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Constructions of p -adic L -functions and admissible measures for Hermitian modular forms

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To Dear Mirjana Vukovic with admiration

For a prime p and a positive integer n , the standard zeta function $L_F(s)$ is considered, attached to an Hermitian modular form $F = \sum_H A(H)q^H$ on the Hermitian upper half plane \mathcal{H}_n of degree n , where H runs through semi-integral positive definite Hermitian matrices of degree n , i.e. $H \in \Lambda_n(\mathcal{O})$ over the integers \mathcal{O} of an imaginary quadratic field K , where $q^H = \exp(2\pi i \text{Tr}(HZ))$. Analytic p -adic continuation of their zeta functions constructed by A.Bouganis in the ordinary case (in [Bou16] is presently extended to the admissible case via growing p -adic measures. Previously this problem was solved for the Siegel modular forms, [CourPa], [BS00]. Present main result is stated in terms of the Hodge polygon $P_H(t) : [0, d] \rightarrow \mathbb{R}$ and the Newton polygon $P_N(t) = P_{N,p}(t) : [0, d] \rightarrow \mathbb{R}$ of the zeta function $L_F(s)$ of degree $d = 4n$. Main theorem gives a p -adic analytic interpolation of the L values in the form of certain integrals with respect to Mazur-type measures.

p -adic zeta functions of modular forms

Since the p -adic zeta function of Kubota-Leopoldt was constructed by p -adic interpolation of zeta-values $\zeta(1-k) = -B_k/k$ ($k \geq 1$) [KuLe64], also p -adic zeta functions of **various modular forms** were constructed, such as p -adic interpolation of the special values

$$L_\Delta(s, \chi) = \sum_{n=1}^{\infty} \chi(n)\tau(n)n^{-s}, \quad (s = 1, 2, \dots, 11), \quad \Delta = \sum_{n=1}^{\infty} \tau(n)q^n,$$

for the Ramanujan function $\tau(n)$ twisted by Dirichlet characters $\chi : (\mathbb{Z}/p^r\mathbb{Z})^* \rightarrow \mathbb{C}^*$. Interpolation done in the elliptic and Hilbert modular cases by Yu.I.Manin and B.Mazur, via modular symbols and p -adic integration, see [Ma73], [Ma76]).

In the Siegel modular case $\text{Sp}(2n, \mathbb{Z})$ the p -adic standard zeta functions were constructed in [Pa88], [Pa91] via **Rankin-Selberg Andrianov's identity** (n even), and [BS00] via **doubling method**.

In 2000-2001 Mirjana Vuković effected an important scientific visit to the Fourier Institut (France) supported by a regional cooperation program, together with Professors A.N.Andrianov (from Sankt-Petersburg, Russia, and Siegfried Boecherer (Mannheim, Germany).

Mirjana Vuković prepared and presented in the Seminar of Number Theory of the Fourier Institute the work *Structures gradués et paragradués.*, see [Vu] developing vastly the ideas of Marc Krasner ([Kr44]-[KrVu87]).

Happy Birthday Dear Mirjana!
Best Wishes from Grenoble, Moscow, and from all friends came to the
conference in Dubrovnik in September 2016!



Figure 1: Mirijana Vukovic with friends in Dubrovnok, September 2016

Hermitian modular group $\Gamma_{n,K}$ and the standard zeta function $\mathcal{Z}(s, f)$ (definitions)

Let $\theta = \theta_K$ be the quadratic character attached to $K, n' = \lfloor \frac{n}{2} \rfloor$.

$$\Gamma_{n,K} = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(\mathcal{O}_K) \mid M\eta_n M^* = \eta_n \right\}, \quad \eta_n = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix},$$

$$\mathcal{Z}(s, \mathbf{f}) = \left(\prod_{i=1}^{2n} L(2s - i + 1, \theta^{i-1}) \right) \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

(defined via Hecke's eigenvalues: $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}, \mathfrak{a} \subset \mathcal{O}_K$)

$$= \prod_{\mathfrak{q}} \mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1} \text{ (an Euler product over primes } \mathfrak{q} \subset \mathcal{O}_K \text{)}$$

with $\deg \mathcal{Z}_{\mathfrak{q}}(X) = 2n$, the Satake parameters $t_{i,\mathfrak{q}}, i = 1, \dots, n$,

$$\boxed{\mathcal{D}(s, \mathbf{f}) = \mathcal{Z}\left(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f}\right)} \quad \text{(Motivically normalized standard zeta function}$$

with a functional equation $s \mapsto \ell - s$; $\mathrm{rk} = 4n$, and motivic weight $\ell - 1$).

Main result: p -adic interpolation of all critical values $\mathcal{D}(s, \mathbf{f}, \chi)$ normalized by $\times \Gamma_{\mathcal{D}}(s) / \Omega_{\mathbf{f}}$, in the critical strip $n \leq s \leq \ell - n$ for all $\chi \bmod p^r$ in both **bounded or unbounded case**, i.e. when the product $\alpha_{\mathbf{f}} = \left(\prod_{\mathfrak{q}|p} \prod_{i=1}^n t_{\mathfrak{q},i} \right) p^{-n(n+1)}$ is **not a p -adic unit**.

The idea of motivic normalization: Ikeda's lifting [Ike08]

The standard Gamma factor of Ikeda's lifting, denoted by \mathbf{f} , of an elliptic modular form f extends to a general (not necessarily lifted) Hermitian modular form \mathbf{f} of weight ℓ , used as a pattern, namely

$$\mathcal{S}_{2k+1}(\Gamma_0(D), \theta) \ni f \rightsquigarrow \mathbf{f} = \mathrm{Lift}(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n}), \text{ if } n = 2n' \text{ is even (E)}$$

$$\mathcal{S}_{2k}(\mathrm{SL}(\mathbb{Z})) \ni f \rightsquigarrow \mathbf{f} = \mathrm{Lift}(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n}), \text{ if } n = 2n' + 1 \text{ is odd (O)}$$

the standard L -function of $\mathbf{f} = \mathrm{Lift}^{(n)}(f)$ is $\mathcal{Z}(s, \mathbf{f}) =$

$$\begin{aligned} & \prod_{i=1}^n L(s + k + n' - i + (1/2), f) L(s + k + n' - i + (1/2), f, \theta) \quad [\text{Ike08}] \\ & = \prod_{i=0}^{n-1} L(s + \ell/2 - i - (1/2), f) L(s + \ell/2 - i - (1/2), f, \theta). \end{aligned}$$

because in the lifted case $k + n' = \ell/2$, and the Gamma factor of the standard zeta function with the symmetry $s \mapsto 1 - s$ becomes (see p.14) $\Gamma_{\mathcal{Z}}(s) = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s + \ell/2 - i - (1/2))^2$. This Gamma factor suggests the following motivic normalization $\mathcal{D}(s) = \mathcal{Z}(s - (\ell/2) + (1/2))$ with the **Gamma factor**

$$\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathcal{Z}}(s - (\ell/2) + (1/2)) = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s - i)^2,$$

and the L -function $\mathcal{D}(s)$ satisfies the symmetry $s \mapsto \ell - s$ of motivic weight $\ell - 1$ with the **slopes** $2 \cdot 0, 2 \cdot 1, \dots, 2 \cdot (n-1), 2 \cdot (\ell - n), \dots, 2 \cdot (\ell - 1)$, so that **Deligne's critical values** are at $s = n, \dots, s = \ell - n$.

General zeta functions: critical values and coefficients

More general zeta functions are Euler products of degree d

$$\mathcal{D}(s, \chi) = \sum_{n=1}^{\infty} \chi(n) a_n n^{-s} = \prod_p \frac{1}{\mathcal{D}_p(\chi(p) p^{-s})}, \quad \Lambda_{\mathcal{D}}(s, \chi) = \Gamma_{\mathcal{D}}(s) \mathcal{D}(s, \chi),$$

where $\deg \mathcal{D}_p(X) = d$ for all but finitely many p , and $\mathcal{D}_p(0) = 1$.

In many cases algebraicity of the zeta values was proven as

$$\frac{\mathcal{D}^*(s_0, \chi)}{\Omega_{\mathcal{D}}^{\pm}} \in \mathbb{Q}(\{\chi(n), a_n\}_n), \quad \text{where } \mathcal{D}^*(s, \chi) \text{ is normalized by } \Gamma_{\mathcal{D}},$$

at critical points $s_0 \in \mathbb{Z}_{crit}$ as linear combinations of **coefficients** a_n dividing out **periods** $\Omega_{\mathcal{D}}^{\pm}$, where $\mathcal{D}^*(s_0, \chi) = \Lambda_{\mathcal{D}}(s_0, \chi)$ if $h^{\ell, \ell} = 0$.

In p -adic analysis, the Tate field is used $\mathbb{C}_p = \hat{\mathbb{Q}}_p$, the completion of an algebraic closure $\bar{\mathbb{Q}}_p$, in place of \mathbb{C} . Let us fix embeddings $\begin{cases} i_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p \\ i_{\infty} : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C} \end{cases}$ and try to continue analytically these zeta values to $s \in \mathbb{Z}_p, \chi \bmod p^r$.

Main result stated with Hodge/Newton polygons of $\mathcal{D}(s)$

The Hodge polygon $P_H(t) : [0, d] \rightarrow \mathbb{R}$ of the function $\mathcal{D}(s)$ and the Newton polygon $P_{N,p}(t) : [0, d] \rightarrow \mathbb{R}$ at p are piecewise linear:

The Hodge polygon of pure weight w has the slopes j of $length_j = h^{j, w-j}$ given by Serre's Gamma factors of the functional equation of the form $s \mapsto w + 1 - s$, relating $\Lambda_{\mathcal{D}}(s, \chi) = \Gamma_{\mathcal{D}}(s) \mathcal{D}(s, \chi)$ and $\Lambda_{\mathcal{D}^{\rho}}(w + 1 - s, \bar{\chi})$, where ρ is the complex conjugation of a_n , and $\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathcal{D}^{\rho}}(s)$ equals to the product $\Gamma_{\mathcal{D}}(s) = \prod_{j \leq \frac{w}{2}} \Gamma_{j, w-j}(s)$, where

$$\Gamma_{j, w-j}(s) = \begin{cases} \Gamma_{\mathbb{C}}(s-j)^{h^{j, w-j}}, & \text{if } j < w, \\ \Gamma_{\mathbb{R}}(s-j)^{h_{+}^{j, j}} \Gamma_{\mathbb{R}}(s-j+1)^{h_{-}^{j, j}}, & \text{if } 2j = w, \text{ where} \end{cases}$$

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right), \quad \Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s) \Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s} \Gamma(s), \quad h^{j, j} = h_{+}^{j, j} + h_{-}^{j, j}, \quad \sum_j h^{j, w-j} = d.$$

The Newton polygon at p is the convex hull of points $(i, \text{ord}_p(a_i))$ ($i = 0, \dots, d$); its slopes λ are the p -adic valuations $\text{ord}_p(\alpha_i)$ of the inverse roots α_i of $\mathcal{D}_p(X) \in \bar{\mathbb{Q}}[X] \subset \mathbb{C}_p[X]$: $length_{\lambda} = \#\{i \mid \text{ord}_p(\alpha_i) = \lambda\}$.

p -adic analytic interpolation of $\mathcal{D}(s, \mathbf{f}, \chi)$

The result expresses the zeta values as integrals with respect to p -adic Mazur-type measures. These measures are constructed from the Fourier coefficients of Hermitian modular forms, and from eigenvalues of Hecke operators on the unitary group.

Pre-ordinary case: $P_H(t) = P_{N,p}(t)$ at $t = \frac{d}{2}$ The integrality of measures is proven by T. Bouganis [Bou16], representing $\mathcal{D}^*(s, \chi) = \Gamma_{\mathcal{D}}(s) \mathcal{D}(s, \chi)$ as a Rankin-Selberg type integral at critical points $s = m$. Coefficients of modular forms in this integral satisfy Kummer-type congruences and produce certain bounded measures $\mu_{\mathcal{D}}$ from integral representations and Petersson product, [CourPa]. For the case of p inert in K , see [Bou16].

Admissible case: $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2}) > 0$ The zeta distributions are unbounded, but their sequence produce h -admissible (growing) measures of Amice-Vélu-type, allowing to integrate any continuous characters $y \in \text{Hom}(\mathbb{Z}_p^*, \mathbb{C}_p^*) = \mathcal{Y}_p$. A general result is used on the existence of h -admissible (growing) measures from binomial congruences for the coefficients of Hermitian modular forms. Their p -adic Mellin transforms $\mathcal{L}_{\mathcal{D}}(y) = \int_{\mathbb{Z}_p^*} y(x) d\mu_{\mathcal{D}}(x)$, $\mathcal{L}_{\mathcal{D}} : \mathcal{Y}_p \rightarrow \mathbb{C}_p$ give p -adic analytic interpolation of growth $\log_p^h(\cdot)$ of the L -values: the values $\mathcal{L}_{\mathcal{D}}(\chi x_p^m)$ are integrals given by $i_p \left(\frac{\mathcal{D}^*(m, \mathbf{f}, \chi)}{\Omega_{\mathbf{f}}} \right) \in \mathbb{C}_p$.

A Hermitian modular form of weight ℓ with character σ

is a holomorphic function \mathbf{f} on \mathcal{H}_n ($n \geq 2$) such that $\mathbf{f}(g\langle Z \rangle) = \sigma(g)\mathbf{f}(Z)j(g, Z)^\ell$ for any $g \in \Gamma_{n, K}$. Here σ be a character of $\Gamma_K^{(n)}$, trivial on $\left\{ \begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix} \right\}$, and for $Z \in \mathcal{H}_n$, put $g\langle Z \rangle = (AZ + B)(CZ + D)^{-1}$, $j(g, Z) = \det(CZ + D)$.

Fourier expansions: a semi-integral Hermitian matrix is a Hermitian matrix $H \in (\sqrt{-D_K})^{-1}M_n(\mathcal{O})$ whose diagonal entries are integral.

Denote the set of semi-integral Hermitian matrices by $\Lambda_n(\mathcal{O})$, the subset of its positive definite elements is $\Lambda_n(\mathcal{O})^+$, with $\mathcal{O} = \mathcal{O}_K$.

A Hermitian modular form \mathbf{f} is called a cusp form if it has a Fourier expansion of the form $\mathbf{f}(Z) = \sum_{H \in \Lambda_n(\mathcal{O})^+} A(H)q^H$. Denote the space of cusp forms of weight ℓ with character σ by $\mathcal{S}_\ell(\Gamma_{n, K}, \sigma)$.

The standard zeta function of a Hermitian modular form

For all integral ideals $\mathfrak{a} \subset \mathcal{O}$ let $T(\mathfrak{a})$ denotes the Hecke operator associated to it as in [Shi00], page 162, using the action of double cosets $\Gamma\xi\Gamma$ with $\xi = \text{diag}(\hat{D}, D)$, $(\det(D)) = (\alpha)$, $\hat{D} = (D^*)^{-1}$, $\alpha \in \mathfrak{a}$.

Consider a non-zero Hermitian modular form $\mathbf{f} \in \mathcal{M}_\ell(\Gamma)$, for a (congruence) subgroup $\Gamma \subset \Gamma_{n, K}$, and assume $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ with $\lambda(\mathfrak{a}) \in \mathbb{C}$ for all integral ideals $\mathfrak{a} \subset \mathcal{O}$. Then

$$\mathcal{Z}(s, \mathbf{f}) = \left(\prod_{i=1}^{2n} L(2s - i + 1, \theta^{i-1}) \right) \sum_{\mathfrak{a}} \lambda(\mathfrak{a})N(\mathfrak{a})^{-s},$$

the sum is over all integral ideals of \mathcal{O}_K .

This series has an Euler product representation $\mathcal{Z}(s, \mathbf{f}) = \prod_{\mathfrak{q}} (\mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1})$, where the product is over all prime ideals of \mathcal{O}_K , $\mathcal{Z}_{\mathfrak{q}}(X)$ is the numerator of the series $\sum_{r \geq 0} \lambda(\mathfrak{q}^r)X^r \in \mathbb{C}(X)$, computed by Shimura as follows.

Euler factors of the standard zeta function, [Shi00], p. 171

The Euler factors $\mathcal{Z}_{\mathfrak{q}}(X)$ in the Hermitian modular case at the prime ideal \mathfrak{q} of \mathcal{O}_K are

- (i) $\mathcal{Z}_{\mathfrak{q}}(X) = \prod_{i=1}^n ((1 - N(\mathfrak{q})^{n-1} t_{\mathfrak{q},i} X)(1 - N(\mathfrak{q})^n t_{\mathfrak{q},i}^{-1} X))^{-1}$,
if $\mathfrak{q}^\rho = \mathfrak{q}$ and $\mathfrak{q} \nmid \mathfrak{c}$, (**the inert case outside level \mathfrak{c}**),
- (ii) $\mathcal{Z}_{\mathfrak{q}_1}(X_1)\mathcal{Z}_{\mathfrak{q}_2}(X_2) = \prod_{i=1}^{2n} ((1 - N(\mathfrak{q}_1)^{2n} t_{\mathfrak{q}_1\mathfrak{q}_2,i}^{-1} X_1)(1 - N(\mathfrak{q}_2)^{-1} t_{\mathfrak{q}_1\mathfrak{q}_2,i} X_2))^{-1}$,
if $\mathfrak{q}_1 \neq \mathfrak{q}_2$, $\mathfrak{q}_1^\rho = \mathfrak{q}_2$ and $\mathfrak{q}_i \nmid \mathfrak{c}$ for $i = 1, 2$ (**the split case outside level**),
- (iii) $\mathcal{Z}_{\mathfrak{q}}(X) = \prod_{i=1}^n (1 - N(\mathfrak{q})^{n-1} t_{\mathfrak{q},i} X)^{-1}$, if $\mathfrak{q}^\rho = \mathfrak{q}$ and $\mathfrak{q}|\mathfrak{c}$ (**inert level divisors**),
- (iv) $\mathcal{Z}_{\mathfrak{q}_1}(X_1)\mathcal{Z}_{\mathfrak{q}_2}(X_2) = \prod_{i=1}^n ((1 - N(\mathfrak{q}_1)^{n-1} t_{\mathfrak{q}_1\mathfrak{q}_2,i}^{-1} X_1)(1 - N(\mathfrak{q}_2)^{n-1} t_{\mathfrak{q}_1\mathfrak{q}_2,i} X_2))^{-1}$,
if $\mathfrak{q}_1 \neq \mathfrak{q}_2$, $\mathfrak{q}_i|\mathfrak{c}$ for $i = 1, 2$ (**split level divisors**).

where the $t_{\mathfrak{q},i}$ above for $? = \mathfrak{q}, \mathfrak{q}_1\mathfrak{q}_2$, are the **Satake parameters** of the eigenform \mathbf{f} .

The standard motivic-normalized zeta $\mathcal{D}(s, \mathbf{f}, \chi)$

The standard zeta function of \mathbf{f} is defined by means of the p -parameters as the following Euler product:

$$\mathcal{D}(s, \mathbf{f}, \chi) = \prod_p \prod_{i=1}^{2n} \left\{ \left(1 - \frac{\chi(p)\alpha_i(p)}{p^s} \right) \left(1 - \frac{\chi(p)\alpha_{4n-i}(p)}{p^s} \right) \right\}^{-1},$$

where χ is an arbitrary Dirichlet character. The p -parameters $\alpha_1(p), \dots, \alpha_{4n}(p)$ of $\mathcal{D}(s, \mathbf{f}, \chi)$ for p not dividing the level C of the form \mathbf{f} are related to the the $4n$ characteristic numbers

$$\alpha_1(p), \dots, \alpha_{2n}(p), \alpha_{2n+1}(p), \dots, \alpha_{4n}(p)$$

of the product of all \mathfrak{q} -factors $\mathcal{Z}_{\mathfrak{q}}(N\mathfrak{q}^{(\ell-1)/2} X)^{-1}$ for all $\mathfrak{q}|p$, which is a polynomial of degree $4n$ of the variable $X = p^{-s}$ (for almost all p) with coefficients in a number field $T = T(\mathbf{f})$.

There is a relation between the two normalizations $\mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f}) = \mathcal{D}(s, \mathbf{f})$ explained in [Ha97] for general zeta functions.

Description of the Main theorem

Let $\Omega_{\mathbf{f}}$ be a period attached to an Hermitian cusp eigenform \mathbf{f} , $\mathcal{D}(s, \mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})$ the standard zeta function, and

$$\alpha_{\mathbf{f}} = \alpha_{\mathbf{f},p} = \left(\prod_{\mathfrak{q}|p} \prod_{i=1}^n t_{\mathfrak{q},i} \right) p^{-n(n+1)}, \quad h = \text{ord}_p(\alpha_{\mathbf{f},p}),$$

The number $\alpha_{\mathbf{f}}$ turns out to be an eigenvalue of Atkin's type operator $U_p : \sum_H A_H q^H \mapsto \sum_H A_p q^H$ on some \mathbf{f}_0 , and $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2})$.

Definition. Let M be a \mathcal{O} -module of finite rank where $\mathcal{O} \subset \mathbb{C}_p$. For $h \geq 1$, consider the following \mathbb{C}_p -vector spaces of functions on \mathbb{Z}_p^* : $\mathcal{E}^h \subset \mathcal{E}^{loc-an} \subset \mathcal{E}$. Then

- a continuous homomorphism $\mu : \mathcal{C} \rightarrow M$ is called a **(bounded) measure** M -valued measure on \mathbb{Z}_p^* .

- $\mu : \mathcal{C}^h \rightarrow M$ is called an **h admissible measure** M -valued measure on \mathbb{Z}_p^* measure if the following growth condition is satisfied

$$\left| \int_{a+(p^v)} (x-a)^j d\mu \right|_p \leq p^{-v(h-j)}$$

for $j = 0, 1, \dots, h-1$, and let $\mathcal{Y}_p = \text{Hom}_{\text{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ be the space of definition of **p -adic Mellin transform**

Theorem ([Am-V], [MTT]) For an h -admissible measure μ , the Mellin transform $\mathcal{L}_\mu : \mathcal{Y}_p \rightarrow \mathbb{C}_p$ exists and has growth $o(\log^h)$ (with infinitely many zeros).

Main Theorem.

Let \mathbf{f} be a Hermitian cusp eigenform of degree $n \geq 2$ and of weight $\ell > 4n + 2$. There exist distributions $\mu_{\mathcal{D},s}$ for $s = n, \dots, \ell - n$ with the properties:

i) for all pairs (s, χ) such that $s \in \mathbb{Z}$ with $n \leq s \leq \ell - n$,

$$\int_{\mathbb{Z}_p^*} \chi d\mu_{\mathcal{D},s} = A_p(s, \chi) \frac{\mathcal{D}^*(s, \mathbf{f}, \bar{\chi})}{\Omega_{\mathbf{f}}}$$

(under the inclusion i_p), with elementary factors $A_p(s, \chi) = \prod_{q|p} A_q(s, \chi)$ including a finite Euler product, Satake parameters $t_{q,i}$, gaussian sums, the conductor of χ ; the integral is a finite sum.

(ii) if $\text{ord}_p \left(\left(\prod_{q|p} \prod_{i=1}^n t_{q,i} \right) p^{-n(n+1)} \right) = 0$ then the above distributions $\mu_{\mathcal{D},s}$ are bounded measures, we set $\mu_{\mathcal{D}} = \mu_{\mathcal{D},s^*}$ and the integral is defined for all continuous characters $y \in \text{Hom}(\mathbb{Z}_p^*, \mathbb{C}_p^*) =: \mathcal{Y}_p$.

Their **Mellin transforms** $\mathcal{L}_{\mu_{\mathcal{D}}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$, $\mathcal{L}_{\mu_{\mathcal{D}}} : \mathcal{Y}_p \rightarrow \mathbb{C}_p$, give bounded p -adic analytic interpolation of the above L -values to on the \mathbb{C}_p -analytic group \mathcal{Y}_p ; and these distributions are **related by**: $\int_X \chi d\mu_{\mathcal{D},s} = \int_X \chi x^{s^*-s} d\mu_{\mathcal{D}}^*$, $X = \mathbb{Z}_p^*$, where $s^* = \ell - n$, $s_* = n$.

Main theorem (continued)

(iii) in the **admissible** case assume that $0 < h \leq \frac{s^* - s_* + 1}{2} = \frac{\ell + 1 - 2n}{2}$, where $h = \text{ord}_p \left(\left(\prod_{q|p} \prod_{i=1}^n t_{q,i} \right) p^{-n(n+1)} \right) > 0$, Then there exist h -admissible measures $\mu_{\mathcal{D}}$ whose integrals

$\int_{\mathbb{Z}_p^*} \chi x_p^s d\mu_{\mathcal{D}}$ are given by $i_p \left(A_p(s, \chi) \frac{\mathcal{D}^*(s, \mathbf{f}, \bar{\chi})}{\Omega_{\mathbf{f}}} \right) \in \mathbb{C}_p$ with $A_p(s, \chi)$ as in (i); their Mellin transforms $\mathcal{L}_{\mathcal{D}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$, belong to the type $o(\log x_p^h)$. (iv) the functions $\mathcal{L}_{\mathcal{D}}$ are determined by

(i)-(iii).

Remarks. (a) Interpretation of s^* : the smallest of the "big slopes" of P_H

(b) Interpretation of $s_* - 1$: the biggest of the "small slopes" of P_H .

Eisenstein series and congruences (KEY POINT!)

The (Siegel-Hermite) Eisenstein series $E_{2\ell,n,K}(Z)$ of weight 2ℓ , character $\det^{-\ell}$, is defined in [Ike08] by $E_{2\ell,n,K}(Z) = \sum_{g \in \Gamma_{n,K,\infty} \setminus \Gamma_{n,K}} (\det g)^\ell j(g, Z)^{-2\ell}$ (converges for $\ell > n$). The **normalized Eisenstein series** is given by $\mathcal{E}_{2\ell,n,K}(Z) = 2^{-n} \prod_{i=1}^n L(i - 2\ell, \theta^{i-1}) \cdot E_{2\ell,n,K}(Z)$. If $H \in \Lambda_n(\mathcal{O})^+$, then the H -th Fourier coefficient of $\mathcal{E}_{2\ell}^{(n)}(Z)$ is **polynomial over \mathbb{Z}** in variables $\{p^{\ell-(n/2)}\}_p$, and equals

$$|\gamma(H)|^{\ell-(n/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H, p^{-\ell+(n/2)}), \gamma(H) = (-D_K)^{[n/2]} \det H.$$

Here, $\tilde{F}_p(H, X)$ is a certain Laurent polynomial in the variables $\{X_p = p^{-s}, X_p^{-1}\}_p$ over \mathbb{Z} . This polynomial is a **key point in proving congruences** for the modular forms in a **Rankin-Selberg integral**. Also, for a certain congruence subgroup $C = \Gamma_{\mathfrak{c}}$, $s \in \mathbb{C}$ and a Hecke ideal character $\psi \bmod \mathfrak{c}$, the series is defined

$$E(Z, s, \ell, \psi) = \sum_{g \in C_\infty \setminus C} \psi(g) (\det g)^\ell j(g, Z)^{-2\ell} |(\det g) j(g, Z)|^{-s}.$$

An integral representation of Rankin-Selberg type

The integral representation of Rankin-Selberg type in the Hermitian modular case: is stated for the level \mathfrak{c} modular forms: **Theorem 4.1 (Shimura, Klosin)**, see [Bou16], p.13. Let $0 \neq \mathbf{f} \in \mathcal{M}_\ell(\Gamma_{\mathfrak{c}}, \psi)$ of scalar weight ℓ , $\psi \bmod \mathfrak{c}$, such that $\forall \mathbf{a}, \mathbf{f}|T(\mathbf{a}) = \lambda(\mathbf{a})\mathbf{f}$, and assume that $2\ell \geq n$, then there exists $\mathcal{T} \in S_+ \cap \mathrm{GL}_n(K)$ and $\mathcal{R} \in \mathrm{GL}_n(K)$ such that

$$\Gamma((s))\psi(\det(\mathcal{T}))\mathcal{Z}(s + 3n/2, \mathbf{f}, \chi) = \Lambda_{\mathfrak{c}}(s + 3n/2, \theta\psi\chi) \cdot C_0 \langle \mathbf{f}, \theta_{\mathcal{T}}(\chi) \mathcal{E}(\bar{s} + n, \ell - \ell_\theta, \chi^\rho \psi) \rangle_{C''},$$

where $\mathcal{E}(Z, s, \ell - \ell_\theta, \psi)_{C''}$ is a normalized group theoretic (or adelic) Eisenstein series with components as above of level \mathfrak{c}'' divisible by \mathfrak{c} , and weight $\ell - \ell_\theta$. Here $\langle \cdot, \cdot \rangle_{C''}$ is the normalized Petersson inner product associated to the congruence subgroup C'' of level \mathfrak{c}'' . $\Gamma((s)) = (4\pi)^{-n(s+h)} \Gamma_n^\ell(s + h)$, $\Gamma_n^\ell(s) = \pi^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \Gamma(s - j)$, where $h = 0$ or 1 , C_0 the index of a subgroup.

Proof of the Main Theorem (ii): Kummer congruences

Let us use the notation $\mathcal{D}_p^{alg}(m, \mathbf{f}, \chi) = A_p(s, \chi) \frac{\mathcal{D}^*(m, \mathbf{f}, \chi)}{\Omega_{\mathbf{f}}}$. The integrality of measures is proven representing $\mathcal{D}_p^{alg}(m, \chi)$ as Rankin-Selberg type integral at critical points $s = m$. Coefficients of modular forms in this integral satisfy Kummer-type congruences and produce bounded measures $\mu_{\mathcal{D}}$ whose construction reduces to congruences of Kummer type between the Fourier coefficients of modular forms, see also [Bou16]. Suppose that we are given infinitely many "critical pairs" (s_j, χ_j) at which one has an integral representation $\mathcal{D}_p^{alg}(s_j, \mathbf{f}, \chi_j) = A_p(s, \chi) \frac{\langle \mathbf{f}, h_j \rangle}{\Omega_{\mathbf{f}}}$ with all $h_j =$

$\sum_{\mathcal{J}} b_{\mathcal{J}} q^{\mathcal{J}} \in \mathcal{M}$ in a certain finite-dimensional space \mathcal{M} containing \mathbf{f} and defined over $\bar{\mathbb{Q}}$. We prove the following Kummer-type congruences:

$$\forall x \in \mathbb{Z}_p^*, \sum_j \beta_j \chi_j x^{k_j} \equiv 0 \pmod{p^N} \implies \sum_j \beta_j \mathcal{D}_p^{alg}(s_j, \mathbf{f}, \chi) \equiv 0 \pmod{p^N}$$

$$\beta_j \in \bar{\mathbb{Q}}, k_j = s^* - s_j, \text{ where } s^* = \ell - n \text{ in our case.}$$

Computing the Petersson products of a given modular form $\mathbf{f}(Z) = \sum_H a_H q^H \in \mathcal{M}_*(\bar{\mathbb{Q}})$ by another modular form $h(Z) = \sum_H b_H q^H \in \mathcal{M}_*(\bar{\mathbb{Q}})$ uses a linear form $\ell_{\mathbf{f}} : h \mapsto \frac{\langle \mathbf{f}, h \rangle}{\langle \mathbf{f}, \mathbf{f} \rangle}$ defined over a subfield $k \subset \bar{\mathbb{Q}}$.

Admissible Hermitian case

Let $\mathbf{f} \in \mathcal{S}_{\ell}(C, \psi)$ be a Hecke eigenform for the congruence subgroup $C = \Gamma_{\mathbf{c}}$ of level \mathbf{c} . Let \mathfrak{q} be a prime of K over p , which is **inert** over \mathbb{Q} . Then we say that \mathbf{f} is **pre-ordinary** at \mathfrak{q} if there exists an eigenform $0 \neq \mathbf{f}_0 \in \mathcal{M}_{\{\mathfrak{p}\}} \subset \mathcal{S}_{\ell}(Cp, \psi)$ with Satake parameters $t_{\mathfrak{q},i}$ such that

$$\left\| \left(\prod_{i=1}^n t_{\mathfrak{q},i} \right) N(\mathfrak{q})^{-\frac{n(n+1)}{2}} \right\|_p = 1,$$

where $\|\cdot\|_p$ the normalized absolute value at p .

The **admissible case** corresponds to

$$\left\| \left(\prod_{\mathfrak{q}|p} \prod_{i=1}^n t_{\mathfrak{q},i} \right) p^{-n(n+1)} \right\|_p = p^{-h} \text{ for a positive } h > 0.$$

An interpretation of h as the difference $h = P_{N,p}(d/2) - P_H(d/2)$ comes from the above explicit relations.

Existence of h -admissible measures

of Amice-Vélu-type gives an unbounded p -adic analytic interpolation of the L -values of growth $\log_p^h(\cdot)$, using the Mellin transform of the constructed measures. This condition says that the product $\prod_{i=1}^n t_{\mathfrak{p},i}$ is nonzero and divisible by a certain power of p in \mathcal{O} :

$$\text{ord}_p \left(\prod_{\mathfrak{q}|p} \left(\prod_{i=1}^n t_{\mathfrak{q},i} \right) p^{-n(n+1)} \right) = h.$$

We use an **easy condition of admissibility** of a sequence of modular distributions Φ_j on $X = \mathbb{Z}_p^*$ with values in the semigroup algebra $\mathcal{O}[[q]] = \mathcal{O}[[q^H]]_{H \in \Lambda(\mathcal{O})^+}$ as in Theorem 4.8 of [CourPa]. It suffices to check **congruences** of the type (with $\varkappa = 4$)

$$U^{\varkappa v} \left(\sum_{j'=0}^j \binom{j}{j'} (-a_p^0)^{j-j'} \Phi_{j'}(a + (p^v)) \right) \in Cp^{vj} \mathcal{O}[[q]]$$

for all $j = 0, 1, \dots, \neq h - 1$. Here $s = s^* - j'$, $\Phi_{j'}(a + (p^v))$ a certain convolution of two Hermitian modular forms, i.e.

$$\Phi_{j'}(\chi) = \theta(\chi) \cdot \mathcal{E}(s, \chi)$$

of a Hermitian theta series $\theta(\chi)$ and an Eisenstein series $\mathcal{E}(s, \chi)$ with any Dirichlet character $\chi \bmod p^r$. We use a general sufficient condition of admissibility of a sequence of modular distributions Φ_j on $X = \mathbb{Z}_p^*$ with values in $\mathcal{O}[[q]]$ as in Theorem 4.8 of [CourPa].

Proof of the Main Theorem (iii): (admissible case)

Using a Rankin-Selberg integral representation for $\mathcal{D}^{alg}(s, \mathbf{f}, \chi)$ and an eigenfunction \mathbf{f}_0 of Atkin's operator $U(p)$ of eigenvalue $\alpha_{\mathbf{f}}$ on \mathbf{f}_0 the Rankin-Selberg integral of $\mathcal{F}_{s, \chi} := \theta(\chi) \cdot \mathcal{E}(s, \chi)$ gives

$$\begin{aligned} \mathcal{D}^{alg}(s, \mathbf{f}, \chi) &= \frac{\langle \mathbf{f}_0, \theta(\chi) \cdot \mathcal{E}(s, \chi) \rangle}{\langle \mathbf{f}, \mathbf{f} \rangle} \quad (\text{the Petersson product on } G = GU(\eta_n)) \\ &= \alpha_{\mathbf{f}}^{-v} \frac{\langle \mathbf{f}_0, U(p^v)(\theta(\chi) \cdot \mathcal{E}(s, \chi)) \rangle}{\langle \mathbf{f}, \mathbf{f} \rangle} = \alpha_{\mathbf{f}}^{-v} \frac{\langle \mathbf{f}_0, U(p^v)(\mathcal{F}_{s, \chi}) \rangle}{\langle \mathbf{f}, \mathbf{f} \rangle}. \end{aligned}$$

Modification in the admissible case: **instead of Kummer congruences**, to **estimate p -adically the**

integrals of test functions: $M = p^v$: $\int_{a+(M)} (x-a)^j d\mathcal{D}^{alg} := \sum_{j'=0}^j \binom{j}{j'} (-a)^{j-j'} \int_{a+(M)} x^{j'} d\mathcal{D}^{alg}$,

using the **orthogonality of characters** and the sequence of zeta distributions $\int_{a+(M)} x^j d\mathcal{D}^{alg} = \frac{1}{\#(\mathcal{O}/M\mathcal{O})^\times} \sum_{\chi \bmod M} \chi^{-1}(a) \int_X \chi(x) x^j d\mathcal{D}^{alg}$, $\int_X \chi d\mathcal{D}_{s^*-j}^{alg} = \mathcal{D}^{alg}(s^* - j, \chi) =: \int_X \chi(x) x^j d\mathcal{D}^{alg}$.

Congruences between the coefficients of the Hermitian modular forms

In order to integrate any locally-analytic function on X , it suffices to check the following binomial congruences for the coefficients of the Hermitian modular form $\mathcal{F}_{s^*-j, \chi} = \sum_{\xi} v(\xi, s^* - j, \chi) q^\xi$: for $v \gg 0$, and a constant C

$$\begin{aligned} &\frac{1}{\#(\mathcal{O}/M\mathcal{O})^\times} \sum_{j'=0}^j \binom{j}{j'} (-a)^{j-j'} \sum_{\chi \bmod M} \chi^{-1}(a) v(p^v \xi, s^* - j', \chi) q^\xi \\ &\in Cp^{vj} \mathcal{O}[[q]] \quad (\text{This is a quasimodular form if } j' \neq s^*) \end{aligned}$$

The resulting measure $\mu_{\mathcal{D}}$ allows to integrate all continuous characters in $\mathcal{Y}_p = \text{Hom}_{cont}(X, \mathbb{C}_p^*)$, including Hecke characters, as they are always locally analytic.

Its p -adic Mellin transform $\mathcal{L}_{\mu_{\mathcal{D}}}$ is an analytic function on \mathcal{Y}_p of the logarithmic growth $\mathcal{O}(\log^h)$, $h = \text{ord}_p(\alpha)$.

Proof of the main congruences

Thus the Petersson product in $\ell_{\mathbf{f}}$ can be expressed through the Fourier coefficients of h in the case when there is a finite basis of the dual space consisting of certain Fourier coefficients: $\ell_{\mathcal{J}_i} : h \mapsto b_{\mathcal{J}_i}(i = 1, \dots, n)$. It follows that $\ell_{\mathbf{f}}(h) = \sum_i \gamma_i b_{\mathcal{J}_i}$, where $\gamma_i \in k$.

Using the expression for $\ell_f(h_j) = \sum_i \gamma_{i,j} b_{j,\tau_i}$, the above congruences reduce to

$$\sum_{i,j} \gamma_{i,j} \beta_j b_{j,\tau_i} \equiv 0 \pmod{p^N}.$$

The last congruence is done by an elementary check on the Fourier coefficients b_{j,τ_i} .

The abstract Kummer congruences are checked for a family of test elements.

In the admissible case it suffices to check **binomial congruences** for the Fourier coefficients as above in place of Kummer congruences.

Appendix A. Rewriting the local factor at p with character θ

Notice that if θ is the quadratic character attached to K/\mathbb{Q} then

$$(1 - \alpha_p X)(1 - \alpha_p \theta(p) X) = \begin{cases} (1 - \alpha_p X)^2 & \text{if } \theta(p) = 1, p\mathfrak{r} = \mathfrak{q}_1 \mathfrak{q}_2, N(\mathfrak{q}_i) = p, \\ (1 - \alpha_p^2 X^2), & \text{if } \theta(p) = -1, p\mathfrak{r} = \mathfrak{q}, N(\mathfrak{q}) = p^2, \\ (1 - \alpha_p X) & \text{if } \theta(p) = 0, p\mathfrak{r} = \mathfrak{q}^2, N(\mathfrak{q}) = p. \end{cases}$$

Thus, if $X = p^{-s}$, $X^2 = p^{-2s}$, $N(\mathfrak{q}) = p$, $\mathcal{Z}_{\mathfrak{q}}(X)^{-1}$

$$\begin{aligned} &= \begin{cases} \prod_{i=1}^{2n} (1 - N(\mathfrak{q}_1)^{2n} t_{\mathfrak{q}_1 \mathfrak{q}_2, i}^{-1} X)(1 - N(\mathfrak{q}_2)^{-1} t_{\mathfrak{q}_1 \mathfrak{q}_2, i} X), & \text{if } \theta(p) = 1, \\ \prod_{i=1}^n (1 - N(\mathfrak{q})^{n-1} t_{\mathfrak{q}, i} X^2)(1 - N(\mathfrak{q})^n t_{\mathfrak{q}, i}^{-1} X^2), & \text{if } \theta(p) = -1, \\ \prod_{i=1}^n (1 - N(\mathfrak{q})^{n-1} t_{\mathfrak{q}, i} X)(1 - N(\mathfrak{q})^n t_{\mathfrak{q}, i}^{-1} X), & \text{if } \theta(p) = 0. \end{cases} \\ &= \begin{cases} \prod_{i=1}^n (1 - \gamma_{p,i} X)^2 \prod_{i=1}^n (1 - \delta_{p,i} X)^2 & \text{if } \theta(p) = 1, \text{ i.e. } p\mathfrak{r} = \mathfrak{q}_1 \mathfrak{q}_2, \\ \prod_{i=1}^n (1 - \alpha_{p,i}^2 X^2) \prod_{i=1}^n (1 - \beta_{p,i}^2 X^2), & \text{if } \theta(p) = -1, \text{ i.e. } p\mathfrak{r} = \mathfrak{q}, \\ \prod_{i=1}^n (1 - \alpha'_{p,i} X) \prod_{i=1}^n (1 - \beta'_{p,i} X) & \text{if } \theta(p) = 0, \text{ i.e. } p\mathfrak{r} = \mathfrak{q}^2, \end{cases} \end{aligned}$$

where $\alpha'_{p,i} = p^{n-1} t_{\mathfrak{q}, i}$, $\beta'_{p,i} p^n t_{\mathfrak{q}, i}^{-1}$, $\gamma_{p,i} = p^{2n} t_{\mathfrak{q}_1 \mathfrak{q}_2, i}^{-1}$, $p^{-1} t_{\mathfrak{q}_1 \mathfrak{q}_2, i}$. It follows that $\prod_{\mathfrak{q}|p} \mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-n-(1/2)} X) = X^{4n} + \dots$

Appendix A (continued). Relations between $\alpha_i(p)$ and $t_{i,\mathfrak{q}}$

were studied and explained by M.Harris [Ha97] for general Hermitian zeta functions $\mathcal{Z}(s, \mathbf{f})$ of type introduced in [Shi00], using representation theory of unitary groups and Deligne's approach to L -functions, see [De79], in terms of a n -dimensional Galois representations $\rho_{\lambda} : \text{Gal}(\bar{K}/K) \rightarrow \text{GL}(M_{\mathbf{f},\lambda}) \cong \text{GL}_n(E_{\lambda})$ over a completion E_{λ} of a number field E containing K and the Hecke eigenvalues of a vector-valued Hermitian modular form \mathbf{f} :

$$\mathcal{Z}(s - n' - \frac{1}{2}, \mathbf{f}) = \mathcal{D}(s, \mathbf{f}) = L(s, M_{\mathbf{f},\lambda} \boxtimes M(\psi))$$

for an algebraic Hecke ideal character ψ as above of the infinity type m_{ψ} , see [GH16], p.20. Here the symbol $L(s, M_{\mathbf{f},\lambda} \boxtimes M(\psi))$ denotes the Rankin-Selberg type convolution (it corresponds to tensor product of Galois representations). Notice that $L(s, M_{\mathbf{f},\lambda})$ is of degree $2n$, and $L(s, M_{\mathbf{f},\lambda} \boxtimes M(\psi))$ is of degree $4n$ because $L(s, \psi) = L(s, R(\psi))$ is of degree 2.

Moreover, M.Harris suggested a general description of $\mathcal{D}(s)$ with given Gamma factors and analytic properties as some $\mathcal{D}(s, \mathbf{f})$ some under natural conditions on Gamma factors, giving higher

versions of Shimura-Taniyama-Weil conjecture (i.e. higher Wiles' modularity theorem). This can be stated also over a totally real field F (instead of \mathbb{Q}), and its quadratic totally imaginary extension K , see [GH16], [Pa94].

Appendix B. Shimura's Theorem: algebraicity of critical values in Cases Sp and UT, p.234 of [Shi00]

Let $\mathbf{f} \in \mathcal{V}(\bar{\mathbb{Q}})$ be a non zero arithmetical automorphic form of type Sp or UT. Let χ be a Hecke character of K such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{\ell} |x_{\mathbf{a}}|^{-\ell}$ with $\ell \in \mathbb{Z}^{\mathbf{a}}$, and let $\sigma_0 \in 2^{-1}\mathbb{Z}$. Assume, in the notations of Chapter 7 of [Shi00] on the weights k_v, μ_v, ℓ_v , that

Case Sp	$2n + 1 - k_v + \mu_v \leq 2\sigma_0 \leq k_v - \mu_v,$ where $\mu_v = 0$ if $[k_v] - l_v \in 2\mathbb{Z}$ and $\mu_v = 1$ if $[k_v] - l_v \notin 2\mathbb{Z}$; $\sigma_0 - k_v + \mu_v$ for every $v \in \mathbf{a}$ if $\sigma_0 > n$ and $\sigma_0 - 1 - k_v + \mu_v \in 2\mathbb{Z}$ for every $v \in \mathbf{a}$ if $\sigma_0 \leq n$.
Case UT	$4n - (2k_{v\rho} + \ell_v) \leq 2\sigma_0 \leq m_v - k_v - k_{v\rho} - \ell_v $ and $2\sigma_0 - \ell_v \in 2\mathbb{Z}$ for every $v \in \mathbf{a}$.

Appendix B. Shimura's Theorem (continued)

Further exclude the following cases

- | | |
|-------------|---|
| (A) Case Sp | $\sigma_0 = n + 1, F = \mathbb{Q}$ and $\chi^2 = 1$; |
| (B) Case Sp | $\sigma_0 = n + (3/2), F = \mathbb{Q}; \chi^2 = 1$ and $[k] - \ell \in 2\mathbb{Z}$ |
| (C) Case Sp | $\sigma_0 = 0, \mathfrak{c} = \mathfrak{g}$ and $\chi = 1$; |
| (D) Case Sp | $0 < \sigma_0 \leq n, \mathfrak{c} = \mathfrak{g}, \chi^2 = 1$ and the conductor of χ is \mathfrak{g} ; |
| (E) Case UT | $2\sigma_0 = 2n + 1, F = \mathbb{Q}, \chi_1 = \theta$, and $k_v - k_{v\rho} = \ell_v$; |
| (F) Case UT | $0 \leq 2\sigma_0 < 2n, \mathfrak{c} = \mathfrak{g}, \chi_1 = \theta^{2\sigma_0}$ and the conductor of χ is \mathfrak{r} |

Then

$$\mathcal{Z}(\sigma_0, \mathbf{f}, \chi) / \langle \mathbf{f}, \mathbf{f} \rangle \in \pi^{n|m|+d\varepsilon} \bar{\mathbb{Q}},$$

where $d = [F : \mathbb{Q}]$, $|m| = \sum_{v \in \mathbf{a}} m_v$, and

$$\varepsilon = \begin{cases} (n+1)\sigma_0 - n^2 - n, & \text{Case Sp, } k \in \mathbb{Z}^{\mathbf{a}}, \text{ and } \sigma_0 > n_0, \\ n\sigma_0 - n^2, & \text{Case Sp, } k \notin \mathbb{Z}^{\mathbf{a}}, \text{ or } \sigma_0 \leq n_0, \\ 2n\sigma_0 - 2n^2 + n & \text{Case UT} \end{cases}$$

Notice that $\pi^{n|m|+d\varepsilon} \in \mathbb{Z}$ in all cases; if $k \notin \mathbb{Z}^{\mathbf{a}}$, the above parity condition on σ_0 shows that $\sigma_0 + k_v \in \mathbb{Z}$, so that $n|m| + d\varepsilon \in \mathbb{Z}$.

Appendix C. Examples of Hermitian cusp forms

The Hermitian Ikeda lift, [Ike08]. Assume $n = 2n'$ even. Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in \mathcal{S}_{2k+1}(\Gamma_0(D_K), \chi)$ be a primitive form, whose L -function is given by

$$L(f, s) = \prod_{p \nmid D_K} (1 - a(p)p^{-s} + \theta(p)p^{2k-2s})^{-1} \prod_{p|D_K} (1 - a(p)p^{-s})^{-1}.$$

For each prime $p \nmid D_K$, define the Satake parameter $\{\alpha_p, \beta_p\} = \{\alpha_p, \theta(p)\alpha_p^{-1}\}$ by

$$(1 - a(p)X + \theta(p)p^{2k}X^2) = (1 - p^k\alpha_p X)(1 - p^k\beta_p X)$$

For $p|D_K$, we put $\alpha_p = p^{-k}a(p)$. Put

$$\begin{aligned} A(H) &= |\gamma(H)|^k \prod_{p|\gamma(H)} \tilde{F}_p(H; \alpha_p), H \in \Lambda_n(\mathcal{O})^+ \\ \mathbf{f}(Z) &= \sum_{H \in \Lambda_n(\mathcal{O})^+} A(H)q^H, Z \in \mathcal{H}_{2n}. \end{aligned}$$

Appendix C (continued). The first theorem (even case)

Theorem 5.1 (Case E) of [Ike08] Assume that $n = 2n'$ is even. Let $f(\tau)$, $A(H)$ and $\mathbf{f}(Z)$ be as above. Then we have $\mathbf{f} \in \mathcal{S}_{2k+2n'}(\Gamma_K^{(n)}, \det^{-k-n'})$.

In the case when n is odd, consider a similar lifting for a normalized Hecke eigenform $n = 2n' + 1$ is odd. Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in \mathcal{S}_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a primitive form, whose L -function is given by

$$L(f, s) = \prod_p (1 - a(p)p^{-s} + p^{2k-1-2s})^{-1}.$$

For each prime p , define the Satake parameter $\{\alpha_p, \alpha_p^{-1}\}$ by

$$(1 - a(p)X + p^{2k-1}X^2) = (1 - p^{k-(1/2)}\alpha_p X)(1 - p^{k-(1/2)}\alpha_p^{-1}X).$$

Put

$$\begin{aligned} A(H) &= |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H; \alpha_p), H \in \Lambda_n(\mathcal{O})^+ \\ \mathbf{f}(Z) &= \sum_{H \in \Lambda_n(\mathcal{O})^+} A(H)q^H, Z \in \mathcal{H}_n. \end{aligned}$$

Appendix C (continued). The second theorem (odd case)

Theorem 5.2 (Case O) of [Ike08]. Assume that $n = 2n' + 1$ is odd. Let $f(\tau)$, $A(H)$ and $\mathbf{f}(Z)$ be as above. Then we have $\mathbf{f} \in \mathcal{S}_{2k+2n'}(\Gamma_K^{(n)}, \det^{-k-n'})$.

The lift $Lift^{(n)}(f)$ of f is a common Hecke eigenform of all Hecke operators of the unitary group, if it is not identically zero (Theorem 13.6).

Theorem 18.1 of [Ike08]. Let n, n' , and f be as in Theorem 5.1 or as in Theorem 5.2. Assume that $Lift^{(n)}(f) \neq 0$. Let $L(s, Lift^{(n)}(f), st)$ be the L -function of $Lift^{(n)}(\mathbf{f})$ associated to $st : {}^L\mathcal{G} \rightarrow \mathrm{GL}_{4n}(\mathbb{C})$. Then up to bad Euler factors, $L(s, Lift^{(n)}(f), st)$ is equal to

$$\prod_{i=1}^n L(s + k + n' - i + \frac{1}{2}, f) L(s + k + n' - i + \frac{1}{2}, f, \theta).$$

Moreover, the $4n$ characteristic roots of $L(s, Lift^{(n)}(f), st)$ given as follows: for $i = 1, \dots, n$

$$\alpha_p p^{-k-n'+i-\frac{1}{2}}, \alpha_p^{-1} p^{-k-n'+i-\frac{1}{2}}, \theta(p) \alpha_p p^{-k-n'+i-\frac{1}{2}}, \theta(p) \alpha_p^{-1} p^{-k-n'+i-\frac{1}{2}}$$

Functional equation of the lift (thanks to Sho Takemori!)

There are two cases [Ike08]: the even case (E) and the odd case (O):

$$\left\{ \begin{array}{l} f \in S_{2k+1}(\Gamma_0(D), \theta), \mathbf{f} = Lift^{(n)}(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n}) \quad (E) \\ \text{(of even degree } n = 2n' \text{ and of weight } 2k + 2n') \\ f \in S_{2k}(\mathrm{SL}(\mathbb{Z})), \mathbf{f} = Lift^{(n)}(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n}) \quad (O) \\ \text{(of odd degree } n = 2n' + 1 \text{ and of weight } 2k + 2n'). \end{array} \right. \quad \text{Then, up to bad Euler factors, the}$$

standard L -function of $\mathbf{f} = Lift^{(n)}(f)$ is given by $\mathcal{Z}(s, \mathbf{f}) = \prod_{i=1}^n L(s + k + n' - i + \frac{1}{2}, f) L(s + k + n' - i + \frac{1}{2}, f, \theta)$

Let us denote $t(s, i) = s + k + n' - i + \frac{1}{2}$ then

$$\begin{aligned} & \left\{ \begin{array}{l} \prod_{i=1}^{2n'} L(s + k + n' - i + \frac{1}{2}, f) L(s + k + n' - i + \frac{1}{2}, f, \theta) \quad (E) \\ \prod_{i=1}^{n'} L(t(s, i), f) L(t(s, n+1-i), f) \\ L(t(s, i), f, \theta) L(t(s, n+1-i), f, \theta) \\ \prod_{i=1}^{2n'+1} L(s + k + n' - i + \frac{1}{2}, f) \\ \times L(s + k + n' - i + \frac{1}{2}, f, \theta) \quad (O) \\ = L(s + k - \frac{1}{2}, f) L(s + k - \frac{1}{2}, f, \theta) \\ \prod_{i=1}^{n'} L(t(s, i), f) L(t(s, n+1-i), f) \\ L(t(s, i), f, \theta) L(t(s, n+1-i), f, \theta). \end{array} \right. \end{aligned}$$

The Gamma factor $\Gamma_{\mathcal{Z}}(s)$ of Ikeda's lift

In the even case $t(1-s, n+1-i) = t(1-s, 2n'+1-i) = (2k+1) - t(s, i)$. The Hecke functional equation $s \mapsto 2k+1-s$ in all symmetric terms of the product, gives the functional equation of the standard L -function of the form $s \mapsto 1-s$, and the gamma factor is then

$$\prod_{i=1}^n \Gamma_{\mathbb{C}}(s + k + n' - i + 1/2)^2 = \Gamma_{\mathcal{D}}(s + n' + \frac{1}{2}).$$

In the odd case $n = 2n' + 1$ when $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$, the $Lift(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n})$. By $2k - t(s, i) = t(1-s, n+1-i)$, the standard L functions has functional equation of the form $s \mapsto 1-s$ and the gamma factor is the same.

Hence the Gamma factor of Ikeda's lifting, denoted by \mathbf{f} , of an elliptic modular form f and **used as a pattern**, extends to a general (not necessarily lifted) Hermitian modular form \mathbf{f} of even weight ℓ , which equals in the lifted case to $\ell = 2k + 2n'$, where $k = (\ell - 2n')/2 = \ell/2 - n' = \ell/2 - n'$, when the Gamma factor of the standard zeta function with the symmetry $s \mapsto 1 - s$ becomes (see p.14) $\prod_{i=1}^n \Gamma_{\mathbb{C}}(s + \ell/2 - n' + n' - i + (1/2))^2 = \prod_{i=1}^n \Gamma_{\mathbb{C}}(s + \ell/2 - i + (1/2))^2 = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s + \ell/2 - i - (1/2))^2$.

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Best wishes to dear Mirjana Vukovic of good health, enthusiasm in life and mathematics, happiness and many new good friends and achievements!

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