



# Summations over generalized ribbon graphs and all genus categorical Gromov-Witten invariants.

Serguei Barannikov

## ► To cite this version:

Serguei Barannikov. Summations over generalized ribbon graphs and all genus categorical Gromov-Witten invariants.. 102e rencontre entre mathématiciens et physiciens théoriciens, Sep 2018, Strasbourg, France. hal-01875023

**HAL Id: hal-01875023**

**<https://hal.science/hal-01875023>**

Submitted on 15 Sep 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Summations over generalized ribbon graphs and all genus categorical Gromov-Witten invariants.

102e Rencontre entre mathématiciens et physiciens

Serguei Barannikov (CNRS, Paris VII, NRU HSE)

15/09/2018

$\psi_i = c_1(T_{p_i}^*)$ ,  $\psi_i \in H^2(\bar{\mathcal{M}}_{g,n})$ . Consider  $H^*(\bar{\mathcal{M}}_{g,n})$ -valued generating function for products of  $\psi_i$

- ▶ Theorem (SB,2009)

$$\sum_{\sum d_i = d} \psi_1^{d_1} \dots \psi_n^{d_n} \prod_{i=1}^n \frac{(2d_i - 1)!!}{\lambda_i^{2d_i + 1}} =$$

$$= \left[ \sum_{[G] \in \Gamma_{g,n}^{dec, odd}} (G, \text{or}(G)) \frac{2^{-\chi(G)}}{|\text{Aut}(G)|} \prod_{e \in \text{Edge}(G)} \frac{1}{\lambda_{i(e)} + \lambda_{j(e)}} \right]$$

- ▶ the sum on the right is over *stable ribbon* graphs of genus  $g$  with  $n$  numbered punctures, with  $2d + n$  edges, and such that the vertices of the graph have cyclically ordered subsets of *arbitrary odd* cardinality
- ▶ In the simplest case, corresponding to the top degree, the cohomology  $H^{6g-6+2n}(\bar{\mathcal{M}}_{g,n})$  is 1-dimensional, the summation is over 3-valent usual ribbon graphs and this formula then reproduces the main identity from Kontsevich proof of Witten conjecture.

# Aim of the talk

This formula is a byproduct of construction of cohomology classes

- ▶  $I$  - an odd derivation acting on cyclic associative  $A_\infty$ -algebra  $A$ , with odd scalar product, in general  $I^2 \neq 0$  (!)
- ▶ **Theorem (S.B.2009)** This data  $\rightarrow$  Cohomology classes in  $H^*(\overline{\mathcal{M}}_{g,n})$
- ▶ This construction defines Cohomological Field Theory
- ▶ Applied to “odd matrix algebra”  $A = q(N)$ ,  
 $q(N) = \{[X, \pi] = 0 \mid X \in gl(N|N)\}$ , where  $\pi$ -odd involution,  $I = [\Xi, \cdot]$ ,  $\Xi$ - odd element  $\rightarrow$  the formula for products of  $\psi_i$
- ▶ This was the first nontrivial computation of categorical Gromov-Witten invariants of higher genus.

# Counterexample to a Theorem of Kontsevich

## Another byproduct is a counterexample to the Theorem 1.3 from Kontsevich “Feynman diagrams in low-dimensional topology” (1993)

### Dual construction

We describe here a way to produce *homology* classes on  $\mathcal{M}_{g,n}$ . The starting ingredient is a finite-dimensional differential associative algebra  $V$  with a non-degenerate *odd* scalar product and trivial cohomology.

The right inverse to the scalar product can be considered as an odd element  $\delta$  of  $V \otimes V$ . It follows from the compatibility of the scalar product with the differential that  $\delta$  is closed. From the triviality of  $H(V)$  it follows that there exists  $\omega \in V \otimes V$  such that  $d\omega = \delta$ . We can use  $\omega$  as a “propagator” and tensors  $T_{(k)} : V^{\otimes k} \rightarrow \mathbb{C}$

$$T(v_1 \otimes \dots \otimes v_k) = (v_1 v_2 \dots v_{k-1}, v_k)$$

as “interactions”. Again, we obtain a state model on ribbon graphs. Now we will consider the complex which is *dual* to the cochain complex from the previous section.

**Theorem 1.3**  $\sum_{\Gamma} Z(\Gamma)\Gamma$  is a well-defined chain and it is closed. Its homology class does not depend on the choice of  $\omega$ .

It turns out this sum is NOT closed

## Counterexample $A = \langle 1, \zeta \rangle / \zeta^2 = 1$

- ▶  $A = \langle 1, \zeta \rangle / \zeta^2 = 1, \bar{\zeta} = 1, \beta(1, \zeta) = 1, d(\zeta) = 1, d^2 = 0, w = \zeta \otimes \zeta.$
- ▶ For this data  $\sum_{\Gamma} Z(\Gamma) \Gamma$  is nonzero on the boundary of the following ribbon graph = a generator of the dual complex:

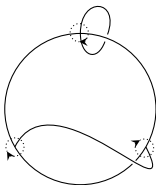


Figure:  $G: Z_{\Gamma}(\partial G) \neq 0$

## Cellular decomposition of $\bar{\mathcal{M}}_{g,n} \times \mathbb{R}^n$

A *stable ribbon graph* is a connected graph  $G$  (recall: a graph  $G$  is a triple  $(\text{Flag}(G), \lambda, \sigma)$ , where  $\text{Flag}(G)$  is a finite set, whose elements are called flags,  $\lambda$  is a partition of  $\text{Flag}(G) \leftrightarrow \text{vertices}$ ,  $\sigma$  is an involution acting on  $\text{Flag}(G) \leftrightarrow \text{edges/legs}$ ) together with:

- ▶ partitions of the set of flags adjacent to every vertex into  $i(v)$  nonempty subsets

$$\text{Leg}(v) = \text{Leg}(v)^{(1)} \sqcup \dots \sqcup \text{Leg}(v)^{(i(v))}, v \in \text{Vert}(G)$$

- ▶ fixed cyclic order on every subset  $\text{Leg}(v)^{(k)}$ ,
- ▶ a number  $g(v) \in \mathbb{Z}_{\geq 0}$  such that  $|\text{Leg}(v)| > 2(2 - i(v) - 2g(v))$ .

- ▶ Define an orientation

$$\text{or}(G) \in \text{Det}(\bigotimes_{v \in \text{Vert}(G)} (k^{\text{Flag}(v)} \oplus k^{\text{Cycle}(v)}))$$

- ▶  $\bigoplus_{\{G\}} k(G, \text{or}(G))$  has natural “generalized contraction of edges” differential  $\delta_{\text{graph}}, \delta_{\text{graph}}^2 = 0$

## Cellular decomposition of $\bar{\mathcal{M}}_{g,n} \times \mathbb{R}^n$

A metric on the stable ribbon graph is a function  $l : \text{Edge}(G) \rightarrow \mathbb{R}_{>0}$ . Given a stable ribbon graph  $G$  w/out legs and a metric on  $G$  one can construct by standard procedure a punctured Riemann surface  $\Sigma(G)$ , which have double points in general.

- ▶ replace every edge by a pair of oriented strips  $[0, l] \times [0 + i\infty[$  one for each flag and glue them side  $0 \times [0 + i\infty[$  to  $l \times [0 + i\infty[$  according to the cyclic order of the cyclically ordered subsets at each vertex. Then glue the two strips for each edge  $[0, l] \leftrightarrow [l, 0]$
- ▶ identify points corresponding to 2 subsets at vertices with  $g(v) = 0, i(v) = 2$ , (double points); for points with  $2g(v) + i(v) > 2$  remplace the vertex by some Riemann surface of genus  $g(v)$ , which does not contain any marked point, connected to the rest via  $i(v)$  double points.
- ▶ This gives an isomorphism of complexes 
$$(\oplus_{\{G\}} k[(G, \text{or}(G)), \delta_{\text{graph}}] \rightarrow \oplus_{g,v} C_*(\bar{\mathcal{M}}_{g,v} \times \mathbb{R}_{>0}^v / \mathbb{S}_v)$$



## Example of the formula for products of $\psi_i$

Example: The class of  $\psi_1$  in  $H^2(\bar{\mathcal{M}}_{g,n})$  is represented as linear combination of stable  $(G, \text{or}(G))$  with  $|\text{Edge}(G)| = 2 + n$ . It is the coefficient in front of  $\frac{1}{\lambda_1^3} \prod_{i=2}^n \frac{1}{\lambda_i}$ .

# The Quantum Master Equation on Cyclic Cochains

- ▶ The stable ribbon graph complex is intimately related with the Quantum Master Equation on Cyclic Cochains (QMCC).
- ▶ Let  $V = V_0 \oplus V_1$  be  $\mathbb{Z}/2\mathbb{Z}$ -graded vector space,  $\dim_k V < \infty$ , scalar product  $\beta : V^{\otimes 2} \rightarrow k[p]$
- ▶  $C^\lambda = (\oplus_{j=1}^{\infty} \text{Hom}((\Pi V^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}})$ ,  
 $S \in \text{Sym}(C^\lambda[1-p])[[\hbar]]$  (symmetric products for odd  $\beta$ , antisymmetric for even  $\beta$ )
- ▶ The Quantum Master Equation on Cyclic Cochains / The noncommutative Batalin-Vilkovisky equation (S.B., 2005)

$$\hbar \Delta_{NC} S + \frac{1}{2} \{S, S\} = 0 \quad \text{QMCC}$$

$$S = \sum_{g \geq 0, i > 0} \hbar^{2g-1+i} S_{g,i}, \quad S_{g,i} \in \text{Sym}^i(C^\lambda[1-p]).$$



$$\{S_{0,1}, S_{0,1}\} = 0,$$

$S_{0,1}$  -  $A_\infty$ -algebra with (even/odd) scalar product, so  $S$ —*multiloop, higher genus* generalization of  $A_\infty$ -algebra.

# The noncommutative BV differential

- ▶ (QMCC)  $\Longleftrightarrow \Delta_{NC}(\exp \frac{1}{\hbar} S) = 0$
- ▶ The noncommutative BV differential on  $F = \text{Sym}(C^\lambda [1 - p])$

$$\begin{aligned} \Delta_{NC}(a_{\rho_1} \dots a_{\rho_r})^\lambda (a_{\tau_1} \dots a_{\tau_t})^\lambda &= \\ &= \sum_{p,q} (-1)^\varepsilon \beta_{\rho_p \tau_q}^\vee (a_{\rho_1} \dots a_{\rho_{p-1}} a_{\tau_{q+1}} \dots a_{\tau_{q-1}} a_{\rho_{p+1}} \dots a_{\rho_r})^\lambda + \end{aligned}$$

$$\sum_{p \pm 1 \neq q} (-1)^{\tilde{\varepsilon}} \beta_{\rho_p \rho_q}^\vee (\dots a_{\rho_{p-1}} a_{\rho_{q+1}} \dots a_{\rho_r})^\lambda (a_{\rho_{p+1}} \dots a_{\rho_{q-1}})^\lambda (a_{\tau_1} \dots a_{\tau_t})^\lambda$$

$$\sum_{p \pm 1 \neq q} (-1)^{\tilde{\varepsilon}} \beta_{\tau_p \tau_q}^\vee (\dots a_{\rho_r})^\lambda (a_{\tau_1} \dots a_{\tau_{p-1}} a_{\tau_{q+1}} \dots a_{\tau_t})^\lambda (a_{\tau_{p+1}} \dots a_{\tau_{q-1}})^\lambda$$

- ▶ signs are the standard Koszul signs taking into account that  $(a_{\rho_1} \dots a_{\rho_r})^\lambda = 1 - p + \sum \bar{a}_{\rho_i}$  ,  $a_i \in \text{Hom}(\Pi V)$  .
- ▶ **Theorem** (S.B.,2006)  $\Delta_{NC}^2 = 0$

$(A, m)$  is a  $d\mathbb{Z}/2\mathbb{Z}$  graded associative algebra with odd scalar product  $\beta$ ,  $\dim_k A < \infty$ .

$H$  is an odd selfadjoint operator  $H : A \rightarrow \Pi A$ ,  $H^\vee = H$ , such that  $Id - [d, H] = P$ ,  $dP = 0$ ,  $P^2 = P$ .  $B$  - the image of  $P$ .

Let  $\Gamma$  be a 3-valent ribbon graph with legs, then put:

- ▶ on every vertex  $v \rightarrow$  the 3-tensors of the cyclic product on  $A$

$$m^v \in ((\Pi A)^{\otimes 3})^\vee$$

- ▶ on every interior edge  $e = (ff') \rightarrow$  the two tensor

$$\beta^\vee(H^\vee u_f, v_{f'}), \beta_{H'}^{\vee, e} \in (\Pi A)^{\otimes 2},$$

- ▶ on every leg  $l \in \text{Leg}(\Gamma) \rightarrow$  element  $a_l \in \Pi B$
- ▶ make the contraction

$$W_\Gamma \left( \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right) = \left\langle \bigotimes_{v \in \text{Vert}(\Gamma)} m^v, \left( \bigotimes_{e \in \text{Edge}(\Gamma)} \beta_{H'}^{\vee, e} \right) \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right\rangle$$



$$W_{\Gamma}(\bigotimes_{l \in \text{Leg}(\Gamma)} a_l) = \langle \bigotimes_{v \in \text{Vert}(\Gamma)} m^v, \left( \bigotimes_{e \in \text{Edge}(\Gamma)} \beta_H^{\vee, e} \right) \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \rangle$$

$$S = \sum_{\{\Gamma\}} \hbar^{1-\chi(\Gamma)} W_{\Gamma}$$

►  $S \in \text{Sym}(\oplus_{j=1}^{\infty} \text{Hom}((\Pi B^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}})[[\hbar]]$

- **Theorem** (S.B. 2009) The sum over ribbon graphs  $S$  satisfy the QMCC/noncommutative Batalin-Vilkovisky equation:

$$\hbar \Delta_{NC} S + \frac{1}{2} \{S, S\} = 0$$

Let  $S \in \text{Sym}(C^\lambda [1 - p])[[\hbar]]$  be a solution to the QMCC, with  $S_{0,1,2} = 0$  ( $d_V = 0$ ), let  $\Gamma$  be a *stable* ribbon graph, then put:

- ▶ on every vertex  $v \rightarrow$  the multi-cyclic tensors

$$S_{g,i}^v \in \text{Sym}(\oplus_{j=1}^i \text{Hom}((\Pi V^{\otimes j}), k))^{\mathbb{Z}/j\mathbb{Z}} [1 - p])$$

- ▶ on every edge  $e = (ff') \rightarrow$  the two tensor

$$\beta^v(u_f, v_{f'}), \beta^{v,e} \in (\Pi V)^{\otimes 2},$$

- ▶ take the contraction  $W_\Gamma = \left\langle \bigotimes_{v \in \text{Vert}(\Gamma)} S_{g,i}^v, \left( \bigotimes_{e \in \text{Edge}(\Gamma)} \beta^{v,e} \right) \right\rangle$

- ▶ **Theorem (S.B.,2006)** For any  $S$ - a solution to the QMCC equation the following chain is a cycle in the *stable ribbon* graph complex

$$W(S) = \sum_{\{\Gamma\} \in \text{SRG}} \hbar^{1-\chi(\Sigma_\Gamma)} W_\Gamma [\Gamma]$$

$$\delta_{\text{graph}} W(S) = 0,$$

therefore  $[W(S)] \in H_* (\overline{\mathcal{M}}_{*,*})$

# More constructions of solutions to the QMCC equation

- ▶ **(Conjecture)** (S.B,2005). Counting of holomorphic curves  $(\Sigma, \partial\Sigma, p_i) \rightarrow (M, \coprod L_i, \oplus H_*(L_i \cap L_j))$  , with  $\mathbb{Z}/2\mathbb{Z}$  -graded local systems, gives solution to the QMCC equation.
- ▶ **Theorem** (S.B. 2013) If  $A$  is an  $A_\infty$ —infinity algebra with the degeneration of the Hodge to de Rham spectral sequence, then the solution to the QMCC is constructed step by step starting from  $\{S_{0,1}, S_{0,1}\} = 0$

# Homotopy theory of the QMCC equation

- ▶ **Theorem (S.B.,2006)** Solutions to the QMCC are in one-to-one correspondence with the structure of algebra over the *Feynman transform* of  $\widehat{k[\mathbb{S}_n]}$
- ▶ **Theorem (S.B.,2006)**  $C^*(\widehat{k[\mathbb{S}_n]}(0, \gamma, \nu)) \simeq C_*(\overline{\mathcal{M}}_{g,\nu}/\mathbb{S}_\nu)$



## Associative algebra plus odd derivation.

- ▶  $A$ - associative algebra , with odd scalar product
- ▶ Assume:  $I$  - an *odd derivation* acting on  $A$ , preserving the scalar product: ,for example  $I = [\Lambda, \cdot]$ ,  $\Lambda \in A^{\text{odd}}$ , in general  $I^2 \neq 0$  (!),  $\exists \tilde{I}$ ,  $[I, \tilde{I}] = 1$ ,  $\text{str}([a, \cdot]) = 0$  for any  $a \in A$ .
- ▶ **Theorem(S.B.2009)** This data  $\rightarrow$  Cohomology classes in  $H^*(\overline{\mathcal{M}}_{g,n})$

## “Odd matrix algebra” $Q(N)$

- ▶ Example  $q(N)$ ,  $q(N) = \{[X, \pi] = 0 | X \in gl(N|N)\}$ , where  $\pi$ -odd involution,  $q(N)$  has *odd trace*  $otr$ ,  $I = [\Xi, \cdot]$ ,  $\Xi$ -odd element  $\Xi = (0 \quad | \text{diag}(\lambda_1, \dots, \lambda_n))$ , ( $I^2 \neq 0$  (!))
- ▶ **Theorem(S.B.2009)** This is the generating function for products of tautological classes  $\psi_i = c_1(T_{p_i}^*)$ .

- ▶ **Conjecture (S.B. 2009)** This construction, applied to  $A_\infty$ -algebra  $A = \text{End}(C)$ ,  $C$  is a generating object of the  $D^b(\text{Coh}(Y))$ ,  $Y$  is the mirror dual Calabi-Yau manifold to  $X$   
 $\rightarrow \dots$
- ▶  $\rightarrow$  all genus Gromov-Witten invariants of  $X$