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# Effects of users' bounded rationality on a traffic network performance: a simulation study

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## Abstract

In this paper, we revisit the principle of bounded rationality applied to dynamic traffic assignment to evaluate its influences on network performance. We investigate the influence of different types of bounded rational user behavior on: (i) route flows at equilibrium; and (ii) network performance in terms of its internal, inflow and outflow capacities. We consider the implementation of a bounded rational framework based on Monte Carlo simulation. A Lighthill-Whitham-Richards (LWR) mesoscopic traffic simulator is considered to calculate time-dependent route costs that account for congestion, spillback and shock-wave effects. Network equilibrium is calculated using the Method of Successive Averages. As a benchmark, the results are compared against both Deterministic and Stochastic User Equilibrium. To model different types of bounded rational user behavior we consider two definitions of user search order (stochastic and strict preferences) and two settings of the indifference band. We also test the framework on a toy Braess network to gain insight into changes in the route flows at equilibrium for both search orders and increasing values of aspiration levels.

*Keywords:* Satisficing behavior, Users preferences, Network performance, LWR model, Traffic simulation.

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Word count: 8923 words

## 1. Introduction

The first notions of traffic assignment were introduced by [Wardrop \(1952\)](#). According to the first Wardrop principle, users aim to minimize their personal route travel times. This leads to a network equilibrium called the Deterministic User Equilibrium (DUE) and it is that most commonly used in dynamic traffic assignment (DTA) problems. Under DUE conditions, no user can decrease his/her own travel time by unilaterally switching routes. However, the first Wardrop principle assumes that users are perfectly rational and perceive all routes and network traffic states perfectly although information on route travel times (i.e., traffic states) is not necessarily perfect. To overcome this problem, [Daganzo and Sheffi \(1977\)](#) and [Daganzo \(1982\)](#) introduced the Stochastic User Equilibrium (SUE), to take into account the uncertainty of route travel times. The Multinomial Logit and C-Logit are the Random Utility models (RUM) most commonly used in DTA problems. Nonetheless, both these models present several limitations when dealing with correlations between routes. In this study we focus in particular on the Probit model solved using Monte Carlo simulations ([Sheffi, 1985](#)).

Revealed ([Zhu, 2011](#)) and stated ([Avineri and Prashker, 2004](#)) preference surveys show that users tend to choose sub-optimal routes instead of optimal ones ([Zhu and Levinson, 2015](#)). We emphasize that a sub-optimal route is understood as a route with a longer travel time than the minimum one for the origin-destination (od) pair. In the literature on static traffic assignment, there are other alternative model frameworks that take into account different types of user behavior. One example is the Prospect Theory ([Kahneman and Tversky, 1979](#); [Tversky and Kahneman, 1992](#)) which consider the users risk-seeking and risk-aversion behavior. It was adapted to the context of route choice

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by Avineri (2006). In the Prospect Theory, users evaluate the different routes in terms of time prospect and choose the route with the maximum prospect. Users are risk-averse when confronted with prospects of gains and risk-seekers when confronted with prospects of losses and are more sensitive to losses than gains (*loss effect*). Another example is the Regret Theory (Bell, 1982; Loomes and Sugden, 1982). The users aim to minimize their regret with respect to the non-selected routes (Chorus, 2012; Li and Huang, 2016). If the users choose the route with the minimum travel time, they will feel joy or feel regret otherwise. Another example is the notion of bounded rationality introduced by the seminal works of Simon (1957, 1966, 1990, 1991). He stated that users choices are driven by aspiration levels (*AL*), which represent a set of goal or target variables that should be achieved or exceeded for the users satisfaction. In his original idea, the user searches until a satisfactory alternative is found. This term used to describe this process was coined by Simon as *satisficing*, which stands for the combination of satisfy and suffice. In this study, we focus on the application of the notion of bounded rationality in a dynamic context, by considering distributions of route travel times and a traffic simulator. The goal of this paper is to investigate the influence of bounded rational user behavior on individual route flows and network performance. This type of study is very important for decision-making in transportation planning.

Mahmassani and Chang (1987) discussed the first notion of bounded rationality applied to traffic assignment, but no mathematical formulation was given. To define users *AL*, Mahmassani and Chang (1987) introduced the concept of *indifference band* (IB), where a route is *satisficing* if the difference between its travel cost and that of the best available route is lower than a given threshold (or IB). The implementation of bounded rationality in traffic assignment is challenging as: (i) the calibration of the *AL* is context dependent (Vreeswijk et al., 2013); and (ii) the BR-UE solutions are not unique (Lou et al., 2010; Di et al., 2013, 2014). Thus, to analyze the BR-UE solutions, some authors have focused on the analysis of the best and worst BR-UE flows of the network (Lou et al., 2010; Di et al., 2013; Eikenbroek et al., 2018). Moreover, the *AL* can change from user to user. A thorough review of bounded rationality in traffic assignment was provided in Di and Liu (2016). There are two main ingredients that dictate bounded rational network equilibrium: (i) the definition of the *AL* that dictates whether a route is *satisficing* or not; and (ii) the users search order that defines how users are guided in their choice of a *satisficing* route.

For a route to be considered as *satisficing*, its route utility must satisfy:

$$U_k \leq AL^{od}, \forall k \in \Omega^{od} \wedge \forall (o, d) \in \Xi \quad (1)$$

where  $U_k$  is the perceived route utility;  $AL^{od}$  is the aspiration level we consider in this paper, to be defined at the od level;  $\Omega^{od}$  is the route choice set for the od pair; and  $\Xi$  is the set of all od pairs of the network.

The  $AL^{od}$  can be calibrated exogenously by route choice surveys or calibrated endogenously by explicit formulations. The most commonly used definition is based on the concept of indifference band (Mahmassani and Chang, 1987; Huang and Lam, 2002; Szeto and Lo, 2006):

$$AL^{od} = \min(\vec{V}) + \Delta^{od}, \forall (o, d) \in \Xi \quad (2)$$

where  $\vec{V}$  is a vector containing all deterministic route utilities  $V_k, \forall k \in \Omega^{od} \wedge \forall (o, d) \in \Xi$  and  $\Omega^{od}$  contains  $N$  routes.  $\Delta^{od}$  is the tolerance or IB at the od level.

Ge and Zhou (2012) proposes a variable definition of the IB ( $\Delta^{od}$ ):

$$AL^{od} = \min(\vec{V}) + \max(|U_p - U_q| \delta_p \delta_q), \forall p, q \in \Omega^{od} \wedge p \neq q \wedge \forall (o, d) \in \Xi \quad (3)$$

where  $\delta_p$  and  $\delta_q$  are dummy variables that equal 1 if routes  $p$  and  $q$  belong to  $\Omega^{od}$ , respectively.

Ge et al. (2015) analyzed the BR-DUE equilibrium, considering exogenously fixed *AL* and fixed and endogenously variable *AL*. In their model framework, the authors showed that the DUE is a special case of the BR-DUE and discussed the existence conditions of the BR-DUE. However, the uncertainty on the travel times was not considered.

Di and Liu (2016) highlighted that a bounded rational behavior can be due to the users habits and inertia, or their cognitive costs or individual preferences. In this paper, we focus our attention on the users preferences as a bounded rational behavior to define the search order for the *satisficing* alternatives. Zhao and Huang (2016) defined a

search order based on a strict preference order for all users sharing the same od pair. This strong assumption allowed obtaining unique BR-UE solutions. To the authors knowledge, the framework of [Zhao and Huang \(2016\)](#) has never been tested in a dynamic context, i.e. considering a traffic simulator and time-dependent path costs. In addition, its dynamic implementation using a traffic simulator is highly challenging because it requires solving sub-optimization problems to calibrate the  $AL$  of the sub-most preferred routes. Thus, a framework capable of solving the global optimization problem is required and discussed further on in this paper. On the other hand, users may also have an indifferent preference for any of the *satisficing* routes (i.e., that satisfy [Eq. 1](#)). This is adopted from the notions discussed in [\(Aguiar et al., 2016\)](#). In this case, we consider that all users sharing the same od pair have a similar indifference preference. The choice is modeled by uniform random sampling of any of the *satisficing* routes. Users are then assigned to the *satisficing* route sampled.

[Szeto and Lo \(2006\)](#)<sup>1</sup> discussed an analytical BR-UE dynamic traffic assignment model. The authors proposed a route swapping algorithm, but no clear definition of the users search order was discussed. Instead, the authors targeted certain users on the most congested routes and switched them to less congested ones for each od pair. Moreover, the BR-UE solutions were not unique. [Han et al. \(2015\)](#) discussed a dynamic simultaneous departure time and route choice bounded rational framework. However, neither of these frameworks included travel time distributions. In this paper, we revisit the notions of bounded rationality by considering the distribution of travel times rather than deterministic values.

The literature includes a large number of applications of a bounded rational framework to static ([Di et al., 2013](#)) and dynamic traffic assignment ([Szeto and Lo, 2006](#); [Han et al., 2015](#)), transportation planning ([Gifford and Checherita, 2007](#)), traffic policy making ([Marsden et al., 2012](#)), congestion pricing ([Lou et al., 2010](#)) and traffic safety ([Sivak, 2002](#)). However, to the authors knowledge, there is no study in the literature that investigates the influence of users preferences (indifferent and strict) for a bounded rational behavior on individual route flows and network performance in terms of the internal level of congestion and inflow and outflow capacities. The goal of this paper is to fill this gap. We consider time-dependent path costs that account for congestion, shock-waves and spillback effects calculated using a mesoscopic Lighthill-Whitham-Richards (LWR) model ([Leclercq and Becarie, 2012](#)). A spillback effect is the reduction of a link capacity that spreads over other connected links in the network. To model bounded rationality behavior, we relax the definition of the search order of the DUE and SUE frameworks ([Sheffi, 1985](#)). In both the DUE and SUE cases, users are assigned to the routes with the minimum travel times based on an all-or-nothing procedure. The search order is relaxed to account for the users indifferent and strict preferences. In the case of the indifferent preference search order, users present indifference behavior when choosing any of the *satisficing* routes, whereas in the case of the strict preference search order ([Zhao and Huang, 2016](#)), users are assigned to the most preferred route if this route is perceived as *satisficing* ([Eq. 1](#)), or to the first sub-most preferred route that satisfies [Eq. 1](#). We make use of Monte Carlo simulations ([Sheffi, 1985](#)) to account for travel times distribution and consider the classical Method of Successive Averages to calculate the network equilibrium. First, we test the bounded rationality methodology in a toy Braess network and consider a simple linear static and flow dependent utility function. We then consider the two settings of the search order previously mentioned and the  $AL^{od}$  defined exogenously. These initial tests allow acquiring insight into how the route flows at equilibrium change according to the two definitions of the search order and increasing values of  $AL^{od}$ . Second, for the dynamic implementation, we also consider the two settings of the users search order (i.e. stochastic and strict preferences) and the concept of the IB ([Eq. 2](#) and [Eq. 3](#)) to define the  $AL^{od}$ . The dynamic tests are performed on a Manhattan network. We investigate the influence of the definition of the search order on the individual route flows and analyze the network performance in terms of the internal, inflow and outflow capacities, given the two search orders and different values of the  $AL^{od}$ . The results are compared against both DUE and SUE as benchmarks.

This paper is organized as follows. In [Sect. 2](#), we discuss the bounded rational model framework considered in this paper. In [Sect. 3](#), we discuss a simple static test scenario on the Braess network, considering both the stochastic and strict preferences search order. In [Sect. 4](#), we discuss the influence of the bounded rationality behavior on the network performance also considering the two search orders. In [Sect. 5](#), we outline the conclusions of this paper.

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<sup>1</sup>The BR-UE ([Mahmassani and Chang, 1987](#)) and Tolerance-based Dynamic User Optimum principle ([Szeto and Lo, 2006](#)) have been used interchangeably in the traffic assignment literature. For the sake of simplicity, we refer to both as BR-UE.

## 2. Bounded rational framework

The analysis of the effect of users' behavior on network performance in terms of its internal inflow and outflow capacities is very important for policy makers, in particular when determining policies aimed at increasing network performance. In this paper, we focus on two types of bounded rational user behavior.

We start by introducing the general formulation of the route utilities. The perceived route utility,  $U_k$ , is:

$$U_k = V_k + \epsilon_k, \forall k \in \Omega^{od} \wedge \forall (o, d) \in \Xi \quad (4)$$

where  $V_k$  is the deterministic route utility; and  $\epsilon_k$  is the uncertainty or error term as often referred to in the literature.

The DUE assumes that users are utility minimizers and the error terms  $\epsilon_k$  are set to 0. Users are assigned based on an all-or-nothing procedure to the route with the minimum travel time. In the case of the SUE, users are also utility minimizers, but they perceive travel times with uncertainty, meaning that the error terms  $\epsilon_k$  are not 0. Theoretically, the Probit model (Daganzo and Sheffi, 1977) is the most attractive model for solving the SUE. However, it requires the computation of a covariance matrix and integrating the multivariate normal distribution. The complexity of the computation increases with the number of routes per od pair. An alternative to this is to use Monte Carlo to consider the distributions of route travel times (Sheffi, 1985). We consider that the error terms are defined at the link level (i.e.,  $\epsilon_a, \forall a \in \Gamma$ ) instead at the route level. This allows capturing existing correlations between different routes sharing the same links. In addition, we consider that the terms  $\epsilon_a$  are gamma distributed (Nielsen, 1997). The principle of the Monte Carlo simulations is to discretize the error terms  $\epsilon_a$  into  $M$  samples or draws and locally solve DUE problems. In this case, for each error draw the deterministic utility for route  $k$  is defined as  $U_k^m, m = 1, \dots, M$ .

Due to the strong assumption of perfect rationality of the DUE with respect to user behavior Mahmassani and Chang (1987) introduced the first notions of bounded rationality applied to route choice. Lou et al. (2010) and Di et al. (2013) formulated the BR-DUE mathematically. Under BR-DUE, all users are satisfied with their choices and no longer consider switching routes. It should be noted that the DUE is an extreme case of the BR-DUE. To extend the BR-DUE in order to include the stochasticity of travel times, we consider Monte Carlo simulations as discussed previously and solve BR-DUE problems locally in a similar spirit. In this case, the network equilibrium corresponds to the Bounded Rational Stochastic User Equilibrium (BR-SUE). Under BR-SUE conditions, users are satisfied with their current choices and no longer consider switching routes given their perception of the route travel times.

In this paper, we relax the assumption of the users search order on the settings of the DUE and SUE. The idea is that users are assigned to *satisficing* routes instead of optimal routes (i.e., routes with the minimum perceived travel times). We reinforce the idea that a *satisficing* route has to satisfy Eq. 1. Note that, in the case of the BR-DUE the condition is  $U_k = V_k \leq AL^{od}, \forall k \in \Omega^{od} \wedge \forall (o, d) \in \Xi$ ; for the BR-SUE the condition is  $U_k^m \leq AL^{od}, \forall m = 1, \dots, M \wedge \forall k \in \Omega^{od} \wedge \forall (o, d) \in \Xi$ . We denote  $\omega^{od}$  as the set of *satisficing* routes for the od pair. We consider two definitions of the user's search order:

- indifferent preferences, where users are randomly assigned based on an all-or-nothing procedure, to one of the routes listed in  $\omega^{od}$ . If  $\omega^{od}$  is empty (i.e., there are no *satisficing* routes), users are assigned to the route with the minimum travel time. In reality, this is consistent with human behavior. If there are no alternatives that satisfy our expectations or goals, we seek to choose the route with the minimum travel time given our perception. This setting of the search order is in line with the original ideas of Simon (1957, 1966, 1990, 1991) and represent the users' indifference to choosing any of the *satisficing* routes. This is modeled by a uniform distribution.
- strict preference order (Zhao and Huang, 2016), where users are assigned based on an all-or-nothing procedure to the most preferred route, if it is *satisficing*. If not, users are assigned to the first sub-most preferred route that is *satisficing*. We define  $\Upsilon^{od}$  as the users strict preference order set.

The idea of the strict preference order was introduced by Zhao and Huang (2016), to deal with the non-uniqueness of the equilibrium solution. However, we highlight two main differences between our methodology and that discussed in Zhao and Huang (2016):

- we consider that routes are *satisficing* if and only if their perceived utility satisfies:  $U_k = V_k \leq AL^{od}, \forall k \in \Omega^{od} \wedge \forall (o, d) \in \Xi$  for the BR-DUE; or  $U_k^m \leq AL^{od}, \forall m = 1, \dots, M \wedge \forall k \in \Omega^{od} \wedge \forall (o, d) \in \Xi$  for the BR-SUE.

Zhao and Huang (2016) consider that routes are *satisficing* according to the strict preference order, i.e. the users are first assigned to the most preferred route and then consecutively to the sub-preferred routes, until all the users are assigned.

- we consider that  $AL$  is defined at the od level (i.e.,  $AL^{od}$ ), while Zhao and Huang (2016) considers its definition at the route level. We also assume that all users sharing the same od pair have the same  $AL^{od}$ . We consider that is more realistic from the user's perspective to set a global  $AL^{od}$  instead of establishing  $AL$  for the sub-preferred routes based on the most preferred ones.

In this paper, we consider the two definitions of  $AL^{od}$  as defined in Eq. 2 and Eq. 3.

To reach a solution for the BR-SUE, we consider Monte Carlo simulations as discussed in Sheffi (1985) and the classical Method of Successive Averages (MSA). The MSA solves a fixed point problem and is commonly used in traffic assignment to solve both the DUE and SUE (Sheffi, 1985). The Monte Carlo simulations consist in discretizing the distributions of the link travel times into  $M$  samples or draws and solving BR-DUE problems locally. For each discretization, we identify the *satisficing* routes and assign the users based on an all-or-nothing assignment following the search order established. If the search order is considered to be the indifferent preferences, all the users are assigned randomly to any of the *satisficing* alternatives. On the other hand, if the search order follows a strict user preference order, all users are assigned to the first *satisficing* alternative found on this strict sequence of preferences. The new temporary route flows,  $Q_k^*$ , correspond to averaging all the local BR-DUE solutions. This corresponds to the temporary route flows  $Q_k^*$ , that will be used to update the new route flows  $Q_k^{j+1}$  at iteration  $j + 1$ , as:

$$Q_k^{j+1} = Q_k^j + \alpha_j \{Q_k^* - Q_k^j\}, \forall k \in \Omega^{od} \wedge \forall (o, d) \in \Xi \quad (5)$$

where  $Q_k^j$  represent the route flows at iteration  $j$  of the MSA; and  $\alpha_j$  is the descent step. This process is repeated at every descent step of the MSA algorithm.

The sequence of descent steps  $\alpha_j$  guarantees the convergence of the MSA. For the theoretical convergence of the algorithm, the following two conditions must be satisfied (Sheffi, 1985):

$$\sum_{j=1}^{\infty} \alpha_j = \infty \quad (6)$$

$$\lim_{j \rightarrow \infty} \alpha_j \rightarrow 0 \quad (7)$$

One definition of  $\alpha_j$  that satisfies both of the previous conditions is:  $\alpha_j = \frac{1}{j}$ . We consider this definition of  $\alpha_j$  in this paper. Other definitions of the descent step size are discussed in the literature (Polyak, 1990; Liu et al., 2007; Taale, 2008; Chen et al., 2011).

A commonly used convergence or stopping criterion is based on the comparison between the current and the previous descent step of the MSA that must be lower to a pre-defined threshold. Instead we consider the number of violations  $N(\lambda)$  and the relative gap (Sbayti et al., 2007).  $N(\lambda)$  represents the number of cases where  $|Q_k^{j+1} - Q_k^j|$  is higher than a pre-defined path convergence threshold  $\Phi$ . Note that  $\Phi$  represents an upper bound. The convergence of the algorithm is achieved if  $N(\lambda) \leq \Phi$ . The relative gap for the DUE is (Sbayti et al., 2007):

$$Gap = \frac{\sum_o \sum_d \sum_{k \in \Omega^{od}} Q_k^{od} (V_k^{od} - \min(V^{od}))}{\sum_o \sum_d D^{od} \min(V^{od})} \quad (8)$$

where  $D^{od}$  is the total demand for the od pair; and  $V_k^{od}$  is the average travel time of route  $k$ ; and  $\min(V^{od})$  is the minimum route travel time for the od pair.

The Gap function (Eq. 8) represents the difference between the travel costs and the equilibrium travel costs. Thus, under perfect DUE conditions,  $Gap = 0$ . This means that all users choose the routes with the minimum travel times.

Under SUE conditions, the Gap is higher than 0, however small. In this case not all users choose the routes with the minimum travel times. In the case of bounded rationality, the Gap value increases as  $AL^{od}$  increases. The Gap function is also a measure of how close users are to the equilibrium route travel times (or  $T^{UE}$ ). The definition of the GAP as defined in Eq. 8 is valid for DUE and SUE and informs on how far we are from the DUE. For both the BR-DUE and BR-SUE convergence, we modify the Gap function as follows:

$$Gap^{BRUE} = \frac{\sum_o \sum_d \sum_{k \in \omega^{od}} Q_k^{od} \max(V_k^{od} - AL^{od}, 0)}{\sum_o \sum_d D^{od} AL^{od}} \quad (9)$$

Thus, under BR-DUE or BR-SUE conditions, the Gap is about 0 if  $\min(V^{od}) \leq V_k^{od} \leq AL^{od}, \forall (o, d) \in \Xi$  and the equilibrium condition is fulfilled. Note that throughout the paper, we use the definition of the Gap as in Eq. 8 as an indicator that measures how far the bounded rational equilibria from the DUE; and Eq. 9 as the equilibrium convergence criterion for the MSA.

We present the solution algorithm of this framework in Algorithm 1. Note that the difference between Algorithm 1 and that proposed by Sheffi (1985) is that we assign the users to *satisficing* routes instead of routes with the minimal travel times. They are assigned to these *satisficing* routes according to one of the search orders discussed previously (i.e. indifferent or strict preferences) at every descent step of the MSA. The first step before entering the MSA loop consists in calculating the route choice set  $\Omega^{od}$ , for each od pair. It defines the set of routes for the users choices. We then perform an initial loading on these routes and consider the number of violations, the GAP (Eq. 9) and the maximum number of iterations for the MSA convergence criteria. The corresponding  $tol$ ,  $\Phi$  and  $N_{max}$  are set. It is also necessary to define the input scale ( $\eta$ ) and shape ( $\zeta$ ) parameters of the link travel time gamma distributions for the first Monte Carlo simulations. We then enter in the MSA loop and the  $AL^{od}$  is first updated based on the average route travel times (see Eq. 2 or Eq. 3). The next step consists in performing the link error sampling considering the  $\eta$  and  $\zeta$  parameters. This is done through Monte Carlo simulations. The algorithm then loops over all the  $M$  error samples and locally solves the BR-DUE problems. For each sample, the route utilities are computed to identify the *satisficing* routes based on  $AL^{od}$ . This defines the *satisficing* set of routes  $\omega^{od}$ . Users are assigned based on the pre-defined search order (indifferent or strict preferences) based on an all-or-nothing procedure to one route in  $\omega^{od}$ . It should be noted that in the case of solving the BR-SUE and taking the indifferent preferences into account, it is necessary to repeat the all-or-nothing assignment on the *satisficing* routes  $A$  times. The users choices for the local BR-DUE correspond to averaging the previous choices. By applying the law of large numbers, when  $A$  is large, we converge to the same average values. The new temporary route flows  $Q_k^*$  correspond to the average of all the local BR-DUE choices. The new route flows  $Q_k^{j+1}$  are updated according to Eq. 5 and the network loading is updated. To determine time-dependent link costs that consider congestion, shock-waves and spillback effects, we run an LWR mesoscopic traffic simulator (Leclercq and Becarie, 2012). The link travel time distributions are obtained based on the simulated vehicle travel times. To update  $\eta$  and  $\zeta$ , we fit a gamma distribution to each link travel time distribution. The updated values of  $\eta$  and  $\zeta$  will be used to perform the error samplings in the next MSA descent step. The  $Gap^{BRUE}$  (Eq. 9) and number of violations  $N(\lambda)$  are updated based on the new average route travel times through the individual vehicles travel times. This process is repeated until convergence is achieved. Note that Algorithm 1 also allows solving the BR-DUE by setting  $\eta = 0$  and  $\zeta = 0$ .

In Tab. 1 we summarize the notations of all symbols and variables used in this paper.

Input the  $AL^{od}$  (if they are set exogenously).  
Input the network, demand scenario and simulation duration  $T$ .  
Calculate the route choice set  $\Omega^{od}$  for each od pair.  
Perform an initial network loading.  
Set  $N(\lambda) > \Phi$  and  $Gap^{BRUE} > tol$ . Initialize  $j = 1$ ,  $\eta$ ,  $\zeta$ ,  $\alpha_j = 1$  and  $Q_k^{j=1} = 0$ .  
Set the MSA stopping criterion  $tol$ .  
**while**  $Gap \geq tol$  **or**  $N(\lambda) \geq \Phi$  **or**  $j \leq N_{max}$  **do**  
    Set  $Q_k^j = Q_k^{j+1}$ .  
    If set endogenously, update the  $AL^{od}$  based on Eq. 2 or Eq. 3.  
    Perform  $M$  error samplings at the link level, based on  $\eta$  and  $\zeta$ .  
    **for**  $m=1$  to  $M$  **do**  
        Compute the route utilities.  
        Determine the *satisficing* routes and update  $\omega^{od}$ ,  $\forall k \in \Omega^{od} \wedge \forall(o, d) \in \Xi$ .  
        Based on the defined search order, perform an all-or-nothing assignment<sup>2</sup>. If  $\omega^{od} = \emptyset$ , all users are assigned to the minimum utility route.  
    **end**  
    Update the new route flows  $Q_k^*$ ,  $\forall k \in \Omega^{od} \wedge \forall(o, d) \in \Xi$ , based on an averaging of the users choices over all error samples.  
    Update the route flows as:  $Q_k^{j+1} = Q_k^j + \alpha_j(Q_k^* - Q_k^j)$ ,  $\forall k \in \Omega^{od} \wedge \forall(o, d) \in \Xi$ .  
    Run the LWR mesoscopic simulator (Leclercq and Becarie, 2012).  
    Based on the link travel times, fit a gamma distribution to update  $\eta$  and  $\zeta$ .  
    Calculate the Gap (Eq. 8) and the number of violations  $N(\lambda)$ .  
    Update  $\alpha_j = \frac{1}{j}$ .  
    Set  $j = j + 1$ .  
**end**  
Save the route flows:  $Q_k^{j+1}$ ,  $\forall k \in \Omega^{od} \wedge \forall(o, d) \in \Xi$ .  
**Algorithm 1:** Dynamic implementation algorithm of the *satisficing* model.



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$k$	Route.
$a$	Link.
$o$	Origin.
$d$	Destination.
$Q^{od}$	Total demand for the od pair.
$\Xi$	Set of all od pairs of the network.
$U_k$	Perceived route utility.
$V_k$	Deterministic route utility.
$\epsilon_k$	Uncertainty or error term.
$\epsilon_a$	Link error term.
$\Omega^{od}$	Route choice set for od pair.
$N$	Number of routes listed in the route choice set per each od pair.
$Q_k$	Flow of route $k$ .
$q_a$	Link flow.
$\delta_{ak}$	Dummy variable that equals 1 if link $a$ belongs to route $k$ .
$\Gamma_a$	Set of links that define the graph.
$AL^{od}$	Aspiration level defined at the od level.
$\vec{V}$	Vector containing all deterministic route utilities.
$\Delta^{od}$	Tolerance or Indifference Band.
$\delta_p$	Dummy variable that equals 1 if route $p$ is listed on the choice set.
$\delta_q$	Dummy variable that equals 1 if route $q$ is listed on the choice set.
$\omega^{od}$	Set of <i>satisficing</i> routes.
$T^{UE}$	User Equilibrium route travel time.
$\Upsilon^{od}$	Users' strict preference order for the <i>od</i> pair.
$M$	Number of draws for the error term discretization.
$N(\lambda)$	Number of violations.
$\Phi$	Pre-defined path convergence threshold.
$\eta$ and $\zeta$	Scale and shape parameters of a gamma distribution.
$j$	Iterative counter of the MSA method.
$tol$	Tolerance for the stopping criterium of the MSWA-I.
$N_{max}$	Maximum number of descent step iterations.
$T_{ff}^a$	Free-flow travel time of link $a$ .
$T$	Simulation period.
$A$	Number of assignment repetitions for BR-SUE and indifferent preferences search order.

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Table 1: Notations used in this paper.

### 3. First tests on a toy network

We first test the bounded rational model framework discussed in the previous section, on a toy Braess network and consider a static flow dependent utility function. The goal of these simple initial tests are to assess and acquire insight into how the route flows at equilibrium change according to the two definitions of the search order (i.e. indifferent and strict preference search order) and increasing values of  $AL^{od}$ . The  $AL^{od}$  are defined exogenously.

#### 3.1. Definition of the test network

For the first test, we consider that the perceived travel times (i.e., route utility) depend only on route flows and route free-flow travel times. We resort to the following definition of the perceived route utility,  $U_k(Q_k)$ :

$$U_k(Q_k) = \sum_{a \in \Gamma_a} (V_a(q_a) + \epsilon_a) \delta_{ak}, \forall k \in \Omega^{od} \quad (10)$$

where  $V_a(q_a) = \theta_1 T_{ff}^a + \theta_2 q_a$  and  $q_a = \sum_{l \in a} \delta_{al} Q_l$ ;  $T_{ff}^a$  is the link free-flow travel time of link  $a$ ;  $q_a$  is the flow of link  $a$ ;  $\delta_{al}$  is a dummy variable that equals 1 if route  $l$  uses link  $a$ , or 0 otherwise;  $Q_l$  is the total flow of route  $l$ ; and  $\epsilon_a$  is the error term associated with link  $a$ .  $\theta_1$  and  $\theta_2$  are parameters set to 1.

For the static tests, we consider the Braess network (Fig. 1). The choice set is:  $\Omega_k = \{1, 4; 2, 5; 1, 3, 5\}$ ,  $\forall k = 1, 2, 3$ . The following sets of link free-flow travel times are considered:  $T_{ff}^1 = 5$ ,  $T_{ff}^2 = 45$ ,  $T_{ff}^3 = 10$ ,  $T_{ff}^4 = 30$ ,  $T_{ff}^5 = 5$ .

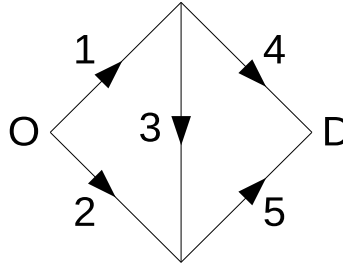


Figure 1: Braess network.

For the MSA convergence, we set  $tol = 10^{-2}$ ,  $\Phi = 0$  and  $N_{max} = 10000$ . A total demand of  $Q^{od} = 10$  is considered. As a reference of user perfect rationality, we consider the DUE. Note that the  $\epsilon_a$  terms are set to 0 for both the DUE and the BR-DUE calculations. For the BR-SUE calculations, we consider a gamma distribution with a shape parameter set to  $\eta = 1$  and a scale parameter set to  $\zeta = 4$ ; and a total of  $M = 2000$  error samples. The BR-DUE and BR-SUE are calculated based on Algorithm 1, except that the link travel times are not updated considering the LWR mesoscopic traffic simulator (Leclercq and Becarie, 2012). Instead, we consider the route utility as defined in Eq. 10, which does not depend on congestion. The link travel times are only sampled at the initiation of the MSA descent procedure.

#### 3.2. Indifferent preference search order and exogenous $AL^{od}$

In this section, we analyze both the BR-DUE and BR-SUE results considering the users indifferent preference search order and an exogenous definition of the  $AL^{od}$ . We also analyze the algorithms convergence towards the equilibrium solution through the Gap function. To define the search order, we consider a uniform distribution to simulate the users choices among the set of *satisficing* routes  $\omega^{od}$ . This procedure must be repeated many times to reach convergence by the Law of large numbers (on average) with the same solution of route flows. Then, for each value of  $AL^{od}$ , we repeat the assignment procedure 1000 times and calculate the average route flows and corresponding standard deviation. We do so for both the BR-DUE and BR-SUE calculations. First, under DUE conditions, only routes 1 and 3 are used. This means that  $T^{UE} = U_1 = U_3 < U_2$  and corresponds to the route flows:  $Q_1 = 1.7$ ,  $Q_2 = 0$  and  $Q_3 = 8.3$ . Note that  $T^{UE}$  is the route travel times at the User Equilibrium.

We first analyze the BR-DUE results, calculated for increasing values of  $AL^{od}$ . These results are listed in Table 2. The first test consists in reproducing the perfect rationality behavior, by setting  $AL^{od} = T^{UE} = 46.7$ . The route flows

under BR-DUE are similar to the DUE. Then, to analyze the equilibrium results for increasing values of  $AL^{od} \in [0, +\infty[$ , we must first identify the critical points for the BR-DUE, that is to say the utility values of each route when the total demand  $Q^{od}$  is assigned to each of the routes. We first consider  $Q_1 = 10, Q_2 = 0$  and  $Q_3 = 0$ , which yields  $U_1 = 55$ . Similarly for route 2,  $U_2(Q_1 = 0, Q_2 = 10, Q_3 = 0) = 60$ , and route 3,  $U_3(Q_1 = 0, Q_2 = 0, Q_3 = 10) = 50$ . These critical points play an important role in analyzing the equilibrium solutions. The minimum of the critical points indicate the value of  $AL^{od}$  from which the objective function is no longer convex. We analyze the BR-DUE route flows for increasing values of  $AL^{od}$  in more detail. For  $AL^{od} \in [T^{UE}, 50]$ , the users switch from route 3 to 1. Note that the users switch from the *satisficing* routes with higher route flows to the ones with lower route flows. For  $AL^{od} \in [50, 55]$ , the algorithm does not converge to the same solution as evidenced by the standard deviation values listed in Table 2. For example, for  $AL^{od} = 50$ , two feasible solutions are found:  $(Q_1 = 5, Q_2 = 0, Q_3 = 5)$  which yields  $(U_1 = 50, U_2 = 55, U_3 = 40)$ ; and  $(Q_1 = 0, Q_2 = 0, Q_3 = 10)$  that yields  $(U_1 = 45, U_2 = 60, U_3 = 50)$ . The convergence of the algorithm to any of these feasible solutions depends on the initial loading of the network for the MSA algorithm. This explains why we do not converge to the same set of route flows for  $AL^{od} \geq 50$ . For  $AL^{od} \in [55, 60]$ , route 2 becomes *satisficing* and users switch from routes 3 and 1 to route 2. For  $AL^{od} \geq 60$ , the route flows will converge to 1/3 as the value of  $AL^{od}$  increases. This represents the users indifference for choosing any of the *satisficing* routes.

$AL^{od}$	$Q_1/Q^{od}$	$Q_2/Q^{od}$	$Q_3/Q^{od}$	$U_1$	$U_2$	$U_3$	$\overline{Gap}$
46.7	0.17 ± 0.00	0.00 ± 0.00	0.83 ± 0.00	46.7	58.2	46.7	0.00
47.5	0.25 ± 0.00	0.00 ± 0.00	0.75 ± 0.00	47.5	57.4	45.0	0.01
48.0	0.30 ± 0.00	0.00 ± 0.00	0.70 ± 0.00	48.0	56.9	44.0	0.03
48.5	0.35 ± 0.00	0.00 ± 0.00	0.65 ± 0.00	48.5	56.4	43.0	0.04
49.0	0.40 ± 0.00	0.00 ± 0.00	0.60 ± 0.00	49.0	55.9	42.0	0.06
49.5	0.45 ± 0.00	0.00 ± 0.00	0.55 ± 0.00	49.5	55.4	41.0	0.09
50.0	0.34 ± 0.23	0.00 ± 0.00	0.66 ± 0.23	48.4	56.6	43.1	0.11
51.0	0.35 ± 0.23	0.00 ± 0.00	0.65 ± 0.23	48.5	56.5	42.9	0.12
52.0	0.35 ± 0.23	0.00 ± 0.00	0.65 ± 0.23	48.4	56.5	43.0	0.12
53.0	0.36 ± 0.23	0.00 ± 0.00	0.64 ± 0.23	48.5	56.4	42.7	0.12
54.0	0.36 ± 0.23	0.00 ± 0.00	0.64 ± 0.23	48.5	56.4	42.8	0.12
55.0	0.43 ± 0.31	0.01 ± 0.01	0.56 ± 0.31	49.2	55.7	41.1	0.24
56.0	0.43 ± 0.32	0.04 ± 0.04	0.53 ± 0.33	48.8	56.1	40.6	0.23
57.0	0.42 ± 0.31	0.08 ± 0.07	0.50 ± 0.32	48.3	56.6	40.0	0.26
58.0	0.41 ± 0.30	0.13 ± 0.10	0.46 ± 0.31	47.8	57.1	39.1	0.28
59.0	0.37 ± 0.31	0.16 ± 0.13	0.47 ± 0.33	47.0	57.9	39.4	0.30
60.0	0.37 ± 0.29	0.21 ± 0.15	0.41 ± 0.30	46.6	58.4	38.2	0.33
65.0	0.40 ± 0.28	0.24 ± 0.15	0.36 ± 0.26	46.5	58.4	37.1	0.34
70.0	0.35 ± 0.29	0.30 ± 0.26	0.35 ± 0.29	45.4	59.5	37.0	0.41
75.0	0.35 ± 0.28	0.33 ± 0.27	0.33 ± 0.27	45.2	59.7	36.5	0.44
100.0	0.32 ± 0.28	0.33 ± 0.28	0.35 ± 0.29	44.9	60.0	36.9	0.45

Table 2: BR-DUE route flows for different values of the  $AL^{od}$ . The Gap values represent average values based on 1000 repetitions of the BR-DUE calculations.

We investigate the algorithms convergence for different values of  $AL^{od}$ , as shown in Fig. 2. To do this, we consider a total of 50 descent steps  $j$  of the MSA algorithm, despite the convergence criterion of  $Gap \leq 10^{-2}$  being verified for a lower number of  $j$ . This allows observing that the solution no longer changes after the convergence criterion is satisfied. In Fig. 2, we show the evolution of the Gap and route flows for increasing values of  $j$ , for the DUE and  $AL^{od} = T^{UE}, 48, 53, 100$ . For all five cases, the Gap value converges to a constant value for increasing values of  $j$  as well as the route flows.

We analyze the BR-SUE results for increasing values of  $AL^{od}$ . These results are listed in Table 3. Under SUE conditions, only routes 1 and 3 are *satisficing*. But, due to the users perception of travel times, there is a residual flow on route 2. For  $AL^{od} \in [48, 55]$ , the users change from route 3 to 1. For  $AL^{od} \geq 60$ , route 2 becomes *satisficing* and

the users also start choosing this route. The comparison of the BR-DUE and BR-SUE results for  $AL^{od} \geq 65$  are of particular interest. In both cases, the route flows converge to  $1/3$  when the value of  $AL^{od}$  is sufficiently large. This represents the users indifference for choosing any of the *satisficing* behaviors, since all the routes comply with the condition defined by Eq. 1. The effect of the perception of the route travel times explains the small differences verified in the route flows between the BR-DUE and BR-SUE for the same value of  $AL^{od}$ . For example, it is interesting to consider the case where  $AL^{od} = 70$ . The BR-DUE route flows are ( $Q_1 = 3.5; Q_2 = 3; Q_3 = 3.5$ ) and the BR-SUE route flows are ( $Q_1 = 3.8; Q_2 = 2.5; Q_3 = 3.7$ ). It can also be seen from the BR-SUE results shown in Table 3, that the MSA algorithm converges to the same solution, as evidenced by the standard deviation of the route flows. To solve the BR-SUE, we solve  $M$  BR-DUE problems locally, and since the search order is the indifference preferences, the users are assigned  $A$  times for each BR-DUE problem. By applying the Law of large numbers, we converge on average to the same solution when  $A$  is sufficiently large.

$AL^{od}$	$Q_1/Q^{od}$	$Q_2/Q^{od}$	$Q_3/Q^{od}$	$U_1$	$U_2$	$U_3$	$\overline{Gap}$
$\sim$	0.35	0.03	0.62	48.2	56.8	42.4	0.005
46.7	$0.35 \pm 0.00$	$0.03 \pm 0.00$	$0.62 \pm 0.00$	48.2	56.8	42.6	0.06
48.0	$0.35 \pm 0.00$	$0.03 \pm 0.00$	$0.62 \pm 0.00$	48.1	56.9	42.4	0.06
50.0	$0.35 \pm 0.00$	$0.03 \pm 0.00$	$0.62 \pm 0.00$	48.2	56.8	42.3	0.08
55.0	$0.40 \pm 0.00$	$0.03 \pm 0.00$	$0.57 \pm 0.00$	48.6	56.4	41.4	0.08
60.0	$0.42 \pm 0.01$	$0.09 \pm 0.01$	$0.49 \pm 0.01$	48.3	56.7	39.7	0.13
65.0	$0.41 \pm 0.01$	$0.18 \pm 0.01$	$0.41 \pm 0.01$	47.2	57.8	38.2	0.19
70.0	$0.38 \pm 0.01$	$0.25 \pm 0.01$	$0.37 \pm 0.01$	46.3	58.7	37.4	0.24
75.0	$0.36 \pm 0.01$	$0.29 \pm 0.01$	$0.35 \pm 0.01$	45.6	59.4	37.0	0.26
100.0	$0.33 \pm 0.00$	$0.33 \pm 0.00$	$0.33 \pm 0.00$	45.0	60.0	36.7	0.29

Table 3: BR-SUE route flows for different values of the  $AL^{od}$ .

We also observe that for both the BR-DUE and BR-SUE results, the average  $\overline{Gap}$  values increase as we increase  $AL^{od}$ , as expected. This represents the effect of the *satisficing* behavior, where users choose *satisficing* routes instead of the routes with the shortest travel times.

### 3.3. Strict preference search order and exogenous $AL^{od}$

In this section, we analyze the BR-DUE results calculated considering an exogenous definition of  $AL^{od}$  and a strict preference search order (Zhao and Huang, 2016). We calculate the BR-DUE results for the Braess network (Fig. 1), considering the six possible strict preference search orders ( $\Upsilon^{od}, \forall k = 1, 2, 3$ ). For the calculations, we consider our bounded rational model framework and the model discussed by Zhao and Huang (2016). The mathematical methodology discussed by Zhao and Huang (2016) is not suitable for a dynamic implementation considering a traffic simulator. This is because it requires solving sub-optimization problems to calibrate the  $AL$  of the sub-preferred routes. In this section, we compare the route flows calculated considering the two frameworks and the six preference search orders. The  $AL^{od}$  are set exogenously according to each strict preference order. We apply the model discussed by Zhao and Huang (2016), considering that the utility of the most preferred route is equal to the value of the aspiration level  $AL^{od}$ ; and the route flows of the remaining routes correspond to the result obtained by solving the sub-UE problem, as done by Zhao and Huang (2016). The results are listed in Tab. 4.

It can be seen that when  $AL^{od} = T^{UE} = 46.6$ , the route flows obtained for both models are equivalent to the DUE result. Thus, both models are able to reproduce the users perfect rationality whatever the preference order.

Consider the first strict preference order  $\Upsilon^{od} = 1, 2, 3$ . For  $47.5 \leq AL^{od} < 50$ , in both models the users switch directly from route 3 to the most preferred route 1. The flows on these two routes are equal for both models. But in our model, route 2 is not selected because it is not considered as *satisficing* whereas in the model of Zhao and Huang (2016), the assignment problem solved is:  $U_1 = A_1$  and  $U_2 = U_3$ . Under sub-User Equilibrium (sub-UE) conditions,  $U_2 > U_3$ . This means that the remaining users that have not chosen the most preferred route 1 will choose route 3. In our model, the users are assigned to the most preferred route until the *satisficing* condition (Eq. 1) is satisfied. Users are then assigned to the sub-preferred routes if and only if they are *satisficing*. Since route 2 is not *satisficing*,

$AL^{od}$	Our model						Zhao and Huang (2016)					
	$Q_1/Q^{od}$	$Q_2/Q^{od}$	$Q_3/Q^{od}$	$U_1$	$U_2$	$U_3$	$Q_1/Q^{od}$	$Q_2/Q^{od}$	$Q_3/Q^{od}$	$U_1$	$U_2$	$U_3$
Preference order $\Upsilon^{od} = 1\ 2\ 3$												
46.6	0.17	0.00	0.83	46.7	58.3	46.7	0.17	0.00	0.83	46.7	58.3	46.6
47.5	0.25	0.00	0.75	47.5	57.5	45.0	0.25	0.00	0.75	47.5	57.5	45.0
50.0	0.50	0.00	0.50	50.0	55.0	40.0	0.50	0.00	0.50	50.0	55.0	40.0
52.5	0.87	0.13	0.00	52.5	52.5	30.0	0.75	0.00	0.25	52.5	52.5	35.0
55.0	1.00	0.00	0.00	55.0	50.0	30.0	1.00	0.00	0.00	55.0	50.0	30.0
Preference order $\Upsilon^{od} = 1\ 3\ 2$												
46.6	0.17	0.00	0.83	46.7	58.3	46.7	0.17	0.00	0.83	46.7	58.3	46.6
47.5	0.25	0.00	0.75	47.5	57.5	45.0	0.25	0.00	0.75	47.5	57.5	45.0
50.0	0.50	0.00	0.50	50.0	55.0	40.0	0.50	0.00	0.50	50.0	55.0	40.0
52.5	0.75	0.00	0.25	52.5	52.5	35.0	0.75	0.00	0.25	52.5	52.5	35.0
55.0	1.00	0.00	0.00	55.0	50.0	30.0	1.00	0.00	0.00	55.0	50.0	30.0
Preference order $\Upsilon^{od} = 2\ 1\ 3$												
46.6	0.17	0.00	0.83	46.7	58.3	46.7	~	~	~	~	~	~
47.5	0.25	0.00	0.75	47.5	57.5	45.0	~	~	~	~	~	~
50.0	0.50	0.00	0.50	50.0	55.0	40.0	~	~	~	~	~	~
55.0	0.75	0.25	0.00	50.0	55.0	30.0	0.50	0.00	0.50	50.0	55.0	40.0
60.0	0.50	0.50	0.00	45.0	60.0	30.0	0.125	0.125	0.75	45.0	60.0	45.0
65.0	0.25	0.75	0.00	40.0	65.0	30.0	0.00	0.50	0.50	40.0	65.0	40.0
70.0	0.00	1.00	0.00	35.0	70.0	30.0	0.00	1.00	0.00	35.0	70.0	30.0
Preference order $\Upsilon^{od} = 2\ 3\ 1$												
46.6	0.17	0.00	0.83	46.7	58.3	46.7	~	~	~	~	~	~
47.5	0.13	0.00	0.87	46.3	58.7	47.5	~	~	~	~	~	~
50.0	0.00	0.00	1.00	45.0	60.0	50.0	~	~	~	~	~	~
55.0	0.00	0.00	1.00	45.0	60.0	50.0	0.50	0.00	0.50	50.0	55.0	40.0
60.0	0.00	0.00	1.00	45.0	60.0	49.9	0.125	0.125	0.75	45.0	60.0	45.0
65.0	0.00	0.50	0.50	40.0	65.0	40.0	0.00	0.50	0.50	40.0	65.0	40.0
70.0	0.00	1.00	0.00	35.0	70.0	30.0	0.00	1.00	0.00	35.0	70.0	30.0
Preference order $\Upsilon^{od} = 3\ 1\ 2$												
46.6	0.17	0.00	0.83	46.7	58.3	46.7	0.17	0.00	0.83	46.7	58.3	46.6
47.0	0.15	0.00	0.85	46.5	58.5	47.0	0.15	0.00	0.85	46.5	58.5	47.0
48.0	0.10	0.00	0.90	46.0	59.0	48.0	0.10	0.00	0.90	46.0	59.0	48.0
49.0	0.05	0.00	0.95	45.5	59.5	49.0	0.05	0.00	0.95	45.5	59.5	49.0
50.0	0.00	0.00	1.00	45.0	60.0	50.0	0.00	0.00	1.00	45.0	60.0	50.0
Preference order $\Upsilon^{od} = 3\ 2\ 1$												
46.6	0.17	0.00	0.83	46.7	58.3	46.7	0.17	0.00	0.83	46.7	58.3	46.6
47.0	0.15	0.00	0.85	46.5	58.5	47.0	0.15	0.00	0.85	46.5	58.5	47.0
48.0	0.10	0.00	0.90	46.0	59.0	48.0	0.10	0.00	0.90	46.0	59.0	48.0
49.0	0.05	0.00	0.95	45.5	59.5	49.0	0.05	0.00	0.95	45.5	59.5	49.0
50.0	0.00	0.00	1.00	45.0	60.0	50.0	0.00	0.00	1.00	45.0	60.0	50.0

Table 4: Route flow distribution for the Braess network for different values of the  $AL$  and under BR-DUE conditions. A set of strict preferences is considered for the search order.

the remaining users choose route 3. Thus, both models yield similar route flows for these two values of  $AL^{od}$ . For  $AL^{od} = 52.5$ , route 2 becomes *satisficing* for our model. Thus the users will switch according to the strict preference order  $\Upsilon^{od}$ . Thus, the users will first switch from route 3 to 2 and then from 2 to 1. In the case of the model of Zhao and Huang (2016), the sub-UE solution corresponds to  $U_2 > U_3$ . Thus, no user chooses route 2. For  $AL^{od} \geq 55$ , all the users choose the most preferred route 1 for both models.

Consider the second strict preference order  $\Upsilon^{od} = 1, 3, 2$ . In this case, the users switch directly from route 3 to the most preferred route 1 as  $AL^{od}$  increases. In the case of our model, the users are assigned to the most preferred route until it is considered as *satisficing*. Then, the remaining users are assigned to the sub-preferred route 3. In the case of the model of Zhao and Huang (2016), the sub-UE solution also corresponds to  $U_2 > U_3$  and thus no users choose route 2. So, for this strict preference order, both models give similar route flows as  $AL^{od}$  increases.

Consider the third strict preference order  $\Upsilon^{od} = 2, 1, 3$ . Given this preference order and for  $AL^{od} \leq 50$ , route 2 is not *satisficing* for our model. Therefore, the users will switch from route 3 (the less preferred) to the second most preferred route 1. The model of Zhao and Huang (2016) cannot be applied for  $AL^{od} < 55$ , since  $AL^{od} \neq U_2$ . This

would lead to a violation of the strict preference order assumption. We make this assumption more flexible. Although route 2 is the most preferred route, it is not considered as *satisficing* and the users choose other most preferred routes that are *satisficing*. This is the case of route 1. In the case of our model, for  $AL^{od} \geq 55$ , all the users switch from route 1 to 2 (the most preferred one). In the case of the model of [Zhao and Huang \(2016\)](#), the users will switch from route 1 to routes 2 and 3 (for  $AL^{od} = 55$  and  $AL^{od} = 60$ ) and then from 3 to 2 (for  $AL^{od} = 65$  and  $AL^{od} = 70$ ). Although route 3 is that least preferred, the users switch from route 1 to 3. Note that we solve the following assignment problem for the model of [Zhao and Huang \(2016\)](#):  $AL^{od} = U_2$  and  $U_1 = U_3$ .

Consider the fourth strict preference order  $\Upsilon^{od} = 2, 3, 1$ . In the case of our model, for  $AL^{od} \leq 60$ , route 2 is not *satisficing*. Therefore, all the users switch from route 1 to 3 in accordance with the strict preference order. For  $AL^{od} \geq 60$ , route 2 becomes *satisficing* and the users switch from route 3 to 2. In the case of the model of [Zhao and Huang \(2016\)](#), the condition  $AL^{od} = U_2$  is satisfied only for  $AL^{od} \geq 55$ . For  $AL^{od} \geq 55$ , we observe a route flow pattern similar to that in the previous strict preference order ( $\Upsilon^{od} = 2, 1, 3$ ) for the same reasons discussed previously.

Consider the fifth and sixth strict preference orders  $\Upsilon^{od} = 3, 1, 2$  and  $\Upsilon^{od} = 3, 2, 1$ . In the case of our model, route 2 is not *satisficing* for these two strict preference orders. Thus, users switch directly from route 1 to 3, as  $AL^{od}$  increases. For  $AL^{od} \geq 50$ , all users choose the most preferred route 3. In the case of the model of [Zhao and Huang \(2016\)](#), we solve the following assignment problem:  $AL^{od} = U_3$  and  $U_1 < U_2$  (sub-UE problem). Since  $U_1 < U_2$  for all the listed values of  $AL^{od}$ , the remaining users that have not chosen route 3 choose route 1 for both strict preference orders. This is why we observe similar route flows for both models and both strict preference orders.

In summary, we validate our bounded rational framework considering a strict users preference search order, by comparing the route flows at equilibrium with the results obtained by the model of [Zhao and Huang \(2016\)](#). Moreover, our bounded rational framework is suitable for dynamic implementation with a traffic simulator and will be tested in the next section.

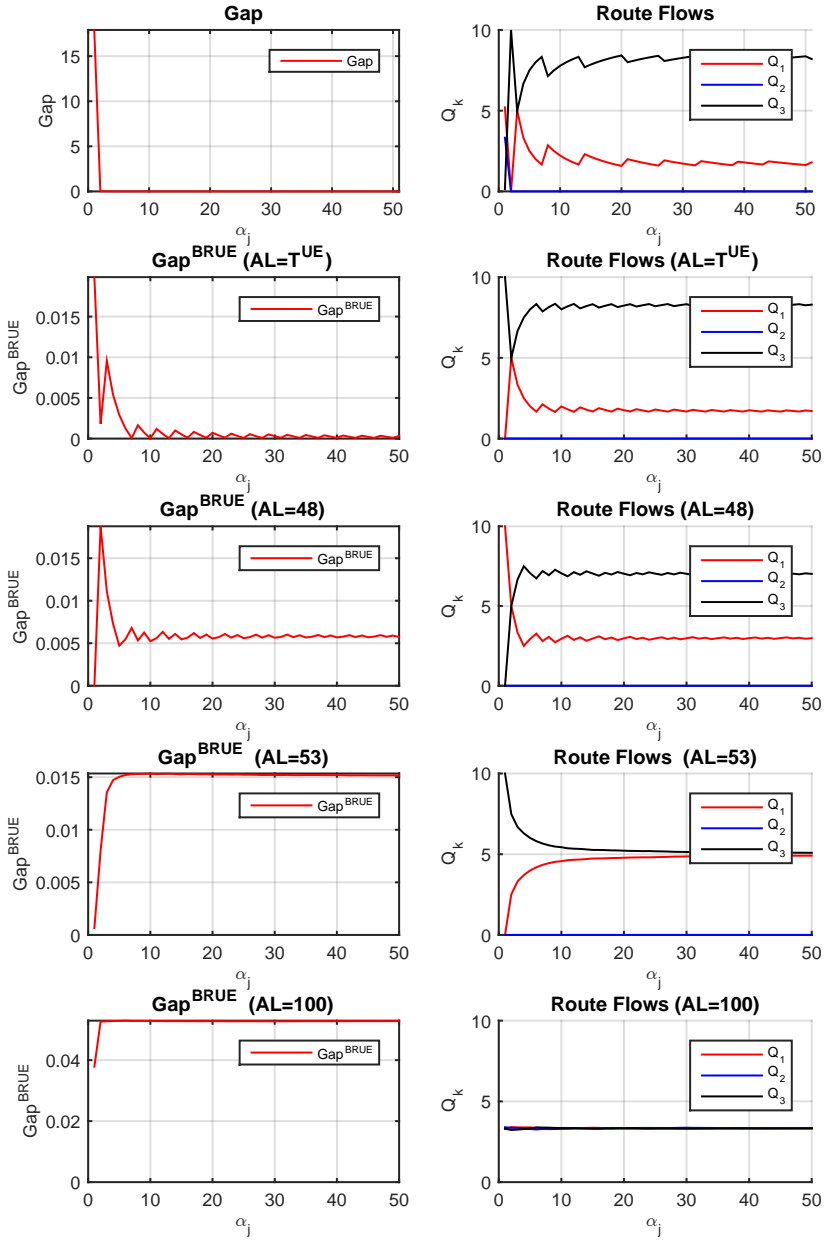


Figure 2: Gap (left) and route flows (right) as a function of the increasing number of MSA descent steps  $j$  for the DUE and several values of  $AL^{od} = T^{UE}, 48, 53, 100$ .

#### 4. Dynamic implementation on a Manhattan network

In this section, we investigate the influence of different types of bounded rational user behavior on: (i) individual route flows; (ii) network performance in terms of its internal, inflow and outflow capacities. To do this, we consider the implementation of the bounded rational framework described in Algorithm 1. To determine the time-dependent cost paths that account for congestion, shock-waves and spillback effects, we consider a mesoscopic LWR traffic simulator (Leclercq and Becarie, 2012). The tests are performed on a Manhattan network. We consider the indifferent and strict preference search orders and both definitions of the  $AL^{od}$ , as in Eq. 2 and Eq. 3.

##### 4.1. Test scenario definition

For the dynamic implementation, we consider the Manhattan network composed of 134 links, as shown in Fig. 3. All the links have the same length of 100 meters. Traffic lights regulate all the intersections. A green light duration of 45 seconds is considered for the traffic lights of the horizontal links whereas a duration of 15 seconds is considered for the traffic lights of the vertical links. Green times are set in the West-East and in the North-South directions. The offsets considered are of 10 and 20 seconds.

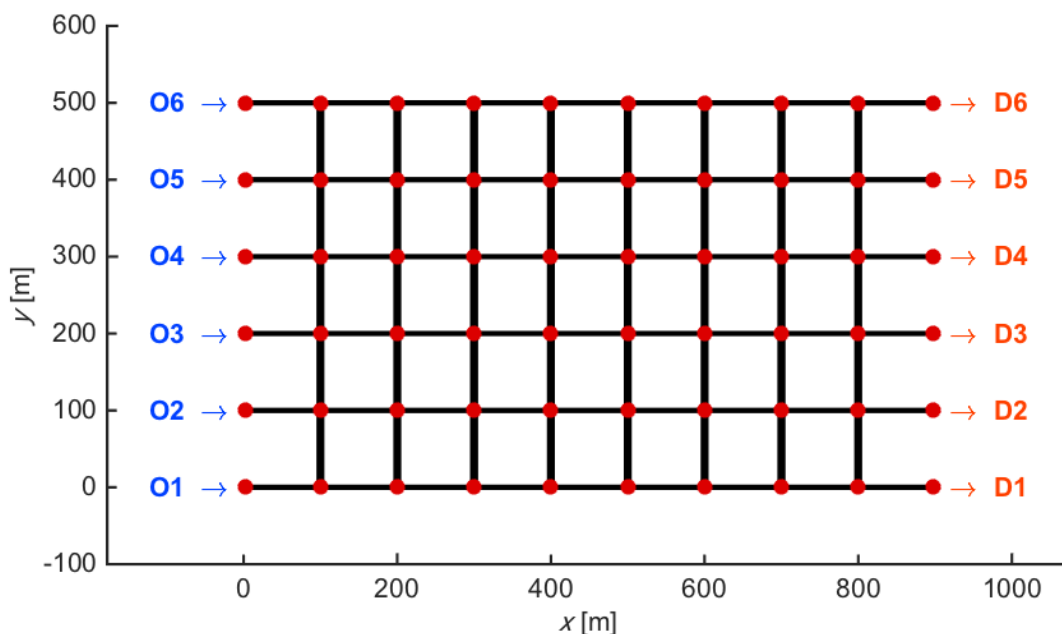


Figure 3: Manhattan network.

A triangular fundamental diagram is considered for each lane of the network, with the following parameters:  $u = 15$  (m/s), for the free-flow speed;  $w = 5$  (m/s) for the wave speed; and  $k_{jam} = 0.2$  (veh/m/lane) for the jam density. The entry links (i.e., from O1 to 6) have two lanes. The total link flow is assigned equally on each lane.

The Manhattan network shown in Fig. 3, has six entries (identified by O1 to O6 in Fig. 3) and exits (identified by D1 to D6 in Fig. 3). For each of the six entries, we consider a constant inflow (demand) of 0.5 (veh/s). There is no capacity restriction at the exits. There is a total of 36 possible od pairs. To define the choice set  $\Omega^{od}$ , we consider 3 paths per od pair. These paths are calculated using a K-shortest path algorithm. This gives a total of 108 routes, considering the 36 possible od pairs.

For the dynamic tests and the bounded rational route choice model, we consider the endogenous definitions of the indifference band for  $AL^{od}$  as defined by Eq. 2 and Eq. 3 and two settings of the search order:



- a *indifferent* preference search order, where users randomly choose any of the *satisficing* alternatives; or, the least worst if there are no *satisficing* alternatives.
- a *strict preference* search order, where users have a strict preference for the routes with the most reliable travel times. We consider the variances of the route travel times as the time reliability indicator. Then, the set of strict preference is built by ordering the routes from the lowest to the highest variance value for each od pair. This set of preferences is updated at every descent step of the MSA, based on the route travel time distributions of the previous simulation.

Considering this search order, users seek *satisficing* alternatives based on this set of strict preferences and on  $AL^{od}$ . Similarly, for the strict preference search order discussed in Sect. 3.3, the users choose only the most preferred route if it is perceived as *satisficing*, i.e. that conforms to Eq. 1. Then, if the most preferred route is not perceived as *satisficing*, the users consider the other most preferred routes until they find one that is *satisficing*. If none of the routes are *satisficing*, the users choose the route with the minimal travel time.

As a reference, we consider the DUE and SUE. To solve the SUE, we consider the Probit model with gamma distributed error terms and use Monte Carlo simulations (Sheffi, 1985). For the indifference band defined by Eq. 2, we consider three exogenous values for  $\Delta^{od}$ : 0; 100; and 500. We have a total of 10 simulation scenarios, considering both search orders defined above. The total simulation period is  $T = 3000$  seconds. For the convergence, we set  $tol = 10^{-2}$ ,  $\Phi = 0$  and  $N_{max} = 250$ .

#### 4.2. Analysis of the individual route flows

In this section, we analyze the individual route flows for the 10 simulation scenarios. Each scenario is identified by one ID number, as listed in Table 5. We also list the Gap values in Table 5, that are calculated using Eq. 8. In Fig. 6, we show the distributions of the average route travel times for the 10 simulation scenarios. Note that, the average travel time per route for each scenario (i.e., the average of these distributions) is also listed in Table 5. In Fig. 4 and Fig. 5, we show the route flow distributions for each od pair of the network and all ten scenarios.

Model	ID Model	Preference	Gap	$\overline{TT}_k$ [s]
DUE	1	~	0.3	315
SUE	2	~	0.3	295
IB ( $\Delta^{od} = 0$ )	3	Indifferent	0.3	294
IB ( $\Delta^{od} = 100$ )	4	Indifferent	7.3	337
IB ( $\Delta^{od} = 500$ )	5	Indifferent	14.7	332
IB (Ge and Zhou, 2012)	6	Indifferent	16.2	350
IB ( $\Delta^{od} = 0$ )	7	Strict	68.0	312
IB ( $\Delta^{od} = 100$ )	8	Strict	68.8	297
IB ( $\Delta^{od} = 500$ )	9	Strict	68.8	297
IB (Ge and Zhou, 2012)	10	Strict	69.1	297

Table 5: The Gap value and the average travel times per route  $\overline{TT}_k$  [in s] calculated from the distributions shown in Fig. 6 are also listed. These values are listed for the DUE, SUE and different settings of the indifference band.

We analyze the individual route flows shown in Fig. 4 and Fig. 5. By setting  $\Delta^{od} = 0$  (Model 3), we observe that for the indifferent preference search order, we obtain similar route flows compared to the SUE (Model 2). However, this is not observed for the strict preference search order, when comparing  $\Delta^{od} = 0$  (Model 7) and the SUE (Model 2). This is due to the specific definition of the search order, where the routes with the most reliable travel times (i.e. with the lowest variances) may not correspond to the routes with the lowest travel times. This is also evidenced by the Gap values listed in Table 5, for the settings of the strict preference search order. Also note that in the case of the indifferent preference search order, setting  $\Delta^{od} = 0$ , only the lowest travel time route per od pair is considered as *satisficing* at each descent step of the MSA. For the indifferent preference search order, the users indifference increases as we increase  $\Delta^{od}$  from 0 to 500,. The route flows will then converge to 1/3 for all the od pairs (Model 5, in Fig. 4 and Fig. 5). For  $\Delta^{od} = 500$ , the indifference band is sufficiently high with the result that all the routes in  $\Omega^{od}$  for all od pairs are *satisficing*. Thus, the users can choose any of the routes. Since the users indifference increases, they

are will choose routes with higher travel times and consequently the Gap value also increases. Note that here, the Gap indicates how far the simulation results are from the DUE; also it is calculated as in Eq. 8. On the other hand, the distributions of the average route travel times (Fig. 6) also shift towards longer travel times due to an increase in user indifference. The average travel times per route also increase from 295 seconds for  $\Delta^{od} = 0$  to 332 seconds for  $\Delta^{od} = 500$ . The strict preference search order reduces the variances of the distributions of the average route travel times compared to the indifferent preference search order.

### 4.3. Analysis of the aggregated traffic stats of the network

In this section, we analyze the network performance in terms of its inflow capacity and internal accumulation of vehicles. In Fig. 7, we show the evolution of the total traveled distance (TTD) as well as the outgoing flow  $Q_{out}$  as a function of the total travel time (TTT), for both definitions of the search orders and all the simulation scenarios. Fig. 7 shows that the network capacity is higher for the strict preference search order case. This is observed by comparing the  $\Delta^{od} = 500$  for both search orders, where the TTD is much lower for the indifferent preference search order compared to the strict preference search order. It can also be seen that the network capacity is approximately similar for the strict preference search order and the different settings of the indifference band. While, for the indifferent preference search order, the network capacity decreases with an increase of the  $\Delta^{od}$ . This also increases the average waiting time for the vehicles to enter the network. The average waiting times per vehicle are: 51 s for the DUE; 52 s for the SUE; 52 s for  $\Delta^{od} = 0$ ; 54 s for  $\Delta^{od} = 100$ ; 61 s for  $\Delta^{od} = 500$ ; and 57 s for the setting of the indifference band defined by Ge and Zhou (2012). Note that these are the averaging waiting times for the indifferent preference search order. On the other hand, since the network capacity is approximately similar for the strict preference search order and the different settings of  $\Delta^{od}$ , the average waiting times per vehicle to enter the network are similar. The average waiting times for the strict preference search order are: 74 s for  $\Delta^{od} = 0$ ; 73 s for  $\Delta^{od} = 100$ ; 75 s for  $\Delta^{od} = 500$ ; and 74 s for the setting of the indifference band defined by Ge and Zhou (2012). From Fig. 7, we can also observe a clear impact of the users search order on the total travel time spent on the network. For example, for  $\Delta^{od} = 500$ , the TTT is larger for the strict preference compared to the indifferent preference search order. This induces a lower internal network performance. Also, in the case of the indifferent preference search order, users will tend to choose routes with higher travel times as  $\Delta^{od}$  increases. This leads to an increase of the accumulation of vehicles inside the network and consequently users spend more time to complete their trips. Also, the outflow  $Q_{out}$  of vehicles decreases as  $\Delta^{od}$  increases (Fig. 7). Note that a lower outflow  $Q_{out}$  means lower system efficiency.

In summary, we show that different types of bounded user rationality have different impacts on the network performance. Considering the indifferent preference search order where users present an indifference behavior for all of the *satisficing* routes, as  $\Delta^{od}$  increases, the internal and outflow capacities of the network decrease. However, when considering the strict user preference order, both the internal and outflow capacities of the network are approximately similar as  $\Delta^{od}$  increases.

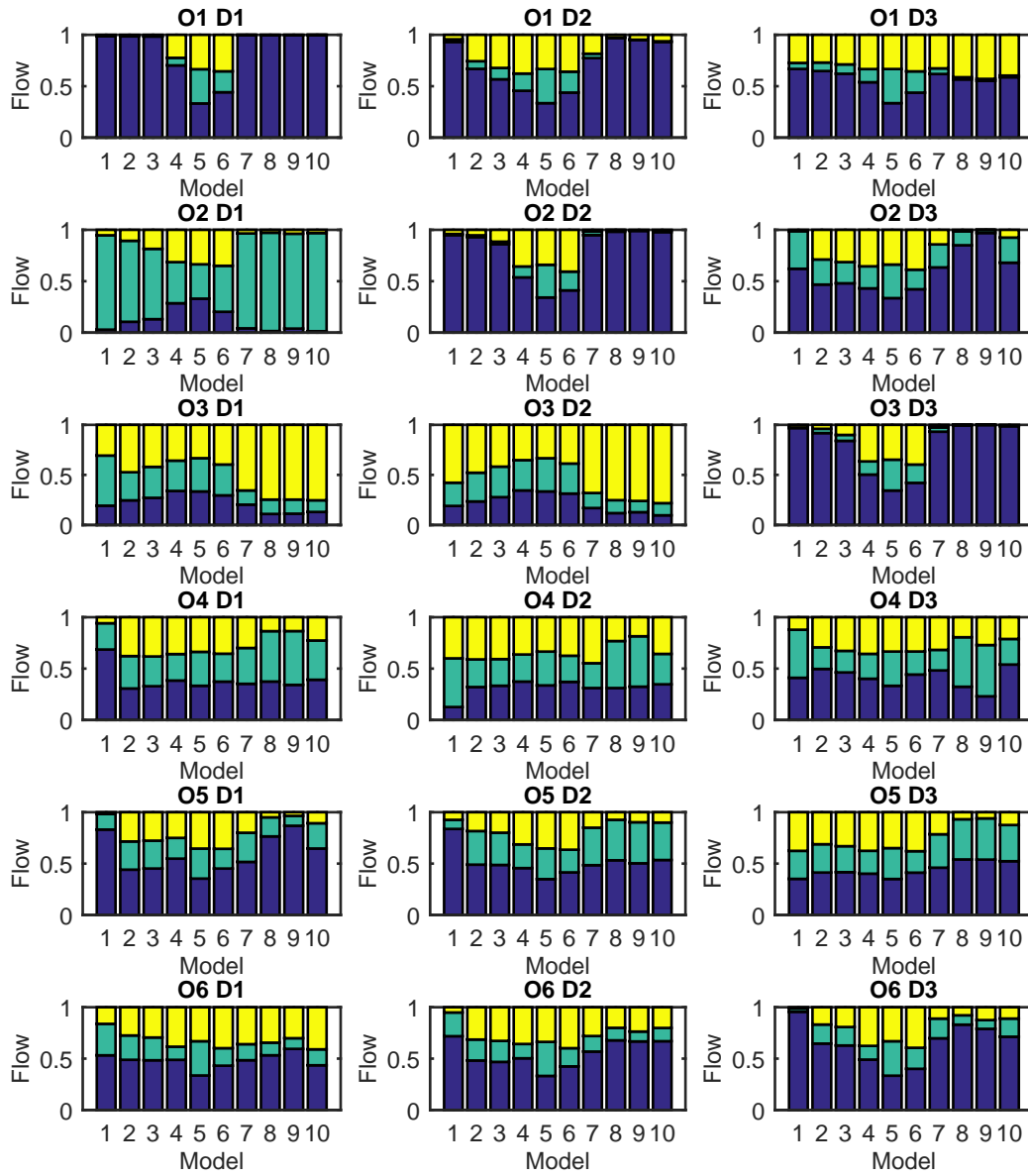


Figure 4: Route flow distributions for the 10 simulation scenarios and for the od pairs:  $o = 1, \dots, 6$ ; and  $d = 1, 2, 3$ . Each simulation scenario is identified by the Model ID equivalent to the ID values listed in Table 5.

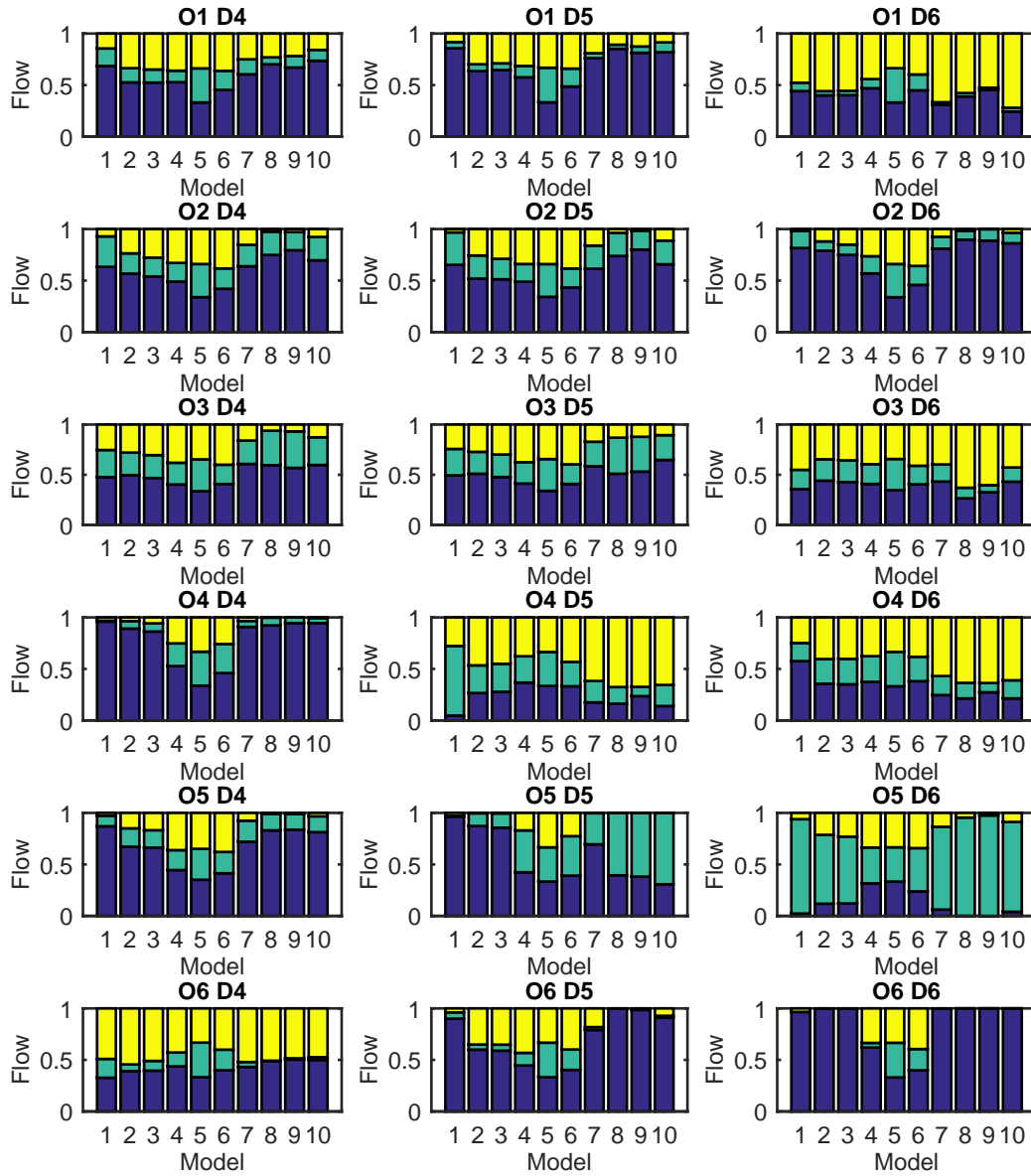


Figure 5: Same as in Fig. 4, but for the od pairs:  $o = 1, \dots, 6$ ; and  $d = 4, 5, 6$ .

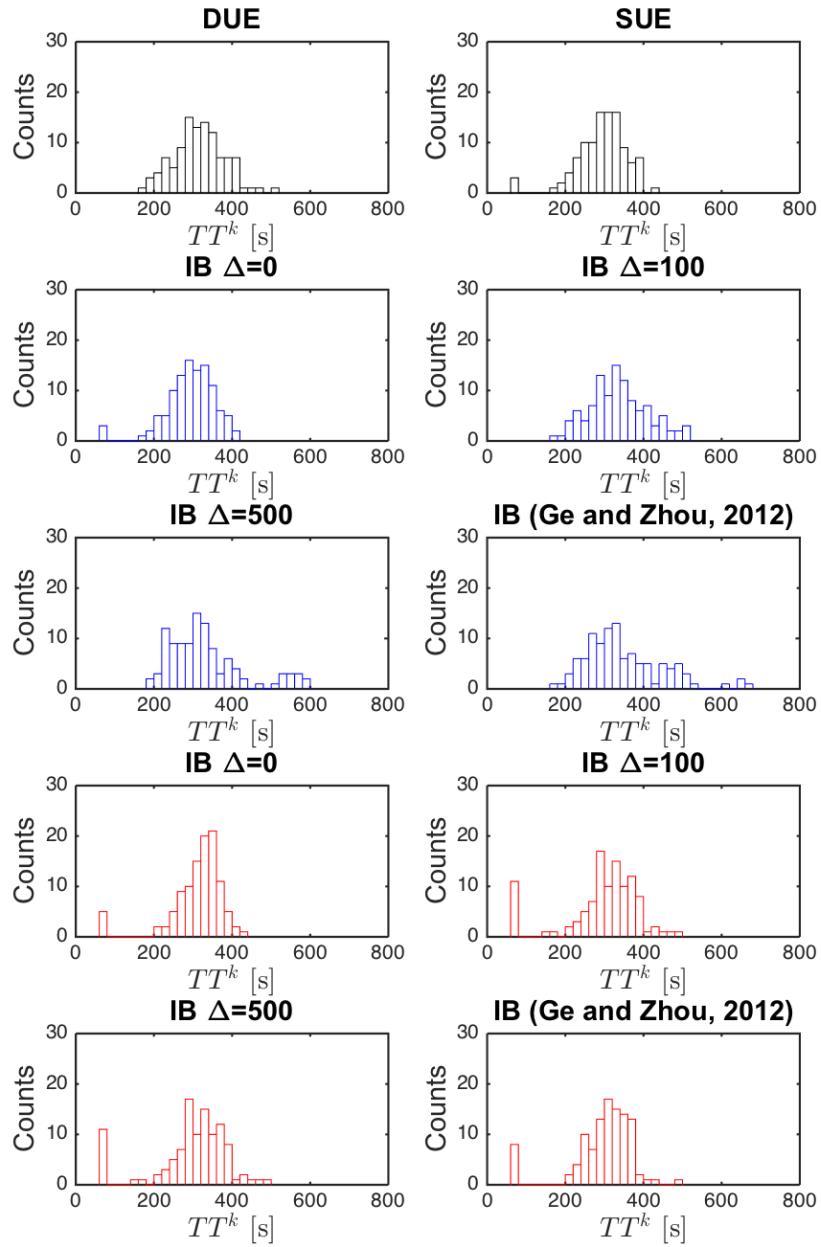


Figure 6: Average route travel time distributions for the DUE, SUE and different settings of the indifference band.

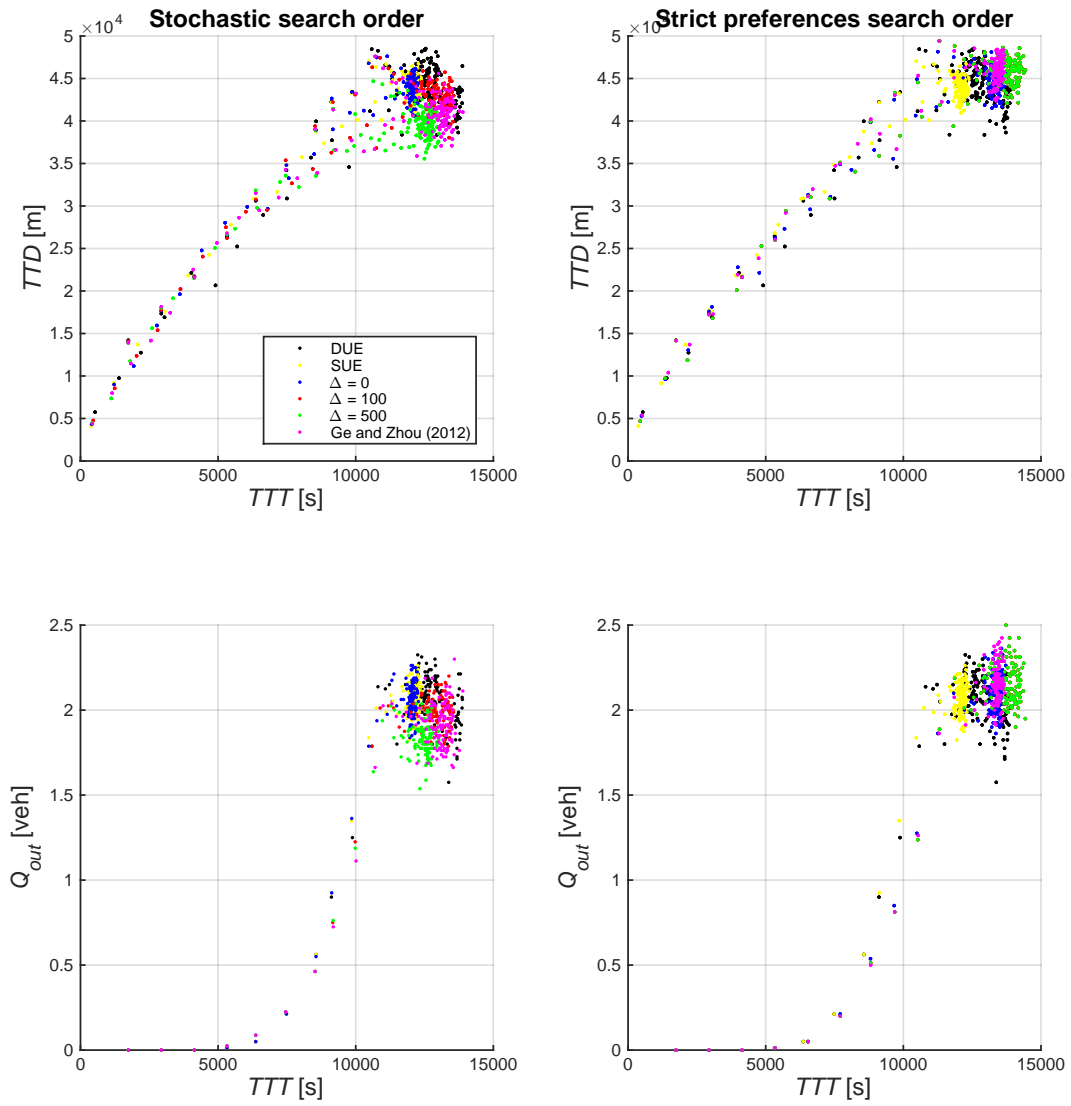


Figure 7: *Top*: Total travel distance (TTD) [in m] as a function of the total travel time (TTT) [in s]. *Bottom*: Total travel time (TTT) [in s] as a function of the vehicle outflow ( $Q_{out}$ ). The results are shown for the 10 simulation scenarios and both the indifferent (*Left*) and strict preference (*Right*) search orders.

## 5. Conclusions

Users route choices determine the level of congestion on a road network. Thus understanding the effects of users behavior is important for transportation network planning policies. In this paper, we investigated the influence of two types of bounded rational behavior, considering users preferences for the search order (i.e. indifferent and strict preferences), on individual route flows and network performance. To do this, we considered a dynamic implementation of a bounded rational framework, using a mesoscopic LWR traffic simulator (Leclercq and Becarie, 2012). The route costs were time-dependent and accounted for congestion, shock-waves and spillback effects. To model the bounded rationality behavior, we relaxed the definition of the search order of the DUE and SUE frameworks (Sheffi, 1985). Thus, instead of using an all-or-nothing procedure to assign the users to the route(s) with the minimum travel time(s), they were assigned according to a more flexible definition of the search order according to user preferences. We also considered both definitions of the indifference band (Eq. 2 and Eq. 3) for  $AL^{od}$ . To account for the distributions of travel times, we used Monte Carlo simulations (Sheffi, 1985) and algorithm based on the Method of Successive Averages was presented to solve the network equilibria.

To first assess and gain insight into the changes of route flows at equilibrium,  $AL^{od}$ , and for both user search orders, we considered a static implementation on the toy Braess network. The results obtained with the indifferent preference search order revealed that: (i) the bounded rational model framework is able to reproduce both DUE and SUE; (ii) when  $AL^{od}$  is sufficiently large, the route flows converge to 1/3, showing the user indifference for the route choice; (iii) the algorithm discussed converges. Also, based on this simple numerical test, we showed that we converge towards the same solution of the BR-SUE calculated, based on averaging over all local BR-DUE problems. In the second test, considering the strict user preference order, we showed that the route flows calculated between our model and the model of Zhao and Huang (2016) reach good agreement. This validated our methodology applied to determine the search order in a dynamic context.

We also investigated the influence of the two settings of the users search order on the individual route flows and network performance, considering both definitions of  $AL^{od}$  as defined in Eq. 2 and Eq. 3. These tests were performed in a dynamic context, using the mesoscopic LWR traffic simulator (Leclercq and Becarie, 2012). We first showed that we were able to reproduce the SUE, by setting  $\Delta^{od} = 0$  for the indifferent preference search order. For the strict preference search order, we did not obtain route flows similar to the SUE when setting  $\Delta^{od} = 0$ . This is due to the fact that routes with the most reliable travel times did not necessarily have the minimal travel time per each od pair. We also showed that for the indifferent preference search order, the route flows also tended to 1/3 as we increased  $\Delta^{od}$ .

We then showed that bounded user rationality had a significant impact on network performance. For the indifferent preference search orders, the network inflow capacity decreased as  $\Delta^{od}$  increased; and the network performance decreased as  $\Delta^{od}$  increased. The outflow  $Q_{out}$  also decreased as  $\Delta^{od}$  increased. For the strict preference search order, the network capacity was approximately similar for the different settings of the indifference band. However, since users were allowed to choose routes with longer travel times as  $\Delta^{od}$  increased, the TTT increased and the internal performance of the network decreased. In brief, we showed that different types of bounded rationality have clearly different influences on network performance. This is very important when guiding policy makers to decide the best measures to implement in order to increase network performance.

As future work, we can extend this work in many directions. We first plan to extend this work to the macroscopic fundamental diagram (MFD) simulation. We also plan to extend this framework to heterogeneous classes of users. The heterogeneity can either be included in the search order or on the setting of the  $AL^{od}$ . And, we also plan to extend this model to a day-to-day assignment, by considering a learning process (e.g., based on reinforcement learning models) either on the  $AL^{od}$  or the users search order definition. We also emphasize that further research in the setting of the  $AL^{od}$  is required. The setting of the preference orders in the search process allows to consider heterogeneous classes of users, with a preference for transportation mode, for example. However, we note that this users heterogeneity can also be included in the setting of the AL instead of the search order. In this case, the total demand  $Q$  of the od pair should be split into homogeneous groups of users with the same preference. The AL should then be defined per od and class of users. For each class of users, they are assigned based on an all-or-nothing assignment to one of the routes listed in  $\omega^{od}$ .

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## Data availability statement

No data were used to support this study.

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