

# Summations over generalized ribbon graphs and all genus categorical Gromov-Witten invariants.

102e Rencontre entre mathématiciens et physiciens

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$\psi_i = c_1(T_{p_i}^*)$ ,  $\psi_i \in H^2(\bar{\mathcal{M}}_{g,n})$ . Consider  $H^*(\bar{\mathcal{M}}_{g,n})$ -valued generating function for products of  $\psi_i$

- ▶ Theorem (SB,2009)

$$\sum_{\sum d_i=d} \psi_1^{d_1} \dots \psi_n^{d_n} \prod_{i=1}^n \frac{(2d_i - 1)!!}{\lambda_i^{2d_i+1}} =$$

$$= \left[ \sum_{[G] \in \Gamma_{g,n}^{dec, odd}} (G, \text{or}(G)) \frac{2^{-\chi(G)}}{|\text{Aut}(G)|} \prod_{e \in \text{Edge}(G)} \frac{1}{\lambda_{i(e)} + \lambda_{j(e)}} \right]$$

- ▶ the sum on the right is over *stable ribbon* graphs of genus  $g$  with  $n$  numbered punctures, with  $2d + n$  edges, and such that the vertices of the graph have cyclically ordered subsets of *arbitrary odd* cardinality
- ▶ In the simplest case, corresponding to the top degree, the cohomology  $H^{6g-6+2n}(\bar{\mathcal{M}}_{g,n})$  is 1-dimensional, the summation is over 3-valent usual ribbon graphs and this formula then reproduces the main identity from Kontsevich proof of Witten conjecture.

## Aim of the talk

This formula is a byproduct of construction of cohomology classes

- ▶  $I$  - an odd derivation acting on cyclic associative  $A_\infty$ -algebra  $A$ , with odd scalar product, in general  $I^2 \neq 0$  (!)
- ▶ **Theorem (S.B.2009)** This data  $\rightarrow$  Cohomology classes in  $H^*(\overline{\mathcal{M}}_{g,n})$
- ▶ This construction defines Cohomological Field Theory
- ▶ Applied to “odd matrix algebra”  $A = q(N)$ ,  
 $q(N) = \{[X, \pi] = 0 \mid X \in gl(N|N)\}$ , where  $\pi$ -odd involution,  $I = [\Xi, \cdot]$ ,  $\Xi$ - odd element  $\rightarrow$  the formula for products of  $\psi_i$
- ▶ This was the first nontrivial computation of categorical Gromov-Witten invariants of higher genus.

# Counterexample to a Theorem of Kontsevich

## Another byproduct is a counterexample to the Theorem 1.3 from Kontsevich “Feynman diagrams in low-dimensional topology” (1993)

### Dual construction

We describe here a way to produce *homology* classes on  $\mathcal{M}_{g,n}$ . The starting ingredient is a finite-dimensional differential associative algebra  $V$  with a non-degenerate *odd* scalar product and trivial cohomology.

The right inverse to the scalar product can be considered as an odd element  $\delta$  of  $V \otimes V$ . It follows from the compatibility of the scalar product with the differential that  $\delta$  is closed. From the triviality of  $H(V)$  it follows that there exists  $\omega \in V \otimes V$  such that  $d\omega = \delta$ . We can use  $\omega$  as a “propagator” and tensors  $T_{(k)} : V^{\otimes k} \rightarrow \mathbf{C}$

$$T(v_1 \otimes \dots \otimes v_k) = (v_1 v_2 \dots v_{k-1}, v_k)$$

as “interactions”. Again, we obtain a state model on ribbon graphs. Now we will consider the complex which is *dual* to the cochain complex from the previous section.

**Theorem 1.3**  $\sum_{\Gamma} Z(\Gamma)\Gamma$  is a well-defined chain and it is closed. Its homology class does not depend on the choice of  $\omega$ .

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It turns out this sum is NOT closed

## Counterexample $A = \langle 1, \zeta \rangle / \zeta^2 = 1$

- ▶  $A = \langle 1, \zeta \rangle / \zeta^2 = 1, \bar{\zeta} = 1, \beta(1, \zeta) = 1, d(\zeta) = 1, d^2 = 0, w = \zeta \otimes \zeta.$
- ▶ For this data  $\sum_{\Gamma} Z(\Gamma)\Gamma$  is nonzero on the boundary of the following ribbon graph = a generator of the dual complex:

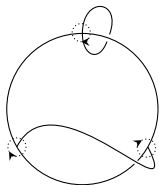


Figure:  $G: Z_{\Gamma}(\partial G) \neq 0$

## Cellular decomposition of $\bar{\mathcal{M}}_{g,n} \times \mathbb{R}^n$

A *stable ribbon graph* is a connected graph  $G$  (recall: a graph  $G$  is a triple  $(Flag(G), \lambda, \sigma)$ , where  $Flag(G)$  is a finite set, whose elements are called flags,  $\lambda$  is a partition of  $Flag(G) \leftrightarrow$  vertices,  $\sigma$  is an involution acting on  $Flag(G) \leftrightarrow$  edges/legs) together with:

- ▶ partitions of the set of flags adjacent to every vertex into  $i(v)$  nonempty subsets

$$Leg(v) = Leg(v)^{(1)} \sqcup \dots \sqcup Leg(v)^{(i(v))}, v \in Vert(G)$$

- ▶ fixed cyclic order on every subset  $Leg(v)^{(k)}$ ,
- ▶ a number  $g(v) \in \mathbb{Z}_{\geq 0}$  such that  $|Leg(v)| > 2(2 - i(v) - 2g(v))$ .

- ▶ Define an orientation

$$or(G) \in Det(\otimes_{v \in Vert(G)} (k^{Flag(v)} \oplus k^{Cycle(v)}))$$

- ▶  $\oplus_{\{G\}} k(G, or(G))$  has natural “generalized contraction of edges” differential  $\delta_{graph}$ ,  $\delta_{graph}^2 = 0$

## Cellular decomposition of $\overline{\mathcal{M}}_{g,n} \times \mathbb{R}^n$

A metric on the stable ribbon graph is a function  $l : \text{Edge}(G) \rightarrow \mathbb{R}_{>0}$ . Given a stable ribbon graph  $G$  w/out legs and a metric on  $G$  one can construct by standard procedure a punctured Riemann surface  $\Sigma(G)$ , which have double points in general.

- ▶ replace every edge by a pair of oriented strips  $[0, l] \times [0 + i\infty[$  one for each flag and glue them side  $0 \times [0 + i\infty[$  to  $l \times [0 + i\infty[$  according to the cyclic order of the cyclically ordered subsets at each vertex. Then glue the two strips for each edge  $[0, l] \leftrightarrow [l, 0]$
- ▶ identify points corresponding to 2 subsets at vertices with  $g(v) = 0, i(v) = 2$ , (double points); for points with  $2g(v) + i(v) > 2$  remplace the vertex by some Riemann surface of genus  $g(v)$ , which does not contain any marked point, connected to the rest via  $i(v)$  double points.
- ▶ This gives an isomorphism of complexes 
$$\left(\bigoplus_{\{G\}} k[(G, \text{or}(G)), \delta_{\text{graph}}]\right) \rightarrow \bigoplus_{g,v} C_*(\overline{\mathcal{M}}_{g,v} \times \mathbb{R}_{>0}^v / \mathbb{S}_v)$$

## Example of the formula for products of $\psi_i$

Example: The class of  $\psi_1$  in  $H^2(\bar{\mathcal{M}}_{g,n})$  is represented as linear combination of stable  $(G, \text{or}(G))$  with  $|\text{Edge}(G)| = 2 + n$ . It is the coefficient in front of  $\frac{1}{\lambda_1^3} \prod_{i=2}^n \frac{1}{\lambda_i}$ .



# The Quantum Master Equation on Cyclic Cochains

- ▶ The stable ribbon graph complex is intimately related with the Quantum Master Equation on Cyclic Cochains (QMCC).
- ▶ Let  $V = V_0 \oplus V_1$  be  $\mathbb{Z}/2\mathbb{Z}$ -graded vector space,  $\dim_k V < \infty$ , scalar product  $\beta : V^{\otimes 2} \rightarrow k[p]$
- ▶  $C^\lambda = (\bigoplus_{j=1}^{\infty} \text{Hom}((\Pi V^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}})$ ,  
 $S \in \text{Sym}(C^\lambda [1-p])[[\hbar]]$  (symmetric products for odd  $\beta$ , antisymmetric for even  $\beta$ )
- ▶ The Quantum Master Equation on Cyclic Cochains / The noncommutative Batalin-Vilkovisky equation (S.B.,2005)

$$\hbar \Delta_{NC} S + \frac{1}{2} \{S, S\} = 0 \quad \text{QMCC}$$

$$S = \sum_{g \geq 0, i > 0} \hbar^{2g-1+i} S_{g,i}, \quad S_{g,i} \in \text{Sym}^i(C^\lambda [1-p]).$$



$$\{S_{0,1}, S_{0,1}\} = 0,$$

$S_{0,1}$  -  $A_\infty$ - algebra with (even/odd) scalar product, so  $S$ -*multiloop*, *higher genus* generalization of  $A_\infty$ - algebra.

# The noncommutative BV differential

- ▶ (QMCC)  $\iff \Delta_{NC}(\exp \frac{1}{\hbar} S) = 0$
- ▶ The noncommutative BV differential on  $F = \text{Sym}(C^\lambda [1 - \rho])$

$$\begin{aligned} \Delta_{NC}(a_{\rho_1} \dots a_{\rho_r})^\lambda (a_{\tau_1} \dots a_{\tau_t})^\lambda &= \\ &= \sum_{p,q} (-1)^\varepsilon \beta_{\rho_p \tau_q}^\vee (a_{\rho_1} \dots a_{\rho_{p-1}} a_{\tau_{q+1}} \dots a_{\tau_{q-1}} a_{\rho_{p+1}} \dots a_{\rho_r})^\lambda + \end{aligned}$$

$$\sum_{p \pm 1 \neq q} (-1)^{\tilde{\varepsilon}} \beta_{\rho_p \rho_q}^\vee (\dots a_{\rho_{p-1}} a_{\rho_{q+1}} \dots a_{\rho_r})^\lambda (a_{\rho_{p+1}} \dots a_{\rho_{q-1}})^\lambda (a_{\tau_1} \dots a_{\tau_t})^\lambda$$

$$\sum_{p \pm 1 \neq q} (-1)^{\tilde{\varepsilon}} \beta_{\tau_p \tau_q}^\vee (\dots a_{\rho_r})^\lambda (a_{\tau_1} \dots a_{\tau_{p-1}} a_{\tau_{q+1}} \dots a_{\tau_t})^\lambda (a_{\tau_{p+1}} \dots a_{\tau_{q-1}})^\lambda$$

- ▶ signs are the standard Koszul signs taking into account that  $(a_{\rho_1} \dots a_{\rho_r})^\lambda = 1 - \rho + \sum \bar{a}_{\rho_i}$ ,  $a_i \in \text{Hom}(\Pi V)$ .
- ▶ **Theorem** (S.B., 2006)  $\Delta_{NC}^2 = 0$

$(A, m)$  is a  $d\mathbb{Z}/2\mathbb{Z}$  graded associative algebra with odd scalar product  $\beta$ ,  $\dim_k A < \infty$ .

$H$  is an odd selfadjoint operator  $H : A \rightarrow \Pi A$ ,  $H^\vee = H$ , such that  $Id - [d, H] = P$ ,  $dP = 0$ ,  $P^2 = P$ .  $B$  - the image of  $P$ .

Let  $\Gamma$  be a 3-valent ribbon graph with legs, then put:

- ▶ on every vertex  $v \rightarrow$  the 3-tensors of the cyclic product on  $A$

$$m^v \in ((\Pi A)^{\otimes 3})^\vee$$

- ▶ on every interior edge  $e = (ff') \rightarrow$  the two tensor

$$\beta^\vee(H^\vee u_f, v_{f'}), \beta_H^{\vee, e} \in (\Pi A)^{\otimes 2},$$

- ▶ on every leg  $l \in \text{Leg}(\Gamma) \rightarrow$  element  $a_l \in \Pi B$

- ▶ make the contraction

$$W_\Gamma \left( \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right) = \left\langle \bigotimes_{v \in \text{Vert}(\Gamma)} m^v, \left( \bigotimes_{e \in \text{Edge}(\Gamma)} \beta_H^{\vee, e} \right) \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right\rangle$$



$$W_{\Gamma} \left( \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right) = \left\langle \bigotimes_{v \in \text{Vert}(\Gamma)} m^v, \left( \bigotimes_{e \in \text{Edge}(\Gamma)} \beta_H^{v,e} \right) \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right\rangle$$

$$S = \sum_{\{\Gamma\}} \hbar^{1-\chi(\Gamma)} W_{\Gamma}$$

▶  $S \in \text{Sym}(\oplus_{j=1}^{\infty} \text{Hom}((\Pi B^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}})[[\hbar]]$

▶ **Theorem** (S.B. 2009) The sum over ribbon graphs  $S$  satisfy the QMCC/noncommutative Batalin-Vilkovisky equation:

$$\hbar \Delta_{NC} S + \frac{1}{2} \{S, S\} = 0$$

Let  $S \in \text{Sym}(C^\lambda [1 - \rho])[[\hbar]]$  be a solution to the QMCC, with  $S_{0,1,2} = 0$  ( $d_V = 0$ ), let  $\Gamma$  be a *stable* ribbon graph, then put:

- ▶ on every vertex  $v \rightarrow$  the multi-cyclic tensors

$$S_{g,i}^v \in \text{Sym}(\bigoplus_{j=1}^i \text{Hom}((\Pi V^{\otimes j}), k))^{\mathbb{Z}/j\mathbb{Z}} [1 - \rho]$$

- ▶ on every edge  $e = (ff') \rightarrow$  the two tensor

$$\beta^v(u_f, v_{f'}), \beta^{v,e} \in (\Pi V)^{\otimes 2},$$

- ▶ take the contraction  $W_\Gamma = \left\langle \bigotimes_{v \in \text{Vert}(\Gamma)} S_{g,i}^v, \left( \bigotimes_{e \in \text{Edge}(\Gamma)} \beta^{v,e} \right) \right\rangle$

- ▶ **Theorem (S.B.,2006)** For any  $S$ - a solution to the QMCC equation the following chain is a cycle in the *stable ribbon* graph complex

$$W(S) = \sum_{\{\Gamma\} \in \text{SRG}} \hbar^{1-\chi(\Sigma_\Gamma)} W_\Gamma [S]$$

$$\delta_{\text{graph}} W(S) = 0,$$

therefore  $[W(S)] \in H_*(\overline{\mathcal{M}}_{*,*})$

## More constructions of solutions to the QMCC equation

- ▶ **(Conjecture)** (S.B,2005). Counting of holomorphic curves  $(\Sigma, \partial\Sigma, p_i) \rightarrow (M, \coprod L_i, \oplus H_*(L_i \cap L_j))$ , with  $\mathbb{Z}/2\mathbb{Z}$ -graded local systems, gives solution to the QMCC equation.
- ▶ **Theorem** (S.B. 2013) If  $A$  is an  $A_\infty$ -infinity algebra with the degeneration of the Hodge to de Rham spectral sequence, then the solution to the QMCC is constructed step by step starting from  $\{S_{0,1}, S_{0,1}\} = 0$

# Homotopy theory of the QMCC equation

- ▶ **Theorem (S.B.,2006)** Solutions to the QMCC are in one-to-one correspondence with the structure of algebra over the *Feynman transform* of  $\widehat{k[\mathbb{S}_n]}$
- ▶ **Theorem (S.B.,2006)**  $C^*(\widehat{k[\mathbb{S}_n]}(0, \gamma, \nu)) \simeq C_*(\overline{\mathcal{M}}_{g,\nu}/\mathbb{S}_\nu)$

## Associative algebra plus odd derivation.

- ▶  $A$ - associative algebra , with odd scalar product
- ▶ Assume:  $I$  - an *odd derivation* acting on  $A$ , preserving the scalar product: ,for example  $I = [\Lambda, \cdot]$ ,  $\Lambda \in A^{\text{odd}}$ , in general  $I^2 \neq 0$  (!),  $\exists \tilde{I}$ ,  $[I, \tilde{I}] = 1$ ,  $str([a, \cdot]) = 0$  for any  $a \in A$ .
- ▶ **Theorem(S.B.2009)** This data  $\rightarrow$  Cohomology classes in  $H^*(\overline{\mathcal{M}}_{g,n})$



## “Odd matrix algebra” $Q(N)$

- ▶ Example  $q(N)$ ,  $q(N) = \{[X, \pi] = 0 \mid X \in \mathfrak{gl}(N|N)\}$ , where  $\pi$ -odd involution,  $q(N)$  has *odd trace*  $otr$ ,  $l = [\Xi, \cdot]$ ,  $\Xi$ -odd element  $\Xi = (0 \mid \text{diag}(\lambda_1, \dots, \lambda_n))$ , ( $l^2 \neq 0$  (!))
- ▶ **Theorem(S.B.2009)** This is the generating function for products of tautological classes  $\psi_i = c_1(T_{p_i}^*)$ .

- ▶ **Conjecture (S.B. 2009)** This construction, applied to  $A_\infty$ -algebra  $A = \text{End}(C)$ ,  $C$  is a generating object of the  $D^b(\text{Coh}(Y))$ ,  $Y$  is the mirror dual Calabi-Yau manifold to  $X$   
 $\rightarrow \dots$
- ▶  $\rightarrow$  all genus Gromov-Witten invariants of  $X$