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Jacques Darné

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On the Andreadakis problem for subgroups of IA_n

Jacques DARNÉ

August 20, 2018

Abstract

Let F_n be the free group on n generators. Consider the group IA_n of automorphisms of F_n acting trivially on its abelianization. There are two canonical filtrations on IA_n : the first one is its lower central series Γ_* ; the second one is the Andreadakis filtration \mathcal{A}_* , defined from the action on F_n . The Andreadakis problem consists in understanding the difference between these filtrations. Here, we show that they coincide when restricted to the subgroup of triangular automorphisms, and to the pure braid group.

Introduction

The group $\text{Aut}(F_n)$ of automorphisms of a free group has a very rich structure, which is somewhat ill-understood. It is linked to various groups appearing in low-dimensional topology : it contains the mapping class group, the braid group, the loop braid group, etc. By looking at its action on $F_n^{ab} \cong \mathbb{Z}^n$, we can decompose $\text{Aut}(F_n)$ as an extension of $GL_n(\mathbb{Z})$ by IA_n , the latter being the subgroup of automorphisms acting trivially on \mathbb{Z}^n . By analogy with the case of the mapping class group, IA_n is known as the *Torelli subgroup* of $\text{Aut}(F_n)$. An explicit finite set of generators of IA_n has been known for a long time [Nie24] – see also [BBM07, 5.6] and our appendix. Nevertheless, the structure of IA_n remains largely mysterious. For instance, IA_3 is not finitely presented [KM97], and it is not known whether IA_n is finitely presented for $n > 3$.

One of the most prominent questions concerning the structure of this Torelli group is the Andreadakis problem. Consider the lower central series $F_n = \Gamma_1(F_n) \supseteq \Gamma_2(F_n) \supseteq \dots$. From it we can define the *Andreadakis filtration* $IA_n = \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \dots$: we define \mathcal{A}_j as the subgroup of automorphisms acting trivially on $F_n/\Gamma_{j+1}(F_n)$ (which is the free nilpotent group of class j on n generators). This filtration of IA_n is central (it is even *strongly central*), so it contains the minimal central filtration on IA_n , its lower central series:

$$\forall k \geq 1, \mathcal{A}_k \supseteq \Gamma_k(IA_n).$$

Problem 1 (Andreadakis). *What is the difference between \mathcal{A}_* and $\Gamma_*(IA_n)$?*

Andreadakis conjectured that the filtrations were the same [And65, p. 253]. In [Bar13], Bartholdi disproved the conjecture, using computer calculations. He then tried to prove that the two filtrations were the same up to finite index, but in the erratum [Bar16], he showed that even this weaker statement cannot be true. His latter proof uses the L -presentation of IA_n given in [DP16], to which he applies algorithmic methods described in [BEH08] to calculate (using the software GAP) the first degrees of the graded groups associated to each filtration.

The present paper is devoted to the study of the Andreadakis problem when restricted to some subgroups of the Torelli group IA_n . Precisely, if G is a subgroup of IA_n , we can consider the two filtrations induced on G by our original filtrations, and we can compare them to the lower central series of G :

$$\Gamma_*(G) \subseteq \Gamma_*(IA_n) \cap G \subseteq \mathcal{A}_* \cap G. \quad (0.1)$$

Problem 2 (Andreadakis problem for subgroups of IA_n). *For which subgroups G of IA_n are the above inclusions equalities ?*

Definition 1. We say that the *Andreadakis equality* holds for a subgroup G of IA_n when $\Gamma_*(G) = \mathcal{A}_* \cap G$.

Obviously, this is not always the case: for instance, the Andreadakis equality does not hold for the cyclic group generated by an element of $\Gamma_2(IA_n)$. However, for some nicely embedded groups, we can hope that it could hold, and it is indeed the case:

Theorem 2 (Th. 5.4, Cor. 5.5 and Th. 6.2). *Let G be the subgroup of triangular automorphisms IA_n^+ , the triangular McCool subgroup $P\Sigma_n^+$, or the pure braid group P_n acting via the Artin action. Then the Andreadakis equality holds for G :*

$$\Gamma_*(G) = \Gamma_*(IA_n) \cap G = \mathcal{A}_* \cap G.$$

The statement about triangular automorphisms has independently been obtained by T. Satoh [Sat17].

The subgroup IA_n^+ is introduced in Definition 5.1. We treat the cases of IA_n^+ and P_n in Sections 5 and 6 ; the proof in the case of $P\Sigma_n^+$ is a straightforward adaptation of the proof for IA_n^+ . The methods used in both cases are very similar: both use a decomposition as an iterated almost-direct product, and fit in the general framework we introduce in Section 4. Sections 1 to 3 consist mainly of reminders from [Dar18], with some additional material, especially in Paragraph 2.1, where we present a general adjunction involving semi-direct products, and in Paragraph 3.1, where we write down a description of the lower central series of a semi-direct product of groups.

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Contents

Introduction	1
1 Strongly central filtrations	3
2 Semi-direct products and actions	4
2.1 Actions: an abstract definition	4
2.2 Representability of actions	6
2.3 Actions in \mathcal{SCF}	6
2.4 The Andreadakis problem	7

3	Exactness of the Lie functor	7
3.1	Lower central series of a semi-direct product of groups	7
3.2	Johnson morphisms	9
4	Decomposition of an induced filtration	10
4.1	General setting	10
4.2	Application to the Andreadakis problem	10
5	First application: triangular automorphisms	11
5.1	Decomposition as an iterated almost-direct product	11
5.2	The Andreadakis equality	12
6	Second application: the pure braid group	13
6.1	Decomposition as an iterated almost-direct product	14
6.2	The Andreadakis equality	14
7	Appendix: Generating sets and relations	15
7.A	Generators of IA_n^+	15
7.B	Presentation of the Drinfeld-Kohno Lie ring	16

1 Strongly central filtrations

Throughout the paper, G will denote an arbitrary group,. The left and right action of G on itself by conjugation are denoted respectively by $x^y = y^{-1}xy$ and ${}^yx = yxy^{-1}$. The *commutator* of two elements x and y in G is $[x, y] := xyx^{-1}y^{-1}$. If A and B are subsets of G , we denote by $[A, B]$ the subgroup generated by the commutators $[a, b]$ with $(a, b) \in A \times B$. We denote the *abelianization* of G by $G^{ab} := G/[G, G]$ and its lower central series by $\Gamma_*(G)$, that is :

$$G =: \Gamma_1(G) \supseteq [G, G] =: \Gamma_2(G) \supseteq [G, \Gamma_2(G)] =: \Gamma_3(G) \supseteq \dots$$

We recall the definition of the category \mathcal{SCF} introduced in [Dar18].

Definition 1.1. A strongly central filtration G_* is a nested sequence of groups $G_1 \supseteq G_2 \supseteq G_3 \dots$ such that $[G_i, G_j] \subseteq G_{i+j}$ for all $i, j \geq 1$. These filtrations are the objects of a category \mathcal{SCF} , where morphisms from G_* to H_* are those group morphisms from G_1 to H_1 sending each G_i into H_i .

Recall that this category has the following features [Dar18]:

- There are forgetful functors $\omega_i : G_* \mapsto G_i$ from \mathcal{SCF} to the category of groups. Since the lower central series $\Gamma_*(G)$ is the minimal strongly central series on a group G , the functor Γ_* is left adjoint to ω_1 .
- There is a functor from the category \mathcal{SCF} to the category $\mathcal{Lie}_{\mathbb{Z}}$ of Lie rings (*i.e.* Lie algebras over \mathbb{Z}), given by: $G_* \mapsto \mathcal{L}(G_*) := \bigoplus G_i/G_{i+1}$, where Lie brackets are induced by group commutators.
- The category \mathcal{SCF} is complete and cocomplete. To compute limits (resp. colimit), just endow the corresponding colimit of groups with the maximal (resp. minimal) compatible filtration.

- It is homological [BB04, def. 4.1.1]. This means essentially that the usual lemmas of homological algebra (the nine lemma, the five lemma, the snake lemma, etc.) are true there.
- It is action-representative (see Paragraph 2.3 below).

In a homological category, we need to distinguish between usual epimorphisms (resp. monomorphisms) and *regular* ones, that is, the ones obtained as coequalizers (resp. equalizers). In \mathcal{SCF} , the former are the u such that $u_1 = \omega_1(u)$ is an epimorphism (resp. a monomorphism), whereas the latter are *surjections* (resp. *injections*):

Definition 1.2. Let $u : G_* \rightarrow H_*$ be a morphism in \mathcal{SCF} . It is called an *injection* (resp. a *surjection*) when u_1 is injective (resp. surjective) and $u^{-1}(H_i) = G_i$ (resp. $u(G_i) = H_i$) for all i .

We can use this to give an explicit interpretation of the general notion of *short exact sequences* [BB04, Def 4.1.5] in \mathcal{SCF} : $1 \rightarrow G_* \xrightarrow{u} H_* \xrightarrow{v} K_* \rightarrow 1$ is a short exact sequence if and only if:

$$\begin{cases} u \text{ is an injection,} \\ v \text{ is a surjection,} \\ u(G_1) = \ker(v). \end{cases} \quad (1.2.1)$$

2 Semi-direct products and actions

An action of group on another one by automorphisms, or of a Lie algebra on another one by derivations, are particular cases of a general notion of actions in a homological (or even only protomodular) category. We recall here the definitions and elementary properties of such actions, and what actions are in \mathcal{SCF} .

2.1 Actions: an abstract definition

In this paragraph, we use the language of [BB04]. A concise and accessible reference on the subject is [HL11]. The reader not familiar with this language can replace the category \mathcal{C} by his favorite example (groups or Lie algebras, for instance).

Definition 2.1. Let \mathcal{C} be a protomodular category. If X and Z are two objects of \mathcal{C} , we define an *action* of Z on X as a split extension (with a given splitting):

$$X \hookrightarrow Y \overset{\curvearrowright}{\twoheadrightarrow} Z.$$

When such an action is given, we will say that Z *acts on* X , and write: $Z \circlearrowleft X$. Actions in \mathcal{C} form a category $\mathcal{Act}(\mathcal{C})$, morphisms between two actions being the obvious ones.

Example 2.2. The category $\mathcal{Act}(\mathcal{Grp})$ is the usual category of group actions on one another by automorphisms: a morphism from $G \circlearrowleft H$ to $G' \circlearrowleft H'$ is given by a morphism $u : G \rightarrow G'$ and a morphism $v : H \rightarrow H'$, v being u -equivariant, that is: $\forall g \in G, \forall h \in H, v(g \cdot h) = u(g) \cdot v(h)$.

Definition 2.3. We denote by \times the functor from $\mathcal{Act}(\mathcal{C})$ to \mathcal{C} sending an action

$$X \hookrightarrow Y \overset{\curvearrowright}{\twoheadrightarrow} Z$$

on Y (which is called $Z \ltimes X$), and by ad the functor from \mathcal{C} to $\mathcal{Act}(\mathcal{C})$ sending an object C on *the adjoint action*:

$$C \xrightarrow{\iota_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} C^2 \xrightleftharpoons[\pi_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}]{\delta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}} C.$$

Proposition 2.4. *The above functors are adjoint to each other:*

$$\ltimes : \mathcal{Act}(\mathcal{C}) \xrightleftharpoons{\text{ad}} \mathcal{C} : \text{ad}.$$

Proof. Let C, X and Z be objects of \mathcal{C} . Let an action of Z on X be given. A morphism $\varphi : Z \ltimes X \rightarrow C$ induces a morphism between actions:

$$\begin{array}{ccccc} X & \xrightarrow{i} & Z \ltimes X & \xrightleftharpoons[p]{s} & Z \\ \varphi i \downarrow & & (\varphi, \varphi s p) \downarrow & & \downarrow \varphi s \\ C & \xrightarrow{\iota_1} & C^2 & \xrightleftharpoons[\pi_2]{\delta} & C. \end{array}$$

Conversely, a morphism between these actions gives in particular a morphism $\varphi : Z \ltimes X \rightarrow C^2 \xrightarrow{\pi_1} C$. One can easily check that these constructions are inverse to each other. \square

Example 2.5. When $\mathcal{C} = \text{Grp}$, the functor ad sends G to the conjugation action $G \circ G$. The semi-direct product can be *defined* as this functor's left adjoint.

Any functor $F : \mathcal{C} \rightarrow \mathcal{D}$ does preserve split epimorphisms. As a consequence:

Fact 2.6. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between protomodular categories. Then, for any action $Z \circ X$ in \mathcal{C} , $F(Z \ltimes X) = F(Z) \ltimes \ker(F(Z \ltimes X \twoheadrightarrow Z))$.*

This allows us to define an induced functor $F_{\#}$ between the categories of actions:

$$F_{\#} \left(X \xrightarrow{i} Y \xrightleftharpoons[p]{s} Z \right) := \ker(Fp) \hookrightarrow F(Y) \xrightleftharpoons[Fp]{Fs} F(Z). \quad (2.6.1)$$

Remark that the description of $F_{\#}$ is particularly simple when F preserves kernels of split epimorphisms: then $\ker(Fp) = F(X)$.

Remark 2.7. This construction $F \mapsto F_{\#}$ makes the construction of the category of actions into a functor from protomodular categories to categories. As one sees easily, it is a 2-functor and, as such, it preserves adjunctions. Moreover, if $F : \mathcal{C} \xrightleftharpoons{G} \mathcal{D} : G$ is an adjunction, then we can write the following diagram:

$$\begin{array}{ccc} \mathcal{Act}(\mathcal{C}) & \xrightleftharpoons[\text{ad}]{\ltimes} & \mathcal{C} \\ F_{\#} \downarrow \uparrow G_{\#} & & \downarrow \uparrow G \\ \mathcal{Act}(\mathcal{D}) & \xrightleftharpoons[\text{ad}]{\ltimes} & \mathcal{D}. \end{array} \quad (2.7.1)$$

Since G commutes to limits, the square of right adjoints is commutative: $G_{\#} \circ \text{ad} = \text{ad} \circ G$. Obviously, the square of left adjoints also commute (which is an equivalent statement).

2.2 Representability of actions

The set $\text{Act}(Z, X)$ of actions of Z on X is a contravariant functor in Z : the restriction of an action along a morphism is defined *via* a pullback. In $\mathcal{G}rp$, as in $\mathcal{L}ie$, this functor is representable, for any X . Indeed, an action of a group K on a group G is given by a morphism $K \rightarrow \text{Aut}(G)$. Similarly, an action of a Lie algebra \mathfrak{k} on a Lie algebra \mathfrak{g} is given by a morphism $\mathfrak{k} \rightarrow \text{Der}(\mathfrak{g})$. The situation when actions are representable has notably been studied in [BJK05]. The following terminology was introduced in [BB07, Def. 1.1]:

Definition 2.8. A protomodular category \mathcal{C} is said to be *action-representative* when the functor $\text{Act}(-, X)$ is representable, for any object $X \in \mathcal{C}$.

A representative for $\text{Act}(-, X)$ is a universal action on X . Explicitly, it is an action of an object $\mathcal{A}(X)$ on X such that any action $Z \curvearrowright X$ is obtained by restriction along a unique morphism $Z \rightarrow \mathcal{A}(X)$.

2.3 Actions in \mathcal{SCF}

Using the explicit description of exact sequences in \mathcal{SCF} given at the end of the first section (1.2.1), we can describe explicitly what an action in \mathcal{SCF} is:

Proposition 2.9. [Dar18, Prop. 1.20]. *An action $K_* \curvearrowright G_*$ in \mathcal{SCF} is the data of a group action of $K = K_1$ on $G = G_1$ satisfying:*

$$\forall i, j \geq 1, [K_i, G_j] \subseteq G_{i+j}.$$

Using this description, one can show that actions are representable in \mathcal{SCF} :

Theorem 2.10. [Dar18, Th. 1.16, Prop. 1.22]. *Let G_* be a strongly central series. Let $j \geq 1$ be an integer. Define $\mathcal{A}_j(G_*) \subseteq \text{Aut}(G_*)$ to be :*

$$\mathcal{A}_j(G_*) = \{ \sigma \in \text{Aut}(G_*) \mid \forall i \geq 1, [\sigma, G_i] \subseteq G_{i+j} \}, \quad (2.10.1)$$

where the commutator is computed in $G_1 \rtimes \text{Aut}(G_*)$, that is: $[\sigma, g] = \sigma(g)g^{-1}$. That is, $\mathcal{A}_j(G_*)$ is the group of automorphisms of G_* acting trivially on every quotient G_i/G_{i+j} . Then $\mathcal{A}_*(G_*)$ is a strongly central series. Moreover, it acts canonically on G_* , and this action is universal. In particular, the category \mathcal{SCF} is action-representative.

If a group K acts on a group G , and G_* is a strongly central filtration on $G = G_1$, we can pull back the canonical filtration $\mathcal{A}_*(G_*)$ by the associated morphism:

$$K \rightarrow \text{Aut}(G).$$

This gives a strongly central filtration $\mathcal{A}_*(K, G_*)$, maximal amongst strongly central filtrations on subgroups of K which act on G_* *via* the given action $K \curvearrowright G$. It can be described explicitly as:

$$\mathcal{A}_j(K, G_*) = \{ k \in K \mid \forall i \geq 1, [k, G_i] \subseteq G_{i+j} \} \subseteq K. \quad (2.10.2)$$

2.4 The Andreadakis problem

Let G be a group. We denote $\mathcal{L}(\Gamma_*(G))$ by $\mathcal{L}(G)$, and we call it *the Lie ring of G* . As products of commutators become sums of brackets inside the Lie algebra, the following fundamental property follows from the definition of the lower central series:

Proposition 2.11. [Dar18, Prop. 1.19]. *The Lie ring $\mathcal{L}(G)$ is generated in degree 1. Precisely, it is generated (as a Lie ring) by $\mathcal{L}_1(G) = G^{ab}$.*

Consider $\mathcal{A}_*(G) := \mathcal{A}_*(\Gamma_*G)$. The group $\mathcal{A}_1(G)$ is the group of automorphisms of G acting trivially on $\mathcal{L}(G)$. But this Lie algebra is generated in degree 1. As a consequence, $\mathcal{A}_1(G)$ is the subgroup of automorphisms acting trivially on the abelianization $G^{ab} = \mathcal{L}_1(G)$, denoted by IA_G . The filtration $\mathcal{A}_*(G)$, being strongly central on $\mathcal{A}_1(G) = IA_G$, contains $\Gamma_*(IA_G)$. We are thus led to the problem of comparing these filtrations ; this is the *Andreadakis problem* (Problem 1), which is a crucial question when trying to understand the structure of automorphism groups of residually nilpotent groups, in particular when trying to understand the structure of $\text{Aut}(F_n)$.

Problem 1 (Andreadakis). *How close is the inclusion $\Gamma_*(IA_G) \subseteq \mathcal{A}_*(G)$ to be an equality ?*

3 Exactness of the Lie functor

In this section, we investigate the Lie algebra of a semi-direct product of groups, and we recall the construction of the Johnson morphism associated with an action in \mathcal{SCF} . Both rely on the following fundamental proposition:

Proposition 3.1. [Dar18, Prop. 1.24]. *The Lie functor $\mathcal{L} : \mathcal{SCF} \rightarrow \text{Lie}_{\mathbb{Z}}$ is exact, i.e. it preserves short exact sequences.*

Corollary 3.2. *The functor \mathcal{L} preserves actions. In other words, if $K_* \circlearrowleft H_*$ is an action in \mathcal{SCF} , then $\mathcal{L}(H_* \rtimes K_*) = \mathcal{L}(H_*) \rtimes \mathcal{L}(K_*)$.*

3.1 Lower central series of a semi-direct product of groups

We can use the tools introduced so far to study the lower central series of a semi-direct product of groups, and its Lie algebra. Precisely, let $G = H \rtimes K$ be a semi-direct product of groups. The functor $F = \Gamma$ preserves split epimorphisms, whence a decomposition into a semi-direct product of strongly central series:

$$\Gamma_*G = H_* \rtimes \Gamma_*K,$$

where H_i is the kernel of the split projection:

$$\Gamma_iG \begin{array}{c} \dashleftarrow \\ \longrightarrow \end{array} \Gamma_iK.$$

The aim of the present paragraph is to give an explicit description of H_* (that is, using the notations of 2.1, to describe $\Gamma_{\#}(K \circlearrowleft H)$) and to identify the conditions under which H_* is equal to Γ_*H .

Let us begin by introducing a general construction:

Proposition-definition 3.3. *Let G be a group, and H a normal subgroup. We define a strongly central filtration $\Gamma_*^G(H)$ on H by:*

$$\begin{cases} \Gamma_1^G(H) := H, \\ \Gamma_{k+1}^G := [G, \Gamma_k^G(H)]. \end{cases}$$

Proof. The inclusions $\Gamma_{k+1}^G \subseteq \Gamma_k^G$ are obtained by induction on k , the first one being the normality of H in G . The strong centrality statement is obtained by induction, using the 3-subgroups lemma. \square

Since $[G, \Gamma_k^G(H)] \subseteq \Gamma_k^G(H)$, the $\Gamma_k^G(H)$ are normal in G . In fact, we have:

$$[G, \Gamma_k^G(H)] \subseteq \Gamma_{k+1}^G(H), \text{ so } G = \mathcal{A}_1(G, \Gamma_*^G(H)).$$

As a consequence, $\mathcal{A}_*(G, \Gamma_*^G(H))$ is a strongly central filtration on the whole of G , so it contains $\Gamma_*(G)$. Thus $\Gamma_*(G)$ acts on $\Gamma_*^G(H)$. Moreover, it is clear that $\Gamma_*^G(H)$ is the minimal strongly central filtration on H such that the action G on H induces an action of $\Gamma_*(G)$.

Now, let $K \curvearrowright H$ be a group action. We can apply the above construction to $G = H \rtimes K \supseteq H$. We will write $\Gamma_*^K(H)$ for $\Gamma_*^{H \rtimes K}(H)$ (this will not cause any confusion: if H is a normal subgroup of a group G , then $\Gamma_*^G(H) = \Gamma_*^{H \rtimes G}(H)$, for the semi-direct product associated to the conjugation action of G on H). Using these constructions, we can identify the filtration H_* defined above:

Proposition 3.4. *If a group K acts on a group H , then:*

$$\Gamma_*(H \rtimes K) = \Gamma_*^K(H) \rtimes \Gamma_*H.$$

Proof. Since $\Gamma_*(K) \subseteq \Gamma_*(H \rtimes K)$ acts on $\Gamma_*^K(H)$, the filtration $\Gamma_*^K(H) \rtimes \Gamma_*(K)$ is strongly central on $H \rtimes K$, so it contains $\Gamma_*(H \rtimes K)$. The other inclusion follows directly from the definitions. \square

Remark 3.5. We have thus identified $\Gamma_{\#} = \mathcal{Act}(\Gamma)$ (see Paragraph 2.1):

$$\Gamma_{\#} : K \curvearrowright H \longmapsto \Gamma_*(K) \curvearrowright \Gamma_*^K(H).$$

In this context, the diagram (2.7.1) reads:

$$\begin{array}{ccc} \mathcal{Act}(\mathcal{SCF}) & \begin{array}{c} \xrightarrow{\times} \\ \xleftarrow{c} \end{array} & \mathcal{SCF} \\ \Gamma_{\#} \downarrow \uparrow \omega_1 & & \Gamma \downarrow \uparrow \omega_1 \\ \mathcal{Act}(\mathcal{Grp}) & \begin{array}{c} \xrightarrow{\times} \\ \xleftarrow{c} \end{array} & \mathcal{Grp}. \end{array}$$

We now describe the conditions under which $\Gamma_*^K(H) = \Gamma_*(H)$:

Proposition-definition 3.6. *Let $H \rtimes K$ be a semi-direct product of groups. Then the following assertions are equivalent:*

1. *The action of K on H^{ab} is trivial,*
2. $[K, H] \subseteq [H, H] = \Gamma_2(H),$

3. $K \circlearrowleft H$ induces an action $\Gamma_*K \circlearrowleft \Gamma_*H$,
4. $\Gamma_*^K(H) = \Gamma_*H$,
5. $\forall i, \Gamma_i(H \rtimes K) = \Gamma_iH \rtimes \Gamma_iK$,
6. $\mathcal{L}(H \rtimes K) \cong \mathcal{L}(H) \rtimes \mathcal{L}(K)$,
7. $(H \rtimes K)^{ab} \cong H^{ab} \times K^{ab}$.

When these conditions are satisfied, we will say that the semi-direct product $H \rtimes K$ is an almost-direct one.

Proof. The second statement means that K acts trivially on $H^{ab} = \mathcal{L}_1(H)$, hence on $\mathcal{L}(H)$, as this algebra is generated in degree one 1 (proposition 2.11). But this means exactly that:

$$\forall i, [K, \Gamma_iH] \subseteq \Gamma_{i+1}H,$$

which is equivalent to K being equal to $\mathcal{A}_1(K, \Gamma_*H)$. Since $\mathcal{A}_*(K, \Gamma_*H)$ is strongly central, this in turn is equivalent to $\Gamma_*K \subseteq \mathcal{A}_*(K, \Gamma_*H)$, which is exactly (3).

The filtration Γ_*H is the minimal strongly central series on H , and $\Gamma_*^K(H)$ is the minimal one on which Γ_*K acts (through the given action of K on H). Hence the equivalence with (4). The assertion (5) is clearly equivalent to (4) and, using the exactness of \mathcal{L} (Proposition (3.1)), we see that it implies (6). The remaining implications (6) \Rightarrow (7) \Rightarrow (2) are straightforward. \square

3.2 Johnson morphisms

In this paragraph also, we recall some material from [Dar18].

As a consequence of Proposition 3.1, the functor \mathcal{L} preserves actions. Precisely, from an action in \mathcal{SCF} :

$$G_* \hookrightarrow H_* \overset{\leftarrow}{\twoheadrightarrow} K_*,$$

we get an action in the category of graded Lie rings:

$$\mathcal{L}(G_*) \hookrightarrow \mathcal{L}(H_*) \overset{\leftarrow}{\twoheadrightarrow} \mathcal{L}(K_*).$$

Such an action is given by a morphism of graded Lie rings:

$$\tau : \mathcal{L}(K_*) \longrightarrow \text{Der}_*(\mathcal{L}(G_*)). \quad (3.6.1)$$

The target is the (graded) Lie algebra of graded derivations: a derivation is of degree k when it raises degrees of homogeneous elements by k .

Definition 3.7. The morphism (3.6.1) is called the *Johnson morphism* associated to the given action $K_* \circlearrowleft G_*$.

We can give an explicit description of this morphism: for $k \in K$, the derivation associated to \bar{k} is induced by $[\bar{k}, -]$ inside $\mathcal{L}(G_* \rtimes K_*) = \mathcal{L}(G_*) \rtimes \mathcal{L}(K_*)$, so it is induced by $[k, -]$ inside $G_* \rtimes K_*$.

Example 3.8. The Johnson morphism associated to the universal action $\mathcal{A}_*(G_*) \circlearrowleft G_*$ is the Lie morphism $\tau : \mathcal{L}(\mathcal{A}_*(G_*)) \longrightarrow \text{Der}_*(\mathcal{L}(G_*))$ induced by $\sigma \mapsto (x \mapsto \sigma(x)x^{-1})$.

The *Johnson morphism* turns out to be a powerful tool in the study of the Andreadakis filtration, thanks to the following injectivity statement:

Lemma 3.9. [Dar18, Lem. 1.28] *Let $K_* \curvearrowright G_*$ be an action in \mathcal{SCF} . The associated Johnson morphism $\tau : \mathcal{L}(K_*) \rightarrow \text{Der}_*(\mathcal{L}(G_*))$ is injective if and only if $K_* = \mathcal{A}_*(K_1, G_*)$.*

Example 3.10. If G is a free group, then $\mathcal{L}(F_n)$ is the free Lie algebra $\mathfrak{L}V$ on the \mathbb{Z} -module $V = G^{ab}$ [Laz54, th. 4.2]. It is also free with respect to derivations, which can be considered as sections of a suitable projection – see for instance [Reu03]. In particular:

$$\text{Der}_k(\mathfrak{L}V) \cong \text{Hom}_{\mathbb{k}}(V, \mathfrak{L}_k V) \cong V^* \otimes \mathfrak{L}_k V.$$

The Andreadakis filtration $\mathcal{A}_* = \mathcal{A}_*(F_n)$ is the universal one acting on $\Gamma_*(F_n)$. Thus the associated Johnson morphism is an embedding:

$$\tau : \mathcal{L}(\mathcal{A}_*) \hookrightarrow \text{Der}(\mathfrak{L}V). \quad (3.10.1)$$

4 Decomposition of an induced filtration

4.1 General setting

Let G be a group endowed with a strongly central filtration \mathcal{A}_* . Let $H \rtimes K$ be a subgroup of G decomposing as a semi-direct product. Then we get, on the one hand, a semi-direct product $(\mathcal{A}_* \cap H) \rtimes (\mathcal{A}_* \cap K)$ of strongly central series. On the other hand, $\mathcal{A}_* \cap (H \rtimes K)$ is a strongly central filtration on $H \rtimes K$ containing the previous one.

Proposition 4.1. *In the above setting, the following assertions are equivalent:*

- (i) $\mathcal{A}_* \cap (H \rtimes K) = (\mathcal{A}_* \cap H) \rtimes (\mathcal{A}_* \cap K)$,
- (ii) Inside $\mathcal{L}(\mathcal{A}_*)$, $\mathcal{L}(\mathcal{A}_* \cap H) \cap \mathcal{L}(\mathcal{A}_* \cap K) = 0$.

When they are satisfied, we will say that H and K are \mathcal{A}_* -disjoint.

Proof. If (i) is true, then the subalgebra $\mathcal{L}(\mathcal{A}_* \cap (H \rtimes K))$ of $\mathcal{L}(\mathcal{A}_*)$ decomposes as a semi-direct product of $\mathcal{L}(\mathcal{A}_* \cap H)$ by $\mathcal{L}(\mathcal{A}_* \cap K)$, thus (ii) holds.

Conversely, suppose (i) false. Then there exists $g = hk \in \mathcal{A}_j \cap (H \rtimes K)$, where neither h nor k belongs to \mathcal{A}_j . This means that $h \equiv k^{-1} \not\equiv 1 \pmod{\mathcal{A}_j}$. Then there exists $i < j$ such that $h, k \in \mathcal{A}_i - \mathcal{A}_{i+1}$, giving a counter-example to our second assertion: $\bar{h} = -\bar{k} \neq 0 \in \mathcal{A}_i / \mathcal{A}_{i+1}$. \square

4.2 Application to the Andreadakis problem

We can apply Proposition 4.1 to the case when $G = IA_n$ and \mathcal{A}_* is the Andreadakis filtration. In that case, the Johnson morphism gives an embedding of $\mathcal{L}(\mathcal{A}_*)$ into $\text{Der}(\mathfrak{L}V)$ (see Example 3.10). Thus, we can check whether H and K are \mathcal{A}_* -disjoint by answering the following question: can an element of K and an element of H induce the same derivation of $\mathfrak{L}V$?

When the subgroups are \mathcal{A}_* -disjoint, then $\mathcal{A}_* \cap (H \rtimes K)$ is the semi-direct product of $\mathcal{A}_* \cap H$ by $\mathcal{A}_* \cap K$. Suppose moreover that the semi-direct product $H \rtimes K$ is an almost-direct one. Then the lower central series $\Gamma_* \cap (H \rtimes K)$ also decomposes as the semi-direct of $\Gamma_* \cap H$ by $\Gamma_* \cap K$. Thus, under these hypotheses, in order to show that $\mathcal{A}_* \cap (H \rtimes K) = \Gamma_* \cap (H \rtimes K)$, we just need to prove that $\mathcal{A}_* H = \Gamma_* H$ and that $\mathcal{A}_* K = \Gamma_* K$. We sum this up in the following:

Theorem 4.2. *Let $H \rtimes K$ be a subgroup of IA_n . Suppose that:*

1. *the semi-direct product is an almost-direct one,*
2. *an element of H and an element of K cannot induce the same derivation of the free Lie algebra through the Johnson morphism,*
3. *the Andreadakis equality holds for H and K ,*

then the Andreadakis equality holds for $H \rtimes K$.

5 First application: triangular automorphisms

Definition 5.1. Fix (x_1, \dots, x_n) an ordered basis of F_n . The subgroup IA_n^+ of IA_n consists of *triangular automorphisms*, i.e. automorphisms φ acting as :

$$\varphi : x_i \mapsto (x_i^{w_i})\gamma_i,$$

where $w_i \in \langle x_j \rangle_{j < i} \cong F_{i-1}$ et $\gamma_i \in \Gamma_2(F_{i-1})$.

5.1 Decomposition as an iterated almost-direct product

Consider the subgroup of IA_n^+ of triangular automorphisms fixing every element of the basis, except for the i -th one. This subgroup is the kernel of the projection $IA_i^+ \rightarrow IA_{i-1}^+$ induced by $x_i \mapsto 1$. It is isomorphic to $\Gamma_2(F_{i-1}) \rtimes F_{i-1}$, the isomorphism being:

$$(\varphi : x_i \mapsto (x_i^w)\gamma) \mapsto (\gamma, w).$$

Thus we obtain a short exact sequence:

$$\Gamma_2(F_{n-1}) \rtimes F_{n-1} \hookrightarrow IA_n^+ \twoheadrightarrow IA_{n-1}^+. \quad (5.1.1)$$

This sequence is split: a section is given by automorphisms fixing x_n . Hence, we get a decomposition into a semi-direct product:

$$IA_n^+ = (\Gamma_2(F_{n-1}) \rtimes F_{n-1}) \rtimes IA_{n-1}^+. \quad (5.1.2)$$

Lemma 5.2. *Let G be a group. For any integer i :*

$$\Gamma_*(\Gamma_i G \rtimes G) = \Gamma_{*+i-1}(G) \rtimes \Gamma_* G.$$

Proof. From the definition of the lower central series and from Definition 3.3 (G acting on $\Gamma_i G$ by conjugation), we immediately deduce the following equality, true for every i :

$$\Gamma_*^G(\Gamma_i G) = \Gamma_{*+i-1}(G).$$

The result follows, by Proposition 3.4. □

In particular, for $G = F_n$ and $i = 2$, this determines the lower central series of $\Gamma_2(F_{n-1}) \rtimes F_{n-1}$:

$$\Gamma_*(\Gamma_2(F_{n-1}) \rtimes F_{n-1}) = \Gamma_{*+1}(F_{n-1}) \rtimes \Gamma_* F_{n-1}.$$

Whence the following description of its abelianization:

$$(\Gamma_2(F_{n-1}) \rtimes F_{n-1})^{ab} = \Gamma_2/\Gamma_3(F_{n-1}) \times F_{n-1}^{ab} = (\mathcal{L}_2 \oplus \mathcal{L}_1)(F_{n-1}).$$

The extension (5.1.1) induces an action of IA_{n-1}^+ on this abelianization. One can easily check that this action is none other than the diagonal action induced by the canonical action of IA_{n-1}^+ on the Lie algebra of F_{n-1} . This action is trivial, by definition of IA_{n-1} , which means that the semi-direct product (5.1.2) is an almost-direct one. Thus, Proposition 3.6 allows us to describe the lower central series of IA_n^+ :

Proposition 5.3. *For every integer n :*

$$\begin{aligned} \Gamma_*(IA_n^+) &= \Gamma_*(\Gamma_2(F_{n-1}) \rtimes F_{n-1}) \rtimes \Gamma_*(IA_{n-1}^+) \\ &= (\Gamma_{*+1}(F_{n-1}) \rtimes \Gamma_*F_{n-1}) \rtimes \Gamma_*(IA_{n-1}^+). \end{aligned}$$

In particular, the Lie algebra of IA_n^+ decomposes as:

$$\mathcal{L}(IA_n^+) = (\mathfrak{L}_{*+1}(\mathbb{Z}^{n-1}) \rtimes \mathfrak{L}_*(\mathbb{Z}^{n-1})) \rtimes \mathcal{L}(IA_{n-1}^+).$$

5.2 The Andreadakis equality

Our aim is to use the decomposition into an almost-direct product described in the previous paragraph to recover the main result of [Sat17]:

Theorem 5.4. *The subgroup of triangular automorphisms satisfies the Andreadakis equality, that is:*

$$\Gamma_*(IA_n^+) = \Gamma_*(IA_n) \cap IA_n^+ = \mathcal{A}_* \cap IA_n^+ = \mathcal{A}_*(IA_n^+, \Gamma_*(F_n)).$$

Proof. Consider the decomposition (5.1.2). We want to apply Theorem 4.2 to it. First, we need to show that the factors are \mathcal{A}_* -disjoint. But this is obvious : each $\varphi \in IA_{n-1}^+$ satisfies $[\varphi, x_n] = 1$, whereas each element ψ in the other factor satisfies $[\psi, x_i] = 1$ for $i < n$. Thus, a derivation d coming from both factors would satisfy $d(x_i) = 0$ for every i , so it must be trivial.

We are thus reduced to showing the Andreadakis equality for the first factor, and the result will follow by induction. Let $\psi : x_n \mapsto x_n^w \cdot \gamma$ be an element of this factor ($w \in F_{n-1}$, $\gamma \in \Gamma_2 F_{n-1}$, and ψ fixes the other generators). Then:

$$\begin{aligned} \varphi \in \mathcal{A}_j &\Leftrightarrow \varphi(x_n) \equiv x_n \pmod{\Gamma_{j+1}(F_n)} \\ &\Leftrightarrow \gamma \equiv [x_n, w] \pmod{\Gamma_{j+1}(F_n)}. \end{aligned}$$

We claim that this is possible only if $\gamma \in \Gamma_{j+1}(F_n)$ and $w \in \Gamma_j(F_n)$. Indeed, let k such that $w \in \Gamma_k - \Gamma_{k+1}$ (such a k exists because F_n is residually nilpotent). If we had $k < j$, then:

$$0 \neq \bar{w} \in \mathcal{L}_k(F_{n-1}) \subseteq \mathcal{L}_k(F_n).$$

Since $\mathcal{L}_*(F_n)$ is the free Lie algebra over the \bar{x}_s , and \bar{x}_n does not appear in \bar{w} , then $[\bar{x}_n, \bar{w}] \neq 0 \in \mathcal{L}_{k+1}(F_n)$, and $[\bar{x}_n, \bar{w}]$, containing \bar{x}_n , cannot be in $\mathcal{L}(F_{n-1})$. In particular, it cannot be equal to $\bar{\gamma}$, which contradicts this hypothesis. Thus we must have $k \geq j$, that is $w \in \Gamma_j$, and $\gamma \equiv [x_n, w] \in \Gamma_{j+1}$.

Using the description of the lower central series of $\Gamma_2(F_{n-1}) \rtimes F_{n-1}$ from the previous paragraph, we see that this means exactly that $\psi \in \Gamma_j(\Gamma_2(F_{n-1}) \rtimes F_{n-1})$, whence the Andreadakis equality for this subgroup, which is the desired conclusion. \square

We can also state the Andreadakis equality for the triangular McCool subgroup, studied in [CPVW08]. It is not a consequence of the previous theorem, but of its proof. It was not obvious from the proof given in [Sat17] ; however, by Lemma 3.9, it is equivalent to the injectivity of the Johnson morphism $\mathcal{L}(P\Sigma_n^+) \rightarrow \text{Der}(LV)$, that was showed in [CHP11, Cor. 6.3].

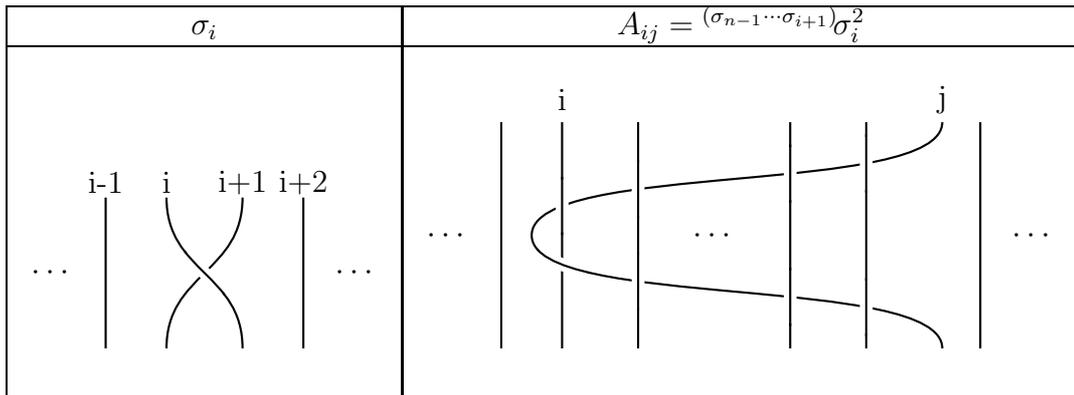
Corollary 5.5. *The subgroup $P\Sigma_n^+$ of triangular basis-conjugating automorphisms satisfies the Andreadakis equality, that is:*

$$\Gamma_*(P\Sigma_n^+) = \Gamma_*(IA_n) \cap P\Sigma_n^+ = \mathcal{A}_* \cap P\Sigma_n^+ = \mathcal{A}_*(P\Sigma_n^+, \Gamma_*(F_n)).$$

Proof. In the proof of Theorem 5.4, consider only those factors corresponding to basis-conjugating automorphisms : take all γ and γ_i to be 1, and forget the factors $\Gamma_2(F_{n-1})$ corresponding to these elements. \square

6 Second application: the pure braid group

We refer to [Bir74] or the more recent [BB05] for a detailed introduction to braid groups. As usual, we denote by B_n Artin's braid group, generated by the σ_i ($1 \leq i < n$), and by P_n the subgroup of pure braids, generated by the A_{ij} ($1 \leq i < j \leq n$). Recall the geometric description of the generators:



An embedding of B_n into B_{n+1} is given by sending σ_i to σ_i (it identifies B_n to the subgroup of braids on the first n strings). Inside B_{n+1} , the $A_{i,n+1} =: x_i$ generate a free group F_n , which is stable under conjugation by elements of B_n . This conjugation action is called the *Artin action* ; explictely, σ_i acts *via* the automorphism:

$$\begin{cases} x_i & \mapsto {}^{x_i}x_{i+1}. \\ x_{i+1} & \mapsto x_i \end{cases} \quad (6.0.1)$$

That $\langle A_{i,n+1} \rangle$ is a free group can be seen using a geometric argument: this subgroup is the kernel of the projection of P_{n+1} onto P_n obtained by forgetting the n -th string, and this kernel identifies canonically with $\pi_1(\mathbb{R}^2 - \{n \text{ points}\})$ [Bir74, th. 1.4]. The above surjection of P_{n+1} onto P_n is split, a splitting been given by the above inclusion of B_n into B_{n+1} . We thus get a decomposition of the pure braid group as a semi-direct product:

$$P_{n+1} = P_n \rtimes F_n. \quad (6.0.2)$$

This decomposition allows us to write any β in P_n uniquely as $\beta' \beta_n$, with $\beta' \in P_n$ and $\beta_n \in \langle A_{1,n}, \dots, A_{n-1,n} \rangle \cong F_{n-1}$. Iterating this, we obtain a unique decomposition of β as:

$$\beta = \beta_1 \cdots \beta_n, \quad \text{avec } \beta_k \in \langle A_{1,k}, \dots, A_{k-1,k} \rangle \cong F_{k-1}.$$

We then say that we have *combed* the braid β . This is key in the proof of the following:

Proposition 6.1. *The Artin action of B_n on F_n is faithful.*

We can thus embed B_n into $\text{Aut}(F_n)$. We will often identify B_n with its image in $\text{Aut}(F_n)$, even if this embedding depends on the choice of an ordered basis of F_n . The corresponding action of B_n on F_n^{ab} is by permutation of the corresponding basis. Thus, for any choice of basis:

$$B_n \cap IA_n = P_n. \quad (6.1.1)$$

6.1 Decomposition as an iterated almost-direct product

Consider the decomposition (6.0.2) of the pure braid group. Since P_n acts through IA_n on F_n (6.1.1), this decomposition is an almost-direct-product. From Proposition 3.6, we deduce that the lower central series also decomposes:

$$\forall j, \Gamma_j(P_n) = \Gamma_j(P_{n-1}) \rtimes \Gamma_j(F_{n-1}),$$

and so does the Lie ring of P_n :

$$\mathcal{L}(P_n) = \mathcal{L}(P_{n-1}) \rtimes \mathcal{L}(F_{n-1}) = \mathcal{L}(P_{n-1}) \rtimes \mathfrak{L}(\mathbb{Z}^{n-1}). \quad (6.1.2)$$

Thus, $\mathcal{L}(P_n)$ decomposes as an iterated semi-direct product of free Lie algebras. With a little more work, using the classical presentation of P_n , we can get a presentation of this Lie ring, which is none other than the Drinfeld-Kohno Lie ring (Proposition 7.3 in the appendix). We refer the reader to the original work [Koh85], or to the recent book [Fre17, section 10.0] for more on this algebra, with a somewhat different point of view.

6.2 The Andreadakis equality

Theorem 6.2. *The pure braid group, embedded into IA_n via the Artin action, satisfies the Andreadakis equality:*

$$\Gamma_*(P_n) = \Gamma_*(IA_n) \cap P_n = \mathcal{A}_* \cap P_n = \mathcal{A}_*(P_n, \Gamma_*(F_n)).$$

Proof. We apply theorem 4.2 to the decomposition $P_n = P_{n-1} \rtimes F_{n-1}$ described above. \mathcal{A}_* -disjointness is easy to verify: any $\beta \in P_{n-1}$ commutes with braids on the strings n and $n+1$, so $[\beta, x_n] = 1$ (where $x_n = A_{n,n+1}$), whereas no w in $F_{n-1} = \langle A_{1,n}, \dots, A_{n-1,n} \rangle$ can commute with x_n , because $\langle A_{1,n}, \dots, A_{n-1,n}, A_{n,n+1} \rangle$ is a free group (it is the one obtained by exchanging the roles of the strings n and $n+1$ in the arguments above). Thus no $[\bar{\beta}, -]$ can coincide with some non-trivial $[\bar{w}, -]$.

In order to show that the factor F_{n-1} satisfies the Andreadakis equality (from which the result follows by induction, using Theorem 4.2), we need the following lemma :

Lemma 6.3. *Let $w \in F_n$ such that for some i , $[w, x_i] \in \Gamma_{j+1}(F_n)$. Then:*

$$\exists n \in \mathbb{Z}, wx_i^n \in \Gamma_j(F_n).$$

In particular, if $w \in \langle x_1, \dots, \hat{x}_i, \dots, x_n \rangle$, or if $w \in \Gamma_2(F_n)$, then n must be 0, so that $w \in \Gamma_j(F_n)$.

Let $w \in \mathcal{A}_j(F_n) \cap \langle A_{1,n}, \dots, A_{n-1,n} \rangle = F_{n-1}$. Then $[w, x_n]$ is in $\Gamma_{j+1}(F_n)$, which is contained in $\Gamma_{j+1}(P_{n+1})$, by the above calculation of the lower central series of P_{n+1} . Observe that w and x_n belong to $\langle A_{1,n}, \dots, A_{n-1,n}, A_{n,n+1} \rangle$, which is another copy of the free group on n generators in the braid group, that will be called \tilde{F}_n . Exchanging the roles of the strings n and $n+1$ in the previous paragraph give an almost-direct product decomposition $P_{n+1} = P_n \times \tilde{F}_n$ and, using Proposition 6.1:

$$\Gamma_{j+1}(P_{n+1}) \cap \tilde{F}_n = \Gamma_{j+1}(\tilde{F}_n).$$

Thus $[w, x_n] \in \Gamma_{j+1}(\tilde{F}_n)$. But since $w \in \langle A_{1,n}, \dots, A_{n-1,n} \rangle = F_{n-1}$, the generator $x_n = A_{n,n+1}$ of \tilde{F}_n does not appear in w . We deduce the conclusion we were looking for, using Lemma 6.3: $w \in \Gamma_j(\tilde{F}_n) \cap F_{n-1} = \Gamma_j(F_{n-1}) \subseteq \Gamma_j(P_n)$. \square

Proof of Lemma 6.3. Let k such that $w \in \Gamma_k - \Gamma_{k+1}$ (such a k exists since F_n is residually nilpotent). If we had $2 \leq k < j$, then $[\bar{w}, x_i] = 0$ inside $\mathcal{L}_{k+1}(F_n) = \mathfrak{L}_{k+1}(V)$, since $k+1 < j+1$. Hence \bar{w} would be a non-trivial element of the centralizer $C(x_i)$ of x_i in the free Lie algebra $\mathfrak{L}(V)$. But $C(x_i) = \mathbb{Z}x_i \subseteq \mathfrak{L}_1(V)$, so k must be 1, which contradicts our hypothesis.

If $w \in F_n - \Gamma_2$, then $\bar{w} \in C(x_i) = \mathbb{Z}x_i$, so there is an n such that $w \equiv x_i^n \pmod{\Gamma_2}$. Then $w x_i^{-n}$ is in Γ_2 and it satisfies the hypothesis of the lemma, so it is in Γ_j , using the first part of the proof. \square

Remark 6.4 (G. Massuyeau). The Andreadakis filtration on braids is none other than the one given by vanishing of the first Milnor invariants. Precisely, a pure braid β acts by conjugation on the (fixed) basis of the free group, x_i being acted upon by the *parallel* w_i [Mil57, Ohk82]. The braid is in \mathcal{A}_j if and only if the parallel is in $\Gamma_j F_n$ (use the above proof), that is, if and only if its Magnus expansion is in $1 + (X_1, \dots, X_n)^k$. This last condition is exactly the vanishing of its Milnor invariants of length less than k . Thus, our Theorem 6.2 can also be interpreted as a consequence of two facts well-known to knot theorists: Milnor invariants of length at most $d+1$ of pure braids generate Vassiliev invariants of degree at most d ; a braid is in $\Gamma_{d+1} P_n$ if and only if it is undetected by Vassiliev invariants of degree at most d [HM00, MW02].

7 Appendix: Generating sets and relations

A finite set of generators of IA_n has been known for a long time [Nie24] – see also [BBM07, 5.6]. These are:

$$K_{ij} : x_t \mapsto \begin{cases} x_j x_i x_j^{-1} & \text{if } t = i \\ x_t & \text{else} \end{cases} \quad \text{and} \quad K_{ijk} : x_t \mapsto \begin{cases} [x_j, x_k] x_i & \text{if } t = i \\ x_t & \text{else.} \end{cases} \quad (7.0.1)$$

7.A Generators of IA_n^+

We give a family of generators of the group of triangular automorphisms:

Lemma 7.1. IA_n^+ is generated by the following elements:

$$\begin{cases} K_{ij} : x_i \mapsto x_i^{x_j} & \text{for } j < i, \\ K_{ijk} : x_i \mapsto x_i[x_j, x_k] & \text{for } j, k < i, \end{cases}$$

Proof. Let G be the subgroup generated by the above elements. Let $\varphi \in IA_n^+$, sending each x_i on some $(x_i^{w_i})\gamma_i$ (as in Definition 5.1). We can decompose φ as $\varphi = \varphi_n \circ \dots \circ \varphi_2$, where φ_i fixes all elements of the basis, except for the i -th one, that is sends to $(x_i^{w_i})\gamma_i$. We claim that each φ_i is in G . Let us fix i , and consider only automorphisms fixing all elements of the basis, save the i -th one. One can check that the subgroup of G generated by the K_{ij} ($j < i$) is the set of automorphism of the form $c_{i,w} : x_i \mapsto x_i^w$ (for $w \in F_{i-1}$). Using the formulas $[a, bc] = [a, b] \cdot ({}^b[a, c])$, $[a, b]^{-1} = [b, a]$ and $[a^{-1}, b] = [b, a]^a$, we can decompose the product of commutators γ_i as:

$$\gamma_i = \prod_{k=1}^m [x_{\alpha_k}, x_{\beta_k}]^{\omega_k},$$

where $\omega_k \in F_{i-1}$. Thus $\varphi_i = c_{i,w_i} \circ K_{i,\alpha_1,\beta_1}^{c_{i,\omega_1}} \circ \dots \circ K_{i,\alpha_m,\beta_m}^{c_{i,\omega_m}}$, whence the desired conclusion. \square

Remark 7.2. Fix $i \geq 1$. From the above proof, we see that the automorphisms:

$$\begin{cases} x_i \mapsto x_i^{x_j} & \text{pour } j < i \quad (x_j \in F_{i-1}), \\ x_i \mapsto x_i[x_j, x_k] & \text{pour } j, k < i \quad ([x_j, x_k] \in \Gamma_2(F_{i-1})), \end{cases}$$

generate the subgroup of IA_n^+ of triangular automorphisms fixing every element of the basis, save the i -th one.

7.B Presentation of the Drinfeld-Kohno Lie ring

The Drinfeld-Kohno Lie ring is the Lie ring $\mathcal{L}(P_n)$. Our methods can be used to recover the usual presentation of this Lie ring:

Proposition 7.3. *The Lie ring of P_n is generated by t_{ij} ($1 \leq i, j \leq n$), under the relations:*

$$\begin{cases} t_{ij} = t_{ji}, t_{ii} = 0 & \forall i, j, \\ [t_{ij}, t_{ik} + t_{kj}] = 0 & \forall i, j, k, \\ [[t_{ij}, t_{kl}] = 0 & \text{si } \{i, j\} \cap \{k, l\} = \emptyset. \end{cases}$$

Proof. We use the classical presentation of the pure braid group (see for instance [BB05]) given, for $r < s < n$ and $i < n$, denoting $A_{\alpha,n}$ by $A_{\alpha,n}$:

$$[A_{rs}, x_i] = [A_{rs}, A_{in}] = \begin{cases} 1 & \text{si } s < i \text{ ou } i < r, \\ [x_i^{-1}, x_r^{-1}] & \text{si } s = i, \\ [x_s^{-1}, x_i] & \text{si } r = i, \\ [[x_r, x_s]^{-1}, x_i] & \text{si } r < i < s. \end{cases}$$

These relations are easily verified ; that they are enough to describe pure braids comes from the decomposition into free factors. We use a similar reasoning to show our proposition.

Let \mathfrak{p}_n be the Lie ring defined by the presentation of the proposition. The relations of the above presentations of P_n imply that the classes of A_{ij} in $\mathcal{L}(P_n)$ (which generate it by 2.11) satisfy these relations, whence a morphism u_n from \mathfrak{p}_n onto $\mathcal{L}(P_n)$. We also can define a split epimorphism π_n from \mathfrak{p}_n onto \mathfrak{p}_{n-1} by sending t_{ij} on 0 if $n \in \{i, j\}$, and on t_{ij} else. Denote by \mathfrak{k}_n the kernel of π_n , and consider the diagram:

$$\begin{array}{ccccc} \mathfrak{k}_n & \hookrightarrow & \mathfrak{p}_n & \xrightarrow{\pi_n} & \mathfrak{p}_{n-1} \\ \downarrow v_n & & \downarrow u_n & & \downarrow u_{n-1} \\ \mathcal{L}(F_{n-1}) & \hookrightarrow & \mathcal{L}(P_n) & \twoheadrightarrow & \mathcal{L}(P_{n-1}) \end{array}$$

where the second line comes from the decomposition (6.1.2). The algebra \mathfrak{k}_n is the ideal of \mathfrak{p}_n generated by the t_{in} . Since the subalgebra generated by the t_{in} already is an ideal, \mathfrak{k}_n is generated by the t_{in} as a Lie algebra. Since v_n sends the t_{in} on a basis of the free Lie algebra $\mathcal{L}(F_{n-1})$, it is an isomorphism (the universal property of the free Lie algebra gives an inverse). Remark that u_1 obviously is an isomorphism. By induction, using the five lemma, we deduce that u_n is an isomorphism. \square

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