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# Asymptotic exponential law for the transition time to equilibrium of the metastable kinetic Ising model with vanishing magnetic field

A. Gaudillière\*      P. Milanesi†      M. E. Vares‡

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## Abstract

We consider a Glauber dynamics associated with the Ising model on a large two-dimensional box with minus boundary conditions and in the limit of a vanishing positive external magnetic field. The volume of this box increases quadratically in the inverse of the magnetic field. We show that at subcritical temperature and for a large class of starting measures, including measures that are supported by configurations with macroscopic plus-spin droplets, the system rapidly relaxes to some metastable equilibrium —with typical configurations made of microscopic plus-phase droplets in a sea of minus spins— before making a transition at an asymptotically exponential random time towards equilibrium —with typical configurations made of microscopic minus-phase droplets in a sea of plus spins inside a large contour that separates this plus phase from the boundary. We get this result by bounding from above the local relaxation times towards metastable and stable equilibria. This makes possible to give a pathwise description of such a transition, to control the asymptotic behaviour of the mixing time in terms of soft capacities and to give estimates of these capacities.

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*Keywords:* Metastability, Glauber dynamics, exponential law, relaxation time, quasi-stationary measures, potential theory.

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## 1 Model and results

### 1.1 Glauber dynamics for the Ising model

For a finite subset  $\Lambda$  of  $\mathbb{Z}^2$  and  $\eta \in \Omega_{\mathbb{Z}^2} = \{-1, +1\}^{\mathbb{Z}^2}$ , the Ising model in the domain  $\Lambda$ , with boundary conditions  $\eta$ , at inverse temperature  $\beta > 0$  and with magnetic field  $h \in \mathbb{R}$ , is associated with the Hamiltonian

$$H_{\Lambda, \eta, h}(\sigma) = -\frac{1}{2} \sum_{\substack{\{x, y\} \subset \Lambda, \\ \|x-y\|_1=1}} \sigma(x)\sigma(y) - \frac{1}{2} \sum_{\substack{x \in \Lambda, y \notin \Lambda, \\ \|x-y\|_1=1}} \sigma(x)\eta(y) - \frac{h}{2} \sum_{x \in \Lambda} \sigma(x), \quad \sigma \in \Omega_{\Lambda} = \{-1, +1\}^{\Lambda}, \quad (1)$$

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the partition function

$$Z_{\Lambda,\eta,h} = \sum_{\sigma \in \Omega_{\Lambda}} e^{-\beta H_{\Lambda,\eta,h}(\sigma)}$$

and the Gibbs measure

$$\mu_{\Lambda,\eta,h}(\sigma) = \frac{e^{-\beta H_{\Lambda,\eta,h}(\sigma)}}{Z_{\Lambda,\eta,h}}, \quad \sigma \in \Omega_{\Lambda}.$$

The maybe unusual factors  $1/2$  in Equation (1) are here to stick to the conventions of [SS98], which is the main reference we will follow.

The associated Glauber dynamics are irreducible continuous time Markov processes

$$X_{\Lambda,\eta,h} = (X_{\Lambda,\eta,h}(t))_{t \geq 0}$$

with a single spin flip generator

$$(\mathcal{L}_{\Lambda,\eta,h}f)(\sigma) = \sum_{x \in \Lambda} w(\sigma, \sigma^x) [f(\sigma^x) - f(\sigma)], \quad f : \Omega_{\Lambda} \rightarrow \mathbb{R}, \quad \sigma \in \Omega_{\Lambda},$$

where the configuration  $\sigma^x$  is obtained from  $\sigma$  by flipping the spin at  $x$ ,

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if } x \neq y, \\ -\sigma(x) & \text{if } x = y, \end{cases}$$

and the transition rates  $w(\sigma, \sigma^x)$  are chosen to satisfy the detailed balance equations

$$\mu_{\Lambda,\eta,h}(\sigma) w(\sigma, \sigma^x) = \mu_{\Lambda,\eta,h}(\sigma^x) w(\sigma^x, \sigma), \quad \sigma \in \Omega_{\Lambda}, \quad x \in \Lambda.$$

One can for example consider a Metropolis dynamics with

$$w(\sigma, \sigma^x) = \exp \left\{ -\beta [H_{\Lambda,\eta,h}(\sigma^x) - H_{\Lambda,\eta,h}(\sigma)]_+ \right\}, \quad \sigma \in \Omega_{\Lambda}, \quad x \in \Lambda,$$

where the brackets  $[\cdot]_+$  stand for the positive part, or a heat bath dynamics

$$w(\sigma, \sigma^x) = \frac{\exp \{ -\beta H_{\Lambda,\eta,h}(\sigma^x) \}}{\exp \{ -\beta H_{\Lambda,\eta,h}(\sigma) \} + \exp \{ -\beta H_{\Lambda,\eta,h}(\sigma^x) \}}, \quad \sigma \in \Omega_{\Lambda}, \quad x \in \Lambda.$$

In this paper we will consider such a dynamics  $X_{\Lambda_h, -, h}$  in the limit of a vanishing positive magnetic field  $h \ll 1$ , with uniform minus boundary conditions and inside a box  $\Lambda_h$ , the volume<sup>1</sup> of which will quadratically diverge in  $1/h$ . As far as the jump rates  $w(\sigma, \sigma^x)$  are concerned, we will only assume that there are two positive constants  $w_{\min}$  and  $w_{\max}$ , possibly depending on our fixed parameter  $\beta$ , such that

$$w_{\min} \leq w(\sigma, \sigma^x) \leq w_{\max}, \quad \sigma \in \Omega_{\Lambda_h}, \quad x \in \Lambda_h,$$

which implies in particular that  $X_{\Lambda_h, -, h}$  is irreducible.

## 1.2 Metastability issues

This kind of evolution is used as a dynamic model to study hysteresis phenomena. The critical temperature of a ferromagnet is the temperature below which, when exposed to a strong negative external magnetic field, it keeps a spontaneous negative magnetization after removing this external field. Then, by exposing the ferromagnet to a small enough positive magnetic field it will keep a higher, but still negative, magnetization for a long time, typically longer than usual experiment times. One gets a positive magnetization only by increasing the value of the external field, or waiting long enough for a relaxation to equilibrium. Then, by removing again the magnetic field before making it decrease back to negative values, the same kind of picture reappears: the ferromagnet gets a spontaneous positive magnetization, then a smaller but still positive magnetization before jumping to an equilibrium negative magnetization after a long enough time or after reaching low enough values for the external field. Two of the main questions associated with such a phenomenon are those of i) describing such a metastable equilibrium

<sup>1</sup>Working in dimension two, the word ‘‘area’’ could have been more appropriate. We will follow the usage by referring to volumes and surfaces rather than areas and perimeters.

and in particular such a higher, but still negative, magnetization; ii) characterizing such a late and abrupt relaxation to equilibrium, and in particular computing the order of magnitude of this relaxation time.

In the fundamental paper [SS98], Schonmann and Shlosman studied such a dynamics  $X_\infty$  in infinite volume and they described the state of the system at time  $t = e^{\alpha/h}$  for positive  $\alpha$ , with vanishing magnetic field  $0 < h \ll 1$ , at any subcritical temperature  $1/\beta < 1/\beta_c$  when starting from any initial measure  $\nu$  stochastically dominated by  $\mu_-$ , which is the thermodynamic limit of the Ising model in a finite box with minus boundary conditions and zero magnetic field. They identified a critical  $\alpha_c$  such that for any  $\alpha < \alpha_c$  the mean value  $\mathbb{E}_\nu[f(X_\infty(t))]$  of any local observable  $f : \{-1, +1\}^{\mathbb{Z}^2} \rightarrow \mathbb{R}$  is close to the  $\mathcal{C}^k$  continuations of its expected values for negative values of the magnetic field  $h < 0 \rightarrow \mu_h(f)$ , with  $\mu_h$  the thermodynamic limit of the Ising model in a finite box with non-zero magnetic field  $h$ . More precisely they answered the first question by proving that, for all  $k > 0$ ,

$$\mathbb{E}_\nu[f(X_\infty(t))] = \sum_{j < k} \frac{h^j}{j!} \left. \frac{d^j \mu_h(f)}{dh^j} \right|_{h=0_-} + O(h^k). \quad (2)$$

As far as the second question is concerned they also proved that for any  $\alpha > \alpha_c$  the mean value  $\mathbb{E}_\nu[f(X_\infty(t))]$  of any local observable  $f$  is close to its expected value  $\mu_h(f)$ . The formula they established for  $\alpha_c$  is particularly remarkable:

$$\alpha_c = \frac{\beta w_\beta^2}{12 m_\beta^*}, \quad (3)$$

where  $m_\beta^*$  is the spontaneous magnetization at inverse temperature  $\beta$ ,

$$m_\beta^* = -\mu_-(\sigma_0)$$

with  $\sigma_0$  the local observable defined by  $\sigma_0 : \omega \in \Omega_{\mathbb{Z}^2} \mapsto \omega(0)$ , and  $w_\beta$  is the surface tension of the unitary volume Wulff shape (see Section 2.1).

At this point it remains to describe the evolution of the system at times of order  $e^{\alpha_c/h}$ , the order of the relaxation time of this dynamics. Since we are in the regime  $h \ll 1$ , for any given  $\alpha \neq \alpha_c$  the two cases  $\alpha < \alpha_c$  and  $\alpha > \alpha_c$  refer to very small and very large times  $t = e^{\alpha/h}$  with respect to  $e^{\alpha_c/h}$ . The  $O(h^k)$  in formula (2) depends on  $\alpha < \alpha_c$  just as, in the case  $\alpha > \alpha_c$ , the “small enough  $h$ ” from which  $\mathbb{E}_\nu[f(X_\infty(e^{\alpha/h}))]$  will be close to  $\mu_h(f)$  depends on  $\alpha$ . More precisely it holds, for any given  $\epsilon > 0$ ,

$$\left| \mathbb{E}_\nu[f(X_\infty(e^{\alpha/h}))] - \mu_h(h) \right| < \epsilon$$

for  $h < h_0(\alpha)$ ; and  $h_0(\alpha)$  vanishes as  $h$  does. One cannot then use these results to describe the system at times  $t$  of order  $e^{\alpha_c/h}$  for small  $h > 0$ . This is the goal of this paper in the simpler case of the dynamics  $X_{\Lambda_h, -, h}$  on, instead of the infinite volume  $\mathbb{Z}^2$ , a Wulff shape domain  $\Lambda_h$  containing around  $(B_{\max}/h)^2$  sites for a large enough  $B_{\max} > 0$ . The box  $\Lambda_h$  is formally defined by

$$\Lambda_h = \left( \frac{B_{\max}}{h} W \right) \cap \mathbb{Z}^2$$

with  $W$  defined after Equation (15) at page 8. As it will be clear from the heuristics of the next section, that goes back to Schonmann and Shlosman indeed, with a small  $B_{\max}$  we would not have any metastable behaviour: equilibrium would look like the minus phase. On the contrary, with a large  $B_{\max}$ , and with such a box shape, the plus phase will invade the whole box at equilibrium, due to the positivity of the magnetic field and despite the minus boundary conditions.

### 1.3 A pathwise description

In this finite volume case, we can give another description, in terms of *restricted ensemble*, of the metastable equilibrium by following [SS98]. The configurations in  $\Omega_{\Lambda_h}$ , which we identify with

$$\Omega_{\Lambda_h, -} = \{ \sigma \in \Omega_{\mathbb{Z}^2} : \sigma(x) = -1 \text{ for all } x \notin \Lambda_h \},$$

can be described as a collection of closed self-avoiding contours on the dual lattice, which separate plus spins from minus spins. In doing so we adopt a standard “splitting rule”, the one used in [DKS92] (Section 3.1 there). We call *external contour* of a given configuration any contour that is not surrounded

by any other contour. We define  $\mathcal{R}_-$  as the set of configurations in  $\Omega_{\Lambda_h}$  such that the volume of each external contour, i.e., the number of sites enclosed in it, is smaller than  $(B_c/h)^2$  with

$$B_c = \frac{w_\beta}{2m_\beta^*}. \quad (4)$$

The expansion (2) is actually an expansion for  $\mu_{\Lambda_h, -, h}(f|\mathcal{R}_-)$ . Our pathwise description will also make use of such a restricted ensemble  $\mu_{\Lambda_h, -, h}(\cdot|\mathcal{R})$  but for another  $\mathcal{R} \neq \mathcal{R}_-$ . The reader can think of  $\mathcal{R}$  as a set that is smaller than  $\mathcal{R}_-$ , since some configurations with limited volume but large perimeter are allowed in the latter and will be excluded from the former. However  $\mathcal{R}$  will not be a subset of  $\mathcal{R}_-$ , since it will include slightly supercritical configurations in the sense of the heuristics of the next paragraph, while all configurations in  $\mathcal{R}_-$  are subcritical.

Before describing the set  $\mathcal{R}$  we will choose, let us first recall the heuristics where Formula (4) comes from. If  $w_\beta$  is the surface free energy of a unitary volume Wulff shape  $W$ , then the free energy of a discrete “plus phase” Wulff shape with a volume of order  $(B/h)^2$  in a “minus phase” can be estimated, for  $h \ll 1$  and up to an additive function that does not depend on  $B$ , by

$$\Phi\left(\frac{B}{h}W\right) = w_\beta \frac{B}{h} - 2\frac{h}{2}\left(\frac{B}{h}\right)^2 m_\beta^* = \frac{1}{h} [w_\beta B - m_\beta^* B^2].$$

We will refer to the quantity  $B/h$  as the *linear size* of such a Wulff shape with volume  $(B/h)^2$ . The  $1/2$  factor in the previous equation comes from the Hamiltonian, while the factor 2 accounts for the volume of the plus phase as well as the volume of the minus phase, which is the volume of  $\Lambda_h$  minus the volume of the Wulff droplet. Let us set

$$\phi(B) = [w_\beta B - m_\beta^* B^2] = \frac{w_\beta^2}{4m_\beta^*} - m_\beta^* \left(B - \frac{w_\beta}{2m_\beta^*}\right)^2 = A - m_\beta^* (B - B_c)^2 \quad (5)$$

with

$$A = \frac{w_\beta^2}{4m_\beta^*}. \quad (6)$$

This computation suggests that a plus phase Wulff droplet of size  $(B/h)^2$  will have a tendency to shrink or grow depending on  $B < B_c$  or  $B > B_c$ . Being the Wulff shape a minimizer of the surface free energy for a given volume, critical Wulff droplets of size  $B_c/h$  will indeed constitute a bottleneck for the dynamics and we will refer to the cases  $B < B_c$  and  $B > B_c$  as the *subcritical* and *supercritical cases*.

To make rigorous such free energy estimates, we will follow [SS98] and use the skeleton description of contours of [DKS92]. Skeletons are associated with long enough contours only. This motivates the following definition inherited from [SS98] and extended to all contours, external or not.

**Definition 1.1.** *Let  $b$  a positive number which is less than  $1/4$ . A contour is said  $b$ -vertebrate, or simply vertebrate, if it encloses more than  $1/h^{2b}$  sites in its interior. A contour is said  $b$ -invertebrate, or simply invertebrate if the number of sites that are enclosed in its interior is less than or equal to  $1/h^{2b}$ .*

We are now ready to define our set  $\mathcal{R}$ . To this end we introduce another parameter  $B_+ > B_c$ , which has to be thought of as close<sup>2</sup> to  $B_c$ , and which, just as  $b$ , will not depend on  $h$ .

**Definition 1.2.** *For  $0 < b < 1/4$  and  $B_+ > B_c$ , we call  $\mathcal{R}$  the set of all configurations  $\sigma$  in  $\Omega_{\Lambda_h}$  for which one can find a collection of at most  $1/h^{(1-b/2)}$  disjoint Wulff shapes and with total linear size less than  $B_+/h$  that contains all the  $b$ -vertebrate contours of  $\sigma$ .*

The reader can think of the relevant configurations in  $\mathcal{R}$  as those with only one large contour enclosed in a subcritical, or slightly supercritical, Wulff shaped box. The reason why we need an upper bound on the number of involved boxes is technical. At some point (see inequality (47) at page 29) we will need to upper bound the number of such possible box arrangements, and this restriction will help.

We define the mixing time of  $X_{\Lambda_h, -, h}$  by

$$t_{\text{mix}, h} = \inf \left\{ t \geq 0 : \forall \sigma \in \Omega_{\Lambda_h}, \forall E \subset \Omega_{\Lambda_h}, \left| \mathbb{P}_\sigma(X_{\Lambda_h, -, h}(t) \in E) - \mu_{\Lambda_h, -, h}(E) \right| \leq \frac{1}{e} \right\},$$

<sup>2</sup>As long as  $\phi(B_+)$  is positive the restricted ensemble  $\mu_{\Lambda_h, -, h}(\cdot|\mathcal{R})$  will be concentrated on the same kind of configurations, but, because some dynamical quantities will also play a role, we will get stronger results by taking  $B_+$  close to  $B_c$  rather than only asking for the positivity of  $\phi(B_+)$ .

with  $\mathbb{P}_\sigma$  the probability measure associated with  $X_{\Lambda_h, -, h}$  started in  $\sigma$ ; so that the total variation distance between  $\mu_{\Lambda_h, -, h}$  and the law of  $X_{\Lambda_h, -, h}(t)$  is exponentially small in  $t$  for  $t$  larger than  $t_{\text{mix}, h}$ . By using techniques from [SS98] one could get the following proposition, that we will obtain as a byproduct of our main results.

**Proposition 1.3.** *For all supercritical  $\beta > \beta_c$  and any  $B_{\text{max}} > 2B_c$  it holds (recall (6))*

$$\lim_{h \rightarrow 0} h \ln(t_{\text{mix}, h}) = \beta A. \quad (7)$$

To describe our dynamics on this time scale  $t_{\text{mix}, h}$  we will use a suitable random time  $T$  so that, starting from the restricted ensemble  $\mu_{\Lambda_h, -, h}(\cdot | \mathcal{R})$ , the rescaled time  $T/t_{\text{mix}, h}$  will converge in law to an exponential random variable of mean one and, for  $t > T$ , the law of  $X_{\Lambda_h, -, h}(t)$  will be close to  $\mu_{\Lambda_h, -, h}$ . The definition of  $T = T_{\lambda_{\mathcal{S}}}$  involves another set of configurations  $\mathcal{S}$  (see Definition 1.4) and a further randomization: it can be interpreted as a killing time under a killing rate  $\lambda_{\mathcal{S}}$  defined below (i.e., rate  $\lambda = \lambda(h)$  effective only when the process is in  $\mathcal{S}$ ). The idea behind the use of such a time  $T_{\lambda_{\mathcal{S}}}$  comes from [BG16], which proposed the use of soft measures and of these random times. In comparison with the plain use of exit times from suitable subsets of the configuration space (approximation to a “metastable basin”) this gives a softer (better) way to deal with the escape from metastability, also allowing a more natural use of potential theoretical tools. For a formal statement of the mentioned convergence in law that does not use stopping times see Definition 1.4, equation (8) and formula (12) below, where  $\nu$  can be taken equal to  $\mu_{\Lambda_h, -, h}(\cdot | \mathcal{R})$  and  $\lambda = \lambda(h) = e^{-\epsilon/h}$  for a small enough  $\epsilon > 0$ .

Now, following [CGOV84], as fully detailed in [OV05], we will use time averages to describe the state of our system at earlier times. We will identify a deterministic time scale  $\theta \ll t_{\text{mix}, h}$  such that, for a large class of starting measures that will be attracted by the restricted ensemble and for all times  $t < T - \theta$ , the time averages of any observable  $f : \Omega_{\Lambda_h} \rightarrow \mathbb{R}$ ,

$$A_\theta(t, f) = \frac{1}{\theta} \int_t^{t+\theta} f(X_{\Lambda_h, -, h}(u)) du,$$

will be close to  $\mu_{\Lambda_h, -, h}(f | \mathcal{R})$  with a probability that goes to 1 for a vanishing magnetic field  $h$ .

Before characterizing this “large class” of starting measures that fall in the basin of attraction of the restricted ensemble, we need to make precise the definitions of  $\mathcal{S} \subset \Omega_{\Lambda_h}$  and of the random time  $T = T_{\lambda_{\mathcal{S}}}$ . The definition  $\mathcal{S}$  is essentially symmetric to that of  $\mathcal{R}$  and uses the symmetric  $B_-$  of  $B_+$  with respect to  $B_c$ :

$$B_- = B_c - (B_+ - B_c).$$

Note that when  $B_+$  is only *slightly* supercritical  $B_-$  too is only *slightly* subcritical.

**Definition 1.4.** *We call  $\mathcal{S}$  the set of all configurations  $\sigma$  in  $\Omega_{\Lambda_h}$  for which there is at least one external contour such that a Wulff shape of volume  $(B_-/h)^2$  can fit in its interior.*

We stress that, while  $\mathcal{R}$  refers to “small enough” contours and  $\mathcal{S}$  refers to “large enough” contours, since  $\mathcal{R}$  allows slightly supercritical contours and  $\mathcal{S}$  allows slightly subcritical contours,  $\mathcal{R}$  and  $\mathcal{S}$  do have a non-empty intersection. These sets are actually tailored to cover all the relevant configurations along typical relaxation paths of the process and allow, at the same time, for some control of the local relaxation times associated with the restricted processes in  $\mathcal{R}$  and  $\mathcal{S}$ . Their non-empty intersection is a corollary of such requirements. As a consequence, we will have to use the results of [BGM18] rather than [BG16]; and [BGM18] will also provide, from such bounds on local relaxation times, the previously mentioned deterministic time scale  $\theta \ll t_{\text{mix}, h}$ .

Let now  $\tau$  be a unit mean exponential time independent of  $X_{\Lambda_h, -, h}$  and let  $\ell_{\mathcal{S}}(t)$  be the local time in  $\mathcal{S}$  up to time  $t$ , i.e., the total time spent in  $\mathcal{S}$  by  $X_{\Lambda_h, -, h}$  up to time  $t$ :

$$\ell_{\mathcal{S}}(t) = \int_0^t \mathbb{1}\{X_{\Lambda_h, -, h}(u) \in \mathcal{S}\} du. \quad (8)$$

(The law of  $\ell_{\mathcal{S}}$ , just as that of  $X_{\Lambda_h, -, h}$ , depends on the starting distribution of  $X_{\Lambda_h, -, h}$ , but, as for  $X_{\Lambda_h, -, h}$ , we omit it in the notation.)  $T_{\lambda_{\mathcal{S}}}$  is the time  $t$  when  $\ell_{\mathcal{S}}(t)$  reaches  $\tau/\lambda$ :

$$T_{\lambda_{\mathcal{S}}} = \min \{t \geq 0 : \lambda \ell_{\mathcal{S}}(t) \geq \tau\}.$$

In other words,  $T_{\lambda_S}$  can be interpreted as the killing time associated with the killing rate defined by

$$\lambda_S(\sigma) = \lambda \mathbf{1}_{\{\sigma \in \mathcal{S}\}}, \quad \sigma \in \Omega_h.$$

The precise value of  $\lambda$  is not relevant, it will be enough to choose it in such a way to have  $1/\lambda$  large, on the one hand, with respect to some “local relaxation time in  $\mathcal{S}$ ” —more precisely, with respect to the mixing time of the “restricted dynamics in  $\mathcal{S}$ ”— and small, on the other hand, with respect to the “global mixing time”  $t_{\text{mix},h}$ .

Let us finally introduce two last stopping times to state our main result. For another parameter  $\kappa > 0$  we define  $T_{\kappa_{\mathcal{R}}}$  in an analogous way, as the killing time associated with a killing rate  $\kappa_{\mathcal{R}}$ , equal to  $\kappa$  in  $\mathcal{R}$  and 0 outside of  $\mathcal{R}$ . With  $\tilde{\tau}$  another unit exponential time independent of  $\tau$  and  $X_{\Lambda_h,-,h}$ ,  $T_{\kappa_{\mathcal{R}}}$  is then the time  $t$  when  $\ell_{\mathcal{R}}(t)$ , local time in  $\mathcal{R}$ , reaches  $\tilde{\tau}/\kappa$ . We call  $T_{\mathcal{X}^c}$  the first time when  $X_{\Lambda_h,-,h}$  goes outside

$$\mathcal{X} = \mathcal{R} \cup \mathcal{S}.$$

Note that  $T_{\lambda_S}$  can also be built from a Poisson clock with rate  $\lambda$  and that is independent from  $X_{\Lambda_h,-,h}$ : it is the first ring time  $T$  for which  $X_{\Lambda_h,-,h}(T)$  is in  $\mathcal{S}$ . Using another independent Poisson clock with rate  $\kappa$  we can also build  $T_{\kappa_{\mathcal{R}}}$  in a similar way.  $T_{\kappa_{\mathcal{R}}}$ ,  $T_{\lambda_S}$  and  $T_{\mathcal{X}^c}$  are stopping times with respect to the natural filtration associated with  $X_{\Lambda_h,-,h}$  and these two independent Poisson processes.

**Theorem 1.** *For any supercritical  $\beta > \beta_c$ , any  $B_{\text{max}} > 2B_c$ , any  $b < 1/4$  and for all small enough  $\epsilon > 0$ , one can choose  $B_+$  close enough to  $B_c$  and  $\lambda = \lambda(h) = e^{-\epsilon/h}$  for which there are  $h_0 > 0$ ,  $\delta > 0$  and  $\delta' < \epsilon$  such that the following holds for  $X_{\Lambda,-,h}$  started from a probability measure  $\nu$  and any observable  $f : \Omega_{\Lambda_h} \rightarrow \mathbb{R}$ .*

- i. *If  $\nu = \mu_{\Lambda_h,-,h}(\cdot | \mathcal{R})$ , then  $T_{\lambda_S}/t_{\text{mix},h}$  converges in law to an exponential random variable of mean 1, i.e., for all  $t > 0$ ,*

$$\lim_{h \rightarrow 0} \mathbb{P}_{\nu} \left( \frac{T_{\lambda_S}}{t_{\text{mix},h}} > t \right) = e^{-t}. \quad (9)$$

Also

$$\lim_{h \rightarrow 0} \mathbb{P}_{\nu} \left( \theta < T_{\lambda_S}, \sup_{t < T_{\lambda_S} - \theta} |A_{\theta}(t, f) - \mu_{\Lambda_h,-,h}(f | \mathcal{R})| \leq \|f\|_{\infty} e^{-\delta/h} \right) = 1, \quad (10)$$

with

$$\theta = \exp \left\{ \frac{1}{h} \left( \frac{\beta A}{2} + \delta' \right) \right\}. \quad (11)$$

- ii. *For all  $h < h_0$  it holds*

$$\left| E_{\nu} \left[ f \left( X_{\Lambda_h,-,h}(T_{\lambda_S}) \right) \right] - \mu_{\Lambda_h,-,h}(f) \right| \leq \|f\|_{\infty} e^{-\delta/h},$$

whatever the starting measure  $\nu$ .

- iii. *If  $\nu$  is such that, with  $\kappa = \lambda$ ,*

$$\lim_{h \rightarrow 0} \mathbb{P}_{\nu} (T_{\kappa_{\mathcal{R}}} < T_{\lambda_S} \wedge T_{\mathcal{X}^c}) = 1,$$

then (9)–(11) are also in force.

### Comments:

- i. Equation (9) can be rewritten without the stopping time  $T_{\lambda_S}$ , i.e., by referring to  $X_{\Lambda_h,-,h}$  only: it reads

$$\lim_{h \rightarrow 0} \mathbb{E}_{\nu} \left[ e^{-\lambda \ell_S(st_{\text{mix},h})} \right] = e^{-s}, \quad s \geq 0. \quad (12)$$

- ii. Since both  $\mu_{\Lambda_h,-,h}(\cdot | \mathcal{R}_-)$ , which does not depend on the parameters  $B_+$  and  $b$ , and  $\mu_{\Lambda_h,-,h}(\cdot | \mathcal{R})$  are concentrated, up to large deviation events, on the subset  $\mathcal{I}$  of  $\mathcal{R}_-$  and  $\mathcal{R}$  that is made of configuration with invertebrate contours only, the same results hold with  $\mu_{\Lambda_h,-,h}(\cdot | \mathcal{R}_-)$  in place of  $\mu_{\Lambda_h,-,h}(\cdot | \mathcal{R})$ . We chose to write them with  $\mu_{\Lambda_h,-,h}(\cdot | \mathcal{R})$  for one main reason only. The key point of the proof will be the derivation of an upper bound for the relaxation time of the dynamics restricted to  $\mathcal{R}$  (as well as the dynamics restricted to  $\mathcal{S}$ ) and we were not able to do the same with the dynamics restricted to  $\mathcal{R}_-$ .

- iii. Such upper bounds will allow us to apply the results of [BGM18]. In particular, given a small enough  $\epsilon > 0$  we will see that one can choose some  $B_+$  sufficiently close to  $B_c$  and  $\lambda = \lambda(h) = e^{-\epsilon/h}$  for which there are constants  $C > 0$  and  $\delta > 0$  such that, if  $\nu = \mu_{\Lambda_h, -, h}(\cdot | \mathcal{R})$  or  $\nu$  satisfies, with  $\kappa = \lambda$  and for  $h$  small enough,

$$\mathbb{P}_\nu(T_{\kappa\mathcal{R}} > T_{\lambda\mathcal{S}} \wedge T_{\mathcal{X}^c}) \leq e^{-2\epsilon/h},$$

then, for all  $a$  such that

$$\epsilon < \beta a < \beta A - \epsilon$$

and all observable  $f : \Omega_\Lambda \rightarrow \mathbb{R}$ , we recover

$$\left| \mathbb{E}_\nu \left[ f \left( X_{\Lambda_h, -, h} \left( e^{\beta a/h} \right) \right) \right] - \mu_{\Lambda_h, -, h}(f | \mathcal{R}) \right| \leq C \|f\|_\infty e^{-\delta/h}. \quad (13)$$

This allows, following Schonmann and Shlosman, for an expansion as in (2).

- iv. The critical value for  $a$  in (13) is  $A$  and not  $\alpha_c/\beta = A/3$  (recall (3) from page 3). The factor  $1/3$  has to do with a different relaxation mechanism in larger boxes. It was first studied in [DS97] and is related both to some spatial entropy associated with the nucleation of a critical droplet and to the time needed for a supercritical droplet to invade a fixed box. In the infinite volume case or already in the case of a large domain  $\Lambda$  of exponentially large volume  $e^{C/h}$  with a large enough  $C$ , not only the asymptotic value of the mean “transition time to equilibrium” would change; it is not clear anymore whether we should expect its law to be asymptotically exponential: to an exponential random time needed to nucleate a critical droplet we should add another time of the same logarithmic scale order (the time needed to invade the given box), and prefactors enter the game at this point. The asymptotic exponential law would survive if the prefactor associated with the nucleation of the critical droplet is dominant.
- v. The condition  $B_{\max} > 2B_c$  ensures that the volume is large enough for the positive magnetic field to overcome the effect of the negative boundary condition, in such a way that the plus phase invades the whole box at equilibrium.
- vi. The restriction on the shape of the domain is technical and will simplify the proof. It avoids in particular a description of typical equilibrium configurations in more general domains.
- vii. Theorem 1 allows us to consider more general starting distributions than in [SS98]. This is due to the fact that controlling the local relaxation time in  $\mathcal{R}$  and  $\mathcal{S}$ , we will not have to rely on the monotonicity of  $X$  in the same way.

Thinking of a slowly changing magnetic field as in the hysteresis phenomena, it is natural to consider starting distributions like  $\mu_{\Lambda_h, -, h'}(\cdot | \mathcal{R}_{h'})$  associated with a different magnetic field  $h'$ , but with the same domain  $\Lambda_h$ . This is one possibility considered in the following corollary of Theorem 1. The other possibility we consider in this corollary is that of the canonical ensemble associated with a small enough magnetization

$$M : \omega \in \Omega_{\Lambda_h} \mapsto \sum_{x \in \Lambda_h} \omega(x),$$

namely  $\mu_{\Lambda_h, -, h}(\cdot | \mathcal{R}$  and  $M > m(B_{\max}/h)^2)$  with  $m < m_\beta^*[2(B_c/B_{\max})^2 - 1]$ . This upper bound corresponds to the magnetization of a critical Wulff shape droplet of plus phase in the minus phase.

**Corollary 1.5.** *Let  $\epsilon > 0$ ,  $c > 0$  and  $m < m_\beta^*[2(B_c/B_{\max})^2 - 1]$  associated with  $\beta > \beta_c$  and  $B_{\max} > 2B_c$ . If  $\nu = \mu_{\Lambda_h, -, h'}(\cdot | \mathcal{R}_{h'})$  associated with  $h' = ch$  or  $\nu = \mu_{\Lambda_h, -, h}(\cdot | \mathcal{R}, M > m(B_{\max}/h)^2)$ , then there are  $B_+ > B_c$ ,  $\lambda = \lambda(h) = e^{-\epsilon/h}$ ,  $\delta > 0$ ,  $\delta' < \epsilon$  and  $C > 0$  such that (9)–(11) and (13) hold for any observable  $f : \Omega_{\Lambda_h} \rightarrow \mathbb{R}$  and if  $\epsilon < \beta a < \beta A - \epsilon$ .*

In the next section we introduce a collection of tools for the proof of Theorem 1, Proposition 1.3 and Corollary 1.5. This includes in particular static estimates, for which the main references are [SS98], [DKS92], [Pfi91], [Iof94] and [Iof95] and dynamical techniques, for which the main references are [Sin92] and [Mar94]. We use the former in Section 3 to give lower bounds on the transition time to equilibrium. We use the latter in Section 4 to give upper bounds on local relaxation times. This is the key point of the proof: we show in the last part of Section 2 how to use the results of [BGM18] to obtain from such estimates an equivalent of Theorem 1, Proposition 1.3 and Estimate (13) for the restriction  $X$  of our process  $X_{\Lambda_h, -, h}$  to  $\mathcal{X} = \mathcal{R} \cup \mathcal{S}$ , and we explain how to reduce the study of  $X_{\Lambda_h, -, h}$  to that of  $X$ . We

finally prove Theorem 1, Proposition 1.3 and Corollary 1.5 in Section 5. From now on we will always assume our fixed parameters  $\beta$  and  $B_{\max}$  to be respectively larger than the critical inverse temperature  $\beta_c$  and  $2B_c$ .

## 2 Tools, notation and strategy

### 2.1 Wulff shape and surface tension

In order to define the surface tension in a direction orthogonal to the unitary vector  $\mathbf{n} = (\cos \theta, \sin \theta)$  for  $\theta \in [0, 2\pi]$ , we have to consider the Ising model in a square box  $\Lambda(L) = [-L, L]^2$  with boundary condition

$$\eta_\theta(x) = \begin{cases} +1 & \text{if } u \cos \theta + v \sin \theta \leq 0, \\ -1 & \text{if } u \cos \theta + v \sin \theta > 0, \end{cases} \quad x = (u, v) \in \mathbb{Z}^2.$$

In a contour description of the configurations that are associated with such a boundary condition, one contour, on the dual lattice, must join two points that are close to  $y(L)$  and  $z(L)$ , which are the two points where the boundary of the box  $[-L, L]^2$  intersects the straight line that goes through the origin and admits  $\mathbf{n}$  as normal vector. The *surface tension* in the direction of this straight line is

$$\tau(\theta) = \lim_{L \rightarrow +\infty} -\frac{1}{\beta \|y(L) - z(L)\|_2} \ln \frac{Z_{\Lambda(L), \eta_\theta, 0}}{Z_{\Lambda(L), +, 0}},$$

with  $Z_{\Lambda(L), +, 0}$  the partition functions associated with the Ising model in  $\Lambda(L)$ , with uniform plus boundary condition and without magnetic field. Thus, the surface tension  $\tau(\theta)$  is the free energy per unit length of an interface between the plus and minus phase in the direction orthogonal to  $\mathbf{n}$ . It is positive and finite for subcritical temperature  $1/\beta < 1/\beta_c$ .

We then define the *surface free energy* of any rectifiable  $\gamma \subset \mathbb{R}^2$  that is the boundary of a simply connected domain  $D \subset \mathbb{R}^2$  by the quantity

$$\mathcal{W}(\gamma) = \oint_\gamma \tau(\theta_s) ds, \quad (14)$$

with  $\theta_s$  the direction of the external normal, i.e., which points outside  $D$ , at the curvilinear abscissa  $s$ . We will refer to  $\mathcal{W}$  as the *Wulff functional*. The *Wulff shape* has a boundary that minimizes this quantity among all the rectifiable boundaries of domains with a given volume. It is defined for  $\rho > 0$  and up to dilatation and translation by

$$W_\rho = \bigcap_{\theta \in [0, 2\pi]} \left\{ x = (u, v) \in \mathbb{R}^2 : u \cos \theta + v \sin \theta \leq \rho \tau(\theta) \right\}. \quad (15)$$

As a consequence of the symmetries of  $\tau$  that are inherited from those of the lattice,  $W_\rho$  is invariant by rotations of angle  $\pi/2$ . We will simply write  $W$ , without the index  $\rho$ , when  $\rho$  is chosen in such a way that  $W_\rho$  has a volume equal to one.

The *support function* with respect to the origin 0 of the convex set  $W_\rho \ni 0$  is actually  $\rho\tau$ , i.e.,

$$\rho\tau(\theta) = \max_{x=(u,v) \in W_\rho} u \cos \theta + v \sin \theta, \quad \theta \in [0, 2\pi].$$

This is a consequence of the triangular inequality: for  $x$ ,  $y$  and  $z$  in  $\mathbb{R}^2$ , if

$$\mathbf{n}_z = (\cos \theta_z, \sin \theta_z), \quad \mathbf{n}_x = (\cos \theta_x, \sin \theta_x), \quad \text{and} \quad \mathbf{n}_y = (\cos \theta_y, \sin \theta_y)$$

are the external normals to the three sides  $[x, y]$ ,  $[y, z]$  and  $[z, x]$  of the triangle  $xyz$ , then

$$\|x - z\|_2 \tau(\theta_y) \leq \|x - y\|_2 \tau(\theta_z) + \|y - z\|_2 \tau(\theta_x)$$

(see Section 4.21 in [DKS92]).

Let us denote by  $|D|$  the volume of any measurable domain  $D \subset \mathbb{R}^2$ . Then Bonnesen's inequality says that for any such domain  $D$  with a rectifiable boundary  $\gamma$ , choosing  $\rho$  in such a way that  $|W_\rho| = |D|$ , it holds

$$\mathcal{W}(\gamma) \geq \mathcal{W}(\partial W_\rho) \sqrt{1 + \left( \frac{\alpha_{out} - \alpha_{in}}{2} \right)^2}, \quad (16)$$

where  $\partial W_\rho$  stands for the boundary of  $W_\rho$ , and  $\alpha_{out}$ , respectively  $\alpha_{in}$ , is the smallest, respectively the largest,  $\alpha$  for which a translate of  $\alpha W_\rho$  contains, respectively is contained in,  $D$ . In the case where  $D$  is a convex set, this is proven in [Fla68] by counting the mean number of intersections between  $\gamma$  and the border of a random translate  $X + \alpha W_\rho$ , for  $\alpha \in [\alpha_{in}, \alpha_{out}]$  and  $X$  uniformly chosen in  $D - \alpha W_\rho$ . Flanders proves in this way Blaschke's inequality

$$\alpha^2 |W_\rho| - \alpha \rho \mathcal{W}(\gamma) + |D| \leq 0, \quad \alpha \in [\alpha_{out}, \alpha_{in}],$$

with equality in the case  $\alpha_{out} = \alpha_{in}$ . This gives a lower bound on the distance between the two roots of this polynomial of degree two in  $\alpha$ , i.e., a lower bound on its discriminant, which leads, together with the equality for  $D = W_\rho$ ,

$$\rho \mathcal{W}(\partial W_\rho) = 2|W_\rho|, \quad (17)$$

to inequality (16). In the case where  $D$  is not a convex set, these inequalities are not a direct consequence of those of the convex case, but the same strategy can be followed even though the computation of this mean intersection number is more delicate. In [DKS92] the authors adapt an argument from [Oss78, Oss79] to cover the case of a non-convex simply connected  $D$  (see Section 2.5 in [DKS92]). We will use this result, rewriting it with the following notation. With  $\rho$  and  $B$  such that  $|D| = |W_\rho| = B^2$ , we set  $B_{in} = \alpha_{in} B$  and  $B_{out} = \alpha_{out} B$ , with  $\alpha_{out}$  and  $\alpha_{in}$  as above so that  $B_{in}^2$  is the volume of the largest Wulff shape that fits in  $D$  and  $B_{out}^2$  that of the smallest Wulff shape that contains it. We denote by  $w_\beta$  the *surface free energy of the unitary volume Wulff shape*  $W$ , so that

$$\mathcal{W}(\partial W_\rho) = w_\beta B$$

and, as a consequence of (17),

$$B = \frac{w_\beta}{2} \rho. \quad (18)$$

**Proposition 2.1** (Blaschke's inequalities [DKS92]). *For any simply connected domain  $D \subset \mathbb{R}^2$  with a rectifiable boundary  $\gamma$  it holds*

$$\mathcal{W}(\gamma) \geq \frac{w_\beta}{2} \left( \frac{|D|}{B_{in}} + B_{in} \right) \quad \text{and} \quad \mathcal{W}(\gamma) \geq \frac{w_\beta}{2} \left( \frac{|D|}{B_{out}} + B_{out} \right).$$

We will also need two simple consequences of the Wulff construction from the support function  $\rho\tau$ .

**Lemma 2.2.** *If two translates of possibly different size Wulff shapes  $x_1 + W_{\rho_1}$  and  $x_2 + W_{\rho_2}$ , of volume  $B_1^2$  and  $B_2^2$ , have a non-empty intersection, then their union is contained in some Wulff shape  $x_0 + W_{(\rho_1 + \rho_2)}$  of volume  $(B_1 + B_2)^2$ .*

**Proof:** Since  $x_1 + W_{\rho_1}$  and  $x_2 + W_{\rho_2}$  have a non-empty intersection, there are  $w_1$  and  $w_2$  in  $W_{\rho_1}$  and  $W_{\rho_2}$  such that  $x_1 + w_1 = x_2 + w_2$ , i.e.,

$$x_1 - w_2 = x_2 - w_1.$$

This means that  $x_1 - W_{\rho_2}$  and  $x_2 - W_{\rho_1}$  also have a non-empty intersection. Let us then choose

$$x_0 \in (x_1 - W_{\rho_2}) \cap (x_2 - W_{\rho_1}).$$

We have  $x_1 - x_0 \in W_{\rho_2}$ , then, writing  $(u_1, v_1)$  and  $(u_0, v_0)$  for the coordinates in  $\mathbb{R}^2$  of  $x_1$  and  $x_0$ , it holds

$$(u_1 - u_0) \cos \theta + (v_1 - v_0) \sin \theta \leq \rho_2 \tau(\theta)$$

for any  $\theta \in [0, 2\pi]$ . For any  $x = (u, v)$  in  $x_1 + W_{\rho_1}$  we also have

$$(u - u_1) \cos \theta + (v - v_1) \sin \theta \leq \rho_1 \tau(\theta),$$

hence

$$(u - u_0) \cos \theta + (v - v_0) \sin \theta \leq (\rho_1 + \rho_2) \tau(\theta).$$

This shows that  $x_1 + W_{\rho_1}$  is contained in  $x_0 + W_{(\rho_1 + \rho_2)}$ , and we can check in the same way that  $x_2 + W_{\rho_2}$  is contained in  $x_0 + W_{(\rho_1 + \rho_2)}$ .  $\square$

The previous proof only use the fact that the Wulff shape is a convex set, to which one can associate a support function to describe it. The last lemma of this section uses by contrast the symmetries of the lattice, namely the fact that  $W = -W$ , i.e., that  $\rho\tau$  is  $\pi$ -periodic.

**Lemma 2.3.** *Given  $B_2 > B_1$ , the largest Wulff shapes to fit in the annulus  $B_2W \setminus B_1W$  of volume  $B_2^2 - B_1^2$  have a volume  $B_0^2 = (B_2 - B_1)^2/4$ .*

**Proof:** The Wulff shape construction from the  $\pi$ -periodical support function  $\rho\tau$  implies that, for any positive  $\rho_1$  and  $\rho_0$  the union of  $W_{\rho_1}$  with all the externally tangent Wulff shapes

$$x + W_{\rho_0}, \quad x \in \partial W_{\rho_1 + \rho_0},$$

is the Wulff shape  $W_{\rho_1 + 2\rho_0}$ . We get the desired result by choosing  $\rho_1$  and  $\rho_0$  in such a way that, with  $\rho_2 = \rho_1 + 2\rho_0$ ,

$$W_{\rho_1} = B_1W \quad \text{and} \quad W_{\rho_2} = B_2W,$$

i.e.,

$$\rho_1 = 2B_1/w_\beta \quad \text{and} \quad \rho_2 = 2B_2/w_\beta$$

so that

$$\rho_0 = \frac{\rho_2 - \rho_1}{2} = \frac{B_2 - B_1}{w_\beta}$$

and

$$B_0^2 = \left(\frac{\rho_0 w_\beta}{2}\right)^2 = \left(\frac{B_2 - B_1}{2}\right)^2.$$

□

## 2.2 Random paths, flows and block flows

Given a generic irreducible Markov process  $Y$  on a finite configuration space  $\mathcal{Y}$  with generator<sup>3</sup>  $\mathcal{L}_Y$

$$(\mathcal{L}_Y f)(\sigma) = \sum_{\sigma' \in \mathcal{Y}} w(\sigma, \sigma') [f(\sigma') - f(\sigma)], \quad f : \mathcal{Y} \mapsto \mathbb{R}, \quad \sigma \in \mathcal{Y},$$

a *path*  $\pi$  is a finite sequence  $(\sigma_0, \sigma_1, \dots, \sigma_l)$  of configurations in  $\mathcal{Y}$  such that  $w(\sigma_k, \sigma_{k+1}) > 0$  for all  $k < l$ . The *length*  $|\pi|$  of such a path  $\pi$  is the integer  $l$ . If  $e = (\sigma, \sigma')$  belongs to the edge set  $\mathcal{E}$  associated with  $Y$ , i.e., if  $\sigma$  and  $\sigma'$  are distinct configurations such that  $w(\sigma, \sigma') > 0$ , we write  $e \in \pi$  if there is  $k < |\pi|$  such that  $e = (\sigma_k, \sigma_{k+1})$ . We will also write  $\sigma \in \pi$  if there is  $k \leq |\pi|$  such that  $\sigma = \sigma_k$ .

Random paths  $\Pi$  are associated with *flows*, i.e., with functions  $\psi : \mathcal{E} \rightarrow \mathbb{R}$ , such that

$$\psi(\sigma, \sigma') = -\psi(\sigma', \sigma), \quad (\sigma, \sigma') \in \mathcal{E}.$$

Indeed, with  $\Pi = (Y_0, \dots, Y_{|\Pi|})$ ,  $\Pi_- = Y_0$  and  $\Pi^+ = Y_{|\Pi|}$  we get such an antisymmetric function by setting

$$\psi(\sigma, \sigma') = E \left[ \sum_{k < |\Pi|} \mathbb{1}\{(\sigma, \sigma') = (Y_k, Y_{k+1})\} - \mathbb{1}\{(\sigma', \sigma) = (Y_k, Y_{k+1})\} \right]$$

and we note that, for all  $\sigma$  in  $\mathcal{Y}$ ,

$$\operatorname{div}_\sigma \psi = \sum_{\sigma' \in \mathcal{Y}} \psi(\sigma, \sigma') = P(\Pi_- = \sigma) - P(\Pi^+ = \sigma).$$

In particular, if there are two disjoint subsets  $\mathcal{A}$  and  $\mathcal{B}$  of  $\mathcal{Y}$  such that  $\Pi_- \in \mathcal{A}$  and  $\Pi^+ \in \mathcal{B}$  with probability one, then  $\psi$  is a *unitary flow from  $\mathcal{A}$  to  $\mathcal{B}$* , i.e., such that

$$\operatorname{div}_\sigma \psi > 0 \Rightarrow \sigma \in \mathcal{A}, \quad \operatorname{div}_\sigma \psi < 0 \Rightarrow \sigma \in \mathcal{B} \quad \text{and} \quad \sum_{\sigma \in \mathcal{A}} \operatorname{div}_\sigma \psi = 1 = - \sum_{\sigma \in \mathcal{B}} \operatorname{div}_\sigma \psi.$$

Sinclair proved in [Sin92] that if  $Y$  is reversible with respect to some probability measure  $\mu_Y$ , i.e., if the *conductances*

$$c(\sigma, \sigma') = \mu_Y(\sigma)w(\sigma, \sigma'), \quad \sigma, \sigma' \in \mathcal{Y},$$

<sup>3</sup>The index  $\mathcal{Y}$ , rather than  $Y$ , in the notation  $\mathcal{L}_Y$  can seem unnatural since the generator depends on the whole process and not only on the configuration space, but we are foreseeing here a later more natural notation, in accordance with [BGM18].

are symmetrical, then for any random path  $\Pi$  with starting and ending configurations that are independently distributed according to  $\mu_{\mathcal{Y}}$ , it holds

$$\frac{1}{\gamma_{\mathcal{Y}}} \leq \max_{e \in \mathcal{E}} \frac{1}{c(e)} P(e \in \Pi) E[|\Pi| \mid e \in \Pi] \leq \max_{e \in \mathcal{E}} \frac{1}{c(e)} P(e \in \Pi) \|\Pi\|_{\infty}$$

with  $1/\gamma_{\mathcal{Y}}$  the relaxation time of  $Y$ , i.e.,

$$\gamma_{\mathcal{Y}} = \min_{\text{Var}_{\mu_{\mathcal{Y}}}(f) \neq 0} \frac{\mathcal{D}(f)}{\text{Var}_{\mu_{\mathcal{Y}}}(f)}, \quad (19)$$

where  $\mathcal{D}$  is the *Dirichlet form* defined by

$$\mathcal{D}(f) = \frac{1}{2} \sum_{\sigma, \sigma' \in \mathcal{Y}} c(\sigma, \sigma') [f(\sigma) - f(\sigma')]^2. \quad (20)$$

In particular, if there is a lower bound

$$w(\sigma, \sigma') \geq w_{\min}, \quad (\sigma, \sigma') \in \mathcal{E},$$

then

$$\frac{1}{\gamma_{\mathcal{Y}}} \leq \frac{\|\Pi\|_{\infty}}{w_{\min}} \max_{(\sigma, \sigma') \in \mathcal{E}} \frac{P((\sigma, \sigma') \in \Pi)}{\mu_{\mathcal{Y}}(\sigma) \vee \mu_{\mathcal{Y}}(\sigma')} \quad (21)$$

The simplest way to obtain upper bounds for relaxation times with a random path  $\Pi$  is to build for each  $\sigma$  and  $\sigma'$  in  $\mathcal{Y}$  a deterministic path  $\pi_{\sigma, \sigma'}$ , usually referred to as *canonical path*, and set  $\Pi = \pi_{\sigma, \sigma'}$  with probability  $\mu_{\mathcal{Y}}(\sigma)\mu_{\mathcal{Y}}(\sigma')$ . Martinelli gave in [Mar94] an upper bound for the relaxation time of the Glauber dynamics  $X_{\Lambda(L), +, 0}$  in the square box  $\Lambda(L) = [-L, L]^2$  by introducing a “block dynamics”, bounding its mixing time by a coupling argument and bounding the relaxation time of the Glauber dynamics in each block with such canonical paths. For a block covering of

$$\Lambda(L) = \bigcup_{j < k} \Lambda_j$$

by partially overlapping rectangular blocks  $\Lambda_j$  of size  $L \times L^{\epsilon+1/2}$ , the associated block dynamics update at rate one the current configuration  $\sigma$  according to  $\mu_{\Lambda_j, \sigma, 0}$ . He bounded the mixing time of this block dynamics by using its monotonicity properties. And as far as the relaxation time of each  $X_{\Lambda_j, \eta, 0}$  is concerned, he built the canonical path  $\pi_{\sigma, \sigma'}$  from any  $\sigma$  in  $\Omega_{\Lambda_j}$  to any  $\sigma'$  in the same configuration space by ordering, independently of  $\sigma$  and  $\sigma'$ , the sites of the rectangle  $\Lambda_j$  and flipping the spins from their value in  $\sigma$  to their value in  $\sigma'$  in this prescribed order. This order  $\preceq$  had the key property that for any  $x$  in  $\Lambda = \Lambda_j$ , with

$$\begin{aligned} \Lambda^{\prec x} &= \{y \in \Lambda : y \prec x\}, \\ \Lambda^{\succeq x} &= \{z \in \Lambda : z \succeq x\} \end{aligned}$$

and

$$\partial\Lambda^{\prec x} = \{(y, z) \in \Lambda^{\prec x} \times \Lambda^{\succeq x} : \|y - z\| = 1\},$$

$|\partial\Lambda^{\prec x}|$  was of the same order as the *shorter* side of the rectangle  $\Lambda$ . Martinelli could then use a practical version of the following abstract lemma.

**Lemma 2.4.** *For any finite box  $\Lambda \subset \mathbb{Z}^2$ , any order  $\preceq$  on  $\Lambda$ , any boundary condition  $\eta \in \Omega_{\mathbb{Z}^2}$ , any configuration  $\sigma_0$  in  $\Omega_{\Lambda}$  and any site  $x$  in  $\Lambda$ , it holds*

$$\frac{1}{\mu_{\Lambda, \eta, h}(\sigma_0)} \sum_{\sigma, \sigma' \in \Omega_{\Lambda}} \mu_{\Lambda, \eta, h}(\sigma) \mu_{\Lambda, \eta, h}(\sigma') \mathbb{1}\left\{(\sigma_0, \sigma_0^x) \in \pi_{\sigma, \sigma'}^{\preceq}\right\} \leq \exp\{2\beta|\partial\Lambda^{\prec x}|\}$$

where  $\pi_{\sigma, \sigma'}^{\preceq}$  stands for the canonical path from  $\sigma$  to  $\sigma'$  associated with the order  $\preceq$ .

**Proof:** Following the computation made in [Mar94], Section 2, denoting, for any  $\sigma^{\prec x} \in \Omega_{\Lambda^{\prec x}}$  and  $\sigma^{\succeq x} \in \Omega_{\Lambda^{\succeq x}}$ , by  $\sigma^{\prec x} \cdot \sigma^{\succeq x}$  the configuration of  $\Omega_{\Lambda}$  that coincides with  $\sigma^{\prec x}$  in  $\Lambda^{\prec x}$  and  $\sigma^{\succeq x}$  in  $\Lambda^{\succeq x}$ , and recalling the presence of the somewhat unusual factor  $1/2$  in our Hamiltonian definition, we have

$$\frac{1}{\mu_{\Lambda, \eta, h}(\sigma_0)} \sum_{\sigma, \sigma' \in \Omega_{\Lambda}} \mu_{\Lambda, \eta, h}(\sigma) \mu_{\Lambda, \eta, h}(\sigma') \mathbb{1}\left\{(\sigma_0, \sigma_0^x) \in \pi_{\sigma, \sigma'}^{\preceq}\right\}$$

$$\begin{aligned}
&\leq \frac{1}{\mu_{\Lambda,\eta,h}(\sigma_0)} \sum_{\sigma,\sigma' \in \Omega_\Lambda} \mu_{\Lambda,\eta,h}(\sigma) \mu_{\Lambda,\eta,h}(\sigma') \mathbb{1}\{\sigma|_{\Lambda^{\geq x}} = \sigma_0|_{\Lambda^{\geq x}}, \sigma'|_{\Lambda^{< x}} = \sigma_0|_{\Lambda^{< x}}\} \\
&= \frac{1}{Z_{\Lambda,\eta,h}} \sum_{\substack{\sigma^{< x} \in \Omega_{\Lambda^{< x}}, \\ \sigma^{\geq x} \in \Omega_{\Lambda^{\geq x}}}} \frac{\exp\{-\beta H_{\Lambda,\eta,h}(\sigma^{< x} \cdot \sigma_0|_{\Lambda^{\geq x}}) - \beta H_{\Lambda,\eta,h}(\sigma_0|_{\Lambda^{< x}} \cdot \sigma^{\geq x})\}}{\exp\{-\beta H_{\Lambda,\eta,h}(\sigma_0|_{\Lambda^{< x}} \cdot \sigma_0|_{\Lambda^{\geq x}})\}} \\
&= \frac{1}{Z_{\Lambda,\eta,h}} \sum_{\substack{\sigma^{< x} \in \Omega_{\Lambda^{< x}}, \\ \sigma^{\geq x} \in \Omega_{\Lambda^{\geq x}}}} \frac{\exp\{-\beta H_{\Lambda,\eta,h}(\sigma^{< x} \cdot \sigma_0|_{\Lambda^{\geq x}}) - \beta H_{\Lambda,\eta,h}(\sigma_0|_{\Lambda^{< x}} \cdot \sigma^{\geq x})\}}{\exp\{-\beta H_{\Lambda,\eta,h}(\sigma_0|_{\Lambda^{< x}} \cdot \sigma_0|_{\Lambda^{\geq x}}) - \beta H_{\Lambda,\eta,h}(\sigma^{< x} \cdot \sigma^{\geq x})\}} e^{-\beta H_{\Lambda,\eta,h}(\sigma^{< x} \cdot \sigma^{\geq x})} \\
&\leq \frac{1}{Z_{\Lambda,\eta,h}} \sum_{\substack{\sigma^{< x} \in \Omega_{\Lambda^{< x}}, \\ \sigma^{\geq x} \in \Omega_{\Lambda^{\geq x}}}} \exp\{2\beta|\partial\Lambda^{< x}|\} e^{-\beta H_{\Lambda,\eta,h}(\sigma^{< x} \cdot \sigma^{\geq x})} \\
&= \exp\{2\beta|\partial\Lambda^{< x}|\}.
\end{aligned}$$

□

The crucial spectral gap estimates of the present paper (see Section 4) rely on the following observation: as far as leading orders are concerned, Martinelli's lower bound on  $\gamma_{\Lambda(L),+,0}$  can be obtained by direct application of formula (21). To do so one has to build a random path  $\Pi$  with starting and ending configurations independently distributed according to  $\mu_{\Lambda(L),+,0}$ . Equivalently one has to build, for each  $\sigma$  and  $\sigma'$  in  $\Omega_{\Lambda(L)}$ , a random path  $\Pi_{\sigma,\sigma'}$  and set  $\Pi = \Pi_{\sigma,\sigma'}$  with probability  $\mu_{\Lambda(L),+,0}(\sigma)\mu_{\Lambda(L),+,0}(\sigma')$ . Here is a block dynamic inspired way to build a suitable  $\Pi_{\sigma,\sigma'}$  from two random paths  $\Pi_\sigma$  and  $\Pi_{\sigma'}$  starting from  $\sigma$  and  $\sigma'$ , respectively. From  $\sigma$  we build a sequence of  $k$  random configurations that we will call "milestones"  $M_1, M_2, \dots, M_k$  in  $\Omega_{\Lambda(L)}$ . We set  $M_0 = \sigma$ , call it our first milestone, and build from each milestone  $M_j$ , with  $j < k$ , the next milestone  $M_{j+1}$  by setting  $M_{j+1}|_{\Lambda_j^c} = M_j|_{\Lambda_j^c}$  and drawing  $M_{j+1}|_{\Lambda_j}$  according to  $\mu_{\Lambda_j, M_j, 0}$ . Next, we use, in each block  $\Lambda_j$ , a canonical path of the single spin flip Glauber dynamics to connect  $M_j$  with  $M_{j+1}$ ; this defines our random path  $\Pi_\sigma$  and we build  $\Pi_{\sigma'}$  in an analogous way from  $\sigma'$ . Consider now, with obvious notation, the event

$$E_{\sigma,\sigma'} = \{M_k = M'_k\}.$$

When  $E_{\sigma,\sigma'}$  occurs we can build  $\Pi_{\sigma,\sigma'}$  by concatenation of  $\Pi_\sigma$ , from  $\sigma$  to  $M_k$ , and the reversed path  $\Pi_{\sigma'}$ , from  $M'_k = M_k$  to  $\sigma'$ . From the conditional probability associated with  $E_{\sigma,\sigma'}$  we get a random path  $\Pi_{\sigma,\sigma'}$  from  $\sigma$  to  $\sigma'$ , then a random path  $\Pi$  with starting and ending configurations independently distributed according to the equilibrium distribution. When used in formula (21), estimating the relaxation time  $1/\gamma_{\Lambda(L),+,0}$  boils down, through DLR equations, to computing a uniform lower bound on  $P(E_{\sigma,\sigma'})$  that, in turns, can be obtained with the very same arguments used by Martinelli for controlling the mixing time of the block dynamics.

This is nothing but an alternative way of articulating Martinelli's ideas. But in doing so we gain some flexibility: there is no need anymore to define any block *dynamic*, we only need to build suitable sequences of milestones for which we can give a uniform lower bound on the probability of such events  $E_{\sigma,\sigma'}$  that are contained in  $\{M_k = M'_k\}$  (we used here the latter event to define  $E_{\sigma,\sigma'}$  but we will later require more from such events; and the inclusion will be again needed for allowing a similar construction of  $\Pi_{\sigma,\sigma'}$  from those of  $\Pi_\sigma$  and  $\Pi_{\sigma'}$ ). In particular the box  $\Lambda_j$  used to build  $M_{j+1}$  from  $M_j$  can now depend in some way of  $M_j$ . We will use this slightly different strategy and the flexibility it allows to control the local relaxation times of  $X_{\Lambda_h,-,h}$  restricted to  $\mathcal{R}$  and  $\mathcal{S}$ . We will refer to such milestone built random paths  $\Pi_\sigma$ ,  $\Pi_{\sigma'}$  or  $\Pi_{\sigma,\sigma'}$  and their associated flows as "block paths" and "block flows". We will also use such a block flow to estimate the soft capacity presented in Section 2.4.

## 2.3 Free energy estimates

In this section we closely follow Schonmann and Shlosman. In [SS98] they derived a number of free energy estimates that we have to slightly adapt to deal with some particular "annular droplets" (see estimate (27) in Lemma 2.9). In this respect we need a slightly stronger theory, but the extension is straightforward and we only write in this section those technical points for which we need a slightly different writing. Also there are a few estimates for which we only need a weaker form than in [SS98] (for example, the probability appearing in estimate (26) of Lemma 2.9 is actually shown to be close to

one in [SS98]). For these estimates, their stronger forms in [SS98] derive from stability results in [DKS92], which, in turn, are based on Blaschke's inequalities of Section 2. We also use Blaschke's inequalities in this paper, but for other purposes, mainly in Section 3 and also in proving estimate (27) in Lemma 2.9.

The key objects introduced in [DKS92] to make sense of a macroscopic (on length scale  $1/h$ ) or even mesoscopic (on length scale  $1/h^b$ , with  $b < 1/4$  as mentioned earlier) notion of free energy are the skeletons associated with vertebrate contours, i.e., contours with more than  $1/h^{2b}$  sites in their interior. Following [SS98] to build them, we will be closer to their construction in [Pfi91].

Let  $r$  be a positive number that is smaller than  $b/2$  and consider a configuration  $\sigma$  in

$$\Omega_{\Lambda,-} = \{\sigma \in \Omega_{\mathbb{Z}^2} : \sigma(x) = -1 \text{ for all } x \notin \Lambda\},$$

which we identified with  $\Omega_{\Lambda}$ , for a finite domain  $\Lambda \subset \mathbb{Z}^2$ . A *skeleton* associated with a vertebrate contour  $\Gamma$  of  $\sigma$  is a possibly self-intersecting polygon  $\gamma \subset \mathbb{R}^2$  such that

- i. the ordered vertices of which are consecutive points on  $\Gamma$  with the same order (for one of the two possible orientation of  $\Gamma$ );
- ii. the side lengths of which lie between  $1/(12h^r)$  and  $1/h^r$ ;
- iii. such that the Hausdorff distance between  $\Gamma$  and  $\gamma$  is smaller than or equal to  $1/h^r$ .

In what follows we will assume that we have an algorithm to assign such a skeleton  $\gamma$  to any vertebrate contour  $\Gamma$ , so that we can refer to *the* collection of skeleton  $S = (\gamma_j : j < k)$  associated with the collection  $G = (\Gamma_j : j < k)$  of the vertebrate contours of a configuration  $\sigma$  in  $\Omega_{\Lambda,-}$ . Such an algorithm is described in [DKS92], Section 5.11, under the assumption that the diameter of  $\Gamma$  is larger than  $1/h^r$ , which is ensured by the fact that  $\Gamma$  is vertebrate. We will refer to this algorithm as the function  $S_{\Lambda,-,h} : \sigma \in \Omega_{\Lambda} \mapsto S$ , which we will see as a random variable on the probability space  $(\Omega_{\Lambda}, \mu_{\Lambda,-,h})$ . Differently from the notation of [SS98],  $G$  and  $S$  are not associated with external only vertebrate contours, but with all the vertebrate contours of a configuration  $\sigma$ . This will lead to some modification in the following definitions, namely in the definition of what will be denoted by  $V(G)$ .

The free energy of a skeleton family  $S = (\gamma_j : j < k)$  will be made of two parts. On the one hand the *surface free energy* of  $S$  is simply defined by

$$\mathcal{W}(S) = \sum_{j < k} \mathcal{W}(\gamma_j),$$

with  $\mathcal{W}(\gamma_j)$  defined by Equation (14). Even if  $\gamma_j$  is self-intersecting and is not the boundary of a simply connected domain, so that the *external* normal can be ill-defined, one can still define some normal with respect to an orientation of  $\gamma_j$  and, since  $\tau$  is  $(\pi/2)$ -periodical, there is no ambiguity for the resulting integral.

The volume free energy, on the other hand, is related with the phase volume of  $S$  introduced in [DKS92], Section 2.10. The *plus-components* of  $S$  are the bounded connected components of  $\mathbb{R}^2 \setminus \cup_{j < k} \gamma_j$  for which there is a continuous path that connects their interior and the unique unbounded component of  $\mathbb{R}^2 \setminus \cup_{j < k} \gamma_j$  with an odd number of crossings of  $\cup_{j < k} \gamma_j$ . The *phase volume* of  $S$  is defined as their joint volume and we denote it by  $\hat{V}(S)$ . The plus-components of  $G$  are defined in the same way we defined those of  $S$  and we call  $V(G)$  the total number of sites they enclose. We define  $\check{V}(S)$  as the number of sites in the plus-components of  $S$  that are at distance larger than  $1/h^r$  from  $\cup_{j < k} \gamma_j$ . The *volume free energy* of  $S$  is the product  $-hm_{\beta}^* \check{V}(S)$ .

Following [SS98] there is a constant  $C > 0$  such that

$$\max\left\{V(G), \hat{V}(S)\right\} - C\mathcal{W}(S)\frac{1}{h^{2r}} \leq \check{V}(S) \leq \min\left\{V(G), \hat{V}(S)\right\}$$

and

$$\left|V(G) - \hat{V}(S)\right| \leq C\mathcal{W}(S)\frac{1}{h^{2r}}.$$

We will denote by  $\{S_{\Lambda,-,h} = S\}$  the set of configurations that are associated with the skeleton family  $S$  and by

$$\mathcal{I} = \{S_{\Lambda,-,h} = \emptyset\}$$

the set of configuration with invertebrate contours only. These are similar to the configuration sets  $\mathcal{S}_S^{h,sk}$  and  $\mathcal{S}_0^{h,sk}$  in [SS98], which are associated with external contours only. Following the proof of Lemma 2.3.6 of [SS98], we have

**Lemma 2.5.** *Given  $\epsilon > 0$ , if  $h$  is small enough and  $\Lambda$  is a simply-connected domain contained in  $\Lambda_h$ , then, for any skeleton family  $S$ ,*

$$\mu_{\Lambda,-,h}(S_{\Lambda,-,h} = S) \leq \mu_{\Lambda,-,h}(\mathcal{I}) \exp \left\{ -\beta \left( (1-\epsilon)W(S) - (1+\epsilon)hm_\beta^* \check{V}(S) \right) \right\}.$$

This result relies on Pfister's low temperature estimate for zero magnetic field ([Pfi91], Lemma 10.1), that was extended in [Iof95] up to critical temperature, and which uses a duality argument that holds for simply connected domains only. This is where the simple connectivity of  $\Lambda$  matters.

To make the volume free energy appear, Schonmann and Shlosman control the derivative with respect to  $h$  of the ratio between  $\mu_{\Lambda,-,h}(S_{\Lambda,-,h} = S)$  and  $\mu_{\Lambda,-,h}(\mathcal{I})$  and they use in particular the fact that, at any subcritical temperature, there is a positive constant  $C$  such that, for all  $h \geq 0$ ,  $\Lambda \subset \mathbb{Z}^2$  and  $x, y \in \mathbb{Z}^2$ ,

$$\mu_{\Lambda,+ ,h}(x \overset{-*}{\longleftrightarrow} y) \leq \mu_h(x \overset{-*}{\longleftrightarrow} y) \leq \mu_+(x \overset{-*}{\longleftrightarrow} y) \leq \exp\{-C\|x-y\|_\infty\}, \quad (22)$$

where the star percolation event  $\{x \overset{-*}{\longleftrightarrow} y\}$  is the set of configurations  $\sigma$  in  $\Omega_{\mathbb{Z}^2}$  for which there is a sequence of sites  $x = z_0, z_1, \dots, z_k = y$  such that  $\|z_j - z_{j+1}\|_\infty = 1$  and  $\sigma(z_j) = \sigma(z_{j+1}) = -1$  for all  $j < k$ . The first two inequalities are a consequence of FKG inequality and the last one is Theorem 1 in [CCS87].

We then get upper bounds on events of type  $\{\mathcal{W}(S_{\Lambda,-,h}) \geq D/h^u, hm_\beta^* \check{V}(S_{\Lambda,-,h}) \leq E/h^v\}$  for  $u, v \geq r$ .

**Lemma 2.6.** *Given  $\epsilon > 0$ ,  $D_0 > 0$  and  $E_0 > 0$ , if  $h$  is small enough and  $\Lambda$  is a simply-connected domain contained in  $\Lambda_h$ , then, for any  $D \geq D_0$ ,  $E \geq E_0$ ,  $F \geq 0$  and  $u, v \geq r$ , it holds*

$$\mu_{\Lambda,-,h} \left( \mathcal{W}(S_{\Lambda,-,h}) \geq \frac{D}{h^u} + (1+\epsilon)F, hm_\beta^* \check{V}(S_{\Lambda,-,h}) = F \right) \leq \mu_{\Lambda,-,h}(\mathcal{I}) \exp \left\{ -\beta(1-\epsilon) \frac{D}{h^u} \right\}$$

and

$$\mu_{\Lambda,-,h} \left( \mathcal{W}(S_{\Lambda,-,h}) \geq \frac{D}{h^u}, hm_\beta^* \check{V}(S_{\Lambda,-,h}) \leq \frac{E}{h^v} \right) \leq \mu_{\Lambda,-,h}(\mathcal{I}) \exp \left\{ -\beta \left( (1-\epsilon) \frac{D}{h^u} - (1+\epsilon) \frac{E}{h^v} \right) \right\}.$$

**Proof:** This is similar to the proof of Lemma 2.3.7 in [SS98]. For any  $D \geq D_0$  and  $F \geq 0$  it holds

$$\begin{aligned} & \mu_{\Lambda,-,h} \left( \mathcal{W}(S_{\Lambda,-,h}) \geq \frac{D}{h^u} + (1+4\epsilon)F, hm_\beta^* \check{V}(S_{\Lambda,-,h}) = F \right) \\ & \leq \sum_{k \geq 0} \mu_{\Lambda,-,h} \left( \mathcal{W}(S_{\Lambda,-,h}) \in (1+4\epsilon)F + \left[ \frac{(1+k)D}{h^u}, \frac{(1+k+1)D}{h^u} \right], hm_\beta^* \check{V}(S_{\Lambda,-,h}) = F \right). \end{aligned}$$

For  $h$  small enough, the number of possible skeleton families  $S$  such that

$$\mathcal{W}(S) \leq (1+4\epsilon)F + \frac{(2+k)D}{h^u}$$

is less than (recall that  $\beta$  is a fixed parameter)

$$\left( 3 \frac{B_{\max}^2}{h^2} \right)^{\frac{(1+4\epsilon)F + (2+k)D/h^u}{\tau(0)^{1/2}} h^r} \leq \exp \left\{ \beta \epsilon \left( (1+4\epsilon)F + \frac{(2+k)D}{h^u} \right) \right\}.$$

Indeed, since

$$\tau(0) = \min_{\theta < 2\pi} \tau(\theta),$$

the second skeleton property implies that, with  $N$  the total number of vertices of a skeleton family  $S$ ,

$$\mathcal{W}(S) \geq N \frac{1}{12h^r} \tau(0),$$

which gives an upper bound on  $N$ . Together with the fact that these vertices have to be in  $\Lambda_h$ , of volume  $(B_{\max}/h)^2$  at most and that each of them can be a first, last or intermediate vertex of a given skeleton, this gives the stated upper bound.

Lemma 2.5 implies then, for any  $\epsilon < 1/8$  and  $h$  smaller than some  $h_0$  that depends on  $\epsilon$ ,  $D_0$  and  $\beta$  only,

$$\begin{aligned} & \mu_{\Lambda, -, h} \left( \mathcal{W}(S_{\Lambda, -, h}) \geq \frac{D}{h^u} + (1 + 4\epsilon)F, hm_{\beta}^* \check{V}(S_{\Lambda, -, h}) = F \right) \\ & \leq \mu_{\Lambda, -, h}(\mathcal{I}) \sum_{k \geq 0} \exp \left\{ \beta \left( \epsilon(1 + 4\epsilon)F + \epsilon \frac{(2+k)D}{h^u} - (1 - \epsilon) \left( \frac{(1+k)D}{h^u} + (1 + 4\epsilon)F \right) + (1 + \epsilon)F \right) \right\} \\ & = \mu_{\Lambda, -, h}(\mathcal{I}) \sum_{k \geq 0} \exp \left\{ -\beta \left( [(1 - 3\epsilon) + (1 - 2\epsilon)k] \frac{D}{h^u} + \epsilon(1 - 8\epsilon)F \right) \right\} \\ & \leq \mu_{\Lambda, -, h}(\mathcal{I}) C \exp \left\{ -\beta(1 - 3\epsilon) \frac{D}{h^u} \right\} \end{aligned}$$

for some constant  $C$  that depends on  $\epsilon$  and  $D_0$  only. This implies the first desired inequality with  $4\epsilon$  in place of  $\epsilon$ .

For  $\epsilon < 1/2$ , any  $D \geq D_0$ ,  $E \geq E_0$  and  $h$  small enough it holds in the same way

$$\begin{aligned} & \mu_{\Lambda, -, h} \left( \mathcal{W}(S_{\Lambda, -, h}) \geq \frac{D}{h^u}, hm_{\beta}^* \check{V}(S_{\Lambda, -, h}) \leq \frac{E}{h^v} \right) \\ & \leq \sum_{k \geq 0} \sum_{j \leq \frac{E/m_{\beta}^*}{h^{1+v}}} \mu_{\Lambda, -, h} \left( \mathcal{W}(S_{\Lambda, -, h}) \in \left[ (1+k) \frac{D}{h^u}, (1+k+1) \frac{D}{h^u} \right], \check{V}(S_{\Lambda, -, h}) = j \right) \\ & \leq \mu_{\Lambda, -, h}(\mathcal{I}) \sum_{k \geq 0} \frac{E/m_{\beta}^*}{h^{1+v}} \exp \left\{ -\beta \left( [(1 - 3\epsilon) + (1 - 2\epsilon)k] \frac{D}{h^u} - (1 + \epsilon) \frac{E}{h^v} \right) \right\} \\ & \leq \mu_{\Lambda, -, h}(\mathcal{I}) C \exp \left\{ -\beta \left( (1 - 3\epsilon) \frac{D}{h^u} - (1 + \epsilon) \frac{E}{h^v} \right) \right\} \end{aligned}$$

for some constant  $C$  that depends on  $\epsilon$ ,  $D_0$  and  $E_0$  only. The thesis follows.  $\square$

For  $\sigma$  in  $\Omega_{\Lambda, -}$  we will also consider the family  $G_{\Lambda, -, h}^{\text{ext}}(\sigma) = (\Gamma_j : j < k)$  of the *external* vertebrate contours of  $\sigma$  as well as the family  $S_{\Lambda, -, h}^{\text{ext}}(\sigma) = (\gamma_j : j < k)$  of their associated skeletons. We will denote by

$$|G_{\Lambda, -, h}^{\text{ext}}(\sigma)| = |S_{\Lambda, -, h}^{\text{ext}}(\sigma)|$$

their number  $k$ . As a first application of the previous upper bounds we have that, conditionally to  $V(G_{\Lambda, -, h}^{\text{ext}}) \leq (B_+/h)^2$  and for  $B_+$  small enough—say  $B_+ \leq 3B_c/2$  and recall that  $B_+$  as to be thought close to  $B_c$ —typical configurations drawn from  $\mu_{\Lambda, -, h}$  are made of invertebrate contours only, i.e., are in  $\mathcal{I}$ . More precisely

**Lemma 2.7.** *There is  $\delta > 0$  such that, if  $h$  is small enough and  $B \leq 3B_c/2$ , then, for all  $k \geq 0$  it holds*

$$\mu_{\Lambda, -, h} \left( |S_{\Lambda, -, h}^{\text{ext}}| = k, V(G_{\Lambda, -, h}^{\text{ext}}) \leq (B/h)^2 \right) \leq \mu_{\Lambda, -, h}(\mathcal{I}) \exp \{ -\delta k/h^b \}.$$

In particular, for  $B_+ \leq 3B_c/2$  and  $h$  small enough, it holds

$$\begin{aligned} \mu_{\Lambda, -, h}(\mathcal{I}^c \mid \mathcal{R}) & \leq \frac{\sum_{k \geq 1} \mu_{\Lambda, -, h} \left( |S_{\Lambda, -, h}^{\text{ext}}| = k, V(G_{\Lambda, -, h}^{\text{ext}}) \leq (B_+/h)^2 \right)}{\mu_{\Lambda, -, h}(\mathcal{I})} \\ & \leq \frac{\mu_{\Lambda, -, h}(\mathcal{I}) 2 \exp \{ -\delta/h^b \}}{\mu_{\Lambda, -, h}(\mathcal{I})} = 2 \exp \{ -\delta/h^b \}. \end{aligned}$$

**Proof:** We will apply the first inequality of the previous lemma with  $\epsilon = 1/8$ . To this end we will give a lower bound on

$$\mathcal{W}(S_{\Lambda, -, h}) - (1 + \epsilon)hm_{\beta}^* \check{V}(S_{\Lambda, -, h}) \geq \mathcal{W}(S_{\Lambda, -, h}^{\text{ext}}) - (1 + \epsilon)hm_{\beta}^* \check{V}(S_{\Lambda, -, h}^{\text{ext}})$$

provided that  $|S_{\Lambda, -, h}^{\text{ext}}| = k$  and  $V(G_{\Lambda_h, -, h}^{\text{ext}}) \leq (B/h)^2$ . If

$$G_{\Lambda_h, -, h}^{\text{ext}} = (\Gamma_j : j < k) \quad \text{and} \quad S_{\Lambda_h, -, h}^{\text{ext}} = (\gamma_j : j < k),$$

we also have

$$\mathcal{W}(S_{\Lambda_h, -, h}^{\text{ext}}) - (1 + \epsilon)hm_\beta^* \check{V}(S_{\Lambda_h, -, h}^{\text{ext}}) \geq \sum_{j < k} \mathcal{W}(\gamma_j) - (1 + \epsilon)hm_\beta^* \check{V}(\gamma_j)$$

with  $-hm_\beta^* \check{V}(\gamma_j)$  the volume free energy of the single skeleton  $\gamma_j$ . To give a lower bound on each term of this sum, we recall that there is  $C > 0$  such that, with  $V(\Gamma_j)$  the number of sites enclosed in  $\Gamma_j$ , it holds

$$V(\Gamma_j) - C\mathcal{W}(\gamma_j)/h^{2r} \leq \check{V}(\gamma_j) \leq V(\Gamma_j)$$

and we separate two cases.

If

$$C\mathcal{W}(\gamma_j)/h^{2r} \geq V(\Gamma_j)/2,$$

then, since  $V(\Gamma_j) \geq 1/h^{2b}$ ,

$$\mathcal{W}(\gamma_j) - (1 + \epsilon)hm_\beta^* \check{V}(\gamma_j) \geq \left[ \frac{h^{2r}}{2C} - (1 + \epsilon)hm_\beta^* \right] V(\Gamma_j) \geq \frac{2\delta}{\beta h^b}$$

for  $h$  small enough and some positive  $\delta$  that depends only on  $\epsilon$ ,  $C$  and  $\beta$ . If instead

$$C\mathcal{W}(\gamma_j)/h^{2r} \leq V(\Gamma_j)/2,$$

then we have on the one hand

$$\frac{1}{2h^{2b}} \leq \frac{1}{2}V(\Gamma_j) \leq \check{V}(\gamma_j) \leq V(\Gamma_j) \leq \left( \frac{3B_c}{2h} \right)^2, \quad (23)$$

and on the other hand, using the isoperimetric property of the Wulff shape,

$$\mathcal{W}(\gamma_j) - (1 + \epsilon)hm_\beta^* \check{V}(\gamma_j) \geq w_\beta \sqrt{\check{V}(\gamma_j)} - (1 + \epsilon)hm_\beta^* \check{V}(\gamma_j).$$

This lower bound is concave in  $\check{V}(\gamma_j)$ . From (23) we need then to evaluate it in  $1/(2h^{2b})$  and  $(3B_c)^2/(2h)^2$  to find its minimum value. Since, for some  $\delta' \leq \delta$  and  $h$  small enough it holds

$$\frac{w_\beta}{\sqrt{2}h^b} - (1 + \epsilon)\frac{m_\beta^*}{2}h^{1-2b} \geq \frac{2\delta'}{\beta h^b}$$

and

$$w_\beta \frac{3B_c/2}{h} - (1 + \epsilon)m_\beta^* \frac{(3B_c/2)^2}{h} \geq \frac{2\delta'}{\beta h^b},$$

this leads to

$$\mathcal{W}(S_{\Lambda_h, -, h}) - (1 + \epsilon)hm_\beta^* \check{V}(S_{\Lambda_h, -, h}) \geq \frac{2k\delta'}{\beta h^b}.$$

We then get the desired estimate by applying Lemma 2.6 and summing on all the possible values of the integer

$$\check{V}(S_{\Lambda, -, h}) = F/(hm_\beta^*) < 2 \left( \frac{B_{\max}}{h} \right)^2.$$

□

We will also need lower bounds based on [Iof94]. For  $B > 0$  and  $\delta > 0$ , let us denote by  $E_{B, \delta}^h$  the event that there is an external contour which surrounds  $(1 - \delta)BW/h$  and is contained in  $(1 + \delta)BW/h$ , and that moreover this is the only external vertebrate contour. With this notation and recalling Equation (5) from page 4, Lemma 3.4.3 in [SS98] gives

**Lemma 2.8.** *There are  $C > 0$  and  $o_h(1)$ , a vanishing function of  $h$  when  $h$  goes to zero, such that, for all  $B > 0$ ,  $\delta > 0$  and all simply-connected  $\Lambda \subset \Lambda_h$  that contains  $(1 + \delta)BW/h$ , it holds*

$$\mu_{\Lambda, -, h}(E_{B, \delta}^h) \geq \mu_{\Lambda, -, h}(\mathcal{I})C \exp \left\{ -\beta(1 + o_h(1)) \frac{\phi(B)}{h} \right\}.$$

This makes possible to give lower bounds on similar events for non simply-connected “Wulff shaped annular domains” of the form

$$A(B_1, B_2) = \left( \frac{B_2}{h}W \setminus \frac{B_1}{h}W \right) \cap \mathbb{Z}^2$$

with  $0 \leq B_1 < B_2$ . (In the case  $B_1 = 0$  this “annular domain” is simply a Wulff shaped box.) For  $\eta$ ,  $\eta_1$  and  $\eta_2$  in  $\Omega_{\mathbb{Z}^2}$  such that  $\eta$  coincides with  $\eta_1$  in  $B_1W/h$  and with  $\eta_2$  outside  $B_2W/h$ , we will write  $\mu_{A,(\eta_1, \eta_2), h}$  for  $\mu_{A, \eta, h}$  with  $A = A(B_1, B_2)$ . Given  $\delta > 0$  we also define  $B_{1, \delta} > B_1$  and  $B_{2, \delta} < B_2$  by the equations

$$(1 - \delta)B_{1, \delta} = B_1 \quad \text{and} \quad (1 + \delta)B_{2, \delta} = B_2$$

and we call  $\tilde{E}_{B, \delta}^h$  the subset of  $E_{B, \delta}^h$  for which there is no vertebrate contour distinct from the external contour which surrounds  $(1 - \delta)BW/h$  and is contained in  $(1 + \delta)BW/h$ .

**Lemma 2.9.** *Given  $\epsilon > 0$ , if  $h$  is small enough, then, for all  $0 \leq B_1 < B_2 \leq B_{\max}$  and  $\delta$  such that  $B_{1, \delta} < B_{2, \delta}$ , it holds, with  $A = A(B_1, B_2)$ ,*

$$\mu_{A, (+, -), h} \left( \tilde{E}_{B_{2, \delta}, \delta}^h \right) \geq \exp \left\{ -\frac{\beta}{h} \left( \epsilon + [\phi(B_2) - \phi(B_1)]_+ \right) \right\}, \quad (24)$$

$$\mu_{A, (+, -), h} \left( E_{B_{1, \delta}, \delta}^h \right) \geq \exp \left\{ -\frac{\beta}{h} \left( \epsilon + [\phi(B_1) - \phi(B_2)]_+ \right) \right\}, \quad (25)$$

$$\mu_{A, (-, +), h} \left( E_{B_{1, \delta}, \delta}^h \right) \geq \exp \left\{ -\frac{\beta}{h} \epsilon \right\} \quad (26)$$

and, if  $B_2 - B_1 < 2B_c$ ,

$$\mu_{A, (-, -), h}(\mathcal{I}) \geq \exp \left\{ -\frac{\beta}{h} \epsilon \right\}. \quad (27)$$

**Proof:** Most of this is already contained in Lemma 3.5.1 of [SS98], which gives stronger lower bounds on similar events, and its proof, which works by conditioning and stochastic domination. We will proceed in the same way. Let us first prove (24). Our event  $\tilde{E}_{B_{2, \delta}, \delta}^h$  is the intersection of the events

$E_0$ : there is a contour  $\Gamma$  that separates interior plus spins from exterior minus spins, that surrounds  $(1 - \delta)B_{2, \delta}W/h$  and that is contained in  $B_2W/h$ ,

$E_1$ : such a contour  $\Gamma$  does not enclose any vertebrate contour

and

$E_2$ : there is no vertebrate contour outside such a contour  $\Gamma$ ,

the first two of which are increasing events. With

$$\Lambda_2 = \frac{B_2}{h}W \cap \mathbb{Z}^2,$$

DLR equations imply

$$\begin{aligned} \mu_{A, (+, -), h}(E_0 \cap E_1 \cap E_2) &= \mu_{A, (+, -), h}(E_0 \cap E_1) \times \mu_{A, (+, -), h}(E_2 \mid E_0 \cap E_1) \\ &= \mu_{A, (+, -), h}(E_0 \cap E_1) \times \mu_{\Lambda_2, -, h}(E_2 \mid E_0 \cap E_1) \end{aligned}$$

and we will use stochastic domination for giving a lower bound of the first factor. Let us denote by  $\tilde{\Lambda}_1$  the set of sites in

$$\Lambda_1 = \frac{B_1}{h}W \cap \mathbb{Z}^2$$

that are at distance  $2/h^{2b}$  from its boundary, and by  $F$  the event that there is a contour  $\Gamma'$  which separates interior plus spins from exterior minus spins, surrounds  $\tilde{\Lambda}_1$  and does not enclose any vertebrate contour that encloses some site in  $\tilde{\Lambda}_1$ . By conditioning on the invertebrate contours enclosed in  $\Gamma'$  and enclosing some site in  $\tilde{\Lambda}_1$ , FKG inequality gives

$$\mu_{A, (-, +), h}(E_0 \cap E_1) \geq \mu_{\Lambda_2, -, h}(E_0 \cap E_1 \mid F) = \frac{\mu_{\Lambda_2, -, h}(E_0 \cap E_1)}{\mu_{\Lambda_2, -, h}(F)}.$$

Together with the previous equality we then have

$$\mu_{A,(+,-),h}(E_0 \cap E_1 \cap E_2) \geq \frac{\mu_{\Lambda_2,-,h}(E_0 \cap E_1 \cap E_2)}{\mu_{\Lambda_2,-,h}(F)}.$$

To get a lower bound on the numerator we use Lemma 2.8 and Estimate (22) from page 14. We observe that  $E_0 \cap E_2 = E_{B_2,\delta,\delta}^h$  and that, conditionally to  $E_{B_2,\delta,\delta}^h$ , a star percolation event involving some sites  $x$  and  $y$  at distance of order  $1/h^b$  has to occur if  $E_1$  does not. Since  $\phi$  is bounded from above, we obtain a constant  $C > 0$  such that for  $h$  and  $\delta$  small enough,

$$\mu_{\Lambda_2,-,h}(E_0 \cap E_1 \cap E_2) \geq \mu_{\Lambda_2,-,h}(\mathcal{I})C \exp \left\{ -\frac{\beta}{h}(\phi(B_2) + \epsilon/2) \right\}.$$

To get an upper bound on the denominator we observe that  $F$  implies, for  $h$  small enough, that  $\check{V}(S_{\Lambda_2,-,h})$  lies between  $(1-\epsilon)(B_1/h)^2$  and  $(B_2/h)^2$  so that the minimal free energy cost is of order  $(\phi(B_1) \wedge \phi(B_2))/h$ . Using Lemma 2.6, we get, for  $h$  small enough

$$\mu_{\Lambda_2,-,h}(F) \leq \mu_{\Lambda_2,-,h}(\mathcal{I}) \exp \left\{ -\frac{\beta}{h}(\phi(B_2) \wedge \phi(B_1) - \epsilon/2) \right\}.$$

This gives the desired estimate.

Inequality (25) is proved in the same way: it holds with  $\tilde{E}_{B_1,\delta,\delta}^h$  in place of  $E_{B_1,\delta,\delta}^h$ , but we will only need an estimate for this larger event. Inequality (26) is then a consequence of (25): the boundary conditions are exchanged and the positive magnetic helps in such a way that there is no size-dependent free energy cost anymore. We refer to the last page of [SS98] for more details.

We finally prove (27). This is the only place where we will make use of the notion of free energy associated with non-external vertebrate contours. Let us now denote by  $E$  the event that there is no vertebrate contour in  $A$  and by  $F$  the event that there is a contour  $\Gamma$  which separates external minus spins from internal plus spins, is enclosed in  $\Lambda_1$  and encloses  $(1-\delta)\Lambda_1/(1+\delta)$ . Since  $\mathcal{I}$  is a decreasing event it holds

$$\mu_{A,(-,-),h}(\mathcal{I}) \geq \mu_{\Lambda_2,-,h}(E | F) = \frac{\mu_{\Lambda_2,-,h}(E_{B_1/(1+\delta),\delta}^h)}{\mu_{\Lambda_2,-,h}(F)}$$

and, using Lemma 2.8, we only need to prove that, for  $h$  and  $\delta$  small enough,

$$\mu_{\Lambda_2,-,h}(F) \leq \mu_{\Lambda_2,-,h}(\mathcal{I}) \exp \left\{ -\frac{\beta}{h}(\phi(B_1) - \epsilon/2) \right\}.$$

In other words we need to show that the free energy of the skeleton families that are compatible with  $F$  cannot macroscopically decrease with respect to that of the skeleton families that are compatible with  $E_{B_1,\delta}^h$ . Like in the proof of Lemma 2.7 we can estimate from below the free energy of the former by the sum of the free energy of the single skeleton associated with the contour  $\Gamma$ , and that of the skeleton family associated with each plus-component outside  $\Gamma$ . Since the former is of order  $\phi(B_1)/h$ , it is sufficient to check that the latter can only have a positive contribution provided that  $B_2 - B_1 < 2B_c$ . Let us denote by  $\mathcal{W}(S)$ ,  $-hm_\beta^* \check{V}(S)$  and  $\check{V}(S) \geq \check{V}(S)$  the surface free energy, the volume free energy and the phase volume of such a skeleton family associated with a single plus-component of the whole contour family. If this single plus-component is simply connected, then, by using Lemma 2.3 and Proposition 2.1, the associated free energy has a lower bound of order

$$\frac{w_\beta}{2} \left( \frac{\check{V}(S)}{(B_2 - B_1)/(2h)} + \frac{B_2 - B_1}{2h} \right) - hm_\beta^* \check{V}(S) \geq h \check{V}(S) \left( \frac{w_\beta}{B_2 - B_1} - m_\beta^* \right).$$

If it is not simply connected but does not enclose  $\Gamma$ , we get a similar lower bound on its associated free energy by estimating it from below with that of the single skeleton associated with its outermost contour. If instead it is not simply connected and it encloses  $\Gamma$ , then, denoting by  $(B/h)^2$  the number of sites enclosed in its outermost contour and taking into account the surface free energy contribution of its innermost contour, the total free energy of this skeleton family has a lower bound of order

$$w_\beta \frac{B}{h} + w_\beta \frac{B_1}{h} - hm_\beta^* \left( \left( \frac{B}{h} \right)^2 - \left( \frac{B_1}{h} \right)^2 \right) \geq \frac{B + B_1}{h} (w_\beta - m_\beta^*(B_2 - B_1)).$$

Provided that

$$B_2 - B_1 < 2B_c = \frac{w_\beta}{m_\beta^*},$$

this gives in all cases a non-negative macroscopic contribution.  $\square$

## 2.4 Exit rates, local relaxation times and soft capacities

We will simply denote by  $X$  the dynamics  $X_{\Lambda_h, -, h}$  restricted to

$$\mathcal{X} = \mathcal{R} \cup \mathcal{S},$$

which is associated with the generator  $\mathcal{L}$  defined by

$$(\mathcal{L}f)(\sigma) = \sum_{\substack{x \in \Lambda_h: \\ \sigma^x \in \mathcal{X}}} w(\sigma, \sigma^x) [f(\sigma^x) - f(\sigma)], \quad \sigma \in \mathcal{X}, \quad f : \mathcal{X} \rightarrow \mathbb{R}.$$

We will also denote by  $\mu$  its reversible measure

$$\mu = \mu_{\Lambda_h, -, h}(\cdot | \mathcal{X}).$$

and by  $\mathcal{D}$  the associated Dirichlet form defined by Equation (20) of Section 2.2. Its spectral gap will be denoted  $\gamma = \gamma_h$ . In this section we briefly recall some definitions from [BGM18] and explain how to use the results of that paper to prove an equivalent of Theorem 1 and Proposition 1.3 for this restricted dynamics  $X$ .

We denote by  $\mathcal{L}_{\mathcal{R}}$  the generator of the dynamics  $X$  restricted to  $\mathcal{R}$ :

$$(\mathcal{L}_{\mathcal{R}}f)(\sigma) = \sum_{\substack{x \in \Lambda_h: \\ \sigma^x \in \mathcal{R}}} w(\sigma, \sigma^x) [f(\sigma^x) - f(\sigma)], \quad \sigma \in \mathcal{R}, \quad f : \mathcal{R} \rightarrow \mathbb{R},$$

and we will denote by  $1/\gamma_{\mathcal{R}}$  the relaxation time of this restricted dynamics. We denote by  $\mu_{\mathcal{R}}$  the *restricted ensemble*

$$\mu_{\mathcal{R}} = \mu(\cdot | \mathcal{R}),$$

with respect to which  $\mathcal{L}_{\mathcal{R}}$  is reversible, and we set

$$\chi_{\mathcal{R}} = \max_{\sigma \in \mathcal{R}} \frac{1}{\mu_{\mathcal{R}}(\sigma)}.$$

We define in the same way  $\mathcal{L}_{\mathcal{S}}$ ,  $1/\gamma_{\mathcal{S}}$ ,  $\mu_{\mathcal{S}}$  and  $\chi_{\mathcal{S}}$ . We will refer to  $1/\gamma_{\mathcal{R}}$  and  $1/\gamma_{\mathcal{S}}$  as *local relaxation times*.

For any  $\lambda \geq 0$  we denote by  $\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*$  the extinction rate from quasi-stationarity of the trace on  $\mathcal{R}$  of our process  $X$  killed at rate  $\lambda$  in  $\mathcal{S}$ , and we set

$$\phi_{\mathcal{R} \setminus \mathcal{S}}^* = \lim_{\lambda \rightarrow \infty} \phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^*.$$

The precise meaning of each of these terms is explained in Section 2.1 of [BGM18], from which we will mainly need the upper bound of Lemma 2.3

$$\phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^* \leq \phi_{\mathcal{R} \setminus \mathcal{S}}^* \leq \mu_{\mathcal{R} \setminus \mathcal{S}}(e_{\mathcal{R} \setminus \mathcal{S}}^*), \quad (28)$$

with  $\mu_{\mathcal{R} \setminus \mathcal{S}} = \mu(\cdot | \mathcal{R} \setminus \mathcal{S})$  and

$$e_{\mathcal{R} \setminus \mathcal{S}}^*(\sigma) = \sum_{\substack{x \in \Lambda_h: \\ \sigma^x \in \mathcal{S}}} w(\sigma, \sigma^x), \quad \sigma \in \mathcal{R} \setminus \mathcal{S}.$$

For any  $\kappa \geq 0$  we define in the same way  $\phi_{\mathcal{S}, \kappa_{\mathcal{R}}}^*$ , then  $\phi_{\mathcal{S} \setminus \mathcal{R}}^*$ ,  $\mu_{\mathcal{S} \setminus \mathcal{R}}$  and  $e_{\mathcal{S} \setminus \mathcal{R}}^*$ . It also holds

$$\phi_{\mathcal{S}, \kappa_{\mathcal{R}}}^* \leq \phi_{\mathcal{S} \setminus \mathcal{R}}^* \leq \mu_{\mathcal{S} \setminus \mathcal{R}}(e_{\mathcal{S} \setminus \mathcal{R}}^*). \quad (29)$$

We will refer to  $\phi_{\mathcal{R} \setminus \mathcal{S}}^*$  and  $\phi_{\mathcal{S} \setminus \mathcal{R}}^*$  as *exit rates* from  $\mathcal{R} \setminus \mathcal{S}$  and  $\mathcal{S} \setminus \mathcal{R}$ .

From Section 2.3 in [BGM18], Dirichlet's and Thomson's principle, the  $(\kappa, \lambda)$ -capacity  $C_\kappa^\lambda(\mathcal{R}, \mathcal{S})$  is the *soft capacity*

$$C_\kappa^\lambda(\mathcal{R}, \mathcal{S}) = \min_{f: \mathcal{X} \rightarrow \mathbb{R}} \left\{ \mathcal{D}(f) + \kappa \sum_{\sigma \in \mathcal{R}} \mu(\sigma) (f(\sigma) - 1)^2 + \lambda \sum_{\sigma \in \mathcal{S}} \mu(\sigma) (f(\sigma) - 0)^2 \right\} \quad (30)$$

$$= \max_{\tilde{\psi} \in \tilde{\Psi}_1(\bar{\mathcal{R}}, \check{\mathcal{S}})} \tilde{\mathcal{D}}(\tilde{\psi})^{-1} \quad (31)$$

where

$$\tilde{\mathcal{D}}(\tilde{\psi}) = \frac{1}{2} \sum_{\sigma \in \mathcal{X}} \sum_{x \in \Lambda} \frac{\tilde{\psi}(\sigma, \sigma^x)^2}{\mu(\sigma) w(\sigma, \sigma^x)} + \sum_{\sigma \in \mathcal{R}} \frac{\tilde{\psi}(\sigma, \bar{\sigma})^2}{\mu(\sigma) \kappa} + \sum_{\sigma \in \mathcal{S}} \frac{\tilde{\psi}(\sigma, \check{\sigma})^2}{\mu(\sigma) \lambda}$$

stands for the energy dissipated by a flow  $\tilde{\psi}$  in the set  $\tilde{\Psi}_1(\bar{\mathcal{R}}, \check{\mathcal{S}})$  of all the unitary flows from  $\bar{\mathcal{R}}$  to  $\check{\mathcal{S}}$  associated with a Markov process  $\tilde{\mathcal{X}}$  on the extended

$$\tilde{\mathcal{X}} = \mathcal{X} \cup \bar{\mathcal{R}} \cup \check{\mathcal{S}}$$

that jumps from any  $\sigma$  in  $\mathcal{R}$  or  $\mathcal{S}$  to  $\bar{\sigma}$  in  $\bar{\mathcal{R}}$  or  $\check{\sigma}$  in  $\check{\mathcal{S}}$  at rate  $\kappa$  or  $\lambda$ .

We will prove in sections 3 and 4 that the following hypothesis ( $\mathcal{H}$ ) is in force:

**Hypothesis ( $\mathcal{H}$ ):** *Given a small enough  $\delta > 0$ , one can choose  $B_+$  close enough to  $B_c$  so that, for all  $h$  small enough, it holds*

$$\frac{1}{\gamma_{\mathcal{R}}} \vee \frac{1}{\gamma_{\mathcal{S}}} \leq \exp \left\{ \frac{\delta}{h} \right\},$$

$$\frac{1}{\phi_{\mathcal{S} \setminus \mathcal{R}}^*} \wedge \frac{1}{\phi_{\mathcal{R} \setminus \mathcal{S}}^*} \geq \exp \left\{ \frac{\beta A - \delta}{h} \right\}$$

and, with  $\kappa = \kappa(h)$  and  $\lambda = \lambda(h)$  such that

$$\lim_{h \rightarrow 0} \kappa(h) e^{\delta/h} = \lim_{h \rightarrow 0} \frac{e^{-(\beta A - \delta)/h}}{\kappa(h)} = \lim_{h \rightarrow 0} \lambda(h) e^{\delta/h} = \lim_{h \rightarrow 0} \frac{e^{-(\beta A - \delta)/h}}{\lambda(h)} = 0,$$

for all  $\epsilon > 0$  and  $h$  small enough

$$\exp \left\{ -\frac{\beta A + \epsilon}{h} \right\} \leq \frac{C_\kappa^\lambda(\mathcal{R}, \mathcal{S})}{\mu(\mathcal{R})} \leq \exp \left\{ -\frac{\beta A - \epsilon}{h} \right\}.$$

This will imply an equivalent of Theorem 1 together with Proposition 1.3 and Estimate (13) for the restricted process  $X$ .

**Lemma 2.10.** *If hypothesis ( $\mathcal{H}$ ) is in force, then, for all small enough  $\delta_0 > 0$ , one can choose  $B_+$  close enough to  $B_c$  such that with  $\kappa = \lambda = e^{-\delta_0/(2h)}$  there is  $h_0 > 0$  for which the following holds for  $X$  started from a probability measure  $\nu$  and any observable  $f: \Omega_{\Lambda_h} \rightarrow \mathbb{R}$ .*

i. *If  $\nu = \mu_{\mathcal{R}}$ , then, for all  $t > 0$ ,*

$$\lim_{h \rightarrow 0} \mathbb{P}_\nu(\gamma T_{\lambda_S} > t) = e^{-t}, \quad (32)$$

and it holds

$$\lim_{h \rightarrow 0} h \ln \frac{1}{\gamma} = \beta A. \quad (33)$$

Also,

$$\lim_{h \rightarrow 0} \mathbb{P}_\nu \left( \theta < T_{\lambda_S} \text{ and } \sup_{t < T_{\lambda_S} - \theta} |A_\theta(t, f) - \mu_{\mathcal{R}}(f)| \leq \|f\|_\infty e^{-\delta_0/(11h)} \right) = 1 \quad (34)$$

with

$$\theta = \exp \left\{ \frac{\beta A}{2h} + \frac{\delta_0}{h} \right\}. \quad (35)$$

ii. *For  $h < h_0$  and whatever the starting measure  $\nu$ , it holds*

$$\left| E_\nu \left[ f(X(T_{\lambda_S})) \right] - \mu(f) \right| \leq \|f\|_\infty e^{-\delta_0/(6h)}.$$

iii. If  $\nu$  is such that

$$\lim_{h \rightarrow 0} \mathbb{P}_\nu(T_{\kappa\mathcal{R}} < T_{\lambda\mathcal{S}}) = 1, \quad (36)$$

then (32)–(35) are also in force. Also if, for  $h$  smaller than some positive  $h_1$ ,

$$\mathbb{P}_\nu(T_{\kappa\mathcal{R}} > T_{\lambda\mathcal{S}}) \leq e^{-\delta_0/h} \quad (37)$$

then

$$|\mathbb{E}_\nu[f(X(t))] - \mu_{\mathcal{R}}(f)| \leq \|f\|_\infty e^{-\delta_0/(6h)} \quad (38)$$

for all small enough  $h$  and all  $t = e^{\beta a/h}$  with  $\delta_0 < \beta a < \beta A - \delta_0$ .

**Proof:** Let  $\delta_0 > 0$  be small enough to have

$$\frac{\beta A}{8} - \frac{\delta_0}{16} > \frac{\delta_0}{9} \quad \text{and} \quad \phi\left(\frac{B_{\max}}{(1 + \delta_0)}\right) < -\delta_0, \quad (39)$$

and choose  $B_+$  as provided by hypothesis ( $\mathcal{H}$ ) with  $\delta_0/4$  in place of  $\delta$ . We use the results of [BGM18], which are based on two hypothesis sets —denoted there by ( $H$ ) and ( $H'$ )— both satisfied with this choice of  $\mathcal{R}$  and  $\mathcal{S}$  associated with  $B_+$ . Indeed, hypotheses ( $H$ ) require

- a)  $\phi_{\mathcal{R}\setminus\mathcal{S}}^*$  to be small with respect to  $\gamma_{\mathcal{R}}$  and  $\gamma_{\mathcal{S}}$  in our considered asymptotic regime  $h \ll 1$ ;
- b)  $\phi_{\mathcal{S}\setminus\mathcal{R}}^*$  to be small with respect to  $\gamma_{\mathcal{S}}$ ;
- c)  $X_{\mathcal{R}}, X_{\mathcal{S}}, X_{\mathcal{R}\setminus\mathcal{S}}$  and  $X_{\mathcal{S}\setminus\mathcal{R}}$  to be all irreducible;
- d)  $\mu(\mathcal{S}) \geq \mu(\mathcal{R})$ ;

( $\mathcal{H}$ ) gives a quantitative of version of a) and b);  $X_{\mathcal{R}}$  and  $X_{\mathcal{R}\setminus\mathcal{S}}$  (as well as, symmetrically,  $X_{\mathcal{S}}$  and  $X_{\mathcal{S}\setminus\mathcal{R}}$ ) are irreducible since, by flipping each plus spin, one gets a path in  $\mathcal{R}$  or  $\mathcal{R}\setminus\mathcal{S}$  from any configuration  $\sigma$  to the uniform minus configuration; and, as a consequence of Lemma 2.7 and Lemma 2.8 with

$$B = \frac{B_{\max}}{(1 + \delta_0)}$$

we have, for all small enough  $h$ ,

$$\frac{\mu(\mathcal{R})}{\mu(\mathcal{S})} \leq \frac{2\mu(\mathcal{I})}{\mu(E_{B,\delta_0}^h)} \leq \exp\left\{-\frac{\beta}{2}\left|\phi\left(\frac{B_{\max}}{(1 + \delta_0)}\right)\right|\right\} \leq \exp\left\{-\frac{\delta_0}{2h}\right\}. \quad (40)$$

Hypotheses ( $H'$ ) require in addition  $\phi_{\mathcal{R}\setminus\mathcal{S}}^*$  to be small with respect to  $\gamma_{\mathcal{R}}/\ln \chi_{\mathcal{R}}$  and  $\phi_{\mathcal{S}\setminus\mathcal{R}}^*$  to be small with respect to  $\gamma_{\mathcal{S}}/\ln \chi_{\mathcal{S}}$ , which is also implied by ( $\mathcal{H}$ ) since there is a positive constant  $C$  such that

$$\ln \chi_{\mathcal{R}} \vee \ln \chi_{\mathcal{S}} \leq C \left(\frac{B_{\max}}{h}\right)^2.$$

These hypothesis sets being satisfied, setting  $\kappa = \lambda = e^{-\delta_0/(2h)}$ ,  $\kappa$  is large with respect to  $\phi_{\mathcal{R}\setminus\mathcal{S}}^*$  and small with respect to  $\gamma_{\mathcal{R}}/\ln \chi_{\mathcal{R}}$ , just as  $\lambda$  is large with respect to  $\phi_{\mathcal{S}\setminus\mathcal{R}}^*$  and  $\phi_{\mathcal{R}\setminus\mathcal{S}}^*$  and small with respect to  $\gamma_{\mathcal{S}}/\ln \chi_{\mathcal{S}}$ .

Proposition 2.8 of [BGM18], with  $\lambda$  and  $\mathcal{S}$  in place of  $\kappa$  and  $\mathcal{R}$ , gives then, whatever the starting distribution  $\nu$ , that the total variation distance between  $\mu_{\mathcal{S}}$  and the law of  $X(T_{\lambda\mathcal{S}})$  is smaller than  $e^{-\delta_0/(5h)}$  for  $h$  small enough. Since, as a consequence of (40), so is that between  $\mu$  and  $\mu_{\mathcal{S}}$ , this gives ii.

Equations (15) and (16) and Proposition 2.8 of [BGM18] also give that  $\phi_{\mathcal{R},\lambda\mathcal{S}}^* T_{\lambda\mathcal{S}}$  converges in law to an exponential random variable of parameter 1 as soon as (36) is ensured. Since (40) implies that  $\mu(\mathcal{S})$  goes to one when  $h$  goes to zero, Theorem 1 of [BGM18] says that the ratios  $\phi_{\mathcal{R},\lambda\mathcal{S}}^*/\gamma$  and  $\phi_{\mathcal{R},\lambda\mathcal{S}}^* \mu(\mathcal{R})/C_\kappa^\lambda(\mathcal{R}, \mathcal{S})$  go to one when  $h$  goes to zero. Then, provided (36),  $\gamma T_{\lambda\mathcal{S}}$  converges in law to an exponential random variable of parameter 1 —this is (32)— and (33) is implied by ( $\mathcal{H}$ ).

As far as the case  $\nu = \mu_{\mathcal{R}}$  is concerned, we simply have to prove that (36) is in force to prove (32). With

$$V_\kappa^\lambda(x) = \mathbb{P}_x(T_{\kappa\mathcal{R}} < T_{\lambda\mathcal{S}}), \quad x \in \mathcal{X},$$

we have

$$\mathbb{P}_{\mu_{\mathcal{R}}}(T_{\kappa_{\mathcal{R}}} < T_{\lambda_{\mathcal{S}}}) = E_{\mu_{\mathcal{R}}}[V_{\kappa}^{\lambda} | \mathcal{R}]$$

and Lemma 3.2 in [BGM18] says that that the latter goes to one when  $h$  goes to zero. Conditions (39) also imply that we can choose  $\eta = e^{-\delta_0/(9h)}$  in Proposition 5.1 of [BGM18] which, together with

$$\lim_{h \rightarrow 0} h \ln \phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^* = \beta A,$$

gives, for some

$$\tilde{\theta} \leq \exp \left\{ \frac{\beta A}{2h} + \frac{3\delta_0}{4h} \right\}$$

and  $h$  small enough,

$$\mathbb{P}_{\mu_{\mathcal{R}}} \left( \tilde{\theta} < T_{\lambda_{\mathcal{S}}} \text{ and } \sup_{t < T_{\lambda_{\mathcal{S}}} - \tilde{\theta}} |A_{\tilde{\theta}}(t, f) - \mu_{\mathcal{R}}(f)| \leq \|f\|_{\infty} e^{-\delta_0/(10h)} \right) \geq 1 - e^{-\delta_0/(10h)}. \quad (41)$$

Since we already now that, starting from  $\mu_{\mathcal{R}}, \phi_{\mathcal{R}, \lambda_{\mathcal{S}}}^* T_{\lambda_{\mathcal{S}}}$  converges in law towards an exponential random variable of parameter one, this implies (34)–(35).

Next, Theorem 3 of [BGM18] says that there is a stopping time  $T^*$ , with

$$\mathbb{E}_{\nu}[T^*] \leq 2e^{\delta_0/(2h)},$$

such that the total variation distance between  $\mu_{\mathcal{R}}$  and the law of  $X(T^*)$  goes to zero as well as the probability that  $T_{\lambda_{\mathcal{S}}} < T^*$  when (36) is in force. The contribution to time averages on time scale  $\theta$  of the trajectories of  $X$  before time  $T^*$  is then negligible and we get, from (41), that (36) implies (34)–(35).

It only remains to prove (38) by assuming (37) for  $h$  small enough. We use to this end optimal couplings associated with total variation estimates provided by [BGM18] to bound the total variation distance between the law of  $X(t)$  and  $\mu_{\mathcal{R}}$ . First, by Markov inequality,

$$\mathbb{P}_{\nu}(T^* \geq t) \leq \frac{2e^{\delta_0/(2h)}}{e^{\beta a/h}} \leq 2e^{-\delta_0/(2h)}, \quad (42)$$

and, assuming  $T^* < t$ , we consider four coupled process  $X_0, X_1, X_2$  and  $X_3$  on the time interval  $[T^*, t]$  with the following marginals:  $X_0(s) = X(s)$  for all  $s \in [T^*, t]$ ;  $X_1(T^*)$  is distributed according to  $\mu_{\mathcal{R}}$  and  $X_1$  evolves according to the restricted dynamics in  $\mathcal{X}$  with generator  $\mathcal{L}$ ;  $X_2$  evolves according to the same dynamics in  $\mathcal{X}$ , but  $X_2(T^*)$  is distributed according to the quasi-stationary distribution  $\mu_{\mathcal{R} \setminus \mathcal{S}}^*$  introduced in Section 2.1 of [BGM18] and for which, with  $T_{\mathcal{S}}$  the hitting time of  $\mathcal{S}$ ,

$$\mathbb{P}_{\mu_{\mathcal{R} \setminus \mathcal{S}}^*}(T_{\mathcal{S}} > s) = e^{-\phi_{\mathcal{R} \setminus \mathcal{S}}^* s} \quad (43)$$

for all  $s \geq 0$ ;  $X_3(T^*) = X_1(T^*)$ , but  $X_3$  evolves according to the restricted dynamics in  $\mathcal{R}$ , so that the law of  $X_3(t)$  is  $\mu_{\mathcal{R}}$ . Then, we simply have to couple these processes in such a way that  $X_0(t) = X_3(t)$  with large probability. Since  $X_1(T^*) = X_3(T^*)$ , it suffices to this end to couple  $X_0(T^*), X_1(T^*)$  and  $X_2(T^*)$  to make them coincide with large probability and use (43) to prove that they will not exit  $\mathcal{R}$  with large probability. Indeed, conditionally to  $\{T^* < t\}$ ,

$$\mathbb{P}(\exists s < t, X_2(s) \notin \mathcal{R}) \leq \mathbb{P}(\exists s < t, X_2(s) \in \mathcal{S}) \leq 1 - e^{-\phi_{\mathcal{R} \setminus \mathcal{S}}^* t} \leq \phi_{\mathcal{R} \setminus \mathcal{S}}^* t \leq e^{-3\delta_0/(4h)}.$$

Conditionally to  $\{T^* < t\}$  and Hypothesis  $(\mathcal{H})$ , Proposition 2.6 in [BGM18] says that, for  $h$  small enough, we can couple  $X_2(T^*)$  and  $X_1(T^*)$  in such a way that

$$\mathbb{P}(X_2(T^*) \neq X_1(T^*)) \leq \exp \left\{ -\frac{1}{h} \left( \frac{\beta A}{2} - \frac{\delta_0}{2} \right) \right\}.$$

From Theorem 3 in [BGM18] and (42) we can couple  $X_1(T^*)$  and  $X_0(T^*)$  in such a way that, for  $h$  small enough,

$$\mathbb{P}(X_1(T^*) \neq X_0(T^*)) \leq \mathbb{P}(T^* \geq t) + \mathbb{P}(T^* \neq T_{\mathcal{R}}^*) + e^{-\delta_0/(5h)} \leq 2e^{-\delta/(2h)} + 3e^{-\delta_0/h} + e^{-\delta_0/(5h)}.$$

With such couplings we get

$$\mathbb{P}(X_3(t) \neq X(t)) \leq \frac{1}{2} e^{-\delta_0/(6h)}$$

for  $h$  small enough, and (38) follows.  $\square$

Assuming hypothesis  $(\mathcal{H})$ , the proof of Theorem 1 and Proposition 1.3 essentially reduces at this point to show that, starting from  $\mu_{\Lambda_{h,-},h}(\cdot|\mathcal{R})$  and with large probability, the system does not leave  $\mathcal{X} = \mathcal{R} \cup \mathcal{S}$  within a time of order  $e^{\beta A/h}$ . We need then a lower bound on an exit time, like are the lower bounds on the inverse exit rates and the upper bound of the soft capacity in hypothesis  $(\mathcal{H})$ . Given the previous free energy estimates and the non-convex Blaschke's inequality, these are standard estimates in the context of metastability studies. They boil down to static estimates (recall in particular (28) and (29)) and we will prove them in the next section. As far as the upper bounds on the local relaxation times and the lower bound on the soft capacity are concerned, we will follow the strategy introduced in Section 2.2, and inspired by the works of Sinclair and Martinelli, to prove them in Section 4.

### 3 Lower bounds for exit times

#### 3.1 Leaving $\mathcal{X}$

Before stating and proving the main lemma of this section we note that, for any  $a > 0$ ,

$$x > 0 \mapsto \frac{a^2}{x} + x$$

is a convex function that reaches its minimum  $2a$  in  $a$ .

**Lemma 3.1.** *Given  $B_+ > B_c$  and  $b < 1/4$ , setting  $\eta > 0$  such that*

$$\frac{B_c^2}{B_+} + B_+ = 2B_c(1 + 2\eta),$$

*it holds*

$$\mu_{\Lambda_{h,-},h}(\mathcal{R} \cup \mathcal{S})^c \leq \mu_{\Lambda_{h,-},h}(\mathcal{I}) \exp \left\{ -\frac{\beta}{h} A(1 + \eta) \right\}$$

*for  $h$  small enough.*

**Proof:** Consider  $\sigma$  in

$$\mathcal{X}^c = (\mathcal{R} \cup \mathcal{S})^c.$$

Let us denote by  $S$  the skeleton collection associated with its vertebrate contours, by  $G^{\text{ext}}$  the collection of its external vertebrate contour, by  $S^{\text{ext}}$  its associated skeleton collection, and set  $B > 0$  such that  $\check{V}(S^{\text{ext}}) = (B/h)^2$ .

Let us first consider the case  $B \geq B_c$ . Since  $\sigma \notin \mathcal{S}$ , the largest Wulff shape enclosed by a contour  $\Gamma$  of  $G^{\text{ext}}$  has a volume smaller than  $(B_-/h)^2$ . Recall Equation (18) of page 9, set  $\rho_- = 2B_-/w_\beta$ , call  $\gamma$  the skeleton of  $\Gamma$  and  $B_2(0, r)$  the Euclidean ball of radius  $r$  centered in the origin. As a consequence of the third skeleton property, the largest Wulff shape contained in a bounded connected component of  $\mathbb{R}^2 \setminus \gamma$  is contained in a translate of

$$W_{\rho_-/h} + B_2(0, 1/h^r) \subset W_{\rho_-/h} + W_{1/(\tau(0)h^r)} = \frac{w_\beta}{2} \left( \frac{\rho_-}{h} + \frac{1}{\tau(0)h^r} \right) W$$

with volume less than

$$\left( \frac{B_-}{h} + \frac{w_\beta}{2\tau(0)h^r} \right)^2 \leq \left( \frac{\tilde{B}_-}{h} \right)^2$$

for any  $\tilde{B}_- > B_-$  and  $h$  small enough. Let us take  $\tilde{B}_-$  close enough to  $B_-$  to have  $\tilde{B}_- < B_c$  and

$$\frac{B_c^2}{\tilde{B}_-} + \tilde{B}_- \geq \left( \frac{B_c^2}{B_-} + B_- \right) \frac{1 + 3\eta/2}{1 + 2\eta}.$$

Since

$$\frac{B_c^2}{B_-} + B_- = \frac{B_c^2}{B_+} + B_+ + 2 \frac{(B_+ - B_c)^3}{B_c B_-} \geq \frac{B_c^2}{B_+} + B_+ = 2B_c(1 + 2\eta),$$

this implies

$$\frac{B_c^2}{\tilde{B}_-} + \tilde{B}_- \geq 2B_c \left(1 + \frac{3\eta}{2}\right).$$

For any positive and small enough  $\epsilon$ , Proposition 2.1 now implies, since  $B \geq B_c > \tilde{B}_-$ ,

$$\begin{aligned} \mathcal{W}(S) - (1 + \epsilon)hm_\beta^* \check{V}(S) &\geq \mathcal{W}(S^{\text{ext}}) - (1 + \epsilon)hm_\beta^* \check{V}(S^{\text{ext}}) \\ &\geq \frac{w_\beta}{2} \left( \frac{(B/h)^2}{\tilde{B}_-/h} + \tilde{B}_-/h \right) - (1 + \epsilon)hm_\beta^* (B/h)^2 \\ &= \frac{w_\beta}{2h} \left[ B^2 \left( \frac{1}{\tilde{B}_-} - \frac{1 + \epsilon}{B_c} \right) + \tilde{B}_- \right] \geq \frac{w_\beta}{2h} \left[ B_c^2 \left( \frac{1}{\tilde{B}_-} - \frac{1 + \epsilon}{B_c} \right) + \tilde{B}_- \right] \\ &\geq \frac{w_\beta}{2h} \left[ 2B_c \left( 1 + \frac{3\eta}{2} \right) - (1 + \epsilon)B_c \right] = \frac{w_\beta B_c}{2h} (1 + 3\eta - \epsilon) \\ &\geq \frac{A}{h} (1 + 2\eta). \end{aligned}$$

We are in shape to use Lemma 2.6, but let us first consider the alternative case  $B \leq B_c$ .

If  $B \leq B_c$ , i.e.,  $\check{V}(S^{\text{ext}}) \leq B_c^2 h^{-2}$ , and  $V(G^{\text{ext}}) \geq (3B_c/2)^2 h^{-2}$ , we also have a lower bound on the free energy. Recalling, indeed, that there is a positive constant  $C$  such that

$$V(G^{\text{ext}}) - C\mathcal{W}(S^{\text{ext}})h^{-2r} \leq \check{V}(S^{\text{ext}})$$

it follows that, for any positive and small enough  $\epsilon$ ,

$$\begin{aligned} \mathcal{W}(S) - (1 + \epsilon)hm_\beta^* \check{V}(S) &\geq \mathcal{W}(S^{\text{ext}}) - (1 + \epsilon)hm_\beta^* \check{V}(S^{\text{ext}}) \\ &\geq \frac{h^{2r-2}}{C} ((3B_c/2)^2 - B_c^2) - (1 + \epsilon)m_\beta^* \frac{B_c^2}{h} \\ &\geq \frac{5B_c^2}{4Ch^{7/4}} - \frac{2m_\beta^* B_c^2}{h}, \end{aligned}$$

so that, for  $h$  small enough,

$$\mathcal{W}(S) - (1 + \epsilon)hm_\beta^* \check{V}(S) \geq \frac{A}{h} (1 + 2\eta).$$

If  $B \leq B_c$ ,  $V(G^{\text{ext}}) \leq (3B_c/2)^2 h^{-2}$  and  $|S^{\text{ext}}| \geq 1/h^{1-b/2}$ , then, by Lemma 2.7, an event which is much more unlikely than  $\mathcal{I}$  has to occur: there is  $\delta > 0$  such that, for  $h$  small enough,

$$\begin{aligned} \mu_{\Lambda_h, -, h} \left( |S^{\text{ext}}| \geq 1/h^{1-b/2}, V(G^{\text{ext}}) \leq (3B_c/2)^2 h^{-2} \right) &\leq \mu_{\Lambda_h, -, h}(\mathcal{I}) \sum_{k \geq 1/h^{1-b/2}} \exp \{ -\delta k/h^b \} \\ &\leq 2\mu_{\Lambda_h, -, h}(\mathcal{I}) \exp \left\{ -\delta/h^{1+b/2} \right\}. \end{aligned}$$

Finally, if  $B \leq B_c$ ,  $V(G^{\text{ext}}) \leq (3B_c/2)^2 h^{-2}$  and  $k = |S^{\text{ext}}| < 1/h^{1-b/2}$ , then, since  $\sigma \notin \mathcal{R}$ , it follows from Lemma 2.2 that the smallest Wulff shapes to contain its external vertebrate contours  $\Gamma_j$  have a total square root volume larger than  $B_+/h$ . Again, using the skeleton properties and the fact that each of these contours encloses a volume which is larger than  $1/h^{2b}$ , we get that the smallest Wulff shapes to contain the associated skeletons  $\gamma_j$  have total volume larger than

$$\sqrt{1 - Ch^{b-r}} \frac{B_+}{h} \geq \sqrt{1 - Ch^{b/2}} \frac{B_+}{h} \geq \frac{\tilde{B}_+}{h}$$

for some positive constant  $C$ , any  $\tilde{B}_+ < B_+$  and  $h$  small enough. We choose  $\tilde{B}_+ > B_c$  such that

$$\frac{B_c^2}{\tilde{B}_+} + \tilde{B}_+ \geq 2B_c \left(1 + \frac{3\eta}{2}\right).$$

Writing  $(B_j/h)^2$  for the phase volume of each single skeleton  $\gamma_j$  and  $(B_{j,\text{out}}/h)^2$  for the volume of the smallest Wulff shape to contain it, we have, using again Proposition 2.1, for any small enough  $\epsilon > 0$  and since  $B \leq B_c < \tilde{B}_+$ ,

$$\mathcal{W}(S) - (1 + \epsilon)hm_\beta^* \check{V}(S) \geq \mathcal{W}(S^{\text{ext}}) - (1 + \epsilon)hm_\beta^* \check{V}(S^{\text{ext}})$$

$$\begin{aligned}
&\geq \sum_{j < k} \frac{w_\beta}{2} \left( \frac{(B_j/h)^2}{B_{j,out}/h} + B_{j,out}/h \right) - (1 + \epsilon) h m_\beta^* (B/h)^2 \\
&\geq \frac{w_\beta}{2h} \left[ \frac{\sum_{j < k} B_j^2}{\sum_{j < k} B_{j,out}} + \sum_{j < k} B_{j,out} - (1 + \epsilon) \frac{B^2}{B_c} \right] \\
&\geq \frac{w_\beta}{2h} \left[ \frac{B^2}{\sum_{j < k} B_{j,out}} + \sum_{j < k} B_{j,out} - (1 + \epsilon) \frac{B^2}{B_c} \right] \\
&\geq \frac{w_\beta}{2h} \left[ \frac{B^2}{\tilde{B}_+} + \tilde{B}_+ - (1 + \epsilon) \frac{B^2}{B_c} \right] = \frac{w_\beta}{2h} \left[ B^2 \left( \frac{1}{\tilde{B}_+} - \frac{1 + \epsilon}{B_c} \right) + \tilde{B}_+ \right] \\
&\geq \frac{w_\beta}{2h} \left[ B_c^2 \left( \frac{1}{\tilde{B}_+} - \frac{1 + \epsilon}{B_c} \right) + \tilde{B}_+ \right] \geq \frac{w_\beta}{2h} \left[ 2B_c \left( 1 + \frac{3\eta}{2} \right) - B_c(1 + \epsilon) \right] \\
&\geq \frac{w_\beta B_c}{2h} (1 + 2\eta) = \frac{A}{h} (1 + 2\eta).
\end{aligned}$$

We conclude with Lemma 2.6 by summing on all the possible values of the integer  $\check{V}(S) < 2(B_{\max}/h)^2$ :

$$\begin{aligned}
\mu_{\Lambda_h, -, h}(\mathcal{X}^c) &\leq \mu_{\Lambda_h, -, h}(\mathcal{I}) \left[ 2 \left( \frac{B_{\max}}{h} \right)^2 \exp \left\{ -(1 - \epsilon) \frac{\beta A}{h} (1 + 2\eta) \right\} + 2 \exp \left\{ -\delta/h^{1+b/2} \right\} \right] \\
&\leq \mu_{\Lambda_h, -, h}(\mathcal{I}) \exp \left\{ -\frac{\beta A}{h} (1 + \eta) \right\}
\end{aligned}$$

for  $\epsilon$  chosen small enough and all small enough  $h$ .  $\square$

It follows that, starting from  $\mu_{\Lambda_h, -, h}(\cdot | \mathcal{R}) = \mu_{\mathcal{R}}$  and with large probability, our process  $X_{\Lambda_h, -, h}$  cannot escape from  $\mathcal{X}$  within time

$$t_1 = \exp \left\{ \frac{\beta A}{h} \left( 1 + \frac{\eta}{2} \right) \right\}$$

for  $h$  large enough. Indeed, since we assumed that for all  $x \in \Lambda_h$  and for all  $\sigma \in \Omega_{\Lambda_h}$

$$w(\sigma, \sigma^x) \leq w_{\max},$$

the number of jumps of the process  $X_{\Lambda_h, -, h}$  within time  $t_1$  is dominated by a Poisson random variable  $N_1$  with mean

$$\lambda_1 = |\Lambda_h| w_{\max} t_1$$

and for which

$$P(N_1 \geq e\lambda_1) \leq e^{-e\lambda_1} E[e^{N_1}] = \exp\{-e\lambda_1 - \lambda_1 + e\lambda_1\} = e^{-\lambda_1} \leq \frac{1}{\lambda_1}.$$

Since from the previous lemma it holds, for  $h$  small enough and all  $t \geq 0$ ,

$$\begin{aligned}
\mathbb{P}_{\mu_{\mathcal{R}}}(X_{\Lambda_h, -, h}(t) \in \mathcal{X}^c) &= \sum_{\sigma \in \mathcal{R}} \sum_{\sigma' \notin \mathcal{X}} \frac{\mu_{\Lambda_h, -, h}(\sigma)}{\mu_{\Lambda_h, -, h}(\mathcal{R})} \mathbb{P}_{\sigma}(X_{\Lambda_h, -, h}(t) = \sigma') \\
&= \sum_{\sigma' \notin \mathcal{X}} \frac{\mu_{\Lambda_h, -, h}(\sigma')}{\mu_{\Lambda_h, -, h}(\mathcal{R})} \sum_{\sigma \in \mathcal{R}} \mathbb{P}_{\sigma'}(X_{\Lambda_h, -, h}(t) = \sigma) \\
&= \sum_{\sigma' \notin \mathcal{X}} \frac{\mu_{\Lambda_h, -, h}(\sigma')}{\mu_{\Lambda_h, -, h}(\mathcal{R})} \mathbb{P}_{\sigma'}(X_{\Lambda_h, -, h}(t) \in \mathcal{R}) \\
&\leq \sum_{\sigma' \notin \mathcal{X}} \frac{\mu_{\Lambda_h, -, h}(\sigma')}{\mu_{\Lambda_h, -, h}(\mathcal{R})} = \frac{\mu_{\Lambda_h, -, h}((\mathcal{R} \cup \mathcal{S})^c)}{\mu_{\Lambda_h, -, h}(\mathcal{R})} \\
&\leq \frac{\mu_{\Lambda_h, -, h}(\mathcal{I}) \exp \left\{ -\frac{\beta A}{h} (1 + \eta) \right\}}{\mu_{\Lambda_h, -, h}(\mathcal{I})} = \exp \left\{ -\frac{\beta A}{h} (1 + \eta) \right\},
\end{aligned}$$

we conclude, with  $T_{\mathcal{X}^c}$  the exit time from  $\mathcal{X}$ ,

$$\mathbb{P}_{\mu_{\mathcal{R}}}(T_{\mathcal{X}^c} \leq t_1) \leq \frac{1}{\lambda_1} + e\lambda_1 \exp\left\{-\frac{\beta A}{h}(1 + \eta)\right\}$$

for  $h$  small enough and

**Lemma 3.2.** *Given  $B_+ > B_c$  and  $b < 1/4$ , setting  $\eta > 0$  such that*

$$\frac{B_c^2}{B_+} + B_+ = 2B_c(1 + 2\eta),$$

it holds

$$\mathbb{P}_{\mu_{\mathcal{R}}}\left(T_{\mathcal{X}^c} \leq \exp\left\{\frac{\beta A}{h}\left(1 + \frac{\eta}{2}\right)\right\}\right) \leq \exp\left\{-\frac{\beta A}{h}\frac{\eta}{3}\right\}$$

for  $h$  small enough.

### 3.2 Entering $\mathcal{S}$ or $\mathcal{R}$

**Lemma 3.3.** *Given  $\delta > 0$ , one can choose  $B_+$  close enough to  $B_c$  to have, for all small enough  $h$ ,*

$$\mu(\mathcal{R} \cap \mathcal{S}) \leq \mu(\mathcal{I}) \exp\left\{-\frac{\beta A - \delta}{h}\right\}$$

and

$$\phi_{\mathcal{S} \setminus \mathcal{R}}^* \vee \phi_{\mathcal{R} \setminus \mathcal{S}}^* \leq \exp\left\{-\frac{\beta A - \delta}{h}\right\}.$$

**Proof:** Consider, for any  $B_+ > B_c$ ,  $\sigma$  in  $\mathcal{R} \cap \mathcal{S}$  and its associated skeleton collection  $S$ . Since  $\sigma \in \mathcal{S}$ , the isoperimetric property of the Wulff shape implies that, for any  $\epsilon > 0$ ,

$$\mathcal{W}(S) \geq (1 - \epsilon)w_\beta \frac{B_-}{h} \tag{44}$$

for all small enough  $h$ . Also, since  $\sigma \in \mathcal{R}$ , it holds

$$\check{V}(S) \leq (1 + \epsilon) \left(\frac{B_+}{h}\right)^2 \tag{45}$$

for all small enough  $h$ . Then, by Lemma 2.6,

$$\begin{aligned} \mu(\mathcal{R} \cap \mathcal{S}) &\leq \mu(\mathcal{I}) \exp\left\{-\frac{\beta}{h}\left[(1 - \epsilon)^2 w_\beta B_- - (1 + \epsilon)^2 m_\beta^* B_+^2\right]\right\} \\ &= \mu(\mathcal{I}) \exp\left\{-\frac{\beta A}{h}\left[2(1 - \epsilon)^2 \frac{B_-}{B_c} - (1 + \epsilon)^2 \left(\frac{B_+}{B_c}\right)^2\right]\right\}. \end{aligned}$$

Choosing  $\epsilon$  small enough and  $B_+$  close enough to  $B_c$  we get

$$\mu(\mathcal{R} \cap \mathcal{S}) \leq \mu(\mathcal{I}) \exp\left\{-\frac{\beta A - \delta}{h}\right\}$$

for all small enough  $h$ .

We proceed in the same way and use inequality (29) from page 19 to bound  $\phi_{\mathcal{S} \setminus \mathcal{R}}^*$ . For all  $\sigma \in \mathcal{S} \setminus \mathcal{R}$  associated with a skeleton family  $S$  it holds

$$e_{\mathcal{S} \setminus \mathcal{R}}^*(\sigma) \leq |\Lambda_h| w_{\max},$$

since  $\sigma \in \mathcal{S}$ , inequality (44) is in force for any  $\epsilon > 0$  and all small enough  $h$ , and  $e_{\mathcal{S} \setminus \mathcal{R}}^*(\sigma) = 0$  unless there is  $x \in \Lambda_h$  such that  $\sigma^x \in \mathcal{R}$  so that inequality (45) is also in force for all small enough  $h$ . Hence, using Lemma 2.8 with a small enough  $\delta'$  in place of  $\delta$ ,

$$\phi_{\mathcal{S} \setminus \mathcal{R}}^* \leq \frac{|\Lambda_h| w_{\max} \mu(\mathcal{I})}{\mu(\mathcal{S} \setminus \mathcal{R})} \exp\left\{-\frac{\beta}{h}\left[(1 - \epsilon)^2 w_\beta B_- - (1 + \epsilon)^2 m_\beta^* B_+^2\right]\right\}$$

$$\begin{aligned}
&\leq \frac{|\Lambda_h| w_{\max} \mu(\mathcal{I})}{\mu\left(E_{B_{\max}/(1+\delta'), \delta'}^h\right)} \exp\left\{-\frac{\beta}{h}\left[(1-\epsilon)^2 w_{\beta} B_- - (1+\epsilon)^2 m_{\beta}^* B_+^2\right]\right\} \\
&\leq \exp\left\{-\frac{\beta}{h}\left[(1-\epsilon)^2 w_{\beta} B_- - (1+\epsilon)^2 m_{\beta}^* B_+^2\right]\right\}
\end{aligned}$$

for all  $h$  small enough, and we conclude in the same way.

Finally, since from inequality (28) it holds —with the convention  $w(\sigma, \sigma') = 0$  for all  $\sigma \neq \sigma'$  such that  $\sigma' \neq \sigma^x$  for all  $x$  in  $\Lambda_h$ —

$$\phi_{\mathcal{R} \setminus \mathcal{S}}^* \leq \sum_{\sigma \in \mathcal{R} \setminus \mathcal{S}} \sum_{\sigma' \in \mathcal{S}} \mu_{\mathcal{R} \setminus \mathcal{S}}(\sigma) w(\sigma, \sigma') \leq \frac{1}{\mu(\mathcal{I})} \sum_{\sigma' \in \mathcal{S}} \mu(\sigma') \sum_{\sigma \in \mathcal{R}} w(\sigma', \sigma)$$

we can use the same arguments to bound  $\phi_{\mathcal{R} \setminus \mathcal{S}}^*$ .  $\square$

### 3.3 Upper bounds for soft capacities

Given  $\delta > \delta' > 0$ , assume that we chose  $B_+ > B_+$  associated with  $\mathcal{R} \supset \mathcal{R}'$  and  $\mathcal{S} \supset \mathcal{S}'$  as in Lemma 3.3. We use the variational principle (30) to get an upper bound on  $C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{S})$ . We build then a test function  $f : \mathcal{X} \rightarrow \mathbb{R}$  with

$$f(\sigma) = \begin{cases} 1 & \text{if } \sigma \in \mathcal{R}' \setminus \mathcal{S}', \\ 1/2 & \text{if } \sigma \in \mathcal{R}' \cap \mathcal{S}', \\ 0 & \text{if } \sigma \in \mathcal{S}' \setminus \mathcal{R}', \\ 1/2 & \text{if } \sigma \notin \mathcal{X}' = \mathcal{R}' \cup \mathcal{S}'. \end{cases}$$

Note that, for all  $x \in \Lambda_h$ , if  $\sigma$  and  $\sigma^x$  both belong to  $\mathcal{X}'$  but neither of them is in  $\mathcal{R}' \cap \mathcal{S}'$ , then  $f(\sigma) = f(\sigma^x)$ . Hence, by Lemma 3.3 and Lemma 3.1 with  $\delta'$  and  $\eta'$  in place of  $\delta$  and  $\eta$ ,

$$\begin{aligned}
\frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{S})}{\mu(\mathcal{R})} &\leq \frac{\mu(\mathcal{R}' \cap \mathcal{S}')}{\mu(\mathcal{I})} \frac{|\Lambda_h| w_{\max}}{4} + \frac{\mu(\mathcal{R} \cap \mathcal{S})}{\mu(\mathcal{I})} (\kappa + \lambda) + \frac{\mu(\mathcal{X} \setminus \mathcal{X}')}{\mu(\mathcal{I})} \left[|\Lambda_h| w_{\max} + \kappa + \lambda\right] \\
&\leq e^{-(\beta A - \delta')/h} \frac{|\Lambda_h| w_{\max}}{4} + e^{-(\beta A - \delta)/h} (\kappa + \lambda) + e^{-\beta A(1+\eta')/h} \left[|\Lambda_h| w_{\max} + \kappa + \lambda\right]
\end{aligned}$$

for all small enough  $h$ . Since  $\delta'$  can be chosen arbitrarily small, we conclude

**Lemma 3.4.** *Given  $\delta > 0$ , choosing  $B_+$  close enough to  $B_c$  to have, for  $h$  small enough,*

$$\phi_{\mathcal{R} \setminus \mathcal{S}}^* \vee \phi_{\mathcal{S}' \setminus \mathcal{R}'}^* \leq \exp\left\{-\frac{\beta A - \delta}{h}\right\},$$

choosing also  $\kappa = \kappa(h)$  and  $\lambda = \lambda(h)$  such that

$$\lim_{h \rightarrow 0} \kappa(h) e^{\delta/h} = \lim_{h \rightarrow 0} \lambda(h) e^{\delta/h} = 0, \quad (46)$$

for all  $\epsilon > 0$ , there is  $h_0 > 0$  such that

$$\frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{S})}{\mu(\mathcal{R})} \leq \exp\left\{-\frac{\beta A - \epsilon}{h}\right\}$$

for all  $h < h_0$ .

## 4 Upper bounds for local relaxation times

### 4.1 On the metastable side

We prove in this section that for any  $\delta > 0$  one can choose  $B_+$  close enough to  $B_c$  in such a way that the local relaxation time  $1/\gamma_{\mathcal{R}}$  is smaller than  $e^{\delta/h}$  for  $h$  small enough. We use to this end a small parameter  $d > 0$ , the value of which will depend on  $\delta$  and will be used to choose  $B_+$ . Given a finite family  $F$  of disjoint Wulff shapes

$$x_j + W_{\rho_j/h} \subset \Lambda_h, \quad j < k,$$

with  $k < 1/h^{1-b/2}$ , we build a sequence of smaller disjoint Wulff shapes

$$x_{j,l} + W_{\rho_{j,l}/h} = x_j + W_{(\rho_j - dl)/h}, \quad j < k, \quad l < l_0,$$

with (recall Equation (18) from page 9)

$$l_0 = \left\lceil \frac{\rho_{\max}}{d} \right\rceil = \left\lceil \frac{2B_{\max}}{dw_\beta} \right\rceil$$

and the convention that for all  $\rho < 0$  and all  $x$  in  $\mathbb{R}^2$ ,  $W_\rho$  and  $x + W_\rho$  both stand for the empty set. For  $l < l_0$  we denote by  $W_l(F)$  their union:

$$W_l(F) = \bigcup_{j < k} x_{j,l} + W_{\rho_{j,l}/h}.$$

Next, we associate with each  $l < l_0$  a family of disjoint annuli of the lattice, the union of which is

$$A_l(F) = \mathbb{Z}^2 \cap \bigcup_{j < k} x_{j,l} + \left( W_{\rho_{j,l}/h} \setminus W_{(\rho_{j,l} - 4d)/h} \right).$$

We also define, independently of  $F$ , a further sequence of Wulff shapes

$$W_{\rho'_l/h} = W_{(\rho_{\max} - (l - l_0)d)/h}, \quad l_0 \leq l < 2l_0,$$

and, for each  $l_0 \leq l < 2l_0$ , we set

$$A_l(F) = \mathbb{Z}^2 \cap \left( W_{\rho'_l/h} \setminus W_{(\rho'_l - 4d)/h} \right).$$

We order the sites of such an annulus  $A_l(F)$  with  $l \geq l_0$  by ordering first the angles, then the radii: for  $x$  and  $y$  in  $A_l(F)$  we say that  $x$  is lower than  $y$  if the angle between the horizontal and the half-line that goes through  $x$  and starts in the annulus center is smaller than the similar angle associated with  $y$  and, if both angles are equal, we say that  $x$  is lower than  $y$  if so are the associated distances to the annulus center. For  $l < l_0$  we order similarly the sites in  $A_l(F)$  by ordering first the annuli, then the angles and the radii.

For  $\sigma \in \Omega_{\Lambda_h}$  and  $l_0 \leq l < 2l_0$ , we consider the collection  $\mathcal{C}$  of the external contours  $\Gamma$  of  $\sigma$  that enclose some  $x$  outside  $A_l(F)$ , we call  $E_l(\sigma)$  the subset of  $\mathbb{Z}^2$  made of all sites enclosed in some  $\Gamma \in \mathcal{C}$  and we call  $\bar{E}_l(\sigma)$  the subset of  $\mathbb{Z}^2$  made of all the sites in  $E_l(\sigma)$  or having a nearest neighbour in  $E_l(\sigma)$ . We define then the ‘‘block’’  $A_l(F, \sigma)$  by

$$A_l(F, \sigma) = A_l(F) \setminus \bar{E}_l(\sigma).$$

To avoid ambiguities, we will denote by  $\nu_{A_l(F, \sigma), \sigma, h}$ , rather than identify with  $\mu_{A_l(F, \sigma), \sigma, h}$ , the law of the  $\Omega_{\Lambda_h}$ -valued random variable  $M$  for which  $M$  and  $\sigma$  coincide outside  $A_l(F, \sigma)$  and the restriction of  $M$  to  $A_l(F, \sigma)$  is drawn according to

$$\mu_{A_l(F, \sigma), \sigma, h} = \mu_{A_l(F, \sigma), -, h}.$$

For  $\sigma \in \Omega_{\Lambda_h}$  and  $l < l_0$ , we make a different block construction by considering the collection  $\mathcal{C}'$  of the external contours  $\Gamma$  of  $\sigma$  that enclose some  $x$  in  $\mathbb{Z}^2 \setminus W_l(F)$ . We call  $E'_l(F, \sigma)$  the subset of  $\mathbb{Z}^2$  made of all sites enclosed in some  $\Gamma \in \mathcal{C}'$  and, similarly, we call  $\bar{E}'_l(F, \sigma)$  the subset of  $\mathbb{Z}^2$  made of all the sites in  $E'_l(F, \sigma)$  or having a nearest neighbour in  $E'_l(F, \sigma)$ . We then set

$$A_l(F, \sigma) = A_l(F) \setminus \bar{E}'_l(F, \sigma)$$

and, similarly, we denote by  $\nu_{A_l(F, \sigma), \sigma, h}$ , the law of the  $\Omega_{\Lambda_h}$ -valued random variable  $M$  for which  $M$  and  $\sigma$  coincide outside  $A_l(F, \sigma)$  and the restriction of  $M$  to  $A_l(F, \sigma)$  is drawn according to  $\mu_{A_l(F, \sigma), \sigma, h}$ . Note that, in both the cases  $l < l_0$  and  $l \geq l_0$ , DLR equations imply that, if  $M$  is drawn according to  $\mu_{\Lambda_h, -, h}$  and  $M'$  is drawn according to  $\nu_{A_l(F, M), M, h}$ , then  $M$  and  $M'$  have the same law.

Given  $F$ , we now associate with each  $\sigma$  in  $\Omega_{\Lambda_h}$  a block path  $\Pi_\sigma$  by setting first  $M_0 = \sigma$ , drawing then, for each  $l < 2l_0$ , the milestone  $M_{l+1}$  according to  $\nu_{A_l(F, M_l), M_l, h}$  and connecting finally each milestone  $M_l$  with  $M_{l+1}$  along the canonical path  $\pi_{M_l, M_{l+1}}^l$  in  $\Omega_{A_l(F)}$  associated with the ordered set  $A_l(F)$ .

**Lemma 4.1.** *There is a positive constant  $C$  such that, for any  $d > 0$ , all  $\sigma_0$  in  $\Omega_{\Lambda_h}$  and all  $x$  in  $\Lambda_h$ ,*

$$\frac{1}{\mu_{\Lambda_h, -, h}(\sigma_0)} \sum_{\sigma \in \Lambda_h} \mu_{\Lambda_h, -, h}(\sigma) P((\sigma_0, \sigma_0^x) \in \Pi_\sigma) \leq 8 \exp \left\{ \frac{Cd}{h} \right\}.$$

**Proof:** We first note that, for  $(\sigma_0, \sigma_0^x)$  to belong to  $\Pi_\sigma$ , there is to be some  $l < 2l_0$  such that  $x$  lies in  $A_l(F)$  and  $(\sigma_0, \sigma_0^x)$  belongs to  $\pi_{M_l, M_{l+1}}^l$ . Since our annuli are of “width”  $4d$  and their linear size decreases by  $d$  in each of our two annulus sequences, there are 8 such  $l$  at most. Now, if  $x \in A_l(F)$ , with

$$\Omega_{l, \sigma_0} = \{ \sigma \in \Omega_{\Lambda_h} : \forall x \notin A_l(F, \sigma_0), \sigma(x) = \sigma_0(x) \}$$

then, by DLR equations and Lemma 2.4, there is  $C > 0$  such that

$$\begin{aligned} & \frac{1}{\mu_{\Lambda_h, -, h}(\sigma_0)} \sum_{\sigma \in \Lambda_h} \mu_{\Lambda_h, -, h}(\sigma) P((\sigma_0, \sigma_0^x) \in \Pi_\sigma) \\ &= \frac{1}{\mu_{\Lambda_h, -, h}(\sigma_0)} \sum_{\sigma_l, \sigma_{l+1} \in \Omega_{l, \sigma_0}} \sum_{\sigma \in \Lambda_h} \mu_{\Lambda_h, -, h}(\sigma) P(M_l = \sigma_l) \nu_{A(F, \sigma_l), \sigma_l, h}(\sigma_{l+1}) \mathbb{1} \left\{ (\sigma_0, \sigma_0^x) \in \pi_{\sigma_l, \sigma_{l+1}}^l \right\} \\ &= \sum_{\sigma_l, \sigma_{l+1} \in \Omega_{l, \sigma_0}} \frac{\mu_{\Lambda_l, -, h}(\sigma_l)}{\mu_{\Lambda_h, -, h}(\sigma_0)} \nu_{A(F, \sigma_l), \sigma_l, h}(\sigma_{l+1}) \mathbb{1} \left\{ (\sigma_0, \sigma_0^x) \in \pi_{\sigma_l, \sigma_{l+1}}^l \right\} \\ &= \frac{1}{\nu_{A(F, \sigma_0), \sigma_0, h}(\sigma_0)} \sum_{\sigma_l, \sigma_{l+1} \in \Omega_{l, \sigma_0}} \nu_{A(F, \sigma_0), \sigma_0, h}(\sigma_l) \nu_{A(F, \sigma_0), \sigma_0, h}(\sigma_{l+1}) \mathbb{1} \left\{ (\sigma_0, \sigma_0^x) \in \pi_{\sigma_l, \sigma_{l+1}}^l \right\} \\ &\leq \exp \left\{ \frac{Cd}{h} \right\}. \end{aligned}$$

□

Given  $\sigma$  and  $\sigma'$  in  $\mathcal{R}$  we will couple two such block paths  $\Pi_\sigma$  and  $\Pi'_{\sigma'}$  associated with two *random* families  $F$  and  $F'$ . We will consider a “good event”  $E_{\sigma, \sigma'}$  for which  $\Pi_\sigma$  and  $\Pi'_{\sigma'}$  will stay in  $\mathcal{R}$  and will end in the same  $M_{2l_0} = M'_{2l_0}$ . Then, conditionally to  $E_{\sigma, \sigma'}$ , we can build a block path  $\Pi_{\sigma, \sigma'}$  in  $\mathcal{R}$  and from  $\sigma$  to  $\sigma'$  by concatenation of  $\Pi_\sigma$ , from  $\sigma$  to  $M_{2l_0}$ , and the reversed image of  $\Pi'_{\sigma'}$ , from  $M'_{2l_0} = M_{2l_0}$  to  $\sigma'$ . Since the previous lemma is uniform in  $F$ , we will get, for all  $\sigma_0$  and  $\sigma_0^x$  in  $\mathcal{R}$

$$\begin{aligned} & \frac{1}{\mu_{\mathcal{R}}(\sigma_0) \vee \mu_{\mathcal{R}}(\sigma_0^x)} \sum_{\sigma, \sigma' \in \mathcal{R}} \mu_{\mathcal{R}}(\sigma) \mu_{\mathcal{R}}(\sigma') P((\sigma_0, \sigma_0^x) \in \Pi_{\sigma, \sigma'} | E_{\sigma, \sigma'}) \\ &\leq \frac{\mu(\mathcal{R})}{\mu(\sigma_0)} \sum_{\sigma \in \mathcal{R}} \frac{\mu(\sigma)}{\mu(\mathcal{R})} \frac{P((\sigma_0, \sigma_0^x) \in \Pi_\sigma)}{\min_{\sigma, \sigma' \in \mathcal{R}} P(E_{\sigma, \sigma'})} + \frac{\mu(\mathcal{R})}{\mu(\sigma_0^x)} \sum_{\sigma' \in \mathcal{R}} \frac{\mu(\sigma')}{\mu(\mathcal{R})} \frac{P((\sigma_0^x, \sigma_0) \in \Pi_{\sigma'})}{\min_{\sigma, \sigma' \in \mathcal{R}} P(E_{\sigma, \sigma'})} \\ &\leq \frac{16e^{Cd/h}}{\min_{\sigma, \sigma' \in \mathcal{R}} P(E_{\sigma, \sigma'})}. \end{aligned}$$

In view of inequality (21) at page 11, we will need a lower bound on  $P(E_{\sigma, \sigma'})$ .

Before building  $E_{\sigma, \sigma'}$  and giving such a lower bound, let us first explain in which sense  $F$  and  $F'$  are random. To sample  $F$  of size  $k < 1/h^{1-b/2}$ , we first sample  $k$  uniformly, then we sample the centers  $x_j$  uniformly in  $B_{\max} W/h$ , and, finally, we sample the  $\rho_j$  uniformly in  $[0, \rho_+]$ , with

$$\rho_+ = \frac{2B_+}{w_\beta},$$

and conditionally to our non-intersection constraint. We sample  $F'$  independently and in the same way.

We say that  $F$  is adapted to  $\sigma$  if the Wulff shapes of  $F$  contain the external vertebrate contours of  $\sigma$ . This is the first requirement for our good event  $E_{\sigma, \sigma'}$  and it happens with a probability larger than

$$\left( \frac{C}{|\Lambda_h|(\rho_{\max}/h)} \right)^{1/h^{1-b/2}} \geq e^{-\delta/(8h)} \quad (47)$$

for some  $C > 0$  and all small enough  $h$ . We assume in what follows that  $F$  is adapted to  $\sigma$ .

The next requirement for  $E_{\sigma,\sigma'}$  is that for each  $l < l_0$ ,  $M_{l+1}$  has no vertebrate contour to enclose a site in the annulus union

$$A_l^2(F) = A_l(F) \setminus W_{l+2}(F),$$

with the convention  $W_{l+2}(F) = \emptyset$  for  $l+2 \geq l_0$ . Provided that  $B_+$  is close enough to  $B_c$  to have

$$\phi(B_+ - 4d) < \phi(B_+), \quad (48)$$

using inductively FKG inequality together with Estimate (25) from page 17 with a small enough  $\epsilon$  depending of  $l_0$ , then  $d$ , this occurs with probability  $e^{-\delta/(8h)}$  at least for all small enough  $h$ .

Provided that the same requirements are satisfied for  $F'$  and  $M'_l$  with  $l < l_0$ , it holds that the milestones  $M_{l_0}$  and  $M'_{l_0}$  are both in  $\mathcal{I}$ . It is also the case that  $\Pi_\sigma$  and  $\Pi'_\sigma$  did not escape  $\mathcal{R}$  up to this point, where we can start to introduce some dependence between them.

Assuming that our previous requirements for  $E_{\sigma,\sigma'}$  were satisfied, the next one is that  $M_{l_0+1}$  and  $M'_{l_0+1}$  are still in  $\mathcal{I}$  and coincide on the annulus

$$A^2 = \mathbb{Z}^2 \cap (W_{\rho_{\max}/h} \setminus W_{(\rho_{\max}-2d)/h}).$$

For  $d$  small enough, inequality (27), DLR equations and FKG inequality show that this happens with a non-negligible probability. Indeed, since  $M_{l_0}$  and  $M'_{l_0}$  are in  $\mathcal{I}$ , the restrictions to

$$A^3 = \mathbb{Z}^2 \cap (W_{\rho_{\max}/h} \setminus W_{(\rho_{\max}-3d)/h})$$

of  $M_{l_0+1}$  and  $M'_{l_0+1}$  are both dominated by a that of a random configuration  $\xi$  drawn according to  $\mu_{A^3 \setminus \bar{E}, -, h}$ , with

$$A^5 = \mathbb{Z}^2 \cap (W_{\rho_{\max}/h} \setminus W_{(\rho_{\max}-5d)/h}).$$

Hence, we can partially sample them first by drawing the external contours  $\Gamma$  of  $\xi$  that will cross the boundary of  $A^3$ , then by drawing the common restriction of  $\xi$ ,  $M_{l_0+1}$  and  $M'_{l_0+1}$  to  $A^3 \setminus \bar{E}$  according to  $\mu_{A^3 \setminus \bar{E}, -, h}$ , with  $\bar{E}$  the set of all sites that are enclosed by one of these  $\Gamma$  or that are a nearest neighbour of such a site. Since, by (27),  $\xi$  is in  $\mathcal{I}$  with a non-negligible probability, larger than  $e^{-\beta\epsilon/h}$ , for all small enough  $h$ , this gives the same lower bound for this new requirement.

Our last requirement, which includes the previous one, is that, for all  $l_0 \leq l < 2l_0$ , the milestones  $M_{l+1}$  and  $M'_{l+1}$  are in  $\mathcal{I}$  and coincide on the annulus

$$A^{l-l_0+2} = \mathbb{Z}^2 \cap (W_{\rho_{\max}/h} \setminus W_{(\rho_{\max}-(l-l_0+2)d)/h}).$$

Provided that our previous set of requirements was satisfied, this implies that the whole paths  $\Pi_\sigma$  and  $\Pi'_\sigma$  all along remain in  $\mathcal{R}$  and end in a same configuration  $M_{2l_0} = M'_{2l_0}$ , and this happens, repeating inductively the previous argument, with a probability  $e^{-\delta/(8h)}$  at least for  $h$  small enough.

Using inequality (21) from page 11, we get that, for any small enough  $d$ , if  $B_+$  is close enough to  $B_c$  for inequality (48) to be in force, then

$$\frac{1}{\gamma_{\mathcal{R}}} \leq \frac{8(B_{\max}/h)^2}{w_{\min}} 16e^{Cd/h} e^{5\delta/(8h)}$$

for some positive constant  $C$  that does not depend on  $d$  and all small enough  $h$ . Choosing  $d$  small enough to have  $Cd < 2/8$  we conclude

**Lemma 4.2.** *Given  $\delta > 0$ , one can choose  $B_+$  close enough to  $B_c$  to have*

$$\frac{1}{\gamma_{\mathcal{R}}} \leq e^{\delta/h}.$$

## 4.2 On the stable side

The goal of this section is to show

**Lemma 4.3.** *Given  $\delta > 0$ , one can choose  $B_+$  close enough to  $B_c$  to have*

$$\frac{1}{\gamma_{\mathcal{S}}} \leq e^{\delta/h}.$$

The proof is similar to that on the metastable side, with some simplifications and some extra complications. We will only indicate the main differences.

Simplifications come from the fact that we will only have to build annular blocks: we will not need union of annuli anymore. Similarly to the previous case, we will use these blocks to build a path of expanding, rather than shrinking, contours, before using the same shrinking blocks to make the final milestones of two block paths coincide.

There are only two kind of complications. We will first need another sequence of shrinking blocks to ensure that, starting from  $\sigma \in \mathcal{S}$  for which there is a large contour that encloses a slightly subcritical Wulff shape, we will only see “the plus-phase” on the internal border of this “large” Wulff shape at the end of the associated first block path. This is needed to use inequality (24) of page 17 with our second, expanding, block sequence—the analogue of the first shrinking sequence on the metastable side—to obtain, as last milestone associated with the last block of this second block sequence, a configuration with only one vertebrate contour, close to the boundary of  $\Lambda_h$ , outside our slightly subcritical Wulff shape. We encounter the second complication in building this second, expanding, block sequence: since our expanding blocks have to be contained in  $\Lambda_h$  and eventually coincide with its boundary, except if we start with an annular block centered on the origin, we cannot have concentric blocks. Because the overlapping properties of our blocks are crucial for the inductive parts of our arguments in giving a lower bound for our good event, there is an issue.

Here is the key lemma we will use to solve it. It says that two non-concentric Wulff shapes on the same side of a common tangent are such that the core of the largest one is contained in the bulk of the smallest one.

**Lemma 4.4.** *Let  $\mathbf{n} = (\cos \theta, \sin \theta)$  be the external normal associated with a Wulff shape  $x + W_\rho$  and  $y$  in  $x + \partial W_\rho$ . For a positive  $d < \rho/3$ , let  $x'$  in  $\mathbb{R}^2$  be such that  $\mathbf{n}$  is also the external normal associated with the Wulff shape  $x' + W_d$  and  $x$  in  $x' + \partial W_d$ . Then the Wulff shapes  $x + W_\rho$  and  $x' + W_{\rho+d}$  are on the same side of a common tangent in  $y$  and it holds*

$$x' + W_{(\rho+d)-4d} = x' + W_{\rho-3d} \subset x + W_{\rho-2d}.$$

**Proof:** By the Wulff shape construction from the support function  $\rho\tau$ , it holds

$$x' + W_d + W_\rho = x' + W_{d+\rho}$$

and, since the perpendicular at distance  $\rho$  of  $x$  to the half-line issued from  $x$  and oriented by  $\mathbf{n}$  is the same as the perpendicular at distance  $\rho + d$  of  $x'$  to the half-line issued from  $x'$  and oriented by  $\mathbf{n}$ , the first part of the thesis follows. Since  $W = -W$  and

$$x = (x_1, x_2) \in x' + W_d,$$

we also have

$$x' = (x'_1, x'_2) \in x + W_d,$$

so that, for all  $\varphi < 2\pi$ ,

$$(x'_1 - x_1) \cos \varphi + (x'_2 - x_2) \sin \varphi \leq d\tau(\varphi)$$

and, for each

$$z = (z_1, z_2) \in x' + W_{\rho-3d},$$

it holds

$$(z_1 - x'_1) \cos \varphi + (z_2 - x'_2) \sin \varphi \leq (\rho - 3d)\tau(\varphi),$$

hence

$$(z_1 - x_1) \cos \varphi + (z_2 - x_2) \sin \varphi \leq (\rho - 2d)\tau(\varphi).$$

We conclude that  $z$  belongs to  $x + W_{\rho-2d}$ . □

Let us now build our three block sequences associated, by analogy with the notation of the previous section, with a Wulff shape

$$F = x^0 + W_{\rho^0/h} \subset \Lambda_h$$

and a small parameter  $d > 0$ . We will only have to consider the case when

$$x^0 \in W_{(\rho_{\max}-2d)/h}$$

and we start with the middle sequence, the expanding one. We set

$$W_k(F) = x_k + W_{\rho_k/h} = x^0 + W_{\rho^0+kd/h} \subset W_{(\rho_{\max}-d)/h}, \quad k < k_1,$$

with

$$k_1 = \left\lceil \frac{\rho^1 - \rho^0}{d} \right\rceil$$

where  $\rho^1$  is the smallest  $\rho$  for which  $x^0 + W_{\rho/h}$  and  $W_{(\rho_{\max}-d)/h}$  have a common tangent. We call  $\mathbf{n} = (\cos \theta, \sin \theta)$  the external normal associated with this common tangent and we define  $y \in \partial W_{k_1-1}(F)$  in such a way that the associated external normal is  $\mathbf{n}$  too. Then, for  $k \geq k_1$ , we inductively define

$$W_k(F) = x_k + W_{\rho_k/h} = x'_{k-1} + W_{(\rho_{k-1}+d)/h}, \quad k < k_0$$

where  $x'_{k-1}$  is associated by the previous lemma with  $\mathbf{n}$ ,  $x_{k-1}$ ,  $\rho_{k-1}/h$ ,  $y$  and  $d/h$  in place of  $\mathbf{n}$ ,  $x$ ,  $\rho$ ,  $y$  and  $d$ , and where

$$k_0 = \left\lceil \frac{\rho_{\max} - \rho^0}{d} \right\rceil.$$

Since  $y \in W_{(\rho_{\max}-d)/h} \setminus W_{(\rho_{\max}-2d)/h}$ , the fact that  $k < k_0$ , together with the common tangent property of the previous lemma, ensure that

$$W_k(F) \subset \Lambda_h.$$

We also have

$$W_{k_0-1} \supset W_{(\rho_{\max}-2d)/h}.$$

We can now define our annuli on the lattice

$$A_k(F) = \mathbb{Z}^2 \cap \left( x_k + (W_{\rho_k/h} \setminus W_{(\rho_k-4d)/h}) \right), \quad k < k_0.$$

For  $\sigma$  in  $\Omega_{\Lambda_h}$  and  $k < k_0$ , we call  $E''_{k,-}(F, \sigma)$  the union of all minus spin percolation clusters that contain a site in  $x_k + W_{(\rho_k-4d)/h}$ . We call  $\bar{E}''_{k,-}(F, \sigma)$  the set made of all the sites in  $E''_{k,-}(F, \sigma)$  and their nearest neighbours. The associated block is

$$A_k(F, \sigma) = A_k(F) \setminus \bar{E}''_{k,-}(F, \sigma).$$

Let us now describe the final, shrinking, annulus sequence. It is the same as in the previous section, with a different indexation only. We set

$$W_{\rho_k/h} = W_{(\rho_{\max}-(k-k_0)d)/h}, \quad k_0 \leq k < k_0 + l_0,$$

with

$$l_0 = \left\lceil \frac{\rho_{\max}}{d} \right\rceil,$$

and, independently of  $F$ ,

$$A_k(F) = \mathbb{Z}^2 \cap (W_{\rho_k/h} \setminus W_{(\rho_k-4d)/h}), \quad k_0 \leq k < k_0 + l_0.$$

To define the initial, shrinking also, annulus sequence, we use negative indices. For  $k \geq -k_0$  we set

$$A_k(F) = A_{k_0-(k+k_0)}(F) = A_{-k}(F), \quad k < 0.$$

We use the same block definition for both the shrinking sequences. For  $\sigma$  in  $\Omega_{\Lambda_h}$  and  $k < 0$  or  $k \geq k_0$  we call  $E'_{k,-}(F, \sigma)$  the union of all minus spin percolation clusters that contain a site outside  $W_k(F)$ . We call  $\bar{E}'_{k,-}(F, \sigma)$  the set made of all the sites in  $E'_{k,-}(F, \sigma)$  and their nearest neighbours. The associated block is

$$A_k(F, \sigma) = A_k(F) \setminus \bar{E}'_{k,-}(F, \sigma).$$

Like in the previous section we call  $\nu_{A_k(F, \sigma), \sigma, h}$  the law of an  $\Omega_{\Lambda_h}$ -valued random variable that coincides with  $\sigma$  outside  $A_k(F, \sigma)$  and for which the restriction to  $A_k(F, \sigma)$  is drawn according to  $\mu_{A_k(F, \sigma), \sigma, h}$ . We associate with  $\sigma \in \Omega_{\Lambda_h}$ , and a random

$$F = x^0 + W_{\rho^0/h}$$

with  $\rho^0 \geq B_-$ , a block path  $\Pi_\sigma$  by setting  $M_{-k_0} = \sigma$ , drawing inductively, for each  $k < k_0 + l_0$ , a milestone  $M_{k+1}$  according to  $\nu_{A_k(F), M_k, h}$  and connecting these milestones by canonical paths. We need then to couple two such block paths  $\Pi_\sigma$  and  $\Pi_{\sigma'}$ , with  $\sigma$  and  $\sigma'$  in  $\mathcal{S}$ , to make them coincide in their final configuration with large enough probability.

Our associated event  $E(\sigma, \sigma')$  is as follows. First we require  $F$  and  $F'$  to be adapted with  $\sigma$  and  $\sigma'$ , i.e., to be enclosed in some of their external contours,  $\Gamma$  and  $\Gamma'$ . The associated probability cost is computed like in the previous section. Then we ask that, for each  $k < 0$ , the only contours of  $M_{k+1}$  and  $M'_{k+1}$  enclosed in  $\Gamma$  and  $\Gamma'$  and that intersect the outer half of  $A_k(F)$  are invertebrate contours. Note that, by construction,  $\Gamma$  and  $\Gamma'$  are contours of each milestone  $M_{k+1}$  and  $M'_{k+1}$  for  $k < 0$ . We use inequality (26) of page 17 together with FKG inequality to control the cost of this event. We also have to use the overlapping properties of our annuli that are implied by Lemma 4.4, but this is not crucial since we could have defined concentric annuli only to deal with this first part. This event implies that, for the milestones  $M_0$  and  $M'_0$ , we only have invertebrate contours enclosed in  $\Gamma$  and  $\Gamma'$  and outside  $W_{(\rho^0 - 3d)/h}$ . Then we require to have, for each milestone  $M_{k+1}$  and  $M'_{k+1}$  with  $0 \leq k < k_0$ , invertebrate contours only in the “inner part” of  $A_k(F)$ , all of them enclosed in some external contour. This is dealt, for  $B_+$  close enough to  $B_c$  to have  $\phi(B_- + d) < \phi(B_-)$  and also  $d$  small enough, with inequality (24) and Lemma 4.4, which says that the bulk of  $A_k(F)$  covers the inner part of  $A_{k+1}(F)$ . Finally we ask for the milestones  $M_{k+1}$  and  $M'_{k+1}$ , with  $k_0 \leq k < k_0 + l_0$ , to coincide in the outer part of  $A_k$ , with one large contour close to the border of  $\Lambda_h$  and that contains only invertebrate contours. The analysis of this last part, with the help of inequality (26) again, and the following conclusions are similar to those of the previous section.

### 4.3 Lower bounds for soft capacities

**Lemma 4.5.** *Given  $\delta > 0$ , choosing  $B_+$  close enough to  $B_c$  to have, for  $h$  small enough,*

$$\frac{1}{\gamma_{\mathcal{R}}} \wedge \frac{1}{\gamma_{\mathcal{S}}} \leq e^{\delta/h},$$

choosing also  $\kappa = \kappa(h)$  and  $\lambda = \lambda(h)$  such that

$$\lim_{h \rightarrow 0} \frac{e^{-(\beta A - \delta)/h}}{\kappa(h)} = \lim_{h \rightarrow 0} \frac{e^{-(\beta A - \delta)/h}}{\lambda(h)} = 0,$$

for all  $\epsilon > 0$ , there is  $h_0 > 0$  such that

$$\frac{C_\kappa^\lambda(\mathcal{R}, \mathcal{S})}{\mu(\mathcal{R})} \geq \exp\left\{-\frac{\beta A + \epsilon}{h}\right\}$$

for all  $h < h_0$ .

**Proof:** For any positive  $\delta' < \delta$ , the proofs of the two previous sections provide us, for  $B'_+ < B_+$  small enough and associated with  $\mathcal{R}' \subset \mathcal{R}$  and  $\mathcal{S}' \subset \mathcal{S}$ , with two random paths  $\Pi_{\mathcal{R}'}$  and  $\Pi_{\mathcal{S}'}$  of length smaller than  $C|\Lambda_h|$  for some constant  $C$ , with starting points  $\Pi_{\mathcal{R}'-}$  and  $\Pi_{\mathcal{S}'-}$  and ending points  $\Pi_{\mathcal{R}'+}$  and  $\Pi_{\mathcal{S}'+}$ , independently distributed according to  $\mu_{\mathcal{R}'}$  and  $\mu_{\mathcal{S}'}$ , and such that

$$\max_{\sigma, \sigma^x \in \mathcal{R}'} \frac{P((\sigma, \sigma^x) \in \Pi_{\mathcal{R}'})}{\mu_{\mathcal{R}'}(\sigma)w(\sigma, \sigma^x)} \leq e^{\delta'/h} \quad \text{and} \quad \max_{\sigma, \sigma^x \in \mathcal{S}'} \frac{P((\sigma, \sigma^x) \in \Pi_{\mathcal{S}'})}{\mu_{\mathcal{S}'}(\sigma)w(\sigma, \sigma^x)} \leq e^{\delta'/h}$$

for  $h$  small enough. Recall the notation of Lemma 2.8, set

$$\mathcal{J} = E_{B_{\max}/(1+\delta'), \delta'}^h$$

and consider the random variables  $\tilde{\Pi}_{\mathcal{R}'}$ , the law of which is that of  $\Pi_{\mathcal{R}'}$  conditioned to  $\{\Pi_{\mathcal{R}'-} \in \mathcal{I}\}$  and  $\{\Pi_{\mathcal{R}'+} \in \mathcal{R}' \cap \mathcal{S}'\}$ , and  $\tilde{\Pi}_{\mathcal{S}'}$ , the law of which is that of  $\Pi_{\mathcal{S}'}$  conditioned to  $\{\Pi_{\mathcal{S}'-} \in \mathcal{R}' \cap \mathcal{S}'\}$  and  $\{\Pi_{\mathcal{S}'+} \in \mathcal{J}\}$ . Since  $\tilde{\Pi}_{\mathcal{R}'}$  and  $\tilde{\Pi}_{\mathcal{S}'}$  have the same law, we can build a new random variable  $\Pi$  by concatenation of  $\tilde{\Pi}_{\mathcal{R}'}$  and  $\tilde{\Pi}_{\mathcal{S}'}$ . Considering the loop erased version of  $\Pi$ , this provide us with a unitary flow  $\psi$  from  $\mathcal{I}$  to  $\mathcal{J}$  and for which, for all  $\sigma$  and  $\sigma^x$  in  $\mathcal{X}$ , it holds

$$|\psi(\sigma, \sigma^x)| \leq P((\sigma, \sigma^x) \in \Pi) + P((\sigma^x, \sigma) \in \Pi) \leq 2e^{\delta'/h} \left( \frac{\mu_{\mathcal{R}'}(\sigma)w(\sigma, \sigma^x)}{\mu_{\mathcal{R}'}(\mathcal{I})\mu_{\mathcal{R}'}(\mathcal{R}' \cap \mathcal{S}')} + \frac{\mu_{\mathcal{S}'}(\sigma)w(\sigma, \sigma^x)}{\mu_{\mathcal{S}'}(\mathcal{R}' \cap \mathcal{S}')\mu_{\mathcal{S}'}(\mathcal{J})} \right)$$

and, recall Lemma 2.7 and Lemma 2.8,

$$|\psi(\sigma, \sigma^x)| \leq 2e^{\delta'/h} \left( \frac{1}{\mu_{\mathcal{R}'(\mathcal{I})}} + \frac{1}{\mu_{\mathcal{S}'(\mathcal{J})}} \right) \frac{\mu(\sigma)w(\sigma, \sigma^x)}{\mu(\mathcal{R}' \cap \mathcal{S}')} \leq \frac{\mu(\sigma)w(\sigma, \sigma^x)}{\mu(\mathcal{R}' \cap \mathcal{S}')} e^{2\delta'/h},$$

so that

$$\frac{|\psi(\sigma, \sigma^x)|^2}{\mu(\sigma)w(\sigma, \sigma^x)} \leq \frac{e^{2\delta'/h}}{\mu(\mathcal{R}' \cap \mathcal{S}')} |\psi(\sigma, \sigma^x)| \leq \frac{e^{2\delta'/h}}{\mu(\mathcal{R}' \cap \mathcal{S}')} \left( P((\sigma, \sigma^x) \in \Pi) + P((\sigma^x, \sigma) \in \Pi) \right),$$

for all small enough  $h$ . By extending each realisation of  $\Pi$  from some  $\sigma_-$  in  $\mathcal{I}$  to some  $\sigma^+$  in  $\mathcal{J}$  into a path from  $\bar{\sigma}_- \in \bar{\mathcal{R}}$  to  $\check{\sigma}^+ \in \check{\mathcal{S}}$ , we obtain, from Thomson's principle (31) at page 20, and Lemma 2.8 again, that there is a positive constant  $C$  such that

$$\begin{aligned} \frac{\mu(\mathcal{R})}{C_\kappa^\lambda(\mathcal{R}, \mathcal{S})} &\leq \frac{\mu(\mathcal{R})e^{2\delta'/h}}{2\mu(\mathcal{R}' \cap \mathcal{S}')} \sum_{\sigma \in \Omega_{\Lambda_h}} \sum_{x \in \Lambda_h} P((\sigma, \sigma^x) \in \Pi) + P((\sigma^x, \sigma) \in \Pi) \\ &\quad + \mu(\mathcal{R}) \sum_{\sigma \in \mathcal{I}} \frac{\mu_{\mathcal{R}'}(\sigma | \mathcal{I})^2}{\kappa\mu(\sigma)} + \mu(\mathcal{R}) \sum_{\sigma \in \mathcal{J}} \frac{\mu_{\mathcal{S}'}(\sigma | \mathcal{J})^2}{\lambda\mu(\sigma)} \\ &\leq \frac{\mu(\mathcal{I})e^{2\delta'/h}}{\mu(\mathcal{R}' \cap \mathcal{S}')} 2E[|\Pi|] + \frac{2}{\kappa} + \frac{2\mu(\mathcal{I})}{\lambda\mu(\mathcal{J})} \\ &\leq 2C|\Lambda_h| \exp\left\{ \frac{(\beta A + 3\delta')}{h} \right\} + \exp\left\{ \frac{(\beta A - \delta/2)}{h} \right\} \leq \exp\left\{ \frac{(\beta A + 4\delta')}{h} \right\}. \end{aligned}$$

for all small enough  $h$ . Since  $\delta'$  is arbitrarily small, this ends the proof.  $\square$

## 5 Proof of the main results

### 5.1 Proof of Theorem 1 and Proposition 1.3

Lemma 3.3, Lemma 4.2 and Lemma 4.3, Lemma 3.4 and Lemma 4.5, Lemma 2.10 and Lemma 3.2 give Theorem 1 and Proposition 1.3 with the relaxation time  $1/\gamma = 1/\gamma_h$  of  $X$  (restricted to  $\mathcal{R} \cup \mathcal{S}$ ) in place of the mixing time  $t_{\text{mix},h}$  of  $X_{\Lambda_h, -, h}$ . We only have to show that for all  $\alpha > 1$  there is a positive  $h_0$  such that, for all positive  $h < h_0$ , it holds

$$\frac{1}{\alpha\gamma_h} \leq t_{\text{mix},h} \leq \frac{\alpha}{\gamma_h}.$$

Let us first show such a lower bound on  $t_{\text{mix},h}$  by contradiction. We assume then the existence of some  $\alpha > 1$  for which there is a decreasing sequence  $h_n \rightarrow 0$  such that  $t_{\text{mix},h_n} \leq 1/(\alpha\gamma_{h_n})$  for all  $n$ . Consider now an optimal coupling between a random variable  $\xi$  with law  $\mu_{\Lambda_h, -, h}$  and our process at time  $t_{\text{mix},h_n} \leq 1/(\alpha\gamma_{h_n})$  and started in  $\mu_{\mathcal{R}}$ . By definition of  $t_{\text{mix},h}$  they will coincide with a probability  $1 - 1/e$  at least. Since  $\mu_{\mathcal{S}}$  is exponentially close to  $\mu_{\Lambda_h, -, h}$ —so that, for any  $\epsilon > 0$  and  $h$  small enough, the total variation distance between  $\mu_{\mathcal{S}}$  and  $\mu_{\Lambda_h, -, h}$  is less than  $\epsilon$ —we can also couple  $X(t_{\text{mix},h_n})$  with a random variable  $\xi_{\mathcal{S}}$  with law  $\mu_{\mathcal{S}}$ :  $\xi$  and  $\xi_{\mathcal{S}}$  will coincide with large probability, larger than  $1 - \epsilon$  for  $n$  large enough. In addition, since  $1/\lambda$  is small with respect to  $1/\gamma_h$ , it holds, for  $n$  large enough,

$$\mathbb{P}_{\mu_{\mathcal{S}}} \left( T_{\lambda_{\mathcal{S}}} > \frac{\epsilon}{\gamma_{h_n}} \right) \leq \epsilon.$$

This gives, for any given  $\epsilon > 0$  and  $n$  large enough,

$$\mathbb{P}_{\mu_{\mathcal{R}}} \left( T_{\lambda_{\mathcal{S}}} > \frac{1}{\alpha\gamma_{h_n}} + \frac{\epsilon}{\gamma_{h_n}} \right) \leq \frac{1}{e} + \epsilon + \epsilon.$$

Since

$$\lim_{h \rightarrow 0} \mathbb{P}_{\mu_{\mathcal{R}}} \left( \gamma_h T_{\lambda_{\mathcal{S}}} > \frac{1}{\alpha} + \epsilon \right) = e^{-(\epsilon+1/\alpha)},$$

we get, for any  $\epsilon > 0$ ,

$$e^{-(\epsilon+1/\alpha)} - \epsilon \leq e^{-1} + 2\epsilon$$

and a contradiction with  $\alpha > 1$ .

As far as the upper bound is concerned, it follows from the second, already proven, point of the theorem that starting from any  $\nu$ , both the distribution of  $X_{\Lambda_h, -, h}$  at time  $T_{\lambda_S}$  and the conditional distribution of  $X(T_{\lambda_S})$  on  $\{T_{\lambda_S} > t\}$ , for any time  $t > 0$ , are exponentially close to equilibrium. Then, so is the conditional distribution of  $X(T_{\lambda_S})$  on  $\{T_{\lambda_S} \leq t\}$ , provided that the probability of this last event is not exponentially small. Indeed, from the equalities

$$\mathbb{P}_\nu(X(T_{\lambda_S}) = \cdot) = \mathbb{P}_\nu(T_{\lambda_S} \leq t)\mathbb{P}_\nu(X(T_{\lambda_S}) = \cdot \mid T_{\lambda_S} \leq t) + \mathbb{P}_\nu(T_{\lambda_S} > t)\mathbb{P}_\nu(X(T_{\lambda_S}) = \cdot \mid T_{\lambda_S} > t)$$

and

$$\mu_{\Lambda_h, -, h} = \mathbb{P}_\nu(T_{\lambda_S} \leq t)\mu_{\Lambda_h, -, h} + \mathbb{P}_\nu(T_{\lambda_S} > t)\mu_{\Lambda_h, -, h}$$

we get, for  $h < h_0$ ,

$$\begin{aligned} & d_{\text{TV}}\left(\mathbb{P}_\nu(X(T_{\lambda_S}) = \cdot \mid T_{\lambda_S} \leq t), \mu_{\Lambda_h, -, h}\right) \\ & \leq \frac{d_{\text{TV}}\left(\mathbb{P}_\nu(X(T_{\lambda_S}) = \cdot), \mu_{\Lambda_h, -, h}\right) + \mathbb{P}_\nu(T_{\lambda_S} > t)d_{\text{TV}}\left(\mathbb{P}_\nu(X(T_{\lambda_S}) = \cdot \mid T_{\lambda_S} > t), \mu_{\Lambda_h, -, h}\right)}{\mathbb{P}_\nu(T_{\lambda_S} \leq t)} \\ & \leq \frac{d_{\text{TV}}\left(\mathbb{P}_\nu(X(T_{\lambda_S}) = \cdot), \mu_{\Lambda_h, -, h}\right) + d_{\text{TV}}\left(\mathbb{P}_\nu(X(T_{\lambda_S}) = \cdot \mid T_{\lambda_S} > t), \mu_{\Lambda_h, -, h}\right)}{\mathbb{P}_\nu(T_{\lambda_S} \leq t)} \\ & \leq \frac{e^{-\delta/h}}{\mathbb{P}_\nu(T_{\lambda_S} \leq t)}. \end{aligned}$$

Our goal is to prove that, with  $t = \alpha/\gamma_h$ , the total variation distance between  $\mu_{\Lambda_h, -, h}$  and the law of  $X(t)$  is smaller than  $1/e$  for  $h$  small enough. The previous observation shows that we just need to this end a uniform upper bound in  $\nu$  on  $\mathbb{P}_\nu(T_{\lambda_S} > t)$ . Indeed, with  $\epsilon$  small enough to have

$$e^{-\alpha} + 3\epsilon < \frac{1}{e},$$

if we show that for all  $\nu$

$$\mathbb{P}_\nu(T_{\lambda_S} > t) \leq e^{-\alpha} + \epsilon, \tag{49}$$

then we have, for  $h$  small enough,

$$d_{\text{TV}}\left(\mathbb{P}_\nu(X(T_{\lambda_S}) = \cdot \mid T_{\lambda_S} \leq t), \mu_{\Lambda_h, -, h}\right) \leq \frac{\epsilon}{1 - e^{-\alpha} - \epsilon} \leq \frac{\epsilon}{1 - 1/e} \leq 2\epsilon;$$

coupling  $X(T_{\lambda_S})$  conditioned to  $\{T_{\lambda_S} \leq t\}$  with a random variable  $\xi$  with law  $\mu_{\Lambda_h, -, h}$  and evolving jointly for a time  $t - T_{\lambda_S}$  two processes with generator  $\mathcal{L}_{\Lambda_h, -, h}$  starting from  $X(T_{\lambda_S})$  and  $\xi$ , we get a coupling between  $X(t)$  conditioned to  $\{T_{\lambda_S} \leq t\}$  with a random variable with law  $\mu_{\Lambda_h, -, h}$  which gives

$$d_{\text{TV}}\left(\mathbb{P}_\nu(X(t) = \cdot \mid T_{\lambda_S} \leq t), \mu_{\Lambda_h, -, h}\right) \leq 2\epsilon;$$

and, from

$$\mathbb{P}_\nu(X(t) = \cdot) = \mathbb{P}_\nu(T_{\lambda_S} \leq t)\mathbb{P}_\nu(X(t) = \cdot \mid T_{\lambda_S} \leq t) + \mathbb{P}_\nu(T_{\lambda_S} > t)\mathbb{P}_\nu(X(t) = \cdot \mid T_{\lambda_S} > t)$$

we get

$$d_{\text{TV}}\left(\mathbb{P}_\nu(X(T_{\lambda_S}) = \cdot), \mu_{\Lambda_h, -, h}\right) \leq 2\epsilon + \mathbb{P}_\nu(T_{\lambda_S} > t) \leq e^{-\alpha} + 3\epsilon \leq \frac{1}{e}.$$

We conclude by proving that, for  $h$  small enough, (49) holds for all  $\nu$ . This is provided by the monotonicity of the dynamics and the already proven part of the theorem. Starting from the uniformly minus configuration, the stopping time  $T_{\lambda_S}$  stochastically dominates all the other  $T_{\lambda_S}$  associated with different starting measures:

$$\mathbb{P}_\nu(T_{\lambda_S} > t) \leq \mathbb{P}_-(T_{\lambda_S} > t) = \mathbb{P}_-\left(T_{\lambda_S} > \frac{\alpha}{\gamma_h}\right).$$

Also,

$$\mathbb{P}_-(T_{\kappa_{\mathcal{R}}} < T_{\lambda_S} \wedge T_{\mathcal{X}^c}) \geq \mathbb{P}_{\mu_{\mathcal{R}}}(T_{\kappa_{\mathcal{R}}} < T_{\lambda_S} \wedge T_{\mathcal{X}^c})$$

and

$$\lim_{h \rightarrow 0} \mathbb{P}_{\mu_{\mathcal{R}}} (T_{\kappa_{\mathcal{R}}} < T_{\lambda_{\mathcal{S}}} \wedge T_{\mathcal{X}^c}) = 1,$$

so that, as a consequence of the third, already proven, point of the theorem,

$$\lim_{h \rightarrow 0} \mathbb{P}_- \left( T_{\lambda_{\mathcal{S}}} > \frac{\alpha}{\gamma_h} \right) = e^{-\alpha},$$

which proves (49) for  $h$  small enough and all starting measure  $\nu$ .  $\square$

## 5.2 Proof of Corollary 1.5

It is sufficient to prove that, starting from  $\nu$ , and for  $B_+$  close enough to  $B_c$ , the event  $\{T_{\kappa_{\mathcal{R}}} > T_1\}$ , with

$$T_1 = T_{\lambda_{\mathcal{S}}} \wedge T_{\mathcal{X}^c},$$

has an exponentially small probability. In the case of the macroscopic droplet, it is proven in the same way that we proved Lemma 3.2: Lemma 2.8 provides the free energy lower bounds on the probability  $\mu_{\Lambda_{h,-,h}}(\mathcal{R}, M > m(B_{\max}/h)^2)$  while Lemma 3.1 and Lemma 3.3 provide the free energy upper bounds on  $\mu((R \cup S)^c)$  and  $\mu(R \cap S)$ . These bounds give that, for  $B_+$  close enough to  $B_c$ , the hitting time of  $\mathcal{S}$  and  $\mathcal{X}^c$  are exponentially larger than  $1/\kappa$  with a probability that is exponentially close to 1 when starting from  $\mu(\cdot \mid \mathcal{R}, M > m(B_{\max}/h)^2)$ .

Then, we only have to deal with the cases  $c < 1$  and  $c > 1$ . We first consider the latter:  $h' > h$ . Using monotonicity we have that  $T'_1$ , obtained by evolving the dynamics with  $h'$  is dominated by  $T_1$ , associated with  $h$ . But  $T'_1$  is asymptotically exponential and of the order of  $t_{\text{mix},h'}$ . This solves the case  $c > 1$  by choosing  $1/\kappa \ll \exp\{A/(ch)\}$ .

In the case  $c < 1$ , so that  $h' < h$ , consider two dynamics starting from  $\mu_{\mathcal{R}}$ , one evolving with  $h'$  the other one with  $h$ . The latter dominates the former, which, as a consequence of the previous case ( $c > 1$ ), will relax towards  $\nu$ , before the escape from metastability for the first system. This shows that  $\mu_{\mathcal{R}}$  dominates  $\nu$ . Then  $T'_1$ , associated with the starting distribution  $\nu$ , dominates  $T_1^{\mu_{\mathcal{R}}}$ , associated with the starting distribution  $\mu_{\mathcal{R}}$ . This provides the required lower bound on  $T_1$ .  $\square$

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