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Summations over generalized ribbon graphs and all genus categorical Gromov-Witten invariants.

102e Rencontre entre mathematiciens et physicists

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15/09/2018

$\psi_i = c_1(T^*_{p_i})$, $\psi_i \in H^2(\bar{\mathcal{M}}_{g,n})$. Consider $H^*(\bar{\mathcal{M}}_{g,n})-$valued generating function for products of ψ_i

- Theorem (SB,2009)

$$\sum_{\sum d_i = d} \psi_1^{d_1} \dots \psi_n^{d_n} \prod_{i=1}^{n} \frac{(2d_i - 1)!!}{\lambda_i^{2d_i+1}} =$$

$$= \left[\sum_{[G] \in \Gamma_{g,n}^{dec,odd}} (G, or(G)) \frac{2^{-\chi(G)}}{|\mathrm{Aut}(G)|} \prod_{e \in \mathrm{Edge}(G)} \frac{1}{\lambda_{i(e)} + \lambda_{j(e)}} \right]$$

- the sum on the right is over *stable ribbon* graphs of genus g with n numbered punctures, with $2d + n$ edges, and such that the vertices of the graph have cyclically ordered subsets of *arbitrary odd* cardinality
- In the simplest case, corresponding to the top degree, the cohomology $H^{6g-6+2n}(\bar{\mathcal{M}}_{g,n})$ is 1-dimensional, the summation is over 3-valent usual ribbon graphs and this formula then reproduces the main identity from Kontsevich proof of Witten conjecture.

Aim of the talk

This formula is a byproduct of construction of cohomology classes

- I - an odd derivation acting on cyclic associative $/A_\infty-$ algebra A, with odd scalar product, in general $I^2 \neq 0$ (!)
- **Theorem (S.B.2009)** This data $\rightarrow$Cohomology classes in $H^*(\overline{\mathcal{M}}_{g,n})$
- This construction defines Cohomological Field Theory
- Applied to "odd matrix algebra" $A = q(N)$, $q(N) = \{[X, \pi] = 0 | X \in gl(N|N)\}$, where $\pi-$odd involution, $I = [\Xi, \cdot]$, Ξ- odd element $\rightarrow$the formula for products of ψ_i
- This was the first nontrivial computation of categorical Gromov-Witten invariants of higher genus.

Counterexample to a Theorem of Kontsevich

Another byproduct is a counterexample to the Theorem 1.3 from Kontsevich "Feynman diagrams in low-dimansional topology" (1993)

Dual construction

We describe here a way to produce *homology* classes on $\mathcal{M}_{g,n}$. The starting ingredient is a finite-dimensional differential associative algebra V with a non-degenerate *odd* scalar product and trivial cohomology.

The right inverse to the scalar product can be considered as an odd element δ of $V \otimes V$. It follows from the compatibility of the scalar product with the differential that δ is closed. From the triviality of $H(V)$ it follows that there exists $\omega \in V \otimes V$ such that $d\omega = \delta$. We can use ω as a "propagator" and tensors $T_{(k)} : V^{\otimes k} \to \mathbf{C}$

$$T(v_1 \otimes \ldots v_k) = (v_1 v_2 \ldots v_{k-1}, v_k)$$

as "interactions". Again, we obtain a state model on ribbon graphs. Now we will consider the complex which is *dual* to the cochain complex from the previous section.

Theorem 1.3 $\sum_{\Gamma} Z(\Gamma)\Gamma$ *is a well-defined chain and it is closed. Its homology class does not depend on the choice of* ω.

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It turns out this sum is NOT closed

Counterexample $A = \langle 1, \xi \rangle / \xi^2 = 1$

- $A = \langle 1, \xi \rangle / \xi^2 = 1, \bar{\xi} = 1,\ \beta(1, \xi) = 1,\ d(\xi) = 1,\ d^2 = 0,$ $w = \xi \otimes \xi$.
- For this data$\sum_\Gamma Z(\Gamma)\Gamma$ is nonzero on the boundary of the following ribbon graph =a generator of the dual complex:

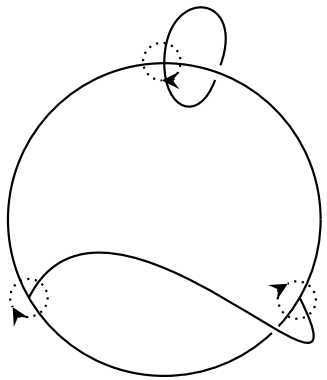

Figure: G: $Z_{\widetilde{I}}(\partial G) \neq 0$

Cellular decomposition of $\bar{\mathcal{M}}_{g,n} \times \mathbb{R}^n$

A *stable ribbon graph* is a connected graph G (recall: a graph G is a triple $(Flag(G), \lambda, \sigma)$, where $Flag(G)$ is a finite set, whose elements are called flags, λ is a partition of $Flag(G) \leftrightarrow$ vertices , σ is an involution acting on $Flag(G) \leftrightarrow$ edges/legs) together with:

- partitions of the set of flags adjacent to every vertex into $i(v)$ nonempty subsets

$$Leg(v) = Leg(v)^{(1)} \sqcup \ldots \sqcup Leg(v)^{(i(v))}, v \in Vert(G)$$

- fixed cyclic order on every subset $Leg(v)^{(k)}$,
- a number $g(v) \in \mathbb{Z}_{\geq 0}$ such that$|Leg(v)| > 2(2 - i(v) - 2g(v))$.
- Define an orientation $or(G) \in Det(\otimes_{v \in Vert(G)}(k^{Flag(v)} \oplus k^{Cycle(v)})$
- $\oplus_{\{G\}} k(G, or(G))$ has natural "generalized contraction of edges" differential δ_{graph}, $\delta^2_{graph} = 0$

Cellular decomposition of $\bar{\mathcal{M}}_{g,n} \times \mathbb{R}^n$

A metric on the stable ribbon graph is a function $l : Edge(G) \to \mathbb{R}_{>0}$. Given a stable ribbon graph G w/out legs and a metric on G one can construct by standard procedure a punctured Riemann surface $\Sigma(G)$, which have double points in general.

- replace every edge by a pair of oriented strips $[0, l] \times [0 + i\infty[$ one for each flag and glue them side $0 \times [0 + i\infty[$ to $l \times [0 + i\infty[$ according to the cyclic order of the cyclically ordered subsets at each vertex. Then glue the two strips for each edge $[0, l] \leftrightarrow [l, 0]$
- identify points corresponding to 2 subsets at vertices with $g(v) = 0, i(v) = 2$, (double points); for points with $2g(v) + i(v) > 2$ remplace the vertex by some Riemann surface of genus $g(v)$, which does not contain any marked point, connected to the rest via $i(v)$ double points.
- This gives an isomorphism of complexes $(\oplus_{\{G\}} k\,[(G, or(G)]\,, \delta_{graph}) \to \oplus_{g,\nu} C_*(\overline{\mathcal{M}}_{g,\nu} \times \mathbb{R}^{\nu}_{>0}/\mathbb{S}_\nu)$

Example of the formula for products of ψ_i

Example: The class of ψ_1 in $H^2(\bar{\mathcal{M}}_{g,n})$ is represented as linear combination of stable $(G, or(G))$ with $|\mathrm{Edge}(G)| = 2 + n$. It is the coefficient in front of $\frac{1}{\lambda_1^3} \prod_{i=2}^{n} \frac{1}{\lambda_i}$.

The Quantum Master Equation on Cyclic Cochains

- The stable ribbon graph complex is intimately related with the Quantum Master Equation on Cyclic Cochains (QMCC).
- Let $V = V_0 \oplus V_1$ be $\mathbb{Z}/2\mathbb{Z}$ -graded vector space, $\dim_k V < \infty$, scalar product $\beta : V^{\otimes 2} \to k\,[p]$
- $C^\lambda = (\oplus_{j=1}^{\infty} Hom((\Pi V^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}})$, $S \in \mathrm{Sym}(C^\lambda\,[1-p])[[\hbar]]$ (symmetric products for odd β , antisymmetric for even β)
- The Quantum Master Equation on Cyclic Cochains / The noncommutative Batalin-Vilkovisky equation (S.B.,2005)

$$\hbar \Delta_{NC} S + \frac{1}{2}\{S, S\} = 0 \qquad \text{QMCC}$$

$S = \sum_{g \geq 0, i > 0} \hbar^{2g-1+i} S_{g,i}$, $S_{g,i} \in \mathrm{Sym}^{\mathrm{i}}(C^\lambda\,[1-p])$.

- $$\{S_{0,1}, S_{0,1}\} = 0,$$

$S_{0,1}$ - $A_\infty-$ algebra with (even/odd) scalar product, so $S-$ *multiloop, higher genus* generalization of $A_\infty-$ algebra.

The noncommutative BV differential

- (QMCC) $\Longleftrightarrow \Delta_{NC}(\exp \frac{1}{\hbar} S) = 0$
- The noncommutative BV differential on $F = \mathrm{Sym}(C^{\lambda}[1-p])$

$$\Delta_{NC}(a_{\rho_1} \dots a_{\rho_r})^{\lambda}(a_{\tau_1} \dots a_{\tau_t})^{\lambda} =$$
$$= \sum_{p,q} (-1)^{\varepsilon} \beta^{\vee}_{\rho_p \tau_q} (a_{\rho_1} \dots a_{\rho_{p-1}} a_{\tau_{q+1}} \dots a_{\tau_{q-1}} a_{\rho_{p+1}} \dots a_{\rho_r})^{\lambda} +$$

$$\sum_{p \pm 1 \neq q} (-1)^{\tilde{\varepsilon}} \beta^{\vee}_{\rho_p \rho_q} (\dots a_{\rho_{p-1}} a_{\rho_{q+1}} \dots a_{\rho_r})^{\lambda} (a_{\rho_{p+1}} \dots a_{\rho_{q-1}})^{\lambda} (a_{\tau_1} \dots a_{\tau_t})^{\lambda}$$

$$\sum_{p \pm 1 \neq q} (-1)^{\tilde{\tilde{\varepsilon}}} \beta^{\vee}_{\tau_p \tau_q} (\dots a_{\rho_r})^{\lambda} (a_{\tau_1} \dots a_{\tau_{p-1}} a_{\tau_{q+1}} \dots a_{\tau_t})^{\lambda} (a_{\tau_{p+1}} \dots a_{\tau_{q-1}})^{\lambda}$$

- signs are the standard Koszul signs taking into account that $(a_{\rho_1} \dots a_{\rho_r})^{\lambda} = 1 - p + \sum \bar{a}_{\rho_i}$, $a_i \in \mathrm{Hom}\,(\Pi V)$.
- **Theorem** (S.B.,2006) $\Delta^2_{NC} = 0$

(A, m) is a d-$\mathbb{Z}/2\mathbb{Z}$ graded *associative* algebra with odd scalar product β, $\dim_k A < \infty$.
H is an odd selfadjoint operator $H : A \to \Pi A$, $H^\vee = H$, such that $Id - [d, H] = P$, $dP = 0$, $P^2 = P$. B - the image of P.
Let Γ be a 3-valent *ribbon* graph with legs, then put:

- on every vertex $v \to$ the 3-tensors of the cyclic product on A

$$m^v \in ((\Pi A)^{\otimes 3})^\vee$$

- on every interieur edge $e = (ff') \to$ the two tensor

$$\beta^\vee(H^\vee u_f, v_{f'}),\ \beta_H^{\vee,e} \in (\Pi A)^{\otimes 2},$$

- on every leg $l \in Leg(\Gamma) \to$ element $a_l \in \Pi B$
- make the contraction

$$W_\Gamma(\bigotimes_{l \in Leg(\Gamma)} a_l) = \langle \bigotimes_{v \in Vert(\Gamma)} m^v, \left(\bigotimes_{e \in Edge(\Gamma)} \beta_H^{\vee,e} \right) \bigotimes_{l \in Leg(\Gamma)} a_l \rangle$$

- $$W_\Gamma(\bigotimes_{l\in Leg(\Gamma)} a_l) = \langle \bigotimes_{v\in Vert(\Gamma)} m^v, \left(\bigotimes_{e\in Edge(\Gamma)} \beta_H^{\vee,e}\right) \bigotimes_{l\in Leg(\Gamma)} a_l\rangle$$

$$S = \sum_{\{\Gamma\}} \hbar^{1-\chi(\Gamma)} W_\Gamma$$

- $S \in \mathrm{Sym}(\oplus_{j=1}^{\infty} Hom((\Pi B^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}})[[\hbar]]$
- **Theorem** (S.B. 2009) The sum over ribbon graphs S satisfy the QMCC/noncommutative Batalin-Vilkovisky equation:

$$\hbar\Delta_{NC} S + \frac{1}{2}\{S, S\} = 0$$

Let $S \in \mathrm{Sym}(C^{\lambda}[1-p])[[\hbar]]$ be a solution to the QMCC, with $S_{0,1,2} = 0$ ($d_V = 0$), let Γ be a *stable* ribbon graph, then put:

- on every vertex $v \to$ the multi-cyclic tensors

$$S^v_{g,i} \in \overset{i}{\mathrm{Sym}}(\oplus_{j=1}^{\infty} Hom((\Pi V^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}}[1-p])$$

- on every edge $e = (ff') \to$ the two tensor

$$\beta^{\vee}(u_f, v_{f'}),\ \beta^{\vee,e} \in (\Pi V)^{\otimes 2},$$

- take the contraction $W_{\Gamma} = \langle \bigotimes_{v \in Vert(\Gamma)} S^v_{g,i}, \left(\bigotimes_{e \in Edge(\Gamma)} \beta^{\vee,e} \right) \rangle$
- **Theorem (S.B.,2006)** For any S- a solution to the QMCC equation the following chain is a cycle in the *stable ribbon* graph complex

$$W(S) = \sum_{\{\Gamma\} \in SRG} \hbar^{1-\chi(\Sigma_{\Gamma})} W_{\Gamma}[\Gamma]$$

$$\delta_{graph} W(S) = 0,$$

therefore $[W(S)] \in H_*\left(\overline{\mathcal{M}}_{*,*}\right)$

More constructions of solutions to the QMCC equation

- **(Conjecture)** (S.B,2005). Counting of holomorphic curves $(\Sigma, \partial\Sigma, p_i) \to (M, \coprod L_i, \oplus H_*(L_i \bigcap L_j))$, with $\mathbb{Z}/2\mathbb{Z}$ -graded local systems, gives solution to the QMCC equation.
- **Theorem** (S.B. 2013) If A is an $A_\infty-$infinity algebra with the degeneration of the Hodge to de Rham spectral sequence, then the solution to the QMCC is constructed step by step starting from $\{S_{0,1}, S_{0,1}\} = 0$

Homotopy theory of the QMCC equation

- **Theorem (S.B.,2006)** Solutions to the QMCC are in one-to-one correspondence with the structure of algebra over the *Feynman transform* of $\widehat{k[\mathbb{S}_n]}$
- **Theorem (S.B.,2006)** C^*(The Feynman transform of $\widehat{k[\mathbb{S}_n]}(0,\gamma,\nu)) \simeq C_*(\overline{\mathcal{M}}_{g,\nu}/\mathbb{S}_\nu)$

Associative algebra plus odd derivation.

- A- associative algebra , with odd scalar product
- Assume: I - an *odd derivation* acting on A, preserving the scalar product: ,for example $I = [\Lambda, \cdot]$, $\Lambda \in A^{\mathrm{odd}}$, in general $I^2 \neq 0$ (!), $\exists \widetilde{I}$, $[I, \widetilde{I}] = 1$, $str([a, \cdot]) = 0$ for any $a \in A$.
- **Theorem(S.B.2009)** This data $\rightarrow$Cohomology classes in $H^*(\overline{\mathcal{M}}_{g,n})$

"Odd matrix algebra" $Q(N)$

- Example $q(N)$, $q(N) = \{[X, \pi] = 0 | X \in gl(N|N)\}$, where $\pi-$odd involution, $q(N)$ has *odd trace otr*, $I = [\Xi, \cdot]$, Ξ-odd element $\Xi = \begin{pmatrix} 0 & |diag(\lambda_1, \ldots, \lambda_n) \end{pmatrix}$, ($I^2 \neq 0$ (!))
- **Theorem(S.B.2009)** This is the generating function for products of tautological classes $\psi_i = c_1(T^*_{p_i})$.

- **Conjecture (S.B. 2009)** This construction, applied to $A_\infty-$algebra $A = End(C)$, C is a generating object of the $D^b(Coh(Y))$, Y is the mirror dual Calabi-Yau manifold to X $\rightarrow$...
- $\rightarrow$all genus Gromov-Witten invariants of X