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L^p NORMS AND SUPPORT OF EIGENFUNCTIONS ON GRAPHS

ETIENNE LE MASSON AND MOSTAFA SABRI

ABSTRACT. This article is concerned with properties of delocalization for eigenfunctions of Schrödinger operators on large finite graphs. More specifically, we provide ℓ^p -norm estimates for such eigenfunctions and show that they have a large support. Our estimates hold for any fixed, possibly irregular graph, in prescribed energy regions, and also for certain sequences of graphs such as N -lifts.

1. INTRODUCTION

Recent years have seen much interest in understanding the spectra and eigenvectors of large finite graphs and random matrices. Concerning the spectrum, central questions are the convergence of the empirical laws (normalized count of eigenvalues in an interval) and the properties of the limiting distribution. If the graphs converge to some random infinite graph (in the sense of Benjamini-Schramm), an important question is also the nature of the spectrum (pure point, absolutely continuous, singularly continuous) of almost every limit graph. Concerning the eigenvectors, one asks whether they become *localized* as the graph grows large (decay exponentially, have small support) or *delocalized*. Eigenvector delocalization is measured by many criteria : large support, uniform distribution of the entries over the graph, and also norm estimates. If the supremum norms of ℓ^2 -normalized eigenvectors decay fast as the graph gets large, this forces the entries of the vector to spread out. Bounds on the ℓ^p norms give further insight into the shape of the eigenvectors.

To present our results, it is instructive to start with very simple graphs : N -cycles. The set of vertices is $\{0, \dots, N-1\}$ and each point has two neighbors. The eigenvalues and eigenvectors of the adjacency matrix on such graphs are completely explicit : we have

$$\lambda_j = 2 \cos \left(\frac{2j\pi}{N} \right) \quad \text{and} \quad \psi_{\lambda_j} = \frac{1}{\sqrt{N}} (1, \omega^j, \omega^{2j}, \dots, \omega^{(N-1)j})$$

for $j = 0, \dots, N-1$, where $\omega = e^{\frac{2\pi i}{N}}$ and the eigenvectors are ℓ^2 -normalized. This is the ideal delocalization one can hope for : all eigenvectors are perfectly uniformly distributed on the graph, in the sense that $|\psi_\lambda(k)|^2 = \frac{1}{N}$ for all k . Here, $\|\psi_\lambda\|_\infty = \frac{1}{|G|^{1/2}}$, $\|\psi_\lambda\|_p = \frac{1}{|G|^{\frac{1}{2} - \frac{1}{p}}}$ for any $p > 2$ and each ψ_λ has full support on the graph.

As a first step towards generalization, one can replace such 2-regular graphs by general $(q+1)$ -regular graphs. This was carried out in a series of papers. It is shown in [8, 22, 7] that most eigenfunctions of such graphs are uniformly distributed in a certain sense, a property known as *quantum ergodicity*. Lower bounds on the support were provided in [20], where it is shown more generally that the eigenfunctions cannot concentrate on small sets. The recent paper [21] provides norm estimates which read as $\|\psi_\lambda\|_p \lesssim \frac{1}{(\log_q |G|)^{1/2}}$ for all $p > 2$, for generic regular graphs. All the preceding results apply to *deterministic* graphs. In case of *random* regular graphs of high degree, much better delocalization properties were obtained in [14]. It is shown essentially that all the ideal properties of N -cycles remain true, modulo logarithmic corrections.

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In this paper, we investigate the eigenvectors of graphs which are more general in two respects : we add potentials and consider irregular graphs. A quantum ergodicity result was recently established in this framework [9, 10, 11], where it is shown essentially that if a sequence of graphs converges in the sense of Benjamini-Schramm¹ to a random tree, and if the spectrum at the limit is purely absolutely continuous almost surely, then most eigenvectors become uniformly distributed in some sense. Our aim here is to provide norm estimates for the eigenfunctions and discuss their support. Our results hold in particular for *all eigenfunctions* in the bulk spectrum, whereas quantum ergodicity is a different delocalization criterion which only assesses *most eigenfunctions* in an interval.

Our theorems are formulated for fixed graphs (G, W) , where $W : V(G) \rightarrow \mathbb{R}$ is a potential and we consider the Schrödinger operator $H_G = \mathcal{A}_G + W$, with \mathcal{A}_G the adjacency matrix. If one wants to consider the asymptotics of a sequence (G_N, W_N) , one must keep track of the corresponding constants in the inequalities, in a common energy region for all N . Our results do not cover all Benjamini-Schramm limits discussed in [9], see § 1.2. Still, we show that if (G_N, W_N) is the set of N -lifts of an arbitrary finite graph (G_1, W_1) with degree ≥ 2 , then our estimates allow to control the whole sequence. An N -lift G_N is an N -cover over G_1 with $W_N(v) = W_1(\pi_N v)$, where $\pi_N : G_N \rightarrow G_1$ is the covering projection. The theory of random N -lift has been extensively studied in the last two decades, as a natural model of random irregular graphs [5, 6, 23, 15, 31, 16]. In particular, such graphs are typically connected (an assumption we make throughout).

1.1. Main results. Our first result concerns Schrödinger operators on N -cycles $G = \{0, \dots, N-1\}$. Each digit j is endowed with a potential W_j and we consider $H_G = \mathcal{A}_G + W$. Our estimates depend on the behavior of the “lifted” Schrödinger operator $H_{\mathbb{Z}}$ on \mathbb{Z} , which is the universal cover of G , with periodic potential $W_{j+kN} := W_j$. The spectrum of $H_{\mathbb{Z}}$ is purely absolutely continuous and consists of at most N bands (see Section 4). We say that λ is in *the bulk of the spectrum* if λ is in the interior of such a band.

Theorem 1.1. *Let ψ_λ be an eigenfunction of H_G on the N -cycle, $\|\psi_\lambda\|_2 = 1$.*

If $\lambda \notin \sigma(H_{\mathbb{Z}})$ and $\delta_\lambda = \text{dist}(\lambda, \sigma(H_{\mathbb{Z}}))$, then

$$\|\psi_\lambda\|_\infty^2 \leq \frac{16\delta_\lambda^{-2}}{N}.$$

If λ is in the bulk of the spectrum, assume moreover that the potential is m -periodic. That is $N = N'm$ and $W_{j+km} = W_j$ on G . Then

$$\|\psi_\lambda\|_\infty^2 \leq \frac{C_{\lambda,m}}{N}.$$

This theorem shows that the ideal decay of the norms on N -cycles that we discussed before remains true if we add periodic potentials.

One can also ask about the support of such eigenfunctions. The answer is easy : if ψ_λ is an eigenfunction of H_G , then $\psi_\lambda(k+1) = (\lambda - W_k)\psi_\lambda(k) - \psi_\lambda(k-1)$. So if ψ_λ vanishes on two consecutive digits, it vanishes identically. This shows the support is at least $\lceil \frac{N}{2} \rceil$. This lower bound is sharp in general.

The theorem remains true for weighted Schrödinger operators $(H_G\psi)(k) = a_k\psi(k+1) + a_{k-1}\psi(k-1) + W_k\psi(k)$, assuming $a_{j+m} = a_j$ in case of the bulk.

We now move to general graphs G , which we always assume to be connected. Our estimates are useful when the graph does not have too many short cycles. We thus define

- ρ_G , the minimal injectivity radius of G , i.e. the largest $\rho \in \mathbb{N}$ such that the ball $B_G(x, \rho)$ is a tree for any $x \in G$,

¹Roughly speaking, (G_N) converges to G in the sense of Benjamini-Schramm if random k -balls in G_N look similar to random k -balls in G , as N gets large. See [13, 4] for details and adaptations to (G_N, W_N) .

- ℓ_G , the largest $\ell \in \mathbb{N}$ such that $B_G(x, \ell)$ has at most one cycle for any $x \in G$.

Equivalently, ρ_G is half the girth (length of the smallest cycle in G). By definition, $\ell_G \geq \rho_G$. In case of random $(q+1)$ -regular graphs, we have $\ell_G \rightarrow +\infty$ almost surely as $|G| \rightarrow +\infty$. In fact, one has $\ell_G \geq \frac{1}{5} \log_q |G|$ almost surely, see [28]. This remains true for certain irregular graphs. More precisely, it follows from [16, Lemma 24] that if we consider a random N -lift G_N of a fixed finite graph G_1 , then with probability converging to one as $N \rightarrow \infty$, we have $\ell_{G_N} \geq \frac{1}{5} \log_{D-1} N$. In contrast, the probability that $\rho_G \geq c \log_q |G|$ is very small, see [28, Corollary 1].

Henceforth we assume that ℓ_G is large.

Our estimates show that eigenvector delocalization is intimately related to the behavior of the Green function on the universal cover of the graph. Let us introduce some notation.

If \mathcal{T} is a tree and $\gamma \in \mathbb{C}^+ = \{\text{Im } z > 0\}$, let

$$\zeta_w^\gamma(v) = -(H_{\mathcal{T}}^{(v|w)} - \gamma)^{-1}(v, v)$$

be the Green function of $\mathcal{T}^{(v|w)} \subset \mathcal{T}$, the subtree containing v if we remove the edge (v, w) from \mathcal{T} . If \mathcal{T} is the universal cover of G , we extend this definition to the finite graph by letting $\zeta_y^\gamma(x) := \zeta_{\tilde{y}}^\gamma(\tilde{x})$ for $(x, y) \in B(G)$, the set of oriented edges of G . Here, \tilde{x}, \tilde{y} are lifts of x, y on the universal cover.

We emphasize that if (x, y) is a directed edge in G , $\zeta_y^\gamma(x)$ is thus a Green function of an infinite tree, not the Green function of the finite graph.

We show in Theorem 2.2 that if G has minimal degree at least 2, then the spectrum of its universal cover (\tilde{G}, \tilde{W}) consists of bands of AC spectrum and possibly some eigenvalues.

Definition 1.2. Fix (G, W) . We say that an eigenvalue λ of H_G belongs to the *bulk of the spectrum* if it lies in the interior of an AC band of $H_{\tilde{G}}$.

If λ is in the bulk spectrum and $s > 1$, we define the parameters

$$(1.1) \quad z_\lambda(G) = \min_{(x,y) \in B(G)} |\text{Im } \zeta_y^{\lambda+i0}(x)|,$$

$$(1.2) \quad Z_{s,\lambda}(G) = \max_{(x_0, x_1) \in B(G)} \sum_{x_2 \in \mathcal{N}_{x_1} \setminus \{x_0\}} \frac{|\zeta_{x_0}^{\lambda+i0}(x_1)|^{2s} |\text{Im } \zeta_{x_1}^{\lambda+i0}(x_2)|^s}{|\text{Im } \zeta_{x_0}^{\lambda+i0}(x_1)|^s},$$

where \mathcal{N}_{x_1} denotes the set of neighbors of x_1 .

For example, if G is $(q+1)$ -regular and $W \equiv 0$, then $\tilde{G} = \mathbb{T}_q$ is the $(q+1)$ -regular tree and the bulk spectrum is $(-2\sqrt{q}, 2\sqrt{q})$ — also known as the *tempered spectrum*.

It follows from Theorem 2.2 that $z_\lambda(G)$ and $Z_{s,\lambda}(G)$ are well-defined, with $z_\lambda(G) > 0$. If moreover $\min \deg G \geq 3$, then $Z_{s,\lambda}(G) < 1$, see § 5.2.

Theorem 1.3. *Let (G, W) be a graph of minimal degree ≥ 2 . Let ψ_λ be an eigenfunction of H_G , $\|\psi_\lambda\|_2 = 1$.*

- (1) *If λ is in the bulk spectrum, we have*

$$\|\psi_\lambda\|_\infty \leq \frac{12Dz_\lambda(G)^{-4}}{\sqrt{\ell_G}},$$

where D is the maximal degree of G .

- (2) *If $\lambda \notin \sigma(H_{\tilde{G}})$, we have the much better bound*

$$\|\psi_\lambda\|_\infty \leq \frac{8\sqrt{2}}{\delta_\lambda} \left(1 + \frac{\delta_\lambda}{2D}\right)^{-\ell_G},$$

where, $\delta_\lambda = \text{dist}(\lambda, \sigma(H_{\tilde{G}}))$.

The fact that eigenfunctions have better delocalization properties outside the spectrum of $H_{\tilde{G}}$ was already known in case of $(q+1)$ -regular graphs with $W = 0$. This region corresponds to the “untempered spectrum”. In this case, [21, Lemma 3.1] implies that

$$(1.3) \quad \|\psi_\lambda\|_\infty \lesssim \frac{1}{q^{c\lambda\ell_G}}.$$

In particular, if $\ell_G \geq c \log_q |G|$, which holds for a typical random graph, this gives

$$(1.4) \quad \|\psi_\lambda\|_\infty \lesssim \frac{1}{|G|^{\varepsilon\lambda}}.$$

This very simple result somehow complements the bounds $\|\psi_\lambda\|_\infty \lesssim \frac{\log |G|}{|G|^{1/2}}$ proved for random graphs in [14] for λ in the bulk of the spectrum (this said, (1.4) is of course much easier to prove). In our case, we know we can still take $\ell_G = \frac{1}{5} \log_{D-1} |G|$ if G is a random N -lift of a base graph G_1 , yielding $\|\psi_\lambda\| \lesssim \frac{1}{|G|^{\frac{1}{5} \log_{D-1} (1 + \frac{\delta_\lambda}{2D})}}$. In contrast, our estimates in the bulk yield $\|\psi_\lambda\|_\infty \lesssim \frac{1}{(\log_{D-1} |G|)^{1/2}}$ generically.

We point out that as the graph grows large, most eigenvalues will lie in $\sigma(H_{\tilde{G}})$. More precisely, if for example G_N is a random N -lift of G_1 , it follows from the estimates in [16, Lemma 24], [19, Lemma 9] that (G_N, W_N) converges to the universal cover $(\tilde{G}_1, \tilde{W}_1)$ in the sense of Benjamini-Schramm with high probability. It is known that Benjamini-Schramm convergence implies the convergence of spectral measures, so the observation follows.

Remark 1.4. In view of Theorem 2.2, there are two finite sets of exceptional energies that are not considered in Theorem 1.3 : the set \mathfrak{F} of infinitely degenerate eigenvalues of $H_{\tilde{G}}$, and the set \mathfrak{F}' of endpoints of the AC spectrum. Note that $\mathfrak{F} = \emptyset$ for $(q+1)$ -regular graphs with $W = 0$ and $\mathfrak{F}' = \{\pm 2\sqrt{q}\}$. We believe it is natural to exclude \mathfrak{F} from our considerations; in fact we expect any eigenfunction ψ_λ of H_G with eigenvalue in \mathfrak{F} to be *localized* (See the end of this section for an example). On the other hand, the fact that we exclude \mathfrak{F}' from our analysis may be an artefact of our method.

As an easy consequence of Theorem 1.3, one gets

Corollary 1.5. (i) (p -norms). For λ in the bulk and $p > 2$,

$$\|\psi_\lambda\|_p \lesssim \frac{1}{(\ell_G)^{\frac{p-2}{2p}}}.$$

For $\lambda \notin \sigma(H_{\tilde{G}})$, we have

$$\|\psi_\lambda\|_p \lesssim \left(1 + \frac{\delta_\lambda}{2D}\right)^{-\frac{p-2}{p}\ell_G}.$$

(ii) (Non-localization). Let $\Lambda \subset G$ and suppose $\|\chi_\Lambda \psi_\lambda\|_2^2 \geq \varepsilon$.

If λ is in the bulk of the spectrum, then $|\Lambda| \geq \ell_G \cdot \frac{z_\lambda(G)^6 \varepsilon}{36D^2}$.

If $\lambda \notin \sigma(H_{\tilde{G}})$, then $|\Lambda| \geq \frac{\delta_\lambda^2}{128} \left(1 + \frac{\delta_\lambda}{2D}\right)^{2\ell_G} \varepsilon$.

This corollary is deduced from the supremum bounds. The results are satisfactory for $\lambda \notin \sigma(H_{\tilde{G}})$. For λ in the bulk however, the estimate on p -norms is not sharp. For example, we have $\|\psi_\lambda\|_p \lesssim \frac{1}{\sqrt{\ell_G}}$ for all $p > 2$ in the regular case [21]. As for non-localization, much better lower bounds $|\Lambda| \gtrsim q^{c\varepsilon\ell_G} \varepsilon$ were established for $(q+1)$ -regular graphs in [20].

We were able obtain sharper p -norm bounds as in [21], but for $p > 4$. As for non-localization, we can also answer the special case where Λ is the support of ψ_λ :

Theorem 1.6. Let (G, W) be a graph of minimal degree ≥ 3 . Let ψ_λ be an eigenfunction of H_G , $\|\psi_\lambda\|_2 = 1$. Assume λ is in the bulk spectrum of H_G . Then

(1) (p -norms) For any $p > 4$,

$$\|\psi_\lambda\|_p \leq \frac{16Dz_\lambda(G)^{-4}}{1 - Z_{p/4,\lambda}^{2/p}(G)} \cdot \frac{1}{\sqrt{\ell_G}}.$$

(2) (Support) If Λ is the support of ψ_λ , let $M_\lambda = Z_{2,\lambda}^{-1/4} > 1$. Then

$$|\Lambda| \geq \frac{1}{4D} M_\lambda^{\ell_G}.$$

Recall that if $(G, W) = (G_N, W_N)$ is an N -lift of some (G_1, W_1) , then $\ell_G \geq \frac{1}{5} \log_{D-1} |G|$ with high probability. In this case we get that the support of any ψ_λ satisfies

$$|\Lambda| \geq \frac{1}{4D} |G|^{\alpha_\lambda},$$

with $\alpha_\lambda = \frac{1}{5 \log_{M_\lambda}(D-1)} = \frac{\ln M_\lambda}{5 \ln(D-1)} > 0$. This shows that the support of eigenfunctions (in the bulk and outside $\sigma(H_{\tilde{G}})$) is generically a fractional power of the number of vertices — a large set.

1.2. Sequences of graphs. The previous results hold for any fixed graph (G, W) , in the prescribed energy regions.

An important question is to study sequences of graphs (G_N, W_N) that grow larger as $N \rightarrow \infty$. In this case, to ensure that $\|\psi_\lambda\|_p \lesssim \frac{1}{\sqrt{\ell_{G_N}}}$ and $|\text{supp } \psi_\lambda| \gtrsim K_\lambda^{\ell_{G_N}}$ with $K_\lambda > 1$, we must keep track of the implicit constants and show they are controlled as N gets large. This is the reason why we presented explicit constants in Theorems 1.3 and 1.6.

If (G_N, W_N) are $(q+1)$ -regular graphs with $W \equiv 0$, the situation is very simple : we have $\sigma(H_{\tilde{G}_N}) = [-2\sqrt{q}, 2\sqrt{q}]$ for all N , $\zeta_y^\lambda(x) = \frac{\exp(i\theta_\lambda)}{\sqrt{q}}$ for any (x, y) , independently of G_N , is the unique solution of $q\zeta^2 - \lambda\zeta + 1 = 0$ with negative imaginary part. In particular, $z_\lambda(G_N) = \frac{\sin \theta_\lambda}{\sqrt{q}}$ independently of N and $Z_{s,\lambda}(G_N) = qq^{-s} = q^{-(s-1)}$. Hence, the constants here are independent of N and we have $\|\psi_\lambda\|_p \lesssim \frac{1}{\sqrt{\ell_{G_N}}}$ and $|\text{supp } \psi_\lambda| \gtrsim M_\lambda^{\ell_{G_N}}$ for the whole sequence.

A more interesting class of graphs we can consider is given by N -lifts $\{(G_N, W_N)\}$, with (G_N, W_N) the N -lift of some base graph (G_1, W_1) say on r vertices, so that G_N has Nr vertices. In this case, all graphs have the same universal cover $(\tilde{G}_1, \tilde{W}_1)$. In particular, $\sigma(H_{\tilde{G}_N}) = \sigma(H_{\tilde{G}_1})$ for all N and we may consider an interval of energy I , say in the common bulk. On the other hand, since (G_N, W_N) covers (G_1, W_1) , $\zeta_y^\gamma(x) = \zeta_{y'}^\gamma(x')$ whenever $\pi_N(x, y) = \pi_N(x', y')$, where $\pi_N : G_N \rightarrow G_1$ is the covering map. Indeed, $\zeta_y^\gamma(x) = \zeta_{\pi_N y}^\gamma(\pi_N x)$, as both are defined by $\zeta_{\tilde{y}}^\gamma(\tilde{x})$ where $\tilde{x}, \tilde{y} \in \tilde{G}_1$ are lifts to the universal cover. Hence, the $\{\zeta_y^\gamma(x)\}_{(x,y) \in B(G_N)}$ are just those of G_1 . There are at most Dr of them. In particular, $z_\lambda(G_N) = z_\lambda(G_1)$ and $Z_{s,\lambda}(G_N) = Z_{s,\lambda}(G_1)$, and we have again $\|\psi_\lambda\|_p \lesssim \frac{1}{\sqrt{\ell_{G_N}}}$ and $|\text{supp } \psi_\lambda| \gtrsim M_\lambda^{\ell_{G_N}}$ for all N .

An example, however, to which our results do not seem to apply, is a sequence of Anderson models (G_N, W_N^ω) . Here W_N^ω is a random i.i.d. potential on a regular graph G_N with few cycles. It is not clear how the parameters $z_\lambda(G_N)$ and $Z_{s,\lambda}(G_N)$ behave. Moreover, there is a distinct universal cover for each N , it may be that we are excluding too many points in \mathfrak{F}_N as N grows large. On the other hand, it is natural that our results do not apply directly in this case, since otherwise we would have a statement for *all* ω instead of an almost-sure statement.

We expect our results to be true *in mean* for the weakly disordered Anderson model, almost surely (e.g. *most* (not all) eigenfunctions in the *limiting AC spectrum* satisfy

$\|\psi_\lambda\|_p \lesssim \frac{1}{\sqrt{\ell_G}}$ a.s.). Here the “limiting AC spectrum” is the AC spectrum of the Benjamini-Schramm limit of the sequence, replacing the bulk spectrum considered here. This different statement however (which would be weaker but apply to more general models) is still open. We mention that there is a heated debate among physicists about the ergodicity versus multi-fractality of the weakly disordered Anderson model on *random* regular graphs [2, 3, 34, 29]. One of the questions is whether $\|\psi_\lambda\|_p \lesssim \frac{1}{|G|^{\frac{1}{2} - \frac{1}{p}}}$ with high probability. This problem is completely out of reach at the moment (at least to us).

1.3. Further remarks.

- The proof of Theorem 1.3 yields more precisely

$$|\psi_\lambda(x)| \leq \frac{12Dz_\lambda^{-4}}{\sqrt{\ell_G(x)}},$$

where $\ell_G(x)$ is the largest $\ell \in \mathbb{N}$ such that $B_G(x, \ell)$ has at most one cycle. So even when ℓ_G is small, we can control the regions of the graph in which $\ell_G(x)$ is large.

- If we introduce $\mathcal{Z}_\lambda = \max_{(x_0, x_1, x_2)} \frac{|\zeta_{x_0}^\lambda(x_1)|^2 |\text{Im} \zeta_{x_1}^\lambda(x_2)|}{|\text{Im} \zeta_{x_0}^\lambda(x_1)|}$ for bulk value λ , we have $\mathcal{Z}_\lambda < 1$ if $\min \deg G \geq 3$, and $Z_{s, \lambda} \leq \mathcal{Z}_\lambda^{s-1}$, cf. § 5.2. In particular, Theorem 1.6 implies $\|\psi_\lambda\|_p \leq \frac{16Dz_\lambda^{-4}}{1 - \mathcal{Z}_\lambda^{\frac{p-4}{2p}}} \frac{1}{\sqrt{\ell_G}}$, and one may take $p \rightarrow \infty$ to deduce a new proof

of Theorem 1.3 part (1). Note however that this only works for $\deg G \geq 3$, while Theorem 1.3 allows for $\deg G \geq 2$. Moreover the proof of Theorem 1.3 yields additional “local” information, as explained in the previous point.

- In the special case of biregular graphs², our estimate on the support gives more precisely $|\Lambda| \geq \frac{[(d_1-1)(d_2-1)]^{\ell_G/4}}{4d_1}$, if $d_1 \geq d_2$. In particular, if G is $(q+1)$ -regular, $|\Lambda| \geq \frac{q^{\ell_G/2}}{4(q+1)}$. This estimate is sharper than [20] (which gives $|\Lambda| \gtrsim q^{2^{-8}\ell_G}$) and the proof is simpler, but the result only holds for the support, not general Λ .
- All the results of Theorem 1.6 also hold if we replace ℓ_G by $n \geq \ell_G$, as long as n satisfies the following condition : for any $\beta > 0$, there are at most $2^{\beta k}$ paths of length k between two edges, for each $k \leq n$.

This condition was used in [21]. While for a given graph, it may occur that such $n > \ell_G$, note that we always have $n \leq \log_{D-1} |G|$, cf. [21]. So for generic graphs, it suffices to consider ℓ_G .

- All results can be adapted to adjacency matrices with weights, namely $(\mathcal{A}_p \psi)(x) = \sum_{y \sim x} p_x(y) \psi(y)$ with $p_x(y)$ positive and symmetric. This can be regarded as putting colors $p(b)$ on the edges. The finite graphs now take the form (G, W, p) , and the covers $(\tilde{G}, \tilde{W}, \tilde{p})$.
- The fact that eigenfunction delocalization depends on the behavior of the Green function was not apparent in [21], but is actually well-known, see [24, 14, 17, 9] to name a few references. For example, note that if ψ_j is an eigenfunction of H_G on G with corresponding eigenvalue λ_j , then for any $\eta > 0$,

$$(1.5) \quad |\psi_j(x)|^2 \leq \eta \sum_{k=1}^n \frac{\eta}{(\lambda_k - \lambda_j)^2 + \eta^2} |\psi_k(x)|^2 = \eta \text{Im} g^{\lambda_j + i\eta}(x, x),$$

where $g^\gamma(x, x) = \langle \delta_x, (H_G - \gamma)^{-1} \delta_x \rangle$ is the Green function of the finite graph. If $|G| = N$ and if one can take $\eta \approx \frac{1}{N}$, and guarantee that $|\text{Im} g^{\lambda_j + \frac{1}{N}}(x, x)|$ stays bounded, this implies that $|\psi_j(x)|^2 \lesssim \frac{1}{N}$, which is the ideal delocalization one can hope to achieve. This idea is implemented in [14] in the framework of random

²bipartite graphs with two types of vertices \bullet, \circ where \bullet has d_1 neighbors \circ , and \circ has d_2 neighbors \bullet .

regular graphs, so that most of the proof is devoted to controlling the Green function g^γ and show that it is close to the Green function G^γ of a tree-like graph, which is well-behaved.

We use a different idea in this paper : we deal *directly* with Green functions on trees (see Section 2), and the bounds come from the exponential decay of $G^{\lambda_j}(x_0, x_k)$ with respect to the length of the path $(x_0; x_k)$, not from the parameter η above.

Finally, we believe the approach of [24, 17] can be used to obtain supremum bounds of the form $\|\psi_\lambda\|_\infty \lesssim \sqrt{\frac{\log \rho_G}{\rho_G}}$ for the graphs considered in this paper. Such bounds are weaker than Theorem 1.3 however, mainly because ℓ_G is generically of order $\log |G|$, while ρ_G is not.

- Our upper bound is in terms of ℓ_G . It is natural to ask if we can go beyond this and prove strong delocalization bounds as in [14]. In our framework of *deterministic* graphs, one cannot hope for an upper bound involving $|G|$ directly. The presence of a few short cycles can create completely localized eigenfunctions, no matter how “nice” the rest of G is. In fact, as shown in [14, Figure 1], already in the context of 3-regular graphs, the presence of a subgraph composed of two inverted triangles creates an eigenfunction supported on two points, no matter how large G is. This shows that an upper bound involving ℓ_G is natural, but leaves the question open whether a decay better than $\ell_G^{-1/2}$ can be achieved for bulk eigenfunctions

In the context of *random* graphs however, we expect much better bounds than the ones achieved here. We believe the present paper could provide a good starting point to prove *complete delocalization* as in [14], but now in the framework of *random N -lifts* of possibly irregular graphs.

We conclude this section with an example of graphs in which functions localize on small regions, justifying why we exclude eigenvalues of the universal cover.

Consider a 6-cycle (x_0, \dots, x_5, x_0) . Let $\psi(x_0) = \psi(x_3) = \frac{1}{2}$, $\psi(x_1) = \psi(x_4) = \frac{-1}{2}$, $\psi(x_2) = \psi(x_5) = 0$. This gives an eigenfunction with eigenvalue -1 . Now consider a long segment (y_1, \dots, y_{3m-1}) . We glue this segment to the cycle, such that $y_1 \sim x_2$ and $y_{3m-1} \sim x_5$. We extend ψ by 0 on this new graph. Then ψ is still an eigenfunction, localized on the small cycle. Here, $\|\psi_\lambda\|_\infty = \frac{1}{2}$ although $\ell_G \gg 4$.

On the other hand, -1 is an eigenvalue of the universal cover. To see this, root the tree at x_0 for example. Let $f(\tilde{x}_0) = 1$, $f(\tilde{x}_1) = -1$, $f(\tilde{x}_5) = 0$. This defines f at the root and its neighbours. For the 2-sphere, $f(\tilde{x}_2) = 0$, $f(\tilde{y}_{3m-1}) = \frac{-1}{2}$, $f(\tilde{x}_4) = \frac{-1}{2}$. For further spheres, we follow this rule : on each line segment, we extend f using the only possible rule, namely $a, -a, 0, a, -a, 0, \dots$. Each time the tree splits (this occurs precisely at \tilde{x}_2 , \tilde{x}_5), we let $f(\tilde{x}_2) = f(\tilde{x}_5) = 0$. If f has value c at the parent of \tilde{x}_2 or \tilde{x}_5 , we let $f(v) = \frac{c}{2}$ on the two children. Note there are only two types of line segments : those of length 2, those of length $3m - 1$. We thus find in any case that if $M = 3m - 1$, so that $2 \leq M$,

$$\|f\|^2 \leq M + 4M \left| \frac{1}{2} \right|^2 + 8M \left| \frac{1}{4} \right|^2 + \dots \leq M \sum_{k=0}^{\infty} 2^{k+1} \cdot 2^{-2k} < \infty,$$

so f is indeed an eigenvector of $H_{\tilde{G}}$ with eigenvalue -1 .

This example shows that energies in \mathfrak{F} can correspond to localized eigenfunctions on the finite graph. It would be interesting to study if this is always the case.

2. LINKING THE EIGENFUNCTIONS TO THE GREEN FUNCTION ON THE COVERING TREE

Let G be a finite graph and \mathcal{A}_G its adjacency matrix. We are interested in eigenfunctions (ψ_λ) of Schrödinger operators $H_G = \mathcal{A}_G + W$ on a G . Our aim in this section is to derive

a convenient representation of ψ_λ using the Green functions of the universal cover. For this, we start by investigating the spectral properties of such covering trees.

Throughout this section, we assume that

(C1) The finite graph G has minimal degree at least 2 and is not a cycle.

Let \mathcal{T} be a tree. If $v, w \in \mathcal{T}$, $v \sim w$, we denote $\mathcal{T}^{(v|w)} \subset \mathcal{T}$ the subtree obtained by removing the branch emanating from v passing by w (keeping $v \in \mathcal{T}^{(v|w)}$). We also denote $\zeta_w^\gamma(v) = -(H_{\mathcal{T}}^{(v|w)} - \gamma)^{-1}(v, v)$, the corresponding Green function for $\gamma \in \mathbb{C}^+ = \{\text{Im } z > 0\}$. As the Green function is a Herglotz function, we have

$$(2.1) \quad |\text{Im } \zeta_w^\gamma(v)| = -\text{Im } \zeta_w^\gamma(v).$$

For future reference, we recall the following classical identities [27, 12] for the Green function on a tree : for any $\gamma \in \mathbb{C}^+$, $v \in \mathcal{T}$, $w \sim v$,

$$(2.2) \quad \frac{1}{G^\gamma(v, v)} = W(v) + \sum_{u \sim v} \zeta_v^\gamma(u) - \gamma, \quad \frac{-1}{\zeta_w^\gamma(v)} = W(v) + \sum_{u \in \mathcal{N}_v \setminus \{w\}} \zeta_v^\gamma(u) - \gamma,$$

$$(2.3) \quad \frac{1}{\zeta_w^\gamma(v)} - \zeta_v^\gamma(w) = \frac{-1}{G^\gamma(v, v)}, \quad \zeta_w^\gamma(v) = \frac{G^\gamma(v, v)}{G^\gamma(w, w)} \zeta_v^\gamma(w),$$

$$(2.4) \quad G^\gamma(v_0; v_k) = G^\gamma(v_0, v_0) \zeta_{v_0}^\gamma(v_1) \cdots \zeta_{v_{k-1}}^\gamma(v_k) \quad G^\gamma(v, w) = G^\gamma(w, v)$$

for any non-backtracking path $(v_0; v_k)$ in a tree \mathcal{T} .

Remark 2.1 (Trees of finite cone type). In this remark, we explain the consequence of assumption **(C1)** on the universal cover.

If \mathbb{T} is a tree, fix an origin $o \in \mathbb{T}$ and regard the rest of the tree as descending from o . So o has \mathcal{N}_o^+ children which are its neighbors in \mathbb{T} , and each $v \neq o$ has a set \mathcal{N}_v^+ of children and a single parent v_- . A *cone* in \mathbb{T} is a subtree $\mathbb{T}^{(v|v_-)}$, i.e. a subtree descending from v . We say that \mathbb{T} is a tree of finite cone type if there are finitely many non-isomorphic cones.

This definition may be extended to allow potentials $W : \mathbb{T} \rightarrow \mathbb{R}$. One regards (\mathbb{T}, W) as a “colored tree”, where each vertex $v \in \mathbb{T}$ has a color $W(v)$. In this case, we say (\mathbb{T}, W) is a tree of finite cone type if there are finitely many non-isomorphic colored subtrees $\mathbb{T}_W^{(v|v_-)}$.

For example, the $(q+1)$ -regular tree \mathbb{T}_q has only 2 cone types : the cone at the origin (which has $q+1$ children) and the other cones (each of which has q children). However, for the colored (\mathbb{T}_q, W) to be of finite cone type, the potential W must in particular take finitely many values.

For our purposes, the trees to keep in mind are the universal covers of finite graphs (G, W) . The potential is lifted to $\mathcal{T} = \tilde{G}$ naturally by $\mathcal{W}(v) := W(\pi v)$, where $\pi : \mathcal{T} \rightarrow G$ is the covering map. It is easy to see that $(\mathcal{T}, \mathcal{W})$ is then a tree of finite cone type. In fact, the cones $\mathcal{T}_{\mathcal{W}}^{(v|v_-)}$ and $\mathcal{T}_{\mathcal{W}}^{(w|w_-)}$ are isomorphic whenever $\pi(v_-, v) = \pi(w_-, w) = (x, y) \in B(G)$, where $B(G)$ is the set of oriented edges of G . This shows in fact that $(\mathcal{T}, \mathcal{W})$ has at most $|B(G)| + 1$ cone types (the +1 comes from the cone at the origin).

For such a universal cover, we use a finite set of labels $\mathfrak{A} = \{1, \dots, m\}$. We label the cone at the origin by 1 and the rest by $j \in \{2, \dots, m\}$, according to the type of the colored cone. Note that the origin has a distinct label (even if some cone is isomorphic to the cone at the origin). We then define a matrix $(M_{i,j})_{i,j \in \mathfrak{A}}$, where a vertex with label i has $M_{i,j}$ children of label j .

Assuming **(C1)** holds, we know by [30, Lemma 3.1] that the non-backtracking matrix \mathcal{B} on G is irreducible. This means that for any directed edges (x_0, x_1) and (y_0, y_1) in G , there is a non-backtracking path (v_0, \dots, v_k) such that $(v_0, v_1) = (x_0, x_1)$ and $(v_{k-1}, v_k) = (y_0, y_1)$. Since the directed edges of G index the different cones, this means that on $(\mathcal{T}, \mathcal{W})$, any colored cone $\mathcal{T}_{\mathcal{W}}^{(w|w_-)}$ appears as a descendant of any $\mathcal{T}_{\mathcal{W}}^{(v|v_-)}$, except for the cone at

the origin which does not appear as it has a distinct label. Hence, in terms of the matrix $M = (M_{i,j})$, we see that **(C1)** implies

(C1') We have $M_{1,1} = 0$. Moreover, for any $k, l \in \{2, \dots, m\}$, there is $n = n(k, l)$ such that $(M^n)_{k,l} \geq 1$.

The fact that $M_{1,1} = 0$ is by definition of the labels, the rest is implied by **(C1)**.

Theorem 2.2. *Assume that **(C1)** holds and let $(\mathcal{T}, \mathcal{W}) = (\tilde{G}, \tilde{W})$. Then*

(i) *The spectrum of $H_{\mathcal{T}}$ consists of a finite union of intervals and points :*

$$(2.5) \quad \sigma(H_{\mathcal{T}}) = (\cup_{r=1}^{\ell} J_r) \sqcup \mathfrak{F} \sqcup \mathfrak{F}' ,$$

where \sqcup denotes the disjoint union. Here J_r are open intervals, \mathfrak{F} is a finite set of points and \mathfrak{F}' is the set of endpoints of the intervals $\{J_r\}$.

The set $\cup_{r=1}^{\ell} J_r$ is never empty.

- (ii) *The limit $\zeta_w^\lambda(v) := \lim_{\eta \downarrow 0} \zeta_w^{\lambda+i\eta}(v)$ exists for λ in J_r , and satisfies $|\operatorname{Im} \zeta_w^\lambda(v)| > 0$ for all (v, w) .*
- (iii) *The limit $G^{\lambda+i0}(v, w)$ exists for any $v, w \in \mathcal{T}$ and $\lambda \in J_r$.*
- (iv) *The spectrum is purely absolutely continuous in closed $I \subset J_r$.*
- (v) *If $\mathfrak{F} \neq \emptyset$, then the points in \mathfrak{F} are eigenvalues of infinite multiplicity.*

Proof. Claims (i), (ii) are proved in [11, Section 4]. There the case $W \equiv 0$ was considered, but the proof remains the same : the fact that $(\mathcal{T}, \mathcal{W})$ is the universal cover of (G, W) is responsible for making the Green function of $H_{\mathcal{T}}$ algebraic. Condition **(C1)** implies condition **(C1')** from Remark 2.1. The role of the latter condition is to link the behaviors of all $\zeta_w^\lambda(v)$ (there are only finitely many distinct such ζ , indexed as ζ_j^γ , $j \in \mathfrak{A}$). The fact that we may take a disjoint union in (2.5) is due to the fact that \mathfrak{F} is by construction a set on which some $\zeta_j^{\lambda+i0}$ has a pole. Such a λ cannot be in J_r by (ii), so it is either isolated from J_r , or an endpoint of J_r .

The fact that $\cup_{r=1}^{\ell} J_r \neq \emptyset$, i.e. that $\sigma(H_{\mathcal{T}})$ always has some continuous spectrum, follows in particular from the results of [18, §1.6], because the tree $\mathcal{T} = \tilde{G}$ always has at least two topological ends under assumption **(C1)**. In that paper, the authors only consider $\mathcal{A}_{\mathcal{T}}$, but their proof remains the same for $H_{\mathcal{T}}$. Simply one replaces [18, eq. (7)] saying that $\lambda f(w) = (\mathcal{A}f)(w)$ by $(\lambda - W(w))f(w) = (\mathcal{A}f)(w)$.

For (iii), first note that by (2.2), $|G^\lambda(v, v)| = \left| \frac{1}{W(v) + \sum_{u \sim v} \zeta_v^\lambda(u) - \lambda} \right| \leq \frac{1}{|\operatorname{Im} \zeta_v^\lambda(u)|} < \infty$ using (ii). We deduce the claim for general w using (2.4).

Item (iv) follows from (iii) and the continuity of $\lambda \mapsto G^{\lambda+i0}(v, v)$, which follows from (2.2) and the continuity of $\lambda \mapsto \zeta_w^{\lambda+i0}(u)$. The latter is proved in [11].

We now prove (v). Since any $\lambda \in \mathfrak{F}$ is an isolated point of the spectrum, it is an eigenvalue. Hence, if $\mu_v(J) = \langle \delta_v, \chi_J(H_{\mathcal{T}}) \delta_v \rangle$ is the spectral measure at $v \in \mathcal{T}$, we know that $\mu_v(\{\lambda\}) > 0$ for some $v \in \mathcal{T}$. By [32, Theorem 1.6], we have $\lim_{\eta \downarrow 0} (-i\eta) G^{\lambda+i\eta}(w, w) = \mu_w(\{\lambda\})$ for any $w \in \mathcal{T}$.

If $\pi : \mathcal{T} \rightarrow G$ is the covering projection, let Γ be the group of automorphisms of \mathcal{T} such that $\pi \circ g = \pi$. This group acts freely on \mathcal{T} and $\mathcal{T}/\Gamma \cong G$. Clearly, $H_{\mathcal{T}}$ is invariant under the action of Γ . It follows that $G^\gamma(gv, gw) = G^\gamma(v, w)$ for any $g \in \Gamma$. In particular, $\mu_{gv}(J) = \mu_v(J)$ for any $g \in \Gamma$.

Showing that λ has infinite multiplicity amounts to proving $\operatorname{rank} \chi_{(\lambda-\epsilon, \lambda+\epsilon)}(H_{\mathcal{T}}) = +\infty$ for all $\epsilon > 0$. If this was not that case, this operator would be of finite rank. In particular it would have a finite trace. But $\operatorname{tr} \chi_{(\lambda-\epsilon, \lambda+\epsilon)}(H_{\mathcal{T}}) = \sum_{w \in \mathcal{T}} \mu_w(\lambda - \epsilon, \lambda + \epsilon) \geq \sum_{g \in \Gamma} \mu_{gv}(\{\lambda\}) = \sum_{g \in \Gamma} \mu_v(\{\lambda\}) = +\infty$. This proves (v). \square

The previous theorem shows that $z_\lambda := z_\lambda(G)$ is well-defined and strictly positive, see (1.1). We deduce some simple bounds. By (2.2), $|G^\gamma(v, v)| \leq \frac{1}{|\operatorname{Im}[W(v) + \sum_{u \sim v} \zeta_v^\gamma(u) - \gamma]|}$.

Since $\min \deg G \geq 2$, it follows that for λ in the bulk spectrum,

$$(2.6) \quad |G^\lambda(v, v)| \leq \frac{1}{2z_\lambda} \quad \text{and} \quad |\zeta_w^\lambda(v)| \leq \frac{1}{z_\lambda},$$

where we used the second part of (2.2) for the other inequality. Similarly, using $|\zeta^\lambda| \geq z_\lambda$ and (2.3), we get

$$(2.7) \quad \frac{1}{|G^\lambda(v, v)|} \leq 2z_\lambda^{-1} \quad \text{and} \quad \frac{|\operatorname{Im} \zeta_w^\lambda(v)|}{|\zeta_w^\lambda(v)|^2} \leq z_\lambda^{-1},$$

where we use $\frac{|\operatorname{Im} \zeta^\lambda|}{|\zeta^\lambda|^2} \leq \frac{1}{|\zeta^\lambda|}$ in the second part.

We now show how we link the behavior of the eigenfunctions of the finite graph to that of the Green function of the universal cover. This can be seen as an alternative to (1.5).

Henceforth, \mathcal{T} will always denote the universal cover (\tilde{G}, \tilde{W}) of (G, W) .

Proposition 2.3. *Let $H_G \psi_\lambda = \lambda \psi_\lambda$. Suppose $\lambda \in \mathbb{R} \setminus (\mathfrak{F} \cup \mathfrak{F}')$. Denote $G^\lambda(v, w) = \lim_{\eta \downarrow 0} G_{\mathcal{T}}^{\lambda+i\eta}(v, w)$. Given $x_0 \in G$, fix any $x_1 \sim x_0$. Then for any $r, k \geq 1$,*

$$(2.8) \quad \psi_\lambda(x_0) = \sum_{(x_2; x_{r+1})} \left[G^\lambda(\tilde{x}_0, \tilde{x}_{r+1}) \psi_\lambda(x_r) - G^\lambda(\tilde{x}_0, \tilde{x}_r) \psi_\lambda(x_{r+1}) \right] \\ + \sum_{(x_{-k}; x_{-1})} \left[G^\lambda(\tilde{x}_0, \tilde{x}_{-k}) \psi_\lambda(x_{-k+1}) - G^\lambda(\tilde{x}_0, \tilde{x}_{-k+1}) \psi_\lambda(x_{-k}) \right],$$

where the sums run over all paths (x_2, \dots, x_{r+1}) (resp. (x_{-k}, \dots, x_{-1})) such that $x_2 \in \mathcal{N}_{x_1} \setminus \{x_0\}$ (resp. $x_{-1} \in \mathcal{N}_{x_0} \setminus \{x_1\}$). Here $\tilde{x}_m \in \mathcal{T}$ is a lift of $x_m \in G$ with $d_{\mathcal{T}}(\tilde{x}_0, \tilde{x}_m) = m$.

If λ is in the bulk $\cup_{r=1}^\ell J_r$, we also have

$$(2.9) \quad \psi_\lambda(x_0) = \frac{1}{|\operatorname{Im} \zeta_{x_0}^\lambda(x_1)|} \sum_{(x_2; x_{r+1})} \left[\operatorname{Im} \left(\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{r-1}}^\lambda(x_r) \right) \psi_\lambda(x_{r+1}) \right. \\ \left. - \operatorname{Im} \left(\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_r}^\lambda(x_{r+1}) \right) \psi_\lambda(x_r) \right].$$

In the above representation, it may happen that $x_r = x_0$ (if there are cycles near x_0), but the corresponding point \tilde{x}_r which appears in the Green function will always satisfy $d_{\mathcal{T}}(\tilde{x}_0, \tilde{x}_r) = r$. The path in \mathcal{T} which links \tilde{x}_0 to \tilde{x}_r is precisely $(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_r)$.

Note that we obtain directly the Green function on the tree, in contrast to (1.5) where one obtains the Green function on the finite graph and then approximates it to a treelike graph. The proof will use the idea of non-backtracking eigenfunctions first introduced in [7] and further developed in [9], along with a simple observation (2.11). The argument would be a bit simpler if we assumed moreover that $G^\lambda(\tilde{x}_0, \tilde{x}_0) \neq 0$. In this case, one can work directly with $\lambda \in \mathbb{R}$ instead of considering $\gamma = \lambda + i\eta$ as we do below.

Proof. Fix $\eta > 0$, denote $\gamma = \lambda + i\eta$ and define

$$(2.10) \quad f_\gamma(x_0, x_1) = \psi_\lambda(x_1) - \zeta_{x_0}^\gamma(x_1) \psi_\lambda(x_0) \quad \text{and} \quad f_\gamma^*(x_0, x_1) = \psi_\lambda(x_0) - \zeta_{x_1}^\gamma(x_0) \psi_\lambda(x_1).$$

Then

$$f_\gamma(x_0, x_1) + \frac{1}{\zeta_{x_1}^\gamma(x_0)} f_\gamma^*(x_0, x_1) = \left[\frac{1}{\zeta_{x_1}^\gamma(x_0)} - \zeta_{x_0}^\gamma(x_1) \right] \psi_\lambda(x_0).$$

Hence, using (2.3),

$$(2.11) \quad \psi_\lambda(x_0) = -G^\gamma(\tilde{x}_0, \tilde{x}_0) \left[f_\gamma(x_0, x_1) + \frac{1}{\zeta_{x_1}^\gamma(x_0)} f_\gamma^*(x_0, x_1) \right].$$

Consider the non-backtracking operator \mathcal{B} on $\ell^2(B)$ defined by

$$(\mathcal{B}f)(b) = \sum_{b^+ \in \mathcal{N}_b^+} f(b^+),$$

where \mathcal{N}_b^+ is the set of outgoing edges from b , i.e. the set of b' such that $t_b = o_{b'}$ and $b' \neq ib$, where ib is the edge reversal of b . Using that $H\psi_\lambda = \psi_\lambda$ along with the recursive identity (2.2), we find that

$$\begin{aligned} (\mathcal{B}f_\gamma)(x_0, x_1) &= [\lambda\psi_\lambda(x_1) - W(x_1)\psi_\lambda(x_1) - \psi_\lambda(x_0)] - \psi_\lambda(x_1) \left[\gamma - W(x_1) - \frac{1}{\zeta_{x_0}^\gamma(x_1)} \right] \\ &= \frac{1}{\zeta_{x_0}^\gamma(x_1)} f_\gamma(x_0, x_1) - i\eta \psi_\lambda(x_1). \end{aligned}$$

Hence, $\zeta^\gamma \mathcal{B}f_\gamma = f_\gamma - i\eta \zeta^\gamma \tau_+ \psi$, where $(\tau_+ \psi)(x_0, x_1) = \psi(x_1)$ and ζ^γ is the multiplication operator by $\zeta^\gamma(x_0, x_1) = \zeta_{x_0}^\gamma(x_1)$. By induction, we get

$$(2.12) \quad (\zeta^\gamma \mathcal{B})^r f_\gamma = f_\gamma - i\eta \sum_{t=1}^r (\zeta^\gamma \mathcal{B})^{t-1} \zeta^\gamma \tau_+ \psi_\lambda.$$

Similarly, if $\iota \zeta^\gamma$ is the multiplication operator by $\iota \zeta^\gamma(x_0, x_1) = \zeta_{x_1}^\gamma(x_0)$, we get

$$(2.13) \quad (\iota \zeta^\gamma \mathcal{B}^*)^k f_\gamma^* = f_\gamma^* - i\eta \sum_{t=1}^k (\iota \zeta^\gamma \mathcal{B}^*)^{t-1} \iota \zeta^\gamma \tau_- \psi_\lambda,$$

where $(\tau_- \psi)(x_0, x_1) = \psi(x_0)$. On the other hand, for any $f \in \ell^2(B)$,

$$[(\zeta^\gamma \mathcal{B})^r f](x_0, x_1) = \sum_{(x_2; x_{r+1})} \zeta_{x_0}^\gamma(x_1) \cdots \zeta_{x_{r-1}}^\gamma(x_r) f(x_r, x_{r+1}),$$

$$[(\iota \zeta^\gamma \mathcal{B}^*)^k f](x_0, x_1) = \sum_{(x_{-k}; x_{-1})} \zeta_{x_1}^\gamma(x_0) \cdots \zeta_{x_{-k+2}}^\gamma(x_{-k+1}) f(x_{-k}, x_{-k+1}).$$

Hence, inserting (2.12) and (2.13) into (2.11) and using (2.4), we get

$$\begin{aligned} \psi_\lambda(x_0) &= - \left[\sum_{(x_2; x_{r+1})} G^\gamma(\tilde{x}_0, \tilde{x}_r) f_\gamma(x_r, x_{r+1}) + \sum_{(x_{-k}; x_{-1})} G^\gamma(\tilde{x}_0, \tilde{x}_{-k+1}) f_\gamma^*(x_{-k}, x_{-k+1}) \right. \\ &\quad \left. + i\eta \sum_{t=1}^r \sum_{(x_2; x_t)} G^\gamma(\tilde{x}_0, \tilde{x}_t) \psi_\lambda(x_t) + i\eta \sum_{t=1}^k \sum_{(x_{-t+1}; x_{-1})} G^\gamma(\tilde{x}_0, \tilde{x}_{-t+1}) \psi_\lambda(x_{-t+1}) \right]. \end{aligned}$$

Finally, we use (2.10) to expand the first two sums and take $\eta \downarrow 0$. Since $\lambda \in \mathbb{R} \setminus (\mathfrak{F} \cup \mathfrak{F}')$, we know from Theorem 2.2 that all $G^\lambda(v, w)$ exist. In particular, the last two error sums vanish, proving the first representation.

For the second one, let

$$(2.14) \quad f_\lambda(x_0, x_1) = \psi_\lambda(x_1) - \zeta_{x_0}^\lambda(x_1) \psi_\lambda(x_0) \quad \text{and} \quad g_\lambda(x_0, x_1) = \psi_\lambda(x_1) - \overline{\zeta_{x_0}^\lambda(x_1)} \psi_\lambda(x_0).$$

Then for $\lambda \in J_r$,

$$(2.15) \quad \psi_\lambda(x_0) = \frac{f_\lambda(x_0, x_1) - g_\lambda(x_0, x_1)}{2i |\operatorname{Im} \zeta_{x_0}^\lambda(x_1)|}.$$

On the other hand, $\mathcal{B}f_\lambda = \frac{1}{\zeta^\lambda} f_\lambda$, $\mathcal{B}g_\lambda = \frac{1}{\overline{\zeta^\lambda}} g_\lambda$. Hence,

$$\psi_\lambda(x_0) = \frac{[(\zeta^\lambda \mathcal{B})^r f_\lambda](x_0, x_1) - [(\overline{\zeta^\lambda} \mathcal{B})^r g_\lambda](x_0, x_1)}{2i |\operatorname{Im} \zeta_{x_0}^\lambda(x_1)|}.$$

Expanding this gives the second representation. □

3. THE PROOF FOR SUPREMUM NORMS

We begin the section by analyzing balls $B_G(x, n)$ with $n \leq \ell_G$.

Remark 3.1. If $n \leq \rho_G$, there is a single path from x to $y \in B_G(x, n)$. If $n \leq \ell_G$, there can be more. If $d_G(x, y) = k$, because there is at most one cycle in $B_G(x, n)$, it is easy to see that there are at most two paths of length k from x to y .

Paths of length $k' > k$ can also reach y by winding around some cycle $C \subset B_G(x, n)$ before terminating at y . The path may loop several times on C , but once it has exited the cycle, it cannot go back to it. In fact, after leaving C , the path has a unique road to reach y . If it were to come back to C , it would have to backtrack on this road, which we exclude. If we fix the length k' , there can be at most two such paths : those traversing the cycle from either direction.

If y is on a cycle, there can also be a path of length $k'' > k$ which does not wind, but traverses the cycle in opposite direction (this situation will not arise in our proof later).

3.1. Within the bulk. Assume $\lambda \in J_r$ and recall the functions f_λ, g_λ in (2.14). As $(\zeta^\lambda \mathcal{B})^r f_\lambda = f_\lambda$ and $(\overline{\zeta^\lambda} \mathcal{B})^r g_\lambda = g_\lambda$, if we define the operators

$$\mathcal{M}_{n,\lambda} = \frac{1}{n} \sum_{r=1}^n \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^r |\operatorname{Im} \zeta^\lambda|^{1/2}$$

and $\overline{\mathcal{M}_{n,\lambda}} = \frac{1}{n} \sum_{r=1}^n \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\overline{\zeta^\lambda} \mathcal{B})^r |\operatorname{Im} \zeta^\lambda|^{1/2}$, we get $\mathcal{M}_{n,\lambda} \frac{f_\lambda}{|\operatorname{Im} \zeta^\lambda|^{1/2}} = \frac{f_\lambda}{|\operatorname{Im} \zeta^\lambda|^{1/2}}$ and $\overline{\mathcal{M}_{n,\lambda}} \frac{g_\lambda}{|\operatorname{Im} \zeta^\lambda|^{1/2}} = \frac{g_\lambda}{|\operatorname{Im} \zeta^\lambda|^{1/2}}$. Recalling (2.15), we thus get

$$(3.1) \quad \psi_\lambda(x_0) = \frac{1}{2i |\operatorname{Im} \zeta_{x_0}^\lambda|^{1/2}} \left[\left(\mathcal{M}_{n,\lambda} \frac{f_\lambda}{|\operatorname{Im} \zeta^\lambda|^{1/2}} \right) (x_0, x_1) - \left(\overline{\mathcal{M}_{n,\lambda}} \frac{g_\lambda}{|\operatorname{Im} \zeta^\lambda|^{1/2}} \right) (x_0, x_1) \right]$$

for any $x_1 \sim x_0$.

We estimate the first sum, the other is similar. Denote $b_j = (x_{j-1}, x_j)$. Write

$$\left(\mathcal{M}_{n,\lambda} \frac{f_\lambda}{|\operatorname{Im} \zeta^\lambda|^{1/2}} \right) (b_1) = \sum_{b' \in \mathcal{B}} \mathcal{M}_{n,\lambda}(b_1, b') \frac{f_\lambda}{|\operatorname{Im} \zeta^\lambda|^{1/2}}(b').$$

To describe the kernel, let $\mathcal{B}^r b_1$ be the set of edges which can be reached by a non-backtracking path of length r from b_1 . For example $\mathcal{B} b_1 = \mathcal{N}_{b_1}^+$ is the set of edges (x_1, x_2) as x_2 varies over $\mathcal{N}_{x_1} \setminus \{x_0\}$. Then we may replace sums over $(x_2; x_{r+1})$ by sums over $b_{r+1} \in \mathcal{B}^r b_1$.

Suppose first the injectivity radius at x_0 satisfies $\rho_G(x_0) \geq n$. If $b_{r+1} \in \mathcal{B}^r b_1$, there is a single path $(b_1, b_2, \dots, b_{r+1})$ from b_1 to b_{r+1} .

Let $\mathcal{K}(b_1, b_2) = \frac{\zeta(b_1) |\operatorname{Im} \zeta(b_2)|^{1/2}}{|\operatorname{Im} \zeta(b_1)|^{1/2}}$ for $b_2 \in \mathcal{B} b_1$, $\mathcal{K}(b_1, b_3) = \frac{\zeta(b_1) \zeta(b_2) |\operatorname{Im} \zeta(b_3)|^{1/2}}{|\operatorname{Im} \zeta(b_1)|^{1/2}}$ for $b_3 \in \mathcal{B}^2 b_1$. More generally, for $r \leq n$ we let $\mathcal{K}(b_1, b_{r+1}) = \frac{\zeta(b_1) \dots \zeta(b_r) |\operatorname{Im} \zeta(b_{r+1})|^{1/2}}{|\operatorname{Im} \zeta(b_1)|^{1/2}}$ for $b_{r+1} \in \mathcal{B}^r b_1$, and $\mathcal{K}(b_1, b') = 0$ otherwise. Then we have precisely $\mathcal{M}_{n,\lambda}(b_1, b') = \frac{1}{n} \mathcal{K}(b_1, b')$. Now by the Cauchy-Schwarz inequality,

$$(3.2) \quad \left| \left(\mathcal{M}_{n,\lambda} \frac{f_\lambda}{|\operatorname{Im} \zeta^\lambda|^{1/2}} \right) (b_1) \right| \leq \left(\sum_{b' \in \mathcal{B}} |\mathcal{M}_{n,\lambda}(b_1, b')|^2 \right)^{1/2} \left\| \frac{f_\lambda}{|\operatorname{Im} \zeta^\lambda|^{1/2}} \right\|_2.$$

On the other hand,

$$\begin{aligned} \sum_{b' \in B} |\mathcal{M}_{n,\lambda}(b_1, b')|^2 &= \frac{1}{n^2} \sum_{r=1}^n \sum_{b_{r+1} \in \mathcal{B}^r b_1} \frac{|\zeta^\lambda(b_1) \cdots \zeta^\lambda(b_r)|^2 |\operatorname{Im} \zeta^\lambda(b_{r+1})|}{|\operatorname{Im} \zeta^\lambda(b_1)|} \\ &= \frac{1}{n^2} \sum_{r=1}^n \sum_{(x_2; x_{r+1})} \frac{|\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{r-1}}^\lambda(x_r)|^2 |\operatorname{Im} \zeta_{x_r}^\lambda(x_{r+1})|}{|\operatorname{Im} \zeta_{x_0}^\lambda(x_1)|}. \end{aligned}$$

It follows from (2.1) and (2.2) that

$$(3.3) \quad \sum_{x_{k+1} \in \mathcal{N}_{x_k} \setminus \{x_{k-1}\}} |\operatorname{Im} \zeta_{x_k}^\lambda(x_{k+1})| = \frac{|\operatorname{Im} \zeta_{x_{k-1}}^\lambda(x_k)|}{|\zeta_{x_{k-1}}^\lambda(x_k)|^2}.$$

Using this identity repeatedly, we deduce that

$$(3.4) \quad \sum_{(x_2; x_{r+1})} |\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{r-1}}^\lambda(x_r)|^2 |\operatorname{Im} \zeta_{x_r}^\lambda(x_{r+1})| = |\operatorname{Im} \zeta_{x_0}^\lambda(x_1)|.$$

Hence, $\sum_{b' \in B} |\mathcal{M}_{n,\lambda}(b_1, b')|^2 = \frac{1}{n}$ (assuming $n \leq \rho_G(x_0)$).

Now assume more generally that $n \leq \ell_G$ and suppose $B_G(x_0, n)$ contains a cycle $C = (u_0, \dots, u_m)$. If $x_0 \notin C$, then there is a unique path from x_0 to a point in C . Indeed, as $n \leq \ell_G$, we know C is the only cycle in $B_G(x_0, n)$. If there were two paths $(x_0, y_1, \dots, y_k, u_i)$ and $(x_0, y'_1, \dots, y'_r, u_j)$ from x_0 to points $u_i, u_j \in C$, we would get an additional cycle $(x_0, y_1, \dots, y_k, u_i, u_{i-1}, \dots, u_j, y'_r, y'_{r-1}, \dots, x_0)$, which is forbidden. Now the nice property is that (3.1) is valid for any $x_1 \sim x_0$. If $(x_0, y_1, \dots, y_k, u_i)$ is the unique path from x_0 to C , we simply choose $x_1 \neq y_1$. Then we are sure that the paths $(x_2; x_{r+1})$ outgoing from (x_0, x_1) will never meet the cycle C . Hence, in this case the values of $\mathcal{M}_{n,\lambda}(b_1, b')$ are the same as the case $\rho_G(x_0) \geq n$ and we get again $\sum_{b' \in B} |\mathcal{M}_{n,\lambda}(b_1, b')|^2 = \frac{1}{n}$.

Now suppose $x_0 \in C$. If $\deg(x_0) \geq 3$, we use the same idea : say $x_0 = u_0$. Then we choose $x_1 \sim x_0$, $x_1 \neq u_1, u_m$ a neighbor not in C . Then again we are sure the paths $(x_2; x_{r+1})$ outgoing from (x_0, x_1) will never meet the cycle, otherwise we would get at least two cycles in $B_G(x_0, n)$. So we get again $\sum_{b' \in B} |\mathcal{M}_{n,\lambda}(b_1, b')|^2 = \frac{1}{n}$.

The case $x_0 \in C$ and $\deg(x_0) = 2$ is more subtle. Here we can no longer avoid the cycle $C = (u_0, \dots, u_m)$. Say $x_0 = u_0$ and let $x_1 = u_1$. Consider $b_{r+1} \in \mathcal{B}^r b_1$, say $b_{r+1} \notin C$. While in the previous cases, $\frac{f_\lambda}{|\operatorname{Im} \zeta^\lambda|^{1/2}}(b_{r+1})$ was simply multiplied by $\frac{\zeta(b_1) \cdots \zeta(b_r) |\operatorname{Im} \zeta(b_{r+1})|^{1/2}}{n |\operatorname{Im} \zeta(b_1)|^{1/2}}$, here elements from $\mathcal{B}^s b_1$, $s > r$ can also reach the same point b_{r+1} and add a contribution to $\frac{f_\lambda}{|\operatorname{Im} \zeta^\lambda|^{1/2}}(b_{r+1})$. This occurs precisely when a long path first makes a few loops around C , then leaves the cycle. As mentioned in Remark 3.1, once it left C , it cannot return to it.

Similarly, any $b_{r+1} \in C$ can be reached through many paths : the shortest one of length r , then the longer ones of length $r + mk \leq n$, which first make k loops around C .

From these considerations, we see that for $e_{r+1} \in \mathcal{B}^r b_1$, $e_{r+1} = (u_r, u_{r+1})$,

$$\mathcal{M}_{n,\lambda}(b_1, e_{r+1}) = \frac{1}{n} \left(\sum_{j=0}^{k_r} (\zeta_{u_0}^\lambda(u_1) \cdots \zeta_{u_m}^\lambda(u_0))^j \right) \frac{\zeta_{u_0}^\lambda(u_1) \cdots \zeta_{u_{r-1}}^\lambda(u_r) |\operatorname{Im} \zeta_{u_r}^\lambda(u_{r+1})|^{1/2}}{|\operatorname{Im} \zeta_{u_0}^\lambda(u_1)|^{1/2}},$$

with $\mathcal{M}_{n,\lambda}(b_1, b_1) = \frac{1}{n} \sum_{j=1}^{k_0} (\zeta_{u_0}^\lambda(u_1) \cdots \zeta_{u_m}^\lambda(u_0))^j$. If $b_{r+1} \in \mathcal{B}^r b_1$, $b_{r+1} \notin C$, let u_i be the unique vertex in C of degree ≥ 3 which is closest to $o(b_{r+1})$. Then

$$(3.5) \quad \mathcal{M}_{n,\lambda}(b_1, b_{r+1}) = \frac{1}{n} \left(\sum_{j=0}^{k_r} (\zeta_{u_0}^\lambda(u_1) \cdots \zeta_{u_m}^\lambda(u_0))^j \right) \cdot \frac{|\zeta_{u_0}^\lambda(u_1) \cdots \zeta_{u_{i-1}}^\lambda(u_i) \zeta_{u_i}^\lambda(v_{i+1}) \cdots \zeta_{v_{r-1}}^\lambda(v_r)| |\operatorname{Im} \zeta_{v_r}^\lambda(v_{r+1})|^{1/2}}{|\operatorname{Im} \zeta_{u_0}^\lambda(u_1)|^{1/2}},$$

where $b_{r+1} = (v_r, v_{r+1})$ and $(u_i, v_{i+1}, \dots, v_{r+1})$ is the unique path from u_i to v_{r+1} .

To handle the above expressions, we must first control the sums over cycles.

Proposition 3.2. *Let G be a connected graph of minimal degree ≥ 2 .*

Suppose G has a cycle (u_0, \dots, u_m, u_0) which does not cover all vertices of G . Then $|\zeta_{u_0}^\lambda(u_1) \cdots \zeta_{u_{m-1}}^\lambda(u_m) \zeta_{u_m}^\lambda(u_0)| \leq (1 - \frac{z_\lambda^2}{4})$ for any λ in the bulk spectrum $\cup_r J_r$.

The proposition is also valid for all $\lambda \in \mathbb{R}$ if we replace the bound by ≤ 1 .

Proof. As the cycle does not cover G , we may assume $\deg(u_0) \geq 3$. Then besides the two neighbors u_1, u_m , we know that u_0 has at least one neighbor v_0 outside the cycle.

Using (3.4), $\sum_{(v_2; v_{m+2})} |\zeta_{u_0}^\lambda(u_1) \zeta_{u_1}^\lambda(v_1) \cdots \zeta_{v_m}^\lambda(v_{m+1})|^2 |\operatorname{Im} \zeta_{v_{m+1}}^\lambda(v_{m+2})| = |\operatorname{Im} \zeta_{u_0}^\lambda(u_1)|$, where the sum is over all non-backtracking paths $(v_2; v_{m+2})$ leaving the oriented edge (u_0, u_1) . There are at least two paths: $(u_2, u_3, \dots, u_m, u_0, u_1)$ and $(u_2, u_3, \dots, u_m, u_0, v_0)$. Hence,

$$|\zeta_{u_0}^\lambda(u_1) \cdots \zeta_{u_m}^\lambda(u_0)|^2 (|\operatorname{Im} \zeta_{u_0}^\lambda(u_1)| + |\operatorname{Im} \zeta_{u_0}^\lambda(v_0)|) \leq |\operatorname{Im} \zeta_{u_0}^\lambda(u_1)|,$$

so $|\zeta_{u_0}^\lambda(u_1) \cdots \zeta_{u_m}^\lambda(u_0)|^2 \leq \frac{|\operatorname{Im} \zeta_{u_0}^\lambda(u_1)|}{|\operatorname{Im} \zeta_{u_0}^\lambda(u_1)| + |\operatorname{Im} \zeta_{u_0}^\lambda(v_0)|}$. Since $\frac{|\operatorname{Im} \zeta_{u_0}^\lambda(v_0)|}{|\operatorname{Im} \zeta_{u_0}^\lambda(u_1)| + |\operatorname{Im} \zeta_{u_0}^\lambda(v_0)|} \geq \frac{z_\lambda}{2z_\lambda^{-1}}$ by (2.6), we get $|\zeta_{u_0}^\lambda(u_1) \cdots \zeta_{u_m}^\lambda(u_0)|^2 \leq 1 - \frac{z_\lambda^2}{2} \leq (1 - \frac{z_\lambda^2}{4})^2$. \square

From this, we get $|\mathcal{M}_{n,\lambda}(b_1, b_1)|^2 \leq \frac{16z_\lambda^{-4}}{n^2}$ and

$$(3.6) \quad |\mathcal{M}_{n,\lambda}(b_1, e_{r+1})|^2 \leq \frac{16z_\lambda^{-4}}{n^2} \cdot \frac{|\zeta_{u_0}^\lambda(u_1) \cdots \zeta_{u_{r-1}}^\lambda(u_r)|^2 |\operatorname{Im} \zeta_{u_r}^\lambda(u_{r+1})|}{|\operatorname{Im} \zeta_{u_0}^\lambda(u_1)|} \leq \frac{16z_\lambda^{-4}}{n^2},$$

where the last inequality is due to (3.4), since $(u_2; u_{r+1})$ is just one of the paths $(x_2; x_{r+1})$.

For general points b_{r+1} (in and out of C), let $\{u_{j_1}, \dots, u_{j_s}\}$ be the vertices in C of degree ≥ 3 . Using (3.5) and Proposition 3.2 again, we have for $C_\lambda = 16z_\lambda^{-4}$,

$$\begin{aligned} \sum_{b' \in B} |\mathcal{M}_{n,\lambda}(b_1, b')|^2 &= \sum_{r=1}^n \left(\sum_{b_{r+1} \in \mathcal{B}^r b_1 \cap C} + \sum_{b_{r+1} \in \mathcal{B}^r b_1 \cap C^c} \right) |\mathcal{M}_{n,\lambda}(b_1, b_{r+1})|^2 \\ &\leq \frac{C_\lambda}{n} + \frac{C_\lambda}{n^2} \sum_{i=1}^s |\zeta_{u_0}^\lambda(u_1) \cdots \zeta_{u_{j_i-1}}^\lambda(u_{j_i})|^2 \sum_{r=j_i}^n \sum_{(v_{j_i+1}; v_{r+1})} \frac{|\zeta_{u_{j_i}}^\lambda(v_{j_i+1}) \cdots \zeta_{v_{r-1}}^\lambda(v_r)|^2 |\operatorname{Im} \zeta_{v_r}^\lambda(v_{r+1})|}{|\operatorname{Im} \zeta_{u_0}^\lambda(u_1)|}, \end{aligned}$$

where we used (3.6) and the fact that $\mathcal{B}^r b_1 \cap C = \{e_{r+1}\}$. The last sum is over all paths outgoing from (u_{j_i-1}, u_{j_i}) and leaving C . In particular, from (3.4),

$$\sum_{b' \in B} |\mathcal{M}_{n,\lambda}(b_1, b')|^2 \leq \frac{C_\lambda}{n} + \frac{C_\lambda}{n} \sum_{i=1}^s \frac{|\zeta_{u_0}^\lambda(u_1) \cdots \zeta_{u_{j_i-2}}^\lambda(u_{j_i-1})|^2 |\operatorname{Im} \zeta_{u_{j_i-1}}^\lambda(u_{j_i})|}{|\operatorname{Im} \zeta_{u_0}^\lambda(u_1)|}.$$

Finally, using (3.4) again, if $(u_0, u_1, x_2; x_{m+1})$ are the m -paths outgoing from (u_0, u_1) , then $\sum_{(x_2; x_{m+1})} |\zeta_{u_0}^\lambda(u_1) \zeta_{u_1}^\lambda(x_2) \cdots \zeta_{x_{m-1}}^\lambda(x_m)|^2 |\operatorname{Im} \zeta_{x_m}^\lambda(x_{m+1})| = |\operatorname{Im} \zeta_{u_0}^\lambda(u_1)|$. In particular,

$$\sum_{i=1}^s |\zeta_{u_0}^\lambda(u_1) \cdots \zeta_{u_{j_i-1}}^\lambda(u_{j_i})|^2 \sum_{(x_{j_i+1}; x_{m+1})} |\zeta_{u_{j_i}}^\lambda(x_{j_i+1}) \cdots \zeta_{x_{m-1}}^\lambda(x_m)|^2 |\operatorname{Im} \zeta_{x_m}^\lambda(x_{m+1})| \leq |\operatorname{Im} \zeta_{u_0}^\lambda(u_1)|,$$

as this is the same sum restricted to paths leaving the cycle. Applying (3.4) to the left-hand side, we finally obtain

$$\sum_{b' \in B} |\mathcal{M}_{n,\lambda}(b_1, b')|^2 \leq \frac{32z_\lambda^{-4}}{n}.$$

Back to (3.2), note that $\|f_\lambda\| \leq D(1 + z_\lambda^{-1})$ by (2.6) and $|\operatorname{Im} \zeta^\lambda| \geq z_\lambda$. Here D is the maximal degree of G . Finally returning to (3.1), estimating the g_λ similarly, we get

$$\|\psi_\lambda\|_\infty \leq \frac{1}{2z_\lambda} \cdot \frac{\sqrt{32}z_\lambda^{-2}}{\sqrt{n}} \cdot (2D(1 + z_\lambda^{-1})) \leq \frac{12Dz_\lambda^{-4}}{\sqrt{n}},$$

where we used that $z_\lambda \leq 1$, as follows from $z_\lambda \leq |\zeta^\lambda| \leq z_\lambda^{-1}$.

3.2. Outside the spectrum. To deal with energies outside the spectrum, we use a Combes-Thomas estimate. This tool is a common ingredient in the theory of random Schrödinger operators to control the Green kernel.

Intuitively, if $\lambda \notin \sigma(H)$, then $\|G^\lambda \delta_x\|^2 = \sum_y |G^\lambda(y, x)|^2 < \infty$, implying that $|G^\lambda(y, x)|$ must decay with $d(x, y)$. The Combes-Thomas estimate says that if H is a Schrödinger operator, then the decay must be exponential.

Lemma 3.3. *Let \mathbb{G} be a countable graph with degree bounded by D and $H = \mathcal{A} + W$ a self-adjoint Schrödinger operator. Suppose $\lambda \notin \sigma(H)$ and let $\delta_\lambda = \operatorname{dist}(\lambda, \sigma(H))$. Then for any $x \in \mathbb{G}$,*

$$\sum_{y: d_{\mathbb{G}}(x,y)=n} |G^\lambda(y, x)|^2 \leq \frac{4}{\delta_\lambda^2} \cdot \frac{1}{(1 + \frac{\delta_\lambda}{2D})^{2n}}.$$

The estimate holds more generally if we replace \mathcal{A} by $(\mathcal{A}_p f)(x) = \sum_{y \sim x} p_y(x) f(y)$, $p_y(x) = p_x(y)$, in which case D should be replaced by $\max_x \sum_{y \sim x} p_x(y)$.

Proof. The proof is contained in [1, Theorem 10.5] or [26, Theorem 11.2].

Fix a cutoff $R > 0$ and consider the multiplication operator $M = e^{\mu \min(d(\cdot, x), R)}$. Then $M\delta_u = c_{u,x,R}\delta_u$, where $c_{u,x,R} = e^{\mu \min(d(u,x), R)}$. In particular, if $d(y, x) = n \leq R$, we get

$$e^{\mu n} G^\lambda(y, x) = \langle \delta_y, M(H - \lambda)^{-1} M^{-1} \delta_x \rangle = \langle \delta_y, (MHM^{-1} - \lambda)^{-1} \delta_x \rangle.$$

Hence, if $B := MHM^{-1} - H$,

$$\begin{aligned} \sum_{y, d(y,x)=n} |G^\lambda(y, x)|^2 &= e^{-2\mu n} \sum_{y, d(y,x)=n} \left| [(MHM^{-1} - \lambda)^{-1} \delta_x](y) \right|^2 \\ &\leq e^{-2\mu n} \|(MHM^{-1} - \lambda)^{-1} \delta_x\|^2 \leq e^{-2\mu n} \|(H + B - \lambda)^{-1}\|^2. \end{aligned}$$

Now $B(u, u') = c_{u,x,R} c_{u',x,R}^{-1} - 1$ if $u \sim u'$ and $B(u, u') = 0$ otherwise. In particular, $|B(u, u')| \leq e^\mu - 1$, $\sum_{u'} |B(u, u')| \leq D(e^\mu - 1)$ and $\sum_u |B(u, u')| \leq D(e^\mu - 1)$. It follows that $\|B\| \leq D(e^\mu - 1)$. Recall $\delta_\lambda = \operatorname{dist}(\lambda, \sigma(H))$. As is well-known, if $\|B\| < \delta_\lambda$, then $\|(H + B - \lambda)^{-1}\| \leq \frac{1}{\delta_\lambda - \|B\|}$. Choosing $\mu = \ln(1 + \frac{\delta_\lambda}{2D})$ proves the claim. \square

We may now prove part (2) of Theorem 1.3. Let $\lambda \notin \sigma(H_{\mathcal{T}})$. We use Proposition 2.3 with $k = r$; focus on the first sum. For $n \leq \ell_G$, we have

$$\left| \sum_{(x_2; x_{r+1})} G^\lambda(\tilde{x}_0, \tilde{x}_{r+1}) \psi_\lambda(x_r) \right|^2 \leq 2 \sum_{(x_2; x_{r+1})} |G^\lambda(\tilde{x}_0, \tilde{x}_{r+1})|^2 \sum_{y: d_G(y, x_0) \leq r} |\psi_\lambda(y)|^2,$$

the factor 2 comes from the second sum involving ψ_λ : as mentioned in Remark 3.1, there can be at most two paths of a fixed length $r + 1$ between x_0 and $y \in B_G(x_0, n)$. The first sum is over a subset of $\{v : d_{\mathcal{T}}(\tilde{x}_0, v) = r + 1\}$, so by Lemma 3.3, we get

$$\left| \sum_{(x_2; x_{r+1})} G^\lambda(\tilde{x}_0, \tilde{x}_{r+1}) \psi_\lambda(x_r) \right|^2 \leq \frac{8}{\delta_\lambda^2} \cdot \frac{1}{(1 + \frac{\delta_\lambda}{2D})^{2r}} \|\psi_\lambda\|_2^2.$$

The other sums in Proposition 2.3 are bounded similarly.

3.3. Proof of the corollary. Finally, since $\|\psi_\lambda\|_2^2 = 1$, we deduce that

$$\|\psi_\lambda\|_p = \left(\sum_{x \in V} |\psi_\lambda(x)|^p \right)^{1/p} \leq \|\psi_\lambda\|_\infty^{\frac{p-2}{p}} \left(\sum_{x \in V} |\psi_\lambda(x)|^2 \right)^{1/p} = \|\psi_\lambda\|_\infty^{\frac{p-2}{p}}.$$

For the second part of the corollary, simply observe that

$$\varepsilon \leq \|\chi_\Lambda \psi_\lambda\|_2^2 = \sum_{x \in \Lambda} |\psi_\lambda(x)|^2 \leq \|\psi_\lambda\|_\infty^2 \cdot |\Lambda|.$$

This completes the proof of Theorem 1.3 and Corollary 1.5.

4. CASE OF SCHRÖDINGER CYCLES

In this section, we consider the case of Schrödinger operators on an N -cycle $C_N = \{0, \dots, N - 1\}$. We denote the potentials by $W_j = W(j)$.

Note that Theorem 2.2 does not apply to this model as **(C1)** does not hold. Fortunately, the analog of this theorem already exists in the literature. In fact, the universal cover of (C_N, W) is just (\mathbb{Z}, W) and $H_{\mathbb{Z}}$ is just a periodic Schrödinger operator on \mathbb{Z} , where $W_{j+kN} = W_j$.

Theorem 4.1 (cf. [33]). *Let (\mathbb{Z}, W) be the universal cover of (C_N, W) .*

- (i) *The spectrum of $H_{\mathbb{Z}}$ is purely absolutely continuous, and consists of at most N bands : $\sigma(H_{\mathbb{Z}}) = \cup_{r=1}^N I_r$.*
- (ii) *If W is periodic on C_N , say $N = mN'$ with $W_{j+m} = W_j$, the spectrum of $H_{\mathbb{Z}}$ has at most m bands. Moreover, $G_{\mathbb{Z}}^{\lambda+i0}(i+km, i+km) = G_{\mathbb{Z}}^{\lambda+i0}(i, i)$ and $\zeta_{i+km}^{\lambda+i0}(i+km \pm 1) = \zeta_i^{\lambda+i0}(i \pm 1)$.*
- (iii) *The limits $\zeta_i^{\lambda+i0}(i \pm 1)$ and $G_{\mathbb{Z}}^{\lambda+i0}(i, j)$ exists for any $i, j \in \mathbb{Z}$ if λ is in the interior \mathring{I}_r . At least one Green function $G^{\lambda+i\eta}(i, i)$ blows up as $\eta \downarrow 0$ at the endpoints of I_r .*
- (iv) *For $\lambda \in \mathring{I}_r$, the Green functions are pure imaginary : $\operatorname{Re} G^{\lambda+i0}(i, i) = 0$.*
- (v) *In general, $G^{\lambda+i0}(i, i)$ only vanishes in the gaps between I_r . More precisely, $G^{\lambda+i0}(i, i)$ has exactly one zero in each gap (this zero can occur at an endpoint of I_r).*
- (vi) *If $\lambda \in \mathring{I}_r$, then $|\operatorname{Im} \zeta_i^{\lambda+i0}(i \pm 1)| > 0$ and $\operatorname{Im} G^{\lambda+i0}(i, i) > 0$ for all i .*

Proof. We simply indicate the references. For (i), see [33, Theorem 5.3.7]. If W is m -periodic, $H_{\mathbb{Z}}$ is also m -periodic, so we have at most m bands. The periodicity of the Green functions simply follows from the isomorphisms of the rooted graphs (\mathbb{Z}, W, i) and $(\mathbb{Z}, W, i + km)$ for any root i . Alternatively, one sees this from [33, eq. (5.4.72)].

For (iii)–(v), see Corollary 5.4.6, Theorem 5.4.15 and Theorem 5.4.17 in [33]. Note that with our notations, $\zeta_i^\gamma(i+1) = \frac{u_{i+1}^+(\gamma)}{u_i^+(\gamma)}$ and $\zeta_{i+1}^\gamma(i) = \frac{u_i^-(\gamma)}{u_{i+1}^-(\gamma)}$ correspond to the m -functions in [33], and $G^{\lambda+i0}(i, i) = \frac{|u_i^+(\lambda)|^2}{2i \operatorname{Im} u_1^+(\lambda)}$. Within \mathring{I}_r , the spectral measure is $d\nu(\lambda) = \frac{1}{\pi N} |\theta'(\lambda)| d\lambda = \frac{1}{N\pi} \sum_{r=0}^{N-1} \operatorname{Im} G^{\lambda+i0}(r, r)$, see [33, eq. (5.3.34), (5.10.14)], with $|\theta'(\lambda)| = |e_j'(\theta_\lambda)| > 0$ for all $\lambda \in \mathring{I}_r$ (see [33, Theorem 5.3.4]). The function e_j is analytic on \mathring{I}_j . All this implies all Green functions $G^{\lambda+i0}(i, i)$ exist on \mathring{I}_r , and at least one of them has a positive imaginary part. In particular, all $u_i^+(\lambda)$ exist on \mathring{I}_r , we have $|\operatorname{Im} u_1^+(\lambda)| > 0$ and $u_0^\pm(\lambda) = 1$ by definition. This shows $\zeta_i^{\lambda+i0}(i+1)$ exists for $i = 0, 1$ and $|\operatorname{Im} \zeta_0^{\lambda+i0}(1)| > 0$. The recursive formula (2.2) here is just $\frac{1}{\zeta_{i-1}^\lambda(i)} = \gamma - W_i - \zeta_i^\gamma(i+1)$. It follows that $|\operatorname{Im} \zeta_1^{\lambda+i0}(2)| > 0$. By induction, if $\zeta_{i-1}^{\lambda+i0}(i)$ exists and $\operatorname{Im} \zeta_{i-1}^{\lambda+i0}(i)$ is nonzero, then in particular $\zeta_{i-1}^{\lambda+i0}(i) \neq 0$, so $\zeta_i^{\lambda+i0}(i+1)$ exists and $|\operatorname{Im} \zeta_i^{\lambda+i0}(i+1)| > 0$ for all i . Using (2.2), this shows all $G^{\lambda+i0}(i, i)$ have a positive imaginary part. Finally, by [33, Theorem 5.4.15], we have $u_i^-(\lambda) = \overline{u_i^+(\lambda)}$ on \mathring{I}_r , so we deduce the same results for $\zeta_{i+1}^{\lambda+i0}(i)$. This completes the proof. \square

We may now proceed with the eigenfunction estimates. Here, the sums $\sum_{(x_2; x_{r+1})}$ in (2.8) reduce to a single path. We take $k = r$ and apply (2.8) for each $r \leq n$ to get

$$\begin{aligned} \psi_\lambda(i) &= \frac{1}{n} \sum_{r=1}^n \left[G^\lambda(i, i+r+1) \psi_\lambda(i+r) - G^\lambda(i, i+r) \psi_\lambda(i+r+1) \right] \\ &\quad + \frac{1}{n} \sum_{r=1}^n \left[G^\lambda(i, i-r) \psi_\lambda(i-r+1) - G^\lambda(i, i-r+1) \psi_\lambda(i-r) \right]. \end{aligned}$$

In fact, here \mathcal{B} is just a shift : $(\mathcal{B}f)(i, i+1) = f(i+1, i+2)$.

We take $n = N$ and use Cauchy-Schwarz :

$$\left| \frac{1}{N} \sum_{r=1}^N G^\lambda(i, i+r+1) \psi_\lambda(i+r) \right|^2 \leq \frac{1}{N} \sum_{r=1}^N |G^\lambda(i, i+r+1)|^2 |\psi_\lambda(i+r)|^2.$$

For $\lambda \notin \sigma(H_{\mathbb{Z}})$, we just bound $|G^\lambda(i, i+r+1)| \leq \frac{1}{\delta_\lambda}$, then the right-hand side is bounded by $\frac{\delta_\lambda^{-2}}{N} \|\psi_\lambda\|_2^2$, proving the claim. For $\lambda \in \mathring{I}_r$, using (2.2), we know $\frac{|\operatorname{Im} \zeta_{i-1}^\lambda(i)|}{|\zeta_{i-1}^\lambda(i)|^2} = |\operatorname{Im} \zeta_i^\lambda(i+1)|$, so by (2.4), we have

$$|G^\lambda(i, i+r+1)|^2 = |G^\lambda(i, i) \zeta_i^\lambda(i+1) \cdots \zeta_{i+r}^\lambda(i+r+1)|^2 = \frac{|G^\lambda(i, i)|^2 \cdot |\operatorname{Im} \zeta_i^\lambda(i+1)|}{|\operatorname{Im} \zeta_{i+r+1}^\lambda(i+r+2)|}.$$

It follows by Theorem 4.1 that $|G^\lambda(i, i+r+1)|^2 \leq C_\lambda$ and thus $\|\psi_\lambda\|_\infty^2 \leq \frac{4C_\lambda}{N} \|\psi_\lambda\|_2^2$. Note that for $\lambda \in \mathring{I}_r$, we are assuming the potential is m -periodic, so all Green functions are those of m -periodic Schrödinger operators, and the constant C_λ may only depend on m (but not on N).

5. THE PROOF FOR P-NORMS

The aim of this section is to prove Theorem 1.6.

5.1. A TT* argument. Recall the operator

$$\mathcal{M}_{n,\lambda} = \frac{1}{n} \sum_{r=1}^n \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^r |\operatorname{Im} \zeta^\lambda|^{1/2}.$$

For λ in the bulk spectrum, we deduce from (3.1) that

$$(5.1) \quad \|\psi_\lambda\|_p \leq \left\| \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} \mathcal{M}_{n,\lambda} \frac{f_\lambda}{|\operatorname{Im} \zeta^\lambda|^{1/2}} \right\|_p + \left\| \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} \overline{\mathcal{M}_{n,\lambda}} \frac{g_\lambda}{|\operatorname{Im} \zeta^\lambda|^{1/2}} \right\|_p,$$

where $\zeta^\lambda(x_0, x_1) := \zeta_{x_0}^\lambda(x_1)$.

We study the first term; the other one is similar. Recalling (1.1), we have

$$(5.2) \quad \left\| \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} \mathcal{M}_{n,\lambda} \frac{f_\lambda}{|\operatorname{Im} \zeta^\lambda|^{1/2}} \right\|_p \leq z_\lambda^{-1} \|\mathcal{M}_{n,\lambda}\|_{2 \rightarrow p} \cdot \|f_\lambda\|_2.$$

On the other hand, we can use a so-called TT^* argument, namely

$$(5.3) \quad \|\mathcal{M}_{n,\lambda}\|_{2 \rightarrow p}^2 = \|\mathcal{M}_{n,\lambda}^*\|_{p' \rightarrow 2}^2 = \|\mathcal{M}_{n,\lambda} \mathcal{M}_{n,\lambda}^*\|_{p' \rightarrow p}.$$

So it suffices to estimate this operator norm to obtain a bound on the p -norm.

We start by calculating

$$(5.4) \quad \mathcal{M}_{n,\lambda} \mathcal{M}_{n,\lambda}^* = \frac{1}{n^2} \sum_{r,r'=1}^n \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^r |\operatorname{Im} \zeta^\lambda| (\mathcal{B}^* \overline{\zeta^\lambda})^{r'} \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}}.$$

We have

$$\left[(\zeta^\lambda \mathcal{B})^r f \right] (x_0, x_1) = \sum_{(x_2; x_{r+1})} \zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{r-1}}^\lambda(x_r) f(x_r, x_{r+1}),$$

where the sum runs over all paths $(x_2; x_{r+1})$ such that $x_2 \in \mathcal{N}_{x_1} \setminus \{x_0\}$. Similarly,

$$\left[(\mathcal{B}^* \overline{\zeta^\lambda})^{r'} g \right] (x_r, x_{r+1}) = \sum_{(y_0; y_{r-1})} \overline{\zeta_{y_0}^\lambda(y_1)} \cdots \overline{\zeta_{y_{r-1}}^\lambda(x_r)} g(y_0, y_1),$$

where the sum runs over all paths $(y_0; y_{r-1})$ such that $y_{r-1} \in \mathcal{N}_{x_r} \setminus \{x_{r+1}\}$. Hence,

$$\begin{aligned} & \left[\frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^r |\operatorname{Im} \zeta^\lambda| (\mathcal{B}^* \overline{\zeta^\lambda})^{r'} \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} f \right] (x_0, x_1) \\ &= \sum_{(x_2; x_{r+1})} \sum_{(y_0; y_{r-1})} \frac{\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{r-1}}^\lambda(x_r) |\operatorname{Im} \zeta_{x_r}^\lambda(x_{r+1})| \overline{\zeta_{y_0}^\lambda(y_1)} \cdots \overline{\zeta_{y_{r-1}}^\lambda(x_r)}}{|\operatorname{Im} \zeta_{x_0}^\lambda(x_1)|^{1/2} |\operatorname{Im} \zeta_{y_0}^\lambda(y_1)|^{1/2}} f(y_0, y_1). \end{aligned}$$

This double sum has a complicated combinatorics : we first sum on r -paths which are rightwards from (x_0, x_1) , then for each edge in this r -sphere, we sum on r -paths which are leftwards. Our aim now is to simplify this expression.

Fix $(x_2; x_{r+1})$. We write $\{(y_0; y_{r-1}) : y_{r-1} \in \mathcal{N}_{x_r} \setminus \{x_{r+1}\}\}$ as

$$\begin{aligned} & \{(x_0; x_{r-1})\} \cup \{(y_0, x_1, \dots, x_{r-1}) : y_0 \in \mathcal{N}_{x_1} \setminus \{x_2\}, y_0 \neq x_0\} \\ & \quad \cup \cdots \cup \{(y_0; y_{r-1}) : y_{r-1} \in \mathcal{N}_{x_r} \setminus \{x_{r+1}\}, y_{r-1} \neq x_{r-1}\}. \end{aligned}$$

For $(y_0; y_{r-1}) = (x_0; x_{r-1})$, we get the term

$$\left[\sum_{(x_2; x_{r+1})} \frac{|\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{r-1}}^\lambda(x_r)|^2 |\operatorname{Im} \zeta_{x_r}^\lambda(x_{r+1})|}{|\operatorname{Im} \zeta_{x_0}^\lambda(x_1)|} \right] \cdot f(x_0, x_1) = f(x_0, x_1).$$

For the terms $(y_0, \dots, y_j, x_{j+1}, \dots, x_{r-1})$, the restriction $y_j \in \mathcal{N}_{x_{j+1}} \setminus \{x_{j+2}\}$, $y_j \neq x_j$ is equivalent to $x_{j+2} \in \mathcal{N}_{x_{j+1}} \setminus \{x_j, y_j\}$, yielding for each $j = 0, \dots, r-1$ the term

$$\begin{aligned} & \sum_{(x_2; x_{j+1})} \sum_{(y_0; y_j)'} \left[\sum_{(x_{j+2}; x_{r+1}), x_{j+2} \in \mathcal{N}_{x_{j+1}} \setminus \{x_j, y_j\}} \frac{\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_j}^\lambda(x_{j+1}) \cdot \overline{\zeta_{y_0}^\lambda(y_1) \cdots \zeta_{y_j}^\lambda(y_{j+1})}}{|\operatorname{Im} \zeta_{x_0}^\lambda(x_1)|^{1/2} |\operatorname{Im} \zeta_{y_0}^\lambda(y_1)|^{1/2}} \right. \\ & \quad \left. \cdot |\zeta_{x_{j+1}}^\lambda(x_{j+2}) \cdots \zeta_{x_{r-1}}^\lambda(x_r)|^2 |\operatorname{Im} \zeta_{x_r}^\lambda(x_{r+1})| \right] \cdot f(y_0, y_1) \\ & = \sum_{(x_2; x_{j+1})} \sum_{(y_0; y_j)'} \frac{\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_j}^\lambda(x_{j+1}) \cdot \overline{\zeta_{y_0}^\lambda(y_1) \cdots \zeta_{y_j}^\lambda(y_{j+1})}}{|\operatorname{Im} \zeta_{x_0}^\lambda(x_1) \operatorname{Im} \zeta_{y_0}^\lambda(y_1)|^{1/2}} \\ & \quad \cdot \left(\frac{|\operatorname{Im} \zeta_{x_j}^\lambda(x_{j+1})|}{|\zeta_{x_j}^\lambda(x_{j+1})|^2} - |\operatorname{Im} \zeta_{x_{j+1}}^\lambda(y_j)| \right) f(y_0, y_1), \end{aligned}$$

where $\sum_{(y_0; y_j)'}$ sums over all paths $(y_0; y_j)$ with $y_j \in \mathcal{N}_{x_{j+1}} \setminus \{x_j\}$.

In the last expression, $\sum_{(x_2; x_{j+1})} \sum_{(y_0; y_j)'}$ can be rewritten as $\sum_{(x_2; x_{2j+2})}$, with y_j relabeled as x_{j+2} , y_{j-1} as x_{j+3} and so on. This gives

$$\begin{aligned} & \sum_{(x_2; x_{2j+2})} \frac{\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_j}^\lambda(x_{j+1}) \cdot \overline{\zeta_{x_{2j+2}}^\lambda(x_{2j+1}) \cdots \zeta_{x_{j+2}}^\lambda(x_{j+1})}}{|\operatorname{Im} \zeta_{x_0}^\lambda(x_1) \operatorname{Im} \zeta_{x_{2j+2}}^\lambda(x_{2j+1})|^{1/2}} \\ & \quad \cdot \left(\frac{|\operatorname{Im} \zeta_{x_j}^\lambda(x_{j+1})|}{|\zeta_{x_j}^\lambda(x_{j+1})|^2} - |\operatorname{Im} \zeta_{x_{j+1}}^\lambda(x_{j+2})| \right) f(x_{2j+2}, x_{2j+1}). \end{aligned}$$

Now note that $[\frac{1}{|\operatorname{Im} \zeta|^{1/2}} (\zeta \mathcal{B})^j \frac{|\operatorname{Im} \zeta|}{\zeta} g](x_0, x_1) = \sum_{(x_2; x_{j+1})} \frac{\zeta_{x_0}(x_1) \cdots \zeta_{x_j}(x_{j+1})}{|\operatorname{Im} \zeta_{x_0}(x_1)|^{1/2}} \frac{|\operatorname{Im} \zeta_{x_j}(x_{j+1})|}{|\zeta_{x_j}(x_{j+1})|^2} g(x_j, x_{j+1})$. So the above is, in operator form,

$$\begin{aligned} & \left(\frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^j \frac{|\operatorname{Im} \zeta^\lambda|}{\zeta^\lambda} (\mathcal{B} \iota \overline{\zeta^\lambda})^{j+1} \frac{1}{|\operatorname{Im} \iota \zeta^\lambda|^{1/2}} \iota f \right) (x_0, x_1) \\ & \quad - \left(\frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^j \zeta^\lambda (\mathcal{B} |\operatorname{Im} \zeta^\lambda| \iota \overline{\zeta^\lambda}) (\mathcal{B} \iota \overline{\zeta^\lambda})^j \frac{1}{|\operatorname{Im} \iota \zeta^\lambda|^{1/2}} \iota f \right) (x_0, x_1), \end{aligned}$$

where ι is edge-reversal. In other words, if

$$(5.5) \quad \mathcal{B}_{\zeta^\lambda} = \frac{|\operatorname{Im} \zeta^\lambda|}{\zeta^\lambda} \mathcal{B} \iota \overline{\zeta^\lambda} - \zeta^\lambda (\mathcal{B} |\operatorname{Im} \zeta^\lambda| \iota \overline{\zeta^\lambda}),$$

then

$$\begin{aligned} & \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^r |\operatorname{Im} \zeta^\lambda| (\mathcal{B}^* \overline{\zeta^\lambda})^r \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} \\ & \quad = I + \sum_{j=0}^{r-1} \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^j \mathcal{B}_{\zeta^\lambda} (\mathcal{B} \iota \overline{\zeta^\lambda})^j \frac{1}{|\operatorname{Im} \iota \zeta^\lambda|^{1/2}} \iota. \end{aligned}$$

It follows that for $r \geq r'$,

$$\begin{aligned} (5.6) \quad & \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^r |\operatorname{Im} \zeta^\lambda| (\mathcal{B}^* \overline{\zeta^\lambda})^{r'} \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} \\ & \quad = \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^{r-r'} |\operatorname{Im} \zeta^\lambda|^{1/2} \\ & \quad \quad + \sum_{j=0}^{r'-1} \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^{j+r-r'} \mathcal{B}_{\zeta^\lambda} (\mathcal{B} \iota \overline{\zeta^\lambda})^j \frac{1}{|\operatorname{Im} \iota \zeta^\lambda|^{1/2}} \iota \end{aligned}$$

and for $r < r'$,

$$\begin{aligned}
(5.7) \quad & \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^r |\operatorname{Im} \zeta^\lambda| (\mathcal{B}^* \bar{\zeta}^\lambda)^{r'} \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} \\
& = |\operatorname{Im} \zeta^\lambda|^{1/2} (\mathcal{B}^* \bar{\zeta}^\lambda)^{r'-r} \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} \\
& \quad + \sum_{j=0}^{r-1} \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^j \mathcal{B}_{\zeta^\lambda} (\mathcal{B} \iota \bar{\zeta}^\lambda)^{j+r'-r} \frac{1}{|\operatorname{Im} \iota \zeta^\lambda|^{1/2}} \iota.
\end{aligned}$$

In the last sum, we used that $\mathcal{B}^* = \iota \mathcal{B} \iota$ and $\iota^2 = I$.

5.2. Some Green function estimates. Let (G, W) be a finite graph of minimal degree ≥ 3 and consider a path $(x_0; x_{r+1})$ in G . Assume λ is in the bulk spectrum of (G, W) . We know from (3.3) that for any $j \geq 1$,

$$\sum_{x_{j+1} \in \mathcal{N}_{x_j} \setminus \{x_{j-1}\}} \frac{|\zeta_{x_{j-1}}^\lambda(x_j)|^2 |\operatorname{Im} \zeta_{x_j}^\lambda(x_{j+1})|}{|\operatorname{Im} \zeta_{x_{j-1}}^\lambda(x_j)|} = 1.$$

Moreover, each term in the sum is strictly positive. If $d(x_j) \geq 3$, this implies that each term (there are at least two) is strictly smaller than 1. It follows that for any $s > 1$,

$$\sum_{x_{j+1} \in \mathcal{N}_{x_j} \setminus \{x_{j-1}\}} \frac{|\zeta_{x_{j-1}}^\lambda(x_j)|^{2s} |\operatorname{Im} \zeta_{x_j}^\lambda(x_{j+1})|^s}{|\operatorname{Im} \zeta_{x_{j-1}}^\lambda(x_j)|^s} < 1.$$

Taking the maximum of the left-hand side over the finite set of oriented edges of G , we obtain $Z_{s,\lambda}(G) < 1$, by definition (1.2). Hence,

$$\begin{aligned}
& \sum_{(x_2; x_{r+1})} \frac{|\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{r-1}}^\lambda(x_r)|^{2s} |\operatorname{Im} \zeta_{x_r}^\lambda(x_{r+1})|^s}{|\operatorname{Im} \zeta_{x_0}^\lambda(x_1)|^s} \\
& \leq Z_{s,\lambda} \sum_{(x_2; x_r)} \frac{|\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{r-2}}^\lambda(x_{r-1})|^{2s} |\operatorname{Im} \zeta_{x_{r-1}}^\lambda(x_r)|^s}{|\operatorname{Im} \zeta_{x_0}^\lambda(x_1)|^s} \leq Z_{s,\lambda}^r.
\end{aligned}$$

On the other hand, $\frac{|\operatorname{Im} \zeta_{x_r}^\lambda(x_{r+1})|^s}{|\operatorname{Im} \zeta_{x_0}^\lambda(x_1)|^s} \geq z_\lambda^{2s}$ by (2.6). We thus get for any $s > 1$ and $(x_0; x_{r+1})$,

$$(5.8) \quad \sum_{(x_2; x_{r+1})} |\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{r-1}}^\lambda(x_r)|^{2s} \leq z_\lambda^{-2s} Z_{s,\lambda}^r.$$

Note that in the case of $(q+1)$ -regular graphs with $W \equiv 0$, we have more precisely $\sum_{(x_2; x_{r+1})} |\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{r-1}}^\lambda(x_r)|^{2s} = q^r q^{-rs} = q^{-(s-1)r}$.

It follows in particular that

$$(5.9) \quad |\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{r-1}}^\lambda(x_r)|^2 \leq z_\lambda^{-2} (Z_{s,\lambda}^{1/s})^r$$

decays exponentially in r .

We also have the following variant of (5.8) :

$$(5.10) \quad \sum_{(x_2; x_{r+1})} |\zeta_{x_1}^\lambda(x_0) \cdots \zeta_{x_r}^\lambda(x_{r-1})|^{2s} \leq z_\lambda^{-6s} Z_{s,\lambda}^r.$$

In fact, $|\zeta_{x_1}^\lambda(x_0) \cdots \zeta_{x_r}^\lambda(x_{r-1})|^{2s} = \left| \frac{G^\lambda(\tilde{x}_0, \tilde{x}_0)}{G^\lambda(\tilde{x}_r, \tilde{x}_r)} \right|^{2s} |\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{r-1}}^\lambda(x_r)|^{2s}$ by (2.3) and we know $\left| \frac{G^\lambda(v,v)}{G^\lambda(w,w)} \right| \leq z_\lambda^{-2}$ by (2.7).

5.3. Proof for p -norms. Now recall Young's inequality : if K is an operator on $\ell^2(B)$, $\frac{1}{q} = \frac{1}{s} + \frac{1}{r} - 1$, $\|K(\cdot, b)\|_r \leq A$ and $\|K(b, \cdot)\|_r \leq A$ for all $b \in B$, then $\|Kf\|_q \leq A\|f\|_s$. We use this with $q = p$ and $s = p'$, so that $r = \frac{p}{2}$. Our operator is $K = \mathcal{M}_{n,\lambda} \mathcal{M}_{n,\lambda}^*$.

Recall that $\mathcal{M}_{n,\lambda} = \frac{1}{n} \sum_{r=1}^n \mathcal{B}_{r,\lambda}$, where $\mathcal{B}_{m,\lambda} = \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^m |\operatorname{Im} \zeta^\lambda|^{1/2}$, so that $\mathcal{M}_{n,\lambda} \mathcal{M}_{n,\lambda}^* = \frac{1}{n^2} \sum_{r,r'=1}^n \mathcal{B}_{r,\lambda} \mathcal{B}_{r',\lambda}^*$. We focus on $\sum_{r=1}^n \sum_{r'=1}^r$, the rest $\sum_{r=1}^n \sum_{r'=r+1}^n$ is bounded similarly.

Using (5.6), if $\mathcal{B}_{m,j,\lambda} = \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^m \mathcal{B}_{\zeta^\lambda} (\mathcal{B} \overline{\zeta^\lambda})^j \frac{1}{|\operatorname{Im} \zeta^\lambda|^{1/2}}$, then for $r \geq r'$,

$$\mathcal{B}_{r,\lambda} \mathcal{B}_{r',\lambda}^* = \mathcal{B}_{r-r',\lambda} + \sum_{j=0}^{r'-1} \mathcal{B}_{r-r',j,\lambda},$$

so for any fixed $b = (x_0, x_1)$,

$$\begin{aligned} & \left\| \frac{1}{n^2} \sum_{r=1}^n \sum_{r'=1}^r \mathcal{B}_{r,\lambda} \mathcal{B}_{r',\lambda}^*(b, \cdot) \right\|_{p/2} \\ & \leq \frac{1}{n^2} \sum_{r=1}^n \sum_{r'=1}^r \|\mathcal{B}_{r-r',\lambda}(b, \cdot)\|_{p/2} + \frac{1}{n^2} \sum_{r=1}^n \sum_{r'=1}^r \sum_{j=0}^{r'-1} \|\mathcal{B}_{r-r',j,\lambda}(b, \cdot)\|_{p/2}. \end{aligned}$$

To calculate these norms, we need to study the kernels of the operators. If $m \leq \rho_G$, we have $\mathcal{B}_{m,\lambda}(b, b') = \frac{1}{|\operatorname{Im} \zeta_{x_0}^\lambda(x_1)|^{1/2}} \zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{m-1}}^\lambda(x_m) |\operatorname{Im} \zeta_{x_m}^\lambda(x_{m+1})|^{1/2}$ if $b' \in \mathcal{B}^m b$, otherwise $\mathcal{B}_{m,\lambda}(b, b') = 0$. Here $(x_2; x_{m+1})$ is the path from (x_0, x_1) with $b' = (x_m, x_{m+1})$. On the other hand, several paths outgoing from b can in principle reach $b' \in \mathcal{B}^m b$, if $m > \rho_G$. As long as $m \leq \ell_G$ however, there is at most one additional path of the same length $(x'_2; x'_{m+1})$ that can reach b' , see Remark 3.1. Hence, if $p > 4$ and $m \leq \ell_G$,

$$\begin{aligned} \|\mathcal{B}_{m,\lambda}(b, \cdot)\|_{p/2} & = \left(\sum_{b' \in \mathcal{B}^m b} |\mathcal{B}_{m,\lambda}(b, b')|^{p/2} \right)^{2/p} \\ & \leq \left(2 \sum_{(x_2; x_{m+1})} \frac{|\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{m-1}}^\lambda(x_m)|^{p/2} |\operatorname{Im} \zeta_{x_m}^\lambda(x_{m+1})|^{p/4}}{|\operatorname{Im} \zeta_{x_0}^\lambda(x_1)|^{p/4}} \right)^{2/p} \leq 2^{2/p} (Z_{p/4,\lambda}^{2/p})^m. \end{aligned}$$

Similarly, we have $\mathcal{B}_{m,j,\lambda}(b, b') = \frac{\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{j+m}}^\lambda(x_{j+m+1}) \cdot \overline{\zeta_{x_{2j+m+2}}^\lambda(x_{2j+m+1}) \cdots \zeta_{x_{j+m+2}}^\lambda(x_{j+m+1})}}{|\operatorname{Im} \zeta_{x_0}^\lambda(x_1) \operatorname{Im} \zeta_{x_{2j+m+1}}^\lambda(x_{2j+m})|^{1/2}}$. $(\frac{|\operatorname{Im} \zeta_{x_{j+m}}^\lambda(x_{j+m+1})|}{|\zeta_{x_{j+m}}^\lambda(x_{j+m+1})|^2} - |\operatorname{Im} \zeta_{x_{j+m+1}}^\lambda(x_{j+m+2})|)$ if $b' \in \mathcal{B}^{2j+m+1}$ and zero otherwise, if $m+2j+1 \leq \rho_G$. We bound the term in parentheses as well as the denominator by $2z_\lambda^{-2}$ using (2.7). This gives

$$\begin{aligned} \|\mathcal{B}_{m,j,\lambda}(b, \cdot)\|_{p/2} & \leq 2z_\lambda^{-2} \left(2 \sum_{(x_2; x_{2j+m+2})} |\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{j+m}}^\lambda(x_{j+m+1}) \right. \\ & \quad \left. \cdot \zeta_{x_{j+m+2}}^\lambda(x_{j+m+1}) \cdots \zeta_{x_{2j+m+1}}^\lambda(x_{2j+m}) \right)^{2/p} \\ & \leq 2^{2/p+1} z_\lambda^{-2} z_\lambda^{-4} (Z_{p/4,\lambda}^{2/p})^j (Z_{p/4,\lambda}^{2/p})^{j+m+1} \leq 4z_\lambda^{-6} (Z_{p/4,\lambda}^{2/p})^{2j+m+1}, \end{aligned}$$

where we first used (5.10) then (5.8). Denoting $\alpha_{p,\lambda} = Z_{p/4,\lambda}^{2/p} < 1$, we get

$$\left\| \frac{1}{n^2} \sum_{r=1}^n \sum_{r'=1}^r \mathcal{B}_{r,\lambda} \mathcal{B}_{r',\lambda}^*(b, \cdot) \right\|_{p/2} \leq \frac{2^{2/p}}{n^2} \sum_{r=1}^n \sum_{r'=1}^r \alpha_{p,\lambda}^{r-r'} + \frac{4z_\lambda^{-6}}{n^2} \sum_{r=1}^n \sum_{r'=1}^r \sum_{j=0}^{r'-1} \alpha_{p,\lambda}^{2j+r-r'+1} \leq \frac{C_{p,\lambda}}{n},$$

since $\sum_{r'=1}^r \alpha^{-r'} = \alpha^{-1} \frac{\alpha^{-r}-1}{\alpha^{-1}-1} \leq \frac{\alpha^{-r}}{1-\alpha}$ and $\sum_{j=0}^{r'-1} \alpha^{2j} \leq \frac{1}{1-\alpha^2}$. To take an explicit $C_{p,\lambda}$, the above is bounded more precisely by $\frac{1}{n} \frac{1}{1-\alpha} (2^{2/p} + \frac{4z_\lambda^{-6}}{1-\alpha^2})$. Now $2^{2/p} \leq 2 \leq \frac{2z_\lambda^{-6}}{1-\alpha^2}$ since $z_\lambda \leq 1$. Also, $\frac{1}{1-\alpha^2} \leq \frac{1}{1-\alpha}$. So we may take $C_{p,\lambda} = \frac{6z_\lambda^{-6}}{(1-\alpha)^2}$.

Similarly, $\|\frac{1}{n^2} \sum_{r=1}^n \sum_{r'=r+1}^n \mathcal{B}_{r,\lambda} \mathcal{B}_{r',\lambda}^*(b, \cdot)\|_{p/2} \leq \frac{C_{p,\lambda}}{n}$. Finally, as $\mathcal{M}_{n,\lambda} \mathcal{M}_{n,\lambda}^*$ is self-adjoint, we deduce from Young's inequality that its $p' \rightarrow p$ norm is bounded by $\frac{2C_{p,\lambda}}{n}$.

Recalling (5.1)–(5.3), since $\|f_\lambda\|_2 \leq D(1 + z_\lambda^{-1})\|\psi_\lambda\|_2$, then arguing similarly for the term containing g_λ , it follows that $\|\psi_\lambda\|_p \leq \frac{2\sqrt{2C_{p,\lambda}(D(1+z_\lambda^{-1}))}}{\sqrt{n}}\|\psi_\lambda\|_2$. As this holds for any $n \leq \ell_G$, the proof is complete.

Remark 5.1. The previous proof continues to hold for $n > \ell_G$, as long as for any $\beta > 0$, there are at most $2^{\beta k}$ paths of length k between two edges, for $k \leq n$. In this case, in the previous argument we obtain $(2^\beta Z_{p/4,\lambda}^{2/p})^k$ instead of $(Z_{p/4,\lambda}^{2/p})^k$, and we choose β small enough so that $2^\beta Z_{p/4,\lambda}^{2/p} < 1$.

5.4. Support of eigenfunctions. We first estimate the support of non-backtracking eigenfunctions f_λ from (2.14). Let S be the support of f_λ .

Recall that we have $(\zeta^\lambda \mathcal{B})^r f_\lambda = f_\lambda$. Now

$$\|f_\lambda\|_2^2 = \langle f_\lambda \chi_S, f_\lambda \rangle = \langle f_\lambda \chi_S, (\zeta^\lambda \mathcal{B})^r f_\lambda \rangle \leq \|f_\lambda \chi_S\|_1 \cdot \|(\zeta^\lambda \mathcal{B})^r f_\lambda\|_\infty,$$

and assuming $r \leq \ell_G$,

$$\begin{aligned} |[(\zeta^\lambda \mathcal{B})^r f_\lambda](x_0, x_1)| &= \left| \sum_{(x_2; x_{r+1})} \zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{r-1}}^\lambda(x_r) f_\lambda(x_r, x_{r+1}) \right| \\ &\leq \sup_{(x_0; x_{r+1})} |\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{r-1}}^\lambda(x_r)| \sum_{(x_2; x_{r+1})} |f_\lambda(x_r, x_{r+1})| \\ &\leq 2z_\lambda^{-1} (Z_{s,\lambda}^{1/2s})^r \|f_\lambda\|_1 = 2z_\lambda^{-1} (Z_{s,\lambda}^{1/2s})^r \|f_\lambda \chi_S\|_1, \end{aligned}$$

where we used (5.9) and the fact that there are at most two paths of length r between two edges if $r \leq \ell_G$. Thus,

$$\|f_\lambda\|_2^2 \leq 2z_\lambda^{-1} (Z_{s,\lambda}^{1/2s})^r \|f_\lambda \chi_S\|_1^2 \leq 2z_\lambda^{-1} (Z_{s,\lambda}^{1/2s})^r \cdot |S| \cdot \|f_\lambda\|_2^2,$$

using the Cauchy-Schwarz inequality.

It follows that³

$$|S| \geq \frac{1}{2} z_\lambda (Z_{s,\lambda}^{-1/2s})^r,$$

with $Z_{s,\lambda}^{-1/2s} > 1$.

To go back to the support of ψ_λ , recall that $f_\lambda(x_0, x_1) = \psi_\lambda(x_1) - \zeta_{x_0}^\lambda(x_1)\psi_\lambda(x_0)$. If $f_\lambda(x_0, x_1) \neq 0$, then either $\psi_\lambda(x_0) \neq 0$ or $\psi_\lambda(x_1) \neq 0$. Hence, if $\Lambda = \text{supp } \psi_\lambda$ and

$$\tilde{S} = \{(x, y), (y, x) : x \in \Lambda, y \sim x\},$$

we have $S \subseteq \tilde{S}$. In particular, $|S| \leq |\tilde{S}| \leq 2D \cdot |\Lambda|$, so

$$|\Lambda| \geq \frac{z_\lambda}{4D} M_\lambda^r$$

with $M_\lambda = Z_{2,\lambda}^{-1/4} > 1$.

To obtain the sharper lower bound (without z_λ), just replace the function f_λ in the previous argument by $\frac{f_\lambda}{|\text{Im } \zeta^\lambda|^{1/2}}$, and the operator $(\zeta^\lambda \mathcal{B})^r$ by $\mathcal{B}_{r,\lambda} = \frac{1}{|\text{Im } \zeta^\lambda|^{1/2}} (\zeta^\lambda \mathcal{B})^r |\text{Im } \zeta^\lambda|^{1/2}$.

³in case of $(q+1)$ -regular graphs, $W \equiv 0$, this argument yields more precisely $|S| \geq \frac{1}{2} q^{r/2}$.

In this case, we get to estimate $\sup_{(x_0; x_{r+1})} \frac{|\zeta_{x_0}^\lambda(x_1) \cdots \zeta_{x_{r-1}}^\lambda(x_r)| \cdot |\operatorname{Im} \zeta_{x_r}^\lambda(x_{r+1})|^{1/2}}{|\operatorname{Im} \zeta_{x_0}^\lambda(x_1)|^{1/2}}$, and from the argument preceding (5.9), we see this is bounded by $Z_{s,\lambda}^{r/2s}$ (instead of $z_\lambda^{-1} Z_{s,\lambda}^{r/2s}$ above).

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