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Two-temperature homogenized eigenfunctions of conduction through domains with jump interfaces

Isabelle Gruais, Dan Poliševski and Alina Ştefan

Abstract. In this paper we study the asymptotic behavior of the eigenvalue problem solutions of the conduction process in an ε -periodic domain formed by two components separated by a first-order jump interface. We prove that when $\varepsilon \rightarrow 0$ the limits of the eigenvalues and eigenfunctions of this problem verify a certain (effective) two-temperature eigenvalue problem. Moreover, we show that the effective eigenvalue problem has only eigenvalues which come from the homogenization process.

Keywords: interfacial jump, conduction, eigenvalue, two-scale convergence, homogenization

MSC 2010: 35B27, 35P15, 49R05, 74A50, 80M40

1 Introduction

During the last decades there was a steady interest for the homogenization of problems with interfacial thermal barriers (see [2], [14], [7], [5]) or equivalent (see [13], [11], [8], [15]). Meanwhile, it was also studied the asymptotic behavior of the eigenvalue problems in ε -periodic domains (see [12]), even for ε -periodically perforated domains (see [17]), where the key ingredient was the prolongation operator which was introduced by [6].

Here we continue our works [9], [10] and [16] by studying the asymptotic behavior of the eigenvalue problem solutions of the conduction process in an ε -periodic domain formed by two components separated by a first-order jump interface. We prove that when $\varepsilon \rightarrow 0$ the limits of the eigenvalues and eigenfunctions of this problem verify a certain (effective) two-temperature eigenvalue problem. Moreover, we show that the effective eigenvalue problem has only eigenvalues which come from the homogenization process. Our key ingredient is a pair of prolongation operators, corresponding to each component of the domain. Otherwise, we mainly follow the methods of the two-scale theory (see [1]). We have to remark that the procedure presented here can straightforwardly be generalized to any n -component ε -periodic domain with interfacial jumps of the first order.

The paper is organized as follows: in Section 2 we study the eigenvalue problem in the ε -periodic domain; in Section 3 we present the key prolongation operators, the a priori estimates and the specific compactness results; Section 4 is devoted to the derivation of the effective eigenvalue problem and its connection with the homogenization process.

2 The eigenvalue problem

Let Ω be an open connected bounded set in \mathbb{R}^N ($N \geq 3$), locally located on one side of the boundary $\partial\Omega$, a Lipschitz manifold composed of a finite number of connected components. For any $\varepsilon \in (0, 1)$, Ω has two ε -periodically distributed components. For convenience, the periodicity is described by using the cube $Y = (0, 1)^N$, as follows:

Let $Y_a \subset\subset Y$ be a Lipschitz open set such that $Y_b = Y \setminus \overline{Y}_a$ is connected. For any $\varepsilon \in (0, 1)$ we denote

$$\mathbb{Z}_\varepsilon = \{k \in \mathbb{Z}^N : \varepsilon k + \varepsilon Y \subseteq \Omega\}. \quad (1)$$

The two ε -periodic components of Ω are defined by:

$$\Omega_{\varepsilon a} = \text{int} \left(\bigcup_{k \in \mathbb{Z}_\varepsilon} (\varepsilon k + \varepsilon \overline{Y}_a) \right) \quad (2)$$

$$\Omega_{\varepsilon b} = \Omega \setminus \overline{\Omega}_{\varepsilon a}. \quad (3)$$

Denoting $\Gamma := \partial Y_a = \partial Y_a \cap \partial Y_b$, the interface between $\Omega_{\varepsilon a}$ and $\Omega_{\varepsilon b}$ have the property:

$$\Gamma_\varepsilon := \bigcup_{k \in \mathbb{Z}_\varepsilon} (\varepsilon k + \varepsilon \Gamma) = \partial \Omega_{\varepsilon a} = \partial \Omega_{\varepsilon a} \cap \partial \Omega_{\varepsilon b}. \quad (4)$$

Let us remark that $\Omega_{\varepsilon b}$ is connected and all the boundaries are at least locally Lipschitz. Also, the inward normal on ∂Y_a , denoted by ν , has the property

$$\nu^\varepsilon(x) = \nu(\{\varepsilon^{-1}x\}), \quad \forall x \in \Gamma_\varepsilon, \quad (5)$$

where $\{\varepsilon^{-1}x\}$ is formed by the fractional parts of the components of $\varepsilon^{-1}x$.

We have to introduce the Hilbert space

$$H_\varepsilon = \left\{ u \in L^2(\Omega) : u|_{\Omega_{\varepsilon a}} \in H^1(\Omega_{\varepsilon a}), u|_{\Omega_{\varepsilon b}} \in H^1(\Omega_{\varepsilon b}), u = 0 \text{ on } \partial\Omega \right\} \quad (6)$$

endowed with the scalar product

$$(u, v)_{H_\varepsilon} = \int_{\Omega_{\varepsilon a}} \nabla u \nabla v + \int_{\Omega_{\varepsilon b}} \nabla u \nabla v + \varepsilon \int_{\Gamma_\varepsilon} [u][v], \quad (7)$$

where $[u] = \gamma_{\varepsilon a} u - \gamma_{\varepsilon b} u$ and $\gamma_{\varepsilon a} u$, $\gamma_{\varepsilon b} u$ are the traces of u on Γ_ε defined in $H^1(\Omega_{\varepsilon a})$ and $H^1(\Omega_{\varepsilon b})$, respectively.

Our domain has the following well-known property [8]:

Lemma 2.1. *For any $v \in H_\varepsilon$ there exists $C > 0$, independent of ε , such that*

$$|v|_{L^2(\Omega_{\varepsilon b})} \leq C |\nabla v|_{L^2(\Omega_{\varepsilon b})}, \quad (8)$$

$$\varepsilon^{1/2} |\gamma_{\varepsilon \alpha} v|_{L^2(\Gamma_\varepsilon)} \leq C \left(|v|_{L^2(\Omega_{\varepsilon \alpha})} + \varepsilon |\nabla v|_{L^2(\Omega_{\varepsilon \alpha})} \right), \quad \alpha \in \{a, b\}, \quad (9)$$

$$|v|_{L^2(\Omega_{\varepsilon a})} \leq C \left(\varepsilon^{1/2} |\gamma_{\varepsilon a} v|_{L^2(\Gamma_\varepsilon)} + \varepsilon |\nabla v|_{L^2(\Omega_{\varepsilon a})} \right). \quad (10)$$

Remark 2.1. *Taking in account the L^2 -norm of the jump on Γ_ε the results of the previous Lemma have important consequences:*

$$\varepsilon^{1/2} |[v]|_{L^2(\Gamma_\varepsilon)} \leq C \left(|v|_{L^2(\Omega)} + \varepsilon |\nabla v|_{L^2(\Omega_{\varepsilon a})} + \varepsilon |\nabla v|_{L^2(\Omega_{\varepsilon b})} \right), \quad (11)$$

$$|v|_{L^2(\Omega_{\varepsilon a})} \leq C |v|_{H_\varepsilon}, \quad \forall v \in H_\varepsilon. \quad (12)$$

Next, we introduce the data of our problem: the transmission factor $h^\varepsilon(x) = h(x/\varepsilon)$ and the symmetric conductivities $a_{ij}^\varepsilon(x) = a_{ij}(x/\varepsilon)$ and $b_{ij}^\varepsilon(x) = b_{ij}(x/\varepsilon)$, where h, a_{ij} and b_{ij} belong to $L_{per}^\infty(Y)$ and have the property that there exists $\delta > 0$ such that

$$h \geq \delta, \quad \text{a.e. on } Y, \quad (13)$$

$$a_{ij} \xi_j \xi_i \geq \delta \xi_i \xi_i \quad \text{and} \quad b_{ij} \xi_j \xi_i \geq \delta \xi_i \xi_i, \quad \forall \xi \in \mathbb{R}^N, \quad \text{a.e. on } Y. \quad (14)$$

We consider the following eigenvalue problem:

Find $\lambda^\varepsilon \in \mathbb{R}^*$ such that $\exists u^\varepsilon \in H_\varepsilon \setminus \{0\}$ verifying the equations

$$-\operatorname{div}(a^\varepsilon \nabla u^\varepsilon) = \lambda^\varepsilon u^\varepsilon, \quad \text{in } \Omega_{\varepsilon a}, \quad (15)$$

$$-\operatorname{div}(b^\varepsilon \nabla u^\varepsilon) = \lambda^\varepsilon u^\varepsilon, \quad \text{in } \Omega_{\varepsilon b}, \quad (16)$$

and the transmission conditions

$$a_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j} \nu_i^\varepsilon = b_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j} \nu_i^\varepsilon = \varepsilon h^\varepsilon (\gamma_{\varepsilon a} u^\varepsilon - \gamma_{\varepsilon b} u^\varepsilon) \quad \text{on } \Gamma_\varepsilon. \quad (17)$$

The variational formulation of the problem (15)-(17) is the following:

Find $\lambda^\varepsilon \in \mathbb{R}^*$ such that $\exists u^\varepsilon \in H_\varepsilon \setminus \{0\}$ verifying

$$G_\varepsilon(u^\varepsilon, v) := \int_{\Omega_{\varepsilon a}} a_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j} \frac{\partial v}{\partial x_i} + \int_{\Omega_{\varepsilon b}} b_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j} \frac{\partial v}{\partial x_i} + \varepsilon \int_{\Gamma_\varepsilon} h^\varepsilon [u^\varepsilon][v] = \lambda^\varepsilon (u^\varepsilon, v), \quad \forall v \in H_\varepsilon \quad (18)$$

where (\cdot, \cdot) denotes the inner product in $L^2(\Omega)$.

Using the procedure of [3], we introduce the operator $T^\varepsilon \in \mathcal{L}(L^2(\Omega), H_\varepsilon)$ by denoting with $T^\varepsilon(u)$, for $u \in L^2(\Omega)$, the unique solution of the problem

$$\int_{\Omega_{\varepsilon a}} a_{ij}^\varepsilon \frac{\partial T^\varepsilon(u)}{\partial x_j} \frac{\partial v}{\partial x_i} + \int_{\Omega_{\varepsilon b}} b_{ij}^\varepsilon \frac{\partial T^\varepsilon(u)}{\partial x_j} \frac{\partial v}{\partial x_i} + \varepsilon \int_{\Gamma_\varepsilon} h^\varepsilon [T^\varepsilon(u)][v] = \int_{\Omega} uv, \quad \forall v \in H_\varepsilon.$$

Defining $\tilde{T}^\varepsilon : L^2(\Omega) \rightarrow L^2(\Omega)$ by $\tilde{T}^\varepsilon = J^\varepsilon \circ T^\varepsilon$, where J^ε is the inclusion of H_ε into $L^2(\Omega)$, we see that the eigenvalue problem (18) is equivalent to the eigenvalue problem

$$\tilde{T}^\varepsilon u^\varepsilon = \mu^\varepsilon u^\varepsilon, \text{ via } \mu^\varepsilon = \frac{1}{\lambda^\varepsilon}.$$

Lemma 2.2. *The inclusion $J^\varepsilon : H_\varepsilon \rightarrow L^2(\Omega)$ is a compact operator.*

Proof. As for any $v \in H_\varepsilon$ we have

$$|J^\varepsilon v|_{L^2(\Omega)} = |v|_{\Omega_{\varepsilon a}} + |v|_{\Omega_{\varepsilon b}} \leq C|v|_{H_\varepsilon},$$

is sufficient to prove that the bounded sequences from H_ε contain a convergent subsequence in $L^2(\Omega)$.

Let $\{v_n\}_n$ a bounded sequence in H_ε . We note $v_n^a = v_n|_{\Omega_{\varepsilon a}}$ and $v_n^b = v_n|_{\Omega_{\varepsilon b}}$. Since $\{v_n^a\}_n$ is a bounded sequence in $H^1(\Omega_{\varepsilon a})$, from the Rellich's theorem there exist $v^a \in H^1(\Omega_{\varepsilon a})$ and a subsequence, still denoted by $\{n\}$, such that

$$v_n^a \rightarrow v^a \text{ strongly in } L^2(\Omega_{\varepsilon a}).$$

Further, $\{v_n^b\}_n$ being bounded in $H^1(\Omega_{\varepsilon b})$, again the Rellich's theorem implies the existence of some $v^b \in H^1(\Omega_{\varepsilon b})$ such that on a sub-subsequence it holds

$$v_n^b \rightarrow v^b \text{ strongly in } L^2(\Omega_{\varepsilon b}).$$

It follows that

$$v_0 = \begin{cases} v^a & \text{in } \Omega_{\varepsilon a}, \\ v^b & \text{in } \Omega_{\varepsilon b} \end{cases} \Rightarrow v_0 \in H_\varepsilon \subset L^2(\Omega).$$

The proof is completed as

$$|v_n - v_0|_{L^2(\Omega)}^2 = |v_n^a - v^a|_{L^2(\Omega_{\varepsilon a})}^2 + |v_n^b - v^b|_{L^2(\Omega_{\varepsilon b})}^2 \rightarrow 0.$$

■

We see now that \tilde{T}^ε is a self-adjoint, compact operator in $L^2(\Omega)$ and recalling for instance [4] it follows that there exist $\{\lambda_k^\varepsilon\}_k$, eigenvalues of the problem (18), with the property

$$0 < \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \dots \rightarrow \infty$$

and $\{u_k^\varepsilon\}_k$, the corresponding eigenfunctions, which are complete and orthonormal in $L^2(\Omega)$.

In the following sections we shall study the behaviour of $(\lambda_k^\varepsilon, u_k^\varepsilon)$ when $\varepsilon \rightarrow 0$.

3 A priori estimates

We begin this section by proving the boundedness of $\{\lambda_k^\varepsilon\}_\varepsilon$, the eigenvalues of (18). For every $k \in \mathbb{N}$ let us denote

$$H_{\varepsilon,k} = \{S \text{ subspace of } H_\varepsilon, \dim S = k\}.$$

Applying the Minimum-maximum principle (see [3]), λ_k^ε can be estimated via the Rayleigh quotient:

$$\begin{aligned} \lambda_k^\varepsilon &= \min_{S \in H_{\varepsilon,k}} \max_{v \in S} \frac{G_\varepsilon(v, v)}{|v|_{L^2(\Omega)}^2} \leq \\ &\leq C \min_{S \in H_{\varepsilon,k}} \max_{v \in S} \frac{|\nabla v|_{L^2(\Omega_{\varepsilon a})}^2 + |\nabla v|_{L^2(\Omega_{\varepsilon b})}^2 + \varepsilon |[v]|_{L^2(\Gamma_\varepsilon)}^2}{|v|_{L^2(\Omega)}^2}. \end{aligned} \quad (19)$$

In order to further estimate λ_k^ε with respect to ε we introduce two prolongation operators.

First, for any $v \in H^1(Y_a)$ let $w \in H^1(Y_b)$ be the only solution of the Dirichlet problem:

$$-\nabla w = 0 \quad \text{in } Y_b, \quad (20)$$

$$w = 0 \quad \text{on } \partial Y, \quad w = v \quad \text{on } \Gamma. \quad (21)$$

We have to introduce here $P_a(v) \in H^1(Y)$ by

$$P_a(v) = \begin{cases} v & \text{in } Y_a, \\ w & \text{in } Y_b. \end{cases}$$

It has the property

$$|P_a(v)|_{H^1(Y)} \leq C |v|_{H^1(Y_a)}. \quad (22)$$

Denoting $u_k^\varepsilon(y) := u(\varepsilon k + \varepsilon y)$, for any $k \in \mathbb{Z}_\varepsilon$, $y \in Y_a$ and $u \in H^1(\Omega_{\varepsilon a})$, we define our first prolongation operator, $P_a^\varepsilon : H^1(\Omega_{\varepsilon a}) \rightarrow H_0^1(\Omega)$, by

$$P_a^\varepsilon(u)(x) = \begin{cases} u(x) & \text{for } x \in \Omega_{\varepsilon a} \\ P_a(u_k^\varepsilon)\left(\{\frac{x}{\varepsilon}\}\right) & \text{for } x \in (\varepsilon k + \varepsilon Y_b), k \in \mathbb{Z}_\varepsilon \\ 0 & \text{for } x \in \Omega \setminus \bigcup_{k \in \mathbb{Z}_\varepsilon} (\varepsilon k + \varepsilon \bar{Y}) \end{cases}$$

Rescaling (22) we easily obtain

$$|P_a^\varepsilon(u)|_{H^1(\Omega)} \leq C \left(|u|_{L^2(\Omega_{\varepsilon a})} + \varepsilon |\nabla u|_{L^2(\Omega_{\varepsilon a})} \right), \quad \forall v \in H^1(\Omega_{\varepsilon a}). \quad (23)$$

Second, for any $v \in H^1(Y_b)$ let $w \in H^1(Y_b)$ be the only solution of the Dirichlet problem:

$$-\nabla w = 0 \quad \text{in } Y_a, \quad (24)$$

$$w = v \text{ on } \Gamma. \quad (25)$$

For this component we introduce $P_b(v) \in H^1(Y)$ by

$$P_b(v) = \begin{cases} w & \text{in } Y_a, \\ v & \text{in } Y_b, \end{cases}$$

which has a property similar to (22):

$$|P_b(v)|_{H^1(Y)} \leq C|v|_{H^1(Y_b)}. \quad (26)$$

Denoting $u_k^\varepsilon(y) := u(\varepsilon k + \varepsilon y)$, for any $k \in \mathbb{Z}_\varepsilon$, $y \in Y_b$, $u \in H^1(\Omega_{\varepsilon b})$, $u = 0$ on $\partial\Omega$, we define our second prolongation operator, $P_b^\varepsilon : \{u \in H^1(\Omega_{\varepsilon b}), u=0 \text{ on } \partial\Omega\} \rightarrow H_0^1(\Omega)$, by

$$P_b^\varepsilon(u)(x) = \begin{cases} u(x) & \text{for } x \in \Omega_{\varepsilon b} \\ P_b(u_k^\varepsilon)\left(\left\{\frac{x}{\varepsilon}\right\}\right) & \text{for } x \in (\varepsilon k + \varepsilon Y_a), k \in \mathbb{Z}_\varepsilon. \end{cases}$$

Rescaling (26) we get

$$|P_b^\varepsilon(u)|_{H^1(\Omega)} \leq C \left(|u|_{L^2(\Omega_{\varepsilon b})} + \varepsilon |\nabla u|_{L^2(\Omega_{\varepsilon b})} \right), \quad \forall u \in H^1(\Omega_{\varepsilon b}), u=0 \text{ on } \partial\Omega. \quad (27)$$

We can now estimate the terms of (19).

The first term gives:

$$\frac{|\nabla v|_{L^2(\Omega_{\varepsilon a})}^2}{|v|_{L^2(\Omega)}^2} \leq \frac{|\nabla P_a^\varepsilon(v)|_{L^2(\Omega_{\varepsilon a})}^2}{|P_a^\varepsilon(v)|_{L^2(\Omega)}^2} \cdot \frac{|P_a^\varepsilon(v)|_{L^2(\Omega)}^2}{|v|_{L^2(\Omega)}^2} < C \frac{|P_a^\varepsilon(v)|_{L^2(\Omega)}^2}{|v|_{L^2(\Omega)}^2},$$

the constant being the first eigenvalue of the problem (20)-(21). Using (23) we obtain

$$\frac{|\nabla v|_{L^2(\Omega_{\varepsilon a})}^2}{|v|_{L^2(\Omega)}^2} < C_1 + \varepsilon^2 C_2 \frac{\int_{\Omega_{\varepsilon a}} a^\varepsilon \nabla v \nabla v}{|v|_{L^2(\Omega)}^2}. \quad (28)$$

The second term gives:

$$\frac{|\nabla v|_{L^2(\Omega_{\varepsilon b})}^2}{|v|_{L^2(\Omega)}^2} \leq \frac{|\nabla P_b^\varepsilon(v)|_{L^2(\Omega_{\varepsilon b})}^2}{|P_b^\varepsilon(v)|_{L^2(\Omega)}^2} \cdot \frac{|P_b^\varepsilon(v)|_{L^2(\Omega)}^2}{|v|_{L^2(\Omega)}^2} < C \frac{|P_b^\varepsilon(v)|_{L^2(\Omega)}^2}{|v|_{L^2(\Omega)}^2},$$

the constant being the first eigenvalue of the problem (24)-(25). Using (27) we obtain

$$\frac{|\nabla v|_{L^2(\Omega_{\varepsilon b})}^2}{|v|_{L^2(\Omega)}^2} < C_1 + \varepsilon^2 C_2 \frac{\int_{\Omega_{\varepsilon b}} b^\varepsilon \nabla v \nabla v}{|v|_{L^2(\Omega)}^2}. \quad (29)$$

The third term can be estimated from (11), that is

$$\frac{|[v]|_{L^2(\Gamma_\varepsilon)}^2}{|v|_{L^2(\Omega)}^2} < C_1 + \varepsilon^2 C_2 \frac{\int_{\Omega_{\varepsilon a}} a^\varepsilon \nabla v \nabla v}{|v|_{L^2(\Omega)}^2} + \varepsilon^2 C_2 \frac{\int_{\Omega_{\varepsilon b}} b^\varepsilon \nabla v \nabla v}{|v|_{L^2(\Omega)}^2}. \quad (30)$$

Adding the estimates (28)-(30) we get

$$\lambda_k^\varepsilon \leq C_1 + \varepsilon^2 C_2 \lambda_k^\varepsilon,$$

which obviously implies that λ_k^ε is bounded for sufficiently small ε .

Setting $v = u_k^\varepsilon$ in (18) and using the coerciveness of $G_\varepsilon(\cdot, \cdot)$ and the orthonormality of $\{u_k^\varepsilon\}_k$ we find

$$\begin{aligned} G_\varepsilon(u_k^\varepsilon, u_k^\varepsilon) &= \int_{\Omega_{\varepsilon a}} a_{ij}^\varepsilon \frac{\partial u_k^\varepsilon}{\partial x_j} \frac{\partial u_k^\varepsilon}{\partial x_i} dx + \int_{\Omega_{\varepsilon b}} b_{ij}^\varepsilon \frac{\partial u_k^\varepsilon}{\partial x_j} \frac{\partial u_k^\varepsilon}{\partial x_i} dx + \\ &+ \varepsilon \int_{\Gamma_\varepsilon} h^\varepsilon [u_k^\varepsilon]^2 ds = \lambda_k^\varepsilon, \quad \forall u^\varepsilon \in H_\varepsilon. \end{aligned} \quad (31)$$

As $\{\lambda_k^\varepsilon\}_\varepsilon$ is bounded, we find that

$$\{u_k^\varepsilon\}_\varepsilon \text{ is bounded in } H_\varepsilon. \quad (32)$$

Using the previously obtained result together with inequalities (8)-(10), it follows that there exists $C > 0$ such that

$$|\nabla u_k^\varepsilon|_{L^2(\Omega_{\varepsilon a})} \leq C, \quad |\nabla u_k^\varepsilon|_{L^2(\Omega_{\varepsilon b})} \leq C, \quad |[u_k^\varepsilon]|_{L^2(\Gamma_\varepsilon)} \leq C. \quad (33)$$

Applying to the properties of the two-scale convergence theory [1], a specific compactness result follows.

Theorem 3.1. *There exist $\lambda_k \in \mathbb{R}_+^*$, $u_k^\alpha \in H^1(\Omega)$, $u_k^b \in H_0^1(\Omega)$ and $\eta_k^\alpha \in L^2(\Omega; \tilde{H}_{per}^1(Y_\alpha))$, $\alpha \in \{a, b\}$, such that the following convergences hold on some subsequence*

$$\chi_\alpha^\varepsilon u_k^\varepsilon \xrightarrow{2s} \chi_\alpha u_k^\alpha, \quad (34)$$

$$\chi_\alpha^\varepsilon \nabla u_k^\varepsilon \xrightarrow{2s} \chi_\alpha (\nabla_x u_k^\alpha + \nabla_y \eta_k^\alpha(\cdot, y)), \quad (35)$$

$$\lambda_k^\varepsilon \rightarrow \lambda_k, \quad (36)$$

where $\chi_\alpha^\varepsilon : L^2(\Omega_{\varepsilon\alpha}) \rightarrow L^2(\Omega)$ and $\chi_\alpha : L^2(\Omega \times Y_\alpha) \rightarrow L^2(\Omega \times Y)$, $\alpha \in \{a, b\}$, denote the straight prolongations with zero; sometimes they can be identified with the corresponding characteristic functions.

4 The homogenization process

Passing (18) to the limit on the subsequence on which the convergences of Theorem 3.1 hold, we obtain like in [8]:

Lemma 4.1. *For any $\varphi_a \in C^\infty(\bar{\Omega})$, $\varphi_b \in \mathcal{D}(\Omega)$ and $\psi_\alpha \in \mathcal{D}(\Omega; C_{per}^\infty(Y_\alpha))$, $\alpha \in \{a, b\}$, it holds*

$$\begin{aligned} \int_{\Omega \times Y_a} a_{ij} \left(\frac{\partial u_k^a}{\partial x_j} + \frac{\partial \eta_k^a}{\partial y_j} \right) \left(\frac{\partial \varphi_a}{\partial x_i} + \frac{\partial \psi_a}{\partial y_i} \right) + \int_{\Omega \times Y_b} b_{ij} \left(\frac{\partial u_k^b}{\partial x_j} + \frac{\partial \eta_k^b}{\partial y_j} \right) \left(\frac{\partial \varphi_b}{\partial x_i} + \frac{\partial \psi_b}{\partial y_i} \right) + \\ + \tilde{h} \int_{\Omega} (u_k^a - u_k^b)(\varphi_a - \varphi_b) = \lambda_k \int_{\Omega \times Y} \chi_a u_k^a \varphi_a + \chi_b u_k^b \varphi_b \end{aligned} \quad (37)$$

where \tilde{h} is defined by

$$\tilde{h} = \int_{\Gamma} h(y) ds. \quad (38)$$

Proof. For φ_α and ψ_α , $\alpha \in \{a, b\}$ like in the hypotheses, we set v in (18) as follows:

$$v(x) = \left(\varphi_a(x) + \varepsilon \psi_a \left(x, \frac{x}{\varepsilon} \right), \varphi_b(x) + \varepsilon \psi_b \left(x, \frac{x}{\varepsilon} \right) \right). \quad (39)$$

We obtain

$$\begin{aligned} \lambda^\varepsilon (\chi_a^\varepsilon u^\varepsilon, \varphi_a) + \lambda^\varepsilon (\chi_b^\varepsilon u^\varepsilon, \varphi_b) + \mathcal{O}(\varepsilon) = \sum_{\alpha \in \{a, b\}} \int_{\Omega_{\varepsilon\alpha}} \alpha_{ij}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j} \left(\frac{\partial \varphi_\alpha}{\partial x_i} + \frac{\partial \psi_\alpha}{\partial y_i} \right) + \\ + \varepsilon \int_{\Gamma_\varepsilon} h^\varepsilon (\gamma_{\varepsilon a} u^\varepsilon - \gamma_{\varepsilon b} u^\varepsilon) [\varphi_a - \varphi_b + \varepsilon(\psi_a - \psi_b)]. \end{aligned} \quad (40)$$

The proof is completed by following the same steps as in the proof presented in [8]. ■

In order to present the next results we have also to introduce the Hilbert space

$$H := [H^1(\Omega) \times H_0^1(\Omega)] \times [L^2(\Omega, \tilde{H}_{per}^1(Y_a) \times L^2(\Omega, \tilde{H}_{per}^1(Y_b))], \quad (41)$$

endowed with the scalar product

$$\begin{aligned} (((u_a, u_b), (\eta_a, \eta_b)), ((\varphi_a, \varphi_b), (\psi_a, \psi_b)))_H = \sum_{\alpha \in \{a, b\}} \int_{\Omega} \nabla u_\alpha \nabla \varphi_\alpha + \\ + \int_{\Omega} (u_a - u_b)(\varphi_a - \varphi_b) + \sum_{\alpha \in \{a, b\}} \int_{\Omega \times Y_\alpha} \nabla_y \eta_\alpha \nabla_y \psi_\alpha. \end{aligned} \quad (42)$$

Using density arguments, Lemma 4.1 yields:

Theorem 4.1. $\lambda_k \in \mathbb{R}_+^*$ and $((u_k^a, u_k^b), (\eta_a, \eta_b)) \in H \setminus \{0\}$ verify:

$$\begin{aligned} & \int_{\Omega \times Y_a} a_{ij} \left(\frac{\partial u_k^a}{\partial x_j} + \frac{\partial \eta_k^a}{\partial y_j} \right) \left(\frac{\partial \varphi_a}{\partial x_i} + \frac{\psi_a}{\partial y_i} \right) + \int_{\Omega \times Y_b} b_{ij} \left(\frac{\partial u_k^b}{\partial x_j} + \frac{\partial \eta_k^b}{\partial y_j} \right) \left(\frac{\partial \varphi_b}{\partial x_i} + \frac{\psi_b}{\partial y_i} \right) + \\ & + \tilde{h} \int_{\Omega} (u_k^a - u_k^b)(\varphi_a - \varphi_b) = \lambda_k \int_{\Omega \times Y} \left(\chi_a u_k^a \varphi_a + \chi_b u_k^b \varphi_b \right), \\ & \forall ((\varphi_a, \varphi_b), (\psi_a, \psi_b)) \in H. \end{aligned} \quad (43)$$

We can present now the main result of this paper.

Theorem 4.2. If $(\lambda_k^\varepsilon, u_k^\varepsilon)$ is a solution of (18) then the limits of the convergences (34)-(36), that is $\lambda_k \in \mathbb{R}_+^*$ and $(u_k^a, u_k^b) \in [H^1(\Omega) \times H_0^1(\Omega)] \setminus \{0\}$, put together a solution of the following effective eigenvalue problem:

Find $(\lambda_k, (u_k^a, u_k^b)) \in \mathbb{R}_+^* \times [H^1(\Omega) \times H_0^1(\Omega)] \setminus \{0\}$ such that

$$\begin{aligned} G^{hom}((u_k^a, u_k^b), (\varphi_a, \varphi_b)) &= \lambda_k \int_{\Omega} m u_k^a \varphi_a + (1-m) u_k^b \varphi_b, \\ \forall (\varphi_a, \varphi_b) &\in H^1(\Omega) \times H_0^1(\Omega), \end{aligned} \quad (44)$$

where $m = |Y_a|$,

$$\begin{aligned} G^{hom}((u_k^a, u_k^b), (\varphi_a, \varphi_b)) &= \\ & \int_{\Omega} \hat{a}_{ij} \frac{\partial u_k^a}{\partial x_j} \frac{\partial \varphi_a}{\partial x_i} + \int_{\Omega} \hat{b}_{ij} \frac{\partial u_k^b}{\partial x_j} \frac{\partial \varphi_b}{\partial x_i} + \tilde{h} \int_{\Omega} (u_k^a - u_k^b)(\varphi_a - \varphi_b), \end{aligned} \quad (45)$$

the effective coefficients \hat{a}_{ij} , $\alpha \in \{a, b\}$, are defined by

$$\hat{\alpha}_{ij} = \int_{Y_\alpha} \alpha_{ij} + \alpha_{ik} \frac{\partial e_j^\alpha}{\partial y_k} dy, \quad \forall i, j \in \{1, 2, \dots, N\}, \quad (46)$$

and $e_k^\alpha \in \tilde{H}_{per}^1(Y_\alpha)$, $k \in \{1, 2, \dots, N\}$, is the unique solution of the local-periodic problem

$$-\frac{\partial}{\partial y_i} \left(\alpha_{ij} \frac{\partial (e_k^\alpha + y_k)}{\partial y_j} \right) = 0 \quad \text{in } Y_\alpha, \quad (47)$$

$$\alpha_{ij} \frac{\partial (e_k^\alpha + y_k)}{\partial y_j} \nu_i = 0 \quad \text{on } \Gamma. \quad (48)$$

Moreover, there are no eigenvalues of the problem (44) except those obtained as limits of the homogenization process.

Proof. The first assertion follows from Theorem 4.1 using standard homogenization procedures.

Next, suppose that there exists an eigenvalue of the problem (44), $\mu \in \mathbb{R}_+^*$, $\mu \neq \lambda_k$, $\forall k \geq 1$, where $\{\lambda_k\}_k$ represents the eigenvalues of the problem (44) obtained as limits of the homogenization process. It follows that there exists k such that $\lambda_k < \mu < \lambda_{k+1}$. Let $w = (w^a, w^b) \in [H^1(\Omega) \times H_0^1(\Omega)] \setminus \{0\}$ be an eigenfunction associated to μ .

Let us introduce $w^\varepsilon \in H_\varepsilon$, the unique the solution to the problem

$$G_\varepsilon(w^\varepsilon, v) = \mu \int_{\Omega_{\varepsilon a}} w^a v + \mu \int_{\Omega_{\varepsilon b}} w^b v, \quad \forall v \in H_\varepsilon. \quad (49)$$

Setting $v = w^\varepsilon$ in the previous relation we find that

$$G_\varepsilon(w^\varepsilon, w^\varepsilon) \leq C \|w^\varepsilon\|_{L^2(\Omega)}.$$

Applying the coercivity property of G_ε , we find that the sequence $\{w^\varepsilon\}_\varepsilon$ is bounded in H_ε . Under these circumstances, there exists $w^* = (w_a^*, w_b^*) \in H^1(\Omega) \times H_0^1(\Omega)$ such that

$$P_a^\varepsilon w^\varepsilon \rightharpoonup w_a^* \text{ in } H^1(\Omega) \text{ and } P_b^\varepsilon w^\varepsilon \rightharpoonup w_b^* \text{ in } H_0^1(\Omega).$$

Homogenizing the problem (49) we obtain

$$\lim_{\varepsilon \rightarrow \infty} G_\varepsilon(w^\varepsilon, v) = G^{hom}(w^*, v) = \mu(w, v) = G^{hom}(w, v), \quad \forall v \in H_0^1(\Omega),$$

from which follows

$$w = w^* \quad \text{and} \quad \int_{\Omega} (w^\varepsilon)^2 \rightarrow \int_{\Omega} m(w^a)^2 + (1-m)(w^b)^2 \neq 0. \quad (50)$$

Next, we define $\hat{w}^\varepsilon = w^\varepsilon - \sum_{i=1}^k \left(\int_{\Omega} w^\varepsilon u_i^\varepsilon \right) u_i^\varepsilon$, where $u_i^\varepsilon \in H_\varepsilon$ is an eigenfunction associated to λ_i . We note that $\int_{\Omega} \hat{w}^\varepsilon u_j^\varepsilon = 0$, $\forall j \leq k$, and hence

$$G^\varepsilon(\hat{w}^\varepsilon, \hat{w}^\varepsilon) \geq \lambda_{k+1}^\varepsilon \int_{\Omega} (\hat{w}^\varepsilon)^2. \quad (51)$$

As

$$\int_{\Omega} w^\varepsilon u_i^\varepsilon = \int_{\Omega} \chi_a^\varepsilon P_a^\varepsilon w^\varepsilon u_i^\varepsilon + \int_{\Omega} (1 - \chi_a^\varepsilon) P_b^\varepsilon w^\varepsilon u_i^\varepsilon, \quad \forall i \in \{1, \dots, k\},$$

the passage to the limit yields

$$\int_{\Omega} w^\varepsilon u_i^\varepsilon \rightarrow \int_{\Omega} m w^a u_i^a + (1-m) w^b u_i^b = 0,$$

which obviously implies

$$\int_{\Omega} (\hat{w}^\varepsilon)^2 \rightarrow \int_{\Omega} m(w^a)^2 + (1-m)(w^b)^2. \quad (52)$$

Passing (49) to the limit in we finally get

$$\begin{aligned} G_\varepsilon(\hat{w}^\varepsilon, \hat{w}^\varepsilon) &\rightarrow G^{hom}(w, w) = \mu \int_{\Omega} m(w^a)^2 + (1-m)(w^b)^2 < \\ &< \lambda_{k+1} \int_{\Omega} m(w^a)^2 + (1-m)(w^b)^2, \end{aligned} \quad (53)$$

which is in contradiction with (51), via (52). ■

Remark 4.1. *The formulation of the effective two-temperature eigenvalue problem follows from Theorem (4.2):*

Find $\lambda \in \mathbb{R}^$ such that $\exists(u^a, u^b) \in [H^1(\Omega) \times H_0^1(\Omega)] \setminus \{0\}$ verifying*

$$-\operatorname{div}(\hat{a}\nabla u^a) = m \lambda u^a \quad \text{in } \Omega, \quad (54)$$

$$-\operatorname{div}(\hat{b}\nabla u^b) = (1-m) \lambda u^b \quad \text{in } \Omega, \quad (55)$$

and the boundary condition

$$\hat{a}_{ij} \frac{\partial u^a}{\partial x_j} n_i = 0 \quad \text{on } \partial\Omega, \quad (56)$$

where n is the outward normal on $\partial\Omega$.

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