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► **To cite this version:**

Hussein Cheikh Ali. Hardy-Sobolev inequalities with singularities on non smooth boundary: Hardy constant and extremals. Part I: Influence of local geometry. *Nonlinear Analysis: Theory, Methods and Applications*, 2019, 10.1016/j.na.2018.12.016 . hal-01875440v2

HAL Id: hal-01875440

<https://hal.science/hal-01875440v2>

Submitted on 17 Sep 2020

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HARDY-SOBOLEV INEQUALITIES WITH SINGULARITIES ON NON SMOOTH BOUNDARY: HARDY CONSTANT AND EXTREMALS. PART I: INFLUENCE OF LOCAL GEOMETRY

HUSSEIN CHEIKH ALI

ABSTRACT. Let Ω be a domain of \mathbb{R}^n , $n \geq 3$. The classical Caffarelli-Kohn-Nirenberg inequality rewrites as the following inequality: for any $s \in [0, 2]$ and any $\gamma < \frac{(n-2)^2}{4}$, there exists a constant $K(\Omega, \gamma, s) > 0$ such that

$$\left(\int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq K(\Omega, \gamma, s) \int_{\Omega} \left(|\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx, \quad (HS)$$

for all $u \in D^{1,2}(\Omega)$ (the completion of $C_c^\infty(\Omega)$ for the relevant norm). When $0 \in \Omega$ is an interior point, the range $(-\infty, \frac{(n-2)^2}{4})$ for γ cannot be improved: moreover, the optimal constant $K(\Omega, \gamma, s)$ is independent of Ω and there is no extremal for (HS). But when $0 \in \partial\Omega$, the situation turns out to be drastically different since the geometry of the domain impacts :

- the range of γ 's for which (HS) holds;
- the value of the optimal constant $K(\Omega, \gamma, s)$;
- the existence of extremals for (HS).

When Ω is smooth, the problem was tackled by Ghoussoub-Robert [16] where the role of the mean curvature was central. In the present paper, we consider nonsmooth domain with a singularity at 0 modeled on a cone. We show how the local geometry induced by the cone around the singularity influences the value of the Hardy constant on Ω . When γ is small, we introduce a new geometric object at the conical singularity that generalizes the "mean curvature": this allows to get extremals for (HS). The case of larger values for γ will be dealt in the forthcoming paper [5]. As an intermediate result, we prove the symmetry of some solutions to singular pdes that has an interest on its own.

Pour ma petite fille Aline

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1. INTRODUCTION

Let Ω be a domain of \mathbb{R}^n , $n \geq 3$, $s \in [0, 2)$ and let us consider the following problem:

$$(1) \quad \begin{cases} -\Delta u - \frac{\gamma}{|x|^2} u = \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\gamma \in \mathbb{R}$, $2^*(s) := \frac{2(n-s)}{n-2}$ is the critical Hardy-Sobolev exponent and Δ is the Euclidean Laplacian that is $\Delta = \operatorname{div}(\nabla)$. This equation makes sense for $u \in D^{1,2}(\Omega)$, that is the completion of $C_c^\infty(\Omega)$ with respect to the norm $u \mapsto \|\nabla u\|_2$. The motivation for considering equation (1) arises from the problem of existence of extremals for the Caffarelli-Kohn-Nirenberg (CKN) inequalities [1]. The Caffarelli-Kohn-Nirenberg inequalities are equivalent to the Hardy-Sobolev inequality (see [16]): for any $\gamma < \frac{(n-2)^2}{4}$ and $s \in [0, 2]$, there exists $K > 0$ such that

$$(2) \quad \left(\int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq K \int_{\Omega} \left(|\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx,$$

for all $u \in D^{1,2}(\Omega)$. More generally, for any $0 \leq s \leq 2$ and any $\gamma \in \mathbb{R}$, we define

$$J_{\gamma,s}^{\Omega}(u) := \frac{\int_{\Omega} |\nabla u|^2 dx - \gamma \int_{\Omega} \frac{u^2}{|x|^2} dx}{\left(\int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}},$$

for $u \in D^{1,2}(\Omega) \setminus \{0\}$, and we define

$$(3) \quad \mu_{\gamma,s}(\Omega) = \inf_{u \in D^{1,2}(\Omega) \setminus \{0\}} J_{\gamma,s}^{\Omega}(u).$$

If $u \in D^{1,2}(\Omega) \setminus \{0\}$ achieves the infimum $\mu_{\gamma,s}(\Omega)$, and if $\mu_{\gamma,s}(\Omega) > 0$, then, up to a constant, u is a solution to (1). We address the following questions:

- (Q1) For which values of $\gamma \in \mathbb{R}$ does (2) hold for some $K > 0$ and all $u \in D^{1,2}(\Omega)$? In other words, when do we have $\mu_{\gamma,s}(\Omega) > 0$?
- (Q2) Is the best constant achieved? In other words, is $\mu_{\gamma,s}(\Omega)$ achieved by some $u \in D^{1,2}(\Omega)$, $u \neq 0$?

The answer to the first question (Q1) depends on the Hardy constant. Define

$$(4) \quad \gamma_H(\Omega) := \mu_{0,2}(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} \frac{u^2}{|x|^2} dx}; u \in C_c^\infty(\Omega) \setminus \{0\} \right\}.$$

The classical Hardy inequality reads $\gamma_H(\mathbb{R}^n) = \frac{(n-2)^2}{4}$ and therefore, we have that $\gamma_H(\Omega) \geq \frac{(n-2)^2}{4}$. As a consequence, interpolating the Hardy inequality (4) and Sobolev inequality ((2) with $\gamma = s = 0$), we get that

$$\gamma < \gamma_H(\Omega) \Rightarrow \mu_{\gamma,s}(\Omega) > 0.$$

When $0 \in \Omega$ is an interior point, it is classical that $\gamma_H(\Omega) = \gamma_H(\mathbb{R}^n) = \frac{(n-2)^2}{4}$. We consider the case $0 \in \partial\Omega$. The study of this type of nonlinear singular problems when $0 \in \partial\Omega$ was initiated by Ghoussoub-Kang [12] and studied by Chern-Lin [6] and Ghoussoub-Robert [16] when Ω is a smooth domain. As a byproduct, we prove the existence of solutions to a perturbation of the initial equation via the Mountain Pass Lemma.

In this work, we tackle the more intricate case of a non smooth domain. We restrict ourselves to domains modeled locally on $\mathbb{R}_+^k \times \mathbb{R}^{n-k}$ for all $k \in \{1, \dots, n\}$. We define the *model cone* at $P \in \bar{\Omega}$ as

$$C_P(\Omega) := \left\{ \lim_{t \rightarrow 0} \frac{1}{t} \overrightarrow{PM_t} / t \mapsto M_t \text{ is a curve of } \Omega \text{ and the limit exists} \right\}.$$

When Ω is smooth, $C_{x_0}(\Omega) = \mathbb{R}^n$ if $x_0 \in \Omega$. Still in the smooth case, $C_{x_0}(\Omega)$ is a half-space bounded by the tangent space at x_0 if $x_0 \in \partial\Omega$. Moreover, when $x_0 \in \partial\Omega$, then $\partial C_{x_0}(\Omega)$ is exactly the tangent space at x_0 .

Definition 1. We fix $1 \leq k \leq n$. Let Ω be a domain of \mathbb{R}^n . We say that $x_0 \in \partial\Omega$ is a singularity of type $(k, n-k)$ if there exist U, V open subsets of \mathbb{R}^n such that $0 \in U$, $0 \in V$ and there exists $\phi \in C^\infty(U, V)$ a diffeomorphism such that $\phi(0) = x_0$ and

$$\phi(U \cap (\mathbb{R}_+^k \times \mathbb{R}^{n-k})) = \phi(U) \cap \Omega \text{ and } \phi(U \cap \partial(\mathbb{R}_+^k \times \mathbb{R}^{n-k})) = \phi(U) \cap \partial\Omega,$$

with the additional hypothesis that the differential at 0 $d\phi_0$ is an isometry.

As one checks, we have that $C_0(\Omega) = d\phi_0(\mathbb{R}_+^k \times \mathbb{R}^{n-k})$, and then $C_0(\Omega)$ is isometric to $\mathbb{R}_+^k \times \mathbb{R}^{n-k}$. In the sequel, we write for convenience

$$\mathbb{R}^{k+,n-k} := \mathbb{R}_+^k \times \mathbb{R}^{n-k} \text{ for all } k \in \{1, \dots, n\}.$$

For example: When Ω is smooth, boundary points are all of type $(1, n-1)$. A general conical singularity is as in

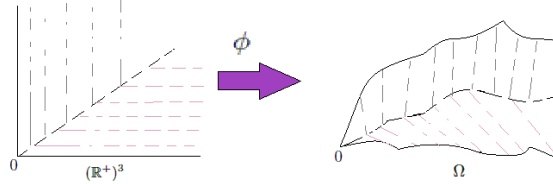


FIGURE 1. Case: $k = 3, n - k = 0$.

Figure 1. We assume that 0 is a singularity of type $(k, n-k)$. We write the cone as $C_0(\Omega) = \{r\sigma; r > 0, \sigma \in D\}$ having 0 as a vertex included in \mathbb{R}^n , where D is the trace of the cone on the sphere S^{n-1} . More generally, given $D \subset S^{n-1}$, define the cone $C := \{r\sigma; r > 0, \sigma \in D\}$. Then we have that

- If D is the sphere S^{n-1} , then $C = \mathbb{R}^n \setminus \{0\}$.
- If D is the half-sphere S_+^{n-1} , then C is the half-space $\mathbb{R}^{1+,n-1} := \mathbb{R}_+^n$.
- If $D = S^{n-1} \cap \mathbb{R}^{k+,n-k}$, then $C = \mathbb{R}^{k+,n-k}$ for all $k \in \{1, \dots, n\}$.

For such cones, see Ghoussoub-Moradifam [13], the Hardy constant is

$$\gamma_H(C) = \frac{(n-2)^2}{4} + \lambda_1(D),$$

such that $\lambda_1(D)$ is the first eigenvalue of Laplacien on $D \subset S^{n-1}$ with Dirichlet boundary condition. In particular, $\gamma_H(\mathbb{R}^{k+,n-k}) = \frac{(n+2k-2)^2}{4}$ where $\lambda_1(D) = k(n+k-2)$ for all $k \in \{1, \dots, n\}$. The model cone is the relevant object to consider to understand the Hardy constant of Ω :

Proposition 1.1. *Let Ω be a bounded domain of \mathbb{R}^n . We assume that $0 \in \partial\Omega$ is a singularity of type $(k, n-k)$ for some $k \in \{1, \dots, n\}$. Then γ_H satisfies the following properties:*

- (i) $\frac{(n-2)^2}{4} < \gamma_H(\Omega) \leq \gamma_H(C_0(\Omega))$.
- (ii) $\gamma_H(\Omega) = \gamma_H(C_0(\Omega))$ for every Ω such that $0 \in \partial\Omega$ and $\Omega \subset C_0(\Omega)$.
- (iii) If $\gamma_H(\Omega) < \gamma_H(C_0(\Omega))$, then it is attained in $D^{1,2}(\Omega)$.
- (iv) For every $\epsilon > 0$, there exists $\mathbb{R}^{k+,n-k} \subsetneq \Omega_\epsilon \subsetneq \mathbb{R}^n$ with a boundary singularity at 0 of type $(k, n-k)$ such that $\gamma_H(\mathbb{R}^{k+,n-k}) - \epsilon \leq \gamma_H(\Omega_\epsilon) \leq \gamma_H(\mathbb{R}^{k+,n-k})$.

The study of the Hardy constant for itself is reminiscent in the litterature. Without being exhaustive, we refer to Fall [8], Fall-Musina [9] and the references therein.

We now tackle the second question **(Q2)**, that is the existence of extremals for (3). In this framework, the following result is classical:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $0 \in \partial\Omega$ is a singularity of type $(k, n-k)$. Assume that $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$, $0 \leq s \leq 2$, and $\mu_{\gamma,s}(\Omega) < \mu_{\gamma,s}(\mathbb{R}^{k+,n-k})$. Then there are extremals for $\mu_{\gamma,s}(\Omega)$. In particular, there exists a minimizer u in $D^{1,2}(\Omega) \setminus \{0\}$ that is a positive solution to the equation*

$$(E) \begin{cases} -\Delta u - \frac{\gamma}{|x|^2} u = \mu_{\gamma,s}(\Omega) \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In other words, being below a critical threshold given by the model cone yields existence of extremals. Such a result is reminiscent in the functional inequalities of elliptic type since the work of Trudinger [19] and Aubin [2] on the Yamabe problem. Related results for Hardy-Sobolev equations are in Bartsch-Peng-Zhang [3] and Pucci-Servadei [18].

We now give sufficient conditions to get the existence condition. As for the Yamabe problem, we need to introduce some test-functions cooked up from a model space: here, it is the model cone. In the smooth case, that is $k = 1$, the test-functions yield a condition on the mean curvature to recover existence. In our non-smooth context, we must tackle two additional difficulties:

- The mean curvature is not defined, and we must define another geometric quantity.
- The extremals for the model space $\mathbb{R}^{k+,n-k}$ are not smooth, and the proof of the symmetry in [16] does not extend to our context.

We are able to recover symmetry via a version of the moving-plane method developed by Berestycki and Nirenberg [4]. Concerning the lack of mean curvature, we introduce a new geometric object.

Definition 2. Let $\Omega \subset \mathbb{R}^n$ be a domain such that $0 \in \partial\Omega$ is a singularity of type $(k, n - k)$. We define

$$(5) \quad \Omega_i := \phi(U \cap \{x_i > 0\}) \text{ for all } i = 1, \dots, k,$$

where (ϕ, U) is a chart as in the Definition 1. We have that:

- (1) For all $i = 1, \dots, k$, Ω_i is smooth around $0 \in \partial\Omega_i$.
- (2) Up to permutation, the Ω_i 's are locally independent of the chart ϕ .
- (3) The Ω_i 's define locally Ω : there exists $\delta > 0$ such that

$$\Omega \cap B_\delta(0) = \bigcap_{i=1}^k \Omega_i \cap B_\delta(0).$$

For example:

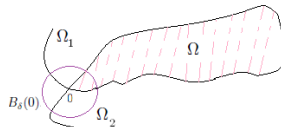


FIGURE 2. Case $k = 2$, $n - k = 0$.

Definition 3. Let S be a submanifold of \mathbb{R}^n . We let $II_{x_0}^S$ be the second fundamental form at x_0 of S , that is

$$\begin{cases} II_{x_0}^S : T_{x_0}S \times T_{x_0}S \times (T_{x_0}S)^\perp & \rightarrow \mathbb{R} \\ (X, Y, \eta) & \mapsto II^S(X, Y, \eta) = \langle \bar{\nabla}_X Y - \nabla_X Y, \eta \rangle_{x_0}. \end{cases}$$

The mean curvature vector at $x_0 \in S$ is the vector $\vec{H}_{x_0}^S \in (T_{x_0}S)^\perp$ such that for all $\eta \in (T_{x_0}S)^\perp$, we have that

$$\langle \vec{H}_{x_0}^S, \eta \rangle_{x_0} = \text{Trace} \left((X, Y) \mapsto II_{x_0}^S(X, Y, \eta) \right).$$

For $k \in \{1, \dots, n\}$ and $m = 1, \dots, k$, we define $\vec{\nu}_m : \partial\Omega_m \rightarrow \mathbb{R}^n$ is the outer unit normal vector of the locally oriented Ω_m around 0 where Ω_m as in (5) (see Definition 2): this definition makes sense locally around 0. In particular, we have $\vec{\nu}_m(0) := (0, \dots, 0, -1, 0, \dots, 0)$ when $d\phi_0 = Id$. We are in position to get an existence result for small values of γ :

Theorem 1.2. Let Ω be a bounded domain in \mathbb{R}^n ($n \geq 3$) such that $0 \in \partial\Omega$ is a singularity of type $(k, n - k)$ for some $k \in \{1, \dots, n\}$. We fix $0 \leq s < 2$ and $0 \leq \gamma < \gamma_H(\Omega)$. Assume that either $s > 0$, or that $\{s = 0, n \geq 4 \text{ and } \gamma > 0\}$. We assume that

$$0 \leq \gamma \leq \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}.$$

Then there are extremals for $\mu_{\gamma,s}(\Omega)$ if

$$(6) \quad GH_{\gamma,s}(\Omega) < 0$$

where, for $\Sigma := \cap_{i=1}^k \partial\Omega_i$, $GH_{\gamma,s}(\Omega)$ is the **generalized mean curvature**

$$(7) \quad GH_{\gamma,s}(\Omega) := c_{\gamma,s}^1 \sum_{m=1}^k \langle \vec{H}_0^\Sigma, \vec{\nu}_m \rangle + c_{\gamma,s}^2 \sum_{i,m=1, i \neq m}^k II_0^{\partial\Omega_m}(\vec{\nu}_i, \vec{\nu}_i) \\ + c_{\gamma,s}^3 \sum_{p,q,m=1, |\{p,q,m\}|=3}^k II_0^{\partial\Omega_m}(\vec{\nu}_p, \vec{\nu}_q)$$

and $c_{\gamma,s}^1, c_{\gamma,s}^2, c_{\gamma,s}^3$ are positive explicit constants. By convention, each of the sums above is zero when empty.

The first term in $GH_{\gamma,s}(\Omega)$ shows the influence of the mean curvature of $\Sigma = \cap_{i=1}^k \partial\Omega_i$ at 0. The second and third sums outline the influence of the positions of the Ω_m 's relatively to each other: these two terms do not appear in the smooth case, that is $k = 1$.

When $k = 1$, condition (6) reads $\langle \vec{H}_0^{\partial\Omega}, \vec{\nu}_{\partial\Omega} \rangle < 0$. We then recover the condition of Ghoussoub-Robert [16]. Our condition is *local*: only the local geometry of the boundary at 0 is relevant here. In the paper [5], we deal with the case $\gamma > \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$: the test-functions then are different, and the existence condition is global.

For the sake of completeness, we now deal with the remaining cases, still for $\gamma \leq \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$.

Theorem 1.3. *Let Ω be a bounded domain in $\mathbb{R}^n (n \geq 3)$ with a singularity of type $(k, n - k)$ at 0 for some $k \in \{1, \dots, n\}$. Then*

- (1) *If $\gamma \leq 0$, then $\mu_{\gamma,0}(\Omega) = \mu_{0,0}(\mathbb{R}^n)$, and there is no extremal.*
- (2) *If $n = 3$, $0 < \gamma \leq \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ and there are extremals for $\mu_{\gamma,0}(\mathbb{R}_+^k \times \mathbb{R}^{3-k})$, then there are extremals for $\mu_{\gamma,0}(\Omega)$ if $GH_{\gamma,0}(\Omega) < 0$.*
- (3) *If $n = 3$, $0 < \gamma$ and there are no extremals for $\mu_{\gamma,0}(\mathbb{R}^{k+,3-k})$, then there are extremals for $\mu_{\gamma,0}(\Omega)$ if $R_\gamma(x_0) > 0$ for some $x_0 \in \Omega$.*

The proof of Theorem 1.3 is similar to what was performed in Ghoussoub-Robert [16], and we will only sketch it in Section 6, where the interior mass $R_\gamma(x_0)$ will be defined in Proposition 6.1.

Our results are summarized in these tables:

Hardy Condition	Dimension	Geometric Condition	Extremal
$0 \leq \gamma \leq \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$	$n \geq 3$	$GH_{\gamma,s}(\Omega) < 0$	Yes

TABLE 1. Case $s > 0$.

Hardy Condition	Dimension	Geometric Condition	Extremal
$0 < \gamma \leq \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$	$n = 3$	$GH_{\gamma,0}(\Omega) < 0$ and $R_\gamma(x_0) < 0$	Yes
	$n \geq 4$	$GH_{\gamma,0}(\Omega) < 0$	Yes
$\gamma \leq 0$	$n \geq 3$		No

TABLE 2. Case $s = 0$.

In this paper, some regularity issues will be used very often. Our main tool will be the article [10] by Felli and Ferrero. We also refer to the historical reference Gmira-Véron [17] and to the monograph [7] by Cirstea. As an intermediate step in our analysis, we will prove a symmetry result for the extremals of $\mu_{\gamma,s}(\mathbb{R}^{k+,n-k})$: with the use of the moving-plane method (see Berestycki-Nirenberg [4]), we will obtain that the symmetries of the domain transfer to the extremals. This will be the object of Theorem 4.1.

2. THE BEST HARDY CONSTANT AND HARDY SOBOLEV INEQUALITY

This section is devoted to the analysis of the Hardy constant $\gamma_H(\Omega)$ and the proof of Proposition 1.1:

Proof of (i) of Proposition 1.1: By definition, $\frac{(n-2)^2}{4} = \gamma_H(\mathbb{R}^n) \leq \gamma_H(\Omega)$. We assume by contradiction that $\gamma_H(\Omega) = \gamma_H(\mathbb{R}^n)$. We that have $\mu_{\gamma,2}(\Omega) = \gamma_H(\Omega) - \gamma = \frac{(n-2)^2}{4} - \gamma < \mu_{\gamma,2}(\mathbb{R}^{k+,n-k}) = \frac{(n+2k-2)^2}{4} - \gamma$. Theorem 1.1 yields $\mu_{\gamma,2}(\Omega)$ is achieved by some $u_0 \in D^{1,2}(\Omega) \setminus \{0\}$. Since $u_0 \in D^{1,2}(\Omega) \subset D^{1,2}(\mathbb{R}^n)$, we get that $\gamma_H(\mathbb{R}^n)$ is achieved in $D^{1,2}(\mathbb{R}^n)$. Replacing u_0 by $|u_0|$, we assume that $u_0 \geq 0$ on \mathbb{R}^n . The Euler-Lagrange equation and the maximum principle yield $u_0 > 0$ on \mathbb{R}^n , contradicting $u_0 = 0$ on $\partial\Omega$. Therefore $\frac{(n-2)^2}{4} < \gamma_H(\Omega)$.

For the other inequality, since Ω is a singularity of type $(k, n-k)$ at 0, we choose a chart (U, ϕ) as in Definition 1. Without loss of generality, we assume that $d\phi_0 = Id$ and that $C_0(\Omega) = \mathbb{R}^{k+,n-k}$. Let $\eta \in C_c^\infty(U)$ such that $\eta(x) = 1$ for $x \in B_\delta(0)$, for some $\delta > 0$ small enough, and consider $(\alpha_\epsilon)_{\epsilon>0} \in]0, +\infty[$ such that $\alpha_\epsilon = o(\epsilon)$ as $\epsilon \rightarrow 0$. We define

$$\rho(x) := |x|^{-k-\frac{n-2}{2}} \prod_{i=1}^k x_i \text{ for all } x \in \mathbb{R}^{k+,n-k}.$$

Note that $\rho \notin D^{1,2}(\mathbb{R}^{k+,n-k})$. We fix $\beta > 1$ and define

$$\rho_\epsilon(x) = \begin{cases} \frac{|x|^\beta}{\epsilon} \rho(x) & \text{if } |x| < \epsilon \\ \rho(x) & \text{if } \epsilon < |x| < \frac{1}{\epsilon} \\ |\epsilon \cdot x|^{-\beta} \rho(x) & \text{if } |x| > \frac{1}{\epsilon}. \end{cases}$$

Note that $\rho_\epsilon \in D^{1,2}(\mathbb{R}^{k+,n-k})$. For $\epsilon > 0$, we define

$$u_\epsilon(y) = \eta(\phi^{-1}(y)) \alpha_\epsilon^{\frac{2-n}{2}} \rho_\epsilon(\alpha_\epsilon^{-1} \phi^{-1}(y)) \text{ for any } y \in \phi(U) \cap \Omega, y = \phi(x)$$

and 0 elsewhere. Immediate computations yield

$$(8) \quad \begin{aligned} \int_{\mathbb{R}^n \setminus B_{\epsilon^{-1}}(0)} \frac{\rho_\epsilon^2}{|x|^2} dx &= O(1) \quad \text{and} \quad \int_{B_\epsilon(0)} \frac{\rho_\epsilon^2}{|x|^2} dx = O(1). \\ \int_{\mathbb{R}^n \setminus B_{\epsilon^{-1}}(0)} |\nabla \rho_\epsilon|^2 dx &= O(1) \quad \text{and} \quad \int_{B_\epsilon(0)} |\nabla \rho_\epsilon|^2 dx = O(1) \end{aligned}$$

when $\epsilon \rightarrow 0$. Since $d\phi_0 = Id$, we have

$$(9) \quad \begin{aligned} \int_{\Omega} \frac{|u_\epsilon(y)|^2}{|y|^2} dy &= \int_{\mathbb{R}^{k+,n-k} \cap U} \frac{|u_\epsilon(\phi(x))|^2}{|\phi(x)|^2} |Jac(\phi(x))| dx \\ &= \int_{B_\delta(0) \cap \mathbb{R}^{k+,n-k}} \frac{|u_\epsilon(\phi(x))|^2}{|x|^2} |1 + O(|x|)| dx + O(1). \end{aligned}$$

Writing $B_\delta(0) = (B_\delta(0) \setminus B_{\epsilon^{-1}\alpha_\epsilon}(0)) \cup (B_{\epsilon^{-1}\alpha_\epsilon}(0) \setminus B_{\epsilon\alpha_\epsilon}(0)) \cup (B_{\epsilon\alpha_\epsilon}(0))$, (8) yields

$$(10) \quad \int_{(B_\delta(0) \setminus B_{\epsilon^{-1}\alpha_\epsilon}(0)) \cap \mathbb{R}^{k+,n-k}} \frac{|u_\epsilon(\phi(x))|^2}{|x|^2} dx = O(1); \quad \int_{B_{\epsilon\alpha_\epsilon}(0) \cap \mathbb{R}^{k+,n-k}} \frac{|u_\epsilon(\phi(x))|^2}{|x|^2} dx = O(1).$$

And,

$$(11) \quad \int_{(B_{\epsilon^{-1}\alpha_\epsilon}(0) \setminus B_{\epsilon\alpha_\epsilon}(0)) \cap \mathbb{R}^{k+,n-k}} \frac{|u_\epsilon(\phi(x))|^2}{|x|^2} (1 + O(|x|)) dx = W_{D,2} \ln \frac{1}{\epsilon^2} + O(1),$$

where $W_{D,2} := 2 \int_D |\prod_{i=1}^k x_i|^2 d\sigma$ with $D = S^{n-1} \cap \mathbb{R}^{k+,n-k}$ for all $k \in \{1, \dots, n\}$. We combine (9), (10) and (11)

$$(12) \quad \int_{\Omega} \frac{|u_\epsilon(y)|^2}{|y|^2} dy = W_{D,2} \ln \frac{1}{\epsilon^2} + O(1) \text{ as } \epsilon \rightarrow 0.$$

Similar arguments yield

$$(13) \quad \int_{\Omega} |\nabla u_\epsilon(y)|^2 dy = W_{D,2} \ln \frac{1}{\epsilon^2} \gamma_H(\mathbb{R}^{k+,n-k}) + O(1) \text{ as } \epsilon \rightarrow 0.$$

By the equations (12), (13), we get that

$$\frac{\int_{\Omega} |\nabla u_{\epsilon}(y)|^2 dy}{\int_{\Omega} \frac{|u_{\epsilon}(y)|^2}{|y|^2} dy} = \gamma_H(\mathbb{R}^{k+,n-k}) + o(1) \text{ as } \epsilon \rightarrow 0,$$

and by the definition of γ_H , we get that $\gamma_H(\Omega) \leq \gamma_H(\mathbb{R}^{k+,n-k})$. This proves (i).

Proof of (ii): If $\Omega \subset C_0(\Omega)$, then the definition yields $\gamma_H(\Omega) \geq \gamma_H(C_0(\Omega))$. The reverse inequality is by (i), which yields (ii).

Proof of (iii): Is a particular case of Theorem 1.1 below when $s = 2$.

Proof of (iv): By Ghoussoub-Robert [16] we have the following lemma:

Lemma 2.1. *Let $(\phi_t)_{t \in \mathbb{N}} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ be such that,*

$$\lim_{t \rightarrow +\infty} (\|\phi_t - Id_{\mathbb{R}^n}\|_{\infty} + \|\nabla(\phi_t - Id_{\mathbb{R}^n})\|_{\infty}) = 0 \text{ and } \phi_t(0) = 0.$$

Let $D \subset \mathbb{R}^n$ be a domain such that $0 \in \partial D$ (not necessarily bounded nor regular), and set $D_t := \phi_t(D)$, $\forall t \in \mathbb{N}$. Then $0 \in \partial D_t$, and $\lim_{t \rightarrow +\infty} \gamma_H(D_t) = \gamma_H(D)$.

Let $\phi \in C^{\infty}(\mathbb{R}^{n-k})$ be such that $0 \leq \phi \leq 1$, $\phi(0) = 0$ et $\phi(x'') = 1$ for all $x'' \in \mathbb{R}^{n-k}$ such that $|x''| \geq 1$. For $t \geq 0$, define $\phi_t(x', x'') := (x_1 - t\phi(x''), \dots, x_k - t\phi(x''), x'')$ for all $(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. Set $\tilde{\Omega}_t := \phi_t(\mathbb{R}^{k+,n-k})$. Lemma 2.1 yields

$$\lim_{t \rightarrow 0} \gamma_H(\tilde{\Omega}_t) = \gamma_H(\mathbb{R}^{k+,n-k}) = \frac{(n+2k-2)^2}{4}.$$

Since $\phi \geq 0$ and $\phi(x'') = 1$ for $x'' \in \mathbb{R}^{n-k}$, $|x''| \geq 1$, we have that $\mathbb{R}^{k+,n-k} \subsetneq \tilde{\Omega}_t$. To finish the proof of (iv), we take $\Omega_{\epsilon} := \tilde{\Omega}_t$ with $\epsilon > 0$, $t > 0$ small enough.

Proposition 2.1. *Let $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ for all $k \in \{1, \dots, n\}$ and $s \in [0, 2]$. Then, for all $\epsilon > 0$ there exists $c_{\epsilon} > 0$ such that for all $u \in D^{1,2}(\Omega)$,*

$$(14) \quad \left[\int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} \leq (\mu_{\gamma,s}(\mathbb{R}^{k+,n-k})^{-1} + \epsilon) \int_{\Omega} \left(|\nabla u|^2 - \frac{\gamma}{|x|^2} u^2 \right) dx + c_{\epsilon} \int_{\Omega} u^2 dx.$$

Proof of Proposition 2.1: We choose a chart (U, ϕ) as in Definition 1. Without loss of generality, we assume that $d\phi_0 = Id$ and then $C = C_0(\Omega) = \mathbb{R}^{k+,n-k}$. Choose $u \in C_c^{\infty}(\phi(B_{\delta}(0)) \cap \Omega)$ and define $v := u \circ \phi$ for all $v \in C_c^1(B_{\delta}(0) \cap C)$. Define the metric $g := \phi^{-1*} Eucl$, where $Eucl$ is the Euclidean metric. We have that $|\phi(x)| = |x|(1 + O(|x|))$ and $|\phi^* Eucl - Eucl|(x) \leq c|x|$ for all $x \in \mathbb{R}^n$ small enough for some $c > 0$.

Step 1: fix $\epsilon > 0$, we first claim that there exists $\delta > 0$ such that for all $u \in C_c^1(\phi(B_{\delta}(0)) \cap \Omega)$,

$$(15) \quad \left[\int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} \leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} \left(|\nabla u|^2 - \frac{\gamma}{|x|^2} u^2 \right) dx.$$

Proof of (15): We have that

$$\begin{aligned} & \left[\int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} = \left[\int_{B_{\delta}(0) \cap C} \frac{|u \circ \phi(x)|^{2^*(s)} |Jac(\phi(x))|}{|\phi(x)|^s} dx \right]^{\frac{2}{2^*(s)}} \\ & \leq (1 + c\delta) \left[\int_{B_{\delta}(0) \cap C} \frac{|v|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} \leq (1 + c\delta) \mu_{\gamma,s}(C)^{-1} \int_{B_{\delta}(0) \cap C} \left(|\nabla v|^2 - \frac{\gamma}{|x|^2} v^2 \right) dx \\ & \leq (1 + c\delta) \mu_{\gamma,s}(C)^{-1} \int \left(|\nabla u|_g^2 - \frac{\gamma}{|\phi^{-1}(x)|^2} u^2 \right) |Jac \phi^{-1}(x)| dx \\ & \leq (1 + c_1\delta) \mu_{\gamma,s}(C)^{-1} \int \left(|\nabla u|^2 - \frac{\gamma}{|x|^2} u^2 \right) dx + c_2\delta \int \left(|\nabla u|^2 + \frac{u^2}{|x|^2} \right) dx \end{aligned}$$

where the last three integrals are taken on $\phi(B_\delta(0)) \cap \Omega$. This give us

$$\left[\int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} \leq (1 + c_1\delta)\mu_{\gamma,s}(C)^{-1} \int_{\Omega} (|\nabla u|^2 - \frac{\gamma}{|x|^2}u^2) dx + c_2\delta \int_{\Omega} (|\nabla u|^2 + \frac{u^2}{|x|^2}) dx.$$

For all $v \in C_c^1(\phi(B_\delta(0)) \cap \Omega)$, we get that

$$(16) \quad \int_{\Omega} \frac{u^2}{|x|^2} dx = \int_{B_\delta(0) \cap C} \frac{v^2}{|x|^2} |1 + O(|x|)| dx \leq (1 + c_1\delta) \int_C \frac{v^2}{|x|^2} dx,$$

and,

$$(17) \quad \int_{\Omega} |\nabla u|^2 dx = \int_{B_\delta(0) \cap C} |\nabla v|_{\phi^* Eucd}^2 |1 + O(|x|)| dx \geq (1 - c_2\delta) \int_C |\nabla v|^2 dx,$$

where $c_1, c_2 > 0$ are independent of δ and v . Since $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$, there exists $c_0 > 0$ for δ small enough,

$$(18) \quad c_0^{-1} \int_{\Omega} |\nabla u|^2 dx \leq \int_{\Omega} \left(|\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx \leq c_0 \int_{\Omega} |\nabla u|^2 dx.$$

With (16), (17) and (18), we get (15) for $\delta > 0$ small enough. This ends Step 1. \square

Step 2: We prove (14) for all $u \in D^{1,2}(\Omega)$.

Let $\eta \in C_c^\infty(\mathbb{R}^n)$ and $\sqrt{\eta}, \sqrt{1-\eta} \in C^2(\mathbb{R}^n)$ be such that $\eta(x) = 1$ if $x \in B_{\delta/2}(0)$ and $\eta(x) = 0$ if $x \notin B_\delta(0)$. We define $\|w\|_{p,|x|^{-s}} = \left[\int_{\Omega} \frac{|w|^p}{|x|^s} dx \right]^{\frac{1}{p}}$. We set $p = 2^*(s)/2$. Hölder's inequality yield

$$\begin{aligned} \|u^2\|_{p,|x|^{-s}} &= \|\eta u^2 + (1-\eta)u^2\|_{p,|x|^{-s}} \\ &\leq \|\eta u^2\|_{p,|x|^{-s}} + \|(1-\eta)u^2\|_{p,|x|^{-s}} \leq \|\sqrt{\eta}u\|_{2^*(s),|x|^{-s}}^2 + \|\sqrt{1-\eta}u\|_{2^*(s),|x|^{-s}}^2, \end{aligned}$$

for all $u \in C_c^\infty(\Omega)$. Since $\sqrt{\eta}u \in C_c^2(B_{\delta_\epsilon} \cap C)$, we use (15) and integrate by parts

$$(19) \quad \begin{aligned} \left[\int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} &\leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} \left(|\nabla \sqrt{\eta}u|^2 - \frac{\gamma}{|x|^2} \eta u^2 \right) dx + \|\sqrt{1-\eta}u\|_{2^*(s),|x|^{-s}}^2 \\ &\leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} \eta \left(|\nabla u|^2 - \frac{\gamma}{|x|^2} u^2 \right) dx + \|\sqrt{1-\eta}u\|_{2^*(s),|x|^{-s}}^2 + c \int_{\Omega} u^2 dx, \end{aligned}$$

where $c > 0$ depends of $\epsilon > 0$.

Case 1: $s = 0$. We claim that

$$(20) \quad \mu_{\gamma,0}(\Omega) \leq \mu_{0,0}(\mathbb{R}^n)$$

We prove the claim. Fix $x_0 \in \Omega$, $x_0 \neq 0$, and take $\eta \in C_c^\infty(\Omega)$ such that $\eta(x) = 1$ around of x_0 . For $x \in \Omega$ and $\epsilon > 0$, we define $u_\epsilon(x) := \eta(x) \left(\frac{\epsilon}{\epsilon^2 + |x-x_0|^2} \right)^{\frac{n-2}{2}}$ for all $x \in \Omega$. Classical computations in the spirit of Aubin [2] yield

$$\lim_{\epsilon \rightarrow 0} \frac{\int_{\Omega} |\nabla u_\epsilon|^2 dx}{\left(\int_{\Omega} u_\epsilon^{2^*} dx \right)^{\frac{2}{2^*}}} = \mu_{0,0}(\mathbb{R}^n)$$

and $\lim_{\epsilon \rightarrow 0} \int_{\Omega} \frac{u_\epsilon^2}{|x|^2} dx = 0$. This yields (20), and the claim is proved.

The Sobolev inequality yields $\|f\|_{2n/(n-2)}^2 \leq \mu_{0,0}(\mathbb{R}^n)^{-1} \|\nabla f\|_2^2$ for all $f \in D^{1,2}(\Omega) \subset D^{1,2}(\mathbb{R}^n)$. We combine these inequalities to get

$$(21) \quad \begin{aligned} \|\sqrt{1-\eta}u\|_{2^*(s),|x|^{-s}}^2 &\leq \mu_{0,0}(\mathbb{R}^n)^{-1} \int_{\Omega} |\nabla(\sqrt{1-\eta}u)|^2 dx \\ &\leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} (1-\eta)|\nabla u|^2 dx + c \int_{\Omega} u^2 dx. \end{aligned}$$

We use the equations (19) and (21)

$$\begin{aligned}
& \left[\int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} \leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} \eta \left(|\nabla u|^2 - \frac{\gamma}{|x|^2} u^2 \right) dx \\
& + (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} (1 - \eta) |\nabla u|^2 dx + c \int_{\Omega} u^2 dx \\
& \leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} |\nabla u|^2 dx - (\mu_{\gamma,s}(C)^{-1} + \epsilon) \gamma \int_{\Omega \setminus B_{\delta_{\frac{\epsilon}{2}}}(0)} \frac{\eta}{|x|^2} u^2 dx \\
& - (\mu_{\gamma,s}(C)^{-1} + \epsilon) \gamma \int_{B_{\delta_{\frac{\epsilon}{2}}}(0)} \frac{\eta}{|x|^2} u^2 dx + c \int_{\Omega} u^2 dx \\
& \leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} |\nabla u|^2 dx - (\mu_{\gamma,s}(C)^{-1} + \epsilon) \gamma \int_{B_{\delta_{\frac{\epsilon}{2}}}(0)} \frac{u^2}{|x|^2} dx + c \int_{\Omega} u^2 dx \\
& \leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} \left(|\nabla u|^2 - \frac{\gamma}{|x|^2} u^2 \right) dx + c \int_{\Omega} u^2 dx.
\end{aligned}$$

Case 2: $0 < s < 2$. We have that $2 < 2^*(s) < 2^*$, let $\nu > 0$ and by interpolation inequality there exists $c_{\nu} > 0$, such that

$$\begin{aligned}
\|\sqrt{1 - \eta}u\|_{2^*(s), |x|^{-s}}^2 & \leq C \left(\nu \|\sqrt{1 - \eta}u\|_{2^*}^2 + c_{\nu} \|\sqrt{1 - \eta}u\|_2^2 \right) \\
& \leq C \left(\nu \mu_{0,0}(\mathbb{R}^n)^{-1} \|\nabla(\sqrt{1 - \eta}u)\|_2^2 + c_{\nu} \|\sqrt{1 - \eta}u\|_2^2 \right).
\end{aligned}$$

We choose ν such that $\nu \mu_{0,0}(\mathbb{R}^n)^{-1} \leq \mu_{\gamma,s}^{-1}(C) + \epsilon$, we get

$$(22) \quad \|\sqrt{1 - \eta}u\|_{2^*(s), |x|^{-s}}^2 \leq (\mu_{\gamma,s}^{-1}(C) + \epsilon) \|\nabla(\sqrt{1 - \eta}u)\|_2^2 + c_{\nu} \|\sqrt{1 - \eta}u\|_2^2.$$

By (19) and (22)

$$\begin{aligned}
& \left[\int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} \leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} \eta \left(|\nabla u|^2 dx - \frac{\gamma}{|x|^2} u^2 \right) dx \\
& + (\mu_{\gamma,s}^{-1}(C) + \epsilon) \|\nabla(\sqrt{1 - \eta}u)\|_2^2 + c_{\nu} \|\sqrt{1 - \eta}u\|_2^2 + c \int_{\Omega} u^2 dx \\
& \leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} \left(|\nabla u|^2 - \frac{\gamma}{|x|^2} u^2 \right) dx + c \int_{\Omega} u^2 dx.
\end{aligned}$$

Cas 3: $s = 2$. We have $2^*(s) = 2$

$$(23) \quad \|\sqrt{1 - \eta}u\|_{2^*(s), |x|^{-s}}^2 = \int_{\Omega \setminus B_{\delta/2}(0)} \frac{1 - \eta}{|x|^2} u^2 dx \leq c_{\delta} \int_{\Omega} u^2 dx,$$

by the equations (19) and (23) we get the result. □

Proposition 2.2. *Let Ω be a bounded domain such that $0 \in \partial\Omega$.*

- (i) *If $\gamma < \gamma_H(\mathbb{R}^{k+, n-k})$, then $\mu_{\gamma,s}(\Omega) > -\infty$.*
- (ii) *If $\gamma > \gamma_H(\mathbb{R}^{k+, n-k})$, then $\mu_{\gamma,s}(\Omega) = -\infty$.*
- (iii) *If $\gamma < \gamma_H(\Omega)$, then $\mu_{\gamma,s}(\Omega) > 0$.*
- (iv) *If $\gamma_H(\Omega) < \gamma < \gamma_H(\mathbb{R}^{k+, n-k})$, then $0 > \mu_{\gamma,s}(\Omega) > -\infty$.*
- (v) *If $\gamma = \gamma_H(\Omega) < \gamma_H(\mathbb{R}^{k+, n-k})$, then $\mu_{\gamma,s}(\Omega) = 0$.*

Proof of Proposition 2.2: Proof of (i): Let $\gamma < \gamma_H(\mathbb{R}^{k+, n-k})$ and $\epsilon > 0$ such that $(1 + \epsilon)\gamma \leq \gamma_H(\mathbb{R}^{k+, n-k})$. By Proposition 2.1 there exist $c_{\epsilon} > 0$ for any $u \in D^{1,2}(\Omega)$ such that

$$\gamma_H(\mathbb{R}^{k+, n-k}) \int_{\Omega} \frac{u^2}{|x|^2} dx \leq (1 + \epsilon) \int_{\Omega} |\nabla u|^2 + c_{\epsilon} \int_{\Omega} u^2 dx.$$

Since $2^*(s) > 2$ and Ω is bounded, Hölder inequality yields $c_1 > 0$ such that

$$(24) \quad \int_{\Omega} u^2 dx \leq c_1 \left(\int_{\Omega} \frac{u^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}.$$

If $\gamma \geq 0$ and since $(1 - \gamma\gamma_H(\mathbb{R}^{k+,n-k})^{-1}(1 + \epsilon)) \geq 0$, by (24), we get

$$J_{\gamma,s}^{\Omega}(u) = \frac{\int_{\Omega} |\nabla u|^2 - \gamma \int_{\Omega} \frac{u^2 dx}{|x|^2}}{\left(\int_{\Omega} \frac{u^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}} \geq \frac{-c_2 c_{\epsilon} \gamma}{\gamma_H(\mathbb{R}^{k+,n-k})},$$

then for any $u \in D^{1,2}(\Omega)$ we have $\mu_{\gamma,s}(\Omega) > -\infty$. If $\gamma < 0$, then $\mu_{\gamma,s}(\Omega) \geq \mu_{0,s}(\Omega) > 0$ by Hardy-Sobolev inequality.

Proof of (ii): We take $(u_{\epsilon})_{\epsilon>0}$ as in the proof of Proposition 1.1 -(i). We get

$$J_{\gamma,s}^{\Omega}(u_{\epsilon}) = \left((\gamma_H(\mathbb{R}^{k+,n-k}) - \gamma) \frac{W_{D,2}}{W_{D,2}^{\frac{2}{2^*(s)}}} + O(1) \right) \left(\ln\left(\frac{1}{\epsilon^2}\right) \right)^{\frac{2-s}{n-s}},$$

As $s < 2$ and $\gamma > \gamma_H(\mathbb{R}^{k+,n-k})$, then $\lim_{\epsilon \rightarrow 0} J_{\gamma,s}^{\Omega}(u_{\epsilon}) = -\infty$, and $\mu_{\gamma,s}(\Omega) = -\infty$.

Proof of (iii): We fix $\gamma < \gamma_H(\Omega)$. For any $u \in D^{1,2}(\Omega) \setminus \{0\}$, we have that

$$J_{\gamma,s}^{\Omega}(u) = \frac{\int_{\Omega} |\nabla u|^2 - \gamma \int_{\Omega} \frac{u^2 dx}{|x|^2}}{\left(\int_{\Omega} \frac{u^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}} \geq \left(1 - \frac{\gamma}{\gamma_H(\Omega)} \right) \mu_{0,s}(\Omega),$$

and then $\mu_{\gamma,s}(\Omega) > 0$.

Proof of (iv): We assume that $\gamma_H(\Omega) < \gamma < \gamma_H(\mathbb{R}^{k+,n-k})$, it follows from Proposition 1.1-(iii) that $\gamma_H(\Omega)$ is attained by some u_0 . We get that $\mu_{\gamma,s}(\Omega) \leq J_{\gamma,s}^{\Omega}(u_0) < 0$.

Proof of (v): We now assume that $\gamma_H(\Omega) = \gamma < \gamma_H(\mathbb{R}^{k+,n-k})$. Then $\mu_{\gamma,s}(\Omega) \geq 0$. Here again, Proposition 1.1 yields an extremal $u_0 \in D^{1,2}(\Omega)$ for $\gamma_H(\Omega)$. We get $J_{\gamma,s}^{\Omega}(u_0) = 0$, and then $\mu_{\gamma,s}(\Omega) = 0$.

Sketch of the proof of Theorem 1.1. The proof is very classical and follows the proof of Proposition 6.2 in [15]. We only sketch it to outline the specific tools we use here. Let $(u_k)_{k \in \mathbb{N}} \in D^{1,2}(\Omega) \setminus \{0\}$ be a minimizing sequence $\mu_{\gamma,s}(\Omega)$ such that $\|u_k\|_{2^*(s),|x|^{-s}}^2 = 1$. Using Proposition 2.1, we get that $(u_k)_{k \in \mathbb{N}}$ is bounded in $D^{1,2}(\Omega)$. As a consequence, up to the extraction of a subsequence, there exists $u \in D^{1,2}(\Omega)$ such that $u_k \rightarrow u$ weakly in $D^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ as $k \rightarrow +\infty$. We write $\theta_k := u_k - u$, so that $\theta_k \rightarrow 0$ weakly in $D^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ as $k \rightarrow +\infty$. We apply the definition (3) of $\mu_{\gamma,s}(\Omega)$ to u and Proposition 2.1 to θ_k for $\epsilon_0 > 0$ small enough. It is then standard to get that $\theta_k \rightarrow 0$ strongly in $D^{1,2}(\Omega)$, and then $u \neq 0$ is a minimizer for $\mu_{\gamma,s}(\Omega)$. As mentioned above, we refer to the proof of Proposition 6.2 in [15] for the method.

3. REGULARITY AND APPROXIMATE SOLUTIONS

We say that $u \in D_{loc,0}^{1,2}(\Omega)$ if there exists $\eta \in C_c^{\infty}(\mathbb{R}^n)$ such that $\eta \equiv 1$ around 0 and $\eta u \in D^{1,2}(\Omega)$. We define $U_{\alpha}(x) := |x|^{-\alpha-k} \prod_{i=1}^k x_i$. As one checks

$$-\Delta U_{\alpha} - \frac{\gamma}{|x|^2} U_{\alpha} = 0 \text{ in } \mathbb{R}^{k+,n-k} \Leftrightarrow \alpha \in \{\alpha_-, \alpha_+\}$$

where

$$\alpha_{\pm} = \frac{n-2}{2} \pm \sqrt{\gamma_H(\mathbb{R}^{k+,n-k}) - \gamma}.$$

Note that $U_{\alpha_-} \in D^{1,2}(\mathbb{R}^{k+,n-k})_{loc,0}$. It is the model for more general equations:

Theorem 3.1 (Felli-Ferrero). *(Optimal regularity) Let Ω be a domain of \mathbb{R}^n with a boundary singularity of type $(k, n-k)$ at 0. We fix $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$. We let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$|f(x, v)| \leq c|v| \left(1 + \frac{|v|^{2^*(s)-2}}{|x|^s} \right) \text{ for all } x \in \Omega, v \in \mathbb{R}.$$

Let $u \in D^{1,2}(\mathbb{R}^{k+,n-k})_{loc,0}$, $u > 0$ be a weak solution to

$$-\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = f(x, u) \text{ in } D^{1,2}(\Omega)_{loc,0}$$

for some $\tau > 0$. Then there exists $K > 0$ such that

$$(25) \quad \lambda^\alpha u(\lambda x) \rightarrow K \left(\prod_{i=1}^k x_i \right) |x|^{-\alpha-k} \text{ in } B_1(0) \cap \mathbb{R}^{k+,n-k},$$

uniformly in C^1 as $\lambda \rightarrow 0$.

This result is essentially in Felli-Ferrero [10]. Applying Theorem 1.1 of Felli-Ferrero [10] to $u \in D^{1,2}(\Omega)$, and since $u > 0$, we get that

$$\lambda^{\frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} + \mu}} u(\lambda x) \rightarrow |x|^{-\frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} + \mu}} \psi \left(\frac{x}{|x|} \right) \text{ as } \lambda \rightarrow 0^+,$$

where μ is an eigenvalue of $L_\gamma := -\Delta_{\mathbb{S}^{n-1}} - \gamma$ on $\mathbb{S}^{n-1} \cap \mathbb{R}^{k+,n-k}$ with Dirichlet boundary condition and $\psi : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ is a nontrivial associated eigenfunction. Since $u > 0$, then $\psi \geq 0$, and then $\psi > 0$, so $\mu = k(n+k-2) - \gamma$ is the first eigenfunction and there exists $K > 0$ such that $\psi(x) = K \prod_{i=1}^k x_i$. This yields (25).

Lemma 3.1. *Assume the $u \in D^{1,2}(\mathbb{R}^{k+,n-k})_{loc,0}$ is a weak solution of*

$$(26) \quad \begin{cases} -\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = 0 & \text{in } D^{1,2}(\mathbb{R}^{k+,n-k})_{loc,0}, \\ u = 0 & \text{on } B_{2\delta} \cap \partial\mathbb{R}^{k+,n-k}, \end{cases}$$

for some $\tau > 0$. Assume there exists $c > 0$ such that

$$(27) \quad |u(x)| \leq c|x|^{-\alpha} \text{ for } x \rightarrow 0, x \in \mathbb{R}^{k+,n-k}.$$

(1) Then, there exists $c_1 > 0$ such that

$$|\nabla u(x)| \leq c_1 |x|^{-\alpha-1} \text{ as } x \rightarrow 0, x \in \mathbb{R}^{k+,n-k}.$$

(2) If $\lim_{x \rightarrow 0} |x|^\alpha u(x) = 0$, then $\lim_{x \rightarrow 0} |x|^{\alpha+1} |\nabla u(x)| = 0$.

Proof. For any $X \in \mathbb{R}^{k+,n-k}$, let $(X_j)_j \in \mathbb{R}^{k+,n-k}$ be such that $\lim X_j = 0$ as $j \rightarrow +\infty$. Take $r_j = |X_j|$ and $\theta_j := \frac{X_j}{|X_j|}$, we have $\lim_{j \rightarrow +\infty} r_j = 0$. Define

$$\tilde{u}_j(X) := r_j^\alpha u(r_j X) \text{ for all } j, X \in (B_R(0) \cap \mathbb{R}^{k+,n-k}) \setminus \{0\}.$$

Since u is a solution of the equation (26), we get

$$\begin{cases} -\Delta \tilde{u}_j - \frac{\gamma + o(1)}{|X|^2} \tilde{u}_j = 0 & \text{in } B_R(0) \cap \mathbb{R}^{k+,n-k}, \\ \tilde{u}_j = 0 & \text{in } B_R(0) \cap \partial\mathbb{R}^{k+,n-k}. \end{cases}$$

Here, $o(1) \rightarrow 0$ in $C_{loc}^0(\overline{\mathbb{R}^{k+,n-k}} \setminus \{0\})$. Since $\lim_{j \rightarrow +\infty} X_j = 0$ and by (27), we get that $|\tilde{u}_j(X)| \leq c|X|^{-\alpha}$ for all $X \in B_R(0) \cap \mathbb{R}^{k+,n-k}$ and all $j \in \mathbb{N}$. It follows from elliptic theory, that there exists $\tilde{u} \in C^2(\overline{\mathbb{R}^{k+,n-k}} \setminus \{0\})$ such that $\tilde{u}_j \rightarrow \tilde{u}$ in $C_{loc}^1(\overline{\mathbb{R}^{k+,n-k}} \setminus \{0\})$. Take $\theta := \lim_{j \rightarrow +\infty} \theta_j$ with $|\theta| = 1$, we have that

$$(28) \quad \lim_{j \rightarrow +\infty} |x_j|^{\alpha+1} \partial_m u(x_j) = \partial_m \tilde{u}(\theta) \text{ for all } m = 1, \dots, n.$$

We assume that there exists $(x_j)_j \in \mathbb{R}^{k+,n-k}$ such that $x_j \rightarrow 0$ and $|x_j|^{\alpha+1} |\nabla u(x_j)| \rightarrow +\infty$ as $j \rightarrow +\infty$. Take $\theta_j = \frac{x_j}{|x_j|}$ and we have $\lim_{j \rightarrow +\infty} |\nabla \tilde{u}_j(\theta_j)| = +\infty$ contradiction with (28). The case when $\lim_{x \rightarrow 0} |x|^\alpha u(x) = 0$ goes similarly. \square

4. SYMMETRY OF THE EXTREMALS FOR $\mu_{\gamma,s}(\mathbb{R}^{k+,n-k})$

In this section we present the symmetry of the extremals for $\mu_{\gamma,s}(\mathbb{R}^{k+,n-k})$. The proof of the symmetry carried out by Ghoussoub-Robert [16] in half space $\{x_1 > 0\}$. For $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$, $s \in [0, 2)$, we consider the problem:

$$(29) \quad \begin{cases} -\Delta u - \frac{\gamma}{|x|^2} u = \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \mathbb{R}^{k+,n-k}, \\ u \geq 0 & \text{in } \mathbb{R}^{k+,n-k}, \\ u = 0 & \text{on } \partial\mathbb{R}^{k+,n-k}. \end{cases}$$

Theorem 4.1. *For $\gamma \geq 0$ and if u it is solution of the equation (29) in $C^2(\mathbb{R}^{k+,n-k}) \cap C(\overline{\mathbb{R}^{k+,n-k}} \setminus \{0\})$ for all $k \in \{1, \dots, n\}$, then $u \circ \sigma = u$ for all isometries of \mathbb{R}^n such that $\sigma(\mathbb{R}^{k+,n-k}) = \mathbb{R}^{k+,n-k}$. In particular:*

- *There exists $w \in C^\infty(]0, \infty[^k \times \mathbb{R}^{n-k})$ such that for all $x_1, \dots, x_k > 0$ and for any $x' \in \mathbb{R}^{n-k}$, we get that $u(x_1, \dots, x_k, x') = w(x_1, \dots, x_k, |x'|)$.*
- *u is a symmetric function of k variables: for all permutation s of the set of indices $\{1, \dots, k\}$, we have $u(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = u(x_{s(1)}, \dots, x_{s(k)}, x_{k+1}, \dots, x_n)$.*

We prove the theorem. We proceed as in Berestycki-Nirenberg [4] (see Ghoussoub-Robert[14] and Fraenkel[11]). We write for convenience $p := 2^*(s) - 1$. We define

$$F := B_{\frac{1}{2}}\left(\frac{1}{2}\vec{e}_1\right) \cap \mathbb{R}^{k+,n-k} \text{ and } v(x) := |x|^{2-n} u\left(-\vec{e}_1 + \frac{x}{|x|^2}\right) \text{ for all } x \in \overline{F} \setminus \{0\},$$

with $v(0) = 0$ and $\vec{e}_1 := (1, 0, \dots, 0)$. Clearly, this is well defined. We have $\partial F = F_1 \cup F_2$ where

$$F_1 := \partial B_{\frac{1}{2}}\left(\frac{1}{2}\vec{e}_1\right) \cap \mathbb{R}^{k+,n-k} \text{ and } F_2 := \cup_{j=2}^k \left(B_{\frac{1}{2}}\left(\frac{1}{2}\vec{e}_1\right) \cap \{x_j = 0\} \right).$$

If $x \in F_1$, then $|x|^2 = x_1$, we have $v(x) = 0$ or if $x \in F_2$, then $v(x) = 0$. Consequently, $v(x) = 0$ for all $x \in \partial F \setminus \{0\}$. We have that $\vec{e}_1 \in \partial F$. Since $|x - |x|^2 \vec{e}_1| = |x||x - \vec{e}_1|$, we have that

$$(30) \quad -\Delta v = \frac{\gamma}{|x|^2 |x - \vec{e}_1|^2} v + \frac{v^p}{|x|^s |x - \vec{e}_1|^s} \text{ in } F.$$

It follows from the assumptions on u that $v \in C^2(F) \cap C(\overline{F} \setminus \{0, \vec{e}_1\})$.

We claim that:

$$(31) \quad v(x'', -x_n) = v(x'', x_n) \text{ for all } x \in F,$$

where $x'' := (x_1, \dots, x_{n-1})$. Theorem 4.1 will be mostly a consequence of this claim.

Proof of (31). For $\lambda \in \mathbb{R}$ we define

$$\begin{aligned} T_\lambda &:= \{x \in \mathbb{R}^n; x_n = \lambda\} & ; & \quad x_\lambda := (x'', 2\lambda - x_n). \\ Z(\lambda) &:= \{x \in F; x_n < \lambda\} & ; & \quad Y(\lambda) := \{x \in \mathbb{R}^n; x_\lambda \in Z(\lambda)\}. \end{aligned}$$

Let $-a := \inf_{x \in F} x_n$, so that $Z(\lambda)$ is empty if and only if $\lambda \leq -a$. Since

$$(32) \quad |x_\lambda|^2 - |x|^2 = 4\lambda(\lambda - x_n),$$

we obtain that $Y(\lambda) \subset F$ if $\lambda \leq 0$. We adapt the moving-plane method. Take $-a < \lambda < 0$ and define

$$g_\lambda(x) := v(x_\lambda) - v(x) \text{ for all } x \in Z(\lambda).$$

We claim that

$$(33) \quad v(x_\lambda) > v(x) \text{ for } \lambda \in (-a, 0) \text{ and } x \in Z(\lambda).$$

We prove the claim (33). Since, $\lambda < 0$, (32) yields $\{x \in Z(\lambda) \Rightarrow x_\lambda \in F\}$. Since

$$|x_\lambda - |x_\lambda|^2 \vec{e}_1|^2 - |x - |x|^2 \vec{e}_1|^2 = (|x_\lambda|^2 - |x|^2) [1 + |x_\lambda|^2 + |x|^2 - 2x_1],$$

for all $x \in \mathbb{R}^n$, $\lambda < 0$ and by (32), we obtain that

$$(34) \quad |x_\lambda - |x_\lambda|^2 \vec{e}_1|^2 - |x - |x|^2 \vec{e}_1|^2 < 0 \text{ in } Z(\lambda).$$

We define

$$c_\lambda(x) := \begin{cases} \frac{v(x_\lambda)^p - v(x)^p}{v(x_\lambda) - v(x)} & \text{if } v(x_\lambda) \neq v(x). \\ pv^{p-1}(x) & \text{if } v(x_\lambda) = v(x). \end{cases}$$

The equation (30) of v , $\gamma \geq 0$ and (34) yield

$$\begin{aligned} -\Delta g_\lambda &= \gamma \left[\frac{v(x)}{|x - |x|^2 \vec{e}_1|^2} - \frac{v(x_\lambda)}{|x_\lambda - |x_\lambda|^2 \vec{e}_1|^2} \right] + \left[\frac{v(x)^p}{|x - |x|^2 \vec{e}_1|^s} - \frac{v(x_\lambda)^p}{|x_\lambda - |x_\lambda|^2 \vec{e}_1|^s} \right] \\ &< -\gamma \frac{g_\lambda}{|x - |x|^2 \vec{e}_1|^2} - c_\lambda(x) \frac{g_\lambda}{|x - |x|^2 \vec{e}_1|^s}, \end{aligned}$$

then,

$$(35) \quad -\Delta g_\lambda + d_\lambda g_\lambda < 0 \text{ in } Z(\lambda),$$

where $d_\lambda(x) := \gamma|x - |x|^2 \vec{e}_1|^{-2} + c_\lambda(x)|x - |x|^2 \vec{e}_1|^{-s}$. We have $Z(\lambda) = F \cap \{x \in \mathbb{R}^n, x_n < \lambda\}$, this gives that $\partial Z(\lambda) \subset \partial F \cup T_\lambda$. Therefore,

$$(36) \quad g_\lambda(x) \geq 0 \text{ if } x \in \partial Z(\lambda),$$

with the strict inequality when $x \in \partial Z(\lambda) \setminus T_\lambda$ and $x_\lambda \in F$ and with equality when $x \in \partial Z(\lambda) \cap T_\lambda$. Again, $g_\lambda(x) = 0$ if x, x_λ in $\partial F \setminus T_\lambda$.

Step 1: We prove (33) for $\lambda + a > 0$ close to 0. Since $x \in Z(\lambda)$, we have $x \in F$ and $x_n < \lambda$. But $\lambda < 0$ thus $x \notin \{0, \vec{e}_1\}$. On the other hand, we have $0 < |x| < 1$ and

$$(37) \quad |d_\lambda(x)| \leq \frac{\gamma}{|x|^2 ||x| - 1|^2} + \left| \frac{c_\lambda(x)}{|x|^s ||x| - 1|^s} \right|.$$

But $v \in C(\overline{F} \setminus \{0, \vec{e}_1\})$, then is a $c_0 > 0$ such that $0 \leq v(x) \leq c_0 \text{ sur } \overline{F} \setminus \{0, \vec{e}_1\}$. The definition of $c_\lambda(x)$ and (37), then there exists $c_1 > 0$ such that $|d_\lambda(x)| \leq c_1$ for all $x \in Z(\lambda)$ and $\lambda < 0$. Next, g_λ verifies (35). For any $\delta > 0$, if $\lambda \in (-a, 0)$ is close to $-a$, then $|Z(\lambda)| \leq \delta$. It follows from Theorem 4.2 that for λ close to $-a$, we have

$$g_\lambda(x) \geq 0 \text{ for } x \in \overline{Z(\lambda)}.$$

We now prove (33) for $x \in Z(\lambda)$. Here again, for any $\delta > 0$, then $|Z(\lambda)| \leq \delta$ for $\lambda \in (-a, 0)$ close to $-a$. Moreover, $Z(\lambda)$ is bounded and g_λ verifies (35). The Maximum principle (Theorem 4.2 below) yields $g_\lambda > 0$ in $Z(\lambda)$ or $g_\lambda \equiv 0$. We assume by contradiction that $g_\lambda \equiv 0$. We fix $x \in \partial F \cap \{x \in \mathbb{R}^n, x_n < \lambda\}$ such that $v(x) = 0$. The definition of g_λ yields $v(x_\lambda) = 0$ and in addition $x_\lambda \in \partial F$. Equation (32) ($4\lambda(\lambda - x_n) = 0$) yields $\lambda = 0$: contradiction with $-a < \lambda < 0$. This yields (33) and Step 1 is proved.

We let $(-a, \beta)$ be the largest open interval in $(-\infty, 0)$ such that

$$g_\lambda > 0 \text{ in } Z(\lambda) \text{ for all } \lambda \in (-a, \beta).$$

Step 2: We claim that $\beta = 0$.

We prove the claim. We assume $\beta < 0$ and we argue by contradiction. Since $g_\lambda(x)$ for all $x \in Z(\lambda)$ and all $\lambda \in (-a, \beta)$, letting $\lambda \rightarrow \beta$, we get that $g_\beta \geq 0$ for $x \in Z(\beta)$. As in the proof of Step 1, the case $g_\beta \equiv 0$ is discarded and the maximum principle yields $g_\beta(x) > 0$ for all $x \in Z(\beta)$.

We fix $\delta > 0$ that will be precised later. We let $D \subset Z(\beta)$ be a smooth domain such that $|Z(\beta) \setminus D| < \frac{\delta}{2}$. Thus $g_\beta(x) > 0$ when $x \in \overline{D}$. For $0 < \epsilon \leq \epsilon_0$, we define $G_\epsilon := Z(\beta + \epsilon) \setminus D$. We let $\epsilon_0 > 0$ small enough such that, for any $\epsilon \in (0, \epsilon_0)$, we have that $|G_\epsilon| < \delta$, $\beta + \epsilon < 0$, and $g_{\beta+\epsilon} > 0$ in D . Equation (35) yields,

$$-\Delta g_{\beta+\epsilon} + d_{\beta+\epsilon} g_{\beta+\epsilon} < 0 \text{ in } G_\epsilon.$$

With (36) and $g_\beta > 0$ in D , we get that $g_{\beta+\epsilon} \geq 0$ on ∂G_ϵ . Then, up to taking $\delta > 0$ small enough, by Theorem 4.2 below, we get $g_{\beta+\epsilon} \geq 0$ for $x \in \overline{G_\epsilon}$. As above, the strong maximum principle yields $g_{\beta+\epsilon} > 0$ for $x \in G_\epsilon$. Consequently, $g_{\beta+\epsilon} > 0$ in $Z(\beta + \epsilon)$. This contradicts the maximality of β . Then $\beta = 0$ and $g_\lambda(x) > 0$ for $\lambda \in (-a, 0)$ and $x \in Z(\lambda)$. This proves (33).

Step 3: Letting $\lambda \rightarrow 0$ in (33), we get that $v(x'', -x_n) \geq v(x'', x_n)$ for all $x \in F$ such that $x_n \leq 0$. By symmetry, we get the reverse inequality. This proves (31).

Proof of the first part of Theorem 4.1: Permuting x_n and any x_j , $j \in \{k+1, \dots, n\}$, it follows from (31) that v is symmetric with respect the hyperplane $\{x_j = 0\}$. Coming back to the definition of u , we get the desired symmetry.

Proof of the second part of Theorem 4.1. As above, this will be a consequence of a claim. We claim that

$$(38) \quad u(x_1, x_2, x') = u(x_2, x_1, x') \text{ in } \mathbb{R}^{k+, n-k}.$$

Proof of (38). We define $E'_{+k} := \{x \in \mathbb{R}^{k+, n-k} ; x_1 - x_2 > 0\} := D'_1 \cap D'_2 \cap (\cap_{i=1}^k D'_i)$ where

$$D'_1 := \{x_1 + x_2 > 0\}, D'_2 := \{x_1 - x_2 > 0\} \text{ et } D'_i := \{x_i > 0\}.$$

We consider the isometry $\sigma(x) := (\frac{x_1+x_2}{\sqrt{2}}, \frac{x_1-x_2}{\sqrt{2}}, x')$ for $x := (x_1, x_2, x') \in \mathbb{R} \times \mathbb{R} \times (\mathbb{R}_+^{k-2} \times \mathbb{R}^{n-k})$. We have that $\sigma(E'_{+k}) = \mathbb{R}^{k+, n-k}$. We define $v(x) := u \circ \sigma(x)$ for all $x \in E'_{+k}$. Equation (29) of u , the isometry σ and the definition of v yield

$$(39) \quad -\Delta v - \frac{\gamma}{|x|^2} v = \frac{v^p}{|x|^s} \text{ in } E'_{+k}.$$

For any $x \in \mathbb{R}^n \setminus \{0\}$, we define the inversion $i(x) = -\vec{e}_1 + \frac{x}{|x|^2}$. We note that: $i^{-1}(D'_i) = D'_i$, and then

$$x \in i^{-1}(D'_1) \Leftrightarrow x \in B_{\frac{1}{\sqrt{2}}} \left(\frac{1}{2}(\vec{e}_1 + \vec{e}_2) \right); \quad x \in i^{-1}(D'_2) \Leftrightarrow x \in B_{\frac{1}{\sqrt{2}}} \left(\frac{1}{2}(\vec{e}_1 - \vec{e}_2) \right).$$

We define $\hat{v}(x) := |x|^{2-n} v(i(x))$ for all $x \in H := i^{-1}(E'_{+k})$, where $v(0) = 0$ and $0, \vec{e}_1 \in \partial H$. Since v verifies (39) and by the definition of \hat{v} , we obtain that

$$-\Delta \hat{v} = \frac{\gamma}{|x|^2 |x - \vec{e}_1|^2} \hat{v} + \frac{\hat{v}^p}{|x|^s |x - \vec{e}_1|^s}.$$

We denote that $\hat{v} \in C^2(H) \cap C(\overline{H} \setminus \{0, \vec{e}_1\})$. Arguing as in the proof of (31), we get that $\hat{v}(x_1, x_2, x') = \hat{v}(x_1, -x_2, x')$ for all $x \in H$. Coming back to v , and then u , we get (38). As noted above, this yields the second part of Theorem 4.1.

Theorem 4.2 (Maximum Principle for small domains). *Let $\Omega \subset \mathbb{R}^n$ be open domain and $a \in L^\infty(\Omega)$ such that $\|a\|_\infty \leq M$. Then there exists $\delta(M, n) > 0$ such that we have the following: if $|\Omega| < \delta$ and $u \in H^1(\Omega)$ satisfies $-\Delta u + au \geq 0$ weakly and $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω .*

Proof. This result is cited in Berestycki-Nirenberg [4] and Fraenkel [11]. We give a short independent proof. Since $-\Delta u + au \geq 0$ weakly, we have that

$$\int_{\Omega} ((\nabla u, \nabla \varphi) + au\varphi) dx \geq 0 \text{ for all } \varphi \in H_0^1(\Omega), \varphi \geq 0.$$

We take $\varphi := u_- := \max\{0, -u\} \in H_0^1(\Omega)$. Since $\nabla u_- = -\mathbf{1}_{u < 0} \nabla u$ a.e, we get

$$\int_{\Omega} (|\nabla u_-|^2 + au_-^2) dx \leq 0.$$

Since $u_-^2 \in L^{\frac{2^*}{2}}(\Omega)$, Hölder's inequality yields

$$(40) \quad \int_{\Omega} |\nabla u_-|^2 dx \leq \|a\|_\infty \text{mes}(\Omega)^{\frac{2}{n}} \|u_-\|_{2^*}^2 \leq \|a\|_\infty \delta^{\frac{2}{n}} \|u_-\|_{2^*}^2.$$

On the other hand, it follows from Sobolev's inequality that $\mu_{0,0}(\mathbb{R}^n) \|u_-\|_{2^*}^2 \leq \|\nabla u_-\|_2^2$. With (40) and $\delta := [\mu_{0,0}(\mathbb{R}^n)^{-1} \|a\|_\infty]^{\frac{n}{2}}$, we obtain $\|u_-\|_{2^*}^2 = 0$. Therefore $u \geq 0$ in Ω . \square

5. EXISTENCE OF EXTREMALS: THE CASE OF SMALL VALUES OF γ

We estimate the functional $J_{\gamma,s}^\Omega$ at some natural test-functions. We let $W \in D^{1,2}(\mathbb{R}^{k+,n-k})$ be a positive extremal for $\mu_{\gamma,s}(\mathbb{R}^{k+,n-k})$. In other words,

$$J_{\gamma,s}^{\mathbb{R}^{k+,n-k}}(W) = \frac{\int_{\mathbb{R}^{k+,n-k}} \left(|\nabla W|^2 - \frac{\gamma}{|x|^2} W^2 \right) dx}{\left(\int_{\mathbb{R}^{k+,n-k}} \frac{|W|^{2^*(s)}}{|x|^s} \right)^{\frac{2}{2^*(s)}}} = \mu_{\gamma,s}(\mathbb{R}^{k+,n-k}).$$

Therefore, there exists $\xi > 0$ such that

$$(41) \quad \begin{cases} -\Delta W - \frac{\gamma}{|x|^2} W = \xi \frac{W^{2^*(s)-1}}{|x|^s} & \text{in } \mathbb{R}^{k+,n-k}, \\ W > 0 & \text{in } \mathbb{R}^{k+,n-k}, \\ W = 0 & \text{on } \partial\mathbb{R}^{k+,n-k}. \end{cases}$$

They exist under the assumption that $s > 0$ or $\{s = 0, \gamma > 0 \text{ and } n \geq 4\}$ (see Ghoussoub-Robert [16]). By Theorem 3.1, there exists $c > 0$ such that

$$(42) \quad W(x) \leq c|x|^{-\alpha_-} \text{ as } x \rightarrow 0.$$

It follows from Lemma 3.1, that there exists $c > 0$ such that

$$(43) \quad |\nabla W(x)| \leq c|x|^{-1-\alpha_-} \text{ as } x \rightarrow 0.$$

Define now the Kelvin transform $\bar{W}(x) := |x|^{2-n} W(\frac{x}{|x|^2})$, since W satisfies (41), then \bar{W} also satisfies (41). By (42), (43) and the definition of \bar{W} we get,

$$(44) \quad W(x) \leq c|x|^{-\alpha_+} \text{ and } |\nabla W(x)| \leq c|x|^{-1-\alpha_+} \text{ as } |x| \rightarrow +\infty.$$

For $r > 0$, we define $\tilde{B}_r := (-r, r)^k \times B_r^{n-k}(0)$, where $B_r^{n-k}(0)$ is the ball of center 0 and radius r in \mathbb{R}^{n-k} . We take the chart (ϕ, U) of Definition 1 so that

$$\phi(\tilde{B}_{3\delta} \cap \mathbb{R}^{k+,n-k}) = \phi(\tilde{B}_{3\delta}) \cap \Omega \text{ and } \phi(\tilde{B}_{3\delta} \cap \partial\mathbb{R}^{k+,n-k}) = \phi(\tilde{B}_{3\delta}) \cap \partial\Omega,$$

where $\delta > 0$. We write the chart $\phi = (\phi^1, \phi^2, \dots, \phi^n)$ and the pull-back metric $g_{ij}(x) := (\phi_{Eucl}^*(x))_{ij} = (\partial_i \phi(x), \partial_j \phi(x))$ for all $i, j = 1, \dots, n$. The Taylor formula of $g_{ij}(x)$ around 0 writes

$$(45) \quad g_{ij}(x) = \delta_{ij} + H_{ij} + O(|x|^2) \text{ with } H_{ij} := \sum_{l=1}^n [\partial_{il} \phi^j(0) + \partial_{jl} \phi^i(0)] x_l.$$

As $x \rightarrow 0$, the inverse metric $g^{-1} = (g^{ij})$ expands as $g^{-1} = Id_n - (H_{ij})_{1 \leq i, j \leq n} + O(|x|^2)$, and the volume element is

$$(46) \quad |Jac(\phi)(x)| = 1 + \sum_{i,j=1}^n \partial_{ji} \phi^j(0) x_i + O(|x|^2),$$

as $x \rightarrow 0$. For any $\epsilon > 0$, we define

$$(47) \quad W_\epsilon(x) := \left(\eta \epsilon^{-\frac{n-2}{2}} W(\epsilon^{-1} \cdot) \right) \circ \phi^{-1}(x) \text{ for all } x \in \phi(\tilde{B}_{3\delta}) \cap \Omega \text{ and } 0 \text{ elsewhere,}$$

where $\eta \in C_c^\infty(\mathbb{R}^n)$ is such that $\eta(x) = 1$ for $x \in \tilde{B}_\delta(0)$ and $\eta(x) = 0$ for $x \notin \tilde{B}_{2\delta}(0)$. Theorem 1.2 will be the consequence of the following estimates:

Proposition 5.1. *Let $0 \leq \gamma < \gamma_H(\mathbb{R}^{k+,n-k}) = \frac{(n+2k-2)^2}{4}$, and assume that there are extremals for $\mu_{\gamma,s}(\mathbb{R}^{k+,n-k})$. Then there exists $c_{\gamma,s}^\beta$ positives constants where $\beta = 1, \dots, 3$ and for all $k \in \{1, \dots, n\}$ and $m = 1, \dots, k$ such that:*

- (1) *For $\gamma < \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$, we have that*

$$(48) \quad J_{\gamma,s}^\Omega(W_\epsilon) = \mu_{\gamma,s}(\mathbb{R}^{k+,n-k}) (1 + GH_{\gamma,s}(\Omega)\epsilon + o(\epsilon)).$$

(2) For $\gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$, we have that

$$(49) \quad J_{\gamma,s}^\Omega(W_\epsilon) = \mu_{\gamma,s}(\mathbb{R}^{k+,n-k}) \left(1 + GH_{\gamma,s}(\Omega) \epsilon \ln \left(\frac{1}{\epsilon} \right) + o \left(\epsilon \ln \left(\frac{1}{\epsilon} \right) \right) \right).$$

with $GH_{\gamma,s}(\Omega)$ as in (7).

Proof. Take $\tilde{B}_{\delta,+k} := \tilde{B}_\delta \cap \mathbb{R}^{k+,n-k}$. For any family $(a_\epsilon)_{\epsilon>0} \in \mathbb{R}$, we define

$$\Theta_\gamma(a_\epsilon) := \begin{cases} o(a_\epsilon) & \text{if } \gamma < \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}, \\ O(a_\epsilon) & \text{if } \gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}. \end{cases} \quad \text{as } \epsilon \rightarrow 0.$$

In order to get lighter computations, we take the following conventions: the integral symbol \int means $\int_{\tilde{B}_{\epsilon^{-1}\delta,+k}}$, and $A_\epsilon^\alpha := \tilde{B}_{\epsilon^{-1}\delta} \cap \{x_\alpha = 0\}$.

Step 1: We claim that

$$\begin{aligned} \int_\Omega |\nabla W_\epsilon|^2 dx &= \int_{\mathbb{R}^{k+,n-k}} |\nabla W|^2 dx + \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int |\nabla W|^2 x_i dx \\ &- 2\epsilon \sum_{m=1}^k (A_{1,m} + A_{2,m} + \partial_{mm} \phi^m(0) \int \partial_m W \partial_m W x_m dx \\ &+ \sum_{i \geq 1; i \neq m} [\partial_{mi} \phi^i(0) \int \partial_i W \partial_m W x_i dx + \partial_{im} \phi^i(0) \int \partial_i W \partial_i W x_m dx] \\ &+ \sum_{p=1; p \neq m} \sum_{q=p+1; q \neq m}^k [\partial_{qp} \phi^m(0) \int \partial_m W \partial_q W x_p dx + \partial_{pq} \phi^m(0) \int \partial_m W \partial_p W x_q dx]) + \Theta_\gamma(\epsilon) \text{ as } \epsilon \rightarrow 0 \end{aligned}$$

where or $m = 1, \dots, k$, we define $x_{0,m} := (x_1, \dots, 0^m, \dots, x_k, x_{k+1}, \dots, x_n)$ and

$$\begin{aligned} A_{1,m} &:= \sum_{i=1; i \neq m}^k \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W x_i \partial_i W dx. \\ A_{2,m} &:= \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W x_i \partial_i W dx. \\ B_{1,m} &:= \sum_{i \geq 1; i \neq m}^k \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_m}{|x|^2} x_i^2 dx. \\ B_{2,m} &:= \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_m}{|x|^2} x_i^2 dx. \end{aligned}$$

Note that

$$(50) \quad \left\{ \begin{array}{l} \alpha_+ - \alpha_- > 1 \Leftrightarrow \gamma < \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4} \\ \alpha_+ - \alpha_- = 1 \Leftrightarrow \gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4} \end{array} \right\}$$

Proof of Step 1: By (43) and (44), there exists $c_1 > 0$ such that

$$(51) \quad |\nabla W_\epsilon(x)| \leq c_1 \epsilon^{\alpha_+ - \frac{n-2}{2}} |x|^{-1-\alpha_+} \text{ for any } x \in \Omega.$$

Therefore,

$$\int_{\phi((\tilde{B}_{3\delta} \setminus \tilde{B}_\delta) \cap \mathbb{R}^{k+,n-k})} |\nabla W_\epsilon|^2 dx \leq c_1^2 \epsilon^{2\alpha_+ - n + 2} \int_{\phi((\tilde{B}_{3\delta} \setminus \tilde{B}_\delta) \cap \mathbb{R}^{k+,n-k})} |x|^{-2-2\alpha_+} dx$$

since $2\alpha_+ - n + 2 = \alpha_+ - \alpha_-$, we get that

$$\int_{\phi((\tilde{B}_{3\delta} \setminus \tilde{B}_\delta) \cap \mathbb{R}^{k+,n-k})} |\nabla W_\epsilon|^2 dx = \Theta_\gamma(\epsilon) \text{ as } \epsilon \rightarrow 0.$$

Then,

$$\int_{\Omega} |\nabla W_{\epsilon}|^2 dx = \int_{\tilde{B}_{\delta,+k}} |\nabla(W_{\epsilon} \circ \phi)|_{\phi^* Eucl}^2 |Jac(\phi)| dx + \Theta_{\gamma}(\epsilon) \text{ as } \epsilon \rightarrow 0.$$

It follows from (45) and for any $\theta \in (0, 1]$ that

$$\begin{aligned} \int_{\Omega} |\nabla W_{\epsilon}|^2 dx &= \int |\nabla(W_{\epsilon} \circ \phi)|_{Eucl}^2 |Jac(\phi)| dx - \sum_{i,j \geq 1} \int H_{ij} \partial_i(W_{\epsilon} \circ \phi) \partial_j(W_{\epsilon} \circ \phi) |Jac(\phi)| dx \\ &\quad + O\left(\int_{\tilde{B}_{\delta,+k}} |x|^{1+\theta} |\nabla(W_{\epsilon} \circ \phi)|^2 dx\right) + \Theta_{\gamma}(\epsilon) \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

Using (45), we get $\sum_{i,j \geq 1} H_{ij} = 2 \sum_{i,j,l \geq 1} \partial_{il} \phi^j(0) x_l$, and then

$$(52) \quad \int_{\Omega} |\nabla W_{\epsilon}|^2 dx = \int_{\tilde{B}_{\delta,+k}} |\nabla(W_{\epsilon} \circ \phi)|_{Eucl}^2 |Jac(\phi)| dx - 2 \sum_{i,j,l \geq 1} \partial_{ij} \phi^l(0) \int_{\tilde{B}_{\delta,+k}} \partial_l(W_{\epsilon} \circ \phi) \partial_i(W_{\epsilon} \circ \phi) x_j |Jac(\phi)| dx + O\left(\int_{\tilde{B}_{\delta,+k}} |x|^{1+\theta} |\nabla(W_{\epsilon} \circ \phi)|^2 dx\right) + \Theta_{\gamma}(\epsilon)$$

as $\epsilon \rightarrow 0$. The two equations (46), (47) and the change of variable $x := \epsilon y$ yield as $\epsilon \rightarrow 0$,

$$(53) \quad \begin{aligned} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} |\nabla(W_{\epsilon} \circ \phi)|_{Eucl}^2 |Jac(\phi)| dx &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} |\nabla W|^2 dx + \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} |\nabla W|^2 x_i dx \\ &+ \epsilon \sum_{k+1 \leq i \leq n; j \geq 1} \partial_{ji} \phi^j(0) \int |\nabla W|^2 x_i dx + O\left(\int |x|^2 |\nabla(W_{\epsilon} \circ \phi)|^2 dx\right), \end{aligned}$$

and

$$(54) \quad \int_{\tilde{B}_{\delta,+k}} \partial_l(W_{\epsilon} \circ \phi) \partial_i(W_{\epsilon} \circ \phi) x_j |Jac(\phi)| dx = \epsilon \int \partial_l W \partial_i W x_j dx + O\left(\int |x|^2 |\nabla(W_{\epsilon} \circ \phi)|^2 dx\right).$$

Plugging together (52), (53), (54) yields

$$\begin{aligned} \int_{\Omega} |\nabla W_{\epsilon}|^2 dx &= \int |\nabla W|^2 dx + \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int |\nabla W|^2 x_i dx \\ &+ \epsilon \sum_{k+1 \leq i \leq n; j \geq 1} \partial_{ji} \phi^j(0) \int |\nabla W|^2 x_i dx - 2\epsilon \sum_{i,j,l \geq 1} \partial_{ij} \phi^l(0) \int \partial_l W \partial_i W x_j dx \\ &\quad + O\left(\int |x|^{1+\theta} |\nabla(W_{\epsilon} \circ \phi)|^2 dx\right) + \Theta_{\gamma}(\epsilon) \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

- If $\gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$, we choose $\theta \in (0, 1)$.
- If $\gamma < \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$, we choose $0 < \theta < \alpha_+ - \alpha_- - 1$ (see (50)).

Therefore, it follows from (51) that we have as $\epsilon \rightarrow 0$ that,

$$(55) \quad \int_{\tilde{B}_{\delta,+k}} |x|^{1+\theta} |\nabla(W_{\epsilon} \circ \phi)|^2 dx = \Theta_{\gamma}(\epsilon).$$

Since $\gamma \geq 0$, we use the symmetry of W (see Theorem 4.1). For $i \geq k+1$, W and $\mathbb{R}^{k+,n-k}$ are invariant by $x \rightarrow (x_1, \dots, -x_i, \dots, x_n)$, then a change of variables yields

$$(56) \quad \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} |\nabla W|^2 x_i dx = - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} |\nabla W|^2 x_i dx = 0.$$

This equality and (55) yield

$$(57) \quad \int_{\Omega} |\nabla W_{\epsilon}|^2 dx = \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} |\nabla W|^2 dx + \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} |\nabla W|^2 x_i dx \\ - 2\epsilon \sum_{i, j, l \geq 1} \partial_{ij} \phi^l(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \partial_l W \partial_i W x_j dx + \Theta_{\gamma}(\epsilon) \text{ as } \epsilon \rightarrow 0.$$

The inequation (44) and $-2 - 2\alpha_+ + n = -(\alpha_+ - \alpha_-)$ yields,

$$\left| \int_{\mathbb{R}^{k+, n-k} \setminus \tilde{B}_{\epsilon^{-1}\delta, +k}} |\nabla W|^2 dx \right| \leq c^2 \int_{\mathbb{R}^{k+, n-k} \setminus \tilde{B}_{\delta, +k}} |x|^{-2-2\alpha_+} dx \leq c_1 \epsilon^{\alpha_+ - \alpha_-},$$

therefore,

$$(58) \quad \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} |\nabla W|^2 dx = \int_{\mathbb{R}^{k+, n-k}} |\nabla W|^2 dx + \Theta_{\gamma}(\epsilon) \text{ as } \epsilon \rightarrow 0.$$

Using again the symmetry of W as in (56), we get

$$\sum_{i, j, l \geq 1} \partial_{ij} \phi^l(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \partial_l W \partial_i W x_j dx \\ = \sum_{m=1}^k (A_{1,m} + A_{2,m} + \partial_{mm} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \partial_m W \partial_m W x_m dx \\ + \sum_{i \geq 1; i \neq m} \left[\partial_{mi} \phi^i(0) \int \partial_i W \partial_m W x_i dx + \partial_{im} \phi^i(0) \int \partial_i W \partial_i W x_m dx \right] \\ + \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \left[\partial_{qp} \phi^m(0) \int \partial_m W \partial_q W x_p dx + \partial_{pq} \phi^m(0) \int \partial_m W \partial_p W x_q dx \right]).$$

Combining (57), (58) and the last equation, we get Step 1. \square

Step 2: We fix $\sigma \in [0, 2]$. We claim that

$$\int_{\Omega} \frac{|W_{\epsilon}|^{2^*(\sigma)}}{|x|^{\sigma}} dx = \int_{\mathbb{R}^{k+, n-k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} dx + \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} x_i dx \\ - \epsilon \frac{\sigma}{2} \sum_{m=1}^k \left(B_{1,m} + B_{2,m} + \partial_{mm} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} \frac{x_m}{|x|^2} x_m^2 dx \right. \\ + 2 \sum_{i \geq 1; i \neq m} \partial_{mi} \phi^i(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} \frac{x_i}{|x|^2} x_m x_i dx \\ \left. + 2 \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} \frac{x_m}{|x|^2} x_q x_p dx \right) + \Theta_{\gamma}(\epsilon).$$

Proof of Step 2: Equations (43) and (44) yield

$$(59) \quad |W_{\epsilon}(x)| \leq c \epsilon^{\alpha_+ - \frac{n-2}{2}} |x|^{-\alpha_+} \text{ for all } \epsilon > 0 \text{ and } x \in \Omega,$$

this implies,

$$\left| \int_{\phi(\tilde{B}_{3\delta} \setminus \tilde{B}_{\delta}) \cap \Omega} \frac{|W_{\epsilon}|^{2^*(\sigma)}}{|x|^{\sigma}} dx \right| \leq c^{2^*(\sigma)} \epsilon^{2^*(\sigma)(\alpha_+ - \frac{n-2}{2})} \int_{\phi(\tilde{B}_{3\delta} \setminus \tilde{B}_{\delta}) \cap \Omega} |x|^{-\alpha_+ + 2^*(\sigma) - \sigma} dx$$

and then, since $2^*(\sigma) \geq 2$ and $\alpha_+ + \alpha_- = n - 2$, we get that

$$\int_{\phi(\bar{B}_{3\delta} \setminus \bar{B}_\delta) \cap \Omega} \frac{|W_\epsilon|^{2^*(\sigma)}}{|x|^\sigma} dx = \Theta_\gamma(\epsilon).$$

Therefore,

$$(60) \quad \int_\Omega \frac{|W_\epsilon|^{2^*(\sigma)}}{|x|^\sigma} dx = \int_{\bar{B}_{\delta,+k}} \frac{|W_\epsilon \circ \phi|^{2^*(\sigma)}}{|\phi(x)|^\sigma} |Jac(\phi)| dx + \Theta_\gamma(\epsilon) \text{ as } \epsilon \rightarrow 0.$$

We choose $\theta \in (0, 1)$ as follows.

- If $\gamma < \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ or $\{\gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4} \text{ and } \sigma < 2\}$ we choose $\theta \in (0, (\alpha_+ - \alpha_-) \frac{2^*(\sigma)}{2} - 1) \cap (0, 1)$.
- If $\gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ and $\sigma = 2$, we choose $0 < \theta < 1$.

This choice makes sense due to (50). Since $d\phi_0 = Id$, a Taylor expansion yields

$$(61) \quad |\phi(x)|^{-\sigma} = |x|^{-\sigma} \left[1 - \frac{\sigma}{2|x|^2} \sum_{i,j,l \geq 1} \partial_{ij} \phi^l(0) x_l x_i x_j + O(|x|^{1+\theta}) \right] \text{ as } \epsilon \rightarrow 0.$$

Inequality (59) yields,

$$\int_{\bar{B}_{\delta,+k}} \frac{|W_\epsilon \circ \phi|^{2^*(\sigma)} |x|^{1+\theta}}{|\phi(x)|^\sigma} dx = \Theta_\gamma(\epsilon).$$

The estimates (60), (61) and the last equation get,

$$(62) \quad \int_\Omega \frac{|W_\epsilon|^{2^*(\sigma)}}{|x|^\sigma} dx = \int_{\bar{B}_{\delta,+k}} \frac{|W_\epsilon \circ \phi|^{2^*(\sigma)}}{|x|^\sigma} |Jac(\phi)| dx - \frac{\sigma}{2} \sum_{i,j,l \geq 1} \partial_{ij} \phi^l(0) \int_{\bar{B}_{\delta,+k}} \frac{|W_\epsilon \circ \phi|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_l}{|x|^2} x_i x_j |Jac(\phi)| dx + \Theta_\gamma(\epsilon).$$

In view of (46), (47) and the change of variable $x := \epsilon y$ yield as $\epsilon \rightarrow 0$,

$$(63) \quad \int_{\bar{B}_{\delta,+k}} \frac{|W_\epsilon|^{2^*(\sigma)}}{|x|^\sigma} |Jac(\phi)| dx = \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} dx + \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} x_i dx + \epsilon \sum_{k+1 \leq i \leq n; j \geq 1} \partial_{ji} \phi^j(0) \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} x_i dx + \Theta_\gamma(\epsilon).$$

And,

$$(64) \quad \int_{\bar{B}_{\delta,+k}} \frac{|W_\epsilon|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_l}{|x|^2} x_i x_j |Jac(\phi)| dx = \epsilon \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_l}{|x|^2} x_i x_j dx + \Theta_\gamma(\epsilon).$$

Plugging together (62), (63), (64) yields,

$$\int_\Omega \frac{|W_\epsilon|^{2^*(\sigma)}}{|x|^\sigma} dx = \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} dx + \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} x_i dx + \epsilon \sum_{k+1 \leq i \leq n; j \geq 1} \partial_{ji} \phi^j(0) \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} x_i dx - \frac{\sigma}{2} \sum_{i,j,l \geq 1} \partial_{ij} \phi^l(0) \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_l}{|x|^2} x_i x_j dx + \Theta_\gamma(\epsilon).$$

By equation (44), we have

$$\int_{\mathbb{R}^{k+,n-k} \setminus \bar{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} dx = \Theta_\gamma(\epsilon).$$

Since $\gamma \geq 0$, using the symmetry of W as in (56) and the last equation,

$$(65) \quad \int_{\Omega} \frac{|W_{\epsilon}|^{2^*(\sigma)}}{|x|^{\sigma}} dx = \int_{\mathbb{R}^{k+n-k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} dx + \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} x_i dx \\ - \epsilon \frac{\sigma}{2} \sum_{i, j, l \geq 1} \partial_{ij} \phi^l(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} \frac{x_l}{|x|^2} x_i x_j dx + \Theta_{\gamma}(\epsilon).$$

We use again the symmetry of W ,

$$\sum_{i, j, l \geq 1} \partial_{ij} \phi^l(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} \frac{x_l}{|x|^2} x_i x_j dx = \sum_{m=1}^k (B_{1,m} + B_{2,m} \\ + \partial_{mm} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} \frac{x_m}{|x|^2} x_m^2 dx \\ + \sum_{i \geq 1; i \neq m} [\partial_{mi} \phi^i(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} \frac{x_i}{|x|^2} x_m x_i dx + \partial_{im} \phi^i(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} \frac{x_i}{|x|^2} x_i x_m dx] \\ + \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k [\partial_{qp} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} \frac{x_m}{|x|^2} x_q x_p dx + \partial_{pq} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} \frac{x_m}{|x|^2} x_p x_q dx]).$$

Replace the last equation in (65), we get Step 2. \square

Step 3: We now prove (48) and (49). We fix $m \in \{1, \dots, k\}$. For: $i = 1, \dots, n$; $l = k+1, \dots, n$; $p = 1, \dots, k$ and $q = p+1, \dots, k$ such that $i, p, q \neq m$, we define

$$M_{p,m} := \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \partial_m W x_p \partial_p W dx \quad \text{and} \quad M_{l,m} := \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \partial_m W x_l \partial_l W dx. \\ K_{i,m} := \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \partial_i W \partial_m W x_i dx \quad \text{and} \quad J_{i,m} := \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \partial_i W \partial_i W x_m dx. \\ L_{m,p,q} := \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \partial_m W \partial_p W x_q dx \quad \text{and} \quad N_{m,p,q} := \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \partial_m W \partial_q W x_p dx. \\ I_m := \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \partial_m W \partial_m W x_m dx.$$

Lemma 5.1. Here $\xi > 0$ and $s \in [0, 2]$, we have as $\epsilon \rightarrow 0$ that:

$$2I_m = \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{x_m^2}{|x|^2} x_m \left(\xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right) dx + \xi \left(1 - \frac{2}{2^*(s)} \right) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx + \Theta_{\gamma}(1). \\ 2M_{p,m} = \int_{\tilde{B}_{\epsilon^{-1}\delta, ++}} \frac{x_p^2}{|x|^2} x_m \left(\xi \frac{s}{2^*(s)} \frac{|W|^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right) dx - \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} \frac{x_p^2 |\partial_m W|^2}{2} d\sigma + \Theta_{\gamma}(1). \\ 2M_{l,m} = \int_{\tilde{B}_{\epsilon^{-1}\delta, ++}} \frac{x_l^2}{|x|^2} x_m \left(\xi \frac{s}{2^*(s)} \frac{|W|^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right) dx - \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} \frac{x_l^2 |\partial_m W|^2}{2} d\sigma + \Theta_{\gamma}(1). \\ K_{i,m} + J_{i,m} = \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{x_i^2}{|x|^2} x_m \left(\xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right) dx + \left(\frac{1}{2} - \frac{1}{2^*(s)} \right) \xi \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx + \Theta_{\gamma}(1). \\ L_{m,p,q} + N_{m,p,q} = \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{x_q x_p}{|x|^2} x_m \left(\xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right) dx - \frac{1}{2} \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_q x_p (\partial_m W)^2 dx + \Theta_{\gamma}(1).$$

Proof of Lemma 5.1. We first state two preliminary remarks. First

$$(66) \quad \int_{\partial \tilde{B}_{\epsilon^{-1}\delta} \cap \mathbb{R}^{k+n-k}} (W^2 + |x|W|\nabla W| + |x|^2|\nabla W|^2) dx = \Theta_\gamma(1).$$

Another remark we will use often is that

$$(67) \quad \partial_i W(x) = 0 \text{ if } x_j = 0, j \neq i, j \leq k$$

We want to calculate the value of

$$I_m = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W \partial_m W x_m dx = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} (\partial_m W)^2 \partial_m \left(\frac{x_m^2}{2} \right) dx.$$

For any domain D , we define ν as the outer normal vector at a boundary point of D when this makes sense. For any $j = 1, \dots, n$, ν_j denote the j th coordinate. In the sequel, the normal vector will be defined except on lower dimensional portions of the boundary and the computations will be valid. On $\{x_\alpha = 0\} = \partial\{x_\alpha > 0\}$, the outer normal vector is $(0, \dots, -1, \dots, 0) = (\nu_{\alpha,i})_{i=1, \dots, n}$ where $\nu_{i,j} := -\delta_{ij}$ for $i = 1, \dots, k$ and $j \geq 1$. Since $W(x_{0,m}) = 0$, (66) and integrations by parts yield

$$\begin{aligned} I_m &= - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W \partial_{mm} W dx + \int_{\partial(\tilde{B}_{\epsilon^{-1}\delta,+k})} \frac{x_m^2 (\partial_m W)^2}{2} \nu_m dx \\ &= - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W [\Delta W - \sum_{i \geq 1; i \neq m} \partial_{ii} W] dx + O \left(\int_{\partial \tilde{B}_{\epsilon^{-1}\delta} \cap \mathbb{R}^{k+n-k}} |x|^2 |\nabla W|^2 dx \right) \\ &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx + \sum_{i \geq 1; i \neq m} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W \partial_{ii} W dx + \Theta_\gamma(1) \\ &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx - \sum_{i \geq 1; i \neq m} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_{im} W \partial_i W dx \\ &\quad + \sum_{i \geq 1; i \neq m} \int_{\partial(\tilde{B}_{\epsilon^{-1}\delta,+k})} x_m^2 \partial_m W \partial_i W \nu_i dx + \Theta_\gamma(1) \\ &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx - \sum_{i \geq 1; i \neq m} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m \left(\frac{(\partial_i W)^2}{2} \right) dx \\ &\quad + \sum_{i \geq 1; i \neq m} \sum_{\alpha=1}^k \int_{B_{\epsilon^{-1}\delta} \cap \{x_\alpha=0\}} x_m^2 \partial_m W \partial_i W \nu_{\alpha,i} d\sigma + \Theta_\gamma(1). \end{aligned}$$

Using again the integrations by parts and (66), we get

$$\begin{aligned} I_m &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx - \sum_{i \geq 1; i \neq m} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m \left(\frac{(\partial_i W)^2}{2} \right) dx + \Theta_\gamma(1) \\ &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx + \sum_{i \geq 1; i \neq m} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m (\partial_i W)^2 dx \\ &\quad - \sum_{i \geq 1; i \neq m} \sum_{\alpha=1}^k \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_\alpha=0\}} x_m^2 \frac{(\partial_i W)^2}{2} \nu_{\alpha,m} dx + \Theta_\gamma(1) \\ &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m (|\nabla W|^2 - (\partial_m W)^2) dx + \Theta_\gamma(1) \\ &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m |\nabla W|^2 dx - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m (\partial_m W)^2 dx + \Theta_\gamma(1) \\ &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m |\nabla W|^2 dx - I_m + \Theta_\gamma(1). \end{aligned}$$

With equation (41), we then get

$$(68) \quad 2I_m = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W \left(\xi \frac{W^{2^*(s)-1}}{|x|^s} + \gamma \frac{W}{|x|^2} \right) dx + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 |\nabla W|^2 dx + \Theta_\gamma(1).$$

Integrating by parts, using that W vanishes on $\partial\mathbb{R}^{k+,n-k}$, we get that

$$(69) \quad \begin{aligned} & \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W \frac{W^{2^*(\sigma)-1}}{|x|^\sigma} dx = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 |x|^{-\sigma} \partial_m \left(\frac{W^{2^*(\sigma)}}{2^*(\sigma)} \right) dx \\ & = - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m (x_m^2 |x|^{-\sigma}) \frac{W^{2^*(\sigma)}}{2^*(\sigma)} dx + \int_{\partial(\tilde{B}_{\epsilon^{-1}\delta,+k})} x_m^2 |x|^{-\sigma} \frac{W^{2^*(\sigma)}}{2^*(\sigma)} \nu_m dx \\ & = - \frac{2}{2^*(\sigma)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(\sigma)}}{|x|^\sigma} dx + \frac{\sigma}{2^*(\sigma)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 x_m \frac{W^{2^*(\sigma)}}{|x|^{\sigma+2}} dx + \Theta_\gamma(1) \end{aligned}$$

as $\epsilon \rightarrow 0$. We claim that

$$(70) \quad \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m |\nabla W|^2 dx = \gamma \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^2}{|x|^2} dx + \xi \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx + \Theta_\gamma(1).$$

Proof of (70). We multiply equation (41) by $x_m W$ and integrate by parts to get

$$(71) \quad \begin{aligned} & \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m |\nabla W|^2 dx = - \int \nabla(x_m) W \nabla W dx + \int x_m W \partial_\nu W dx + \gamma \int x_m \frac{W^2}{|x|^2} dx + \xi \int x_m \frac{W^{2^*(s)}}{|x|^s} dx \\ & = - \int \nabla(x_m) \nabla \left(\frac{W^2}{2} \right) dx + \int x_m W \partial_\nu W dx + \gamma \int x_m \frac{W^2}{|x|^2} dx + \xi \int x_m \frac{W^{2^*(s)}}{|x|^s} dx \\ & = - \int_{\partial} \frac{W^2}{2} \partial_\nu x_m dx + \int x_m W \partial_\nu W dx + \gamma \int x_m \frac{W^2}{|x|^2} dx + \xi \int x_m \frac{W^{2^*(s)}}{|x|^s} dx \end{aligned}$$

where all integrals are taken on $\tilde{B}_{\epsilon^{-1}\delta,+k}$ or $\partial\tilde{B}_{\epsilon^{-1}\delta,+k}$. Since W vanishes on $\partial\mathbb{R}^{k+,n-k}$ and by (66), we have

$$(72) \quad \int_{\partial\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m W \partial_\nu W dx = \int_{\partial\tilde{B}_{\epsilon^{-1}\delta} \cap \mathbb{R}^{k+,n-k}} x_m W \partial_\nu W dx = \Theta_\gamma(1).$$

And,

$$(73) \quad \int_{\partial\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{W^2}{2} \partial_\nu x_m dx = \int_{\partial\tilde{B}_{\epsilon^{-1}\delta} \cap \mathbb{R}^{k+,n-k}} \frac{W^2}{2} \partial_\nu x_m dx = \Theta_\gamma(1).$$

Then (71), (72) and (73) yields (70). □

Combining (68), (69) and (70), we obtain

$$\begin{aligned} 2I_m &= \xi \left[- \frac{2}{2^*(s)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx + \frac{s}{2^*(s)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 x_m \frac{W^{2^*(s)}}{|x|^{s+2}} dx \right] \\ &+ \gamma \left[- \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^2}{|x|^2} dx + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 x_m \frac{W^2}{|x|^{2+2}} dx \right] + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m |\nabla W|^2 dx + \Theta_\gamma(1) \\ &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_m^2}{|x|^2} x_m \left[\xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right] dx - \xi \frac{2}{2^*(s)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx \\ &- \gamma \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^2}{|x|^2} dx + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m |\nabla W|^2 dx + \Theta_\gamma(1) \end{aligned}$$

And then

$$\begin{aligned} 2I_m &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_m^2}{|x|^2} x_m \left[\xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right] dx - \xi \frac{2}{2^*(s)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx \\ &\quad - \gamma \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^2}{|x|^2} dx + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \left[\xi \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right] dx + \Theta_\gamma(1), \end{aligned}$$

by the last equality, we obtain the value of I_m . We now fix $m, p \in \{1, \dots, k\}$ such that $p \neq m$. Integrating by parts, we get that

$$\begin{aligned} M_{p,m} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W x_p \partial_p W dx = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W \partial_p \left(\frac{x_p^2}{2} \right) \partial_p W dx \\ &= - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_p (\partial_m W \partial_p W) dx + \int_{\partial(\tilde{B}_{\epsilon^{-1}\delta,+k})} \partial_m W \frac{x_p^2}{2} \partial_p W \nu_p dx, \end{aligned}$$

with $\nu_{i,j} := -\delta_{ij}$ for $i = 1, \dots, k$ and $j \geq 1$, since $W(x_{0,m}) = 0$, we have that

$$\begin{aligned} M_{p,m} &= - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_p (\partial_m W \partial_p W) dx + \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \partial \mathbb{R}^{k+n-k}} \partial_m W \frac{x_p^2}{2} \partial_p W \nu_p d\sigma \\ &\quad + O \left(\int_{\mathbb{R}^{k+n-k} \cap \partial \tilde{B}_{\epsilon^{-1}\delta}} |x|^2 |\nabla W|^2 d\sigma \right) \\ &= - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_p (\partial_m W \partial_p W) dx + \sum_{\alpha=1}^k \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_\alpha=0\}} \partial_m W \frac{x_p^2}{2} \partial_p W \nu_{\alpha,p} d\sigma \\ &\quad + O \left(\int_{\mathbb{R}^{k+n-k} \cap \partial \tilde{B}_{\epsilon^{-1}\delta}} |x|^2 |\nabla W|^2 d\sigma \right) \\ &= - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} [\partial_{mp} W \partial_p W + \partial_m W \partial_{pp} W] dx + \Theta_\gamma(1) \\ &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W [-\Delta W] dx - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m \left(\frac{|\partial_p W|^2}{2} \right) dx \\ &\quad + \sum_{j \geq 1; j \neq p} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W \partial_{jj} W dx + \Theta_\gamma(1) \\ &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W [-\Delta W] dx - \int_{\partial(\tilde{B}_{\epsilon^{-1}\delta,+k})} \frac{x_p^2}{4} |\partial_p W|^2 \nu_m d\sigma + \sum_{j \geq 1; j \neq p} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W \partial_{jj} W dx + \Theta_\gamma(1) \\ &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W [-\Delta W] dx - \sum_{\alpha=1}^k \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_\alpha=0\}} \frac{x_p^2}{4} |\partial_p W|^2 \nu_{\alpha,m} d\sigma \\ &\quad + \sum_{j \geq 1; j \neq p} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W \partial_{jj} W dx + \Theta_\gamma(1) \end{aligned}$$

And then

$$\begin{aligned} M_{p,m} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W [-\Delta W] dx - \sum_{j \geq 1; j \neq p} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_{jm} W \partial_j W dx \\ &\quad + \sum_{j \geq 1; j \neq p} \sum_{\alpha=1}^k \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_\alpha=0\}} \frac{x_p^2}{2} \partial_m W \partial_j W \nu_{\alpha,j} d\sigma + \Theta_\gamma(1). \end{aligned}$$

So we have

$$\begin{aligned}
M_{p,m} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W [-\Delta W] dx - \sum_{j \geq 1; j \neq p} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m \left(\frac{x_p^2}{4} |\partial_j W|^2 \right) dx \\
&+ \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} \frac{x_p^2}{2} |\partial_m W|^2 \nu_{m,m} d\sigma + \Theta_\gamma(1) \\
&= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W [-\Delta W] dx - \sum_{j \geq 1; j \neq p} \sum_{\alpha=1}^k \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_\alpha=0\}} \frac{x_p^2}{4} |\partial_j W|^2 \nu_{\alpha,m} d\sigma \\
&+ \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} \frac{x_p^2}{2} |\partial_m W|^2 \nu_{m,m} d\sigma + \Theta_\gamma(1) \\
&= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W [-\Delta W] dx - \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} \frac{x_p^2}{4} |\partial_m W|^2 d\sigma + \Theta_\gamma(1).
\end{aligned}$$

Moreover, using (41), we have that

$$M_{p,m} = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W \left(\frac{\gamma}{|x|^2} W + \xi \frac{W^{2^*(s)-1}}{|x|^s} \right) dx - \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} \frac{x_p^2 |\partial_m W|^2}{4} d\sigma + \Theta_\gamma(1).$$

Using again that W vanishes on $\partial\mathbb{R}^{k+,n-k}$, we get that

$$\begin{aligned}
&\int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_p^2 \partial_m W \frac{W^{2^*(\sigma)-1}}{|x|^\sigma} dx = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_p^2 |x|^{-\sigma} \partial_m \left(\frac{W^{2^*(\sigma)}}{2^*(\sigma)} \right) dx \\
&= \frac{\sigma}{2^*(\sigma)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2 x_m}{|x|^{\sigma+2}} W^{2^*(\sigma)} dx + O \left(\int_{\partial\tilde{B}_{\epsilon^{-1}\delta} \cap \mathbb{R}^{k+,n-k}} |x|^{2-\sigma} W^{2^*(\sigma)} d\sigma \right) \\
&= \frac{\sigma}{2^*(\sigma)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2 x_m}{|x|^{\sigma+2}} W^{2^*(\sigma)} dx + \Theta_\gamma(1) \text{ as } \epsilon \rightarrow 0.
\end{aligned}$$

Moreover,

$$M_{p,m} = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2 x_m}{2|x|^2} \left(\xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right) - \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} \frac{x_p^2 |\partial_m W|^2}{4} d\sigma + \Theta_\gamma(1).$$

The proof is similiar for $M_{l,m}$ for all $l \geq k+1$. Fix $m \in \{1, \dots, k\}$ and $i \geq 1$ such that $i \neq m$, we have that

$$K_{i,m} : = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_i W \partial_m W x_i dx = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_i W \partial_m W x_i \partial_m x_m dx.$$

Integrating by parts again and using (67), we get

$$\begin{aligned}
K_{i,m} &= - \int x_i x_m \partial_i W \partial_{mm} W dx - \int x_i x_m \partial_m W \partial_{mi} W dx + \sum_{\alpha=1}^k \int_{A_\epsilon^\alpha} x_i x_m \partial_m W \partial_i W \nu_{\alpha,m} dx + \Theta_\gamma(1) \\
&= \int x_i x_m \partial_i W (-\Delta W) dx + \sum_{j \geq 1; j \neq m} \int x_i x_m \partial_i W \partial_{jj} W dx - \frac{1}{2} \int x_i x_m \partial_i (\partial_m W)^2 dx + \Theta_\gamma(1) \\
&= \int x_i x_m \partial_i W (-\Delta W) dx - \int x_m \partial_i W \partial_i W dx - \sum_{j \geq 1; j \neq m} \int x_i x_m \partial_{ij} W \partial_j W dx \\
&\quad + \frac{1}{2} \int x_m (\partial_m W)^2 dx - \frac{1}{2} \sum_{\alpha=1}^k \int_{A_\epsilon^\alpha} x_i x_m (\partial_m W)^2 \nu_{\alpha,i} dx + \Theta_\gamma(1) \\
&= \int x_i x_m \partial_i W (-\Delta W) dx - J_{i,m} - \frac{1}{2} \sum_{j \geq 1; j \neq m} \int x_i x_m \partial_i (\partial_j W)^2 dx + \frac{1}{2} \int x_m (\partial_m W)^2 dx + \Theta_\gamma(1) \\
&= \int x_i x_m \partial_i W (-\Delta W) dx - J_{i,m} + \frac{1}{2} \sum_{j \geq 1; j \neq m} \int x_m (\partial_j W)^2 dx \\
&\quad - \frac{1}{2} \sum_{j \geq 1; j \neq m} \sum_{\alpha=1}^k \int_{A_\epsilon^\alpha} x_i x_m (\partial_j W)^2 \nu_{\alpha,i} dx + \frac{1}{2} \int x_m (\partial_m W)^2 dx + \Theta_\gamma(1) \\
&= \int x_i x_m \partial_i W (-\Delta W) dx - J_{i,m} + \frac{1}{2} \sum_{j \geq 1; j \neq m} \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} x_m (\partial_j W)^2 dx + \frac{1}{2} \int x_m (\partial_m W)^2 dx + \Theta_\gamma(1) \\
&= \int x_i x_m \partial_i W (-\Delta W) dx - J_{i,m} + \frac{1}{2} \int x_m |\nabla W|^2 dx + \Theta_\gamma(1),
\end{aligned}$$

since W is a solution to (41), then there exists $\xi > 0$ such that

$$K_{i,m} + J_{i,m} = \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} x_i x_m \partial_i W \left(\xi \frac{W^{2^*(s)-1}}{|x|^s} + \gamma \frac{W}{|x|^2} \right) dx + \frac{1}{2} \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} x_m |\nabla W|^2 dx + \Theta_\gamma(1).$$

Since W vanishes on $\partial \mathbb{R}^{k+,n-k}$, we get

$$\begin{aligned}
&\int_{\bar{B}_{\epsilon^{-1}\delta,+k}} x_i x_m \partial_i W \frac{W^{2^*(\sigma)-1}}{|x|^\sigma} dx = \frac{1}{2^*(\sigma)} \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} \frac{x_i x_m}{|x|^\sigma} \partial_i (W^{2^*(\sigma)}) dx \\
&= -\frac{1}{2^*(\sigma)} \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(\sigma)}}{|x|^\sigma} dx + \frac{\sigma}{2^*(\sigma)} \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} x_i^2 x_m \frac{W^{2^*(\sigma)}}{|x|^{\sigma+2}} dx + \Theta_\gamma(1).
\end{aligned}$$

Then with (70)

$$\begin{aligned}
K_{i,m} + J_{i,m} &= \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} \frac{x_i^2}{|x|^2} x_m \left[\xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right] dx \\
&\quad + \left(\frac{1}{2} - \frac{1}{2^*(s)} \right) \xi \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx + \Theta_\gamma(1).
\end{aligned}$$

Fix $m \in \{1, \dots, k\}$, $p \in \{1, \dots, k\}$ and $q \in \{p+1, \dots, k\}$ such that $p, q \neq m$. We get

$$L_{m,p,q} := \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} \partial_m W \partial_p W x_q dx = \int_{\bar{B}_{\epsilon^{-1}\delta,+k}} \partial_m W \partial_p W x_q \partial_p x_p dx.$$

Using again the integrations by parts, (66) and (67), we get

$$\begin{aligned}
L_{m,p,q} &= - \int x_q x_p \partial_m W \partial_{pp} W dx - \int x_p x_q \partial_p W \partial_{mp} W dx + \sum_{\alpha=1}^k \int_{A_\epsilon^\alpha} x_q x_p \partial_m W \partial_p W \nu_{\alpha,p} dx + \Theta_\gamma(1) \\
&= \int x_q x_p \partial_m W (-\Delta W) dx + \sum_{j \geq 1; j \neq p} \int x_q x_p \partial_m W \partial_{jj} W dx - \frac{1}{2} \int x_q x_p \partial_m (\partial_p W)^2 dx + \Theta_\gamma(1) \\
&= \int x_q x_p \partial_m W (-\Delta W) dx - \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} x_p \partial_m W \partial_q W dx - \sum_{j \geq 1; j \neq p} \int x_p x_q \partial_{jm} W \partial_j W dx \\
&\quad + \sum_{j \geq 1; j \neq p} \sum_{\alpha=1}^k \int_{A_\epsilon^\alpha} x_q x_p \partial_m W \partial_j W \nu_{\alpha,j} dx - \frac{1}{2} \sum_{\alpha=1}^k \int_{A_\epsilon^\alpha} x_q x_p (\partial_p W)^2 \nu_{\alpha,m} dx + \Theta_\gamma(1) \\
&= \int x_q x_p \partial_m W (-\Delta W) dx - N_{m,p,q} - \frac{1}{2} \sum_{j \geq 1; j \neq p} \int x_p x_q \partial_m (\partial_j W)^2 dx \\
&\quad + \int_{A_\epsilon^m} x_q x_p (\partial_m W)^2 \nu_{m,m} dx + \Theta_\gamma(1) \\
&= \int x_q x_p \partial_m W (-\Delta W) dx - N_{m,p,q} - \frac{1}{2} \sum_{j \geq 1; j \neq p} \sum_{\alpha=1}^k \int_{A_\epsilon^\alpha} x_q x_p (\partial_j W)^2 \nu_{\alpha,m} dx \\
&\quad + \int_{A_\epsilon^m} x_q x_p (\partial_m W)^2 \nu_{m,m} dx + \Theta_\gamma(1) \\
&= \int x_q x_p \partial_m W (-\Delta W) dx - N_{m,p,q} + \frac{1}{2} \int_{A_\epsilon^m} x_q x_p (\partial_m W)^2 \nu_{m,m} dx + \Theta_\gamma(1),
\end{aligned}$$

with $A_\epsilon^\alpha := \tilde{B}_{\epsilon^{-1}\delta} \cap \{x_\alpha = 0\}$, other integrals being taken on $\tilde{B}_{\epsilon^{-1}\delta, +k}$. With (41), we then get

$$L_{m,p,q} + N_{m,p,q} = \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} x_q x_p \partial_m W \left(\xi \frac{W^{2^*(s)-1}}{|x|^s} + \gamma \frac{W}{|x|^2} \right) dx + \frac{1}{2} \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_q x_p (\partial_m W)^2 \nu_{m,m} dx + \Theta_\gamma(1).$$

Integrating by parts, using that W vanishes on $\partial \mathbb{R}^{k+, n-k}$, for $\sigma \in [0, 2]$, we get that

$$\begin{aligned}
\int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} x_q x_p \partial_m W \frac{W^{2^*(\sigma)-1}}{|x|^\sigma} dx &= \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} x_q x_p |x|^{-\sigma} \partial_m \left(\frac{W^{2^*(\sigma)}}{2^*(\sigma)} \right) dx \\
&= \frac{\sigma}{2^*(\sigma)} \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} x_q x_p x_m \frac{W^{2^*(\sigma)}}{|x|^{\sigma+2}} dx + \Theta_\gamma(1) \text{ as } \epsilon \rightarrow 0.
\end{aligned}$$

And then

$$L_{m,p,q} + N_{m,p,q} = \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{x_q x_p x_m}{|x|^2} \left(\xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right) - \frac{1}{2} \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_q x_p (\partial_m W)^2 dx + \Theta_\gamma(1).$$

This ends the proof of Lemma 5.1. □

We define (all integrals are taken on $\tilde{B}_{\epsilon^{-1}\delta,+k}$)

$$\begin{aligned}
A_\epsilon := & \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \left(\int |\nabla W|^2 x_i dx - \gamma \int \frac{|W|^2}{|x|^2} x_i dx \right) \\
& - 2 \sum_{m=1}^k \sum_{i=1; i \neq m}^k \partial_{ii} \phi^m(0) \left(\int \partial_m W x_i \partial_i W dx + \gamma \int \frac{|W|^2}{|x|^2} \frac{x_m}{|x|^2} x_i^2 dx \right) \\
& - 2 \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \left(\int \partial_m W x_i \partial_i W dx + \gamma \int \frac{|W|^2}{|x|^2} \frac{x_m}{|x|^2} x_i^2 dx \right) \\
& - 2 \sum_{m=1}^k \partial_{mm} \phi^m(0) \left(\int \partial_m W \partial_m W x_m dx + \gamma \int \frac{|W|^2}{|x|^2} \frac{x_m}{|x|^2} x_m x_m dx \right) \\
& - 2 \sum_{m=1}^k \sum_{i \geq 1; i \neq m} \partial_{mi} \phi^i(0) \left(\int \partial_i W \partial_m W x_i dx + \gamma \int \frac{|W|^2}{|x|^2} \frac{x_i}{|x|^2} x_m x_i dx \right) \\
& - 2 \sum_{m=1}^k \sum_{i \geq 1; i \neq m} \partial_{im} \phi^i(0) \left(\int \partial_i W \partial_i W x_m dx + \gamma \int \frac{|W|^2}{|x|^2} \frac{x_i}{|x|^2} x_i x_m dx \right) \\
& + \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \left(-2 \int \partial_m W \partial_q W x_p dx + \gamma \int \frac{|W|^2}{|x|^2} \frac{x_m}{|x|^2} x_q x_p dx \right) \\
& + \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \left(-2 \int \partial_m W \partial_p W x_q dx + \gamma \int \frac{|W|^2}{|x|^2} \frac{x_m}{|x|^2} x_p x_q dx \right)
\end{aligned}$$

and

$$\begin{aligned}
B_\epsilon := & \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int \frac{|W|^{2^*(s)}}{|x|^s} x_i dx - \frac{s}{2} \sum_{m=1}^k \sum_{i=1; i \neq m}^k \partial_{ii} \phi^m(0) \int \frac{|W|^{2^*(s)}}{|x|^s} \frac{x_m}{|x|^2} x_i^2 dx \\
& - \frac{s}{2} \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \int \frac{|W|^{2^*(s)}}{|x|^s} \frac{x_m}{|x|^2} x_i^2 dx - \frac{s}{2} \sum_{m=1}^k \partial_{mm} \phi^m(0) \int \frac{|W|^{2^*(s)}}{|x|^s} \frac{x_m}{|x|^2} x_m^2 dx \\
& - s \sum_{m=1}^k \sum_{i \geq 1; i \neq m} \partial_{mi} \phi^i(0) \int \frac{|W|^{2^*(s)}}{|x|^s} \frac{x_i^2}{|x|^2} x_m dx - s \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \int \frac{|W|^{2^*(s)}}{|x|^s} \frac{x_m}{|x|^2} x_q x_p dx
\end{aligned}$$

Steps 1 and 2 and (59) yield

$$\begin{aligned}
\int_{\Omega} \left(|\nabla W_\epsilon|^2 - \gamma \frac{|W_\epsilon|^2}{|x|^2} \right) dx &= \int_{\mathbb{R}^{k+n-k}} \left(|\nabla W|^2 - \gamma \frac{|W|^2}{|x|^2} \right) dx + A_\epsilon \epsilon + \Theta_\gamma(\epsilon), \\
\int_{\Omega} \frac{W_\epsilon^{2^*(s)}}{|x|^s} dx &= \int_{\mathbb{R}^{k+n-k}} \frac{W^{2^*(s)}}{|x|^s} dx + \epsilon B_\epsilon + \Theta_\gamma(\epsilon)
\end{aligned}$$

It follows from (41) that

$$\int_{\mathbb{R}^{k+n-k}} \left(|\nabla W|^2 - \gamma \frac{W^2}{|x|^2} \right) dx = \xi \int_{\mathbb{R}^{k+n-k}} \frac{W^{2^*(s)}}{|x|^s} dx.$$

Since W is an extremal for the Euclidean inequality, we have that

$$\frac{\int_{\mathbb{R}^{k+n-k}} (|\nabla W|^2 - \frac{\gamma}{|x|^2} W^2) dx}{\left(\int_{\mathbb{R}^{k+n-k}} \frac{W^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}} = \mu_{\gamma,s}(\mathbb{R}^{k+n-k}).$$

Note that, for $\gamma \leq \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$, we have that $\lim_{\epsilon \rightarrow 0} A_\epsilon \epsilon = \lim_{\epsilon \rightarrow 0} B_\epsilon \epsilon = 0$. Therefore, the above estimates yield

$$J_{\gamma,s}^\Omega(W_\epsilon) = \mu_{\gamma,s}(\mathbb{R}^{k+,n-k}) \left(1 + \frac{1}{\xi \int_{\mathbb{R}^{k+,n-k}} \frac{W^{2^*(s)}}{|x|^s} dx} \left(A_\epsilon - \frac{2\xi}{2^*(s)} B_\epsilon \right) \epsilon + \Theta_\gamma(\epsilon) \right).$$

In the following formula, all the integrals are on $\tilde{B}_{\epsilon^{-1}\delta,+k}$ and $F(x) := \gamma \frac{W^2}{|x|^2} + \xi \frac{W^{2^*(s)}}{|x|^s}$. Using the notations of Step 3 and Lemma 5.1, we get

$$\begin{aligned} A_\epsilon - \frac{2\xi}{2^*(s)} B_\epsilon &= \sum_{i=1}^k \sum_j \partial_{ji} \phi^j(0) \xi \left(1 - \frac{2}{2^*(s)} \right) \int \frac{|W|^{2^*(s)}}{|x|^s} x_i dx \\ &+ \sum_{m=1}^k \sum_{i=1, i \neq m}^k \partial_{ii} \phi^m(0) \left(-2M_{im} + \int \frac{x_i^2}{|x|^2} x_m F(x) dx \right) \\ &+ \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \left(-2M_{im} + \int \frac{x_i^2}{|x|^2} x_m F(x) dx \right) + \sum_{m=1}^k \partial_{mm} \phi^m(0) \left(-2I_m + \int \frac{x_m^2}{|x|^2} x_m F(x) dx \right) \\ &+ \sum_{m=1}^k \sum_{i \geq 1; i \neq m} \partial_{mi} \phi^i(0) \left(-2K_{im} + \int \frac{x_i^2}{|x|^2} x_m F(x) dx \right) + \sum_{m=1}^k \sum_{i \geq 1; i \neq m} \partial_{mi} \phi^i(0) \left(-2J_{im} + \int \frac{x_i^2}{|x|^2} x_m F(x) dx \right) \\ &+ \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \left(-2N_{m,p,q} + \int \frac{x_p x_q}{|x|^2} x_m F(x) dx \right) \\ &+ \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \left(-2L_{m,p,q} + \int \frac{x_p x_q}{|x|^2} x_m F(x) dx \right) \\ &= \sum_{i=1}^k \sum_j \partial_{ji} \phi^j(0) \xi \left(1 - \frac{2}{2^*(s)} \right) \int \frac{|W|^{2^*(s)}}{|x|^s} x_i dx + \frac{1}{2} \sum_{m=1}^k \sum_{i=1, i \neq m}^k \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma \\ &+ \frac{1}{2} \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma - \xi \left(1 - \frac{2}{2^*(s)} \right) \partial_{mm} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx \\ &- \xi \left(1 - \frac{2}{2^*(s)} \right) \sum_{i \geq 1; i \neq m} \partial_{im} \phi^i(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx \\ &+ \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_q x_p |\partial_m W|^2 d\sigma \\ &= \frac{1}{2} \sum_{m=1}^k \sum_{i=1, i \neq m}^k \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma + \frac{1}{2} \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma \\ &+ \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_q x_p |\partial_m W|^2 d\sigma \end{aligned}$$

With the symmetries of W (see Theorem 4.1), there exists $\alpha_\epsilon, \beta_\epsilon, \tau_\epsilon > 0$ such that

$$\begin{aligned} \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma &= \alpha_\epsilon && \text{if } i = 1, \dots, k, i \neq m \\ \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma &= \beta_\epsilon && \text{if } i = k+1, \dots, n \\ \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_q x_p |\partial_m W|^2 d\sigma &= \tau_\epsilon && \text{if } p, q, m \in \{1, \dots, k\} \text{ are distinct} \end{aligned}$$

Then, we get that

$$\begin{aligned} A_\epsilon - \frac{2\xi}{2^*(s)} B_\epsilon &= \frac{\alpha_\epsilon}{2} \sum_{m=1}^k \sum_{i=1, i \neq m}^k \partial_{ii} \phi^m(0) \\ &+ \frac{\beta_\epsilon}{2} \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) + \tau_\epsilon \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0). \end{aligned}$$

We distinguish two cases:

Case 1: $\gamma < \gamma_H(\mathbb{R}^{k+, n-k}) - \frac{1}{4}$, that is $\alpha_+ - \alpha_- > 1$. It follows from the pointwise control (43) that $x \mapsto |x|^2 |\nabla W|^2 \in L^1(\mathbb{R}^{k+, n-k} \cap \{x_m = 0\})$, therefore

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \alpha_\epsilon &= 2c_{\gamma, s}^2 := \int_{\mathbb{R}^{k+, n-k} \cap \{x_m = 0\}} x_i^2 |\partial_m W|^2 d\sigma > 0 & \text{if } i = 1, \dots, k, i \neq m \\ \lim_{\epsilon \rightarrow 0} \beta_\epsilon &= 2c_{\gamma, s}^1 := \int_{\mathbb{R}^{k+, n-k} \cap \{x_m = 0\}} x_i^2 |\partial_m W|^2 d\sigma > 0 & \text{if } i = k+1, \dots, n \\ \lim_{\epsilon \rightarrow 0} \tau_\epsilon &= c_{\gamma, s}^3 := \int_{\mathbb{R}^{k+, n-k} \cap \{x_m = 0\}} x_q x_p |\partial_m W|^2 d\sigma > 0 & \text{if } p, q, m \in \{1, \dots, k\} \text{ distinct.} \end{aligned}$$

Consequently,

$$\begin{aligned} A_\epsilon - \frac{2\xi}{2^*(s)} B_\epsilon &= c_{\gamma, s}^2 \sum_{m=1}^k \sum_{i=1, i \neq m}^k \partial_{ii} \phi^m(0) \\ &+ c_{\gamma, s}^1 \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) + c_{\gamma, s}^3 \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) + o(1) \end{aligned}$$

Case 2: $\gamma = \gamma_H(\mathbb{R}^{k+, n-k}) - \frac{1}{4}$, that is $\alpha_+ - \alpha_- = 1$. It follows from (25) that

$$\lim_{\lambda \rightarrow 0} \lambda^{\alpha_-} |x|^{\alpha_- + k} \partial_m W(\lambda x) = K \left(\prod_{j=1, j \neq m}^k x_j - (\alpha_- + k) \frac{p(x) x_m}{|x|^2} \right),$$

where $p(x) := \prod_{j=1}^k x_j$. As in the proof of (44), a Kelvin transform yields

$$(74) \quad \lim_{\lambda \rightarrow +\infty} \lambda^{\alpha_+} |x|^{\alpha_+ + k} \partial_m W(\lambda x) = K \prod_{j=1, j \neq m}^k x_j \text{ on } \{x_m = 0\}.$$

We claim that

$$(75) \quad \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m = 0\}} x_i^2 |\partial_m W|^2 dx = 2c_{\gamma, s}^2 \ln \left(\frac{1}{\epsilon} \right) + o \left(\ln \left(\frac{1}{\epsilon} \right) \right) \text{ as } \epsilon \rightarrow 0,$$

where,

$$c_{\gamma, s}^2 := \frac{K^2}{2} \int_{S^{n-2} \cap (\{x_m = 0\} \cap \mathbb{R}^{k+, n-k})} \sigma_i^2 \left(\prod_{j=1, j \neq m}^k \sigma_j \right)^2 d\sigma$$

is independent of $i \in \{1, \dots, k\}$, $i \neq m$. We prove the claim. Since $n - 2 - 2\alpha_+ = -1$, we have

$$(76) \quad \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m = 0\}} x_i^2 |\partial_m W|^2 dx = \int_{(\tilde{B}_{\epsilon^{-1}\delta} \setminus \tilde{B}_1) \cap \{x_m = 0\}} x_i^2 |\partial_m W|^2 dx + O(1) = \int_1^{\epsilon^{-1}\delta} \frac{f(r)}{r} dr + O(1),$$

where

$$f(r) := \int_{S^{n-2} \cap (\{x_m = 0\} \cap \mathbb{R}^{k+, n-k})} r^{2\alpha_+} \sigma_i^2 |\partial_{\sigma, m} W(r\sigma)|^2 d\sigma.$$

It follows from the uniform convergence in (74) that $\lim_{r \rightarrow +\infty} f(r) = 2c_{\gamma, s}^2$. Then (74) and (76) yield (75) and then the claim.

Similarly, there exists explicit constants $c_{\gamma,s}^2, c_{\gamma,s}^3 > 0$ such that

$$\begin{aligned} \int_{\bar{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma &= 2c_{\gamma,s}^1 \ln\left(\frac{1}{\epsilon}\right) + o\left(\ln\left(\frac{1}{\epsilon}\right)\right); \\ \int_{\bar{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_q x_p |\partial_m W|^2 d\sigma &= c_{\gamma,s}^3 \ln\left(\frac{1}{\epsilon}\right) + o\left(\ln\left(\frac{1}{\epsilon}\right)\right) \end{aligned}$$

for $i \geq k+1$ and $p, q, m \in \{1, \dots, k\}$ all distinct. Therefore

$$\begin{aligned} A_\epsilon - \frac{2\xi}{2^*(s)} B_\epsilon &= \left(c_{\gamma,s}^2 \sum_{m=1}^k \sum_{i=1, i \neq m}^k \partial_{ii} \phi^m(0) + c_{\gamma,s}^1 \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \right. \\ &\quad \left. + c_{\gamma,s}^3 \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \right) \ln\left(\frac{1}{\epsilon}\right) + o\left(\ln\left(\frac{1}{\epsilon}\right)\right) \end{aligned}$$

We are left with writing the expressions of Cases 1 and 2 intrinsically. We refer to Definition 3. For any $1 \leq i_1, i_2 \leq n$ such that $i_1, i_2 \neq m$, we have

$$\begin{aligned} \partial_{i_1 i_2} \phi^m(0) &= -\langle \vec{\nu}_m(0), \partial_{i_1 i_2} \phi(0) \rangle = \langle \partial_{i_1}(\vec{\nu}_m \circ \phi)(0), \partial_{i_2} \phi(0) \rangle \\ &= II_0^{\partial\Omega_m}(\partial_{i_1} \phi, \partial_{i_2} \phi) := II_{i_1 i_2}^m. \end{aligned}$$

For $p \neq m$, we have $\vec{\nu}_p \in (T_0 \partial\Omega_m)^\perp$ and

$$\sum_{p,q,m=1, |\{p,q,m\}|=3}^k II_0^{\partial\Omega_m}(\vec{\nu}_p, \vec{\nu}_q) = \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{pq} \phi^m(0).$$

Define $\Sigma := \cap_{j=1}^k \partial\Omega_j$. We have that

$$\sum_{m=1}^k \langle \vec{H}_0^\Sigma, \vec{\nu}_m \rangle = \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \text{ and } \sum_{m=1}^k \sum_{i=1, i \neq m}^k \partial_{ii} \phi^m(0) = \sum_{i,m=1, i \neq m}^k II_0^{\partial\Omega_m}(\vec{\nu}_i, \vec{\nu}_i)$$

□

Theorem 1.2 is a straightforward application of Theorem 1.1 and Proposition 5.1.

6. PROOF OF THEOREM 1.3

Point (1): we assume that $s = 0$ and $\gamma \leq 0$. It follows from the definition that $\mu_{\gamma,0}(\Omega) \geq \mu_{0,0}(\mathbb{R}^n)$. With the reverse inequality (20), we get that $\mu_{\gamma,0}(\Omega) = \mu_{0,0}(\mathbb{R}^n)$. If there was an extremal for $\mu_{\gamma,0}(\Omega)$, it would also be an extremal for $\mu_{0,0}(\mathbb{R}^n)$, with no compact support, contradicting the boundedness of Ω . This proves (1) of Theorem 1.3.

Point (2): Point (2) of Theorem 1.3 is a straightforward application of Theorem 1.1 and Proposition 5.1.

Point (3): We assume that $n = 3$, $s = 0$, $\gamma > 0$ and there is no extremal for $\mu_{\gamma,0}(\mathbb{R}_+^k \times \mathbb{R}^{3-k})$. In this situation, see Proposition 1.3 of [16], we have that $\mu_{\gamma,0}(\mathbb{R}_+^k \times \mathbb{R}^{3-k}) = \mu_{0,0}(\mathbb{R}^3)$. The following proposition is as in [16]:

Proposition 6.1. *Let $\Omega \subset \mathbb{R}^3$ be an open domain such that $0 \in \partial\Omega$. Fix $x_0 \in \Omega$. If $\gamma \in (0, \gamma_H(\Omega))$, then the equation*

$$\begin{cases} -\Delta G - \frac{\gamma}{|x|^2} G = 0; G > 0 & \text{in } \Omega \setminus \{x_0\} \\ G = 0 & \text{on } \partial\Omega \setminus \{0\} \end{cases}$$

has a solution $G \in C^2(\Omega \setminus \{x_0\}) \cap D_1^2(\Omega \setminus \{x_0\})_{loc,0}$, that is unique up to multiplication by a constant. Moreover, for any $x_0 \in \Omega$, there exists a unique $R_\gamma(x_0) \in \mathbb{R}$ independent of the choice of G and $c_G > 0$ such that

$$G(x) = c_G \left(\frac{1}{|x - x_0|} + R_\gamma(x_0) \right) + o(1) \text{ as } x \rightarrow x_0.$$

The proof is similar to the proof of Proposition 10.1 in [16]. Cooking-up some test-functions $(u_\epsilon)_{\epsilon>0}$ as in Lemma 10.2 of [16], we get that $\mu_{\gamma,0}(\Omega) \leq J_{\gamma,s}^\Omega(u_\epsilon) < \mu_{0,0}(\mathbb{R}^3) = \mu_{\gamma,0}(\mathbb{R}^3)$ when $R_\gamma(x_0) > 0$ for some $x_0 \in \Omega$. Point (3) of Theorem 1.3 is then a consequence of Theorem 1.1.

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