\}_{l \leq N} \in \mathcal{AL}(N)$ such that $L_1 \notin \tilde{V}$, we have $|\langle \phi_1, B\phi_k \rangle| \neq 0$ for every $k \in \mathbb{N}$. From [DZ06, *Remark A.4*], the point **1.** of Assumptions I($3 + \epsilon$) follows thanks to (6) and (7), *i.e*

$$\forall \epsilon > 0, \exists C > 0 \quad : \quad |\langle \phi_1, B\phi_k \rangle| \geq C_3/k^{5+\epsilon}, \quad \forall k \in \mathbb{N}.$$

Let $(k, j), (m, n) \in I$, $(k, j) \neq (m, n)$ for $I := \{(j, k) \in \mathbb{N}^2 : j \neq k\}$. Again as in [Duc18a, *Example 1.2*], we compute $F_k(\cdot)$ with $k \in \mathbb{N}$, analytic in \mathbb{R}^+ , such that $\langle \phi_k, B\phi_k \rangle = F_k(L_1)$. Each $F_{j,k,l,m}(\cdot) := F_j(\cdot) - F_k(\cdot) - F_l(\cdot) + F_m(\cdot)$ is non-constant and analytic in \mathbb{R}^+ , the set of its positive zeros $V_{j,k,l,m}$ is discrete and $V := \bigcup_{(j,k),(l,m) \in I : (j,k) \neq (l,m)} V_{j,k,l,m}$ is a countable. For $\{L_l\}_{l \leq N} \in \mathcal{AL}(N)$ so that $L_1 \notin V \cup \tilde{V}$, Assumptions I($3 + \epsilon$) are verified.

Let $P(x) := (5x^6 - 24x^5 L_1 + 45x^4 L_1^2 - 40x^3 L_1^3 + 15x^2 L_1^4 - L_1^6)$. We prove the validity of the point **2.** of Assumptions II($3 + \epsilon_1, \epsilon_2$) with $\epsilon_1, \epsilon_2 > 0$ so that $\epsilon_1 + \epsilon_2 \in (0, 1/2)$. For $m > 0$, we have $B : H^m \rightarrow H^m$ and $\partial_x P(0) = \partial_x P(L_1) = 0$ that imply $B : H_{\mathcal{G}}^2 \rightarrow H_{\mathcal{G}}^2$. In addition, it holds

$$P(L_1) = 0, \quad \partial_x^2 P(L_1) = 0, \quad \partial_x^3 P(L_1) = 0 \implies B : H_{\mathcal{NK}}^m \quad \forall m \in (0, 9/2).$$

Given $d \in [3 + \epsilon_1 + \epsilon_2, 7/2)$ and $d_1 \in (d, 7/2)$, we have $\text{Ran}(B|_{H_{\mathcal{G}}^{2+d}}) \subseteq H^{2+d} \cap H_{\mathcal{NK}}^{1+d} \cap H_{\mathcal{G}}^2$ and $\text{Ran}(B|_{H_{\mathcal{NK}}^{d_1}}) \subseteq H_{\mathcal{NK}}^{d_1}$, which lead to the point **2.** of Assumptions II($3 + \epsilon_1, \epsilon_2$). Theorem 3.3 guarantees the statement. \square

Remark 4.2. *Example 1.2 is proved as [Duc18a, Example 1.2] that ensures the same result with $N = 4$. The only difference is that Example 1.2 follows from Theorem 3.3 instead of [Duc18a, Theorem 2.4] (valid for $N \leq 4$).*

5 Examples: Energetic controllability

Proof of Example 1.3. Let us assume $N = 3$. The (\mathcal{D}) conditions lead to

$$\exists (a_k^1, a_k^2, a_k^3) \in \mathbb{C}^3 \quad : \quad \phi_k = (a_k^1 \sin(\sqrt{\lambda_k}x), a_k^2 \sin(\sqrt{\lambda_k}x), a_k^3 \sin(\sqrt{\lambda_k}x)).$$

By imposing (\mathcal{NK}) in the internal vertex v_0 , we obtain $\sum_{l \leq 3} a_k^l \cos(\sqrt{\lambda_k}L) = 0$ and $a_k^m \sin(\sqrt{\lambda_k}L) = c \in \mathbb{R}$ for every $m \leq 3$. When $c \neq 0$, we have

the eigenvalues $\left\{\frac{(2k-1)^2\pi^2}{4L^2}\right\}_{k\in\mathbb{N}}$ corresponding to the eigenfunctions $\{g_k\}_{k\in\mathbb{N}}$ so that $g_k = \left(\sqrt{\frac{2}{3L}} \sin\left(\frac{(2k-1)\pi}{2L}x\right), \sqrt{\frac{2}{3L}} \sin\left(\frac{(2k-1)\pi}{2L}x\right), \sqrt{\frac{2}{3L}} \sin\left(\frac{(2k-1)\pi}{2L}x\right)\right)$. When $c = 0$, we obtain the eigenvalues $\left\{\frac{k^2\pi^2}{L^2}\right\}_{k\in\mathbb{N}}$ of multiplicity two that we associate to couples of sequences of eigenfunctions $\{f_k^1\}_{k\in\mathbb{N}}$ and $\{f_k^2\}_{k\in\mathbb{N}}$ such that every $f_k^1 := \left(-\sqrt{\frac{4}{3L}} \sin\left(\frac{k\pi}{L}x\right), \sqrt{\frac{1}{3L}} \sin\left(\frac{k\pi}{L}x\right), \sqrt{\frac{1}{3L}} \sin\left(\frac{k\pi}{L}x\right)\right)$ and $f_k^2 := \left(0, -\sqrt{\frac{1}{L}} \sin\left(\frac{k\pi}{L}x\right), \sqrt{\frac{1}{L}} \sin\left(\frac{k\pi}{L}x\right)\right)$. Moreover, $\{f_k^1\}_{k\in\mathbb{N}} \cup \{f_k^2\}_{k\in\mathbb{N}} \cup \{g_k\}_{k\in\mathbb{N}}$ is an Hilbert basis of \mathcal{H} and the eigenvalues of A (not considering their multiplicity) are $\left\{\frac{k^2\pi^2}{L^2}\right\}_{k\in\mathbb{N}} \cup \left\{\frac{(2k-1)^2\pi^2}{4L^2}\right\}_{k\in\mathbb{N}}$

We reorder $\{f_k^1\}_{k\in\mathbb{N}} \cup \{g_k\}_{k\in\mathbb{N}}$ in $\varphi = \{\varphi_k\}_{k\in\mathbb{N}}$. The point **2.** of Assumptions I($\varphi, 1$) follows since there exist $C_3, C_4 > 0$ such that, for every $k \in \mathbb{N}$, we have $B_{k,k} := \langle \varphi_k, B\varphi_k \rangle = C_3 + C_4k^{-2} \in \mathbb{R}^+$ and $\mu_k = \frac{\pi^2 k^2}{4L}$. If

$$\mu_j - \mu_k - \mu_l + \mu_m = 0 \quad \implies \quad B_{j,j} - B_{k,k} - B_{l,l} + B_{m,m} \neq 0.$$

The point **1.** of Assumptions I($\varphi, 1$) is verified as there exists $C_1, C_2 > 0$ such that $|\langle \varphi_1, B\varphi_k \rangle| \geq \frac{C_1\sqrt{\lambda_k}\sqrt{\lambda_1}}{(\lambda_k - \lambda_1)^2} \geq \frac{C_2}{k^3}$ for every $k \in \mathbb{N}$.

Now, B stabilizes $\overline{\text{span}\{\varphi_k : k \in \mathbb{N}\}}^{L^2} \cap H_{\mathcal{G}}^2$ and H^m for every $m > 0$, ensuring the point **1.** of Assumptions II($\varphi, 1, 0$). Since $\inf_{j,k \in \mathbb{N}} |\mu_k - \mu_j| = \frac{\pi^2}{4L^2}$, the point **2.** of Theorem 3.7 holds and the result is proved.

When $N > 3$, the spectrum contains simple eigenvalues relative to some eigenfunctions $\{g_k\}_{k\in\mathbb{N}}$ and multiple eigenvalues each one corresponding to $N - 1$ eigenfunctions $\{f_{k;j}\}_{l \leq N-1}$ with $k \in \mathbb{N}$ (see [BK13, p. 15]). For each $k \in \mathbb{N}$, we construct $\{f_{k;j}\}_{l \leq N-1}$ such that only the functions $\{f_{k;j}\}_{l \leq N-2}$ have null component in e_1 . We reorder $\{f_{k;N-1}\}_{k\in\mathbb{N}} \cup \{g_k\}_{k\in\mathbb{N}}$ in $\varphi = \{\varphi_k\}_{k\in\mathbb{N}}$ and the proof is achieved as done for $N = 3$. \square

Example 5.1. Let \mathcal{G} be a star graph containing two edges e_1 and e_2 long 1. Let e_1 and e_2 connect the internal vertex of \mathcal{G} , equipped with (NK) , with two external vertices both equipped with (\mathcal{D}) .

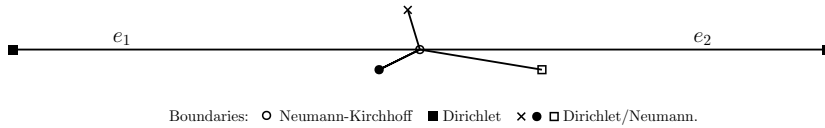


Figure 7: Example of star graph described by Example 5.1 with $N = 5$.

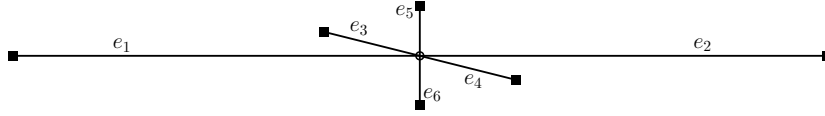
Let $B\psi = (x^2(\psi^1(x) - \psi^2(x)), x^2(\psi^2(x) - \psi^1(x)), 0, \dots, 0)$ for every $\psi \in \mathcal{H}$. There exists $\{\varphi_k\}_{k\in\mathbb{N}} \subset \{\phi_k\}_{k\in\mathbb{N}}$ such that the (BSE) is globally exactly controllable in $H_{\mathcal{G}}^3 \cap \mathcal{H}$ and energetically controllable in $\{k^2\pi^2\}_{k\in\mathbb{N}}$.

Proof. Let $\varphi = \{\varphi_k\}_{k\in\mathbb{N}}$ and $\mu = \{\mu_k\}_{k\in\mathbb{N}}$ be such that

$$\mu_k = k^2\pi^2, \quad \varphi_k^1 = -\varphi_k^2 = \sin(k\pi x), \quad \varphi_k^l = 0, \quad \forall 3 \leq l \leq N.$$

The point **2.** of Theorem 3.7 is valid for $\tilde{d} = 0$. In addition, $\langle \varphi_k, B\varphi_1 \rangle = 4\langle \varphi_k^1, x^2 \varphi_1^1 \rangle_{L^2(e_1, \mathbb{C})}$ with $k \in \mathbb{N}$ is the integral treated in [Duc18c, Example 1.1], where Assumptions I($\varphi, 1$) are ensured. As in Example 1.3, B stabilizes $\mathcal{H} \cap H_{\mathcal{G}}^2$ and $B : H^m \rightarrow H^m$ for every $m > 0$, which validate the point **1.** of Assumptions II($\varphi, 1, 0$). The claim follows from Theorem 3.7. \square

Example 5.2. Let \mathcal{G} be a star graph equipped with (\mathcal{D}) and composed by $\frac{N}{2}$ couples of edges $\{e_{2j-1}, e_{2j}\}_{j \leq \frac{N}{2}}$ long $\{L_j\}_{j \leq \frac{N}{2}} \in \mathcal{AL}(\frac{N}{2})$ with $N \in 2\mathbb{N}$.



Boundaries: \circ Neumann-Kirchhoff \blacksquare Dirichlet.

Figure 8: Example of star graph described by Example 5.2 with $N = 6$.

Let B be such that $B\psi = ((B\psi)^1, \dots, (B\psi)^N)$ for every $\psi \in \mathcal{H}$ and

$$(B\psi)^{2j} = -(B\psi)^{2j-1} = \sum_{l=1}^{N/2} \frac{L_l^{1/2}}{L_j^{1/2}} x^2 \left(\psi^{2l} \left(\frac{L_l}{L_j} x \right) - \psi^{2l-1} \left(\frac{L_l}{L_j} x \right) \right), \quad \forall j \leq \frac{N}{2}.$$

It exists $\mathcal{C} \subset (\mathbb{R}^+)^N$ countable so that, for every $\{L_j\}_{j \leq N} \in \mathcal{AL}(N) \setminus \mathcal{C}$, there exists $\{\varphi_k\}_{k \in \mathbb{N}} \subseteq \{\phi_k\}_{k \in \mathbb{N}}$ such that (BSE) is globally exactly controllable in $H_{\mathcal{G}}^{3+\epsilon} \cap \widetilde{\mathcal{H}}$ with $\epsilon \in (0, 1/2)$ and energetically controllable in $\{\frac{k^2 \pi^2}{L_j^2}\}_{k, j \in \mathbb{N}, j \leq N/2}$.

Proof. The example follows as Example 5.1. For every couple of edges long L_j with $j \in \mathbb{N}$, we have $\{\frac{k^2 \pi^2}{L_j^2}\}_{k \in \mathbb{N}} \subset \{\lambda_k\}_{k \in \mathbb{N}}$. Let $\{\mu_k\}_{k \in \mathbb{N}}$ be the reordering of $\{\frac{k^2 \pi^2}{L_j^2}\}_{k \in \mathbb{N}}$ for every $j \leq N/2$ and $\{\varphi_k\}_{k \in \mathbb{N}}$ be an orthonormal system of \mathcal{H} made by corresponding eigenfunctions. For $k \in \mathbb{N}$, there exist $m(k) \in \mathbb{N}$ and $l(k) \leq N/2$ so that $\varphi_k^n \equiv 0$ for $n \neq 2l(k), 2l(k) - 1$ and

$$\mu_k = m(k)^2 \pi^2 L_{l(k)}^{-2}, \quad \varphi_k^{2l(k)-1}(x) = -\varphi_k^{2l(k)}(x) = \sqrt{L_{l(k)}^{-1}} \sin(\sqrt{\mu_k} x).$$

Let $[r]$ be the entire part of $r \in \mathbb{R}^+$. For $k \in \mathbb{N}$, $|\langle \varphi_1, B\varphi_k \rangle|$ corresponds to

$$\begin{aligned} & \left| \sum_{l=1}^N \left\langle \varphi_k^l(x), \sum_{n=1}^{N/2} \frac{L_n^{1/2} x^2}{L_{[(l+1)/2]}^{1/2}} \left(\varphi_1^{2n-1} \left(\frac{L_n}{L_{[(l+1)/2]} x \right) - \varphi_1^{2n} \left(\frac{L_n}{L_{[(l+1)/2]} x \right) \right) \right\rangle_{L^2(e_l, \mathbb{C})} \right| \\ &= \left| \int_0^{L_{l(k)}} \frac{4x^2}{L_{l(k)}} \sin\left(\frac{m(1)\pi x}{L_{l(k)}}\right) \sin\left(\frac{m(k)\pi x}{L_{l(k)}}\right) dx \right| \geq C \left| \int_0^1 x^2 \sin(m(1)\pi x) \sin(m(k)\pi x) dx \right| \end{aligned}$$

with $C = 4 \min_{l \leq N} L_l$. The point **1.** of Assumptions I($\varphi, 1$) holds as in Example 5.1. Instead, the point **2.** and the point **1.** of Assumptions II($\varphi, 1, \epsilon$) with $\epsilon \in (0, \frac{1}{2})$ hold thanks to the analyticity argument from Example

1.2. Thus, there exists $V \subset (\mathbb{R}^+)^N$ countable so that, for each $\{L_l\}_{l \leq N} \in \mathcal{AL}(N) \setminus V$, Assumptions I($\varphi, 1$) and Assumptions II($\varphi, 1, \epsilon$) are satisfied.

As in the proof of [Duc18a, Lemma A.2], thanks [Duc18a, Proposition A.1], for every $\epsilon > 0$, there exist $C > 0$ and $\tilde{d} \geq 0$ so that $|\mu_{k+1} - \mu_k| \geq Ck^{-\tilde{d}}$ for each $k \in \mathbb{N}$. Theorem 3.7 leads to the statement. \square

Example 5.3. Let \mathcal{G} be a compact quantum graph. Let $\{e_j\}_{j \leq \tilde{N}} \subset \mathcal{G}$ be self-closing edges of lengths $\{L_j\}_{j \leq \tilde{N}}$ with $\tilde{N} \leq N$. For $\psi = (\psi^1, \dots, \psi^N)$, let

$$B\psi = ((B\psi)^1, \dots, (B\psi)^N), \quad (B\psi)^l = \sum_{j \leq \tilde{N}} x^2 \left(\frac{L_j x}{L_l} - L_j \right) \psi^j \left(\frac{L_j}{L_l} x \right), \quad \forall l \leq N.$$

It exists $\mathcal{C} \subset (\mathbb{R}^+)^{\tilde{N}}$ countable so that, if $\{L_j\}_{j \leq \tilde{N}} \in \mathcal{AL}(\tilde{N}) \setminus \mathcal{C}$, then there exists $\{\varphi_k\}_{k \in \mathbb{N}} \subseteq \{\phi_k\}_{k \in \mathbb{N}}$ such that (BSE) is globally exactly controllable in $H_{\mathcal{G}}^{3+\epsilon} \cup \mathcal{H}$ with $\epsilon \in (0, 1/2)$ and energetically controllable in $\left\{ \frac{k^2 \pi^2}{L_j^2} \right\}_{\substack{k, j \in \mathbb{N} \\ j \leq \tilde{N}}}$.

Proof. Let $\{\varphi_k\}_{k \in \mathbb{N}}$ be such that, for each $k \in \mathbb{N}$, there exist $m(k) \in \mathbb{N}$ and $l(k) \leq \tilde{N}$ such that, for every $n \neq l(k)$ and $n \leq N$,

$$\mu_k = m(k)^2 \pi^2 L_{l(k)}^{-2}, \quad \varphi_k^{l(k)}(x) = \sqrt{2L_{l(k)}^{-1}} \sin(\sqrt{\lambda_k} x), \quad \phi_k^n \equiv 0.$$

Now, $\{\varphi_k\}_{k \in \mathbb{N}}$ is an orthonormal system made by eigenfunctions of A and relative to the eigenvalues $\{\mu_k\}_{k \in \mathbb{N}}$. The claim follows as Example 5.2. \square

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A Appendix: Spectral properties

Let us consider the notation introduced in [DZ06, Appendix A] as $\| \cdot \|$, $E(\cdot)$ and $F(\cdot)$. For $x \in \mathbb{R}$, $\{L_j\}_{j \leq N} \in (\mathbb{R}^+)^N$ and $i \leq N$, we also denote

$$n(x) := E\left(x - \frac{1}{2}\right), \quad r(x) := F\left(x - \frac{1}{2}\right), \quad d(x) := \left\| x - \frac{1}{2} \right\|, \quad \tilde{m}^i(x) := n\left(\frac{L_i}{\pi} x\right).$$

In this appendix, we pursue [Duc18a, Appendix A], which is based on the techniques developed in [DZ06, Appendix A].

Lemma A.1. Let $\{L_k\}_{k \leq N} \subset \mathbb{R}^+$, $I_1 \subseteq \{1, \dots, N\}$, $I_2 := \{1, \dots, N\} \setminus I_1$ and

$$a(\cdot) := \prod_{i \in I_2} |\sin(\cdot L_i)| \sum_{i \in I_1} \prod_{\substack{j \in I_1 \\ j \neq i}} |\cos(\cdot L_j)| + \prod_{i \in I_1} |\cos(\cdot L_i)| \sum_{i \in I_2} \prod_{\substack{j \in I_2 \\ j \neq i}} |\sin(\cdot L_j)|.$$

Let $\{\tilde{L}_j\}_{j \leq N} \subset \mathbb{R}^+$ be such that $\tilde{L}_j = 2L_j$ when $j \in I_1$ and $\tilde{L}_j = L_j$ when $j \in I_2$. There exists $C > 0$ such that, for every $x \in \mathbb{R}$, there holds

$$a(x) \geq C \min \left(\min_{i \leq N} \prod_{j \neq i} \left\| \left(\tilde{m}^i(x) + \frac{1}{2} \right) \frac{\tilde{L}_j}{L_i} \right\|, \min_{i \leq N} \prod_{j \neq i} \left\| m^i(x) \frac{\tilde{L}_j}{L_i} \right\| \right).$$

Proof. From [DZ06, relation (A.3)], for every $x \in \mathbb{R}$, there follows

$$(8) \quad 2d(x) \leq |\cos(\pi x)| \leq \pi d(x).$$

As $2d\left(\left(\tilde{m}^i(x) + \frac{1}{2}\right)\frac{L_j}{L_i}\right) \leq \left|\cos\left(\left(\tilde{m}^i(x) + \frac{1}{2}\right)\frac{L_j}{L_i}\pi\right)\right|$ and $\tilde{m}^i(x) + \frac{1}{2} = \frac{L_i x}{\pi} - r\left(\frac{L_i}{\pi}x\right)$ for $x \in \mathbb{R}$ and $i, j \leq N$, we have

$$(9) \quad 2d\left(\left(\tilde{m}^i(x) + \frac{1}{2}\right)\frac{L_j}{L_i}\right) \leq \left|\cos(L_j x)\right| + \left|\sin\left(\pi \frac{L_j}{L_i} \left|r\left(\frac{L_i}{\pi}x\right)\right|\right)\right|.$$

Now, $|\sin(\pi|r(\cdot)|)| \leq \pi \| |r(\cdot)| \| \leq \pi|r(\cdot)| = \pi d(\cdot) \leq \frac{\pi}{2} |\cos(\pi(\cdot))|$ thanks to [DZ06, relation (A.3)] and (8). For every $x \in \mathbb{R}$, it holds

$$(10) \quad \left|\sin\left(\pi \frac{L_j}{L_i} \left|r\left(\frac{L_i}{\pi}x\right)\right|\right)\right| \leq \pi \frac{L_j}{L_i} \left|r\left(\frac{L_i}{\pi}x\right)\right| \leq \frac{\pi L_j}{2L_i} |\cos(L_i x)|.$$

From (9) and (10), there exists $C_1 > 0$ such that, for every $i, j \leq N$,

$$(11) \quad 2d\left(\left(\tilde{m}^i(x) + \frac{1}{2}\right)\frac{L_j}{L_i}\right) \leq |\cos(L_j x)| + \frac{\pi L_j}{2L_i} |\cos(L_i x)|, \quad \forall x \in \mathbb{R}^+, \\ \implies C_1 \prod_{\substack{j \in I_1 \\ j \neq i}} d\left(\left(\tilde{m}^i(x) + \frac{1}{2}\right)\frac{L_j}{L_i}\right) \leq \prod_{\substack{j \in I_1 \\ j \neq i}} |\cos(L_j x)| + |\cos(L_i x)|.$$

From [DZ06, relation (A.3)], as (9) and (10), there exists $C_2 > 0$ such that

$$(12) \quad 2 \left\| \left(\tilde{m}^i(x) + \frac{1}{2}\right)\frac{L_j}{L_i} \right\| \leq |\sin(L_j x)| + \frac{\pi L_j}{2L_i} |\cos(L_i x)|, \quad \forall x \in \mathbb{R}, \\ \implies C_2 \prod_{\substack{j \in I_1 \\ j \neq i}} d\left(\left(\tilde{m}^i(x) + \frac{1}{2}\right)\frac{L_j}{L_i}\right) \prod_{\substack{j \in I_2 \\ j \neq i}} \left\| \left(\tilde{m}^i(x) + \frac{1}{2}\right)\frac{L_j}{L_i} \right\| \\ \leq \prod_{\substack{j \in I_2 \\ j \neq i}} |\sin(L_j x)| \prod_{\substack{j \in I_1 \\ j \neq i}} |\cos(L_j x)| + |\cos(L_i x)|.$$

Now, $d(x) = \left\| \frac{1}{2}(2x - 1) \right\| \geq \frac{1}{2} \|2x - 1\| = \frac{1}{2} \|2x\|$ for every $x \in \mathbb{R}$ and $d\left(\left(\tilde{m}^i(x) + \frac{1}{2}\right)\frac{L_j}{L_i}\right) \geq \frac{1}{2} \left\| \left(\tilde{m}^i(x) + \frac{1}{2}\right)\frac{2L_j}{L_i} \right\|$, which imply

$$(13) \quad C_2 \prod_{\substack{j \leq N \\ j \neq i}} \frac{1}{2} \left\| \left(\tilde{m}^i(\cdot) + \frac{1}{2}\right)\frac{\tilde{L}_j}{L_i} \right\| \leq a(\cdot) + |\cos(L_i(\cdot))|.$$

Equivalently, from the proof of [DZ06, *Proposition A.1*], for every $x \in \mathbb{R}$,

$$(14) \quad 2 \left\| m^i(x) \frac{L_j}{L_i} \right\| \leq |\sin(L_j x)| + \frac{\pi L_j}{2L_i} |\sin(L_i x)|,$$

$$(15) \quad 2d\left(m^i(x) \frac{L_j}{L_i}\right) \leq |\cos(L_j x)| + \frac{\pi L_j}{2L_i} |\sin(L_i x)|,$$

$$(16) \quad \implies C_2 \prod_{\substack{j \leq N \\ j \neq i}} \frac{1}{2} \left\| m^i(\cdot) \frac{\tilde{L}_j}{L_i} \right\| \leq a(\cdot) + |\sin(L_i(\cdot))|.$$

The claim follows as [DZ06, *Proposition A.1*]. Indeed, if $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^+$ is so that $a(\lambda_k) \xrightarrow{k \rightarrow \infty} 0$, then there exist some $i_0 \leq N$ such that $|\sin(\lambda_k L_{i_0})| \xrightarrow{k \rightarrow \infty} 0$ or $|\cos(\lambda_k L_{i_0})| \xrightarrow{k \rightarrow \infty} 0$. By considering (13) and (16) with $i = i_0$, we have

$$z(\lambda_k) := \min \left(\min_{i \leq N} \prod_{j \neq i} \left\| \tilde{m}^i(\lambda_k) + \frac{1}{2} \right\| \frac{\tilde{L}_j}{L_i}, \min_{i \leq N} \prod_{j \neq i} \left\| m^i(\lambda_k) \frac{\tilde{L}_j}{L_i} \right\| \right) \xrightarrow{k \rightarrow \infty} 0.$$

As [DZ06, *Proposition A.1*], the lemma is proved since $z(\lambda_k)$ converges to 0 at least as fast as $a(\lambda_k)$ thanks to the identities (11), (12), (14) and (15). \square

Proposition A.2. *Let $\{L_j\}_{j \leq N} \subset \mathbb{R}$, $I_1 \subseteq \{1, \dots, N\}$ and $I_2 := \{1, \dots, N\} \setminus I_1$. If $\{L_j\}_{j \leq N} \in \mathcal{AL}(N)$, then, for every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that, for every $x > \max\{\pi/2L_j : j \leq N\}$, we have*

$$\prod_{j \in I_2} |\sin(xL_j)| \sum_{j \in I_1} \prod_{\substack{k \in I_1 \\ k \neq j}} |\cos(xL_k)| + \prod_{j \in I_1} |\cos(xL_j)| \sum_{j \in I_2} \prod_{\substack{k \in I_2 \\ k \neq j}} |\sin(xL_k)| \geq \frac{C_\epsilon}{x^{1+\epsilon}}.$$

Proof. The claim is due to Lemma A.1 and to the Schmidt's Theorem [DZ06, *Theorem A.8*], which implies that, for every $\epsilon > 0$ and $i \leq N$, there exist $C_1(i), C_2(i), C_3(i) > 0$ such that, for every $x \in \mathbb{R}$, there holds

$$\prod_{\substack{j \leq N \\ j \neq i}} \left\| \tilde{m}^i(x) + \frac{1}{2} \right\| \frac{\tilde{L}_j}{L_i} \geq \frac{C_1(i)}{(2\tilde{m}^i(x) + 1)^{1+\epsilon}} \geq \frac{C_1(i)}{\left(\frac{2L_i}{\pi}x + 1\right)^{1+\epsilon}} \geq \frac{C_2(i)}{x^{1+\epsilon}}$$

and $\prod_{\substack{j \leq N \\ j \neq i}} \left\| m^i(x) \frac{\tilde{L}_j}{L_i} \right\| \geq C_3(i)x^{-1-\epsilon}$ for every $x > \frac{\pi}{2} \max\{1/L_j : j \leq N\}$.

The statement follows with $C_\epsilon := \min(\min_{i \leq N} C_2(i), \min_{i \leq N} C_3(i))$. \square

B Appendix: Moments problems

Let $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ and $\Lambda = \{\lambda_k\}_{k \in \mathbb{Z}^*} \subset \mathbb{R}^+$ be an ordered sequence of pairwise distinct numbers such that there exist $\mathcal{M} \in \mathbb{N} \setminus \{1\}$ and $\delta > 0$ such that

$$(17) \quad \inf_{\{k \in \mathbb{Z}^* : k + \mathcal{M} \neq 0\}} |\lambda_{k+\mathcal{M}} - \lambda_k| \geq \delta \mathcal{M}.$$

From (17), there does not exist \mathcal{M} consecutive $k, k+1 \in \mathbb{Z}^*$ such that $|\lambda_{k+1} - \lambda_k| < \delta$. This leads to a partition of \mathbb{Z}^* in subsets that we call E_m with $m \in \mathbb{Z}^*$. This partition also defines an equivalence relation in \mathbb{Z}^* such that $k \sim n$ if and only if there exists $m \in \mathbb{Z}^*$ such that $k, n \in E_m$. Now, $\{E_m\}_{m \in \mathbb{Z}^*}$ are the corresponding equivalence classes and $i(m) := |E_m| \leq \mathcal{M} - 1$. For every $\mathbf{x} := \{x_k\}_{k \in \mathbb{Z}^*}$, we define $\mathbf{x}^m := \{x_l\}_{l \in E_m}$ for $m \in \mathbb{Z}^*$.

Let $\hat{\mathbf{h}} = \{h_j\}_{j \leq i(m)} \in \mathbb{C}^{i(m)}$ with $m \in \mathbb{Z}^*$. For every $m \in \mathbb{Z}^*$, we denote $F_m(\hat{\mathbf{h}}) : \mathbb{C}^{i(m)} \rightarrow \mathbb{C}^{i(m)}$ the matrix with elements, for every $j, k \leq i(m)$,

$$F_{m;j,k}(\hat{\mathbf{h}}) := \begin{cases} \prod_{\substack{l \neq j \\ l \leq k}} (h_j - h_l)^{-1}, & j \leq k, \\ 1, & j = k = 1, \\ 0, & j > k. \end{cases}$$

For each $k \in \mathbb{Z}^*$, there exists $m(k) \in \mathbb{Z}^*$ such that $k \in E_{m(k)}$. Let $F(\mathbf{\Lambda})$ be the linear operator on $\ell^2(\mathbb{Z}^*, \mathbb{C})$ such that $F(\mathbf{\Lambda}) : D(F(\mathbf{\Lambda})) \rightarrow \ell^2(\mathbb{Z}^*, \mathbb{C})$ and

$$(F(\mathbf{\Lambda})\mathbf{x})_k = \left(F_{m(k)}(\mathbf{\Lambda}^{m(k)})\mathbf{x}^{m(k)} \right)_k, \quad \forall \mathbf{x} = \{x_k\}_{k \in \mathbb{Z}^*} \in D(F(\mathbf{\Lambda})),$$

$$H(\mathbf{\Lambda}) := D(F(\mathbf{\Lambda})) = \{ \mathbf{x} := \{x_k\}_{k \in \mathbb{Z}^*} \in \ell^2(\mathbb{Z}^*, \mathbb{C}) : F(\mathbf{\Lambda})\mathbf{x} \in \ell^2(\mathbb{Z}^*, \mathbb{C}) \}.$$

Remark B.1. We call $F_m(\mathbf{\Lambda}^m)^{-1}$ the inverse matrix of $F_m(\mathbf{\Lambda}^m)$ for $m \in \mathbb{Z}^*$. Now, $F(\mathbf{\Lambda}) : H(\mathbf{\Lambda}) \rightarrow \text{Ran}(F(\mathbf{\Lambda}))$ is invertible and $F(\mathbf{\Lambda})^{-1}$ is so that

$$(F(\mathbf{\Lambda})^{-1}\mathbf{x})_k = \left(F_{m(k)}(\mathbf{\Lambda}^{m(k)})^{-1}\mathbf{x}^{m(k)} \right)_k, \quad \forall \mathbf{x} \in \text{Ran}(F(\mathbf{\Lambda})), \quad k \in \mathbb{Z}^*.$$

Let $F(\mathbf{\Lambda})^*$ be the infinite matrix so that $(F(\mathbf{\Lambda})^*\mathbf{x})_k = \left(F_{m(k)}(\mathbf{\Lambda}^{m(k)})^*\mathbf{x}^{m(k)} \right)_k$ for any $\mathbf{x} = \{x_k\}_{k \in \mathbb{Z}^*}$ and $k \in \mathbb{Z}^*$, where $F_{m(k)}(\mathbf{\Lambda}^{m(k)})^*$ is the transposed of $F_{m(k)}(\mathbf{\Lambda}^{m(k)})$. For $T > 0$, let the sequences of functions in $L^2((0, T), \mathbb{C})$

$$\mathbf{e} := \{e^{i\lambda_k(\cdot)}\}_{k \in \mathbb{Z}^*}, \quad \mathbf{\Xi} := \{\xi_k(\cdot)\}_{k \in \mathbb{Z}^*} = F(\mathbf{\Lambda})^*\mathbf{e}.$$

Remark B.2. When $H(\mathbf{\Lambda})$ is dense in $\ell^2(\mathbb{Z}^*, \mathbb{C})$, we consider $F(\mathbf{\Lambda})^*$ as the unique adjoint operator of $F(\mathbf{\Lambda})$ in $\ell^2(\mathbb{Z}^*, \mathbb{C})$ with domain $H(\mathbf{\Lambda})^* := D(F(\mathbf{\Lambda})^*)$. As in Remark B.1, we define $(F(\mathbf{\Lambda})^*)^{-1}$ the inverse operator of $F(\mathbf{\Lambda})^* : H(\mathbf{\Lambda})^* \rightarrow \text{Ran}(F(\mathbf{\Lambda})^*)$ and $(F(\mathbf{\Lambda})^*)^{-1} = (F(\mathbf{\Lambda})^{-1})^*$.

Theorem B.3 (Theorem 3.29; [DZ06]). Let $\{\lambda_k\}_{k \in \mathbb{Z}^*}$ be an ordered sequence of pairwise distinct real numbers satisfying (17). If $T > 2\pi/\delta$, then $\{\xi_k\}_{k \in \mathbb{Z}^*}$ forms a Riesz Basis in the space $X := \overline{\text{span}\{\xi_k \mid k \in \mathbb{Z}^*\}}^{L^2}$.

Lemma B.4. Let $\nu := \{\nu_k\}_{k \in \mathbb{Z}^*}$ be an ordered sequence of pairwise distinct real numbers satisfying (17). Let G be an entire function such that $G \in L^\infty(\mathbb{R})$ and there exist $J, I > 0$ such that $|G(z)| \leq J e^{I|z|}$ for every $z \in \mathbb{C}$. If $\{\nu_k\}_{k \in \mathbb{Z}^*}$ are simple zeros of G such that there exist $\tilde{d} \geq 0, C > 0$ such that

$$(18) \quad |G'(\nu_k)| \geq C|k|^{-\tilde{d}}, \quad \forall k \in \mathbb{Z}^*, \nu_k \neq 0,$$

$$\implies \exists C > 0 \quad : \quad \text{Tr}\left(F_m(\mathbf{\nu}^m)^* F_m(\mathbf{\nu}^m)\right) \leq C \min\{|l| \in E_m\}^{2\tilde{d}}, \quad \forall m \in \mathbb{Z}^*.$$

Proof. Construction of a biorthogonal sequence to $\{e^{i\nu_k(\cdot)}\}_{k \in \mathbb{Z}^}$.* Let $T > \max(2\pi/\delta, 2I)$. For every $k \in \mathbb{Z}^*$, we define $G_k(z) := G(z)(z - \nu_k)^{-1}$. Thanks to the Paley-Wiener's Theorem [DZ06, *Theorem 3.19*], for every $k \in \mathbb{Z}^*$, there exists $w_k \in L^2(\mathbb{R}, \mathbb{R})$ with support in $[-I, I]$ such that

$$G_k(z) = \int_{-I}^I e^{izs} w_k(s) ds = \int_{-T/2}^{T/2} e^{izs} w_k(s) ds = \int_0^T e^{izt} e^{-iz\frac{T}{2}} w_k(t - T/2) dt.$$

For $j, k \in \mathbb{Z}^*$ and $c_k := G'(\nu_k)$, we call $v_k(t) := e^{i\nu_k\frac{T}{2}} w_k(t - T/2)$ and $\langle v_k, e^{i\nu_j(\cdot)} \rangle_{L^2((0, T), \mathbb{C})} = \delta_{k,j} G_k(\nu_k) = \delta_{k,j} G'(\nu_k) = \delta_{k,j} c_k$. The sequence $\{v_k\}_{k \in \mathbb{Z}^*}$ is biorthogonal to $\{e^{i\nu_k(\cdot)}/c_k\}_{k \in \mathbb{Z}^*}$ and $\{v_k/c_k\}_{k \in \mathbb{Z}^*}$ is biorthogonal to $\{e^{i\nu_k(\cdot)}\}_{k \in \mathbb{Z}^*}$. Thanks to the Plancherel's identity, $\|v_k\|_{L^2((0, T), \mathbb{C})} = \|G_k\|_{L^2(\mathbb{R}, \mathbb{R})}$. We show that from the Phragmén-Lindelöf Theorem (*e.g.* [You80, *p. 82; Theorem 11*]),

$$(19) \quad \exists C_1 > 0 \quad : \quad \|v_k\|_{L^2((0, T), \mathbb{C})} = \|G_k\|_{L^2(\mathbb{R}, \mathbb{R})} \leq C_1, \quad \forall k \in \mathbb{Z}^*.$$

First, G is entire, while there exist I and J such that $|G(z)| \leq J e^{I|z|}$ for every $z \in \mathbb{C}$. Second, there exists $M > 0$ such that $|G(x)| \leq M$ for every $x \in \mathbb{R}$. From [You80, *p. 82; Theorem 11*], we have $|G(x + iy)| \leq M e^{I|y|}$ for every $x, y \in \mathbb{R}$. For every $k \in \mathbb{Z}^*$, we consider $\|G_k\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \overline{G_k(x)} G_k(x) dx = \int_{\mathbb{R}} \frac{\overline{G(x)} G(x)}{(x - \nu_k)^2} dx$ and there exists $c_1 > 0$ not depending on k such that

$$\|G_k\|_{L^2(\mathbb{R})}^2 \leq \int_{|x - \nu_k| \leq 1} \overline{G(x)} G(x) (x - \nu_k)^{-2} dx + M^2 c_1.$$

The validity of (19) follows since $\int_{|x - \nu_k| \leq 1} \frac{\overline{G(x)} G(x)}{(x - \nu_k)^2} dx \leq M^2 c_2$ with $c_2 > 0$ not depending on k . Indeed, the Cauchy Integral Theorem implies

$$\int_{|x - \nu_k| \leq 1} \frac{\overline{G(x)} G(x)}{(x - \nu_k)^2} dx \leq \int_0^\pi |\overline{G(\nu_k + e^{i\theta})} G(\nu_k + e^{i\theta})| d\theta \leq M^2 \int_0^\pi e^{2I \sin(\theta)} d\theta.$$

Construction of a Riesz basis. Let $\mathbf{v} := \{\nu_k\}_{k \in \mathbb{Z}^*}$ and $\mathbf{e} := \{e^{i\nu_k(\cdot)}\}_{k \in \mathbb{Z}^*} \subset L^2((0, T), \mathbb{C})$. Thanks to Proposition B.3, $\Xi = \{\xi_k\}_{k \in \mathbb{Z}^*} := \{(F(\mathbf{v})^* \mathbf{e})_k\}_{k \in \mathbb{Z}^*}$ forms a Riesz basis in $X := \overline{\text{span}\{\xi_k : k \in \mathbb{Z}^*\}}^{L^2}$. We call $\tilde{\mathbf{v}} := \{\tilde{v}_k\}_{k \in \mathbb{Z}^*}$ the corresponding biorthogonal sequence which is also a Riesz basis of X . From Remark B.2, the map $F(\mathbf{v})$ is invertible from $H(\mathbf{v})^*$ to $\text{Ran}(F(\mathbf{v})^*)$ and $(F(\mathbf{v})^*)^{-1} = (F(\mathbf{v})^{-1})^*$. As $\mathbf{v}/\mathbf{c} = \{v_k/c_k\}_{k \in \mathbb{Z}^*}$ is biorthogonal to $\{e^{i\nu_k(\cdot)}\}_{k \in \mathbb{Z}^*}$, we have $\{v_k/c_k\}_{k \in \mathbb{Z}^*} = F(\mathbf{v})\tilde{\mathbf{v}}$. Indeed, for every $j, k \in \mathbb{N}$,

$$\delta_{k,j} = \langle v_k/c_k, ((F(\mathbf{v})^*)^{-1} \Xi)_j \rangle_{L^2((0, T), \mathbb{C})} = \langle (F(\mathbf{v})^{-1} \mathbf{v}/\mathbf{c})_k, \xi_j \rangle_{L^2((0, T), \mathbb{C})},$$

which implies that $(F(\mathbf{v})^{-1} \mathbf{v}/\mathbf{c})_k = \tilde{v}_k$. The uniqueness of the biorthogonal family to Ξ implies the uniqueness of the biorthogonal family to \mathbf{e} . From [BL10, *Appendix B; Proposition 19.(2)*], there exist $C_2, C_3 > 0$ such that

$$C_2 \|\mathbf{x}\|_{\ell^2}^2 \leq \int_0^T |u(s)|^2 ds \leq C_3 \|\mathbf{x}\|_{\ell^2}^2, \quad \forall u(t) = \sum_{k \in \mathbb{Z}^*} \xi_k x_k, \quad \mathbf{x} \in \ell^2(\mathbb{Z}^*, \mathbb{C}).$$

Thanks to the biorthogonality, $x_k = \langle \tilde{v}_k, u \rangle_{L^2((0,T),\mathbb{C})} = \int_0^T \overline{\tilde{v}_k(\tau)} u(\tau) d\tau$ for every $k \in \mathbb{Z}^*$. For every $k \in \mathbb{Z}^*$, we call $m(k) \in \mathbb{Z}^*$ the number such that $k \in E_{m(k)}$. Thanks to (18) and (19), there exist $C_4, C_5 > 0$ such that, for every $k \in \mathbb{Z}^*$, we have $|(F(\mathbf{v})\mathbf{x})_k| \leq C_4 \min\{|l| \in E_{m(k)}\}^{\tilde{d}} \|\mathbf{x}\|_{\ell^2}$ since

$$\begin{aligned} |(F(\mathbf{v})\mathbf{x})_k| &= |\langle (F(\mathbf{v}))\{\langle \tilde{v}_l, u \rangle_{L^2((0,T),\mathbb{C})}\}_{l \in \mathbb{Z}}\}_k| = |\langle v_k/c_k, u \rangle_{L^2((0,T),\mathbb{C})}| \\ &\leq \|v_k\|_{L^2((0,T),\mathbb{C})} \|u\|_{L^2((0,T),\mathbb{C})} |c_k|^{-1} \leq C_3^{\frac{1}{2}} \|G_k\|_{L^2(\mathbb{R},\mathbb{R})} \|\mathbf{x}\|_{\ell^2} |G'(\nu_k)|^{-1} \leq C_5 |k|^{\tilde{d}} \|\mathbf{x}\|_{\ell^2}. \end{aligned}$$

In conclusion, there exists $C_6 > 0$ so that $|(F_{m;j,k}(\mathbf{v}^m))| \leq C_6 \min_{l \in E_m} |l|^{\tilde{d}}$ for every $j, k \leq i(m)$, which leads to the statement. \square

Proposition B.5. *Let $\{\lambda_k\}_{k \in \mathbb{Z}^*}$ be an ordered sequence of pairwise distinct real numbers such that $\{\nu_k\}_{k \in \mathbb{Z}^*} = \{\text{sgn}(\lambda_k) \sqrt{|\lambda_k|}\}_{k \in \mathbb{Z}^*}$ satisfies (17). Let*

$$(20) \quad \exists C_1, C_2 > 0 \quad C_1 |k| \leq |\nu_k| \leq C_2 |k|, \quad \forall k \in \mathbb{Z}^*, \quad \nu_k \neq 0.$$

Let G be an entire function so that $\{\nu_k\}_{k \in \mathbb{Z}^}$ are its simple zeros, $G \in L^\infty(\mathbb{R})$ and there exist $J, I > 0$ such that $|G(z)| \leq J e^{I|z|}$ for every $z \in \mathbb{C}$. If there exist $\tilde{d} \geq 1$ and $C > 0$ such that $|G'(\nu_k)| \geq C |k|^{-\tilde{d}}$ for every $k \in \mathbb{Z}^*$ such that $\nu_k \neq 0$, then the space $H(\mathbf{\Lambda})$ contains $h^{\tilde{d}-1}(\mathbb{Z}^*, \mathbb{C})$.*

Proof. From the hypotheses, we notice that there exist $\delta > 0$ and $\mathcal{M} \in \mathbb{N} \setminus \{1\}$ so that $\inf_{\substack{k \in \mathbb{Z}^* \\ k+\mathcal{M} \neq 0}} |\nu_{k+\mathcal{M}} - \nu_k| \geq \delta \mathcal{M} \min_{\substack{k \in \mathbb{Z}^* \\ \nu_k \neq 0}} (|\nu_k|, 1)$ and we have

$$\inf_{\substack{k \in \mathbb{Z}^* \\ k+\mathcal{M} \neq 0}} |\lambda_{k+\mathcal{M}} - \lambda_k| = \inf_{\substack{k \in \mathbb{Z}^* \\ k+\mathcal{M} \neq 0}} \left| |\nu_{k+\mathcal{M}}| - |\nu_k| \right| \left| |\nu_{k+\mathcal{M}}| + |\nu_k| \right| \geq \min_{\substack{k \in \mathbb{Z}^* \\ \nu_k \neq 0}} (|\nu_k|, 1) \delta \mathcal{M}$$

since $\{\lambda_k\}_{k \in \mathbb{Z}^*} = \{\text{sgn}(\nu_k) \nu_k^2\}_{k \in \mathbb{Z}^*}$. Now, $\mathbf{\Lambda} := \{\lambda_k\}_{k \in \mathbb{Z}^*}$ and $\mathbf{v} := \{\nu_k\}_{k \in \mathbb{Z}^*}$ satisfy (17) with respect to $\delta' := \min_{\substack{k \in \mathbb{Z}^* \\ \nu_k \neq 0}} \{|\nu_k|, 1\} \delta$ and \mathcal{M} . This implies that

the theory exposed in this appendix and the definitions of the equivalence classes E_m in \mathbb{Z}^* is valid for both the sequences $\mathbf{\Lambda}$ and \mathbf{v} . From Lemma B.4,

$$\text{Tr} \left(F_m(\mathbf{v}^m)^* F_m(\mathbf{v}^m) \right) \leq C \min\{|l| \in E_m\}^{2\tilde{d}}, \quad \forall m \in \mathbb{Z}^*,$$

and $|\lambda_l - \lambda_k| \geq \min\{|\nu_l|, |\nu_k|\} |\nu_l - \nu_k|$ for $l, k \in \mathbb{Z}^*$. Let $m \in \mathbb{Z}^*$ and $I \subseteq E_m$ so that $I \neq \emptyset$. Now, $|I| \leq |E_m| \leq \mathcal{M} - 1$ and

$$\prod_{j,k \in I} |\lambda_k - \lambda_j| \geq \min_{\substack{l \in I \\ \nu_l \neq 0}} |\nu_l|^{2|I|} \prod_{j,k \in I} |\nu_k - \nu_j| \geq C_1 \min_{\substack{l \in I \\ \nu_l \neq 0}} |\nu_l| \prod_{j,k \in I} |\nu_k - \nu_j|$$

for $C_1 = \min_{\substack{l \in \mathbb{Z}^* \\ \nu_l \neq 0}} (|\nu_l|^{\mathcal{M}-2}, 1)$. Thus, there exists $C_2 > 0$ so that, for every

m and $j, k \in E_m$, we have $|F_{m;j,k}(\mathbf{\Lambda}^m)| \leq C_2 |F_{m;j,k}(\mathbf{v}^m)| \min\{|\nu_l|^{-1} : l \in E_m, \nu_l \neq 0\}$. Thanks to (20) and Lemma B.4, there exists $C_3 > 0$ such that

$$\text{Tr} \left(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m) \right) \leq C_2^2 \min_{\substack{l \in E_m \\ \nu_l \neq 0}} |\nu_l|^{-2} \text{Tr} \left(F_m(\mathbf{v}^m)^* F_m(\mathbf{v}^m) \right) \leq C_3 \min_{l \in E_m} |l|^{2\tilde{d}-2}.$$

Let $\rho(M)$ be the spectral radius of a matrix M and let $\|M\| = \sqrt{\rho(M^*M)}$ be its euclidean norm. As $(F_m(\mathbf{\Lambda}^m)^*F_m(\mathbf{\Lambda}^m))$ is positive-definite,

$$\|F_m(\mathbf{\Lambda}^m)\|^2 = \rho(F_m(\mathbf{\Lambda}^m)^*F_m(\mathbf{\Lambda}^m)) \leq C_3 \min_{l \in E_m} |l|^{2\tilde{d}-2}, \quad m \in \mathbb{Z}^*.$$

In conclusion, $h^{\tilde{d}-1}(\mathbb{Z}^*, \mathbb{C}) \subset H(\mathbf{\Lambda})$ as, for every $\mathbf{x} = \{x_k\}_{k \in \mathbb{Z}^*} \in h^{\tilde{d}-1}(\mathbb{Z}^*, \mathbb{C})$,

$$\begin{aligned} \|F(\mathbf{\Lambda})\mathbf{x}\|_{\ell^2}^2 &\leq \sum_{m \in \mathbb{Z}^*} \|F_m(\mathbf{\Lambda}^m)\|^2 \sum_{l \in E_m} |x_l|^2 \leq C_3 \sum_{m \in \mathbb{Z}^*} \min_{l \in E_m} |l|^{2\tilde{d}-2} \sum_{l \in E_m} |x_l|^2, \\ \implies \|F(\mathbf{\Lambda})\mathbf{x}\|_{\ell^2}^2 &\leq C_3 \|\mathbf{x}\|_{h^{\tilde{d}-1}}^2 < +\infty. \quad \square \end{aligned}$$

Remark B.6. *If Proposition B.5 is satisfied with $\mathbf{\Lambda} = \{\lambda_k\}_{k \in \mathbb{Z}^*}$ and $\tilde{d} \geq 1$, then $H(\mathbf{\Lambda}) \supseteq h^{\tilde{d}-1}(\mathbb{Z}^*, \mathbb{C})$, which is dense in $\ell^2(\mathbb{Z}^*, \mathbb{C})$. Thanks to Remark B.2, we consider $F(\mathbf{\Lambda})^*$ as the unique adjoint operator of $F(\mathbf{\Lambda})$. As $\text{Tr}(F_m(\mathbf{\Lambda}^m)^*F_m(\mathbf{\Lambda}^m)) = \text{Tr}(F_m(\mathbf{\Lambda}^m)F_m(\mathbf{\Lambda}^m)^*)$ for every $m \in \mathbb{Z}^*$, the techniques developed in the proof of Proposition B.5 lead to $H(\mathbf{\Lambda})^* \supseteq h^{\tilde{d}-1}(\mathbb{Z}^*, \mathbb{C})$.*

Proposition B.7. *Let $\{\omega_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^+ \cup \{0\}$ be an ordered sequence of pairwise distinct numbers so that there exist $\delta, C_1, C_2 > 0$ and $\mathcal{M} \in \mathbb{N} \setminus \{1\}$ such that*

$$\inf_{k \in \mathbb{N}} |\omega_{k+\mathcal{M}} - \omega_k| \geq \delta \mathcal{M}, \quad C_1 k^2 \leq |\omega_k| \leq C_2 k^2, \quad \forall k \in \mathbb{N} \setminus \{1\}.$$

Let G be an entire function so that $\{\pm\sqrt{\omega_k}\}_{k \in \mathbb{N}}$ are its simple zeros, $G \in L^\infty(\mathbb{R})$ and there exist $J, I > 0$ such that $|G(z)| \leq J e^{I|z|}$ for every $z \in \mathbb{C}$. If

$$\exists \tilde{d} \geq 1, C > 0 \quad : \quad |G'(\pm\sqrt{\omega_k})| \geq C k^{-\tilde{d}}, \quad \forall j \in \mathbb{N},$$

then, for $T > 2\pi/\delta$ and for every $\{x_k\}_{k \in \mathbb{N}} \in h^{\tilde{d}-1}(\mathbb{N}, \mathbb{C})$ with $x_1 \in \mathbb{R}$,

$$(21) \quad \exists u \in L^2((0, T), \mathbb{R}) \quad : \quad x_k = \int_0^T u(\tau) e^{i(\omega_k - \omega_1)\tau} d\tau, \quad \forall k \in \mathbb{N}.$$

Proof. Let $\nu := \{\nu_k\}_{k \in \mathbb{Z}^*}$ be such that $\nu_k = -\sqrt{\omega_k}$ for $k > 0$ and $\nu_k = \sqrt{\omega_{-k}}$ for $k < 0$. We call $\mathbf{\Lambda} := \{\lambda_k\}_{k \in \mathbb{Z}^*}$ such that $\lambda_k = -\omega_k$ for $k > 0$ and $\lambda_k = \omega_{-k}$ for $k < 0$. Let $\Theta := \{\theta_k\}_{k \in \mathbb{Z}^* \setminus \{-1\}}$ be such that $\theta_k = -\omega_k + \omega_1$ for $k > 0$ and $\theta_k = \omega_{-k} - \omega_1$ for $k < -1$. We consider $\mathcal{M}' \in \mathbb{N} \setminus \{1\}$ and $\delta' > 0$ so that ν and $\mathbf{\Lambda}$ satisfy (17) with respect to \mathcal{M}' and δ' , while

$$(22) \quad \inf_{\{k \in \mathbb{Z}^* \setminus \{-1\} : k + \mathcal{M}' \in \mathbb{Z}^* \setminus \{-1\}\}} |\theta_{k+\mathcal{M}'} - \theta_k| \geq \delta' \mathcal{M}'.$$

Let $\{E_m\}_{m \in \mathbb{Z}^*}$ be the equivalence classes in \mathbb{Z}^* defined by ν and $\mathbf{\Lambda}$ as in the proof of Proposition B.5. Let $-1 \in E_{-1}$. Now, $\{E_m\}_{m \in \mathbb{Z}^* \setminus \{-1\}} \cup \{E_{-1} \setminus \{-1\}\}$ are the equivalence classes in $\mathbb{Z}^* \setminus \{-1\}$ defined by (22). Remark B.6 implies

$H(\Lambda)^* \supseteq h^{\tilde{d}-1}(\mathbb{Z}^*, \mathbb{C})$. Let $F(\Theta)$ be defined in $\ell^2(\mathbb{Z}^* \setminus \{-1\}, \mathbb{C})$. For $m \neq -1$, $F_m(\Theta^m) = F_m(\Lambda^m)$ and $F_m(\Theta^m)^* = F_m(\Lambda^m)^*$. As in Remark B.6,

$$H(\Theta) \supseteq h^{\tilde{d}-1}(\mathbb{Z}^* \setminus \{-1\}, \mathbb{C}) \quad \implies \quad H(\Theta)^* \supseteq h^{\tilde{d}-1}(\mathbb{Z}^* \setminus \{-1\}, \mathbb{C}).$$

For $T > 0$, we define in $L^2 := L^2((0, T), \mathbb{C})$ the sequences of functions

$$\mathbf{e} := \{e^{i\theta_k(\cdot)}\}_{k \in \mathbb{Z}^* \setminus \{-1\}}, \quad \Xi := \{\xi_k(\cdot)\}_{k \in \mathbb{Z}^* \setminus \{-1\}} = F(\Theta)^* \mathbf{e}.$$

When $T > 2\pi/\delta$, Theorem B.3 ensures that $\{\xi_k\}_{k \in \mathbb{Z}^* \setminus \{-1\}}$ is a Riesz Basis in $X := \overline{\text{span}_{k \in \mathbb{Z}^* \setminus \{-1\}} \{\xi_k\}}^{L^2}$, $M : g \in X \mapsto \{\langle \xi_k, g \rangle_{L^2}\}_{k \in \mathbb{Z}^* \setminus \{-1\}} \in \ell^2(\mathbb{Z}^* \setminus \{-1\}, \mathbb{C})$ is invertible from [BL10, Appendix B; Proposition 19.(2)] and

$$\langle \xi_k, g \rangle_{L^2} = (F(\Theta)^* \langle \mathbf{e}, g \rangle_{L^2})_k, \quad \forall k \in \mathbb{Z}^* \setminus \{-1\}.$$

Let $\tilde{X} := M^{-1} \circ F(\Theta)^*(h^{\tilde{d}-1}(\mathbb{Z}^* \setminus \{-1\}, \mathbb{C}))$. The map $(F(\Theta)^*)^{-1} \circ M : g \in \tilde{X} \mapsto \{\langle \mathbf{e}, g \rangle_{L^2}\}_{k \in \mathbb{Z}^* \setminus \{-1\}} \in h^{\tilde{d}-1}(\mathbb{Z}^* \setminus \{-1\}, \mathbb{C})$ is invertible. For every $\{x_k\}_{k \in \mathbb{Z}^* \setminus \{-1\}} \in h^{\tilde{d}-1}(\mathbb{Z}^* \setminus \{-1\}, \mathbb{C})$, there exists $u \in L^2$ such that

$$(23) \quad x_k = \int_0^T u(\tau) e^{-i\theta_k \tau} d\tau, \quad \forall k \in \mathbb{Z}^* \setminus \{-1\}.$$

Given $\{x_k\}_{k \in \mathbb{N}} \in h^{\tilde{d}-1}(\mathbb{N}, \mathbb{C})$, we introduce $\{\tilde{x}_k\}_{k \in \mathbb{Z}^* \setminus \{-1\}} \in h^{\tilde{d}-1}(\mathbb{Z}^* \setminus \{-1\}, \mathbb{C})$ such that $\tilde{x}_k = x_k$ for $k > 0$, while $\tilde{x}_k = \bar{x}_{-k}$ for $k < -1$. Thanks to (23) and to the definition of Θ , there exists $u \in L^2$ such that

$$\int_0^T u(s) e^{i(\omega_k - \omega_1)s} ds = x_k = \int_0^T \bar{u}(s) e^{i(\omega_k - \omega_1)s} ds, \quad k \in \mathbb{N} \setminus \{1\}.$$

If $x_1 \in \mathbb{R}$, then u is real and (21) is solvable with $u \in L^2((0, T), \mathbb{R})$. \square

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