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A revisit of the Borch rule for the Principal-Agent Risk-Sharing problem

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Abstract

In this paper we provide a new approach to tackle the Principal-Agent Risk-Sharing problem using optimal stochastic control technics. Our analysis relies on an optimal decomposition of the expected utility of the Principal in terms of the reservation utility of the Agent. In particular, this allows us to derive the Borch rule as a necessary optimality condition for this decomposition to hold, which sheds a new light on this economic concept. As a by-product, this approach provides a class of risk-sharing plans that satisfy the Borch rule; class to which the optimal plan belongs.

Keywords : Principal Agent problem; Risk-Sharing; Borch rule; Reverse Hölder inequality; Optimal Contracting Theory.

1 Introduction

Many economic situations in the framework of optimal contracting or incentive policy design can be gathered under the Principal-Agent formulation, where an Agent is asked to perform an action on behalf of a Principal in exchange for a wage. A huge part of the literature is dedicated to situations where the Principal and the Agent do not have access to the same information such as the Moral Hazard problem. To measure the impact of the asymmetry of information, it is enlightening to compare the optimal contract with the optimal Risk-Sharing rule where the Principal dictates the Agent's actions and guarantees a reservation utility to him. The Risk-Sharing problem is often thought to be easier than the Moral Hazard one since a classical methodology whose main ingredient is the Borch rule (which has been introduced by Borch in [1] in a reinsurance setting) is available in most simple models. However, the derivation of this rule (in the context of the Principal-Agent formulation) is generally obtained through an optimisation problem which is a much stronger version of the original one. Incidentally, both problems share the same solution in spite of their intrinsic differences (we will elaborate on this point below).

In this paper, we provide a purely stochastic control theory approach to tackle the Risk-Sharing

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Principal-Agent problem (without any use of an alternative a priori ill-posed optimisation formulation), and derive the Borch rule as an optimality condition for an optimal decomposition of the utility of the Principal in terms of the reservation utility of the Agent. Before presenting our analysis we briefly sketch in the next paragraph, the classical methodology for obtaining the optimal Risk-Sharing plan.

As described in Chapter 2 of [3], we study the case of a Principal wanting to delegate a task to an Agent through a bilateral one-shot relationship, regulated through a legally enforced and binding contract. Consider the simple one-period risk sharing problem between a Principal who owns a firm (or a portfolio) whose wealth at maturity $t = 1$ is subject to uncertainty, and an Agent to whom a wage is offered in exchange for a participation to the firm. More precisely, in order to reduce his exposure to uncertainty, the Principal hires the Agent at time $t = 0$ in a take-it-or-leave-it contract in which (if accepted) the Agent is asked to produce an effort $a \geq 0$ at time $t = 0$ in return for the payment of a wage W at time $t = 1$. The wealth of the Agent then becomes X^a at time $t = 1$ with

$$X^a = x_0 + a + B,$$

where x_0 is the initial wealth at time $t = 0$ and B is a standard Gaussian random variable modelling the stochastic exposure of the Principal. Note that in this simple model, we assume that the Principal fully observes both the outcome X^a (at time $t = 1$) and the action a of the Agent, meaning that the Principal actually dictates this action to the Agent.¹

To model the cost of effort for the Agent, we introduce a function κ defined on \mathbb{R}_+ and chosen to be strictly convex, continuous, non-decreasing and such that its first derivative is invertible (the reason for these hypotheses will become clear in Section 2). A simple example of such a function is the quadratic cost function which, for a fixed constant $K > 0$, is defined for any x in \mathbb{R}_+ as $\kappa(x) = K \frac{|x|^2}{2}$.

As specified earlier, the contract (which from now will be modelled by the pair (action,wage)=(a, W)) is a take-it-or-leave-it contract that will be accepted by the Agent if a Participation Constraint (PC) condition (or reservation utility constraint) given below is satisfied :

$$\mathbb{E}[U_A(W - \kappa(a))] \geq U_A(y_0), \quad (1.1)$$

where : U_A stands for the utility function of the Agent, κ models the quadratic cost of effort for the Agent as described above, and $y_0 \geq 0$ represents the level of requirement for the Agent to accept the contract.

Hence, the Risk-Sharing problem (which is nothing else than the design of an optimal contract by the Principal) simply writes as :

$$\sup_{(a,W) \text{ subject to (1.1)}} \mathbb{E}[U_P(X^a - W)], \quad (1.2)$$

where U_P is the utility function of the Principal. We choose to work with exponential utility functions both for the Principal and the Agent with respective relative risk aversion coefficients $\gamma_P > 0$ and $\gamma_A > 0$,

$$U_P(x) := -\exp(-\gamma_P x), \quad U_A(x) := -\exp(-\gamma_A x), \quad x \in \mathbb{R}.$$

¹A typical example of such a situation is when the Agent is the Principal himself, meaning that as a manager of the firm, the Principal decides his level of work and the salary he pays himself for it. Note that as the Principal decides of the action of the agent, this action must be a positive effort in the firm as $a < 0$ would mean that the Agent would sabotage the firm, which of course does not make any sense.

Now that a simple context of the Risk-Sharing problem (1.2) has been recalled we move to its resolution. A classical way of tackling it, lies in a rewriting as :

$$\sup_{(a,W)} \{ \mathbb{E} [U_P(X^a - W)] + \lambda (\mathbb{E} [U_A(W - \kappa(a))] - U_A(y_0)) \}, \quad (1.3)$$

where $\lambda > 0$ is a Lagrange multiplier. A very tempting method consists in maximising inside the expectation, by introducing a Lagrangian \mathcal{L}

$$\mathcal{L}(a, W) := U_P(X^a - W) + \lambda (U_A(W - \kappa(a)) - U_A(y_0))$$

and considering the problem

$$\sup_{(a,W)} \mathcal{L}(a, W). \quad (1.4)$$

A first order optimality condition in the W variable immediately gives that :

$$\frac{U'_P(X^{a^*} - W^*)}{U'_A(W^* - \kappa(a))} = \lambda, \quad (1.5)$$

where (a^*, W^*) is the optimal contract (if it exists).

Relation (1.5) is known as the Borch rule and was introduced by Karl Borch in a reinsurance context ([1]). Indeed, as a step away from the classical actuarial models used at the time to design insurance contracts, Karl Borch tackled the optimal reinsurance problem using tools from game theory. He considered that both parties (insurer and reinsurer) were risk-averse and used a risk sharing setting, stating in [2]: "Reinsurance contracts are based on complete confidence between both parties. If one party has information that may be relevant in the estimation of the probability distributions, it is considered as fraud, or breach of faith, if he does not make this information available to the other party." One key result from his works on the reinsurance problem is this so-called Borch rule which, from the economic point of view, states that the marginal utilities of the Principal and of the Agent are proportional for the optimal contract when in a Risk-Sharing situation.

This feature is well-known in the Economic literature. But if we look at it mathematically, it seems less transparent. Indeed, whilst the Lagrange multiplier λ in both formulations (1.3) and (1.4) agree, in (1.3) it is by nature deterministic (as we maximise expectations) whereas in (1.4) we optimise for any possible outcomes of the contingent claim B making λ dependent of these outcomes (roughly speaking λ is a function of B). The same comments apply to the effort a . Indeed, by the nature of the problem, the Principal imposes an acceptable effort at time $t = 0$ to the Agent, so a is a positive number independent of the outcomes of B . But the maximisation problem (1.4) does not take into account this constraint and could possibly lead to an optimal effort a^* which could non-trivially depend on B . This fact is indeed well-known as a measurability issue and justifies theory of stochastic optimal control as in general maximising an expectation and maximising inside an expectation do not lead to the same optima. However, in this very particular case, both problems coincide. Indeed, a first order optimality condition in a on the Lagrangian \mathcal{L} together with the Borch rule (1.5) gives that $a^* = \operatorname{argsup}_{x \in \mathbb{R}_+} (x - \kappa(x))$ is the optimal effort. For the quadratic effort function mentioned above, we obtain $a^* = \frac{1}{R}$. And then, the Borch rule again imposes the optimal contract W^* to be affine in the optimal wealth that is :

$$W^* = \frac{\gamma_P}{\gamma_A + \gamma_P} X^{a^*} + \beta^*, \quad (1.6)$$

where β^* is an explicit constant depending on the parameters of the model, *i.e.* $x_0, y_0, \gamma_P, \gamma_A$ and κ . Note that the form of the optimal wage in (1.6) is exactly a Risk-Sharing rule as the Agent

gets a fraction of the wealth of the firm plus a fixed premium β^* which ensures the participation constraint (1.1) to be in force.

As previously mentioned, we aim in this paper to provide a new approach for obtaining the Borch rule in the context of the Risk-Sharing problem (with exponential utilities) and for deriving the optimal plan. Our contribution lies in a new interpretation of the Borch rule as an optimality condition for the original problem (1.2) without using the a priori ill-posed auxiliary formulation (1.4). More precisely, it comes as a condition to optimally decompose the utility of the Principal in terms of the one of the Agent, and is mathematically a geometric relation linked to the reverse Hölder inequality. Note that we find that a family of affine contracts enjoys the geometric property and that the optimal contract is the one which minimises the fixed salary β .

We proceed as follows. First in Section 2, we describe the Principal-Agent model in continuous time that we consider. Our main results and our approach are given in Section 3. Finally, we collect the proofs of our results in Section 5.

2 The continuous time risk-sharing model

We now specify the model under interest which is a a continuous-time version of the simple one introduced in the previous section. More precisely we consider one Principal and one Agent. The Principal will provide a unique cash flow (wage) W at maturity (denoted T) to the Agent and requires in exchange an action $a = (a_t)_{t \in [0, T]}$ (that is completely monitored by the Principal) continuously in time according to the random fluctuations of the wealth of the firm. A contract will thus be a pair (Wage, Action) = (W, a) . The rigorous demonstration of our results calls for a precise mathematical framework that we give below. In particular we will impose some integrability conditions on admissible contracts that might seem at first glance a bit artificial. Hopefully, the optimal contract that we will derive (and that coincides with the one presented in with the classical economic literature) belongs to this class.

We start with the probabilistic structure that is required to define the random fluctuations of the wealth of the Principal. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a Brownian motion $B := (B_t)_{t \in [0, T]}$ is defined with its natural and completed filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$. We denote by $\mathbb{E}[\cdot]$ the expectation with respect to the probability measure \mathbb{P} .

The Agent will be asked to perform an action a continuously in time, according to the performances of the firm. Hence we introduce the set \mathcal{P} of \mathbb{F} -predictable stochastic processes $a = (a_t)_{t \in [0, T]}$ and the set of actions is given as :

$$\mathbb{H}_2 := \left\{ a = (a_t)_{t \in [0, T]} \in \mathcal{P}, \text{ s.t. } \mathbb{E} \left[\exp \left(q \int_0^T |a_t|^2 dt \right) \right] < +\infty, \forall q > 0 \right\}.$$

As we will work with exponential preferences for the Agent and the Principal, we require so-called "exponential moments" on the actions and wages. As mentioned previously, this is a technical assumption. Given a in \mathbb{H}_2 , the wealth of the principal at any intermediate t between 0 and the maturity T is given by :

$$X_t^a = x_0 + \int_0^t a_s ds + B_t, \quad t \in [0, T], \mathbb{P} - a.s., \quad (2.1)$$

where $x_0 \in \mathbb{R}$ is a fixed real number. For any action a in \mathbb{H}_2 , we set $\mathbb{F}^{X^a} := (\mathcal{F}_t^{X^a})_{t \in [0, T]}$ the natural filtration generated by X^a . In particular, we are interested in the set of $\mathcal{F}_T^{X^a}$ -measurable random

variables which provides the natural set for the wage W paid by the Principal to the Agent. More precisely, we set²

$$\mathcal{W} := \{\mathcal{F}_T - \text{measurable random variables } W, \mathbb{E}[\exp(qW)] < +\infty, \forall q \in \mathbb{R}^*\}.$$

Once again, the fact that we ask for so-called finite exponential moments of any (positive, respectively negative) order for the action (respectively for the wage) is purely technical. As we will see, the optimal contract will satisfy these technical assumptions.

As mentioned in the introduction, to model the cost of effort for the Agent we introduce a function κ defined on \mathbb{R}_+ and chosen to be strictly convex, continuous, non-decreasing and such that its first derivative is invertible. The reason for these hypotheses is the fact that the Legendre transform of this function is key in the problem's resolution. We denote it as $\tilde{\kappa}$ and recall that it is defined by the relation

$$\tilde{\kappa}(p) = \sup_{x \in \mathbb{R}_+} (px - \kappa(x)), \text{ for any } p \geq 0.$$

We now define

$$\kappa^*(p) = \operatorname{argsup}_{x \in \mathbb{R}_+} (px - \kappa(x)), \text{ for any } p \geq 0,$$

and a quick computation tells us that

$$\forall p \in \mathbb{R}_+, \quad \kappa^*(p) = (\kappa')^{-1}(p) \geq 0.$$

Two values will appear many times during the paper:

$$\kappa^*(1) = (\kappa')^{-1}(1) \quad \text{and} \quad \tilde{\kappa}(1) = (\kappa')^{-1}(1) - \kappa((\kappa')^{-1}(1)).$$

A simple example of such a function, that we mentioned in the introduction, is the quadratic cost function which, for a fixed constant $K > 0$, is defined for any x in \mathbb{R}_+ as $\kappa(x) = K \frac{|x|^2}{2}$. Using the above formulae, we know that for this function

$$\kappa^*(1) = \frac{1}{K} \quad \text{and} \quad \tilde{\kappa}(1) = \frac{1}{2K}.$$

Finally we define the set of admissible contracts :

$$\mathcal{C} := \{(W, a), W \in \mathcal{W}, a \in \mathbb{H}_2\}.$$

As explained in the introduction, the Agent will accept a contract (W, a) in \mathcal{C} if and only if the following participation constraint (PC) is satisfied :

$$\mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right] \geq U_A(y_0), \quad (2.2)$$

where y_0 is a given real number, $\kappa : \mathbb{R}^+ \rightarrow \mathbb{R}$ models the cost of effort for the agent and is as discussed above, and $U_A(x) := -\exp(-\gamma_A x)$ with $\gamma_A > 0$ the risk aversion parameter for the Agent. From now on we assume that parameters (y, γ_A) are fixed. With these notations at hand, we can state the Principal's problem which writes down in term of a classical Risk-Sharing problem as follows:

$$\sup_{(W, a) \in \mathcal{C}_{PC}} \mathbb{E} [U_P(X_T^a - W)], \quad (2.3)$$

where $U_P(x) := -\exp(-\gamma_P x)$ with $\gamma_P > 0$ fixed and where

$$\mathcal{C}_{PC} := \{(W, a) \in \mathcal{C}, (2.2) \text{ is in force}\}$$

is the set of admissible contracts satisfying the participation constraint (2.2) (we will refer to these contracts as *acceptable* contracts or acceptable Risk-Sharing plans).

² $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$

3 Main results

In this section we present our approach to tackle the Risk-Sharing problem (2.3) which relies on a decomposition of the expected utility of the Principal in terms of the one of the Agent. In particular, we derive the Borch rule as an optimal way for performing this decomposition. We first give in the section below a bound on the utility of the Principal and then derive the Borch rule which comes as a necessary condition to attain this bound.

3.1 Obtaining an upper bound for the expected utility associated to any acceptable contract

Throughout this section we fix an acceptable contract (W, a) in \mathcal{C}_{PC} . Our goal in this section is to give an upper bound on the expected utility of the Principal. The main idea is to make appear the PC (2.2) in this quantity. We proceed in several steps.

Step 1 : expressing the utility of the Principal in terms of the one of the Agent and the reserve Hölder inequality

In a first step, we express the (expected) utility of the Principal in terms of the one of the Agent. We have :

$$\begin{aligned} & \mathbb{E}[U_P(X_T^a - W)] \\ &= \mathbb{E}\left[U_P\left(X_T^a - \int_0^T \kappa(a_t)dt\right) \times \exp\left(\gamma_P\left(W - \int_0^T \kappa(a_t)dt\right)\right)\right] \\ &= \mathbb{E}\left[U_P\left(X_T^a - \int_0^T \kappa(a_t)dt\right) \times \left|U_A\left(W - \int_0^T \kappa(a_t)dt\right)\right|^{-\frac{\gamma_P}{\gamma_A}}\right]. \end{aligned} \quad (3.1)$$

We want to exploit the PC as it stands, that is in expectation. To get at least an upper bound of this quantity where the left hand side of (2.2) (that is the expected utility relative to the Agent) appears, we need some kind of Hölder inequality. However the classical Hölder inequality cannot be applied for two reasons : first the exponent $-\frac{\gamma_P}{\gamma_A}$ of the utility of the Agent is negative; and then the negativity of the mapping U_P calls for the use of a Hölder inequality in the reverse direction. These two features are taken into account in the so-called *Reverse Hölder inequality* which can be seen as a counterpart to the classical Hölder inequality. For more clarity we recall this result below in its general form.

Proposition 3.1 (Reverse Hölder inequality). *Let $p \in (1, +\infty]$. Let F and G be two random variables such that $G \neq 0$, \mathbb{P} -a.s.. Then :*

(i) *The reverse Hölder inequality holds, that is,*

$$\mathbb{E}[|F \times G|] \geq \mathbb{E}\left[|F|^{\frac{1}{p}}\right]^p \times \mathbb{E}\left[|G|^{\frac{-1}{p-1}}\right]^{-p+1}.$$

(ii) *in addition, the inequality is an equality, that is,*

$$\mathbb{E}[|F \times G|] = \mathbb{E}\left[|F|^{\frac{1}{p}}\right]^p \times \mathbb{E}\left[|G|^{\frac{-1}{p-1}}\right]^{-p+1}$$

if and only if there exists some constant (that is non-random) $\alpha \geq 0$ such that $|F| = \alpha|G|^{-\frac{p}{p-1}}$.

We will come back to the equality case given in Item (ii) later on, but at the minute we focus on the first one to proceed with our analysis. More precisely, let :

$$F := U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right), \quad G := \left| U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right|^{-\frac{\gamma_P}{\gamma_A}}. \quad (3.2)$$

Note naturally, that these two random variables depend on the contract (W, a) under interest.

We wish to apply (i) of Proposition 3.1 to F and G with some exponent p that we calibrate so that $|G|^{-\frac{1}{p-1}} = \left| U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right|$; which immediately gives $p = 1 + \frac{\gamma_P}{\gamma_A} = \frac{\gamma_A + \gamma_P}{\gamma_A} > 1$. We thus immediately obtain:

$$\begin{aligned} \mathbb{E} \left[|F|^{\frac{1}{p}} \right] &= \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_P + \gamma_A}} \right], \\ \mathbb{E} \left[|G|^{\frac{-1}{p-1}} \right] &= -\mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right]. \end{aligned}$$

Applying (i) of Proposition 3.1 to F and G with this particular choice of p in Relation (3.1) gives

$$\begin{aligned} &\mathbb{E} [U_P (X_T^a - W)] \\ &= -\mathbb{E} [|FG|] \\ &\leq \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}} \times \mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right]^{-\frac{\gamma_P}{\gamma_A}}. \end{aligned} \quad (3.3)$$

Step 2 : use of the PC (2.2)

From (3.3), the PC (2.2) gives

$$\mathbb{E} [U_P (X_T^a - W)] \leq U_A(y_0)^{-\frac{\gamma_P}{\gamma_A}} \times \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}}. \quad (3.4)$$

Step 3 : optimisation in a

From here, we obtain an upper bound for the utility of the Principal which is free of W but which still depends on a . We thus perform an optimisation in a which will provide a candidate for the optimal effort. This result (whose proof is postponed to Section 5.1) is presented in the following proposition.

Proposition 3.2. *For any (W, a) in \mathcal{C}_{PC} it holds that :*

$$\begin{aligned} &\mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}} \\ &\geq \exp(-\gamma_P x_0) \exp \left(T \gamma_P \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)} \right) \right), \end{aligned} \quad (3.5)$$

and thus

$$\mathbb{E} [U_P (X_T^a - W)] \leq U_P(x_0 - y_0) \exp \left(\gamma_P T \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)} \right) \right). \quad (3.6)$$

In addition

$$\begin{aligned} & \mathbb{E} \left[\left| U_P \left(X_T^{a^*} - \int_0^T \kappa(a_t^*) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}} \\ &= \exp(-\gamma_P x_0) \exp \left(\gamma_P T \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)} \right) \right), \end{aligned} \quad (3.7)$$

where $a_t^* := \kappa^*(1)$ for any t in $[0, T]$.

3.2 A revisit of the Borch rule and derivation of the optimal Risk-Sharing plan

We have obtained as the right-hand side of (3.6) an upper bound for the utility of the Principal for any acceptable contract. Now we would like to examine which are the contracts that attain this bound. The bound has been derived using three successive estimates : the first one is the reverse Hölder inequality (Step 1 above); the second is just the use of the PC (2.2) (Step 2); and finally the last one is to impose the constant action $a_t^* = \kappa^*(1)$ (for any t in $[0, T]$) to the Agent.

We start by examining for which contract Inequality (3.3) is indeed an equality. In fact, a necessary and sufficient condition for this to happen is given in the classical statement of the reverse Hölder inequality as Part (ii) of Proposition 3.1. It states that for a contract (W, a) , inequality in (3.3) is an equality if and only if the utilities of the Principal and the Agent to some well-chosen powers (which are given by the random variables F and G in (3.1)) are equal up to a positive deterministic constant. Rewriting this expression, we recover the Borch rule as follows :

Theorem 3.1 (Borch rule). *The inequality (3.3) is an equality for a contract (W, a) in \mathcal{C} if and only if there exists a positive constant α such that :*

$$\frac{U'_P(X_T^a - W)}{U'_A\left(W - \int_0^T \kappa(a_t) dt\right)} = \alpha, \quad \mathbb{P} - a.s.. \quad (3.8)$$

The main ideas behind the proof (which is a rewriting of Condition (ii) of Proposition 3.1 in our setting) have already been discussed above. The reader can find the details in Section 5.2. We make several remarks on this result.

Remark 3.1. *Note first that in Borch's original approach ([1]), the Borch rule appears as a necessary optimality condition for a Pareto optimal Risk-Sharing plan in a reinsurance framework. In the classical Principal-Agent formulation (as recalled in the introduction), it comes as a necessary "compatibility" condition which guarantees that the Lagrange multipliers in problems (1.3) and (1.4) are the same and, in particular, that they are deterministic. In our approach, the Borch rule comes as some sort of geometric necessary condition which allows one to optimally "extract" the PC in its expectation form from the expectation of the utility of the Principal.*

Remark 3.2. *The Borch rule above characterises the "optimal" contracts in some way. However note that it is valid for any contract in \mathcal{C} and not necessarily in \mathcal{C}_{PC} . In other words, at this stage we do not require the PC (2.2) to be satisfied. This will come in a second stage that we describe in the next section.*

3.3 From the Borch rule to the optimal contract

As in the classical approach described in the introduction, the Borch rule is a central ingredient in deriving the optimal contract. From the previous section we know that if we consider a contract

(W, a) that enjoys the Borch rule (3.8) then we have that

$$\begin{aligned} & \mathbb{E} [U_P (X_T^a - W)] \\ &= \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}} \times \mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right]^{-\frac{\gamma_P}{\gamma_A}}, \end{aligned}$$

and that

$$\begin{aligned} & \mathbb{E} [U_P (X_T^a - W)] \\ & \leq U_A(y_0)^{-\frac{\gamma_P}{\gamma_A}} \times \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}}, \end{aligned}$$

with an equality if the PC is bound by the contract. Finally, we know from Relations (3.5), (3.6) and (3.7) that choosing $a_t = a_t^* = \kappa^*(1)$ (for any t in $[0, T]$) gives

$$\begin{aligned} & \mathbb{E} \left[U_P \left(X_T^{a^*} - W \right) \right] \\ & \leq U_P(x_0 - y_0) \exp \left(\gamma_P T \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)} \right) \right), \end{aligned}$$

with equality if and only if there exists a contract of the form (W, a^*) that binds the PC (2.2). This brings us to the following question : "Can we find a wage W such that the contract (W, a^*) satisfies the Borch rule (3.8) and that satisfies the PC (2.2); and eventually that binds it ?" If yes , this contract will be an optimal Risk-Sharing plan. This is the purpose of the last result of the paper.

Theorem 3.2. *We set*

$$\beta^* := \frac{T\gamma_A|\gamma_P|^2}{2|\gamma_A + \gamma_P|^2} + y_0 + T\kappa(\kappa^*(1)) - \frac{\gamma_P}{\gamma_A + \gamma_P} (x_0 + T\kappa^*(1)).$$

(i) *Let (W, a^*) in \mathcal{C} be an acceptable contract. Then, this contract satisfies the Borch rule (3.8) if and only if there exists $\beta \in \mathbb{R}$ such that*

$$W = \frac{\gamma_P}{\gamma_A + \gamma_P} X_T^{a^*} + \beta.$$

In addition, such a contract satisfies the PC (2.2) (in other words it belongs to \mathcal{C}_{PC}) if and only if $\beta \geq \beta^$.*

(ii) *The contract (W^*, a^*) is optimal for the problem (2.3) with*

$$W^* = \frac{\gamma_P}{\gamma_A + \gamma_P} X_T^{a^*} + \beta^*.$$

Once again we postpone the proof to Section 5.3.

Remark 3.3. *Note that the optimal Risk-Sharing rule is to give the proportion $\frac{\gamma_P}{\gamma_A + \gamma_P}$ of wealth to the Agent plus the smallest fixed premium β^* which makes the contract acceptable for the Agent (that is such that the PC (2.2) is satisfied). However our approach allows us to find that not only the optimal affine contract satisfies the Borch rule.*

4 The case of a risk neutral Principal

We have considered the case of a Principal and an Agent who are risk averse. However, an important case in the literature consists in a risk neutral Principal and a risk averse Agent. More precisely, the problem (2.3)-(2.2) becomes :

$$\sup_{(W,a) \in \mathcal{C}_{PC}} \mathbb{E}[X_T^a - W], \quad (4.1)$$

where we use the same notations as previously (in particular the PC (2.2) is in force for the Agent with utility function $U_A(x) = -\exp(-\gamma_A x)$). Since our approach relies on the exponential structure of functions U_P and U_A , we cannot carry it directly in the risk neutral case. However, as it is well-known, the risk neutral framework can be seen as a limit case with formally $\gamma_P = 0$ by rescaling the mapping U_P to become $\tilde{U}_P(x) := -\frac{\exp(-\gamma_P x) - 1}{\gamma_P}$ and by letting γ_P go to 0. Hence, we can use our approach with \tilde{U}_P and U_A to derive the optimal contract for a risk neutral Principal.

Consider a contract (W, a) that satisfies the (PC). Then by Lemma 5.1 (in Section 5.4),

$$\begin{aligned} \mathbb{E}[X_T^a - W] &= \mathbb{E}\left[\lim_{\gamma_P \rightarrow 0} \tilde{U}_P(X_T^a - W)\right] \\ &= \lim_{\gamma_P \rightarrow 0} \mathbb{E}\left[\tilde{U}_P(X_T^a - W)\right] \\ &= \lim_{\gamma_P \rightarrow 0} \gamma_P^{-1} (\mathbb{E}[U_P(X_T^a - W)] + 1) \\ &\leq \lim_{\gamma_P \rightarrow 0} \gamma_P^{-1} (\mathbb{E}[U_P(X_T^{a^*} - W^*)] + 1), \end{aligned}$$

according to (ii) of Theorem 3.2. Using the explicit computation (5.1), we have that

$$\begin{aligned} \mathbb{E}[X_T^a - W] &= \mathbb{E}\left[\lim_{\gamma_P \rightarrow 0} \tilde{U}_P(X_T^a - W)\right] \\ &\leq \lim_{\gamma_P \rightarrow 0} \gamma_P^{-1} \left(U_P(x_0 - y_0) \exp\left(\gamma_P T \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)}\right)\right) + 1 \right) \\ &= \lim_{\gamma_P \rightarrow 0} \frac{\exp\left(\frac{\gamma_A \gamma_P^2 T}{2(\gamma_A + \gamma_P)}\right) \exp(-\gamma_P(x_0 - y_0 + T\tilde{\kappa}(1))) - 1}{\gamma_P} \\ &= x_0 - y_0 + T\tilde{\kappa}(1). \end{aligned}$$

So we have given the upper bound $x_0 - y_0 + T\tilde{\kappa}(1)$ to the value problem of the Risk Neutral Principal. An explicit computation gives that this upper bound can be attained by choosing the contract (a_{RN}^*, W_{RN}^*) with

$$a_{RN}^* = \kappa^*(1) \text{ and } W_{RN}^* = y_0 + T\kappa(\kappa^*(1)),$$

which is formally the optimal contract found in in Theorem 3.2 with $\gamma_P = 0$.

So as it is well-known, the Risk Sharing problem with a Risk Averse principal extends to the Risk Neutral principal by choosing the optimal wage to be a pure premium allowing the PC. This has economic meaning : the Principal is neutral to risk and is thus willing to give a fixed wage to his Agent regardless of the performance of the output process.

5 Proofs

In this section we collect the proofs of the technical results we made use of to proceed with our analysis.

5.1 Proof of Proposition 3.2

Our objective is to give an upper bound independent of a of the right hand side of Inequality (3.4). We have

$$\begin{aligned}
& \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right] \\
&= \mathbb{E} \left[\exp \left(-\gamma_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right)^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right] \\
&= \mathbb{E} \left[\exp \left(-\gamma_P \left(x_0 + \int_0^T (a_t - \kappa(a_t)) dt + B_T \right) \right)^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right] \\
&= \exp \left(-\frac{\gamma_P \gamma_A}{\gamma_A + \gamma_P} x_0 \right) \mathbb{E} \left[\exp \left(-\frac{\gamma_P \gamma_A}{\gamma_A + \gamma_P} B_T - \frac{T}{2} \frac{|\gamma_P \gamma_A|^2}{|\gamma_A + \gamma_P|^2} \right) \exp \left(\int_0^T \Phi(a_t) dt \right) \right],
\end{aligned}$$

where

$$c \mapsto \Phi(c) := -\frac{\gamma_P \gamma_A}{\gamma_A + \gamma_P} (c - \kappa(c)) + \frac{|\gamma_P \gamma_A|^2}{2|\gamma_A + \gamma_P|^2}.$$

Note that the mapping Φ is convex on \mathbb{R}_+ , and letting $a^* := \kappa^*(1)$,

$$\Phi(c) \geq \Phi(a^*) = \frac{\gamma_P \gamma_A}{\gamma_A + \gamma_P} \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)} \right), \quad \forall c \geq 0.$$

So,

$$\begin{aligned}
& \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_P + \gamma_A}} \right] \\
&\geq \exp \left(-\frac{\gamma_P \gamma_A}{\gamma_A + \gamma_P} x_0 \right) \exp \left(\frac{T \gamma_P \gamma_A}{\gamma_A + \gamma_P} \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)} \right) \right),
\end{aligned}$$

as $\mathbb{E} \left[\exp \left(-\frac{\gamma_P \gamma_A}{\gamma_A + \gamma_P} B_T - \frac{T}{2} \frac{|\gamma_P \gamma_A|^2}{|\gamma_A + \gamma_P|^2} \right) \right] = 1$ since B_T is a centered Gaussian random variable with variance T . Plugging this expression in Inequality (3.4) gives the result.

5.2 Proof of the Borch rule : Theorem 3.1

By (ii) of Proposition 3.1, Inequality (3.3) is an equality if and only the contract (W, a) is such that there exists a positive constant α such that the random variables F and G defined in (3.2) enjoys :

$$|F| = \alpha |G|^{-\frac{p}{p-1}}.$$

By definition of F , G and $p = \frac{\gamma_A + \gamma_P}{\gamma_A}$ this condition reads as :

$$\frac{\exp \left(-\gamma_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right)}{\exp \left(-(\gamma_A + \gamma_P) \left(W - \int_0^T \kappa(a_t) dt \right) \right)} = \alpha.$$

Thus

$$\alpha = \frac{\exp \left(-\gamma_P (X_T^a - W) \right)}{\exp \left(-\gamma_A \left(W - \int_0^T \kappa(a_t) dt \right) \right)}$$

$$= \frac{U'_P(X_T^a - W)}{U'_A\left(W - \int_0^T \kappa(a_t) dt\right)} \times \frac{\gamma_A}{\gamma_P},$$

which is the result by changing α to $\frac{\alpha \gamma_P}{\gamma_A}$.

5.3 Proof of Theorem 3.2

We start with an intermediate technical result, namely that any affine contract of the form (W, a^*) with

$$W = \rho X_T^{a^*} + \beta,$$

with $\rho \geq 0$ and β in \mathbb{R} is in \mathcal{C} . In other words the technical integrability conditions of an admissible contract are satisfied. Obviously, as a^* is deterministic and constant, a^* belongs to \mathbb{H}_2 . To prove that W belongs to \mathcal{W} (whose definition is given in Section 2), as β is a constant we only need to prove that $\rho X_T^{a^*}$ belongs to \mathcal{W} . Let q in \mathbb{R}^* . By definition, $X_T^{a^*} = x_0 + T\kappa^*(1) + B_T$, where B_T is a centered Gaussian random variable with variance T . Hence

$$\begin{aligned} \mathbb{E} \left[\exp(q\rho X_T^{a^*}) \right] &= \exp(q\rho(x_0 + T\kappa^*(1))) \mathbb{E} [\exp(q\rho B_T)] \\ &= \exp(q\rho(x_0 + T\kappa^*(1))) \exp\left(\frac{T|q\rho|^2}{2}\right) < +\infty. \end{aligned}$$

With this result at hand, we now prove Statement (i). Let (W, a^*) be a contract in \mathcal{C} that satisfies the Borch rule (3.8). Then, there exists a positive constant α such that

$$\frac{U'_P(X_T^a - W)}{U'_A\left(W - \int_0^T \kappa(a_t) dt\right)} = \alpha, \mathbb{P} - a.s..$$

Hence since a^* is deterministic, the quantity $\exp(W(\gamma_A + \gamma_P) - \gamma_P X_T^{a^*})$ is deterministic. In other words the random quantities in the exponential must vanish, which means that there exists a constant β such that $W = \frac{\gamma_P}{\gamma_A + \gamma_P} X_T^{a^*} + \beta$. Conversely, a direct computation proves that any contract of this form satisfies the Borch rule (3.8).

In addition let (W, a^*) of the form $W = \frac{\gamma_P}{\gamma_A + \gamma_P} X_T^{a^*} + \beta$ with β some constant. Then using that $a_t^* = \kappa^*(1)$ for any t , this contract satisfies the PC (2.2) if and only if :

$$\mathbb{E} \left[\exp \left(-\gamma_A \left(\frac{\gamma_P}{\gamma_A + \gamma_P} X_T^{a^*} + \beta \right) \right) \right] \leq \exp(-\gamma_A (y_0 + T\kappa(\kappa^*(1)))).$$

Since $X_T^{a^*} = x_0 + \frac{T}{\kappa} + B_T$, we have that :

$$\begin{aligned} &\mathbb{E} \left[\exp \left(-\gamma_A \left(\frac{\gamma_P}{\gamma_A + \gamma_P} X_T^{a^*} + \beta \right) \right) \right] \leq \exp(-\gamma_A (y_0 + T\kappa(\kappa^*(1)))) \\ \Leftrightarrow &\mathbb{E} \left[\exp \left(-\frac{\gamma_A \gamma_P}{\gamma_A + \gamma_P} B_T \right) \right] \leq \exp \left(-\gamma_A \left(y_0 + T\kappa(\kappa^*(1)) - \beta - \frac{\gamma_P}{\gamma_A + \gamma_P} (x_0 + T\kappa^*(1)) \right) \right) \\ \Leftrightarrow &\exp \left(\frac{T|\gamma_A \gamma_P|^2}{2|\gamma_A + \gamma_P|^2} \right) \leq \exp \left(-\gamma_A \left(y_0 + T\kappa(\kappa^*(1)) - \beta - \frac{\gamma_P}{\gamma_A + \gamma_P} (x_0 + T\kappa^*(1)) \right) \right), \end{aligned}$$

since B_T is a centered Gaussian random variable with variance T . So the PC is satisfied for this contract if and only if

$$\beta \geq \frac{T\gamma_A|\gamma_P|^2}{2|\gamma_A + \gamma_P|^2} + y_0 + T\kappa(\kappa^*(1)) - \frac{\gamma_P}{\gamma_A + \gamma_P} (x_0 + T\kappa^*(1)) = \beta^*.$$

We turn to Point (ii). The previous computations prove that for $\beta = \beta^*$ the PC (2.2) is bound. Hence the contract (W^*, a^*) with the wage $W^* = \frac{\gamma_P}{\gamma_A + \gamma_P} X_T^{a^*} + \beta^*$ satisfies the Borch rule and binds the PC (and the action is a^*) so by the considerations of Section 3 it is the optimal contract, or put it differently we have that

$$\mathbb{E} \left[U_P \left(X_T^{a^*} - W^* \right) \right] = U_P(x_0 - y_0) \exp \left(\gamma_P T \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)} \right) \right), \quad (5.1)$$

so the upper bound given by the right-hand side (3.6) is attained.

Remark 5.1. *The optimum binds the participation constraint. Indeed when choosing β^* in such a way as to reach the lower bound, we are led to choose the smallest β such that the participation constraint holds and this means choosing β such that the constraint holds in equality, i.e.*

$$\mathbb{E} \left[\exp \left(-\gamma_A \left(W^* - \int_0^T \kappa(a^*) dt \right) \right) \right] = \exp(-\gamma_A y).$$

This makes sense from an economic point of view: without the constraint, the Principal would probably want the Agent to work as much as possible for as little as possible so any global optimum would not satisfy the participation constraint. The optimum for our problem therefore binds the PC.

5.4 A technical lemma

Lemma 5.1. *Let (a, W) be an admissible contract in \mathcal{C} . The sequence of random variables $\left(\tilde{U}_P(X_T^a - W) \right)_{0 < \gamma_P < 1}$ is uniformly integrable. And so :*

$$\mathbb{E} [X_T^a - W] = \mathbb{E} \left[\lim_{\gamma_P \rightarrow 0} \tilde{U}_P(X_T^a - W) \right] = \lim_{\gamma_P \rightarrow 0} \mathbb{E} \left[\tilde{U}_P(X_T^a - W) \right].$$

Proof. The second part of the statement is a consequence of the uniform integrability (UI) and of the fact that the identity mapping is the limit (as γ_P goes to 0) of \tilde{U}_P . So we focus on the UI property and apply de la Vallée-Poussin criterion. We have :

$$\begin{aligned} & \sup_{0 < \gamma_P < 1} \mathbb{E} \left[|\tilde{U}_P(X_T^a - W)|^2 \right] \\ &= \gamma_P^{-2} \sup_{0 < \gamma_P < 1} \mathbb{E} \left[|\exp(-\gamma_P(X_T^a - W)) - 1|^2 \right] \\ &= \sup_{0 < \gamma_P < 1} \mathbb{E} \left[|\bar{X}|^2 |\exp(-\gamma_P \bar{X})|^2 \right], \end{aligned}$$

where \bar{X} is a random point between 0 and $X_T^a - W$ (using mean value theorem). By Cauchy-Schwarz's inequality we have,

$$\begin{aligned} & \sup_{0 < \gamma_P < 1} \mathbb{E} \left[|\tilde{U}_P(X_T^a - W)|^2 \right] \\ & \leq \mathbb{E} \left[|\bar{X}|^4 \right]^{1/2} \sup_{0 < \gamma_P < 1} \mathbb{E} \left[\exp(-4\gamma_P \bar{X}) \right]^{1/2}. \end{aligned}$$

As $|\bar{X}| \leq |X_T^a - W|$, \mathbb{P} -a.s., we have that $\mathbb{E} [|\bar{X}|^4] < +\infty$. Regarding the second term,

$$\begin{aligned} & \sup_{0 < \gamma_P < 1} \mathbb{E} \left[\exp(-4\gamma_P \bar{X}) \right] \\ & \leq \sup_{0 < \gamma_P < 1} \left(\mathbb{P} [\bar{X} \geq 0] + \mathbb{E} \left[\exp(-4\gamma_P \bar{X}) \mathbf{1}_{\bar{X} < 0} \right] \right) \\ & \leq 1 + \mathbb{E} \left[\exp(-4R\bar{X}) \mathbf{1}_{\bar{X} < 0} \right] \\ & \leq 1 + \mathbb{E} \left[\exp(4R|X_T^a - W|) \right] < +\infty. \end{aligned}$$

□

6 Conclusion

This paper uses the Reverse Hölder inequality to derive a new approach to the Risk-Sharing Principal-Agent problem. Through a specific decomposition of the Principal's expected utility (that relies of the multiplicative property of exponential utility functions) we are able to extract the participation constraint in its expectation form. We are then able to compute the optimal risk-sharing plan whilst also making the Borch rule appear. During this reasoning, we actually show that a whole class of risk-sharing plans satisfies the Borch rule. The optimal plan amongst this class is the one which binds the Agent's participation constraint.

We note that this new approach gives a new meaning to the Borch rule: it appears as some sort of geometric condition allowing us to extract the PC in its expectation form from the expectation of the utility of the Principal.

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