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GEOMETRIC GENERATION OF THE WRAPPED FUKAYA CATEGORY OF WEINSTEIN MANIFOLDS AND SECTORS

BAPTISTE CHANTRAINE, GEORGIOS DIMITROGLOU RIZELL,
PAOLO GHIGGINI, AND ROMAN GOLOVKO

ABSTRACT. We prove that the wrapped Fukaya category of any $2n$ -dimensional Weinstein manifold (or, more generally, Weinstein sector) W is generated by the unstable manifolds of the index n critical points of its Liouville vector field. Our proof is geometric in nature, relying on a surgery formula for Floer homology and the fairly simple observation that Floer homology vanishes for Lagrangian submanifolds that can be disjointed from the isotropic skeleton of the Weinstein manifold. Note that we do not need any additional assumptions on this skeleton. By applying our generation result to the diagonal in the product $W \times W$, we obtain as a corollary that the open-closed map from the Hochschild homology of the wrapped Fukaya category of W to its symplectic cohomology is an isomorphism, proving a conjecture of Seidel.

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1. INTRODUCTION

The *wrapped Fukaya category* is an A_∞ -category associated to any Liouville manifold. Its objects are exact Lagrangian submanifolds which are either compact or cylindrical at infinity, possibly equipped with extra structure, the morphism spaces are wrapped Floer chain complexes, and the A_∞ operations are defined by counting perturbed holomorphic polygons with Lagrangian boundary conditions. Wrapped Floer cohomology was defined by A. Abbondandolo and M. Schwarz [1], at least for cotangent fibres, but the general definition and the chain level construction needed to define an A_∞ -category is due to M. Abouzaid and P. Seidel [3]. The definition of the wrapped Fukaya category was further extended to the relative case by Z. Sylvan, who introduced the notions of *stop* and *partially wrapped Fukaya category* in [34], and by S. Ganatra, J. Pardon and V. Shende, who later introduced the similar notion of *Liouville sector* in [23].

In this article we study the wrapped Fukaya category of Weinstein manifolds and sectors. In the absolute case our main result is the following.

Theorem 1.1. *If $(W, \theta, \mathfrak{f})$ is a $2n$ -dimensional Weinstein manifold of finite type, then its wrapped Fukaya category $\mathcal{WF}(W, \theta)$ is generated by the Lagrangian cocore planes of the index n critical points of \mathfrak{f} .*

In the relative case (i.e. for sectors) our main result is the following. We refer to Section 2.3 for the definition of the terminology used in the statement.

Theorem 1.2. *The wrapped Fukaya category of the Weinstein sector $(S, \theta, \mathfrak{f})$ is generated by the Lagrangian cocore planes of its completion $(W, \theta_W, \mathfrak{f}_W)$ and by the spreading of the Lagrangian cocore planes of its belt $(F, \theta_F, \mathfrak{f}_F)$.*

Remark 1.3. Exact Lagrangian submanifolds are often enriched with some extra structure: Spin structures, grading or local systems. We ignore them for simplicity, but the same arguments carry over also when that extra structure is considered.

Generators of the wrapped Fukaya category are known in many particular cases. We will not try to give a comprehensive overview of the history of this recent but active subject because we would not be able to make justice to everybody who has contributed to it. However, it is important to mention that F. Bourgeois, T. Ekholm and Y. Eliashberg in [8] sketch a proof that the Lagrangian cocore discs split-generate the wrapped Fukaya category of a Weinstein manifold of finite type. Split-generation is a weaker notion than generation, but is sufficient for most applications. Unfortunately however, Bourgeois, Ekholm and Eliashberg's proposed proof relies on their Legendrian surgery formula, whose analytic details are still in progress. Most generation results so far, including that of Bourgeois, Ekholm and Eliashberg, rely on Abouzaid's split-generation criterion [2]. On the contrary,

our proof is more direct and similar in spirit to Seidel's proof in [32] that the Lagrangian thimbles generate the Fukaya-Seidel category of a Lefschetz fibration or to Biran and Cornea's cone decomposition of Arnol'd type Lagrangian cobordisms [6].

A product of Weinstein manifolds is a Weinstein manifold. Therefore, by applying Theorem 1.1 to the diagonal in a twisted product, and using results of S. Ganatra [22] and Y. Gao [24], we obtain the following result.

Theorem 1.4. *Let $(W, \theta, \mathfrak{f})$ be a Weinstein manifold of finite type. Let \mathcal{D} the full A_∞ subcategory of $\mathcal{WF}(W, \theta)$ whose objects are the Lagrangian cocore planes. Then the open-closed map*

$$(1) \quad \mathcal{OC}: HH_*(\mathcal{D}, \mathcal{D}) \rightarrow SH^*(W)$$

is an isomorphism.

In Equation 1 HH_* denotes Hochschild homology, SH^* denotes symplectic cohomology and \mathcal{OC} is the open-closed map defined in [2]. Theorem 1.4 in particular proves that

$$(2) \quad \mathcal{OC}: HH_*(\mathcal{WF}(W, \theta), \mathcal{WF}(W, \theta)) \rightarrow SH^*(W)$$

is an isomorphism. This proves a conjecture of Seidel in [33] for Weinstein manifolds of finite type. Note that a proof of this conjecture, assuming the Legendrian surgery formula of Bourgeois, Ekholm and Eliashberg was given by S. Ganatra and M. Maydanskiy in the appendix of [8].

The above result implies in particular that Abouzaid's generation criterion [2] is satisfied for the subcategory consisting of the cocore planes of a Weinstein manifold, from which one can conclude that the cocores split-generate the wrapped Fukaya category. In the exact setting under consideration this of course follows a fortiori from Theorem 1.1, but there are extensions of the Fukaya category in which this generation criterion has nontrivial implications. Notably, this is the case for the version of the wrapped Fukaya category for monotone Lagrangians, as we proceed to explain.

The wrapped Fukaya category as well as symplectic cohomology were defined in the monotone symplectic setting in [30] using coefficients in the Novikov field. When this construction is applied to exact Lagrangians in an exact symplectic manifold, a change of variables $x \mapsto t^{-\mathcal{A}(x)}x$, where $\mathcal{A}(x)$ is the action of the generator x and t is the formal Novikov parameter, makes the Floer complexes and the open-closed map become identified with the original complexes and map tensored with the Novikov field. The generalisation of Abouzaid's generation criterion to the monotone setting established in [30] thus shows that

Corollary 1.5. *The wrapped Fukaya category of monotone Lagrangian submanifolds of a Weinstein manifold which are unobstructed in the strong*

sense (i.e. with $\mu^0 = 0$, where μ^0 is the number of Maslov index two holomorphic discs passing through a generic point) is split-generated by the Lagrangian cocore planes of the Weinstein manifold.

Remark 1.6. The strategy employed in the proof of Theorem 1.1 for showing generation fails for non-exact Lagrangian submanifolds in two crucial steps: in Section 7 and Section 8. First, there are well known examples of unobstructed monotone Lagrangian submanifolds in a Weinstein manifold which are Floer homologically nontrivial even if they are disjoint from the skeleton. Second, our treatment of Lagrangian surgeries requires that we lift the Lagrangian submanifolds in W to Legendrian submanifolds of $W \times \mathbb{R}$, and this is possible only for exact Lagrangian submanifolds. It is unclear to us whether it is true that the cocores generate (and not merely split-generate) the $\mu^0 = 0$ part of the monotone wrapped Fukaya category.

The strategy of the proof of Theorem 1.1 is the following. Given a cylindrical Lagrangian submanifold L , by a compactly supported Hamiltonian isotopy we make it transverse to the stable manifolds of the Liouville flow. Thus, by dimension reason, it will be disjoint from the stable manifolds of the critical points of index less than n and will intersect the stable manifolds of the critical points of index n in finitely many points a_1, \dots, a_k . For each a_i we consider a Lagrangian plane D_{a_i} passing through a_i , transverse both to L and to the stable manifold, and Hamiltonian isotopic to the unstable manifold of the same critical point. The Lagrangian planes D_{a_1}, \dots, D_{a_k} are what we call the Lagrangian cocore planes. We can assume that they are all pairwise disjoint.

At each a_i we perform a Lagrangian surgery between L and D_{a_i} so that the resulting Lagrangian \bar{L} is disjoint from the skeleton of W . Since \bar{L} will be in general immersed, we have to develop a version of wrapped Floer cohomology for immersed Lagrangian submanifolds. To do that we borrow heavily from the construction of Legendrian contact homology in [17]. In particular our wrapped Floer cohomology between immersed Lagrangian submanifolds uses augmentations of the Chekanov-Eliashberg algebras of the Legendrian lifts as bounding cochains. A priori there is no reason why such a bounding cochain should exist for \bar{L} , but it turns out that we can define it inductively provided that D_{a_1}, \dots, D_{a_k} are isotoped in a suitable way. A large part of the technical work in this paper is devoted to the proof of this claim.

Then we prove a correspondence between twisted complexes in the wrapped Fukaya category and Lagrangian surgeries by realising a Lagrangian surgery as a Lagrangian cobordism between the Legendrian lifts and applying the Floer theory for Lagrangian cobordisms we defined in [9]. This result can have an independent interest. Then we can conclude that \bar{L} is isomorphic, in an appropriated triangulated completion of the wrapped Fukaya category, to a twisted complex \mathbb{L} built from $L, D_{a_1}, \dots, D_{a_k}$.

Finally, we prove that the wrapped Floer cohomology of \bar{L} with any other cylindrical Lagrangian is trivial. This is done by a fairly simple action argument based on the fact that the Liouville flow displaces \bar{L} from any compact set because \bar{L} is disjoint from the skeleton of W . Then the twisted complex \mathbb{L} is a trivial object, and therefore some simple homological algebra shows that L is isomorphic to a twisted complex built from D_{a_1}, \dots, D_{a_k} .

This article is organised as follows. In Section 2 we recall some generalities about Weinstein manifolds and sectors. In Section 3 we recall the definition and the basic properties of Legendrian contact homology. In Sections 4 and 5 we define the version of Floer homology for Lagrangian immersions that we will use in the rest of the article. Despite their length, these sections contain mostly routine verifications and can be skipped by the readers who are willing to accept that such a theory exists. In Section 6 we define wrapped Floer cohomology for Lagrangian immersions using the constructions of the previous two sections. In Section 7 we prove that an immersed Lagrangian submanifold which is disjoint from the skeleton is Floer homologically trivial. In Section 8 we prove that Lagrangian surgeries correspond to twisted complexes in the wrapped Fukaya category. In Section 9 we finish the proof of the generation results, and in particular we construct the bounding cochain for \bar{L} . Finally, in Section 10 we prove the isomorphism between Hochschild homology and symplectic cohomology.

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2. GEOMETRIC SETUP

In this section we revise some elementary symplectic geometry with the purpose of fixing notation and conventions.

2.1. Liouville manifolds. Let (W, θ) be a Liouville manifold of finite type, from now on called simply a Liouville manifold. This means that $d\theta$ is a symplectic form, the Liouville vector field \mathcal{L} defined by the equation

$$\iota_{\mathcal{L}} d\theta = \theta$$

is complete and, for some $R_0 < 0$, there exists a proper smooth function $\mathfrak{r}: W \rightarrow [R_0, +\infty)$ such that, for $w \in W$,

- (i) $d_w \mathfrak{r}(\mathcal{L}_w) > 0$ if $\mathfrak{r}(w) > R_0$, and
- (ii) $d\mathfrak{r}_w(\mathcal{L}_w) = 1$ if $\mathfrak{r}(w) \geq R_0 + 1$.

In particular, R_0 is the unique critical value of \mathfrak{r} , which is of course highly nondegenerate, and every other level set is a contact type hypersurface.

We use the function \mathfrak{r} to define some useful subsets of W .

Definition 2.1. For every $R \in [R_0, +\infty)$ we denote $W_R = \mathfrak{r}^{-1}([R_0, R])$, $W_R^e = W \setminus \text{int}(W_R)$ and $V_R = \mathfrak{r}^{-1}(R)$.

The subsets W_R^e will be called the *ends* of W . The Liouville flow of (W, θ) induces an identification

$$(3) \quad ([R_0 + 1, +\infty) \times V, e^r \alpha) \cong (W_{R_0+1}^e, \theta),$$

where $V = V_0$ and α is the pull-back of θ to V_0 . More precisely, if ϕ denotes the flow of the Liouville vector field, the identification (3) is given by $(r, v) \mapsto \phi_r(v)$. Let $\xi = \ker \alpha$ be the contact structure defined by α . Every V_R , for $R > R_0$, is contactomorphic to (V, ξ) . Under the identification (3), the function \mathfrak{r} , restricted to $W_{R_0+1}^e$, corresponds to the projection to $[R_0 + 1, +\infty)$ in the sense that the following diagram commutes

$$\begin{array}{ccc} ([R_0 + 1, +\infty) \times V & \xrightarrow{\phi} & W_{R_0+1}^e \\ & \searrow & \swarrow \mathfrak{r} \\ & [R_0 + 1, +\infty) & \end{array}$$

Remark 2.2. The choice of R_0 in the definition of \mathfrak{r} is purely arbitrary because the Liouville flow is complete. In fact, for every map $\mathfrak{r}: W \rightarrow [R_0, +\infty)$ as above and for any $R'_0 < R_0$ there is a map $\mathfrak{r}': W \rightarrow [R'_0, +\infty)$ satisfying (i) and (ii), which moreover coincides with \mathfrak{r} on $\mathfrak{r}^{-1}([R_0 + 1, +\infty))$.

A diffeomorphism $\psi: W \rightarrow W$ is an *exact symplectomorphism* if $\psi^* \theta = \theta + dq$ for some function $q: W \rightarrow \mathbb{R}$. Flows of Hamiltonian vector fields are, of course, the main source of exact symplectomorphisms. Given a function $H: [-t_-, t_+] \times W \rightarrow \mathbb{R}$, where $t_{\pm} \geq 0$ and are allowed to be infinite, we define the Hamiltonian vector field X_H by

$$\iota_{X_H} d\theta = -dH.$$

Here dH denotes the differential in the directions tangent to W , and therefore X_H is a time-dependent vector field on W .

We spell out the change in the Liouville form induced by a Hamiltonian flow because it is a computation that will be needed repeatedly.

Lemma 2.3. *Let $H: [-t_-, t_+] \times W \rightarrow \mathbb{R}$ be a Hamiltonian function and φ_t its Hamiltonian flow. Then, for all $t \in [t_-, t_+]$, we have $\varphi_t^* \theta = \theta + dq_t$, where*

$$q_t = \int_0^t (-H_\sigma + \theta(X_{H_\sigma})) \circ \varphi_\sigma d\sigma.$$

Proof. We compute

$$\begin{aligned} \varphi_t^* \theta - \theta &= \int_0^t \frac{d}{d\sigma} (\varphi_\sigma^* \theta) d\sigma = \int_0^t \varphi_\sigma^* (L_{X_{H_\sigma}} \theta) d\sigma = \\ &= \int_0^t \varphi_\sigma^* (\iota_{X_{H_\sigma}} d\theta + d\iota_{X_{H_\sigma}} \theta) d\sigma = \int_0^t \varphi_\sigma^* (-dH_\sigma + d(\theta(X_{H_\sigma}))) d\sigma. \end{aligned}$$

□

2.2. Weinstein manifolds. In this article we will be concerned mostly with Weinstein manifolds of finite type. We recall their definition, referring to [11] for further details.

Definition 2.4. A Weinstein manifold $(W, \theta, \mathfrak{f})$ consists of:

- (i) an even dimensional smooth manifold W without boundary,
- (ii) a one-form θ on W such that $d\theta$ is a symplectic form and the Liouville vector field \mathcal{L} associated to θ is complete, and
- (iii) a proper Morse function $\mathfrak{f}: W \rightarrow \mathbb{R}$ bounded from below such that \mathcal{L} is a pseudogradient of \mathfrak{f} in the sense of [11, Equation (9.9)]: i.e.

$$d\mathfrak{f}(\mathcal{L}) \geq \delta(\|\mathcal{L}\|^2 + \|d\mathfrak{f}\|^2),$$

where $\delta > 0$ and the norms are computed with respect to some Riemannian metric on W .

The function \mathfrak{f} is called a *Lyapunov function* (for \mathcal{L}).

If \mathfrak{f} has finitely many critical points, then $(W, \theta, \mathfrak{f})$ is a Weinstein manifold of finite type. From now on, Weinstein manifold will always mean Weinstein manifold of finite type.

Given a regular value M of \mathfrak{f} the compact manifold $\{\mathfrak{f} \leq M\}$ is called a *Weinstein domain*. Any Weinstein domain can be completed to a Weinstein manifold in a canonical way by adding half a symplectisation of the contact boundary.

By Condition (iii) in Definition 2.4, the zeroes of \mathcal{L} coincide with the critical points of \mathfrak{f} . If W has dimension $2n$, the critical points of \mathfrak{f} have index at most

n . For each critical point p of \mathfrak{f} of index n , there is a stable manifold Δ_p and an unstable manifold D_p which are both exact Lagrangian submanifolds. We will call the unstable manifolds Δ_p of the critical points of index n the *Lagrangian cocore planes*.

Definition 2.5. Let $W_0 \subset W$ be a Weinstein domain containing all critical points of \mathfrak{f} . The *Lagrangian skeleton* of $(W, \theta, \mathfrak{f})$ is the attractor of the negative flow of the Liouville vector field on the compact part of W , i.e.

$$W^{sk} := \bigcap_{t>0} \phi^{-t}(W_0),$$

where ϕ denotes the flow of the Liouville vector field \mathcal{L} . Alternatively, W^{sk} can be defined as the union of unstable manifolds of all critical points of \mathfrak{f} .

The stable manifolds of the index n critical points form the top dimensional stratum of the Lagrangian skeleton.

A Morse function gives rise to a handle decomposition. In the case of a Weinstein manifold $(W, \theta, \mathfrak{f})$, the handle decomposition induced by \mathfrak{f} is compatible with the symplectic structure and is called the Weinstein handle decomposition of $(W, \theta, \mathfrak{f})$. By the combination of [11, Lemma 12.18] and [11, Corollary 12.21] we can assume that \mathcal{L} is Morse-Smale. This implies that we can assume that handles of higher index are attached after handles of lower index. The deformation making \mathcal{L} Morse-Smale can be performed without changing the symplectic form $d\theta$ and so that the unstable manifolds corresponding to the critical points of index n before and after such a deformation are Hamiltonian isotopic.

We will denote the union of the handles of index strictly less than n by W^{sc} . This will be called the *subcritical subdomain* of W . By construction, ∂W^{sc} is a contact type hypersurface in W .

We choose $\mathfrak{r}: W \rightarrow [R_0, +\infty)$, and we homotope the Weinstein structure so that

$$W_{R_0} = W^{sc} \cup \mathcal{H}_1 \cup \dots \cup \mathcal{H}_l,$$

where $\mathcal{H}_1, \dots, \mathcal{H}_l$ all are *standard* Weinstein handles corresponding to the critical points p_1, \dots, p_l of \mathfrak{f} of Morse index n ; see [35] for the description of the standard model, and [11, Section 12.5] for how to produce the Weinstein homotopy.

Remark 2.6. We could easily modify \mathfrak{f} so that it agrees with \mathfrak{r} on W_0^e . However, this will not be necessary.

The *core* of the Weinstein handle \mathcal{H}_i is the Lagrangian disc $C_i = \Delta_{p_i} \cap \mathcal{H}_i$. Let $D_\delta T^*C_i$ denote the disc cotangent bundle of C_i of radius $\delta > 0$. By the Weinstein neighbourhood theorem, there is a symplectic identification $\mathcal{H}_i \cong D_\delta T^*C_i$ for some δ . However, the restriction of θ to \mathcal{H}_i does not correspond to the restriction of the canonical Liouville form to $D_\delta T^*C_i$.

2.3. Weinstein sectors. In this section we introduce Weinstein sectors. These will be particular cases of Liouville sectors as defined in [23] characterised, roughly speaking, by retracting over a Lagrangian skeleton with boundary. In Section 2.4 below we will then show that any Weinstein pair as introduced in [19, Section 2] can be completed to a Weinstein sector.

Definition 2.7. A *Weinstein sector* $(S, \theta, I, \mathfrak{f})$ consists of:

- (1) an even dimensional smooth manifold with boundary S ;
- (2) a one-form θ on S such that $d\theta$ is a symplectic form and the associated Liouville vector field \mathcal{L} is complete and everywhere tangent to ∂S ;
- (3) a smooth function $I: \partial S \rightarrow \mathbb{R}$ which satisfies
 - (a) $dI(\mathcal{L}) = \alpha I$ for some function $\alpha: \partial S \rightarrow \mathbb{R}_+$ which is constant outside of a compact set and
 - (b) $dI(C) > 0$, where C is a tangent vector field on ∂S such that $\iota_C d\theta|_{\partial S} = 0$ and $d\theta(C, N) > 0$ for an outward pointing normal vector field N ;
- (4) a proper Morse function $\mathfrak{f}: S \rightarrow \mathbb{R}$ bounded from below having finitely many critical points, such that \mathcal{L} is a pseudogradient of \mathfrak{f} and satisfying moreover
 - (a) $d\mathfrak{f}(C) > 0$ on $\{I > 0\}$ and $d\mathfrak{f}(C) < 0$ on $\{I < 0\}$,
 - (b) the Hessian of a critical point of \mathfrak{f} on ∂S evaluates negatively on the normal direction N , and
 - (c) there is a constant $c \in \mathbb{R}$ whose sublevel set satisfies $\{\mathfrak{f} \leq c\} \subset S \setminus \partial S$ and contains all interior critical points of \mathfrak{f} .

For simplicity we will often drop part of the data from the notation. We will always assume that S is a Weinstein sector of *finite type*, i.e. that \mathfrak{f} has only finitely many critical points. A Weinstein sector is a particular case of an exact Liouville sector in the sense of [23].

Example 2.8. After perturbing the canonical Liouville form, the cotangent bundle of a smooth manifold Q with boundary admits the structure of a Weinstein sector.

To a Weinstein sector $(S, \theta, I, \mathfrak{f})$ we can associate two Weinstein manifolds in a canonical way up to deformation: the *completion* and the *belt*. The completion of S is the Weinstein manifold $(W, \theta_W, \mathfrak{f}_W)$ obtained by completing the Weinstein domain $W_0 = \{\mathfrak{f} \leq c\}$, which contains all interior critical points of \mathfrak{f} . The belt of S is the Weinstein manifold $(F, \theta_F, \mathfrak{f}_F)$ where $F = I^{-1}(0)$, $\theta_F = \theta|_F$ ¹ and $\mathfrak{f}_F = \mathfrak{f}|_F$. To show that the belt is actually a Weinstein manifold it is enough to observe that $d\theta_F$ is a symplectic form because F is transverse to the vector field C , and that the Liouville vector

¹We abuse the notation by denoting the pull back by the inclusion as a restriction.

field \mathcal{L} is tangent to F because $dI(\mathcal{L}) = \alpha I$, and therefore the Liouville vector field of θ_F is $\mathcal{L}_F = \mathcal{L}|_F$.

Let $\kappa \in \mathbb{R}$ be a number such that all critical points of \mathfrak{f} are contained in $\{\mathfrak{f} \leq \kappa\}$. We denote $S_0 = \{\mathfrak{f} \leq \kappa\}$ and $F_0 = F \cap S_0 = \{\mathfrak{f}_F \leq \kappa\}$. By Condition (4a) of Definition 2.7, the boundary ∂S_0 is a contact manifold with convex boundary with dividing set ∂F_0 . Moreover $S \setminus S_0$ can be identified to a half symplectisation. Thus, given $R_0 \ll 0$, we can define a function $\mathfrak{r}: S \rightarrow [R_0, +\infty)$ satisfying the properties analogous to those in Section 2.1. We then write $S_R := \mathfrak{r}^{-1}[R_0, R]$ and $S_R^e = S \setminus \text{int}(S_R)$.

Definition 2.9. Let ϕ be the flow of \mathcal{L} . The *skeleton* $S^{sk} \subset S$ of a Weinstein sector $(S, \theta, \mathfrak{f})$ is given by

$$S^{sk} := \bigcap_{t>0} \phi^{-t}(S_0).$$

Remark 2.10. Let W and F the completion and the belt, respectively, of the Weinstein sector S . To understand the skeleton S^{sk} it is useful to note the following:

- (1) critical points of \mathfrak{f} on ∂S are also critical points of $\mathfrak{f}|_{\partial S}$ and vice versa,
- (2) any critical point $p \in \partial S$ of \mathfrak{f} lies inside $\{I = 0\} = F$ and is also a critical point of \mathfrak{f}_F ,
- (3) the Morse indices of the two functions satisfy the relation

$$\text{ind}_{\mathfrak{f}}(p) = \text{ind}_{\mathfrak{f}_F}(p) + 1,$$

- (4) the skeleton satisfies $S^{sk} \cap \partial S = F^{sk}$.

The top stratum of the skeleton of $(S, \theta, \mathfrak{f})$ is given by the union of the stable manifolds of the critical points of \mathfrak{f} of index n , where $2n$ is the dimension of S . Those are of two types: the stable manifolds Δ_p where p is an interior critical point of \mathfrak{f} , which are also stable manifolds for \mathfrak{f}_W in the completion, and the stable manifolds Θ_p where p is a boundary critical point of \mathfrak{f} , for which $\Delta'_p = \Theta_p \cap \partial S$ is the stable manifold of p for \mathfrak{f}_F in F .

Thus the Weinstein sector S can be obtained by attaching Weinstein handles, corresponding to the critical points of \mathfrak{f} in the interior of ∂S , and Weinstein half-handles, corresponding to the critical points of \mathfrak{f} in the boundary ∂S . We denote by S^{sc} the subcritical part of S , i.e. the union of the handles and half-handles of index less than n (where $2n$ is the dimension of S), by $\{\mathcal{H}_i\}$ the critical handle corresponding to Δ_i and by $\{\mathcal{H}'_j\}$ the half-handle corresponding to Θ_j . Finally we also choose the function $\mathfrak{r}: S \rightarrow [R_0, \infty)$ as in Section 2.1 which furthermore satisfies

$$S_{R_0} = S^{sc} \cup \mathcal{H}_1 \cup \dots \cup \mathcal{H}_l \cup \mathcal{H}'_1 \cup \dots \cup \mathcal{H}'_{l'}$$

and modify the Liouville form θ so that $\mathcal{H}_1, \dots, \mathcal{H}_l, \mathcal{H}'_1, \dots, \mathcal{H}'_{l'}$ are *standard* Weinstein handles.

It follows from the symplectic standard neighbourhood theorem that a collar neighbourhood of ∂S is symplectomorphic to

$$(F \times T^*(-2\varepsilon, 0], d\theta_F + dp \wedge dq).$$

Definition 2.11. Let S be a Weinstein sector and let L be a Lagrangian submanifold of its belt F . The *spreading* of L is

$$\text{spr}(L) = L \times T_{-\varepsilon}^*(-2\varepsilon, 0] \subset F \times T^*(-2\varepsilon, 0) \subset S.$$

Remark 2.12. The spreading of L depends on the choice of symplectic standard neighbourhood of the collar. However, given two different choices, the corresponding spreadings are Lagrangian isotopic. Furthermore, if L is exact in F , then $\text{spr}(L)$ is exact in S , and thus two different spreadings are Hamiltonian isotopic.

Example 2.13. When the Weinstein sector is the cotangent bundle of a manifold with boundary, the spreading of a cotangent fibre of $T^*\partial Q$ is simply a cotangent fibre of T^*Q .

The proof of the following lemma is immediate.

Lemma 2.14. *The cocore planes of the index n half-handles of S are the spreading of the cocore planes of the corresponding index $n - 1$ handles of F .*

2.4. Going from a Weinstein pair to a Weinstein sector. In this section we describe how to associate a Weinstein sector to a Weinstein pair. We recall the definition of Weinstein pair, originally introduced in [19].

Definition 2.15. A *Weinstein pair* (W_0, F_0) is a pair of Weinstein domains $(W_0, \theta_0, \mathfrak{f}_0)$ and $(F_0, \theta_F, \mathfrak{f}_F)$ together with a codimension one Liouville embedding of F_0 into ∂W_0 .

We denote the completions of $(W_0, \theta_0, \mathfrak{f}_0)$ and $(F_0, \theta_F, \mathfrak{f}_F)$ by $(W, \theta_0, \mathfrak{f}_0)$ and $(F, \theta_F, \mathfrak{f}_F)$ respectively. Let $F_1 \subset F$ be a Weinstein domain retracting on F_0 . If F_1 is close enough to F_0 , the symplectic standard neighbourhood theorem provides us with a Liouville embedding

$$(4) \quad ((1 - 3\epsilon, 1] \times [-3\delta, 3\delta] \times F_1, sdu + s\theta_F) \hookrightarrow (W_0, \theta_0).$$

Here s and u are coordinates on the first and second factors, respectively, and we require that the preimage of ∂W_0 is $\{s = 1\}$ and $F_1 \subset \partial W_0$ is identified with $\{(1, 0, x) : x \in F_1\}$. We denote \mathcal{U} the image of the embedding (4). After deforming \mathfrak{f}_0 we may assume that it is of the form $\mathfrak{f}_0(s, u, x) = s$ in the same coordinates.

Let \mathcal{L}_F be the Liouville vector field of (F, θ_F) . We choose a smooth function $\mathfrak{r}_F : F \rightarrow [R_0, +\infty)$, $R_0 \ll 0$, such that

$$\bullet \quad F_0 = \mathfrak{r}_F^{-1}([R_0, 0]),$$

- $d\mathfrak{r}_F(\mathcal{L}_F) > 0$ holds inside $\mathfrak{r}_F^{-1}(R_0, +\infty)$, and
- $d\mathfrak{r}_F(\mathcal{L}_F) = 1$ holds inside $\mathfrak{r}_F^{-1}(R_0 + 1, +\infty)$.

For simplicity of notation we also assume that

- $F_1 = \mathfrak{r}_F^{-1}([R_0, 1])$,

where F_1 denotes the manifold appearing in Formula (4). This condition is apparently a loss of generality because it cannot be satisfied for every Liouville form on F . However, the general case can be treated with minimal changes.

Consider the smooth function

$$r: [-3\delta, 3\delta] \times F_1 \rightarrow \mathbb{R},$$

$$r(u, x) := 2 - \left(\frac{u}{3\delta}\right)^2 - \mathfrak{r}_F(x) - c$$

for some small number $c > 0$.

Lemma 2.16. *There exists a Weinstein domain $\widetilde{W}_0 \subset W$ containing all critical points of \mathfrak{f}_0 and which intersects some collar neighbourhood of W_0 containing \mathcal{U} precisely in the subset*

$$\mathcal{C} := \{s \leq r(u, x)\} \subset \mathcal{U}.$$

The goal is now to deform the Liouville form θ_0 on

$$S_0 := W_0 \cap \widetilde{W}_0$$

to obtain a Liouville form θ so that the completion of (S_0, θ) is the sought Weinstein sector. The deformation will be performed inside \mathcal{C} . Given a smooth function $\rho: [1 - 3\epsilon, 1] \rightarrow \mathbb{R}$ such that:

- $\rho(s) = 0$ for $s \in [1 - 3\epsilon, 1 - 2\epsilon]$,
- $\rho(s) = 2s - 1$ for $s \in [1 - \epsilon, 1]$, and
- $\rho'(s) \geq 0$ for $s \in [1 - 3\epsilon, 1]$,

we define a Liouville form $\theta_{\mathcal{U}}$ on \mathcal{U} by

$$\theta_{\mathcal{U}} = s(du + \theta_F) - d(\rho(s)u).$$

The proof of the following lemma is a simple computation.

Lemma 2.17. *Let $\rho: [1 - 3\epsilon, 1] \rightarrow \mathbb{R}$ be a smooth function such that:*

The Liouville vector field $\mathcal{L}_{\mathcal{U}}$ corresponding to the Liouville form $\theta_{\mathcal{U}}$ on \mathcal{U} is equal to

$$\mathcal{L}_{\mathcal{U}} = (s - \rho(s))\partial_s + \rho'(s)u\partial_u + \frac{\rho(s)}{s}\mathcal{L}_F.$$

We define the Liouville form θ on S_0 as $\theta|_{\mathcal{C}} = \theta_{\mathcal{U}}$ and $\theta|_{S_0 \setminus \mathcal{C}} = \theta_0$. By Lemma 2.17 the Liouville vector field \mathcal{L} of θ is transverse to $\partial S_0 \setminus \partial W_0$ and is equal to

$$(1-s)\partial_s + 2u\partial_u + \frac{2s-1}{s}\mathcal{L}_F$$

in a neighbourhood of $\partial S_0 \setminus \partial W_0$; in particular it is tangent to $\partial S_0 \setminus \partial W_0$. Thus we can complete (S_0, θ) to (S, θ) by adding a half-symplectisation of $\partial S_0 \setminus \text{int}(\partial W_0)$. We define a function $I: \partial S \rightarrow \mathbb{R}$ by setting $I = u$ on $\partial S_0 \cap \partial W_0$ and extending it to ∂S so that $dI(\mathcal{L}) = 2I$ everywhere. It is easy to check that (S, θ, I) is an exact Liouville sector.

A Lyapunov function $\mathfrak{f}: S_0 \rightarrow \mathbb{R}_+$ for \mathcal{L} can be obtained by interpolating between \mathfrak{f}_0 on $S_0 \setminus \mathcal{C}$ and $u^2 - (s-1)^2 + \mathfrak{f}_F + C$ on $\mathcal{C} \cap \{s \in [1-\epsilon, 1]\}$ for sufficiently large C . The Lyapunov function \mathfrak{f} can then be extended to a Lyapunov function $\mathfrak{f}: S \rightarrow \mathbb{R}_+$ in a straightforward way. The easy verification that $(S, \theta, I, \mathfrak{f})$ is a Weinstein sector is left to the reader.

2.5. Exact Lagrangian immersions with cylindrical end.

Definition 2.18. Let (W, θ) be a $2n$ -dimensional Liouville manifold. An *exact Lagrangian immersion with cylindrical end* (or, alternately, an *immersed exact Lagrangian submanifold with cylindrical end*) is an immersion $\iota: L \rightarrow W$ such that:

- (1) L is an n -dimensional manifold and ι is a proper immersion which is an embedding outside finitely many points,
- (2) $\iota^*\theta = df$ for some function $f: L \rightarrow \mathbb{R}$, called the *potential* of (L, ι) , and
- (3) the image of ι is tangent to the Liouville vector field of (W, θ) outside of a compact set of L .

In the rest of the article, *immersed exact Lagrangian submanifold* will always mean immersed exact Lagrangian submanifold with cylindrical end. Note that L is allowed to be compact, and in that case Condition (3) is empty: a closed immersed Lagrangian submanifold is a particular case of immersed Lagrangian submanifolds with cylindrical ends. With an abuse of notation, we will often write L either for the pair (L, ι) or for the image $\iota(L)$.

Example 2.19. Let $(W, \theta, \mathfrak{f})$ be a Weinstein manifold. The Lagrangian cocore planes D_p introduced in Section 2.2 are Lagrangian submanifolds with cylindrical ends.

Properness of ι and Condition (3) imply that for every immersed exact Lagrangian submanifold with cylindrical end $\iota: L \rightarrow W$ there is $R > 0$ sufficiently large such that $\iota(L) \cap W_R^e$ corresponds to $[R, +\infty) \times \Lambda$ under the identification

$$(W_R^e, \theta) \cong ([R, +\infty) \times V, e^r \alpha),$$

where Λ is a Legendrian submanifold of (V, ξ) . Then we say that L is *cylindrical over* Λ . Here Λ can be empty (if L is compact) or disconnected.

There are different natural notions of equivalence between immersed exact Lagrangian submanifolds. The stronger one is Hamiltonian isotopy.

Definition 2.20. Two exact Legendrian immersions (L, ι_0) and (L, ι_1) with cylindrical ends are Hamiltonian isotopic if there exists a function $H: [0, 1] \times W \rightarrow \mathbb{R}$ with Hamiltonian flow φ_t such that $\iota_1 = \varphi_1 \circ \iota_0$, and moreover $\iota_t = \varphi_t \circ \iota_0$ has cylindrical ends for all $t \in [0, 1]$.

Remark 2.21. If $f_0: L \rightarrow \mathbb{R}$ is the potential of (L, ι_0) , by Lemma 2.3 we can choose

$$(5) \quad f_1 = f_0 + \int_0^1 (-H_\sigma + \theta(X_{H_\sigma})) \circ \varphi_\sigma d\sigma.$$

as potential for (L, ι_1) .

The weakest one is exact Lagrangian regular homotopy.

Definition 2.22. Two exact Legendrian immersions (L, ι_0) and (L, ι_1) with cylindrical ends are exact Lagrangian regular homotopic if there exists a smooth path of immersions $\iota_t: L \rightarrow W$ for $t \in [0, 1]$ such that (L, ι_t) is an exact Lagrangian immersion with cylindrical ends for every $t \in [0, 1]$.

We recall that any exact regular homotopy $\iota_t: L \rightarrow W$ can be generated by a local Hamiltonian defined on L in the following sense.

Lemma 2.23. *An exact regular Lagrangian homotopy ι_t induces a smooth family of functions $G_t: L \rightarrow \mathbb{R}$ determined uniquely, up to a constant depending on t , by the requirement that the equation*

$$\iota_t^*(d\theta(\cdot, X_t)) = dG_t$$

be satisfied, where $X_t: L \rightarrow TW$ is the vector field along the immersion that generates ι_t . When ι_t has compact support, then dG_t has compact support as well.

Conversely, any Hamiltonian $G_t: L \rightarrow \mathbb{R}$ generates an exact Lagrangian isotopy $\iota_t: L \rightarrow (W, \theta)$ for any initial choice of exact immersion $\iota = \iota_0$.

Remark 2.24. If ι_t is generated by an ambient Hamiltonian isotopy, then H extends to a single-valued Hamiltonian on W itself. However, this is not necessarily the case for an arbitrary exact Lagrangian regular homotopy.

The limitations of our approach to define Floer homology for exact Lagrangian immersions require that we work with a restricted class of exact immersed Lagrangian submanifolds.

Definition 2.25. We say that a Lagrangian immersion (L, ι) is *nice* if the singularities of $\iota(L)$ are all transverse double points, and for every double point p the values of the potential at the two points in the preimage of p are distinct. Then, given a double point p , we will denote $\iota^{-1}(p) = \{p^+, p^-\}$, where $f(p^+) > f(p^-)$.

Remark 2.26. If L is not connected, we can shift the potential on different connected components by independent constants. If $\iota^{-1}(p)$ is contained in a connected component of L , then $f(p^+) - f(p^-)$ is still well defined. However, if the points in $\iota^{-1}(p)$ belong to different connected components, the choice of p^+ and p^- in $\iota^{-1}(p)$, and $f(p^+) - f(p^-)$, depend of the choice of potential. For technical reasons related to our definition of Floer homology, it seems useful, although unnatural, to consider the potential (up to shift by an overall constant) as part of the data of an exact Lagrangian immersion.

For nice immersed exact Lagrangian submanifolds we define a stronger form of exact Lagrangian regular homotopy.

Definition 2.27. Let (L, ι_0) and (L, ι_1) be nice exact Lagrangian immersed submanifolds with cylindrical ends. An exact Lagrangian regular homotopy (L, ι_t) is a *safe isotopy* if (L, ι_t) is nice for every $t \in [0, 1]$.

Niceness can always be achieved after a C^1 -small exact Lagrangian regular homotopy. In the rest of this article exact Lagrangian immersions will always be assumed nice.

2.6. Contactisation and Legendrian lifts. We define a contact manifold (M, β) , where $M = W \times \mathbb{R}$, with a coordinate z on \mathbb{R} , and $\beta = dz + \theta$. We call (M, β) the *contactisation* of (W, θ) . A Hamiltonian isotopy $\varphi_t: W \rightarrow W$ which is generated by a Hamiltonian function $H: [0, 1] \times W \rightarrow \mathbb{R}$ lifts to a contact isotopy $\psi_t^+: M \rightarrow M$ such that

$$\psi_t^+(x, z) = (\psi_t(x), z - q_t(x)),$$

where $q_t: W \rightarrow \mathbb{R}$ is the function defined in Lemma 2.3.

An immersed exact Lagrangian (L, ι) with potential $f: L \rightarrow \mathbb{R}$ uniquely defines a Legendrian immersion

$$\iota^+: L \rightarrow W \times \mathbb{R}, \quad \iota^+(x) = (\iota(x), -f(x)).$$

Moreover ι^+ is an embedding when (L, ι) is nice. We denote the image of ι^+ by L^+ and call it the *Legendrian lift* of L . On the other hand, any Legendrian submanifold of (M, β) projects to an immersed Lagrangian in W . This projection is called the *Lagrangian projection* of the Legendrian submanifold.

Double points of L correspond to Reeb chords of L^+ , and the action (i.e. length) of the Reeb chord projecting to a double point p is $f(p^+) - f(p^-)$. If L is

connected, different potentials induce Legendrian lifts which are contact isotopic by a translation in the z -direction. In particular, the action of Reeb chords is independent of the lift. On the other hand, if L is disconnected, different potentials can induce non-contactomorphic Legendrian lifts and the action of Reeb chords between different connected components depends on the potential.

3. LEGENDRIAN CONTACT HOMOLOGY

3.1. The Chekanov-Eliashberg algebra. In view of the correspondence between Legendrian submanifolds of (M, β) and exact Lagrangian immersions in (W, θ) , Floer homology for Lagrangian immersions will be a variation on the theme of Legendrian contact homology. The latter was proposed by Eliashberg and Hofer and later defined rigorously by Chekanov, combinatorially, in \mathbb{R}^3 with its standard contact structure in [10], and by Ekholm, Etnyre and Sullivan, analytically, in the contactisation of any Liouville manifold in [17]. In this subsection we summarise the analytical definition.

For $d > 0$, let $\tilde{\mathcal{R}}^{d+1} = \text{Conf}^{d+1}(\partial D^2)$ be the space of parametrised discs with $d+1$ punctures on the boundary. The automorphism group $\text{Aut}(D^2)$ acts on $\tilde{\mathcal{R}}^{d+1}$ and its quotient is the Deligne-Mumford moduli space \mathcal{R}^{d+1} . Given $\zeta = (\zeta_0, \dots, \zeta_d) \in \tilde{\mathcal{R}}^{d+1}$, we will denote

$$\Delta_\zeta = D^2 \setminus \{\zeta_0, \dots, \zeta_d\}.$$

Following [32], near every puncture ζ_i we will define positive and negative universal striplike ends with coordinates $(\sigma_i^+, \tau_i^+) \in (0, +\infty) \times [0, 1]$ and $(\sigma_i^-, \tau_i^-) \in (-\infty, 0) \times [0, 1]$ respectively. We will assume that $\sigma_i^- = -\sigma_i^+$ and $\tau_i^- = 1 - \tau_i^+$.

Remark 3.1. Putting both positive and negative strip-like ends near each puncture could be useful to compare wrapped Floer cohomology and contact homology, which use different conventions for positive and negative punctures.

Definition 3.2. Let (V, α) be a contact manifold with contact structure ξ and Reeb vector field \mathcal{R} . An almost complex structure J on $\mathbb{R} \times V$ is *cylindrical* if

- (1) J is invariant under translations in \mathbb{R} ,
- (2) $J(\partial_r) = \mathcal{R}$, where r is the coordinate in \mathbb{R} ,
- (3) $J(\xi) \subset \xi$, and $J|_\xi$ is compatible with $d\alpha|_\xi$.

Definition 3.3. An almost complex structure J on a Liouville manifold W is *compatible with θ* if it is compatible with $d\theta$ and, outside a compact set, corresponds to a cylindrical almost complex structure under the identification (3). We denote by $\mathcal{J}(\theta)$ the set of almost complex structures on W which are compatible with θ .

It is well known that $\mathcal{J}(\theta)$ is a contractible space.

Given an exact Lagrangian immersion (L, ι) in W , we will consider almost complex structures J on W which satisfy the following

- (†) J is compatible with θ , integrable in a neighbourhood of the double points of (L, ι) , and for which L moreover is real-analytic near the double points.

We will denote the set of double points of (L, ι) by D .

Let $u: \Delta_\zeta \rightarrow W$ be a J -holomorphic map with boundary in L . If u has finite area and no puncture at which the lift of $u|_{\partial\Delta_\zeta}$ to L has a continuous extension, then $\lim_{z \rightarrow \zeta_i} u(z) = p_i$ for some $p_i \in D$. Since the boundary of u switches branch near p_i , the following dichotomy thus makes sense:

Definition 3.4. We say that ζ_i is a *positive puncture* at p_i if

$$\lim_{\sigma_i^+ \rightarrow +\infty} (\iota^{-1} \circ u)(\sigma_i^+, 0) = p_i^+$$

and that ζ_i is a *negative puncture* at p_i if

$$\lim_{\sigma_i^- \rightarrow -\infty} (\iota^{-1} \circ u)(\sigma_i^-, 0) = p_i^+.$$

Let L be an immersed exact Lagrangian. If p_1, \dots, p_d are double points of L (possibly with repetitions), we denote by $\mathfrak{N}_L(p_0; p_1, \dots, p_d; J)$ the set of pairs (ζ, u) where:

- (1) $\zeta \in \tilde{R}^{d+1}$ and $u: \Delta_\zeta \rightarrow W$ is a J -holomorphic map,
- (2) $u(\partial\Delta_\zeta) \subset L$, and
- (3) ζ_0 is a positive puncture at p_0 and ζ_i , for $i = 1, \dots, d$, is a negative puncture at p_i .

The group $\text{Aut}(D^2)$ acts on $\tilde{\mathfrak{N}}_L(p_0; p_1, \dots, p_d; J)$ by reparametrisations; the quotient is the moduli space $\mathfrak{N}_L(p_0; p_1, \dots, p_d; J)$. Note that the set p_1, \dots, p_d can be empty. In this case, the elements of the moduli spaces $\mathfrak{N}(p_0; J)$ are called *teardrops*.

Given $u \in \tilde{\mathfrak{N}}_L(p_0; p_1, \dots, p_d; J)$, let D_u be the linearisation of the Cauchy-Riemann operator at u . By standard Fredholm theory, D_u is a Fredholm operator with index $\text{ind}(D_u)$. We define the *index* of u as

$$\text{ind}(u) = \text{ind}(D_u) + d - 2.$$

The index is locally constant, and we denote by $\mathfrak{N}_L^k(p_0; p_1, \dots, p_d; J)$ the subset of $\mathfrak{N}_L(p_0; p_1, \dots, p_d; J)$ consisting of classes of maps u with $\text{ind}(u) = k$.

The following proposition is a version of [17, Proposition 2.3]:

Proposition 3.5. *For a generic J satisfying the condition (\dagger) , the moduli space $\mathfrak{N}_L^k(p_0; p_1, \dots, p_d; J)$ is a transversely cut out manifold of dimension k . In particular, if $k < 0$ it is empty; if $k = 0$ it is compact, and therefore consists of a finite number of points; and if $k = 1$, it can be compactified in the sense of Gromov, see [17, Section 2.2].*

The boundary of the compactification of the moduli space $\mathfrak{N}_L^1(p_0; p_1, \dots, p_d; J)$ is

$$(6) \quad \bigsqcup_{q \in D} \bigsqcup_{0 \leq i < j \leq d} \mathfrak{N}_L^0(p_0; p_1, \dots, p_i, q, p_{j+1}, \dots, p_d; J) \times \mathfrak{N}_L^0(q; p_{i+1}, \dots, p_j; J).$$

If L is spin, the moduli spaces are orientable and a choice of spin structure induces a coherent orientation of the moduli spaces; see [16].

Definition 3.6. We say that an almost complex structure J on W is L -regular if it satisfies (\dagger) for L and all moduli spaces $\mathfrak{N}_L(p_0; p_1, \dots, p_d; J)$ are transversely cut out.

To a Legendrian submanifold L^+ of (M, β) we can associate a differential graded algebra $(\mathfrak{A}, \mathfrak{d})$ called the *Chekanov-Eliashberg algebra* (or *Legendrian contact homology algebra*) of L^+ . As an algebra, \mathfrak{A} is the free unital noncommutative algebra generated by the double points of the Lagrangian projection L or, equivalently, by the Reeb chords of L^+ . The grading takes values in $\mathbb{Z}/2\mathbb{Z}$ and is simply given by the self-intersection of the double points. If $2c_1(W) = 0$ and the Maslov class of L^+ vanishes, it can be lifted to an integer valued grading by the Conley-Zehnder index. We will not make explicit use of the integer grading, and therefore we will not describe it further, referring the interested reader to [17, Section 2.2] instead.

The differential \mathfrak{d} is defined on the generators as:

$$\mathfrak{d}(p_0) = \sum_{d \geq 0} \sum_{p_1, \dots, p_d} \# \mathfrak{N}_L^0(p_0; p_1, \dots, p_d; J) p_1 \dots p_d.$$

According to [17, Proposition 2.6], the Chekanov-Eliashberg algebra is a Legendrian invariant:

Theorem 3.7 ([17]). *If L_0^+ and L_1^+ are Legendrian isotopic Legendrian submanifolds of (M, β) , then their Chekanov-Eliashberg algebras $(\mathfrak{A}^0, \mathfrak{d}_0)$ and $(\mathfrak{A}^1, \mathfrak{d}_1)$ are stably tame isomorphic.*

The definition of stable tame isomorphism of DGAs was introduced by Chekanov in [10], and then discussed by Ekholm, Etnyre and Sullivan in [17]. We will not use it in this article but note that on the homological level a stable tame isomorphism induces an isomorphism.

3.2. Bilinearised Legendrian contact cohomology. Differential graded algebras are difficult objects to manipulate, and therefore Chekanov introduced a linearisation procedure. The starting point of this procedure is the existence of an augmentation.

Definition 3.8. Let \mathfrak{A} be a differential graded algebra over a commutative ring \mathbb{F} . An *augmentation* of \mathfrak{A} is a unital differential graded algebra morphism $\varepsilon: \mathfrak{A} \rightarrow \mathbb{F}$.

Let L_0^+ and L_1^+ be Legendrian submanifolds of (M, β) with Lagrangian projections L_0 and L_1 with potentials f_0 and f_1 respectively. We recall that the potential is the negative of the z coordinate. We will assume that L_0^+ and L_1^+ are *chord generic*, which means in this case that $L_0^+ \cap L_1^+ = \emptyset$ and all singularities of $L_0 \cup L_1$ are transverse double points.

Let \mathfrak{A}_0 and \mathfrak{A}_1 be the Chekanov-Eliashberg algebras of L_0^+ and L_1^+ respectively. Let $\varepsilon_0: \mathfrak{A}_0 \rightarrow \mathbb{F}$ and $\varepsilon_1: \mathfrak{A}_1 \rightarrow \mathbb{F}$ be augmentations. Now we describe the construction of the *bilinearised Legendrian contact cohomology* complex $LCC_{\varepsilon_0, \varepsilon_1}(L_0^+, L_1^+; J)$.

First, we introduce some notation. We denote by D_i the set of double points of L_i (for $i = 1, 2$) and by \mathcal{C} the intersection points of L_0 and L_1 such that $f_0(q) < f_1(q)$. Double points in D_i correspond to Reeb chords of L_i^+ , and double points in \mathcal{C} corresponds to Reeb chords from L_1^+ to L_0^+ (note the order!). We define the exact immersed Lagrangian $L = L_0 \cup L_1$ and, for $\mathbf{p}^0 = (p_1^0, \dots, p_{l_0}^0) \in D_0^{l_0}$, $\mathbf{p}^1 = (p_1^1, \dots, p_{l_1}^1) \in D_1^{l_1}$ and $q_{\pm} \in \mathcal{C}$, we denote

$$\mathfrak{N}_L^i(q_+; \mathbf{p}^0, q_-, \mathbf{p}^1; J) := \mathfrak{N}_L^i(q_+; p_1^0, \dots, p_{l_0}^0, q_-, p_1^1, \dots, p_{l_1}^1; J),$$

where J is an L -regular almost complex structure. If ε_i is an augmentation of \mathfrak{A}_i , we denote $\varepsilon_i(\mathbf{p}^i) := \varepsilon_i(p_1^i) \cdots \varepsilon_i(p_{l_i}^i)$.

As an \mathbb{F} -module, $LCC_{\varepsilon_0, \varepsilon_1}(L_0^+, L_1^+; J)$ is freely generated by the set \mathcal{C} and the differential of a generator $q_- \in \mathcal{C}$ is defined as

$$(7) \quad \partial_{\varepsilon_0, \varepsilon_1}(q_-) = \sum_{q_+ \in \mathcal{C}} \sum_{l_0, l_1 \in \mathbb{N}} \sum_{\mathbf{p}^i \in D_i^{l_i}} \# \mathfrak{N}_L^0(q_+; \mathbf{p}^0, q_-, \mathbf{p}^1; J) \varepsilon_0(\mathbf{p}^0) \varepsilon_1(\mathbf{p}^1) q_+.$$

The bilinearised Legendrian contact cohomology $LCH_{\varepsilon_0, \varepsilon_1}(L_0^+, L_1^+)$ is the homology of this complex. The set of isomorphism classes of bilinearised Legendrian contact homology groups is independent of the choice of J and is a Legendrian isotopy invariant by the adaptation of Chekanov's argument from [10] due to the first author and Bourgeois [7].

4. FLOER HOMOLOGY FOR EXACT LAGRANGIAN IMMERSIONS

In this section we define a version of Floer homology for exact Lagrangian immersions. Recall that our exact Lagrangian immersions are equipped

with choices of potentials making their Legendrian lifts embedded. This is not new material; similar or even more general accounts can be found, for example, in [4], [5] and [27].

4.1. Cylindrical Hamiltonians.

Definition 4.1. Let W be a Liouville manifold. A Hamiltonian function $H: [0, 1] \times W \rightarrow \mathbb{R}$ is *cylindrical* if there is a function $h: \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $H(t, w) = h(e^{\tau(w)})$ outside a compact set of W .

The following example describes the behaviour of the Hamiltonian vector field of a cylindrical Hamiltonian in an end of W , after taking into account the identification (3).

Example 4.2. Let (V, α) be a contact manifold with Reeb vector field \mathcal{R} and let $(\mathbb{R} \times V, d(e^r \alpha))$ be its symplectisation. Given a smooth function $h: \mathbb{R}^+ \rightarrow \mathbb{R}$, we define the autonomous Hamiltonian $H: \mathbb{R} \times V \rightarrow \mathbb{R}$ by $H(r, v) = h(e^r)$. Then the Hamiltonian vector field of H is

$$X_H(r, v) = h'(e^r) \mathcal{R}(r, v).$$

Let (L_0, ι_0) and (L_1, ι_1) be two immersed exact Lagrangian submanifolds of W with cylindrical ends over Legendrian submanifolds Λ_0 and Λ_1 of (V, α) . Given a cylindrical Hamiltonian $H: [0, 1] \times W \rightarrow \mathbb{R}$, we denote by \mathcal{C}_H — or simply \mathcal{C} when there is no risk of confusion — the set of Hamiltonian chords $x: [0, 1] \rightarrow W$ of H such that $x(0) \in L_0$ and $x(1) \in L_1$. If φ_t denotes the Hamiltonian flow of H , then \mathcal{C}_H is in bijection with $\varphi_1(L_0) \cap L_1$.

Definition 4.3. A cylindrical Hamiltonian $H: [0, 1] \times W \rightarrow \mathbb{R}$, with Hamiltonian flow φ_t , is *compatible* with L_0 and L_1 if

- (i) no starting point or endpoint of a chord $x \in \mathcal{C}_H$ is a double point of (L_0, ι_0) or (L_1, ι_1) ,
- (ii) $\varphi_1(L_0)$ intersects L_1 transversely,
- (iii) for ρ large enough $h'(\rho) = \lambda$ is constant, and
- (iv) all time-one Hamiltonian chords from L_0 to L_1 are contained in a compact region.

Condition (iv) is equivalent to asking that λ should not be the length of a Reeb chord from Λ_0 to Λ_1 .

Remark 4.4. If cylindrical Hamiltonian H is *compatible* with L_0 and L_1 , then \mathcal{C}_H is a finite set.

For Liouville sectors we need to modify the definition of a cylindrical Hamiltonian in order to prevent the trajectories from hitting the boundary in finite time. To that aim, the crucial notion is that of *excellent dynamics*, for which we refer to [23, Section 2.9]. We introduce the following terminology.

Definition 4.5. Let X be a vector field on a manifold V . We say that a subset $\mathcal{N} \subset V$ is *coconvex* (for X) if every finite time trajectory of the flow of X with initial and final point in $V \setminus \mathcal{N}$ is contained in $V \setminus \mathcal{N}$.

Excellent dynamics implies that ∂V has an coconvex collar neighbourhood \mathcal{N} , but the converse is not true. The following lemma is proved in [23, Section 2.9]. Also, see [13, Section 4.2] for a related construction.

Lemma 4.6. *Given a contact manifold (V, α) , it is possible to find a function $g: V \rightarrow \mathbb{R}_{\geq 0}$ such that*

- (1) $g > 0$ outside the boundary ∂V , and $g \equiv 1$ outside a collar neighbourhood $\partial V \times [0, \delta)$,
- (2) $g = t^2 G$ on $\partial V \times [0, \delta)$, where $G > 0$ and t is the coordinate of $[0, \delta)$, and
- (3) the contact vector field X_g with contact Hamiltonian g has excellent dynamics.

Note that X_g vanishes along ∂V . It is called a *cut off Reeb vector field* in [23] because it is the Reeb vector field of the contact form $g^{-1}\alpha$ on $\text{int}(V)$.

Definition 4.7. Let S be a Liouville sector. A Hamiltonian function $H: [0, 1] \times S \rightarrow \mathbb{R}$ is *cylindrical* if there is a function $h: \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $H(t, w) = g(w)h(e^{\mathfrak{r}(w)})$ outside a compact set of S .

The definition of cylindrical Hamiltonian compatible with two immersed exact Lagrangian submanifolds in the case of sectors is the same as Definition 4.3. Condition (iv) is equivalent to asking that λ should not be the length of a chord of the cut off Reeb vector field.

4.2. Obstructions. If one tries to define Floer homology for immersed Lagrangian submanifolds by extending the usual definition naively, one runs into the problem that the “differential” might not square to zero because of the bubbling of teardrops in one-dimensional families of Floer strips. Thus, if (L, ι) is an immersed Lagrangian submanifold and J is L -regular, we define a map $\mathfrak{d}^0: D \rightarrow \mathbb{Z}$ by

$$\mathfrak{d}^0(p) = \#\mathfrak{N}_L^0(p; J)$$

and extend it by linearity to the free module generated by D . The map \mathfrak{d}^0 is called the *obstruction* of (L, ι) . If $\mathfrak{d}^0 = 0$ we say that (L, ι) is *unobstructed*.

Typically, asking that an immersed Lagrangian submanifold be unobstructed is too much, and a weaker condition will ensure that Floer homology can be defined. We observe that \mathfrak{d}^0 is a component of the Chekanov-Eliashberg algebra of the Legendrian lift of L , and make the following definition.

Definition 4.8. Let (L, ι) be an immersed exact Lagrangian submanifold. The *obstruction algebra* $(\mathfrak{D}, \mathfrak{d})$ of (L, ι) — or of (L, ι, J) when the almost complex structure is not clear from the context — is the Chekanov-Eliashberg algebra of the Legendrian lift L^+ .

If L is connected, its obstruction algebra $(\mathfrak{D}, \mathfrak{d})$ does not depend on the potential. On the other hand, if L is disconnected, the potential differences at the double points of L involving different connected components, and therefore what holomorphic curves are counted in $(\mathfrak{D}, \mathfrak{d})$, depend on the choice of the potential.

Definition 4.9. An exact immersed Lagrangian (L, ι) is *unobstructed* if $(\mathfrak{D}, \mathfrak{d})$ admits an augmentation.

Weak unobstructedness does not depend on the choice of L -regular almost complex structure and is invariant under general Legendrian isotopies as a consequence of Theorem 3.7 (this fact will not be needed). However, we will need the invariance statement from the following proposition.

Proposition 4.10. *If L_0 and L_1 are safe isotopic exact Lagrangian immersions, and J_0 and J_1 are L_0 -regular and L_1 -regular almost complex structures respectively, then the obstruction algebras $(\mathfrak{D}_0, \mathfrak{d}_0)$ of L_0 and $(\mathfrak{D}_1, \mathfrak{d}_1)$ of L_1 are isomorphic. In particular, there is a bijection between the augmentations of $(\mathfrak{D}_0, \mathfrak{d}_0)$ and the augmentations of $(\mathfrak{D}_1, \mathfrak{d}_1)$.*

If in addition $J_1 = (\varphi_1)_ J_0$ is satisfied for a diffeomorphism φ_1 , then the bijection of double points of L_0 and L_1 induced by φ_1 moreover induces an isomorphism of the obstruction algebras.*

Sketch of proof. Let φ_t be a smooth isotopy such that $\varphi_t(L_0) = L_t$ is the projection of a Legendrian isotopy. For any t and $\epsilon > 0$ there is a bijection between the double points of L_t and the double points of $L_{t+\epsilon} = \psi(L_t)$, where ψ is a diffeomorphism which may be assumed to be arbitrarily C^2 -small for sufficiently small $\epsilon > 0$. This diffeomorphism induces an *isomorphism* between the obstruction algebra $(\mathfrak{D}_1, \mathfrak{d}_1)$ and the obstruction algebra $(\tilde{\mathfrak{D}}_0, \tilde{\mathfrak{d}}_0)$ of L_0 defined using the pulled-back almost complex structure $d\psi^{-1} \circ J_1 \circ d\psi$, which is tame since ψ is C^2 -small. Thus, to prove the proposition, it is enough to compare the obstruction algebras of L defined using different L -regular almost complex structures.

Let $(\mathfrak{D}, \mathfrak{d})$ and $(\mathfrak{D}, \tilde{\mathfrak{d}})$ be obstruction algebras of L defined using L -regular almost complex structures J and \tilde{J} respectively. We will sketch a bifurcation analysis argument to define an isomorphism

$$\mathfrak{Y}: (\mathfrak{D}, \mathfrak{d}) \rightarrow (\mathfrak{D}, \tilde{\mathfrak{d}});$$

for more details see [16, Lemma 4.21]). We fix a path J^\bullet of almost complex structures satisfying (\dagger) such that $J^0 = J$ and $J^1 = \tilde{J}$, and define the

parametrised moduli spaces $\mathfrak{N}_L^k(p_o; p_1, \dots, p_d; J^\bullet)$ consisting of pairs (δ, u) such that $\delta \in [0, 1]$ and $u \in \mathfrak{N}_L^k(p_o; p_1, \dots, p_d; J^\delta)$.

For a generic path J^\bullet there is a finite set of “bifurcation instants” $\delta_1 < \dots < \delta_n$ and intersection points $q_0^i, \dots, q_{d_i}^i$ (for $i = 1, \dots, n$) such that the unique nonempty moduli spaces of negative index are $\mathfrak{N}_L^{-1}(q_0^i; q_1^i, \dots, q_{d_i}^i; J^{\delta_i})$ for $i = 1, \dots, n$, each of which consists of a finite number of points.

First, assume that there is a unique bifurcation point $\delta \in (0, 1)$ in the path J^\bullet . Then \mathfrak{Y} is defined on the generators as

$$(8) \quad \mathfrak{Y}(p) = \begin{cases} p & \text{if } p \neq q_0, \\ q_0 + \#\mathfrak{N}_L^{-1}(q_0; q_1, \dots, q_d; J^\delta) q_1 \dots q_d & \text{if } p = q_0. \end{cases}$$

If J^\bullet has more than one bifurcation instant, we split it into pieces, each of which has a unique bifurcation instant, and we compose the isomorphisms induced by each piece. \square

Proposition 4.10 is the main reason why we have made the choice of distinguishing between the obstruction algebra of a Lagrangian immersion and the Chekanov-Eliashberg algebra of its Legendrian lift. In fact Legendrian submanifolds are more naturally considered up to Legendrian isotopy. However, in this article we will consider immersed Lagrangian submanifolds only up to the weaker notion of safe isotopy.

Remark 4.11. Presumably one can prove that, up to homotopy, \mathfrak{Y} is independent of the choice of path between J and \tilde{J} by doing bifurcation analysis for homotopies of paths. However, this might require some form of abstract perturbations as in [15].

4.3. The differential. We denote by Z the strip $\mathbb{R} \times [0, 1]$ with coordinates (s, t) . Let $\tilde{\mathcal{R}}^{l_0|l_1} \cong \text{Conf}^{l_0}(\mathbb{R}) \times \text{Conf}^{l_1}(\mathbb{R})$ be the set of pairs (ζ^0, ζ^1) such that $\zeta^0 = \{\zeta_1^0, \dots, \zeta_{l_0}^0\} \subset \mathbb{R} \times \{0\}$ and $\zeta^1 = \{\zeta_1^1, \dots, \zeta_{l_1}^1\} \subset \mathbb{R} \times \{1\}$. We assume that the s -coordinates of ζ_j^i are increasing in j for ζ_j^0 and decreasing for ζ_j^1 . We define

$$Z_{\zeta^0, \zeta^1} := Z \setminus \{\zeta_1^0, \dots, \zeta_{l_0}^0, \zeta_1^1, \dots, \zeta_{l_1}^1\}.$$

The group $\text{Aut}(Z) = \mathbb{R}$ acts on $\tilde{\mathcal{R}}^{l_0|l_1}$.

In the rest of this section we will assume that H is compatible with (L_0, ι_0) and (L_1, ι_1) . We will consider time-dependent almost complex structure J_\bullet on W which satisfy the following

- ($\dagger\dagger$) (i) J_t is compatible with θ for all t ,
- (ii) $J_t = J_0$ for $t \in [0, 1/4]$ in a neighbourhood of the double points of L_0 and $J_t = J_1$ for $t \in [3/4, 1]$ in a neighbourhood of the double points of L_1 ,

(iii) J_0 satisfies (\dagger) for L_0 and J_1 satisfies (\dagger) for L_1 .

Condition (ii) is necessary to ensure that J_\bullet is independent of the coordinate $\sigma_{i,j}^-$ in some neighbourhoods of the boundary punctures ζ_j^i , so that we can apply standard analytical results.

For the same reason we fix once and for all a nondecreasing function $\chi: \mathbb{R} \rightarrow [0, 1]$ such that

$$\chi(t) = \begin{cases} 0 & \text{for } t \in [0, 1/4], \\ 1 & \text{for } t \in [3/4, 1] \end{cases}$$

and $\chi'(t) \leq 3$ for all t . It is easy to see that, if φ_t is the Hamiltonian flow of a Hamiltonian function H , then $\varphi_{\chi(t)}$ is the Hamiltonian flow of the Hamiltonian function H' such that $H'(t, w) = \chi'(t)H(\chi(t), w)$. We will use χ' to cut off the Hamiltonian vector field in the Floer equation to ensure that it has the right invariance properties in the strip-like ends corresponding to the boundary punctures ζ_j^i .

Given Hamiltonian chords $x_+, x_- \in \mathcal{C}$ and self-intersections $p_1^0, \dots, p_{l_0}^0$ of L_0 and $p_1^1, \dots, p_{l_1}^1$ of L_1 we define the moduli space

$$\widetilde{\mathfrak{M}}_{L_0, L_1}(p_1^1, \dots, p_{l_1}^1, x_-, p_1^0, \dots, p_{l_0}^0, x_+; H, J_\bullet)$$

of triples (ζ^0, ζ^1, u) such that

- $(\zeta^0, \zeta^1) \in \widetilde{\mathcal{R}}^{l_0|l_1}$ and $u: Z_{\zeta^0, \zeta^1} \rightarrow W$ is a map satisfying the Floer equation
- $$(9) \quad \frac{\partial u}{\partial s} + J_t \left(\frac{\partial u}{\partial t} - \chi'(t) X_H(\chi(t), u) \right) = 0,$$
- $\lim_{s \rightarrow \pm\infty} u(s, t) = x_\pm(\chi(t))$ uniformly in t ,
 - $u(s, 0) \in L_0$ for all $(s, 0) \in Z_{\zeta^0, \zeta^1}$,
 - $u(s, 1) \in L_1$ for all $(s, 1) \in Z_{\zeta^0, \zeta^1}$,
 - $\lim_{z \rightarrow \zeta_j^i} u(z) = p_j^i$, and
 - ζ_j^i is a negative puncture at p_j^i for $i = 0, 1$ and $j = 1, \dots, l_i$.

The group $\text{Aut}(Z) = \mathbb{R}$ acts on $\widetilde{\mathfrak{M}}_{L_0, L_1}(p_1^1, \dots, p_{l_1}^1, x_-, p_1^0, \dots, p_{l_0}^0, x_+; H, J_\bullet)$ by reparametrisations. We will denote the quotient by

$$\mathfrak{M}_{L_0, L_1}(p_1^1, \dots, p_{l_1}^1, x_-, p_1^0, \dots, p_{l_0}^0, x_+; H, J_\bullet).$$

For $u \in \widetilde{\mathfrak{M}}_{L_0, L_1}(p_1^1, \dots, p_{l_1}^1, x_-, p_1^0, \dots, p_{l_0}^0, x_+; H, J_\bullet)$, we denote by F_u the linearisation of the Floer operator

$$\mathcal{F}(u) = \frac{\partial u}{\partial s} + J_t \left(\frac{\partial u}{\partial t} - \chi'(t) X_H(\chi(t), u) \right)$$

at u . By standard Fredholm theory, F_u is a Fredholm operator with index $\text{ind}(F_u)$. We define

$$\text{ind}(u) = \text{ind}(F_u) + l_0 + l_1.$$

The index is locally constant, and we denote by

$$\mathfrak{M}_{L_0, L_1}^k(p_1^1, \dots, p_{l_1}^1, x_-, p_1^0, \dots, p_{l_0}^0, x_+; H, J_\bullet)$$

the subset of $\mathfrak{M}_{L_0, L_1}(p_1^1, \dots, p_{l_1}^1, x_-, p_1^0, \dots, p_{l_0}^0, x_+; H, J_\bullet)$ consisting of classes of maps u with index $\text{ind}(u) = k$.

Observe that similar construction for closed, exact, graded, immersed Lagrangian submanifolds was considered by Alston and Bao in [5], where the regularity statement appears in [5, Proposition 5.2] and compactness is discussed in [5, Section 4]. In addition, the corresponding statement in the case of Legendrian contact homology in $P \times \mathbb{R}$ was proven by Ekholm, Etnyre and Sullivan, see [17, Proposition 2.3]. The following proposition translates those compactness and regularity statements to the settings under consideration.

Proposition 4.12. *For a generic time-dependent almost complex structure J_\bullet satisfying $(\dagger\dagger)$, for which moreover J_0 is L_0 -regular and J_1 is L_1 -regular, the moduli space*

$$\mathfrak{M}_{L_0, L_1}^k(p_1^1, \dots, p_{l_1}^1, x_-, p_1^0, \dots, p_{l_0}^0, x_+; H, J_\bullet)$$

is a transversely cut-out manifold of dimension $k - 1$. If $k = 1$ it is compact, and therefore consists of a finite number of points. If $k = 2$ it can be compactified in the sense of Gromov-Floer.

The boundary of the compactification of the one-dimensional moduli space $\mathfrak{M}_{L_0, L_1}^2(p_1^1, \dots, p_{l_1}^1, x_-, p_1^0, \dots, p_{l_0}^0, x_+; H, J_\bullet)$ is

$$\begin{aligned} (10) \quad & \bigsqcup_{y \in C_H} \bigsqcup_{0 \leq h_i \leq l_i} \mathfrak{M}_{L_0, L_1}^1(p_{h_1+1}^1, \dots, p_{l_1}^1, y, p_1^0, \dots, p_{h_0}^0, x_+; H, J_\bullet) \\ & \quad \times \mathfrak{M}_{L_0, L_1}^1(p_1^1, \dots, p_{h_1}^1, x_-, p_{h_0+1}^0, \dots, p_{l_0}^0, y; H, J_\bullet) \\ & \bigsqcup_{q \in D_1} \bigsqcup_{0 \leq i \leq j \leq l_1} \mathfrak{M}_{L_0, L_1}^1(p_1^1, \dots, p_i^1, q, p_{j+1}^1, \dots, p_{l_1}^1, x_-, p_1^0, \dots, p_{l_0}^0, x_+; H, J_\bullet) \\ & \quad \times \mathfrak{N}_{L_1}^0(q; p_{i+1}^1, \dots, p_j^1; J_1) \\ & \bigsqcup_{q \in D_0} \bigsqcup_{0 \leq i \leq j \leq l_0} \mathfrak{M}_{L_0, L_1}^1(p_1^1, \dots, p_{l_1}^1, x_-, p_1^0, \dots, p_i^0, q, p_{j+1}^0, \dots, p_{l_0}^0, x_+; H, J_\bullet) \\ & \quad \times \mathfrak{N}_{L_0}^0(q; p_{i+1}^0, \dots, p_j^0; J_0). \end{aligned}$$

If both L_0 and L_1 are spin the moduli spaces are orientable, and a choice of spin structure on each Lagrangian submanifold induces a coherent orientation on the moduli spaces.

Remark 4.13. We use different conventions for the index of maps involved in the definition of the obstruction algebra and for maps involved in the definition of Floer homology. Unfortunately this can cause some confusion,

but it is necessary to remain consistent with the standard conventions in the literature.

Definition 4.14. We say that a time-dependent almost complex structure J_\bullet on W is (L_0, L_1) -regular if it satisfies $(\dagger\dagger)$ for L_0 and L_1 and all moduli spaces $\mathfrak{M}_{L_0, L_1}^k(p_1^1, \dots, p_{l_1}^1, x_-, p_1^0, \dots, p_{l_0}^0, x_+; H, J_\bullet)$ are transversely cut out.

Note that, strictly speaking, the condition of being (L_0, L_1) -regular depends also on the Hamiltonian, even if we have decided to suppress it from the notation.

Suppose that the obstruction algebras \mathfrak{D}_0 and \mathfrak{D}_1 admit augmentations ε_0 and ε_1 respectively. To simplify the notation we write $\mathbf{p}^i = (p_1^i, \dots, p_{l_i}^i)$,

$$\mathfrak{M}_{L_0, L_1}^k(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, J_\bullet) = \mathfrak{M}_{L_0, L_1}^k(p_1^1, \dots, p_{l_1}^1, x_-, p_1^0, \dots, p_{l_0}^0, x_+; H, J_\bullet)$$

and $\varepsilon_i(\mathbf{p}^i) = \varepsilon_i(p_1^i) \dots \varepsilon_i(p_{l_i}^i)$ for $i = 0, 1$. We also introduce the weighted count

$$\mathfrak{m}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+) = \# \mathfrak{M}_{L_0, L_1}^0(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, J_\bullet) \varepsilon_0(\mathbf{p}^0) \varepsilon_1(\mathbf{p}^1).$$

Then we define the Floer complex over the commutative ring \mathbb{F}

$$CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet) = \bigoplus_{x \in \mathcal{C}} \mathbb{F}x$$

with differential

$$\partial: CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet) \rightarrow CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet)$$

defined on the generators by

$$(11) \quad \partial x_+ = \sum_{x_- \in \mathcal{C}_H} \sum_{l_0, l_1 \in \mathbb{N}} \sum_{\mathbf{p}^i \in D_i^{l_i}} \mathfrak{m}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+) x_-.$$

The algebraic interpretation of the Gromov-Floer compactification of the one-dimensional moduli spaces in Proposition 4.12 is that $\partial^2 = 0$. We will denote the homology by $HF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H)$. The suppression of J_\bullet from the notation is justified by Subsection 5.1.

4.4. Comparison with bilinearised Legendrian contact cohomology.

In this subsection we compare the Lagrangian Floer homology of a pair of immersed exact Lagrangian submanifolds with the bilinearised Legendrian contact cohomology of a particular Legendrian lift of theirs.

Let L_0 and L_1 be exact Lagrangian immersions, $H: [0, 1] \times W \rightarrow \mathbb{R}$ a Hamiltonian function compatible with L_0 and L_1 with Hamiltonian flow φ_t , and J_\bullet an (L_0, L_1) -regular almost complex structure. We will introduce the “backward” isotopy $\bar{\varphi}_t = \varphi_1 \circ \varphi_t^{-1}$, where $\varphi_1 \circ \varphi_t^{-1} \circ \varphi_1^{-1}$ can be generated by the Hamiltonian $-H(t, \varphi_t^{-1} \circ \varphi_1^{-1})$.

Given an almost complex structure J_\bullet and an arbitrary path ϕ_t of symplectomorphisms, we denote by $\phi_* J_\bullet$ the almost complex structure defined as $\phi_* J_t = d\phi_{\chi(t)} \circ J_t \circ d\phi_{\chi(t)}^{-1}$. The time rescaling by χ ensures that $\phi_* J_\bullet$ satisfies $(\dagger\dagger)$ for $\phi_1(L_0)$ and L_1 if and only if J_\bullet does for L_0 and L_1 .

Lemma 4.15. *Denote by $\mathbf{0}$ the constantly zero function on W and set $q_\pm = x_\pm(1) \in \varphi_1(L_0) \cap L_1$, regarded as Hamiltonian chords of $\mathbf{0}$. There is a bijection*

$\bar{\varphi}_* : \mathfrak{M}_{L_0, L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, J_\bullet) \rightarrow \mathfrak{M}_{\varphi_1(L_0), L_1}(\mathbf{p}^1, q_-, \varphi_1(\mathbf{p}^0), q_+; \mathbf{0}, \bar{\varphi}_* J_\bullet)$
defined by $(\bar{\varphi}_* u)(s, t) = \bar{\varphi}_t(u(s, t))$, and moreover J_\bullet is (L_0, L_1) -regular if and only if $\bar{\varphi}_* J_\bullet$ is $(\varphi_1(L_0), L_1)$ -regular.

Proof. Since

$$\begin{aligned} \frac{\partial v}{\partial s}(s, t) &= d\bar{\varphi}_t \left(\frac{\partial u}{\partial s}(s, t) \right) \quad \text{and} \\ \frac{\partial v}{\partial t}(s, t) &= d\bar{\varphi}_t \left(\frac{\partial u}{\partial t}(s, t) - \chi'(t)X(\chi(t), u(s, t)) \right), \end{aligned}$$

u satisfies the Floer equation with Hamiltonian H if and only if $\bar{\varphi}_* u$ satisfies the Floer equation with Hamiltonian $\mathbf{0}$. The map $\bar{\varphi}_*$ is invertible because $\bar{\varphi}_t$ is for each t . Finally, we observe that $d\bar{\varphi}$ intertwines the linearised Floer operators at u and v . \square

Given two Legendrian submanifolds Λ_0 and Λ_1 in $(W \times \mathbb{R}, dz + \theta) = (M, \beta)$, we say that Λ_0 is *above* Λ_1 if the z -coordinate of any point of Λ_0 is larger than the z -coordinate of any point of Λ_1 .

Lemma 4.16. *Let L_0 and L_1 be immersed exact Lagrangian submanifolds of (W, θ) and H a cylindrical Hamiltonian compatible with L_0 and L_1 . We denote by φ_t the Hamiltonian flow of H and $\tilde{L}_0 = \varphi_1(L_0)$. We choose Legendrian lifts of \tilde{L}_0 and L_1 to Legendrian submanifolds \tilde{L}_0^+ and L_1^+ of (M, β) such that \tilde{L}_0^+ is above L_1^+ . If \tilde{J} is an L -regular almost complex structure on W for $L = \tilde{L}_0 \cup L_1$, let $J_\bullet = (\bar{\varphi}_t^{-1})_* \tilde{J}$. For every pair of augmentations ε_0 and ε_1 of the obstruction algebras of (L_0, J_0) and (L_1, J_1) respectively, there is an isomorphism of complexes*

$$CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet) \cong LCC_{\tilde{\varepsilon}_0, \varepsilon_1}(\tilde{L}_0^+, L_1^+; \tilde{J})$$

where $\tilde{\varepsilon}_0 = \varepsilon_0 \circ \varphi_1^{-1}$ is an augmentation of the obstruction algebra of (\tilde{L}_0, \tilde{J}) .

Proof. By Lemma 4.15 there is an isomorphism of complexes

$$CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet) \cong CF((\tilde{L}_0, \tilde{\varepsilon}_0), (L_1, \varepsilon_1); \mathbf{0}, \tilde{J}).$$

By definition the obstruction algebras of \tilde{L}_0 and L_1 are isomorphic to the Chekanov-Eliashberg algebras of \tilde{L}_0^+ and L_1^+ . Moreover, the chain complexes $CF((\tilde{L}_0, \tilde{\varepsilon}_0), (L_1, \varepsilon_1); \mathbf{0}, \tilde{J})$ and $LCC_{\tilde{\varepsilon}_0, \varepsilon_1}(\tilde{L}_0^+, L_1^+; \tilde{J})$ are both generated by

intersection points between \tilde{L}_0 and L_1 . Thus it remains only to match the differentials.

For any $i = 1, \dots, d$ and $\zeta = \{\zeta_0, \dots, \zeta_d\} \in \tilde{\mathcal{R}}^d$ there is a biholomorphism $\Delta_\zeta \cong Z_{\zeta^0, \zeta^1}$, where $\zeta^0 = \{\zeta_1, \dots, \zeta_{i-1}\}$, $\zeta^1 = \{\zeta_{i+1}, \dots, \zeta_d\}$, ζ_i is mapped to $s = +\infty$ and ζ_0 is mapped to $s = -\infty$. Such biholomorphisms induce bijections between the moduli spaces defining the boundary of $CF((\tilde{L}_0, \tilde{\varepsilon}_0), (L_1, \varepsilon_1); \mathbf{0}, \tilde{J})$ and the moduli spaces defining the boundary of $LCC_{\tilde{\varepsilon}_0, \varepsilon_1}(\tilde{L}_0^+, L_1^+; \tilde{J})$. \square

4.5. Products. After the work done for the differential, the higher order products can be easily defined. For simplicity, in this section we will consider immersed exact Lagrangian submanifolds L_0, \dots, L_d which are pairwise transverse and cylindrical over chord generic Legendrian submanifolds. Thus the generators of the Floer complexes will be intersection points, which we will assume to be disjoint from the double points. The routine modifications needed to introduce Hamiltonian functions into the picture are left to the reader.

Given $d \leq 2$ and $l_i \geq 0$ for $i = 0, \dots, d$ we define

$$\tilde{\mathcal{R}}^{l_0|\dots|l_d} = \text{Conf}^{l_0+\dots+l_d+d+1}(\partial D^2)$$

where $d+1$ points $\zeta_0^m, \dots, \zeta_d^m$ (ordered counterclockwise) are labelled as *mixed* and the other ones ζ_j^i , with $i = 0, \dots, d$ and $j = 1, \dots, l_i$ (ordered counterclockwise and contained in the sector from ζ_i^m to ζ_{i+1}^m) are labelled as *pure*. Given $\zeta \in \tilde{\mathcal{R}}^{l_0|\dots|l_d}$, we denote $\Delta_\zeta = D^2 \setminus \zeta$. For $i = 0, \dots, d$ let $\partial_i \Delta_\zeta$ be the subset of $\partial \Delta_\zeta$ whose closure in ∂D^2 is the counterclockwise arc from ζ_i^m to ζ_{i+1}^m .

We will consider also a (generic) domain dependent almost complex structure J_\bullet such that every J_z , $z \in \Delta_\zeta$ satisfies (\dagger) , and moreover is of the form $(\dagger\dagger)$ at the strip-like ends of the mixed punctures and is constant in a neighbourhood of the arcs $\partial_i \Delta_\zeta$ outside those strip-like ends.

Finally we define the moduli spaces $\mathfrak{M}_{L_0, \dots, L_d}(\mathbf{p}^d, x_0, \mathbf{p}^0, x_1, \dots, \mathbf{p}^{d-1}, x_d; J_\bullet)$ of pairs (u, ζ) (up to action of $\text{Aut}(D^2)$), where:

- $\zeta \in \tilde{\mathcal{R}}^{l_0|\dots|l_d}$ and $u: \Delta_\zeta \rightarrow W$ satisfies $du + J_\bullet \circ du \circ i = 0$,
- $u(\partial_i \Delta_\zeta) \subset L_i$,
- $\lim_{z \rightarrow \zeta_i^m} u(z) = x_i$,
- $\lim_{z \rightarrow \zeta_j^i} u(z) = p_j^i$, and
- p_j^i is a negative puncture at ζ_j^i for $i = 0, \dots, d$ and $j = 1, \dots, l_i$.

As usual, we denote by $\mathfrak{M}_{L_0, \dots, L_d}^0(\mathbf{p}^d, x_0, \mathbf{p}^0, x_1, \dots, \mathbf{p}^{d-1}, x_d; J_\bullet)$ the zero-dimensional part of the moduli spaces.

If $\varepsilon_0, \dots, \varepsilon_d$ are augmentations for the corresponding Lagrangian immersions, we define the weighted count

$$\begin{aligned} \mathfrak{m}(\mathbf{p}^d, x_0, \mathbf{p}^0, x_1, \dots, \mathbf{p}^{d-1}, x_d) = \\ \# \mathfrak{M}_{L_0, \dots, L_d}^0(\mathbf{p}^d, x_0, \mathbf{p}^0, x_1, \dots, \mathbf{p}^{d-1}, x_d; J_\bullet) \varepsilon_0(\mathbf{p}^0) \dots \varepsilon_d(\mathbf{p}^d) \end{aligned}$$

and define a product

$$(12) \quad \mu^d: CF(L_{d-1}, L_d) \otimes \dots \otimes CF(L_0, L_1) \rightarrow CF(L_0, L_d),$$

where we wrote $CF(L_i, L_j)$ instead of $CF((L_i, \varepsilon_i), (L_j, \varepsilon_j))$ for brevity sake, via the formula

$$\mu^d(x_1, \dots, x_d) = \sum_{x_0 \in L_0 \cap L_d} \sum_{l_i \geq 0} \sum_{p_j^i \in D_i^{l_i}} \mathfrak{m}(\mathbf{p}^d, x_0, \mathbf{p}^0, x_1, \dots, \mathbf{p}^{d-1}, x_d) x_0.$$

Remark 4.17. The operations μ^d satisfy the A_∞ relations.

The following lemma will be useful in Section 8. It is a straight forward corollary of the existence of pseudoholomorphic triangles supplied by Corollary 4.22 below. The only point where we need to take some care is due to the fact that the Weinstein neighbourhood considered is immersed.

Lemma 4.18. *Let (L, ι) be an exact Lagrangian immersion. We extend ι to a symplectic immersion $\iota_*: (D_\delta T^*L, d\mathbf{q} \wedge d\mathbf{p}) \rightarrow (W, d\theta)$. Let (L, ι') be safe isotopic to (L, ι) and, moreover, assume that*

- *there exists a sufficiently C^1 -small Morse function $g: L \rightarrow \mathbb{R}$ with local minima e_i , all whose critical points are disjoint from the double points of L , such that $\iota' = \iota_* \circ dg$, and*
- *outside of some compact subset L' is obtained by a small perturbation of L by the negative Reeb flow.*

We will denote $L = (L, \iota)$ and $L' = (L, \iota')$. Then, if L admits an augmentation ε and ε' is the corresponding augmentation of L' , for every cylindrical exact Lagrangian submanifold T such that $\iota_^{-1}(T)$ is a union of cotangent disc fibres, the map*

$$\begin{aligned} \mu^2(e, \cdot): CF(T, (L, \varepsilon)) &\rightarrow CF(T, (L', \varepsilon')), \\ e &:= \sum_i e_i \in CF((L, \varepsilon), (L', \varepsilon')), \end{aligned}$$

is an isomorphism of complexes for a suitable almost complex structure on W as in Corollary 4.22.

In the case when L is closed and embedded, the element e is always a cycle which is nontrivial in homology as was shown by Floer (it is identified with the minimum class in the Morse cohomology of L). In general the following holds.

Lemma 4.19. *Under the hypotheses of Lemma 4.18, the element*

$$e \in CF((L, \varepsilon), (L', \varepsilon'))$$

is a cycle. Furthermore, e is a boundary if and only if $CF(T, (L, \varepsilon)) = 0$ for every Lagrangian T .

Proof. The assumption that the augmentation ε' is identified canonically with the augmentation ε implies that e is a cycle by the count of pseudo-holomorphic discs with a negative puncture at e supplied by Lemma 4.21.

Assume that $\partial E = e$. The last property is then an algebraic consequence of the Leibniz rule $\partial\mu^2(E, x) = \mu^2(\partial E, x)$ in the case when $\partial x = 0$, combined with the fact that $\mu^2(e, \cdot)$ is a quasi-isomorphism as established by the previous lemma. \square

Later it will be useful to switch perspectives slightly, and instead of with the chain e , work with an augmentation induced by that chain. In general, given (L_0, ε_0) , (L_1, ε_1) and a chain $c \in CF((L_0, \varepsilon_0), (L_1, \varepsilon_1))$ we can consider Legendrian lifts L_0^+ and L_1^+ such that L_0^+ is above L_1^+ and the unital algebra morphism $\varepsilon_c: \mathfrak{A}(L_0^+ \cup L_1^+) \rightarrow \mathbb{F}$ uniquely determined

$$\varepsilon_c(x) := \begin{cases} \varepsilon_i(x), & \text{if } x \in \mathfrak{A}(L_i), \\ \langle c, x \rangle, & \text{if } x \in L_0 \cap L_1, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ is the coefficient of x in c .

Lemma 4.20. *The element*

$$c \in CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); \mathbf{0}, J)$$

is a cycle if and only if

$$\varepsilon_c: \mathfrak{A}(L_0^+ \cup L_1^+) \rightarrow \mathbb{F}$$

is an augmentation, where the almost complex structure J has been used to define the latter algebra as well, and the Legendrian lifts have been chosen so that no Reeb chord starts on L_0^+ and ends on L_1^+ .

Proof. Note that the Floer complex under consideration has a differential which counts J -pseudoholomorphic strips, and that the obstruction algebra has a differential counting pseudoholomorphic discs with at least one boundary puncture. Identifying the appropriate counts of discs, the statement can be seen to follow by pure algebra, together with the fact that the differential of the DGA counts punctured pseudoholomorphic discs, and thus respects the filtration induced by the different components. The crucial property that is needed here is that, under the assumptions made on the Legendrian lifts, the differential of the Chekanov-Eliashberg algebra applied to a mixed chord is a sum of words, each of which contains precisely one mixed chord. \square

4.6. Existence of triangles. In this section we prove an existence result for small pseudoholomorphic triangles with boundary on an exact Lagrangian cobordism and a small push-off. The existence of these triangles can be deduced as a consequence of the fact that the wrapped Fukaya category is homologically unital. Here we take a more direct approach based upon the adiabatic limit of pseudoholomorphic discs on a Lagrangian and its push-off from [18]; when the latter push-off becomes sufficiently small, these discs converge to pseudoholomorphic discs on the single Lagrangian with gradient-flow lines attached (called *generalised pseudoholomorphic discs* in the same paper).

Let $L \subset W$ be an exact immersed Lagrangian submanifold with cylindrical end. We recall that, as usual, we assume that every immersed Lagrangian submanifold is nice. Consider the Hamiltonian push-off $L_{\epsilon f}$, which we require to be again an exact immersed Lagrangian submanifold with cylindrical end, which is identified with the graph of $d(\epsilon f)$ for a Morse function $f: L \rightarrow \mathbb{R}$ inside a Weinstein neighbourhood $(T_\delta^* L, -d(pdq)) \looparrowright (W, d\theta)$ of L . We further assume that $df(\mathcal{L}) > 0$ outside of a compact subset. (The assumption that the push-off is cylindrical at infinity does of course impose additional constraints on the precise behaviour of the Morse function outside of a compact subset.)

Now consider a Legendrian lift $L^+ \cup L_{\epsilon f}^+$ for which $(L_{\epsilon f})^+$ is above L^+ . For $\epsilon > 0$ sufficiently small, it is the case that $L \cup L_{\epsilon f}$ again has only transverse double points. Moreover, the Reeb chords on the Legendrian lift can be classified as follows, using the notation from [18, Section 3.1]:

- Reeb chords $\mathcal{Q}(L) \cong \mathcal{Q}(L_{\epsilon f})$ on the lifts of L and $L_{\epsilon f}$ respectively, which stand in a canonical bijection;
- Reeb chords \mathcal{C} being in a canonical bijection with the critical points of f ; and
- two sets \mathcal{Q} and \mathcal{P} of Reeb chords from L to $L_{\epsilon f}$, each in canonical bijection with $\mathcal{Q}(L)$, and where the lengths of any Reeb chord in \mathcal{Q} is greater than the length of any Reeb chord in \mathcal{P} .

See the aforementioned reference for more details, as well as Figure 1.

Lemma 4.21 ([18]). *For a suitable generic Riemannian metric g on L for which (f, g) constitutes a Morse-Smale pair and associated almost complex structure, which can be made to coincide with an arbitrary cylindrical almost complex structure outside of a compact subset, there is a bijection between the set of pseudoholomorphic discs which have*

- boundary on $L^+ \cup L_{\epsilon f}^+$ and precisely one positive puncture,
- at least one negative puncture at a local minimum $e \in \mathcal{C}$ of f , and
- form a moduli space of expected dimension zero,

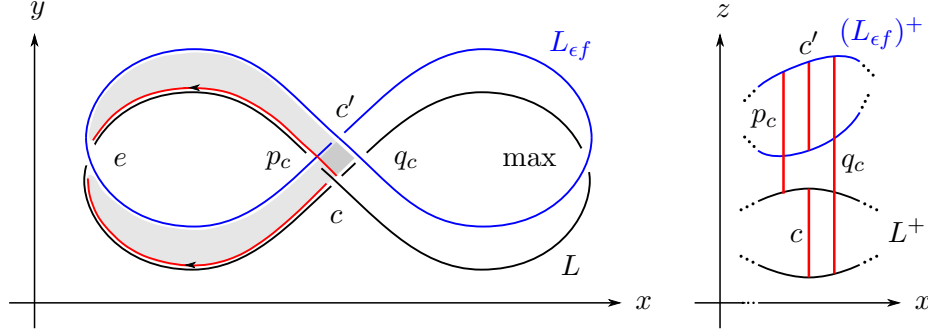


FIGURE 1. The small triangles on the two-copy living near gradient flow-lines of $-\nabla f$ shown in red. The upper copy with respect to the z -coordinate is shown in blue, while the lower copy is in black. The contact form used here is $dz - ydx$.

and the set of negative gradient flow-lines on (L, g) that connect either the starting point or the end point of a Reeb chord $c \in \mathcal{Q}(L)$ with the local minimum e , together with the set of negative gradient flow-lines that connect some critical point of index one with e .

More precisely, each such pseudoholomorphic disc lives in a small neighbourhood of the aforementioned flow-line. In the first case, it is a triangle with a positive puncture at the Reeb chord $q_c \in \mathcal{Q}$ corresponding to c , and its additional negative punctures at e and either c (for the flow-line from the starting point of c) or c' (for the flow-line from the endpoint of c); see Figure 1. In the second case, it is a Floer strip corresponding to the negative gradient flow-line connecting the saddle point and the local minimum.

Proof. This is an immediate application of Parts (3) and (4) of [18, Theorem 5.5]. A generalised pseudoholomorphic disc with a negative puncture at a local minimum can be rigid only if it is a flow-line connecting a saddle point to the minimum, or consists of a constant pseudoholomorphic disc located at one of the Reeb chords at $c \in \mathcal{Q}(L)$ together with a flow-line from that double point to the local minimum. The aforementioned result gives a bijection between such generalised pseudoholomorphic discs and pseudoholomorphic strips and triangles on the two-copy. \square

Now consider an auxiliary exact immersed Lagrangian L' intersecting $L \cup L_{\epsilon f}$ transversely. For $\epsilon > 0$ sufficiently small there is a bijection between the intersection points $L \cap L'$ and $L_{\epsilon f} \cap L'$.

Corollary 4.22. *For a suitable Morse-Smale pair (f, g) and almost complex structure as in Lemma 4.21 there is a unique rigid and transversely cut out pseudoholomorphic triangle with corners at $e, c \in L \cap L_{\epsilon f}$ and the corresponding double point $c' \in L_{\epsilon f}$ for any connected gradient flow-line from*

$c \in L \cap L_{\epsilon f}$ to the local minimum $e \in \mathcal{C}$. The triangle is moreover contained inside a small neighbourhood of the same flow-line.

Proof. We need to apply Lemma 4.21 in the case when L^+ is taken to be the Legendrian lift $(L \cup L_{\epsilon f})^+$, where $L_{\epsilon f}^+$ is above L^+ , and the push-off is taken to be $(L \cup L')_{\epsilon F}^+$ for a Morse function $F: L \cup L' \rightarrow \mathbb{R}$ that restricts to f on L . \square

5. CONTINUATION MAPS

In this section we analyse what happens to the Floer homology when we change J , H (in some suitable way) or move the Lagrangian submanifolds by a compactly supported safe exact isotopy.

5.1. Changing the almost complex structure. Following [17] (see also [20]) we will use the bifurcation method to prove invariance of Floer homology for Lagrangian intersection under change of almost complex structure. It seems, in fact, that the more usual continuation method is not well suited to describe how the obstruction algebras change when the almost complex structure changes.

Let us fix Lagrangian immersions (L_0, ι_0) and (L_1, ι_1) and a cylindrical Hamiltonian H compatible with L_0 and L_1 . For a generic one-parameter family of time-dependent almost complex structure J_\bullet^δ parametrised by an interval $[\delta_-, \delta_+]$ such that

- the extrema $J_\bullet^{\delta_-}$ and $J_\bullet^{\delta_+}$ are (L_0, L_1) -compatible, and
- J_\bullet^δ satisfies $(\dagger\dagger)$ for all $\delta \in [\delta_-, \delta_+]$

we define the *parametrised* moduli spaces

$$\mathfrak{M}_{L_0, L_1}^k(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, J_\bullet^\delta)$$

consisting of pairs (δ, u) such that $\delta \in [\delta_-, \delta_+]$ and

$$u \in \mathfrak{M}_{L_0, L_1}^k(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, J_\bullet^\delta).$$

Using the zero-dimensional parametrised moduli spaces, we will define a continuation map

$$\Upsilon_{J_\bullet^\delta}: LCC((L_0, \varepsilon_0^+), (L_1, \varepsilon_1^+); H, J_\bullet^{\delta_+}) \rightarrow LCC((L_0, \varepsilon_0^-), (L_1, \varepsilon_1^-); H, J_\bullet^{\delta_-}).$$

Proposition 5.1. *For a generic one-parameter family J_\bullet^δ of time-dependent almost complex structures as above, the parametrised moduli space*

$$\mathfrak{M}_{L_0, L_1}^k(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H, J_\bullet^\delta)$$

is a transversely cut-out manifold of dimension k . If $k = 0$ it is compact, and therefore consists of a finite number of points. If $k = 1$ it can be compactified in the sense of Gromov-Floer.

If both L_0 and L_1 are spin, the moduli spaces are orientable, and a choice of spin structure on each Lagrangian submanifold induces a coherent orientation on the parametrised moduli spaces.

In the following lemma we look more closely at the structure of the zero-dimensional parametrised moduli spaces. The analogous statement in the setting of Lagrangian Floer homology (for Lagrangian submanifolds) appears in [20, Section 3]. In the case of Legendrian contact homology, the corresponding construction appears in [17, Section 2.4].

Lemma 5.2. *For a generic J_\bullet° there is a finite set $\Delta \subset (\delta_-, \delta_+)$ such that for $\delta \in \Delta$ exactly one of the following cases holds:*

- (i) *there is a unique nonempty moduli space $\mathfrak{M}_{L_0}^{-1}(q_0^0; q_1^0, \dots, q_d^0; J_0^\delta)$ and all other moduli spaces are transversely cut out,*
- (ii) *there is a unique nonempty moduli space $\mathfrak{M}_{L_1}^{-1}(q_0^1; q_1^1, \dots, q_d^1; J_1^\delta)$ and all other moduli spaces are transversely cut out, or*
- (iii) *there is a unique nonempty moduli space $\mathfrak{M}_{L_0, L_1}^0(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H, J_\bullet^\delta)$ and all other moduli spaces are transversely cut out,*

while for every $\delta \in [\delta_-, \delta_+] \setminus \Delta$ the moduli spaces of negative virtual dimension are empty.

(Of course, the self-intersection points q_j^i appearing in the three cases above have nothing to do with each other.) Note that the lemma does not claim that J_\bullet^δ is (L_0, L_1) -regular for $\delta \notin \Delta$: for example, if δ is a critical value of the projection

$$\mathfrak{M}_{L_0, L_1}^1(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H, J_\bullet^\circ) \rightarrow [\delta_-, \delta_+],$$

then J_\bullet^δ is not (L_0, L_1) -regular, but $\delta \notin \Delta$.

Remark 5.3. If $\Delta = \emptyset$, then the complexes $CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet^{\delta_-})$ and $CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet^{\delta_+})$ are isomorphic. In fact the one-dimensional parametrised moduli spaces have boundary points only at $J_\bullet^{\delta_-}$ and $J_\bullet^{\delta_+}$, and this implies that the algebraic count of elements of the zero-dimensional moduli spaces is the same for $J_\bullet^{\delta_-}$ and $J_\bullet^{\delta_+}$.

Next we describe what happens when we cross $\delta \in \Delta$ of type (i) or (ii). Since the cases are symmetric, we will describe only (i).

Lemma 5.4. *Suppose that $\Delta = \{\delta_0\}$ and that the unique nontransversely cut out moduli space for $J_\bullet^{\delta_0}$ is $\mathfrak{M}_{L_0}^{-1}(q_0^0; q_1^0, \dots, q_d^0; J_0^{\delta_0})$. Then the identity is an isomorphism of complexes between $CF((L_0, \varepsilon_0^-), (L_1, \varepsilon_1^-); H, J_\bullet^{\delta_-})$ and $CF((L_0, \varepsilon_0^+), (L_1, \varepsilon_1^+); H, J_\bullet^{\delta_+})$.*

Proof. By Proposition 4.10 there are isomorphisms

$$\mathfrak{Y}_i: (\mathfrak{D}_i^-, \mathfrak{d}_i^-) \rightarrow (\mathfrak{D}_i^+, \mathfrak{d}_i^+)$$

and, given augmentations ε_i^+ for $(\mathfrak{D}_i^+, \mathfrak{d}_i^+)$, we define $\varepsilon_i^- = \varepsilon_i^+ \circ \mathfrak{Y}_i$. For brevity, we denote

$$CF_{\pm} = CF((L_0, \varepsilon_0^{\pm}), (L_1, \varepsilon_1^{\pm}); H, J_{\bullet}^{\delta_{\pm}}).$$

We decompose the differential $\partial_{\varepsilon_0^{\pm}, \varepsilon_1^{\pm}}: CF_{\pm} \rightarrow CF_{\pm}$ as

$$\partial_{\varepsilon_0^{\pm}, \varepsilon_1^{\pm}} = (\varepsilon_0^{\pm} \otimes Id \otimes \varepsilon_1^{\pm}) \circ \tilde{\partial}_{\pm},$$

where $\tilde{\partial}_{\pm}: CF_{\pm} \rightarrow \mathfrak{D}_0^{\pm} \otimes CF_{\pm} \otimes \mathfrak{D}_1^{\pm}$ is the antecedent of the differential defined by counting the same moduli spaces as in Equation (11), but without augmenting the self-intersection points. The lemma then follows from the commutativity of the following diagram

$$\begin{array}{ccccc} CF_- & \xrightarrow{\tilde{\partial}_-} & \mathfrak{D}_1^- \otimes CF_- \otimes \mathfrak{D}_0^- & \xrightarrow{\varepsilon_1^- \otimes Id \otimes \varepsilon_0^-} & CF_- \\ \downarrow Id & & \downarrow \mathfrak{Y}_1 \otimes Id \otimes \mathfrak{Y}_0 & & \downarrow Id \\ CF_+ & \xrightarrow{\tilde{\partial}_+} & \mathfrak{D}_1^+ \otimes CF_+ \otimes \mathfrak{D}_0^+ & \xrightarrow{\varepsilon_1^+ \otimes Id \otimes \varepsilon_0^+} & CF_+ \end{array}$$

where the first square is commutative because of the structure of the boundary of the one-dimensional parametrised moduli spaces for J_{\bullet}^{δ} , while the second square commutes because of the relationship between the augmentations. \square

Finally, we analyse how the complex changes when we cross $\delta \in \Delta$ of type (iii).

Lemma 5.5. *Suppose that $\Delta = \{\delta_0\}$ and that the unique nontransversely cut out moduli space for $J_{\bullet}^{\delta_0}$ is $\mathfrak{M}_{L_0, L_1}^0(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H, J_{\bullet}^{\delta_0})$. Then the map*

$$\Upsilon_{J_{\bullet}}: CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_{\bullet}^{\delta_0^+}) \rightarrow CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_{\bullet}^{\delta_0^-})$$

defined as

$$\Upsilon_{J_{\bullet}}(x) = \begin{cases} x & \text{if } x \neq y_+, \\ y_+ + \#\mathfrak{M}_{L_0, L_1}^0(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H, J_{\bullet}^{\delta_0})\varepsilon_0(\mathbf{q}^0)\varepsilon_1(\mathbf{q}^1)y_- & \text{if } x = y_+ \end{cases}$$

is an isomorphism of complexes.

Proof. The proof is the same as in [20]. However, the proof in [20] holds only in the case of \mathbb{Z}_2 -coefficients. For more general coefficients, we rely on the discussion in [16, 17]. \square

Given a generic homotopy J_{\bullet}^{δ} , we split it into pieces containing only one point of Δ and compose the maps obtained in Lemma 5.4 and 5.5.

5.2. Changing the Hamiltonian. In this section we will keep the almost complex structure fixed. Let H_- and H_+ be time-dependent cylindrical Hamiltonian functions which are compatible with immersed Lagrangian submanifolds L_0 and L_1 . From a one-parameter family of cylindrical Hamiltonians H_s such that

- (i) $H_s = H_-$ for $s \ll 0$,
- (ii) $H_s = H_+$ for all $s \gg 0$, and
- (iii) $\partial_s h'_s(e^{\mathfrak{r}(w)}) \leq 0$ if $\mathfrak{r}(w)$ is sufficiently large,

we will define a continuation map

$$\Phi_{H_s}: CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_+) \rightarrow CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_-).$$

Given a time-dependent almost complex structure J_\bullet , an H_- -Hamiltonian chord x_- , an H_+ -Hamiltonian chord x_+ and double points $\mathbf{p}^0 = (p_1^0, \dots, p_{l_0}^0)$ of L_0 and $\mathbf{p}^1 = (p_1^1, \dots, p_{l_1}^1)$ of L_1 , we define the moduli spaces

$$\mathfrak{M}_{L_0, L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet)$$

as the set of triples (ζ^0, ζ^1, u) such that:

- $(\zeta^0, \zeta^1) \in \tilde{\mathcal{R}}^{l_0|l_1}$ and $u: Z_{\zeta^0, \zeta^1} \rightarrow W$ is a map satisfying the Floer equation

$$(13) \quad \frac{\partial u}{\partial s} + J_t \left(\frac{\partial u}{\partial t} - \chi'(t) X_{H_s}(\chi(t), u) \right) = 0,$$

- $\lim_{s \rightarrow \pm\infty} u(s, t) = x_\pm(\chi(t))$,
- $u(s, 0) \in L_0$ for all $(s, 0) \in Z_{\zeta^0, \zeta^1}$,
- $u(s, 1) \in L_1$ for all $(s, 1) \in Z_{\zeta^0, \zeta^1}$, and
- each ζ_j^i is a negative puncture at p_j^i for $i = 0, 1$ and $j = 1, \dots, l_i$.

Note the only difference between Equation (13) and Equation (9) is that we made X_{H_s} depend on s in Equation (13). For this reason there is no action of $\text{Aut}(Z)$ on the moduli spaces $\mathfrak{M}_{L_0, L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet)$.

Let F_u be the linearisation at u of the Floer operator with s -dependent Hamiltonian. We define

$$\text{ind}(u) = \text{ind}(F_u) + l_0 + l_1,$$

and define $\mathfrak{M}_{L_0, L_1}^k(\mathbf{p}^0, x_-, \mathbf{p}^1, x_+; H_s, J_\bullet)$ as the subset of

$$\mathfrak{M}_{L_0, L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet)$$

consisting of the maps u with $\text{ind}(u) = k$.

The following statement is analogous to the statement in Morse theory, [20, Section 3]. A similar boundary degeneration statement in the case of Legendrian contact homology appears in [17, Section 2.4].

Proposition 5.6. *Given H_s , for a generic time-dependent almost complex structure J_\bullet satisfying $(\dagger\dagger)$ with respect to both H_+ and H_- , the moduli space $\mathfrak{M}_{L_0, L_1}^k(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet)$ is a transversely cut-out manifold of dimension k . If $k = 0$ it is compact, and therefore consists of a finite set of points. If $k = 1$ it can be compactified in the sense of Gromov-Floer.*

If both L_0 and L_1 are spin, the choice of a spin structure on each induces a coherent orientation.

We denote \mathcal{C}_- the set of Hamiltonian chords of H_- and \mathcal{C}_+ the set of Hamiltonian chords of H_+ . We also introduce the weighted count

$$\mathfrak{m}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s) = \# \mathfrak{M}_{L_0, L_1}^0(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet) \varepsilon_0(\mathbf{p}^0) \varepsilon_1(\mathbf{p}^1).$$

Given $x_+ \in \mathcal{C}_+$, we define the continuation map

$$(14) \quad \Phi_{H_s}(x_+) = \sum_{x_- \in \mathcal{C}_-} \sum_{l_0, l_1 \in \mathbb{N}} \sum_{\mathbf{p}^i \in D_i^{l_i}} \mathfrak{m}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s) x_-.$$

The Gromov-Floer compactification of the one-dimensional moduli spaces implies the following lemma.

Lemma 5.7. *The map Φ_{H_s} is a chain map.*

We denote by

$$\Phi_{H_-, H_+}^* : HF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_+) \rightarrow HF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_-)$$

the map induced in homology by Φ_{H_s} — soon it will be apparent that the notation is justified. As it happens in the more standard Floer homology for Lagrangian submanifolds, the continuation maps satisfy the following properties.

Lemma 5.8. (1) *Up to homotopy, Φ_{H_s} depends only on the endpoints H_+ and H_- of H_s ,*
 (2) *$\Phi_{H, H}^*$ is the identity for every H , and*
 (3) *$\Phi_{H_-, H}^* \circ \Phi_{H, H_+}^* = \Phi_{H_-, H_+}^*$.*

Sketch of proof. In order to prove (1), we follow the standard procedure for defining chain homotopies in Floer theory; see [20] for more details. Given a homotopy H_s^δ , $\delta \in [0, 1]$, between s -dependent Hamiltonian functions H_s^0 and H_s^1 with $H_s^\delta \equiv H_-$ for $s \ll 0$ and $H_s^\delta \equiv H_+$ for $s \gg 0$, we define the parametrised moduli spaces $\mathfrak{M}_{L_0, L_1}^k(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s^\delta, J_\bullet)$ of pairs (δ, u) such $\delta \in [0, 1]$ and $u \in \mathfrak{M}_{L_0, L_1}^k(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s^\delta, J_\bullet)$. We define the weighted count

$$\mathfrak{m}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s^\bullet) = \# \mathfrak{M}_{L_0, L_1}^{-1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s^\bullet, J_\bullet) \varepsilon_0(\mathbf{p}^0) \varepsilon_1(\mathbf{p}^1).$$

Then the chain homotopy

$$K : CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_+, J_\bullet) \rightarrow CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_-, J_\bullet)$$

between $\Phi_{H_s^0}$ and $\Phi_{H_s^1}$ is defined as

$$K(x_+) = \sum_{x_- \in \mathcal{C}_-} \sum_{l_0, l_1 \in \mathbb{N}} \sum_{\mathbf{p}^i \in D_i^{l_i}} \mathfrak{m}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s^\bullet) x_-.$$

In order to prove (2) we can choose $H_s \equiv H$: then the moduli space $\mathfrak{M}_{L_0, L_1}^0(\mathbf{p}^0, x_-, \mathbf{p}^1, x_+; H_s, J_\bullet)$ consists of constant strips.

We fix s -dependent Hamiltonian function H_s^+ and H_s^- such that $H_s^+ = H_+$ for $s \geq 1$ and $H_s^+ = H$ for $s \leq 0$, and $H_s^- = H$ for $s \geq 0$ and $H_s^- = H_-$ for $s \leq -1$. In order to prove (3) we introduce the family of Hamiltonian functions

$$H_s^R = \begin{cases} H_{s-R}^+ & \text{for } s \geq 0, \text{ and} \\ H_{s+R} & \text{for } s \leq 0 \end{cases}$$

with $R > 0$. By (1), $\Phi_{H_s^R}$ induces Φ_{H_+, H_-}^* for all R . For $R \gg 0$ there is an identification

$$(15) \quad \begin{aligned} & \mathfrak{M}_{L_0, L_1}^0(p_1^1, \dots, p_{l_1}^1, x_-, p_1^0, \dots, p_{l_0}^0, x_+; H_s^R, J_\bullet) \cong \\ & \bigsqcup_{x \in \mathcal{C}_H} \bigsqcup_{0 \leq h_i \leq l_i} \mathfrak{M}_{L_0, L_1}^0(p_{h_1+1}^1, \dots, p_{l_1}^1, x, p_1^0, \dots, p_{h_0}^0, x_+; H_s^+, J_\bullet) \times \\ & \mathfrak{M}_{L_0, L_1}^0(p_1^1, \dots, p_{h_1}^1, x_-, p_{1+h_0}^0, \dots, p_{l_0}^0, x; H_s^-, J_\bullet) \end{aligned}$$

which follows from standard compactness and gluing techniques, once we know that, for any $R' > 0$, there is R_0 such that, for all $R \geq R_0$, if

$$(\zeta^0, \zeta^1, u) \in \mathfrak{M}_{L_0, L_1}^0(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s^R, J_\bullet),$$

then $\zeta_j^i \notin [-R', R']$ for $i = 0, 1$ and $j = 1, \dots, l_i$.

This follows from a simple compactness argument: if there is R' and a sequence R_n with $(\zeta_n^0, \zeta_n^1, u_n) \in \mathfrak{M}_{L_0, L_1}^0(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_{s^{R_n}}^R, J_\bullet)$ and for every n there is some $\zeta_j^i \in [-R', R']$, then the limit for $n \rightarrow \infty$ has one level which is a solution of a Floer equation with s -invariant data and at least one boundary puncture. By index reasons this level must have index zero, but it cannot be constant because of the boundary puncture. This is a contradiction. \square

With lemma 5.8 at hand, we can prove the following invariance property in the usual formal way.

Corollary 5.9. *If H_0 and H_1 are cylindrical Hamiltonian functions which are compatible with L_0 and L_1 and such that $h'_0(\mathfrak{r}(w)) = h'_1(\mathfrak{r}(w))$ for w outside of a compact set, then the continuation map*

$$\Phi_{H_0, H_1}^* : HF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_0) \rightarrow HF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_1)$$

is an isomorphism.

5.3. Compactly supported safe isotopies. Let $\psi_t: W \rightarrow W$ be a compactly supported smooth isotopy such that $\iota_t = \psi_t \circ \iota_1: L_1 \rightarrow W$ is a safe isotopy. By Lemma 2.23 there exists a local Hamiltonian G_t defined on L_1 which generates the ι_t and for which dG_t has compact support. (Recall that G_t may not extend to a single-valued function on W .)

In the following we will make the further assumption that the path

$$(\psi_t)_* J_1 = d\psi_t \circ J_1 \circ d\psi_t^{-1}, \quad t \in [0, 1],$$

consists of compatible almost complex structures. This will cause no restriction, since we only need the case when ψ_t is equal to the Liouville flow, which is conformally symplectic.

Remark 5.10. In the following manner more general safe isotopies can be considered. Since it is possible to present any smooth isotopy as a concatenation of C^2 -small isotopies, it then suffices to carry out the constructions here for each step separately. Namely, since tameness is an open condition, sufficiently C^2 -small isotopies may be assumed to preserve any given tame almost complex structure. Further control near the double points can then be obtained by assuming that ψ_t actually is conformally symplectic there, which can be assumed without loss of generality.

Denote by $L'_1 = \psi_1 \circ \iota_1(L_1)$ the image. By the usual abuse of notation, we will write L'_1 or $\psi_1(L_1)$ instead of $(L_1, \psi_1 \circ \iota)$. From now on we will assume that the Hamiltonian H is compatible both with L_0 and L_1 and with L_0 and L'_1 .

By Proposition 4.10 the obstruction algebras \mathfrak{D}_1 of (L_1, J_1) as well as \mathfrak{D}'_1 of $(L'_1, (\psi_1)_* J_1)$ are isomorphic by a canonical isomorphism induced by ψ_1 , and therefore any augmentation ε_1 of \mathfrak{D}_1 corresponds to an augmentation ε'_1 of \mathfrak{D}'_1 .

We fix time-dependent almost complex structures J_\bullet^+ and J_\bullet^- such that

- $J_t^\pm = J_0$ for $t \in [0, 1/4]$,
- $J_t^+ = J_1$ and $J_t^- = (\psi_1)_* J_1$ for $t \in [3/4, 1]$,
- J_\bullet^+ is (L_0, L_1) -regular and J_\bullet^- is (L_0, L'_1) -regular.

Given augmentations ε_0 for L_0 and ε_1 for L_1 , we will define a chain map

$$\Psi_G: CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet^+) \rightarrow CF((L_0, \varepsilon_0), (L'_1, \varepsilon'_1); H, J_\bullet^-)$$

using a Floer equation with moving boundary conditions. The presence of self-intersection points of L_1 makes the construction of the moduli spaces more subtle than in the usual case because, in order to have strip-like ends, we need to make the moving boundary conditions constant near the boundary punctures, and therefore domain dependent.

Recall the sets

$$\begin{aligned}\text{Conf}^n(\mathbb{R}) &= \{(\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n \mid \zeta_1 < \dots < \zeta_n\} \quad \text{and} \\ \overline{\text{Conf}}^n(\mathbb{R}) &= \{(\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n \mid \zeta_1 \leq \dots \leq \zeta_n\}.\end{aligned}$$

(Note that this is not how configuration spaces are usually compactified.) Given $n \in \mathbb{N}$, we denote $\mathbf{n} = \{1, \dots, n\}$ and for $m < n$ we denote $\text{hom}(\mathbf{n}, \mathbf{m})$ the set of nondecreasing and surjective function $\phi: \mathbf{n} \rightarrow \mathbf{m}$. Every $\phi \in \text{hom}(\mathbf{n}, \mathbf{m})$ induces an embedding $\phi^*: \overline{\text{Conf}}^m(\mathbb{R}) \rightarrow \overline{\text{Conf}}^n(\mathbb{R})$ defined by

$$\phi^*(\zeta_1, \dots, \zeta_m) = (\zeta_{\phi(1)}, \dots, \zeta_{\phi(m)}).$$

The boundary of $\overline{\text{Conf}}^n(\mathbb{R})$ is a stratified space with dimension m stratum

$$\bigsqcup_{\phi \in \text{hom}(\mathbf{n}, \mathbf{m})} \phi^*(\text{Conf}^m(\mathbb{R})).$$

The embeddings ϕ defined above extend to diffeomorphisms

$$\bar{\phi}^*: \overline{\text{Conf}}^m(\mathbb{R}) \times \mathbb{R}_+^{n-m} \rightarrow \overline{\text{Conf}}^n(\mathbb{R})$$

such that

$$(\zeta'_1, \dots, \zeta'_n) = \bar{\phi}^*((\zeta_1, \dots, \zeta_m), (\epsilon_1, \dots, \epsilon_{n-m}))$$

if

$$\zeta'_i = \zeta_{\phi(i)} + \sum_{k=0}^{i-\phi(i)} \epsilon_k,$$

where $\epsilon_0 = 0$ for the sake of the formula.

Lemma 5.11. *Fix $\delta > 0$. There is a family of constants $\kappa_n > 0$ and smooth functions*

$$\nu_n: \overline{\text{Conf}}^n(\mathbb{R}) \times \mathbb{R} \rightarrow [0, 1]$$

such that, denoting by s the coordinate in the second factor,

- (1) $\nu_n(\zeta_1, \dots, \zeta_n, s) = 0$ for $s > \kappa_n$,
- (2) $\nu_n(\zeta_1, \dots, \zeta_n, s) = 1$ for $s < -\kappa_n$,
- (3) $\partial_s \nu_n(\zeta_1, \dots, \zeta_n, s) \in [-2, 0]$ for all $(\zeta_1, \dots, \zeta_n, s) \in \overline{\text{Conf}}^n(\mathbb{R}) \times \mathbb{R}$,
- (4) $\partial_s \nu_n(\zeta_1, \dots, \zeta_n, s) = 0$ if $|s - \zeta_i| \leq \frac{\delta}{2}$ for some $i = 1, \dots, n$, and
- (5) $\nu_n \circ \phi^* = \nu_m$ for all $m < n$ and all $\phi \in \text{hom}(\mathbf{n}, \mathbf{m})$.

Proof. We can construct the sequences κ_n and ν_n inductively over n using the fact that the set of functions satisfying (1)–(4) is convex. \square

Given $\zeta = (\zeta_1, \dots, \zeta_n) \in \text{Conf}^n(\mathbb{R})$, we define the function $\nu_\zeta: \mathbb{R} \rightarrow [0, 1]$ by $\nu_\zeta(s) = \nu_n(\zeta_1, \dots, \zeta_n, s)$.

Lemma 5.12. *For every n , there is a contractible set of smooth maps*

$$\tilde{J}_n: \overline{\text{Conf}}^n(\mathbb{R}) \times Z \rightarrow \mathcal{J}(\theta)$$

such that

- (1) $\tilde{J}_n(\zeta, s, t) = J_t^+$ if $s > \kappa_n + 1$,
- (2) $\tilde{J}_n(\zeta, s, t) = J_t^-$ if $s < -\kappa_n - 1$,
- (3) $\tilde{J}_n(\zeta, s, t) = J_0$ if $t \in [0, 1/4]$,
- (4) $\tilde{J}_n(\zeta, s, t) = d\psi_{\nu_\zeta(s)} \circ J_1 \circ d\psi_{\nu_\zeta(s)}^{-1}$ if $t \in [3/4, 1]$, and
- (5) for all $\phi \in \text{hom}(\mathbf{n}, \mathbf{n} - \mathbf{1})$, $\tilde{J}_n(\phi^*(\zeta), s, t) = \tilde{J}_{n-1}(\zeta, s, t)$.

Proof. We build \tilde{J}_n inductively on n . At each step, the map \tilde{J} is determined in the complement of $\text{Conf}^n(\mathbb{R}) \times [-\kappa_n - 1, \kappa_n + 1] \times [1/4, 3/4]$. We can extend it to $\overline{\text{Conf}}^n(\mathbb{R}) \times Z$ because $\mathcal{J}(\theta)$ is contractible. \square

Given $\zeta \in \text{Conf}^n(\mathbb{R})$, we will denote \tilde{J}_ζ the s - and t -dependent almost complex structure obtained by restricting \tilde{J} to $\{\zeta\} \times Z$. Given $(\zeta^0, \zeta^1) \in \tilde{\mathcal{R}}^{l_0|l_1}$, we will not distinguish between $\zeta_j^1 \in \mathbb{R} \times \{1\}$ and its s -coordinate, and by this abuse of notation, to $(\zeta^0, \zeta^1) \in \tilde{\mathcal{R}}^{l_0|l_1}$ we will associate ν_{ζ^1} and \tilde{J}_{ζ^1} . For simplicity, the s - and t -dependence of \tilde{J}_{ζ^1} will be omitted in writing the Floer equation.

Consider the sets

$$\begin{aligned} \mathcal{C}_H &= \{x: [0, 1] \rightarrow W : x(0) \in L_0, x(1) \in L_1\}, \\ \mathcal{C}'_H &= \{x: [0, 1] \rightarrow W : x(0) \in L_0, x(1) \in L'_1\}. \end{aligned}$$

Definition 5.13. Given $x_+ \in \mathcal{C}_H$, $x_- \in \mathcal{C}'_H$, and $\mathbf{p}^i \in D_i^{l_i}$ for $i = 0, 1$ and $l_i \geq 0$ we define the moduli space

$$\mathfrak{M}_{L_0, L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G, \tilde{J}, \nu)$$

as the set of triples (ζ^0, ζ^1, u) such that:

- $(\zeta^0, \zeta^1) \in \mathcal{R}^{l_0|l_1}$ and $u: Z_{\zeta^0, \zeta^1} \rightarrow W$ satisfies the Floer equation
- $$(16) \quad \frac{\partial u}{\partial s} + \tilde{J}_{\zeta^1} \left(\frac{\partial u}{\partial t} - \chi'(t) X_H(\chi(t), u) \right) = 0,$$
- $\lim_{s \rightarrow \pm\infty} u(s, t) = x_\pm(\chi(t))$,
 - $u(s, 0) \in L_0$ for all $(s, 0) \in Z_{\zeta^0, \zeta^1}$,
 - $u(s, 1) \in \psi_{\nu_{\zeta^1}(s)}(L_1)$ for all $(s, 1) \in Z_{\zeta^0, \zeta^1}$, and
 - each ζ_j^0 is a negative puncture at p_j^0 and each ζ_j^1 is a negative puncture at $\psi_{\nu_{\zeta^1}(\zeta_j^1)}(L_1)$.

(Recall that $\psi_t: W \rightarrow W$ here is a smooth isotopy satisfying the assumptions made in the beginning of this section, whose restriction to L_1 in particular is the compactly supported safe isotopy generated by $G: \mathbb{R} \times L \rightarrow \mathbb{R}$.) We denote by $\mathfrak{M}_{L_0, L_1}^k(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G, \tilde{J}, \nu)$ the set of triples (ζ^0, ζ^1, u) where $\text{ind}(u) = k$.

Proposition 5.14. *For a generic \tilde{J} as in Lemma 5.12, the moduli space $\mathfrak{M}_{L_0, L_1}^k(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G, \tilde{J}, \nu)$ is a transversely cut out manifold of dimension k . If $k = 0$ it is compact, and therefore consists of a finite set of points. If $k = 1$ it can be compactified in the Gromov-Floer sense.*

If L_0 and L_1 are spin, a choice of a spin structure on each induces a coherent orientation.

Definition 5.15. We denote the weighted count

$$\mathbf{m}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G) = \#\mathfrak{M}_{L_0, L_1}^0(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G, \tilde{J}, \nu) \varepsilon_0(\mathbf{p}^0) \varepsilon_1(\mathbf{p}^1)$$

and define Ψ_G as

$$\Psi_G(x_+) = \sum_{x_- \in \mathcal{C}_H} \sum_{l_0, l_1 \in \mathbb{N}} \sum_{\mathbf{p}^i \in D_i^{l_i}} \mathbf{m}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G) x_-.$$

Remark 5.16. The word \mathbf{p}^1 consists of double points living on the different Lagrangian immersions $\psi_{\nu_{\zeta^1}(\zeta_j^1)}(L_1)$. However, when using the pushed forward almost complex structures $(\psi_{\nu_{\zeta^1}(\zeta_j^1)})_* J_1$, their obstruction algebras all become canonically identified with $(\mathfrak{A}(L_1), \mathfrak{d})$ defined using J_1 . This motivates our abuse of notation ε_1 for an augmentation induced by these canonical identifications.

Lemma 5.17. *The map Ψ_G is a chain map. Moreover, up to chain homotopy, it does not depend on the choice of ν and on the homotopy class of ψ_t relative to the endpoints.*

Proof. Proposition 5.14 implies that Ψ_G is a chain map. The chain homotopies between the continuation maps defined using different choices of ν and ψ_t are defined, as usual, by counting index -1 elements in parametric moduli spaces. We leave the standard details to the reader. \square

Lemma 5.18. *Let $G^0, G^1: L_1 \rightarrow \mathbb{R}$ be local Hamiltonian functions generating the safe isotopies $\psi_t^0 \circ \iota_1$ and $\psi_t^1 \circ \psi_1^0 \circ \iota_1$ respectively, and let $G^2: L_1 \rightarrow \mathbb{R}$ be a local Hamiltonian function generating*

$$\psi_t^2 = \begin{cases} \psi_{2t}^1 & \text{for } t \in [0, 1/2], \\ \psi_{2t-1}^2 \circ \psi_1^1 & \text{for } t \in [1/2, 1]. \end{cases}$$

Then Ψ_{G^2} is chain homotopic to $\Psi_{G^0} \circ \Psi_{G^1}$.

The proof of Lemma 5.18 is analogous to the proof of Lemma 5.8 and is therefore left to the reader.

Corollary 5.19. *If $G: \mathbb{R} \times L \rightarrow \mathbb{R}$ satisfies $dG_t = 0$ outside a compact subset of $(0, 1) \times L$, then the map Ψ_G induces an isomorphism in homology.*

If \bar{J}_\bullet^+ is one (L_0, L_1) -regular almost complex structure and \bar{J}_\bullet^- is another $(L_0, \psi_1(L_1))$ -regular almost complex structure, instead of repeating the construction of \tilde{J} with \bar{J}_\bullet^\pm as starting point, we prefer to consider \tilde{J} assigned once and for all to the triple (L_0, L_1, G) and define the continuation map

$$CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, \bar{J}_\bullet^+) \rightarrow CF((L_0, \varepsilon_0), (L'_1, \varepsilon_1); H, \bar{J}_\bullet^-)$$

as the composition $\Upsilon_- \circ \Psi_G \circ \Upsilon_+$, where

$$\Upsilon_+ : CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, \bar{J}_\bullet^+) \rightarrow CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H, J_\bullet^+),$$

$$\Upsilon_- : CF((L_0, \varepsilon_0), (L'_1, \varepsilon'_1); H, J_\bullet^-) \rightarrow CF((L_0, \varepsilon_0), (L'_1, \varepsilon'_1); H, \bar{J}_\bullet^-)$$

are the maps defined in Section 5.1.

5.4. Commutation between continuation maps. Let H_s be a homotopy from H_- to H_+ as in Section 5.2 and J_\bullet a homotopy from J_\bullet^{-1} to J_\bullet^{+1} as in Section 5.1. For simplicity we will denote

$$CF(H_\pm, J_\bullet^\pm) = CF((L_0, \varepsilon_0^\pm), (L_1, \varepsilon_1^\pm); H_\pm, J_\bullet^{\pm 1}).$$

If $\varepsilon_i^+ = \varepsilon_i^- \circ \mathfrak{Y}_i$, we have defined there continuation maps

$$\Upsilon_\pm : CF(H_\pm, J_\bullet^+) \rightarrow CF(H_\pm, J_\bullet^-),$$

$$\Phi_\pm : CF(H_\pm, J_\bullet^\pm) \rightarrow CF(H_\mp, J_\bullet^\pm)$$

and now we will to prove that they are compatible in the following sense.

Proposition 5.20. *The diagram*

$$(17) \quad \begin{array}{ccc} CF(H_+, J_\bullet^+) & \xrightarrow{\Phi_+} & CF(H_-, J_\bullet^+) \\ \Upsilon_+ \downarrow & & \downarrow \Upsilon_- \\ CF(H_+, J_\bullet^-) & \xrightarrow{\Phi_-} & CF(H_-, J_\bullet^-) \end{array}$$

commutes up to homotopy.

Proposition 5.20 will be proved by applying the bifurcation method to the definition of the continuation maps Φ_\pm : i.e. we will study the parametrised moduli spaces $\mathfrak{M}_{L_0, L_1}^0(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet)$ consisting of pairs (δ, u) such that $\delta \in [0, 1]$ and $u \in \mathfrak{M}_{L_0, L_1}^0(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet^\delta)$. For a generic homotopy J_\bullet , these parametrised moduli spaces are transversely cut out manifolds of dimension one. As before, there is a finite set Δ of bifurcation points such that, for all $\delta \in \Delta$, there is a unique nonempty moduli space of one of the following types:

- (i) $\mathfrak{M}_{L_0}^{-1}(q_0^0; q_1^0, \dots, q_d^0; J_0^\delta)$ or $\mathfrak{M}_{L_1}^{-1}(q_0^1; q_1^1, \dots, q_d^1; J_1^\delta)$,
- (ii) $\mathfrak{M}_{L_0, L_1}^0(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_-, J_\bullet^\delta)$ or $\mathfrak{M}_{L_0, L_1}^0(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_+, J_\bullet^\delta)$,
- (iii) $\mathfrak{M}_{L_0, L_1}^{-1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_s, J_\bullet^\delta)$.

To these moduli spaces correspond four types of boundary configuration for the compactification of the one-dimensional parametrised moduli spaces $\mathfrak{M}_{L_0, L_1}^0(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet^\delta)$, which we write schematically as:

- (i) $\mathfrak{N}_{L_0}^{-1}(J_0^\delta) \times \mathfrak{M}^0(H_s, J_\bullet^\delta)$ or $\mathfrak{N}_{L_1}^{-1}(J_1^\delta) \times \mathfrak{M}^0(H_s, J_\bullet^\delta)$,
- (ii) $\mathfrak{M}^0(H_-, J_\bullet^\delta) \times \mathfrak{M}^0(H_s, J_\bullet^\delta)$ or $\mathfrak{M}^0(H_s, J_\bullet^\delta) \times \mathfrak{M}^0(H_+, J_\bullet^\delta)$,
- (iii) $\mathfrak{M}^1(H_-, J_\bullet^\delta) \times \mathfrak{M}^{-1}(H_s, J_\bullet^\delta)$ or $\mathfrak{M}^{-1}(H_s, J_\bullet^\delta) \times \mathfrak{M}^1(H_+, J_\bullet^\delta)$,
- (iii)' $\mathfrak{N}_{L_0}^1(J_0^\delta) \times \mathfrak{M}^{-1}(H_s, J_\bullet^\delta)$ or $\mathfrak{N}_{L_1}^1(J_1^\delta) \times \mathfrak{M}^{-1}(H_s, J_\bullet^\delta)$.

There is also one fifth type of boundary configuration:

- (iv) $\mathfrak{M}^0(H_s, J_\bullet^{-1})$ or $\mathfrak{M}^0(H_s, J_\bullet^1)$.

In order to prove Proposition 5.20 we split the homotopy J_\bullet^δ into pieces, each of which contains only one element of Δ and we prove that, for each pieces, the corresponding diagram (17) commutes up to homotopy. Putting every piece together, we will obtain the result. We rescale each piece of homotopy so that it is parametrised by $[-1, 1]$ and the bifurcation point is 0.

Lemma 5.21. *Let $\Delta = \{0\}$ be of type (i). Then Diagram (17) commutes.*

Proof. Assume without loss of generality that the nonempty moduli space of negative formal dimension is $\mathfrak{N}_{L_1}^{-1}(q_0^1; q_1^1, \dots, q_d^1; J_1^0)$. We have proved in Lemma 5.4 that Υ_\pm are the identity map. We define a map

$$\tilde{\Phi}_\pm: CF(H_+, J^\pm) \rightarrow \mathfrak{D}_1^\pm \otimes CF(H_-, J^\pm) \otimes \mathfrak{D}_0^\pm$$

such that Φ_\pm decomposes as

$$\Phi_\pm = \varepsilon_1^\pm \otimes Id \otimes \varepsilon_0^\pm,$$

and consider the diagram

$$\begin{array}{ccccc} CF(H_+, J_\bullet^+) & \xrightarrow{\tilde{\Phi}_+} & \mathfrak{D}_1^+ \otimes CF(H_-, J_\bullet^+) \otimes \mathfrak{D}_0^+ & \xrightarrow{\varepsilon_1^+ \otimes Id \otimes \varepsilon_0^+} & CF(H_-, J_\bullet^+) \\ \downarrow \Upsilon_+ & & \downarrow \mathfrak{Y}_1 \otimes \Upsilon_- \otimes \mathfrak{Y}_0 & & \downarrow \Upsilon_- \\ CF(H_+, J_\bullet^-) & \xrightarrow{\tilde{\Phi}_-} & \mathfrak{D}_1^- \otimes CF(H_-, J_\bullet^-) \otimes \mathfrak{D}_0^- & \xrightarrow{\varepsilon_1^- \otimes Id \otimes \varepsilon_0^-} & CF(H_-, J_\bullet^-). \end{array}$$

The first square commutes because of the structure of the compactification of the one-dimensional parametrised moduli spaces and the second by the choice of augmentations. (Recall that Υ_\pm simply are the identity maps here.) \square

Lemma 5.22. *Let $\Delta = \{0\}$ be of type (ii). Then Diagram (17) commutes.*

Proof. We assume, without loss of generality, that the moduli space of negative formal dimension is $\mathfrak{M}_{L_1, L_1}^0(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_+, J_\bullet^0)$. By Lemma 5.5 the

continuation maps for the change of almost complex structure are $\Upsilon_- = Id$ and

$$\Upsilon_+(x) = \begin{cases} x & \text{if } x \neq y_+, \\ y_+ + \#\mathfrak{M}_{L_0, L_1}^0(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_+, J_\bullet^\delta) \varepsilon_0(\mathbf{q}^0) \varepsilon_1(\mathbf{q}^1) y_- & \text{if } x = y_+. \end{cases}$$

The structure of the compactification of one-dimensional parametrised moduli spaces implies that

$$\begin{aligned} & \#\mathfrak{M}^0(\mathbf{q}^1 \mathbf{p}^1, x_-, \mathbf{p}^0 \mathbf{q}^0, y_+; H_s, J_\bullet^1) - \#\mathfrak{M}^0(\mathbf{q}^1 \mathbf{p}^1, x_-, \mathbf{p}^0 \mathbf{q}^0, y_+; H_s, J_\bullet^0) \\ &= \#\mathfrak{M}^0(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_+, J_\bullet^\delta) \#\mathfrak{M}^0(\mathbf{p}^1, x_-, \mathbf{p}^0, y_-; H_s, J_\bullet^0), \end{aligned}$$

while the cardinality of all other moduli spaces remains unchanged. This implies that Diagram (17) commutes. \square

We have dropped the Lagrangian labels from the notation in order to keep the formulas compact. We will do the same in the proofs of the following lemma.

Lemma 5.23. *Let $\Delta = \{0\}$ be of type (iii). Then Diagram (17) commutes up to homotopy.*

Proof. Let $\mathfrak{M}_{L_0, L_1}^{-1}(\mathbf{q}^1, y_-, \mathbf{q}^+, y_+; H_s, J_\bullet^0)$ be the nonempty moduli space of negative formal dimension. In this case $\Upsilon_\pm = Id$ and we define a linear map $K: CF(H_+, J_\bullet^{+1}) \rightarrow CF(H_-, J_\bullet^{-1})$ by

$$K(x) = \begin{cases} 0 & \text{if } x \neq y_+, \\ \#\mathfrak{M}_{L_0, L_1}^{-1}(\mathbf{q}^1, y_-, \mathbf{q}^+, y_+; H_s, J_\bullet^\delta) \varepsilon_0(\mathbf{q}^0) \varepsilon_1(\mathbf{q}^1) y_- & \text{if } x = y_+. \end{cases}$$

The structure of the boundary of the compactification of the one-dimensional parametrised moduli spaces implies that

$$\begin{aligned} & \#\mathfrak{M}^0(\mathbf{q}^1 \mathbf{p}^1, x_-, \mathbf{p}^0 \mathbf{q}^0, y_+; H_s, J_\bullet^1) - \#\mathfrak{M}^0(\mathbf{q}^1 \mathbf{p}^1, x_-, \mathbf{p}^0 \mathbf{q}^0, y_+; H_s, J_\bullet^0) \\ &= \#\mathfrak{M}^{-1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_s, J_\bullet^\delta) \#\mathfrak{M}^1(\mathbf{p}^1, x_-, \mathbf{p}^0, y_-; H_-, J_\bullet^0), \text{ and} \\ & \#\mathfrak{M}^0(\mathbf{q}^1 \mathbf{p}^1, y_-, \mathbf{p}^0 \mathbf{q}^0, x_+; H_s, J_\bullet^1) - \#\mathfrak{M}^0(\mathbf{q}^1 \mathbf{p}^1, y_-, \mathbf{p}^0 \mathbf{q}^0, x_+; H_s, J_\bullet^0) \\ &= \#\mathfrak{M}^{-1}(\mathbf{q}^1, y_-, \mathbf{q}^0, y_+; H_s, J_\bullet^\delta) \#\mathfrak{M}^1(\mathbf{p}^1, x_-, \mathbf{p}^0, y_-; H_-, J_\bullet^0). \end{aligned}$$

From this it follows that $\Phi_+ - \Phi_- = \partial K + K\partial$. \square

The degenerations of type (iii)' are cancelled algebraically by the augmentations, and therefore we obtain the commutativity of the diagram (17) for a generic homotopy J_\bullet^δ .

Now we compare the continuation maps Φ for the change of Hamiltonian and the continuation maps Ψ for compactly supported safe isotopies of L_1 . Let $G: \mathbb{R} \times L_1 \rightarrow \mathbb{R}$ be a local Hamiltonian function satisfying $dG_t = 0$ outside a compact subset of $(0, 1) \times L_1$ which generates the safe isotopy $\psi_t \circ \iota_1$, and

denote $L'_1 = \psi_1(L_1)$. If H_+ and H_- are two Hamiltonian functions which are compatible both with L_0 and L_1 and with L_0 and L'_1 , then we have continuation maps

$$\Psi_G^\pm: CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_\pm, J_\bullet^\pm) \rightarrow CF((L_0, \varepsilon_0), (L'_1, \varepsilon'_1); H_\pm, J_\bullet^\pm)$$

and continuation maps

$$\Phi: CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_+, J_\bullet^+) \rightarrow CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_-, J_\bullet^+),$$

$$\Phi': CF((L_0, \varepsilon_0), (L'_1, \varepsilon'_1); H_+, J_\bullet^+) \rightarrow CF((L_0, \varepsilon_0), (L'_1, \varepsilon'_1); H_-, J_\bullet^+)$$

induced by a homotopy of Hamiltonians H_s with $H_s = H_+$ for $s \geq 1$ and $H_s = H_-$ for $s \leq -1$.

Lemma 5.24. *The diagram*

$$\begin{array}{ccc} CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_+, J_\bullet^+) & \xrightarrow{\Phi} & CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_-, J_\bullet^+) \\ \Psi_G^+ \downarrow & & \downarrow \Psi_G^- \\ CF((L_0, \varepsilon_0), (L'_1, \varepsilon'_1); H_+, J_\bullet^+) & \xrightarrow{\Phi'} & CF((L_0, \varepsilon_0), (L'_1, \varepsilon'_1); H_-, J_\bullet^+) \end{array}$$

commutes up to homotopy.

Sketch of proof. For $R \in \mathbb{R}$ we define $H_s^R = H_{s-R}$. We define the moduli spaces $\mathfrak{M}_{L_0, L_1}^k(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s^R, G, \tilde{J}, \nu)$ as in Definition 5.13 by replacing the Hamiltonian H by the s -dependent Hamiltonian H_s^R . Counting pairs (R, u) where $R \in \mathbb{R}$ and

$$u \in \mathfrak{M}_{L_0, L_1}^{-1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s^R, G, \tilde{J}, \nu),$$

weighted by the augmentations, we obtain a homotopy between $\Psi_G^- \circ \Phi$ and $\Phi' \circ \Psi_G^+$. \square

6. WRAPPED FLOER COHOMOLOGY FOR EXACT LAGRANGIAN IMMERSIONS

In this section we define wrapped Floer cohomology for unobstructed immersed exact Lagrangian submanifolds. With the preparation of the previous sections in place, the definition is not different from the usual one for Lagrangian submanifolds.

6.1. Wrapped Floer cohomology as direct limit. We start by defining wrapped Floer cohomology as a direct limit. This point of view will be useful in the vanishing theorem of the following section. A sketched of the chain level construction, following [3], will be given in the next subsection.

Let (L_0, ι_0) and (L_1, ι_1) be immersed exact Lagrangian submanifolds with augmentations ε_0 and ε_1 respectively. We assume that all intersection points between L_0 and L_1 are transverse, L_0 and L_1 are cylindrical over Legendrian

submanifolds Λ_0 and Λ_1 respectively, and all Reeb chords between Λ_0 and Λ_1 are nondegenerate.

For every $\lambda \in \mathbb{R}$ we denote by $h_\lambda: \mathbb{R}^+ \rightarrow \mathbb{R}$ the function

$$(18) \quad h_\lambda(\rho) = \begin{cases} 0 & \text{if } \rho \in (0, 1], \\ \lambda\rho - \lambda & \text{if } \rho \geq 1. \end{cases}$$

We smooth h_λ inside the interval $[4/5, 6/5]$ (or any sufficiently small neighbourhood of 1 independent of λ) and, by abuse of notation, we still denote the resulting function by h_λ . We assume also that the resulting smooth function satisfies $h''(\rho) \geq 0$ for all $\rho \in \mathbb{R}^+$. We define time-independent cylindrical Hamiltonians $H_\lambda: W \rightarrow \mathbb{R}$ by

$$(19) \quad H_\lambda(w) = h_\lambda(e^{\tau(w)}).$$

For sectors, we multiply $h_\lambda(e^{\tau(w)})$ by the function g as in Definition 4.7. Hamiltonian functions of this form will be called *wrapping Hamiltonian functions*.

We fix a sequence of positive real number λ_n such that $\lim_{n \rightarrow +\infty} \lambda_n = +\infty$ and, for any n , λ_n is not the length of a Reeb chord from Λ_0 to Λ_1 . The set of (L_0, L_1) -regular almost complex structures for every H_{λ_n} is dense, and we pick an element J_\bullet .

By Subsection 5.2, for every $m \geq n$ there are continuation maps

$$\Phi_{\lambda_n, \lambda_m}: HF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda_n}, J_\bullet) \rightarrow HF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda_m}, J_\bullet)$$

forming a direct system.

Definition 6.1. The wrapped Floer cohomology of (L_0, ε_0) and (L_1, ε_1) is defined as

$$(20) \quad HW((L_0, \varepsilon_0), (L_1, \varepsilon_1); J_\bullet) = \varinjlim HF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda_n}, J_\bullet).$$

Wrapped Floer cohomology is well defined, in the sense that it is independent of the choice of the almost complex structure J_\bullet , and of the smoothing of the piecewise linear functions H_{λ_n} and of the sequence λ_n . Invariance of the almost complex structure follows from Proposition 5.20. Invariance of the smoothing of H_{λ_n} follows from Lemma 5.8 and Corollary 5.9. Finally, if $\lambda'_n \rightarrow +\infty$ is another sequence such that no λ'_n is not the length of a Reeb chord from Λ_0 to Λ_1 , we can make both λ_n and λ'_n subsequences of a diverging sequences λ''_n and standard properties of the direct limit give canonical isomorphisms

$$\begin{aligned} \varinjlim HF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda''_n}, J_\bullet) &\cong \varinjlim HF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda_n}, J_\bullet) \\ &\cong \varinjlim HF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda'_n}, J_\bullet). \end{aligned}$$

Therefore $HW((L_0, \varepsilon_0), (L_1, \varepsilon_1))$ does not depend on the sequence λ_n up to isomorphism. It can also be proved that it is invariant under safe isotopies,

but we will need, and prove, only invariance under compactly supported ones.

Lemma 6.2. *Let $G: \mathbb{R} \times L_1 \rightarrow \mathbb{R}$ be a local Hamiltonian function which satisfies $dG_t = 0$ outside a compact subset of $(0, 1) \times L_1$ and let $\psi_t \circ \iota_1$ be the exact regular homotopy it generates, which is assumed to be a safe isotopy. If $L'_1 = \psi_1(L_1)$, J'_\bullet is an (L_0, L'_1) -regular almost complex structure and ε'_1 is the augmentation for L'_1 with respect to J'_\bullet corresponding to ε_1 , then there is an isomorphism*

$$HW((L_0, \varepsilon_0), (L_1, \varepsilon_1); J_\bullet) \cong HW((L_0, \varepsilon_0), (L'_1, \varepsilon'_1); J'_\bullet).$$

Proof. It is enough to observe that, for every n , the isomorphisms

$$CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda_n}, J_\bullet) \cong CF((L_0, \varepsilon_0), (L'_1, \varepsilon'_1); H_{\lambda_n}, J'_\bullet)$$

defined in Subsection 5.3 commute with the continuation maps $\Phi_{\lambda_n, \lambda_m}$ and therefore define isomorphisms of direct systems. This follows from Lemma 5.24 and Proposition 5.20. \square

6.2. A sketch of the chain level construction. Here we recall very briefly the definition of the wrapped Floer complex and the A_∞ -operations. Since Lagrangian immersions will appear only in an intermediate step of the proof of the main theorem, we will not try to make them objects of an enlarged wrapped Fukaya category. Presumably this can be done as in the embedded case, but we have not checked the details of the construction of the necessary coherent Hamiltonian perturbations.

Let L_0 and L_1 be exact Lagrangian immersions which intersect transversely and are cylindrical over chord generic Legendrian submanifolds. We fix a wrapping Hamiltonian $H \geq 0$ as in Equation (18) such that, for every $w \in \mathbb{N}$, the Hamiltonian wH is compatible with L_0 and L_1 (in the sense of Definition 4.3). We also fix an (L_0, L_1) -regular almost complex structure J_\bullet .

Let ε_0 and ε_1 be augmentations of the obstruction algebras of (L_0, J_0) and (L_1, J_1) respectively. Following [3] we define the *wrapped Floer chain complex* as the $\mathbb{F}[q]/(q^2)$ -module

$$(21) \quad CW((L_0, \varepsilon_0), (L_1, \varepsilon_1); J_\bullet) = \bigoplus_{w=0}^{\infty} CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); wH, J_\bullet)[q]$$

with a differential μ^1 such that, on $x + yq \in CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); wH, J_\bullet)[q]$, it is defined as

$$\mu^1(x + yq) = \partial x + y + \Phi_w(y) + (\partial y)q$$

where

$$\Phi_w: CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); wH, J_\bullet) \rightarrow CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); (w+1)H, J_\bullet)$$

is the continuation map for the change of Hamiltonian defined in Subsection 5.2.

Remark 6.3. The endomorphism ι_q (denoted ∂_q in [3]) defined as

$$\iota_q(x + yq) = y$$

is a chain map. However, its action in homology is trivial.

Remark 6.4. The direct sum (21) starts from $w = 1$ in [3]. It is equivalent to start from $w = 0$, when possible, by [3, Lemma 3.11]. The homology of $CW((L_0, \varepsilon_0), (L_1, \varepsilon_1); J_\bullet)$ is isomorphic to $HW((L_0, \varepsilon_0), (L_1, \varepsilon_1))$ defined as the direct limit in Equation (20) by [3, Lemma 3.12].

The A_∞ -operations between wrapped Floer complexes are defined by counting pseudoholomorphic polygons with carefully constructed Hamiltonian perturbations. In the immersed case, those polygons will be allowed to have boundary punctures at double points and, as usual, must be counted with a weight coming from the augmentations. The only thing we need to know about the operations between wrapped Floer cohomology groups is that the component

$$\mu^d: CF(L_{d-1}, L_d) \otimes \dots \otimes CF(L_0, L_1) \rightarrow CF(L_0, L_d)$$

of the operation

$$\mu_{WF}^d: CW(L_{d-1}, L_d) \otimes \dots \otimes CW(L_0, L_1) \rightarrow CW(L_0, L_d)$$

coincides with the operation μ^d defined in Equation (12). For simplicity of notation we have dropped the augmentations from the above formulas.

7. A TRIVIAL TRIVIALITY RESULT

An exact Lagrangian embedding with cylindrical ends which is disjoint from the skeleton is known to have vanishing wrapped Floer cohomology. This was proven in [12, Theorem 9.11(b)] but also follows from e.g. [3, Section 5.1]. Note that the statement is false in the more general case when the Lagrangian merely is monotone. In this section we extend this classical vanishing result to our setting of *exact* Lagrangian immersions. We will write the details of the proof for a Liouville manifold and explain the necessary modifications for Liouville sectors in Remark 7.9.

7.1. Action and energy. In this subsection we define an action for double points of immersed exact Lagrangian submanifolds and for Hamiltonian chords and prove action estimates for various continuation maps. Let $p \in W$ be a double point of a Lagrangian immersion (L, ι) with potential f . We recall that there are points $p_\pm \in L$ characterised by $\iota^{-1}(p) = \{p_+, p_-\}$ and $f(p_+) > f(p_-)$. We define the *action* of p as

$$\mathfrak{a}(p) = f(p_+) - f(p_-).$$

If L is disconnected and p is in the intersection between the images of two connected components, then $\mathfrak{a}(p)$ depends on the choice of the potential function f , otherwise it is independent of it.

Given a holomorphic map $(\zeta, u) \in \tilde{\mathfrak{N}}_L(p_0; p_1, \dots, p_d; J)$, Stokes's theorem immediately yields

$$\int_{\Delta_\zeta} u^* d\theta = \mathfrak{a}(p_0) - \sum_{i=1}^d \mathfrak{a}(p_i).$$

Since $\int_{\Delta_\zeta} u^* d\theta > 0$ for a nonconstant J -holomorphic map, if

$$\tilde{\mathfrak{N}}_L(p_0; p_1, \dots, p_d; J) \neq \emptyset,$$

we obtain

$$(22) \quad \mathfrak{a}(p_0) - \sum_{i=1}^d \mathfrak{a}(p_i) > 0.$$

Given two Lagrangian submanifolds L_0 and L_1 with potentials f_0 and f_1 and a Hamiltonian function H , we define the action of a Hamiltonian chord $x: [0, 1] \rightarrow W$ from L_0 to L_1 as

$$(23) \quad \mathcal{A}(x) = \int_0^1 x^* \theta - \int_0^1 H(x(t)) dt + f_0(x(0)) - f_1(x(1)).$$

Note that this is the negative of the action used in [29].

Example 7.1. Let $H: W \rightarrow \mathbb{R}$ be a cylindrical Hamiltonian such that $H(w) = h(e^{\mathfrak{r}(w)})$, where $h: \mathbb{R}^+ \rightarrow \mathbb{R}$. Then a Hamiltonian chord $x: [0, 1] \rightarrow W$ from L_0 to L_1 is contained in a level set $\mathfrak{r}^{-1}(r)$ and has action

$$(24) \quad \mathcal{A}(x) = h'(e^r) e^r - h(e^r) + f_0(x(0)) - f_1(x(1)).$$

The following lemma, which we prove in the more general case of the moduli spaces of Floer solutions with an s -dependent Hamiltonian, applies equally to the particular case of moduli spaces used in the definition of the Floer differential. We introduce the following notation. Given a set A and a function $f: A \rightarrow \mathbb{R}$, we denote $\|f\|_\infty^+ := \sup_{a \in A} \max\{f(a), 0\}$.

Lemma 7.2. *Let $H_s: \mathbb{R} \times [0, 1] \times W \rightarrow \mathbb{R}$ be an s -dependent cylindrical Hamiltonian function satisfying conditions (i), (ii), and (iii) of Subsection 5.2. We make the simplifying hypothesis that $\partial_s H_s \equiv 0$ if $s \notin [-1, 1]$. If $\mathfrak{M}_{L_0, L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet) \neq \emptyset$, then*

$$(25) \quad \mathcal{A}(x_-) \leq \mathcal{A}(x_+) + 6\|\partial_s H_s\|_\infty^+.$$

Note that Equation (25) is far from being sharp, but there will be no need for a sharper estimate.

Proof. Let $(u, \zeta^0, \zeta^1) \in \mathfrak{M}_{L_0, L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H_s, J_\bullet)$. Then, in a metric on u^*TW induced by $d\theta$ and J_\bullet ,

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_0^1 |\partial_s u|^2 dt ds &= \int_{-\infty}^{+\infty} \int_0^1 d\theta(\partial_s u, \partial_t u - \chi'(t)X_{H_s}(\chi(t), u)) dt ds \\ &= \int_{Z_{\zeta^0, \zeta^1}} u^* d\theta - \int_{-\infty}^{+\infty} \int_0^1 \chi'(t) dH_s(\chi(t) \partial_s u) dt ds. \end{aligned}$$

Using Stokes's theorem we obtain

$$\begin{aligned} \int_{Z_{\zeta^0, \zeta^1}} u^* d\theta &= f_1(x_-(1)) - f_1(x_+(1)) - \\ &\quad f_0(x_-(0)) + f_0(x_+(0)) - \sum_{i=0}^1 \sum_{j=1}^{l_i} \mathfrak{a}(p_j^i). \end{aligned}$$

Using the equality $\partial_s(H_s \circ u) = (\partial_s H_s) \circ u + dH_s(\partial_s u)$ we obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_0^1 \chi'(t) dH_s(\chi(t), (\partial_s u)) dt ds &= \\ \int_{-\infty}^{+\infty} \int_0^1 \chi'(t) \partial_s(H_s(\chi(t), u(s, t))) dt ds &- \\ \int_{-\infty}^{+\infty} \int_0^1 \chi'(t) (\partial_s H_s)(\chi(t), u(s, t)) dt ds. \end{aligned}$$

We can compute

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_0^1 \chi'(t) \partial_s(H_s(\chi(t), u(s, t))) dt ds &= \\ \int_0^1 \chi'(t) H_+(\chi(t), x_+(\chi(t))) dt - \int_0^1 \chi'(t) H_-(\chi(t), x_-(\chi(t))) dt &= \\ \int_0^1 H_+(t, x_+(t)) dt - \int_0^1 H_-(t, x_-(t)) dt. \end{aligned}$$

Thus, rearranging the equalities, we have

$$\begin{aligned} \mathcal{A}(x_+) - \mathcal{A}(x_-) &= \int_{-\infty}^{+\infty} \int_0^1 |\partial_s u|^2 dt ds + \sum_{i=0}^1 \sum_{j=1}^{l_i} \mathfrak{a}(p_j^i) - \\ &\quad \int_{-\infty}^{+\infty} \int_0^1 \chi'(t) \partial_s H_s(\chi(t) u(s, t)) ds dt. \end{aligned}$$

Finally, we estimate

$$\int_{-\infty}^{+\infty} \int_0^1 \chi'(t) (\partial_s H_s)(\chi(t), u(s, t)) dt ds \leq 6 \|\partial_s H_s\|_\infty^+$$

and obtain Equation (25). \square

Corollary 7.3. *The differential in $CF(L_0, L_1; H, J_\bullet)$ decreases the action. If $\partial_s H_s \leq 0$, then the continuation map Φ_{H_s} also decreases the action.*

Now we turn our attention at the continuation map Ψ_G defined in Subsection 5.3. Let $G: \mathbb{R} \times L_1 \rightarrow \mathbb{R}$ be a local Hamiltonian function such that $dG_t = 0$ outside a compact subset of $(0, 1) \times L_1$ and let $\psi_t \circ \iota_1$ be the compactly supported exact regular homotopy it generates. Now assume that ψ_t is a safe isotopy. Denote, as usual, $L'_1 = \psi_1(L_1)$.

First we make the following remark about a special type of safe isotopy and the action of the image of the double points.

Remark 7.4. Let (L, ι) be an exact Lagrangian immersion and $\psi_t: W \rightarrow W$ a smooth isotopy. If $\psi_t^* \theta = e^{c(t)} \theta$, then

- (1) $\psi_t(L)$ is a safe isotopy, and
- (2) if p is a double point of (L, ι) , then $\psi_t(p)$ is a double point of $(L, \psi_t \circ \iota)$ whose action satisfies $\mathfrak{a}(p) = e^{c(t)} \mathfrak{a}(\psi_t(p))$.

Given a Hamiltonian function $H: [0, 1] \times W \rightarrow \mathbb{R}$ which is compatible both with L_0 and L_1 , as well as with L_0 and L'_1 , let \mathcal{C}_H be the set of Hamiltonian chords of H from L_0 to L_1 and let \mathcal{C}'_H be the set of Hamiltonian chords of H from L_0 to L'_1 .

Observe that any safe Lagrangian isotopy from L_1 to L'_1 induces a continuous family of potentials f_1^s . Fixing a choice of local Hamiltonian $G: \mathbb{R} \times L_1 \rightarrow \mathbb{R}$ generating the safe isotopy makes the potential f'_1 on L'_1 determined by the choice of potential f_1 on L_1 via a computation as in the proof of Lemma 2.3.

Lemma 7.5. *For every chords $\mathbf{x}_- \in \mathcal{C}'_H$ and $\mathbf{x}_+ \in \mathcal{C}_H$ and for every sets of self-intersection points $\mathbf{p}^0 = (p_1^0, \dots, p_{l_0}^0)$ of L_0 and $\mathbf{p}^1 = (p_1^1, \dots, p_{l_1}^1)$ of L_1 , if $\mathfrak{M}_{L_0, L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G, \tilde{J}, \nu) \neq \emptyset$, then*

$$\mathcal{A}(x_-) \leq \mathcal{A}(x_+) + 2\mu \|G\|_\infty,$$

where $\|G\|_\infty$ is the supremum norm of G and $\mu \geq 0$ is the measure of the subset $\{s \in \mathbb{R}\}$ for which $G_s: L \rightarrow \mathbb{R}$ is not constantly zero.

Proof. Consider $(\zeta^0, \zeta^1, u) \in \mathfrak{M}_{L_0, L_1}(\mathbf{p}^1, x_-, \mathbf{p}^0, x_+; H, G, \tilde{J}, \nu)$. Observe that the map $u: Z_{\zeta^0, \zeta^1} \rightarrow W$ extends to a continuous map $u: Z \rightarrow W$. We have:

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_0^1 |\partial_s u|^2 dt ds &= \int_Z du^* \theta - \int_0^1 \left(\int_{-\infty}^{+\infty} \left(\frac{\partial}{\partial s} (H \circ u) \right) ds \right) dt = \\ &= \int_Z u^* d\theta - \int_0^1 H(x_+(t)) dt + \int_0^1 H(x_-(t)) dt. \end{aligned}$$

We denote by $u_i: \mathbb{R} \rightarrow W$, for $i = 0, 1$, the continuous and piecewise smooth maps $u_i(s) = u(s, i)$ and use Stokes theorem:

$$\int_Z u^* d\theta = \int_0^1 x_+^* \theta - \int_0^1 x_-^* \theta + \int_{\mathbb{R}} u_0^* \theta - \int_{\mathbb{R}} u_1^* \theta.$$

Let f_0 and f_1 be the potentials of L_0 and L_1 respectively, and \tilde{f}_1 the potential of $\psi_1(L_1)$. The map u_0 takes values in L_0 , and therefore

$$\int_{\mathbb{R}} u_0^* \theta = f_0(x_+(0)) - \sum_{j=1}^{l_0} \mathfrak{a}(p_j^0) - f_0(x_-(0)).$$

We are left with the problem of estimating $\int_{\mathbb{R}} u_1^* \theta$, which is slightly more complicated here because $u_1(s) \in \psi_{\nu_{\zeta^1}(s)}(L_1)$. We will denote $\psi_s^\nu := \psi_{\nu_{\zeta^1}(s)}$. Recall that ψ_s^ν is a smooth isotopy inducing a safe isotopy of L_1 generated by the local Hamiltonian function $G^\nu(s, w) = \nu_{\zeta^1}'(s)G(\nu_{\zeta^1}(s), w)$, $w \in L_1$.

We use the following trick. Consider $W \times \mathbb{R} \times \mathbb{R}$ with the Liouville form $\Theta := \theta + \tau d\sigma$, where σ is the coordinate of the first copy of \mathbb{R} and τ is the coordinate in the second copy. The notation here conflicts with the use of (σ, τ) as coordinates in the strip-like ends near the boundary punctures, but this will not cause confusion.

Consider the symplectic suspension

$$\Sigma := \{(x, s, t) \in W \times \mathbb{R}^2; x \in \psi_s^\nu(y), y \in L_1, t = -G^\nu(s, y)\}.$$

of the isotopy $\psi_s^\nu(L_1)$, which is an exact Lagrangian immersion. This should be seen as a corrected version of the trace of the isotopy, in order to make it Lagrangian.

Lift $u_1: \mathbb{R} \rightarrow W$ to $\tilde{u}_1: \mathbb{R} \rightarrow W \times \mathbb{R} \times \mathbb{R}$ by defining

$$\tilde{u}_1(s) = (u_1(s), s, -G^\nu(s, \bar{u}_1(s))) \in \Sigma,$$

and where \bar{u}_1 is the lift of u_1 to L which is smooth away from the punctures.

Using the computation in the proof of Lemma 2.3, together with the Lagrangian condition satisfied by Σ , we obtain

$$\int_{-\infty}^{+\infty} \tilde{u}_1^* \Theta = f_1'(x_+(1)) - \sum_{j=1}^{l_1} \mathfrak{a}(p_j^1) - f_1(x_-(1)),$$

as well as

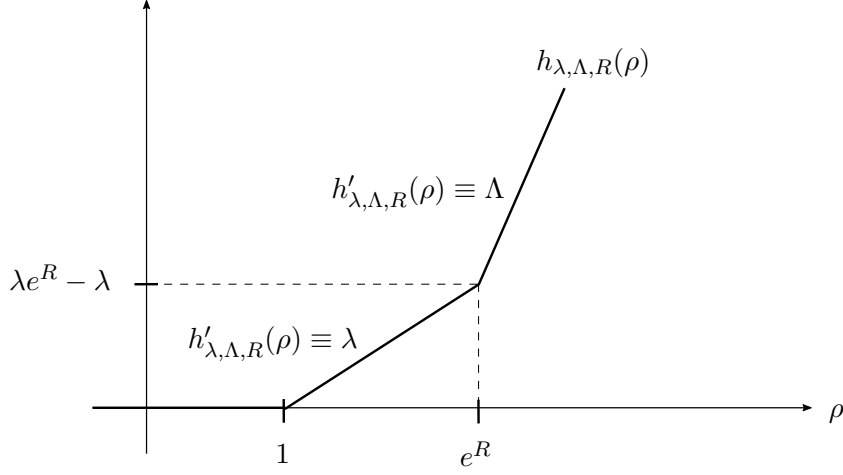
$$u_1^* \theta - \tilde{u}_1^* \Theta = G^\nu(s, \bar{u}_1(s)) d\sigma.$$

Observe that we here abuse notation, and use $\mathfrak{a}(p_j^1) > 0$ for the action computed with respect to the induced potential function on $\psi_s^\nu(L_1)$ for the corresponding value of $s \in \mathbb{R}$.

Since $\|\nu_{\zeta^1}'\|_\infty \leq 2$, we finally obtain

$$\mathcal{A}(x_-) \leq \mathcal{A}(x_+) + 2\mu\|G\|_\infty,$$

where $\mu \geq 0$ is as required. \square

FIGURE 2. The graph of $h_{\lambda, \Lambda, R}$.

7.2. Pushing up. In this subsection we prove the following proposition.

Proposition 7.6. *Let (L_0, ι_0) and (L_1, ι_1) be exact Lagrangian immersions in a Liouville manifold (W, θ) and let J_\bullet be an (L_0, L_1) -regular almost complex structure. If the Liouville flow of (W, θ) displaces L_1 from any compact set, then, for all pair of augmentations ε_0 and ε_1 of the obstruction algebras of (L_0, J_0) and (L_1, J_1) respectively,*

$$HW((L_0, \varepsilon_0), (L_1, \varepsilon_1), J_\bullet) = 0.$$

We postpone the proof after a couple of lemmas. Given $\Lambda > \lambda > 0$ and $L > 0$, we define a function $h_{\lambda, \Lambda, R}: \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$h_{\lambda, \Lambda, R}(\rho) = \begin{cases} 0 & \text{for } \rho \leq 1, \\ \lambda\rho - \lambda & \text{for } \rho \in [1, e^R], \\ \Lambda\rho - (\Lambda - \lambda)e^R - \lambda & \text{for } \rho \geq e^R. \end{cases}$$

See Figure 2 for the graph of $h_{\lambda, \Lambda, R}$.

The function $h_{\lambda, \Lambda, R}$ has two corners: one at $(1, 0)$ and one at $(e^R, \lambda e^R - \lambda)$. We smooth $h_{\lambda, \Lambda, R}$ in a small neighbourhood of these corners so that the new function (which we still denote by $h_{\lambda, \Lambda, R}$) satisfies $h''_{\lambda, \Lambda, R}(\rho) \geq 0$ for all ρ . We define the (time independent) cylindrical Hamiltonian $H_{\lambda, \Lambda, R}: W \rightarrow \mathbb{R}$ by $H_{\lambda, \Lambda, R}(w) = h_{\lambda, \Lambda, R}(e^{\tau(w)})$. We make the assumption that there is no Hamiltonian time-1 chord from L_0 to L_1 on ∂W_r when r satisfies either $h'_{\lambda, \Lambda, R}(e^r) = \lambda$ or $h'_{\lambda, \Lambda, R}(e^r) = \Lambda$. This is equivalent to assuming that there is no Reeb chord from Λ_0 to Λ_1 of length either λ or Λ .

We assume, without loss of generation, that L_0 and L_1 intersect transversely, that $L_i \cap W_1^e$ is a cylinder over a Legendrian submanifold Λ_i and that all

Reeb chords from Λ_0 to Λ_1 are nondegenerate. Then we have three types of chords:

- constant chords, i.e. intersection points between L_0 and L_1 , which are contained in W_0 ,
- chords coming from smoothing the corner of $h_{\lambda,\Lambda,R}$, which are concentrated around ∂W_1 , and
- chords coming from smoothing the corner of $h_{\lambda,\Lambda,R}$ at $(e^R, \lambda e^R - \lambda)$, which are concentrated around ∂W_L .

Constant chords and chords coming from smoothing the first corner will be called *type I chords*, while chords coming from smoothing the second corner will be called *type II chords*. We say that a chord of $H_{\lambda,\Lambda,L}$ *appears at slope* s if it is contained in ∂W_r for r such that $h'_{\lambda,\Lambda,L}(e^r) = s$. By abuse of terminology, we will consider the intersection points between L_0 and L_1 as chords appearing at slope zero.

Lemma 7.7. *Given $\lambda > 0$, there exists $C > 0$ such that, for every $\Lambda > \lambda$ and every $R \geq C$, every chord of type II of $H_{\lambda,\Lambda,R}$ has larger action than any chord of type I.*

Proof. If x is a Hamiltonian chord contained in ∂W_r , then the action of x is

$$(26) \quad \mathcal{A}(x) = h'_{\lambda,\Lambda,R}(e^r)e^r - h_{\lambda,\Lambda,R}(e^r) + f_0(x(0)) - f_1(x(1)).$$

Observe that $|f_0(x(0)) - f_1(x(1))|$ is uniformly bounded because f_0 and f_1 are locally constant outside of a compact set. The Hamiltonian chords of type I appear at slope $\lambda_- < \lambda$ and near ∂W_0 , and therefore r in Equation (26) is close to zero. Then, there is a constant C_- , depending on f_0 , f_1 and the smoothing procedure at the first corner such that, if x is a chord of type I, then $\mathcal{A}(x) \leq \lambda_- + C_-$.

On the other hand, if x is a chord of type II, then it appears at slope $\lambda_+ > \lambda$ and around $r = \log(R)$. Then there is a constant C_+ , depending on f_0 , f_1 and the smoothing procedure at the second corner such that $\mathcal{A}(x) \geq R\lambda_+ - \lambda R + \lambda - C_+ = R(\lambda_+ - \lambda) + \lambda - C_+$. The lemma follows from $\lambda_+ - \lambda > 0$ and the fact that chords arise at a discrete set of slopes. \square

From now on we will always take $R \geq C$. The consequence of Lemma 7.7 is that the chords of type I generate a subcomplex of

$$CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda,\Lambda,R}, J_\bullet)$$

which we will denote by $CF^I((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda,\Lambda,R}, J_\bullet)$. The main ingredient in the proof of Proposition 7.6 is the following lemma.

Lemma 7.8. *If the Liouville flow of (W, θ) displaces L_1 from any compact set, then the inclusion map*

$$CF^I((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda,\Lambda,R}, J_\bullet) \hookrightarrow CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda,\Lambda,R}, J_\bullet)$$

is trivial in homology whenever $\Lambda \gg 0$ is sufficiently large.

Proof. The Liouville flow applied to L_1 gives rise to a compactly supported safe isotopy from L_1 to L'_1 , and is generated by the time-dependent local Hamiltonian $G: \mathbb{R} \times L \rightarrow \mathbb{R}$ for which $dG_t = 0$ outside of a compact subset of $(0, 1) \times L \rightarrow \mathbb{R}$; see Lemma 2.23. Since the Liouville form is *conformally* symplectic, it actually preserves the space of compatible almost complex structure cylindrical at infinity.

We will choose to apply the Liouville flow so that $L'_1 \subset \{\rho \geq e^R\}$; recall that this is possible by our assumptions.

The continuation map

$$\Psi_G: CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda, \Lambda, R}, J_\bullet) \rightarrow CF((L_0, \varepsilon_0), (L'_1, \varepsilon'_1); H_{\lambda, \Lambda, R}, J'_\bullet)$$

defined in Subsection 5.3 induces an isomorphism in homology if J'_\bullet is an (L_0, L'_1) -regular almost complex structure such that $J'_0 = J_0$ and $J'_1 = (\psi_1)_* J_1$, and ε'_1 is the augmentation of the obstruction algebra of (L'_1, J'_1) defined by $\varepsilon'_1 = \varepsilon_1 \circ \psi_1^{-1}$.

By Lemma 7.5, there is a constant C , independent of Λ , such that $\Psi_G(x)$ is a linear combination of chords of action at most C whenever x is a chord from L_0 to L_1 of type I. On the other hand, the complex $CF((L_0, \varepsilon_0), (L'_1, \varepsilon_1); H_{\lambda, \Lambda, R})$ is generated by Hamiltonian chords x of action

$$\mathcal{A}(x) = (\Lambda - \lambda)e^R + \lambda + f_0(x(0)) - f_1(x(1)).$$

Here we have used $L'_1 \subset \{\rho \geq e^R\}$, together with the particular form of $H_{\lambda, \Lambda, R}$ in the same subset. This implies that, for Λ large enough, $\Psi_G(x) = 0$ for all chords x of type I. \square

Proof of Proposition 7.6. For every λ and Λ there are continuation maps

$$\begin{aligned} \Phi_{\lambda, \Lambda}^{(1)}: CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_\lambda, J_\bullet) &\rightarrow CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda, \Lambda, R}, J_\bullet), \\ \Phi_{\lambda, \Lambda}^{(2)}: CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda, \Lambda, R}) &\rightarrow CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_\Lambda, J_\bullet) \text{ and} \\ \Phi_{\lambda, \Lambda}: CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_\lambda, J_\bullet) &\rightarrow CF((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_\Lambda, J_\bullet) \end{aligned}$$

such that there is a chain homotopy between $\Phi_{\lambda, \Lambda}$ and $\Phi_{\lambda, \Lambda}^{(2)} \circ \Phi_{\lambda, \Lambda}^{(1)}$.

We can assume that $\Phi_{\lambda, \Lambda}$, $\Phi_{\lambda, \Lambda}^{(1)}$ and $\Phi_{\lambda, \Lambda}^{(2)}$ are defined using s -dependent Hamiltonians H_s^* ($s = \emptyset, (1), (2)$) such that $\partial_s H_s^* \leq 0$, and therefore they decrease the action. Hence, the image of $\Phi_{\lambda, \Lambda}^{(1)}$ is contained in

$$CF^I((L_0, \varepsilon_0), (L_1, \varepsilon_1); H_{\lambda, \Lambda, R}, J_\bullet)$$

(here we use Lemma 7.7) and therefore it follows from Lemma 7.8 that $\Phi_{\lambda, \Lambda} = 0$ in homology. By the definition of wrapped Floer cohomology as a direct limit, this implies that $HW((L_0, \varepsilon_0), (L_1, \varepsilon_1), J_\bullet) = 0$. \square

Remark 7.9. The arguments of this section apply equally well to Liouville sectors, provided that the following precautions are taken. If the cylindrical Hamiltonian is chosen as in Definition 4.7, then its Hamiltonian vector field has excellent dynamics on V . We will assume that the Legendrian end of L_0 and L_1 are disjoint from the coconvex neighbourhood \mathcal{N} of Definition 4.5. This condition is easy to obtain up to safe isotopy. Then the excellent dynamics of the Hamiltonian vector field implies that Hamiltonian chords between L_0 and L_1 are contained in a compact subset of $\text{int}(W)$. Then we can use the contact form $g^{-1}\alpha$ instead of α to estimate actions, and the proof goes through.

8. FLOER HOMOLOGY AND LAGRANGIAN SURGERY

Lalonde and Sikorav in [26] and then Polterovich in [28] defined a surgery operation on Lagrangian submanifolds. It is expected that Lagrangian surgery should correspond to a twisted complex (i.e. an iterated mapping cone) in the Fukaya category. Results in this direction have been proved by Seidel in [31], Fukaya, Oh, Ohta and Ono in [21] and by Biran and Cornea in [6]. Our goal in this section is to establish Proposition 8.16, which provides us with a result along these lines in the generality that we need.

The difficult point in handling the Lagrangian surgery from the Floer theoretic perspective is that, except in very favourable situations, the Lagrangian submanifolds produced are not well behaved from the point-of-view of pseudoholomorphic discs. In our situation, we turn out to be lucky, since only surgeries that preserve exactness are needed. Nevertheless, there still is a complication stemming from the fact that the resulting Lagrangian only is immersed (as opposed to embedded). This is the main reason for the extra work needed, and here we rely on the theory developed in the previous sections.

The bounding cochains that we will consider in this exact immersed setting are those corresponding to augmentations of the corresponding obstruction algebras introduced in Section 4.2. This turns out to be a very useful perspective, since it enables us to apply techniques from Legendrian contact homology in order to study them.

8.1. The Cthulhu complex. In this subsection we recall, and slightly generalise, the definition of Floer homology for Lagrangian cobordisms we defined in [9].

Definition 8.1. Given cylindrical exact Lagrangian immersions L_+ and L_- in (W, θ) which coincide outside of a compact set, an exact Lagrangian cobordism Σ from L_- to L_+ is a properly embedded submanifold

$$\Sigma \subset (\mathbb{R} \times M, d(e^t \beta)) = (\mathbb{R} \times W \times \mathbb{R}, d(e^t(dz + \theta)))$$

such that, for C and R sufficiently large,

- (1) $\Sigma \cap (-\infty, -C] \times W \times \mathbb{R} = (-\infty, -C] \times L_-^+$,
- (2) $\Sigma \cap [C, +\infty) \times W \times \mathbb{R} = [C, +\infty) \times L_+^+$,
- (3) $\Sigma \cap (\mathbb{R} \times W_R^e \times \mathbb{R})$ is tangent to both ∂_s and the lift of the Liouville vector field \mathcal{L} of (W, θ) , and
- (4) $e^s \alpha|_\Sigma = dh$ for a function $h: \Sigma \rightarrow \mathbb{R}$ which is constant on $\Sigma \cap (-\infty, -C] \times W \times \mathbb{R}$.

The intersection

$$\Sigma \cap (\mathbb{R} \times W_R^e \times \mathbb{R}) = \mathbb{R} \times (L_\pm^\pm \cap (W_R^e \times \mathbb{R}))$$

of (3) is called the *lateral end* of Σ .

The surgery cobordism $\Sigma(a_1, \dots, a_k)$ defined in the previous subsection clearly satisfies all these properties when $\mathbb{L}(a_1, \dots, a_k)$ is connected.

Given two exact Lagrangian cobordisms Σ^0 and Σ^1 from $(L_-^0)^+$ to $(L_+^0)^+$ and from $(L_-^1)^+$ to $(L_+^1)^+$ with augmentations ε_-^0 and ε_-^1 of $(L_-^0)^+$ and $(L_-^1)^+$ respectively, we define the *Cthulhu complex* $Cth_{\varepsilon_-^0, \varepsilon_-^1}(\Sigma^0, \Sigma^1)$ which, as an \mathbb{F} -module, splits as a direct sum

$$\begin{aligned} Cth_{\varepsilon_-^0, \varepsilon_-^1}(\Sigma^0, \Sigma^1) = \\ LCC_{\varepsilon_+^0, \varepsilon_+^1}((L_+^0)^+, (L_+^1)^+) \oplus CF_{\varepsilon_-^0, \varepsilon_-^1}(\Sigma^0, \Sigma^1) \oplus LCC_{\varepsilon_-^0, \varepsilon_-^1}((L_-^0)^+, (L_-^1)^+), \end{aligned}$$

where ε_+^i is the augmentations of $\mathfrak{A}((L_+^i)^+)$ induced by ε_-^i and Σ^i , and $CF_{\varepsilon_-^0, \varepsilon_-^1}(\Sigma^0, \Sigma^1)$ is the \mathbb{F} -module freely generated by the intersection points $\Sigma^0 \cap \Sigma^1$, which we assume to be transverse. Furthermore, we assume that $L_\pm^0 \cap L_\pm^1 \cap W_R^e = \emptyset$, which is not a restriction since the ends are cylinders over Legendrian submanifolds.

The differential on the Cthulhu complex can be written as a matrix

$$\mathfrak{d}_{\varepsilon_-^0, \varepsilon_-^1} = \begin{pmatrix} d_{++} & d_{+0} & d_{+-} \\ 0 & d_{00} & d_{0-} \\ 0 & d_{-0} & d_{--} \end{pmatrix}$$

where d_{++} and d_{--} are the differentials of $LCC_{\varepsilon_+^0, \varepsilon_+^1}((L_+^0)^+, (L_+^1)^+)$ and $LCC_{\varepsilon_-^0, \varepsilon_-^1}((L_-^0)^+, (L_-^1)^+)$ respectively, and the other maps are defined by counting J -holomorphic discs in $\mathbb{R} \times M$ with boundary on $\Sigma^0 \cup \Sigma^1$ and boundary punctures asymptotic to Reeb chords from $(L_1^\pm)^+$ to $(L_1^\pm)^+$ and intersection points between Σ^0 and Σ^1 . See [9, Section 6] for the detailed definition. The cobordisms considered in [9] have the property that $\Sigma^i \cap [-C, C] \times M$ is compact for every $C > 0$, while here we consider cobordisms with a lateral end. The theory developed in [9] can be extended to the present situation thank to the following maximum principle.

Lemma 8.2. *Let J and \tilde{J} be almost complex structures on $\mathbb{R} \times W \times \mathbb{R}$ and W , respectively, each cylindrical inside the respective symplectisation*

$\mathbb{R} \times W_R^e$ and half-symplectisation W_R^e for some $R > 0$. We moreover require that the canonical projection

$$(\mathbb{R} \times W_R^e \times \mathbb{R}, J) \rightarrow (W_R^e, \tilde{J})$$

is holomorphic. Then every J -holomorphic map $u: \Delta \rightarrow \mathbb{R} \times W \times \mathbb{R}$ with

- $\Delta = D^2 \setminus \{\zeta_0, \dots, \zeta_d\}$ where $(\zeta_0, \dots, \zeta_d) \in \text{Conf}^{d+1}(\partial D^2)$,
- $u(\partial\Delta) \subset \Sigma^0 \cup \Sigma^1$, and
- u maps some neighbourhood of the punctures $\{\zeta_0, \dots, \zeta_d\}$ into $\mathbb{R} \times W_R \times \mathbb{R}$,

has its entire image contained inside $\mathbb{R} \times W_R \times \mathbb{R}$.

Proof. By the assumptions the image of the curve $u|_{u^{-1}(\mathbb{R} \times W_R^e \times \mathbb{R})}$ under the canonical projection

$$(\mathbb{R} \times W_R^e \times \mathbb{R}, J) \rightarrow (W_R^e, \tilde{J})$$

is compact with boundary on $\mathbb{R} \times \partial W_R^e \times \mathbb{R}$. The statement is now a consequence of the maximum principle for pseudoholomorphic curves inside $W_R^e \cong [R, +\infty) \times V$ which

- satisfy a cylindrical boundary condition, and
- are pseudoholomorphic for a cylindrical almost complex structure.

Namely, by e.g. [25, Lemma 5.5], the symplectisation coordinate $\mathfrak{r}: W_R^e \rightarrow [R, +\infty)$ restricted to such a curve cannot have a local maximum. \square

With Lemma 8.2 at hand, the arguments of [9] go through, and therefore we have the following result.

Theorem 8.3 ([9]). *The map $\mathfrak{d}_{\varepsilon_-^0, \varepsilon_-^1}$ is a differential ad the Cthulhu complex*

$$(Cth_{\varepsilon_-^0, \varepsilon_-^1}(\Sigma^0, \Sigma^1), \mathfrak{d}_{\varepsilon_-^0, \varepsilon_-^1})$$

is acyclic.

The consequence of interest for us is the following.

Corollary 8.4. *If $\Sigma^0 \cap \Sigma^1 = \emptyset$, then the map*

$$d_{+-}: LCC_{\varepsilon_-^0, \varepsilon_-^1}((L_-^0)^+, (L_-^1)^+) \rightarrow LCC_{\varepsilon_+^0, \varepsilon_+^1}((L_+^0)^+, (L_+^1)^+)$$

is a quasi-isomorphism.

Proof. If $\Sigma^0 \cap \Sigma^1 = \emptyset$, the Cthulhu differential simplify as follows:

$$\mathfrak{d}_{\varepsilon_-^0, \varepsilon_-^1} = \begin{pmatrix} d_{++} & 0 & d_{+-} \\ 0 & 0 & 0 \\ 0 & 0 & d_{--} \end{pmatrix}$$

and thus the Cthulhu complex becomes the cone of d_{+-} . Since it is acyclic, it follows that d_{+-} is a quasi-isomorphism. \square

8.2. The surgery cobordism. In this subsection we describe the Lagrangian surgery of [26] and [28] from the Legendrian viewpoint. In particular, we interpret it as a Lagrangian cobordism between the Legendrian lifts of the Lagrangian submanifolds before and after the surgery. We refer to [14] for more details.

We first describe the local model for Lagrangian surgery. Given $\eta, \delta > 0$, we consider the open subset

$$\mathcal{V}_{\eta,\delta} := \{|q| < \eta, |p| < 2\delta, z \in \mathbb{R}\}$$

of $\mathcal{J}^1(\mathbb{R}^n)$. Given $\zeta > 0$, we denote by $\Lambda_{\eta,\delta,\zeta}^+$ the (disconnected) Legendrian submanifold of $\mathcal{V}_{\eta,\delta}$ given by the two sheets

$$\{(q, \pm df_{\eta,\delta,\zeta}(|q|), \pm f_{\eta,\delta,\zeta}(|q|)) : |q| < \eta\},$$

where

$$f_{\eta,\delta,\zeta}(s) = \frac{\delta}{2\eta}s^2 + \frac{\zeta}{2}.$$

This is a Legendrian submanifold with a single Reeb chord of length ζ . Note that $\Lambda_{\eta,\delta,\zeta}^+$ is described by the generating family $F_{\eta,\delta,\zeta}^+ : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$F_{\eta,\delta,\zeta}^+(q, \xi) = \frac{\xi^3}{3} - g^+(|q|)\xi,$$

where

$$g^+(s) = \left(\frac{3}{2}f_{\eta,\delta,\zeta}(s)\right)^{\frac{2}{3}}.$$

Note that g^+ is smooth because $g^+(s) > 0$ holds for every s . Let $g^- : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a function such that

- (i) $g^-(s) = \left(\frac{3}{2}f_{\eta,\delta,\zeta}(s)\right)^{\frac{2}{3}}$ for $s > 3\eta/4$,
- (ii) $g^-(s) < 0$ for $s < \eta/2$, and
- (iii) $0 < (g^-)'(s) < 2\frac{\delta\eta}{\delta\eta+\zeta}$.

Note that Condition (iii) can be achieved if $\zeta < \frac{7\delta\eta}{16}$. The Legendrian submanifold $\Lambda_{\eta,\delta,\zeta}^-$ of $\mathcal{V}_{\delta,\eta}$ generated by

$$F_{\eta,\delta,\zeta}^-(q, \xi) = \frac{\xi^3}{3} - g^-(|q|)\xi$$

coincides with $\Lambda_{\eta,\delta,\zeta}^+$ near $|q| = \varepsilon$ and has no Reeb chords (see Figure 3). Note that indeed $\Lambda_{\eta,\delta,\zeta}^- \subset \mathcal{V}_{\delta,\eta}$ because Condition (iii) ensures that the p coordinates of $\Lambda_{\eta,\delta,\zeta}^-$, given by $\frac{\partial F_{\eta,\delta,\zeta}^-}{\partial p_i}$ along critical values of $F(q, \cdot)$, are smaller than 2δ .

On Figure 4 we see the front and Lagrangian projections of the one dimensional version of Λ^+ and Λ^- .

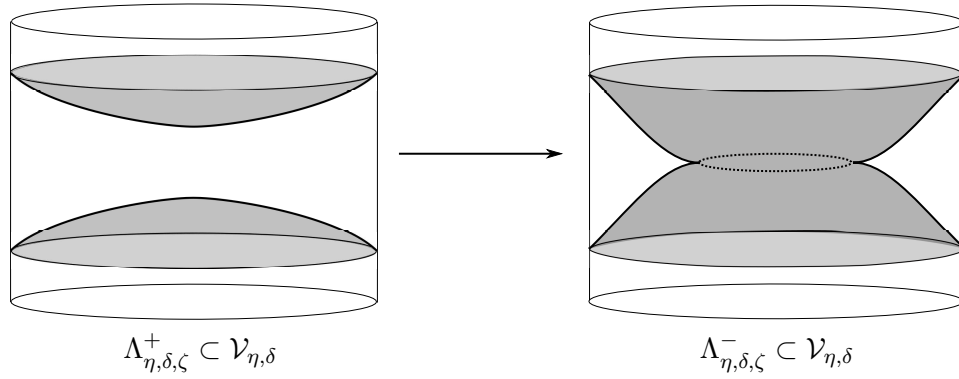
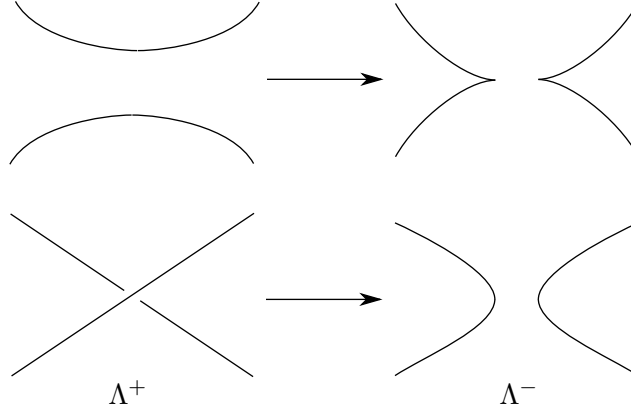
FIGURE 3. The front projections of Λ^+ and Λ^- .

FIGURE 4. Front (top) and Lagrangian (bottom) projections of the Lagrangian surgery

Let \mathbb{L} be an exact Lagrangian immersion in (W, θ) with double points a_1, \dots, a_k , and let \mathbb{L}^+ be a Legendrian lift of \mathbb{L} . The double points of \mathbb{L} lift to Reeb chords of \mathbb{L}^+ which we will denote with the same name by an abuse of notation.

Definition 8.5. A set of Reeb chords $\{a_1, \dots, a_k\}$ on \mathbb{L}^+ is called *contractible* if, for all $i = 1, \dots, k$, there is a neighbourhood \mathcal{U}_i of the Reeb chord a_i in the contactisation (M, β) of (W, θ) and a strict contactomorphism $(\mathcal{U}_i, \mathcal{U}_i \cap \mathbb{L}^+) \cong (\mathcal{V}_{\eta_i, \delta_i}, \Lambda^1_{\eta_i, \delta_i, \zeta_i})$ for numbers $\eta_i, \delta_i, \zeta_i$ satisfying $\zeta_i < \frac{7\delta_i\eta_i}{16}$.

Remark 8.6. This is a restrictive assumption because, in general, the lengths of the chords a_1, \dots, a_k cannot be modified independently. An example when this is possible, and which will be the case in our main theorem, is when \mathbb{L}^+ is a link with $k+1$ components, all a_i are mixed chords, and each component contains either the starting point or the end point of at least one of the a_i . In this situation we can indeed modify the Legendrian link by

Legendrian isotopies of each his components so that its Lagrangian projection is unchanged and all the previous conditions on the neighbourhoods are satisfied. (Warning: this might not be an isotopy of the Legendrian *link*.)

In the following we assume that $\{a_1, \dots, a_k\}$ is a set of contractible Reeb chords on \mathbb{L}^+ . We denote by $\mathbb{L}^+(a_1, \dots, a_k)$ the Legendrian submanifold of (M, β) obtained by replacing each of the $\Lambda_{\eta_i, \delta_i, \zeta_i}^+$ by the corresponding $\Lambda_{\eta_i, \delta_i, \zeta_i}^-$ and by $\mathbb{L}(a_1, \dots, a_k)$ the Lagrangian projection of $\mathbb{L}^+(a_1, \dots, a_k)$. Observe that we here need to make use of the identifications with the standard model, which exist by the contractibility condition.

Then $\mathbb{L}(a_1, \dots, a_k)$ is an exact Lagrangian immersion in (W, θ) which is the result of Lagrangian surgery on \mathbb{L} along the self-intersection points a_1, \dots, a_k . It is evident from the construction that $\mathbb{L}(a_1, \dots, a_k)$ coincides with \mathbb{L} outside of a neighbourhood of the a_i 's and has k self-intersection points removed. The latter fact follows from the fact that since ζ_i can be chosen arbitrarily small, no Reeb chords are created when going from $\Lambda_{\eta_i, \delta_i, \zeta_i}^+$ to $\Lambda_{\eta_i, \delta_i, \zeta_i}^-$.

Next we construct an exact Lagrangian cobordism $\Sigma(a_1, \dots, a_k)$ in the symplectisation of (M, β) with \mathbb{L} at the positive end and $\mathbb{L}(a_1, \dots, a_k)$ at the negative end. Fix $T > 0$ and choose a function $G: (0, \varepsilon) \times \mathbb{R}_+ \rightarrow \mathbb{R}$ such that:

- $G(t, s) = g^-(s)$ for $t < 1/T$,
- $G(t, s) = g^+(s)$ for $t > T$,
- $\frac{\partial G}{\partial t}(t, 0) > 0$, and
- $G(t, s) = g^+(s) = g^-(s)$ for $s > 3\eta/4$.

We consider the Lagrangian submanifold of $T^*(\mathbb{R}^+ \times B^n(\eta))$ described by the generating family

$$F(t, q, \xi) = t \cdot \left(\frac{\xi^3}{3} + G(t, |q|)\xi \right),$$

which is mapped by the symplectomorphism $T^*(\mathbb{R}^+ \times B^n(\eta)) \cong \mathbb{R} \times \mathcal{J}^1(B^n(\eta))$ to a Lagrangian cobordisms $\Sigma_{\eta, \delta, \zeta}$ in the symplectisation of (M, β) from $\Lambda_{\eta_k, \delta_k, \zeta_k}^-$ at the negative end to $\Lambda_{\eta_k, \delta_k, \zeta_k}^+$ at the positive end. Self-intersections of $\Sigma_{\eta, \delta, \zeta}$ are given by the critical points of the function

$$\Delta_F(t, q, \xi_1, \xi_2) = F(t, q, \xi_1) - F(t, q, \xi_2)$$

with non 0 critical value, and such points do not exist because of the third condition on G . Thus this cobordism is embedded.

In the trivial cobordism $\mathbb{R} \times \mathbb{L}^+$ we replace $\mathbb{R} \times (\mathcal{U}_i \cap \mathbb{L}^+)$ with $\Sigma_{\eta_i, \delta_i, \zeta_i}$, for all $i = 1, \dots, k$, to get a a cobordism $\Sigma(a_1, \dots, a_k)$ from $\mathbb{L}^+(a_1, \dots, a_k)$ at the negative end to \mathbb{L}^+ at the positive end.

8.3. Effect of surgery on Floer homology. In this subsection we use $\Sigma(a_1, \dots, a_k)$ and our Floer theory for Lagrangian cobordisms to relate the Floer homology of \mathbb{L} with the Floer homology of $\mathbb{L}(a_1, \dots, a_k)$. The Lagrangian cobordism $\Sigma(a_1, \dots, a_k)$ induces a dga morphism

$$\Phi_\Sigma: \mathfrak{A}(\mathbb{L}^+) \rightarrow \mathfrak{A}(\mathbb{L}^+(a_1, \dots, a_k)).$$

It follows from [14, Theorem 1.1] that, for a suitable almost complex structure on the cobordism that has been obtained by perturbing an arbitrary cylindrical almost complex structure, we have

$$(27) \quad \begin{aligned} \Phi_\Sigma(a_i) &= 1 && \text{for } i = 1, \dots, k, \\ \Phi_\Sigma(c) &= c + \mathbf{w} && \text{if } c \neq a_i, \end{aligned}$$

where \mathbf{w} is a linear combination of products $c_1 \dots c_m$ with

$$\mathfrak{a}(c_1) + \dots + \mathfrak{a}(c_m) < \mathfrak{a}(c).$$

Lemma 8.7. *If $\varepsilon: \mathfrak{A}(\mathbb{L}^+) \rightarrow \mathbb{F}$ is an augmentation such that $\varepsilon(a_i) = 1$ for $i = 1, \dots, k$, then there is an augmentation $\bar{\varepsilon}: \mathfrak{A}(\mathbb{L}^+(a_1, \dots, a_k)) \rightarrow \mathbb{F}$ such that $\varepsilon = \bar{\varepsilon} \circ \Phi_\Sigma$.*

Proof. Let \mathfrak{I} be the bilateral ideal generated by $a_i - 1, \dots, a_k - 1$: then ε induces an augmentation $\bar{\varepsilon}: \mathfrak{A}(\mathbb{L}^+)/\mathfrak{I} \rightarrow \mathbb{F}$. By Equations (27) Φ_Σ is surjective and its kernel is \mathfrak{I} . Surjectivity is proved by a sort of Gauss elimination using the action filtration. Then there is an isomorphism between $\mathfrak{A}(\mathbb{L}^+)/\mathfrak{I}$ and $\mathfrak{A}(\mathbb{L}^+(a_1, \dots, a_k))$, and therefore the augmentation $\bar{\varepsilon}: \mathfrak{A}(\mathbb{L}^+)/\mathfrak{I} \rightarrow \mathbb{F}$ induces an augmentation on $\mathfrak{A}(\mathbb{L}^+(a_1, \dots, a_k))$, which we still denote by $\bar{\varepsilon}$. \square

Note that the construction of $\bar{\varepsilon}$ is not explicit because the isomorphism $\mathfrak{A}(\mathbb{L}^+)/\mathfrak{I} \cong \mathfrak{A}(\mathbb{L}^+(a_1, \dots, a_k))$ is not explicit.

Proposition 8.8. *For any immersed cylindrical exact Lagrangian submanifold $T \subset W$ with augmentation ε' there is a quasi-isomorphism*

$$LCC_{\varepsilon', \bar{\varepsilon}}(T^+, \mathbb{L}^+(a_1, \dots, a_k)) \xrightarrow{\simeq} LCC_{\varepsilon', \varepsilon}(T^+, \mathbb{L}^+),$$

under the assumption that the augmentations ε and $\bar{\varepsilon}$ are as in Lemma 8.7.

Proof. We denote by Σ_T the trivial cobordism $\Sigma_T = \mathbb{R} \times T^+ \subset \mathbb{R} \times M$. Recall that the surgery cobordism goes from $\mathbb{L}^+(a_1, \dots, a_k)$ to \mathbb{L}^+ . Since the surgery is localised in a neighbourhood of the intersection points a_1, \dots, a_k , by a Hamiltonian isotopy we can assume that

$$\Sigma_T \cap \Sigma(a_1, \dots, a_k) = \emptyset.$$

Then Corollary 8.4 implies that the map d_{+-} in the Cthulhu differential for the cobordisms Σ_T and $\Sigma(a_1, \dots, a_k)$ is a quasi-isomorphism. \square

Lemma 8.9. *Let \mathbb{L}' be an immersed exact Lagrangian submanifold with an augmentation ε' and let $\mathbb{L}^+, (\mathbb{L}')^+$ be Legendrian lifts such that \mathbb{L}^+ is above $(\mathbb{L}')^+$. When Lemma 8.7 is applied to an augmentation*

$$\varepsilon_c: \mathfrak{A}(\mathbb{L}^+ \cup (\mathbb{L}')^+) \rightarrow \mathbb{F}$$

induced by the cycle

$$c \in CF((\mathbb{L}, \varepsilon), (\mathbb{L}', \varepsilon'))$$

as in Lemma 4.20, then the push-forward of the augmentation under the DGA morphism

$$\overline{\varepsilon}_c = \overline{\varepsilon}_{\bar{c}}: \mathfrak{A}(\mathbb{L}^+(a_1, \dots, a_k) \cup (\mathbb{L}')^+) \rightarrow \mathbb{F}$$

is induced by a cycle

$$\bar{c} \in CF((\mathbb{L}(a_1, \dots, a_k), \bar{\varepsilon}), (\mathbb{L}', \bar{\varepsilon}'))$$

which moreover is mapped to c under the quasi-isomorphism from Proposition 8.8.

Proof. There is no Reeb chord starting on either \mathbb{L}^+ or $\mathbb{L}^+(a_1, \dots, a_k)$ and ending on $(\mathbb{L}')^+$, so the pushed-forward augmentation is automatically of the form $\bar{\varepsilon}_{\bar{c}}$. Lemma 4.20 then implies that \bar{c} is a cycle.

The last statement is an algebraic consequence of the fact that the discs counted by the DGA morphism Φ_Σ induced by the surgery cobordism can be identified with the discs counted by the quasi-isomorphism from Proposition 8.8. \square

Now assume that \mathbb{L}' is a push off of \mathbb{L} as constructed in Lemma 4.18, and let $e \in CF(\mathbb{L}, \mathbb{L}')$ be the “unit” defined by the sum of the local minima e_i of the Morse function on the connected components of \mathbb{L} ; i.e. $e = \sum e_i$.

Corollary 8.10. *The cycle*

$$\bar{e} \in LCC_{\bar{\varepsilon}, \varepsilon'}(\mathbb{L}^+(a_1, \dots, a_k), (\mathbb{L}')^+)$$

provided by Lemma 8.9 (which is mapped to e under the quasi-isomorphism by Proposition 8.8) satisfies the property that

$$\mu^2(\bar{e}, \cdot): CF(T, (\mathbb{L}(a_1, \dots, a_k), \bar{\varepsilon})) \rightarrow CF(T, (\mathbb{L}', \varepsilon'))$$

is a quasi-isomorphism for any exact Lagrangian submanifold T with cylindrical end.

Proof. Consider the Legendrian lift $\mathbf{L}^+ = \mathbb{L}^+ \cup (\mathbb{L}')^+$ such that \mathbb{L}^+ is above $(\mathbb{L}')^+$. Then let the lift $\mathbf{L}^+(a_1, \dots, a_k) = \mathbb{L}^+(a_1, \dots, a_k) \cup (\mathbb{L}')^+$, be specified uniquely by the requirement that it coincides with the first lift outside of a compact subset.

Recall that e is closed by Lemma 4.19 and by Lemma 4.20 there is thus an induced augmentation ε_e of $\mathfrak{A}(\mathbf{L}^+)$. Recall that this augmentation coincides with ε and ε' when restricted to the generators on the components \mathbb{L}^+ and

$(\mathbb{L}')^+$, respectively, while $\varepsilon_e(e_i) = 1$ holds for any chord corresponding to a local minimum and $\varepsilon_e(c) = 0$ for every other chord c between \mathbb{L}^+ and $(\mathbb{L}')^+$.

Applying Proposition 8.8 to the Legendrian $\mathbf{L}^+(a_1, \dots, a_k)$ obtained by surgery on \mathbf{L} , yields a quasi-isomorphism

$$LCC_{\bar{\varepsilon}}(T^+, \mathbf{L}^+(a_1, \dots, a_k)) \xrightarrow{\cong} LCC_{\varepsilon_e}(T^+, \mathbf{L}^+).$$

(Here we use that $\bar{\varepsilon} = \bar{\varepsilon}_e$ by Lemma 8.9.) The complex on the right-hand side is acyclic by Lemma 4.18, and hence so is the complex on the left-hand side. The sought statement is now a consequence of the straight-forward algebraic fact that the complex

$$LCC_{\bar{\varepsilon}}(T^+, \mathbf{L}^+(a_1, \dots, a_k))$$

is equal to the mapping cone of

$$\mu^2(\bar{e}, \cdot): CF(T, (\mathbb{L}(a_1, \dots, a_k), \bar{\varepsilon})) \rightarrow CF(T, (\mathbb{L}', \varepsilon')).$$

□

8.4. Twisted complexes. The aim of this section is to relate the geometric notion of Lagrangian surgery to the algebraic notion of twisted complex in the wrapped Fukaya category. We first recall the definition of a twisted complex in an A_∞ -category.

Given a unital A_∞ -category \mathcal{A} , we describe the category $Tw\mathcal{A}$ of twisted complexes over \mathcal{A} and recall its basic properties. We introduce the following notation: given a number d of matrices A_i with coefficients in the morphism spaces of an A_∞ -algebra, we denote by $\mu_{\mathcal{A}}^d(A_d, \dots, A_1)$ the matrix whose entries are obtained by applying $\mu_{\mathcal{A}}^d$ to the entries of the formal product of the A_i 's.

Definition 8.11. A *twisted complex* over \mathcal{A} is given by the following data:

- a finite collection of objects L_0, \dots, L_k of \mathcal{A} for some k ,
- integers κ_i for $i = 0, \dots, k$, and
- a matrix $X = (x_{ij})_{0 \leq i, j \leq k}$ such that $x_{ij} \in \text{hom}_{\mathcal{A}}(L_i, L_j)$ and $x_{ij} = 0$ if $i \geq j$, which satisfies the *Maurer-Cartan equation*

$$\sum_{d=1}^k \mu_{\mathcal{A}}^d(\underbrace{X, \dots, X}_{d \text{ times}}) = 0.$$

The integers κ_i are degree shifts and are part of the definition only if the morphism spaces $\text{hom}_{\mathcal{A}}(L_i, L_j)$ are graded, and otherwise are suppressed.

Given two twisted complexes $\mathfrak{L} = (\{L_i\}, \{\kappa_i\}, X)$ and $\mathfrak{L}' = (\{L'_i\}, \{\kappa'_i\}, X')$ we define $\text{hom}_{Tw\mathcal{A}}(\mathfrak{L}, \mathfrak{L}') := \bigoplus_{i,j} \text{hom}_{\mathcal{A}}(L_i, L'_j)[\kappa_i - \kappa'_j]$ and, given $d+1$ twisted complexes $\mathfrak{L}_0, \dots, \mathfrak{L}_d$, we define A_∞ operations

$$\mu_{Tw\mathcal{A}}^d: \text{hom}_{Tw\mathcal{A}}(\mathfrak{L}_{d-1}, \mathfrak{L}_d) \otimes \dots \otimes \text{hom}_{Tw\mathcal{A}}(\mathfrak{L}_0, \mathfrak{L}_1) \rightarrow \text{hom}_{Tw\mathcal{A}}(\mathfrak{L}_0, \mathfrak{L}_d)$$

by

$$(28) \quad \mu_{Tw\mathcal{A}}^d(q_d, \dots, q_1) = \sum_{k_1, \dots, k_d \geq 0} \mu_{\mathcal{A}}^{k_1 + \dots + k_d + d}(\underbrace{X_d, \dots, X_d}_{k_d}, q_d, X_{d-1}, \dots, X_1, q_1, \underbrace{X_0, \dots, X_0}_{k_0}).$$

It is shown in [32, Section 3.k] that the set of twisted complexes with operations $\mu_{Tw\mathcal{A}}^d$ constitutes an A_∞ -category $Tw\mathcal{A}$ which contains \mathcal{A} as a full subcategory. Furthermore it is shown in [32, Lemma 3.32 and Lemma 3.33] that $Tw\mathcal{A}$ is the triangulated envelope of \mathcal{A} and thus $H^0 Tw(\mathcal{A})$ is the derived category of \mathcal{A} .

Definition 8.12. We say that a collection of objects L_1, \dots, L_k of \mathcal{A} generates \mathcal{A} if and only if any object L of \mathcal{A} is quasi-isomorphic in $Tw\mathcal{A}$ to a twisted complex built from the object L_i 's.

Lemma 8.13. *If there is a twisted complex \mathfrak{L} built from L_0, \dots, L_k such that, for every object T of \mathcal{A} we have $H \operatorname{hom}_{Tw\mathcal{A}}(T, \mathfrak{L}) = 0$, then L_0 is quasi-isomorphic in $Tw\mathcal{A}$ to a twisted complex built from L_1, \dots, L_k .*

Proof. This follows from the iterated cone description of twisted complexes from [32, Lemma 3.32]. More precisely, from the definition of twisted complexes, for any object T we have that $\operatorname{hom}_{\mathcal{A}}(T, L_0)$ is a quotient complex of $\operatorname{hom}_{Tw\mathcal{A}}(T, \mathfrak{L})$ by the twisted complex \mathfrak{L}' built from \mathfrak{L} starting at L_1 (i.e. “chopping” out L_0 from the twisted complex \mathfrak{L}), and thus those three objects fit in an exact triangle. The vanishing of $H \operatorname{hom}_{Tw\mathcal{A}}(T, \mathfrak{L})$ implies then that

$$H \operatorname{hom}_{\mathcal{A}}(T, L_0) \cong H \operatorname{hom}_{Tw\mathcal{A}}(T, \mathfrak{L}').$$

The result follows now because the map from L_0 to \mathfrak{L}' , which is given by the maps (x_{0j}) , is a map of twisted complexes. \square

We now relate twisted complexes in the wrapped Fukaya category with certain augmentations of the Chekanov-Eliashberg algebra of the Legendrian lift of the involved Lagrangian submanifolds.

Remark 8.14. In the following lemma we will make a slight abuse of notation by building twisted complexes from immersed exact Lagrangian submanifolds: to our knowledge, the wrapped Fukaya category has not yet been extended to include also exact *immersed* Lagrangian submanifolds. However, since the statements and proofs only concern transversely intersecting Lagrangian submanifolds, there are no additional subtleties arising when considering the A_∞ operations. In other words, we only consider morphisms between *different* objects in the category. The worried reader can thus think of twisted complexes in the “Fukaya pre-category”. Of course if all Lagrangian submanifolds L_i involved are embedded, the statements make sense also in the ordinary wrapped Fukaya category.

Lemma 8.15. *Let (L_i, ϵ_i) , for $i = 0, \dots, k$, be unobstructed exact immersed Lagrangian submanifolds which are assumed to be equipped with fixed potentials f_i . We denote $\mathbb{L} = L_1 \cup \dots \cup L_k$ and \mathbb{L}^+ its Legendrian lift determined by the given potentials. We assume that \mathbb{L}^+ is embedded.*

If $\epsilon: \mathfrak{A}(\mathbb{L}^+) \rightarrow \mathbb{F}$ is an augmentation such that:

- (1) $\epsilon(p) = \epsilon_i(p)$ for every pure chord p of L_i^+ , and
- (2) $\epsilon(a) = 0$ for every mixed chord a from L_i^+ to L_j^+ such that $i > j$,

we define

$$x_{ij} := \begin{cases} \sum_{a \in L_i \cap L_j} \epsilon(a)a & \text{if } i < j, \\ 0 & \text{if } i \geq j, \end{cases}$$

and $X = (x_{ij})_{0 \leq i, j \leq k}$, where the double point a is considered as an element in the summand with wrapping parameter $w = 0$ (see Section 6.2). Then (ignoring the degrees for simplicity) the pair $\mathfrak{L} = (\{(L_i, \epsilon_i)\}, X)$ is a twisted complex in the wrapped Fukaya category. Moreover, for any test Lagrangian submanifold T ,

$$H \operatorname{hom}_{T_w \mathcal{W}F}(T, \mathfrak{L}) = HW(T, (\mathbb{L}, \epsilon)).$$

Proof. Denote by ϵ_0 the augmentation of $\mathfrak{A}(\mathbb{L})$ which vanishes on the mixed chords, while taking the value ϵ_i on the generators living on the component L_i . Recall the chain model for wrapped Floer complex described in Subsection 6.2, where the homotopy direct limit $CW((\mathbb{L}, \epsilon_0), (\mathbb{L}, \epsilon_0); J_\bullet)$ is an infinite direct sum starting with the term with wrapping parameter $w = 0$, i.e. the complex

$$CF((\mathbb{L}, \epsilon_0), (\mathbb{L}, \epsilon_0)); \mathbf{0}, J_\bullet) \oplus CF((\mathbb{L}, \epsilon_0), (\mathbb{L}, \epsilon_0); \mathbf{0}, J_\bullet)q.$$

The bounding cochain X can be identified to a sum of elements in the leftmost summand by definition.

Note that \mathbb{L} , of course, is only immersed. However, in the case when it consists of a union of embeddings, it still represents an object in the twisted complexes of the ordinary wrapped Fukaya category; namely, it is the “direct sum” of the Lagrangian submanifolds L_i , $i = 1, \dots, m$.

First we prove that X satisfies the Maurer-Cartan equation. The Maurer-Cartan equation involves a count of holomorphic polygons in moduli spaces $\mathfrak{M}_{L_0, \dots, L_d}^0(\mathbf{p}^d, a_0, \mathbf{p}^1, a_1, \dots, \mathbf{p}^{d-1}, a_d; J)$ as in Section 4.5. On the other hand, the equation $\epsilon \circ \mathfrak{d} = 0$ counts holomorphic polygons in the moduli spaces $\mathfrak{N}_{\mathbb{L}}(a_0; \mathbf{p}^1, a_1, \dots, a_d, \mathbf{p}^d)$, which are the subset of the previous moduli spaces consisting of those holomorphic polygons which satisfy the extra requirement that the intersection points a_1, \dots, a_d should be negative punctures (in the sense of Definition 3.4). Condition (2) in the definition of ϵ however implies

that $\#\mathfrak{M}_{L_0, \dots, L_d}^0(\mathbf{p}^d, a_0, \mathbf{p}^1, a_1, \dots, \mathbf{p}^{d-1}, a_d; J)$ is multiplied by a nonzero coefficient only if a_1, \dots, a_d are negative punctures. This proves that X satisfies the Maurer-Cartan equation.

For the second part, note that the differential in $\text{hom}_{Tw\mathcal{WF}}(T, \mathfrak{L})$ counts the same holomorphic polygons (with Hamiltonian perturbations) as the differential in $CW(T, (\mathbb{L}, \varepsilon))$ because the Maurer-Cartan element X involves only elements in the Floer complexes defined with wrapping parameter $w = 0$, and hence vanishing Hamiltonian term. \square

The previous lemma together with Proposition 8.8 implies the following result, which is the main result of this section:

Proposition 8.16. *Let $(L_1, \varepsilon_1), \dots, (L_m, \varepsilon_m)$ be unobstructed immersed exact Lagrangian submanifolds with preferred choices of potentials f_i , and let a_1, \dots, a_k be a set of intersection points lifting to contractible Reeb chords on the induced Legendrian lift \mathbb{L}^+ , where $\mathbb{L} := L_1 \cup \dots \cup L_m$. Assume that there is an augmentation ε of the Chekanov-Eliashberg algebra of \mathbb{L}^+ such that:*

- (1) $\varepsilon(c) = \varepsilon_i(c)$ if c is a double point of L_i ,
- (2) $\varepsilon(a_i) = 1$ for $i = 1, \dots, k$, and
- (3) $\varepsilon(q) = 0$ if $q \in L_i \cap L_j$ is an intersection point, with $i > j$, at which $f_i(q) > f_j(q)$ (i.e. q corresponds to a Reeb chord from L_i^+ to L_j^+).

Then for any other exact Lagrangian submanifold T there is a quasi-isomorphism

$$CW(T, (\mathbb{L}(a_1, \dots, a_k), \bar{\varepsilon})) \cong \text{hom}(T, \mathfrak{L}),$$

with $\bar{\varepsilon}$ induced by ε as in Lemma 8.7, and where \mathfrak{L} is a twisted complex built from the L_i with $i = 1, \dots, m$.

Remark 8.17. Conditions (2) and (3) of Proposition 8.16 imply that if a_k is an intersection point between different Lagrangians L_i and L_j for $i < j$, then $f_i(a_k) < f_j(a_k)$. Conditions (1) and (2) of Proposition 8.16 imply that if a_k is a self-intersection point of L_i , then augmentation ε_i evaluates to 1 on a_k .

Proof of Proposition 8.16. We consider the twisted complex \mathfrak{L} built from L_i , $i = 1, \dots, m$, that is constructed by an application of Lemma 8.15 with the augmentation ε . In other words, the twisted complex is defined using the Maurer-Cartan element

$$X := a_1 + \dots + a_k \in CF((\mathbb{L}, \varepsilon_0), (\mathbb{L}, \varepsilon_0); \mathbf{0}, J_\bullet) \subset CW((\mathbb{L}, \varepsilon_0), (\mathbb{L}, \varepsilon_0); J_\bullet)$$

living in the summand with wrapping parameter $w = 0$. The quasi-isomorphism

$$\text{hom}(T, \mathfrak{L}) \cong CW(T, (\mathbb{L}, \varepsilon))$$

is then a consequence of the same lemma.

What remains is constructing a quasi-isomorphism

$$CW(T, (\mathbb{L}(a_1, \dots, a_k), \bar{\epsilon})) \cong \text{hom}(T, \mathfrak{L})$$

for all test Lagrangian submanifolds T . This is done by considering the twisted complex corresponding to the cone of the “unit” $\bar{\epsilon}$ from Corollary 8.10. We proceed to give the details.

Let \mathbb{L}' be the push-off of $L_1 \cup \dots \cup L_m$ as considered in Lemma 4.18. Consider the cycle $\bar{\epsilon} \in CW((\mathbb{L}(a_1, \dots, a_k), \bar{\epsilon}), (\mathbb{L}', \epsilon'))$ supplied by Corollary 8.10. As above, $\bar{\epsilon}$ is an element in the summand

$$\begin{aligned} & CF((\mathbb{L}(a_1, \dots, a_k), \bar{\epsilon}), (\mathbb{L}', \epsilon'); \mathbf{0}, J_\bullet) \\ & \subset CW((\mathbb{L}(a_1, \dots, a_k), \bar{\epsilon}), (\mathbb{L}', \epsilon'); J_\bullet) \end{aligned}$$

with wrapping parameter $w = 0$. Then

$$(\{(\mathbb{L}(a_1, \dots, a_k), \bar{\epsilon}), (\mathbb{L}', \epsilon')\}, \bar{\epsilon})$$

is a twisted complex \mathfrak{L}' corresponding to the cone of $\mu^2(\bar{\epsilon}, \cdot)$. The last part of Corollary 8.10 combined with Lemma 8.13 then establishes the sought quasi-isomorphism. Indeed, every summand

$$CF(T, (\mathbf{L}, \epsilon_{\bar{\epsilon}}); w \cdot H, J_\bullet) \subset CW(T, (\mathbf{L}, \epsilon_{\bar{\epsilon}}); J_\bullet)$$

in the homotopy direct limit which computes the homology of the cone is acyclic by Corollary 8.10. (Here $\mathbf{L} = \mathbb{L}(a_1, \dots, a_k) \cup \mathbb{L}'$ as in the proof of the latter corollary.)

Here we remind the reader that the components of $\mathbb{L}(a_1, \dots, a_k)$ typically are only immersed, as opposed to embedded. The latter statement established is hence on the level of twisted complexes on the pre-category level; c.f. Remark 8.14. \square

9. GENERATING THE WRAPPED FUKAYA CATEGORY

In this section we prove Theorem 1.1 and Theorem 1.2. Most of the proof is the same in the two cases, and for simplicity we will write the details for a Weinstein manifold. In Section 9.4 we present the modifications needed to extend the proof to Weinstein sectors.

9.1. Geometric preparations. Before proving the main theorems we need some geometric preparations which will be used in the technical work of Section 9.3. Recall that the Liouville form θ has been modified in order to make $(\mathcal{H}_1 \cup \dots \cup \mathcal{H}_l, \theta, \mathfrak{f})$ into a union of standard critical Weinstein handles. After adding the differential of a function supported in a small neighbourhood of $\mathcal{H}_1 \cup \dots \cup \mathcal{H}_l$, we change the Liouville form once again so that the symplectomorphism between $(\mathcal{H}_i, d\theta)$ and $(D_\delta T^*C_i, d\mathbf{p} \wedge d\mathbf{q})$ maps the new Liouville form θ_c to $\mathbf{p}d\mathbf{q}$. We make the modification so that the new Liouville vector field \mathcal{L}_c is still positively transverse to ∂W_0 , has no zeros outside W_0 , and so that the new and old Lagrangian skeleta coincide. (On

the other hand \mathcal{L}_c it is no longer a pseudo-gradient vector field for \mathfrak{f} , but this will not impair the proof of Theorem 1.1.) Note that the above identification maps the core of a handle to the zero section and the cocore into a cotangent fibre. Further, we perform the construction of the new Liouville form so that the corresponding Liouville vector field is still everywhere tangent to D_i . The reason for changing θ to θ_c is to simplify the arguments of Subsection 9.3.

The set of cylindrical exact Lagrangian submanifolds of (W, θ) coincide with that of (W, θ_c) and the wrapped Floer cohomology between any two such Lagrangian submanifolds is unaffected by the modification of θ by the invariance properties of wrapped Floer homology; see [3, Section 5]. This means that $\mathcal{WF}(W, \theta)$ is quasi-equivalent to $\mathcal{WF}(W, \theta_c)$.

With a new Liouville vector field we will choose a new function $\mathfrak{r}: W \rightarrow [R_0, +\infty)$ satisfying Conditions (i) and (ii) of Section 2.1 for $R_0 \ll 0$ such that, on ∂W_0 , the old and new \mathfrak{r} coincide. From now on, \mathfrak{r} will always be defined using the new Liouville vector field \mathcal{L}_c . Later in the proof of Proposition 9.3, we will modify \mathfrak{r} so that the new $R_0 \ll 0$ becomes sufficiently small, while keeping \mathfrak{r} fixed outside of a compact subset.

Let ψ_t be the Liouville flow of (W, θ_c) and let

$$\widehat{\mathcal{H}}_i := \bigcup_{t \geq 0} \psi_t(\mathcal{H}_i).$$

It follows that $\widehat{\mathcal{H}}_i \subset W$ are pairwise disjoint, embedded codimension zero manifolds. Moreover, there are exact symplectomorphisms

$$(\widehat{\mathcal{H}}_i, \theta_c) \cong (T^*C_i, \mathbf{p}d\mathbf{q})$$

with the standard symplectic cotangent bundles.

Recall Conditions (i) and (ii) from Subsection 2.1. In particular, $\mathfrak{r}^{-1}(R_0) = W^{sc} \cup \mathcal{H}_1 \cup \dots \cup \mathcal{H}_l$, while $\mathfrak{r}|_{\mathfrak{r}^{-1}[R_0+1, +\infty)}$ is a symplectisation coordinate induced by the hypersurface $\mathfrak{r}^{-1}(R_0 + 1)$ of contact type. In the following we make the further assumption that

$$(29) \quad \mathfrak{r}^{-1}(R_0 + 1) \cap \widehat{\mathcal{H}}_i = S_{r_0}^* T^* C_i$$

for some $r_0 > 0$, where the latter radius- r spherical cotangent bundle is induced by the flat metric on C_i . This means that

$$(30) \quad \mathfrak{r}(\mathbf{p}, \mathbf{q}) = \log \|\mathbf{p}\| - \log r_0 + R_0 + 1, \quad \|\mathbf{p}\| \geq r_0,$$

holds in the above canonical coordinates.

Given a point $a \in C_i$ (for some $i = 1, \dots, l$), we denote by D_a the Lagrangian plane which satisfies $D_a \cap C_i = \{a\}$ while being everywhere tangent to the Liouville vector field. In particular, $D_a \cap \mathcal{H}_i$ corresponds to the cotangent fibre $D_\delta T_a^* C_i \subset D_\delta T^* C_i$ under the identification $\mathcal{H}_i \cong D_\delta T^* C_i$.

Lemma 9.1. *For every $i = 1, \dots, l$ and $a \in C_i$, the Lagrangian plane D_a is isotopic to D_i by a cylindrical Hamiltonian isotopy.*

Proof. Recall that $(\widehat{\mathcal{H}}_i, \theta_c)$ is isomorphic to $(T^*C_i, \mathbf{pd}\mathbf{q})$ as a Liouville manifold and D_a and D_i correspond to two cotangent fibres. Therefore they are clearly isotopic by a cylindrical Hamiltonian isotopy. \square

In particular, D_a and D_i are isomorphic objects in the wrapped Fukaya category when $a \in C_i$.

The next lemma is immediate.

Lemma 9.2. *Let $L \subset W$ be a cylindrical exact Lagrangian submanifold. Then, up to a (compactly supported) Hamiltonian isotopy, we can assume that $L \cap (C_1 \cup \dots \cup C_l) = \{a_1, \dots, a_k\}$, the intersections are transverse and $L \cap W^{sc} = \emptyset$.*

Now we are going to normalise the intersections between L and the planes D_{a_i} . For every a_i we choose the natural symplectomorphism between a neighbourhood

$$\mathcal{D}_{a_i} \subset (\widehat{\mathcal{H}}_i, \theta_c) \cong (T^*C_i, \mathbf{pd}\mathbf{q})$$

of $D_{a_i} \cong T_{a_i}^*C_i$ and $(D_\eta T^*D_{a_i}, -d\tilde{\mathbf{p}} \wedge d\tilde{\mathbf{q}})$ for some $\eta > 0$ small, where $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ are the canonical coordinates on $T^*D_{a_i}$. It is clearly possible to make this identification so that

$$(31) \quad \mathfrak{r}(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}) = \log \|\tilde{\mathbf{q}}\| - \log r_0 + R_0 + 1, \quad \|\tilde{\mathbf{q}}\| \geq r_0,$$

is satisfied.

We redefine \mathfrak{r} as in Remark 2.2, without deforming it outside of a compact subset. After making $R_0 \ll 0$ sufficiently small in this manner, we may assume that:

- $R_0 + k + 3 \leq 0$,
- $L \cap W_{R_0+k+3}$ is the union of k disjoint discs with centres at a_1, \dots, a_k , and
- the connected component of $L \cap W_{R_0+k+3}$ containing a_i is identified inside $\mathcal{D}_{a_i} \cong D_\eta T^*D_{a_i}$ with the graph of the differential of a function $g_{a_i}: D_{a_i} \rightarrow \mathbb{R}$ for $i = 1, \dots, k$.

Then we modify L by a compactly supported Hamiltonian isotopy so that it satisfies the following properties:

- (L1) The connected component of $L \cap W_{R_0+k+3}$ containing a_i is contained inside the Weinstein neighbourhood

$$\mathcal{D}_{a_i} \cap W_{R_0+k+3} \cong D_\eta T^*(D_{a_i} \cap \{\|\tilde{\mathbf{q}}\| \leq e^{k+2}r_0\}),$$

where it is described by the graph of the differential of a function $g_{a_i}: D_{a_i} \cap \{\|\tilde{\mathbf{q}}\| \leq e^{k+2}r_0\} \rightarrow \mathbb{R}$ with a nondegenerate minimum at a_i and no other critical points,

- (L2) the connected components of $L \cap W_{R_0+k+3} \setminus W_{R_0+k+2}$ are cylinders which are disjoint from all the cocores D_{a_i} ; moreover, these cylinders are tangent to the Liouville vector field \mathcal{L}_c in the same subset; and
- (L3) $\|g_{a_i}\|_{C^2} \leq \epsilon'$ for $i = 1, \dots, k$ and $\epsilon' > 0$ small which will be specified in Lemma 9.4.

Conditions (L1)—(L3) provides sufficient control of the intersections of L and the Lagrangian skeleton. Later in Lemma 9.4 we will use this in order to perform a deformation of the immersed Lagrangian submanifold

$$L \cup D_{a_1} \cup \dots \cup D_{a_k}$$

by Hamiltonian isotopies applied to the different components D_{a_i} . The goal is to obtain an exact Lagrangian immersion admitting a suitable augmentation; the corresponding bounding cochain (see Lemma 8.15) will then give us the twisted complex which exhibits L as an object built out of the different D_{a_i} .

9.2. Proof of Theorem 1.1. In this section we prove Theorem 1.1 assuming the results of Section 9.3. The proof of Theorem 1.2 follows similarly with just a bit of additional work; see Subsection 9.4. The result is a corollary of the following proposition.

Proposition 9.3. *Let $L \subset W$ be an exact Lagrangian submanifold with cylindrical end. If $L \cap W^{sk} = L \cap (C_1 \cup \dots \cup C_l) = \{a_1, \dots, a_k\}$ and the intersections are transverse, then L is isomorphic in $\text{Tw}\mathcal{WF}(W, \theta)$ to a twisted complex built from the objects D_{a_1}, \dots, D_{a_k} .*

Proof. We assume that L satisfies Conditions (L1), (L2) and (L3) from the previous section. Then by Lemma 9.4 combined with Lemma 9.5 there exist Lagrangian planes $D_{a_1}^w, \dots, D_{a_k}^w$ satisfying the following properties. First $D_{a_i}^w$ is Hamiltonian isotopic to D_{a_i} (possibly after re-indexing the a_1, \dots, a_k) by a cylindrical Hamiltonian isotopy supported in $W \setminus W_{R_0+i}$. Second, for an appropriate Legendrian lift \mathbb{L}^+ of $\mathbb{L} = L \cup D_{a_1}^w \cup \dots \cup D_{a_k}^w$ to $(W \times \mathbb{R}, \theta + dz)$ such that the intersection point a_i lifts to a Reeb chord from $(D_{a_i}^w)^+$ to L^+ of length $\epsilon > 0$ for $i = 1, \dots, k$ — see Lemma 9.4 for more details — there exists an augmentation $\epsilon: \mathfrak{A}(\mathbb{L}^+) \rightarrow \mathbb{F}$ for which

- (1) $\epsilon(a_i) = 1$ for $i = 1, \dots, k$, and
- (2) $\epsilon(d) = 0$ if d is a chord from L^+ to $(D_{a_i}^w)^+$ for $i = 1, \dots, k$, or a chord from $(D_{a_i}^w)^+$ to $(D_{a_j}^w)^+$ with $i > j$.

Moreover, using Property (L3) above for $\epsilon' > 0$ sufficiently small, it follows that the Reeb chords a_i all are contractible (c.f. Definition 8.5).

By Proposition 8.16 the augmentation ϵ induces a twisted complex \mathfrak{L} built from $L_1 = D_{a_1}^w, \dots, L_k = D_{a_k}^w, L_{k+1} = L$, for which

$$\text{hom}(T, \mathfrak{L}) \cong CW(T, (\mathbb{L}(a_1, \dots, a_k), \bar{\epsilon})).$$

The right-hand side is an acyclic complex by Proposition 7.6. Using this acyclicity, the aforementioned proposition implies the sought statement that L is quasi-isomorphic to a twisted complex built from the different $D_{a_i}^w \cong D_{a_i}$. For the last statement, we have used the invariance properties for wrapped Floer cohomology under cylindrical Hamiltonian isotopy; see e.g. [3, Section 5]. \square

The proof of Theorem 1.1 is now a simple combination of Lemma 9.2, Proposition 9.3 and Lemma 9.1. To that end, we again need the fact that Hamiltonian isotopies by cylindrical Hamiltonians induce isomorphisms in the wrapped Fukaya category; see e.g. [3, Section 5].

9.3. Constructing the augmentation. We start by assuming that the modifications from Section 9.1 have been performed, so that in particular (L1)–(L3) are satisfied. When considering potentials in this subsection, recall that we have modified the Liouville form from θ to θ_c .

Let $f: L \rightarrow \mathbb{R}$ be a potential function for L . We order the intersection points a_1, \dots, a_k such that

$$f(a_k) \leq \dots \leq f(a_1).$$

The Morse function $g_{a_i}: D_{a_i} \cap \{\tilde{\mathbf{q}} \leq e^{k+2}r_0\} \rightarrow \mathbb{R}$ from (L1) can be assumed to be sufficiently small by (L3), so that df is almost zero inside $L \cap W_{R_0+k+1}$.

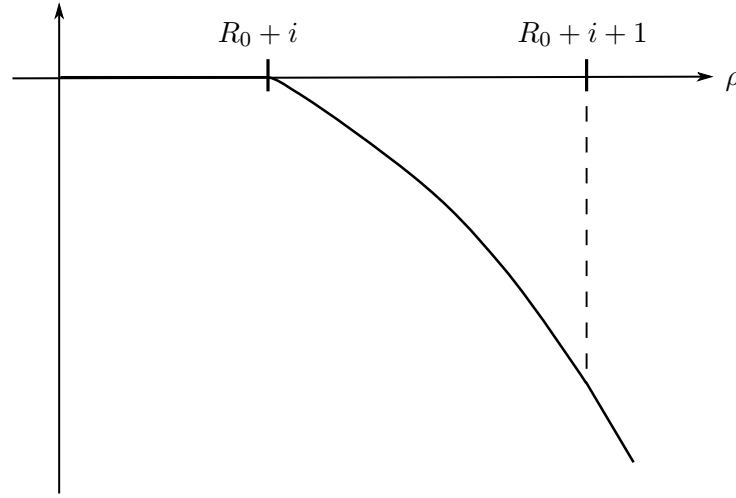


FIGURE 5. The graph of h_i .

We fix functions $h_i: \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$h_i(\rho) = \begin{cases} 0, & \text{if } \rho \leq R_0 + i, \\ -\rho + R_0 + i + \frac{1}{2}, & \text{if } \rho \geq R_0 + i + 1, \end{cases}$$

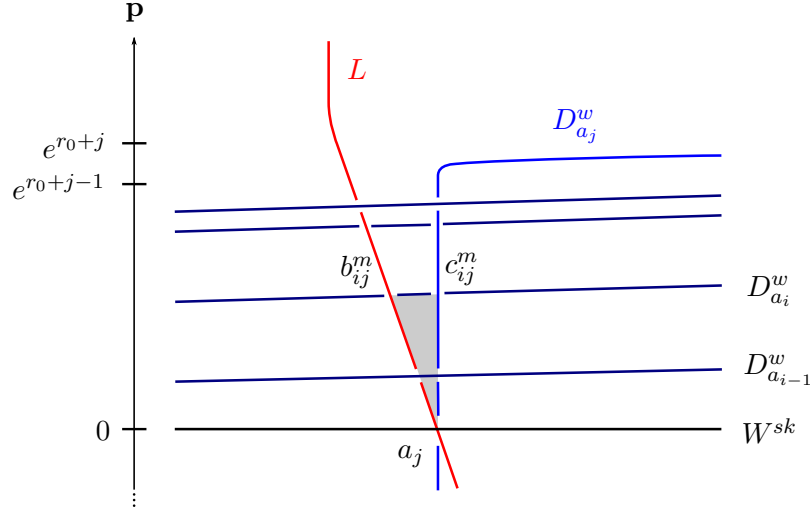


FIGURE 6. A schematic picture of the wrapping and of the small triangle with $i < j$. (Also c.f. Equation (30) combined with Figure 5.)

and $\mathfrak{h}_i''(\rho) \leq 0$ for all $\rho \in \mathbb{R}^+$. Then we define the cylindrical Hamiltonians $H^i: W \rightarrow \mathbb{R}$, $i = 1, \dots, k$, by

$$H^i(w) = \mathfrak{h}_i(e^{\mathfrak{r}(w)}).$$

The graph of \mathfrak{h}_i appears in Figure 5.

We will denote by ϕ_t^i the flow of the Hamiltonian vector field of H^i . Given $T_i \in \mathbb{R}$, we denote $D_{a_i}^w = \phi_{T_i}^i(D_{a_i})$.

We fix $\epsilon > 0$, and on each Lagrangian plane D_{a_i} we choose the potential function $f_i: D_{a_i} \rightarrow \mathbb{R}$ such that

$$f_i = f(a_i) + \epsilon.$$

Note that the functions f_i indeed are constant, since the Liouville vector field is tangent to the planes D_{a_i} . Let $f_i^w: D_{a_i}^w \rightarrow \mathbb{R}$ be the potential function on $D_{a_i}^w$ induced by f_i using Equation (5).

We denote by $\mathbb{L} = L \cup D_{a_1}^w \cup \dots \cup D_{a_k}^w$, which we regard as an exact Lagrangian immersion, and by \mathbb{L}^+ the Legendrian lift of \mathbb{L} to $(W \times \mathbb{R}, \theta + dz)$ defined using the potential functions f, f_1^w, \dots, f_k^w . Note that an intersection point $d \in D_{a_i}^w \cap D_{a_j}^w$ lifts to a chord starting on $D_{a_i}^w$ and ending on $D_{a_j}^w$ if and only if $f_i^w(d) > f_j^w(d)$, and similarly if one of the two discs is replaced by L and its potential is replaced by f .

Lemma 9.4. *There exist real numbers $0 < T_k < \dots < T_1$ and $\epsilon, \epsilon' > 0$ such that, if L satisfies (L1)–(L3), then each chord of \mathbb{L}^+ is of one of the following types:*

- (1) type a: the chords a_i , going from $(D_{a_i}^w)^+$ to L^+ for $i = 1, \dots, k$, of length ϵ ,
- (2) type b: chords b_{ij}^m consisting of all other chords from $(D_{a_i}^w)^+$ to L^+ for $1 \leq i < j \leq k$ and $1 \leq m \leq m_0(i, j)$ for some $m_0(i, j)$,
- (3) type c: chords c_{ij}^m from $(D_{a_i}^w)^+$ to $(D_{a_j}^w)^+$ for $1 \leq i < j \leq k$ and $1 \leq m \leq m_0(i, j)$, and
- (4) “order-reversing” type: chords from L^+ to $(D_{a_i}^w)^+$ for $i = 1, \dots, k$ or chords from $(D_{a_i}^w)^+$ to $(D_{a_j}^w)^+$ for $i = 1, \dots, k$ and $i > j$.

(see Figure 6). Moreover, for every $i < j$ and m , there exists a unique rigid and transversely cut out pseudoholomorphic triangle in W having boundary on $L \cup D_{a_i}^w \cup D_{a_j}^w$, a positive puncture at b_{ij}^m , and negative punctures at a_j and c_{ij}^m , in the order following the boundary orientation. (Positivity and negativity is determined by our choice of Legendrian lift.)

Note that the set $\{c_{ij}^m\}$ could be empty for some i, j . In that case, we say that $m_0(i, j) = 0$.

Proof. Recall that Properties (L1)–(L3) from Subsection 9.1 have been made to hold; in particular $L \cap W_{R_0+k+2}$ consists of a k number of discs which may be assumed to be close to the discs D_{a_i} , $i = 1, \dots, k$.

The proof of the lemma at hand is easier to see if one starts by Hamiltonian isotoping L to make it *coincide* with $D_{a_1} \cup \dots \cup D_{a_k}$ inside W_{R_0+k+2} . (Thus, we can argue about the intersection points of the deformations $D_{a_i}^w$ and $D_{a_j}^w$, as opposed to the intersection points of $D_{a_i}^w$ and the different parts of L .) By Property (L2) it suffices to deform L in such a way that it becomes the graph dg_{a_i} for a function satisfying $g_{a_i} \equiv 0$ inside the subsets $D_{a_i} \cap W_{R_0+k+2}$. Note that, in the case when L and $D_{a_j} \cap W_{R_0+k+2}$ *coincide*, the intersection points b_{ij}^m and c_{ij}^m coincide as well.

First, we observe that $L, D_{a_1}^w, \dots, D_{a_k}^w$ are embedded exact Lagrangian submanifolds, and therefore there is no Reeb chord either from L^+ to L^+ or from $(D_{a_i}^w)^+$ to $(D_{a_i}^w)^+$ for any $i = 1, \dots, k$.

From Equation (5), the potential of $D_{a_i}^w$ is

$$f_i^w = f(a_i) + \epsilon + T_i(\mathfrak{h}'_i(e^r)e^r - \mathfrak{h}_i(e^r)).$$

Note that the quantity $T_i(\mathfrak{h}'_i(e^r)e^r - \mathfrak{h}_i(e^r))$ is nonincreasing in r because $\mathfrak{h}''_i \leq 0$. Therefore f_i^w satisfies

$$\begin{aligned} f_i^w(w) &= f(a_i) + \epsilon && \text{if } w \in D_{a_i}^w \cap W_{R_0+i}, \\ f_i^w(w) &\in [f(a_i) + \epsilon - T_i(R_0 + i + \tfrac{1}{2}), f(a_i) + \epsilon] && \text{if } w \in D_{a_i}^w \cap (W_{R_0+i+1} \setminus W_{R_0+i}), \\ f_i^w(w) &= f(a_i) + \epsilon - T_i(R_0 + i + \tfrac{1}{2}) && \text{if } w \in D_{a_i}^w \cap W_{R_0+i+1}^e. \end{aligned}$$

Note that $D_{a_i}^w \cap W_{R_0+i} = D_{a_i} \cap W_{R_0+i}$ and that $D_{a_i}^w \cap W_{R_0+i+1}^e$ is a cylinder over a Legendrian submanifold.

We choose positive numbers $0 < T_k < \dots < T_1$ such that

- (1) $f(a_1) + \epsilon - T_1(R_0 + 1 + \frac{1}{2}) < \dots < f(a_k) + \epsilon - T_k(R_0 + k + \frac{1}{2})$,
- (2) $f(a_i) + \epsilon - T_i(R_0 + i + \frac{1}{2}) < \min_L f$ for all $i = 1, \dots, k$,
- (3) there are no intersection points between $L, D_{a_1}, \dots, D_{a_k}, D_{a_1}^w, \dots, D_{a_k}^w$ in their cylindrical parts, and
- (4) at every intersection point between $L, D_{a_1}, \dots, D_{a_k}, D_{a_1}^w, \dots, D_{a_k}^w$ the respective potential functions are different, except for intersection points $p \in L \cap D_{a_i}^w$ where $H^i(p) = \mathfrak{h}_i(e^r) = 0$.

The last two conditions are achieved by choosing T_1, \dots, T_k generically.

We observe that, for any point $c \in D_{a_i}^w \cap D_{a_j}$ and any $i, j = 1, \dots, k$, the quantity $\mathfrak{a}(c) = |f_i^w(c) - f_j(c)|$ is independent of ϵ . Then we choose $\epsilon > 0$ sufficiently small so that

$$\epsilon < \min\{\mathfrak{a}(c) : c \in D_{a_i}^w \cap D_{a_j} \text{ and } \mathfrak{a}(c) \neq 0 \text{ for } i, j = 1, \dots, k\}.$$

This implies that, for all $c \in D_{a_i}^w \cap D_{a_j}$ such that $\mathfrak{a}(c) \neq 0$, the signs of $f_i^w(c) - f_j(c)$ and of $f_i^w(c) - f(c)$ are equal. (Recall that $f = f_j - \epsilon$ holds there by construction.)

Consider the set of points $c_{ij}^m \in D_{a_i}^w \cap D_{a_j}$ with positive action difference

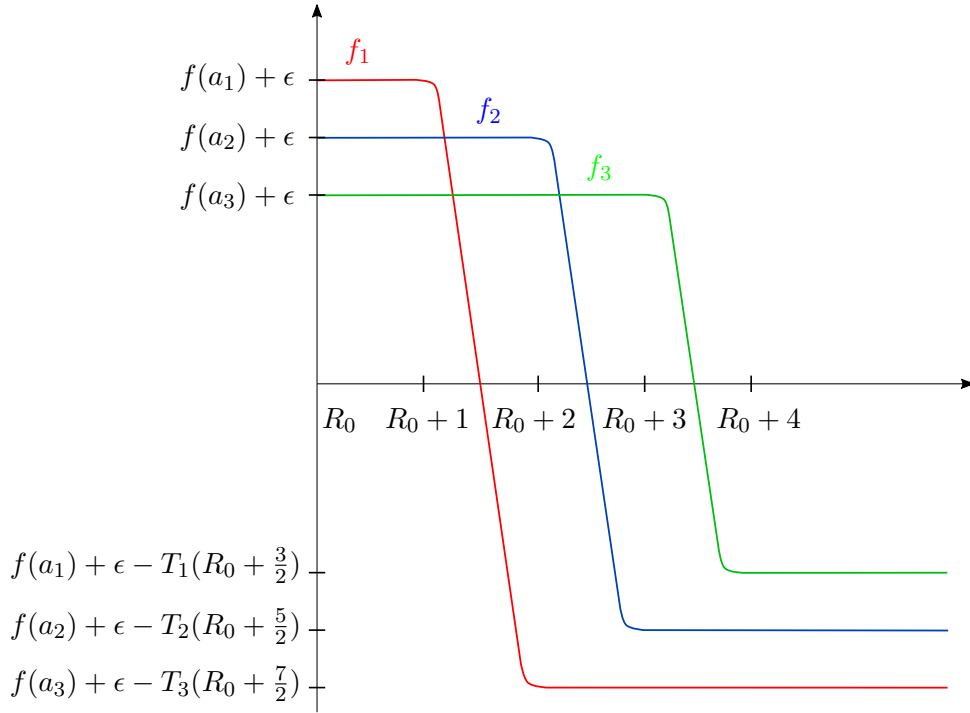
$$0 < f_i^w(c_{ij}^m) - f_j(c_{ij}^m) = f_i^w(c_{ij}^m) - (f(a_j) + \epsilon).$$

(Here m is an index distinguishing the various points with the required property.) Then $i < j$ and $c_{ij}^m \in W_{R_0+i+1} \setminus W_{R_0+i}$; in particular $c_{ij}^m \in D_{a_i}^w \cap D_{a_j}^w$. See Figure 7.

The intersection points b_{ij}^m now coincide with c_{ij}^m , but seen as intersections of $L = D_{a_j}$ and $D_{a_i}^w$. We now perturb L back to make it coincide with the graph of dg_{a_i} of a sufficiently small Morse function g_{a_i} near each D_{a_i} having a unique critical point consisting of a global minimum. Recall that this global minimum corresponds to the intersection point $a_i \in L \cap D_{a_i}$.

We can make the Morse function satisfy $\|g_{a_i}\|_{C^2} \leq \epsilon'$ for $\epsilon' > 0$ sufficiently small. In particular, this means that each intersection point c_{ij}^m still corresponds to a unique intersection point $b_{ij}^m \in D_{a_i}^w \cap L$, such that moreover $f_i^w(b_{ij}^m) - f(b_{ij}^m) > 0$ is satisfied. Conversely, any intersection point $d \in D_{a_i}^w \cap L$ with $f_i^w(d) - f(d) > 0$ is either a_i or one of the b_{ij}^m . This is the case because the only intersection point in $D_{a_i}^w \cap L \cap W_{R_0+i}$ is a_i and for any intersection point $d \in D_{a_i}^w \cap L \cap W_{R_0+i+1}^e$ we must have $f_i^w(d) - f(d) < 0$.

The existence of the triangle follows now by applying Corollary 4.22 to $L \cup D_{a_i}^w \cup D_{a_j}^w$ intersected with the subset $W_{R_0+j} \subset W$. Note that, inside this Liouville subdomain, our deformed Lagrangian L is given as the graph of the differential of a small Morse function g_{a_j} on D_{a_j} (using a Weinstein neighbourhood of the latter); hence the lemma indeed applies. Here the monotonicity property for the symplectic area of a pseudoholomorphic disc must be used in order to deduce that the triangles of interest can be a priori confined to the same Liouville subdomain.

FIGURE 7. The profiles of f_i^w .

□

The triangle provided by the previous lemma is the stepping stone in the inductive construction of an augmentation for $\mathfrak{A}(\mathbb{L}^+)$.

Lemma 9.5. *The Chekanov-Eliashberg algebra $(\mathfrak{A}(\mathbb{L}^+), \mathfrak{d})$ of \mathbb{L}^+ admits an augmentation $\varepsilon: \mathfrak{A}(\mathbb{L}^+) \rightarrow \mathbb{F}$ such that $\varepsilon(a_i) = 1$ for all $i = 1, \dots, k$. Moreover, this augmentation vanishes on the order reversing chords.*

Proof. Set $L_i := D_{a_i}^w$ and $L_{k+1} := L$. recall that each of the L_i is embedded, and therefore there is no Reeb chord from L_i^+ to itself for any i . Thus all Reeb chords go between different connected components of \mathbb{L}^+ and are as described in Lemma 9.4.

The bilateral ideal of $\mathfrak{A}(\mathbb{L})$ generated by the order reversing chords is preserved by the differential, and therefore the quotient algebra, which we will denote by \mathfrak{A}^\rightarrow , inherits a differential \mathfrak{d}^\rightarrow . We can identify \mathfrak{A}^\rightarrow to the subalgebra of \mathfrak{A} generated by the chords of type a , b and c , and \mathfrak{d}^\rightarrow to the portion of the differential of $\mathfrak{A}(\mathbb{L})$ involving only generators of \mathfrak{A}^\rightarrow .

On \mathfrak{A}^\rightarrow we define a filtration of algebras

$$(32) \quad \mathbb{Z} = \mathfrak{A}_{k+1}^\rightarrow \subset \mathfrak{A}_k^\rightarrow \subset \dots \subset \mathfrak{A}_0^\rightarrow = \mathfrak{A}^\rightarrow,$$

where $\mathfrak{A}_i^\rightarrow$ is generated by all chords a_s, b_{sj}^m, c_{sj}^m with $s \geq i$.

Given a chord c of \mathbb{L}^+ , we denote its action by $\mathfrak{a}(c)$. The differential \mathfrak{d}^\rightarrow preserves the action filtration on \mathfrak{A}^\rightarrow (and on all its subalgebras). We assume that

- (i) the actions of all chords b_{ij}^m and c_{ij}^m are pairwise distinct and,
- (ii) for all i, j, m , the actions $\mathfrak{a}(b_{ij}^m)$ and $\mathfrak{a}(c_{ij}^m)$ are close enough that, whenever $\mathfrak{a}(c_{i-j-}^{m-}) < \mathfrak{a}(c_{ij}^m) < \mathfrak{a}(c_{i+j+}^{m+})$, we also have $\mathfrak{a}(c_{i-j-}^{m-}) < \mathfrak{a}(b_{ij}^m) < \mathfrak{a}(c_{i+j+}^{m+})$.

The first is a generic assumption, and the second is achieved by choosing $\epsilon' > 0$ sufficiently small in Lemma 9.4.

For each fixed i we define a total order on the pairs (j, m) by declaring that $(h, l) \prec_i (j, m)$ if $\mathfrak{a}(c_{ih}^l) < \mathfrak{a}(c_{ij}^m)$. When the index i is clear from the context, we will simply write \prec .

We know that $\mathfrak{d}^\rightarrow a_i = 0$ by action reasons and $\langle \mathfrak{d}^\rightarrow b_{ij}^m, a_j c_{ij}^m \rangle = 1$ by the last part of Lemma 9.4. Combining this partial information on the differential \mathfrak{d}^\rightarrow and the assumptions (i) and (ii) above with the action filtration, we obtain the following structure for the differential:

$$\begin{aligned} \mathfrak{d}^\rightarrow a_i &= 0, \\ \mathfrak{d}^\rightarrow b_{ij}^m &= \alpha_j^m a_i + \sum_{(h,l) \prec_i (j,m)} \beta_{jl}^{mh} b_{ih}^l + a_j c_{ij}^m + \sum_{(l,h) \prec_i (j,m)} w_{lj}^{hm} c_{ih}^l, \\ \mathfrak{d}^\rightarrow c_{ij}^m &= \sum_{(l,h) \prec_i (j,m)} \tilde{w}_{lj}^{hm} c_{ih}^l \end{aligned}$$

with $\alpha_j^m, \beta_{jl}^{mh} \in \mathbb{Z}$ and $w_{lj}^{hm}, \tilde{w}_{lj}^{hm} \in \mathfrak{A}_{i+1}^\rightarrow$.

Then the filtration (32) is preserved by \mathfrak{d}^\rightarrow . We want to define an augmentation $\varepsilon: \mathfrak{A}^\rightarrow \rightarrow \mathbb{Z}$ such that $\varepsilon(a_i) = 1$ for all $i = 1, \dots, k$ working by induction on i .

For $i = k + 1$, there is nothing to prove since $\mathfrak{A}_{k+1}^\rightarrow = \mathbb{Z}$.

Suppose now we have defined an augmentation $\varepsilon: \mathfrak{A}_{i+1}^\rightarrow \rightarrow \mathbb{Z}$. We will extend it to an augmentation $\varepsilon: \mathfrak{A}_i^\rightarrow \rightarrow \mathbb{Z}$ by an inductive argument over the action of the chords c_{ij}^m . For this reason in the following discussion i will be fixed.

We define $\varepsilon(a_i) = 1$ and $\varepsilon(b_{ij}^m) = 0$ for all j and m . To define ε on c_{ij}^m we work inductively with respect to the order \prec induced by the action. Suppose that we have defined $\varepsilon(c_{ih}^l)$ for all c_{ih}^l such that $(h, l) \prec (j, m)$. Then we can achieve $\varepsilon(\mathfrak{d}^\rightarrow b_{ij}^m) = 0$ by prescribing an appropriate value to

$$\varepsilon(c_{ij}^m) = \varepsilon(a_j c_{ij}^m),$$

since the values of ε on all other chords appearing in the expression of $\mathfrak{d}^\rightarrow b_{ij}^m$ already have been determined.

Now we have defined ε on all generators of $\mathfrak{A}_i^\rightarrow$ and, by construction, $\varepsilon(\mathfrak{d}^\rightarrow d) = 0$ for every chord d in $\mathfrak{A}_i^\rightarrow$ except possibly for the chords c_{ij}^m . We will prove that in fact $\varepsilon(\mathfrak{d}^\rightarrow c_{ij}^m) = 0$ holds as well, and thus show that ε is an augmentation on $\mathfrak{A}_i^\rightarrow$. Once again we will argue by induction on the action of the chords c_{ij}^m .

If (j, m) is the minimal element for the order \prec , then $\mathfrak{d}^\rightarrow c_{ij}^m = 0$ and therefore $\varepsilon(\mathfrak{d}^\rightarrow c_{ij}^k) = 0$. Suppose now that we have verified that $\varepsilon(\mathfrak{d}^\rightarrow c_{ih}^l) = 0$ for all $(h, l) \prec (j, m)$. From $\mathfrak{d}^\rightarrow(\mathfrak{d}^\rightarrow b_{ij}^m) = 0$ and $\varepsilon(\mathfrak{d}^\rightarrow(a_j c_{ij}^m)) = \varepsilon(\mathfrak{d}^\rightarrow c_{ij}^m)$ we obtain

$$\varepsilon(\mathfrak{d}^\rightarrow c_{ij}^m) + \sum_{(h,l) \prec (j,m)} \beta_{jl}^{mh} \varepsilon(\mathfrak{d}^\rightarrow b_{ih}^l) + \sum_{(l,h) \prec (j,m)} \varepsilon(\mathfrak{d}^\rightarrow(w_{lj}^{hm} c_{ih}^l)) = 0.$$

We have $\varepsilon(\mathfrak{d}^\rightarrow b_{ih}^l) = 0$ by construction and $\varepsilon(\mathfrak{d}^\rightarrow(w_{lj}^{hm} c_{ih}^l)) = 0$ by the induction hypothesis. From this we conclude that $\varepsilon(\mathfrak{d}^\rightarrow c_{ij}^m) = 0$.

Finally we simply precompose ε with the projection $\mathfrak{A}(\mathbb{L}) \rightarrow \mathfrak{A}^\rightarrow$ and obtain an augmentation of $\mathfrak{A}(\mathbb{L})$ satisfying the required conditions. \square

9.4. Proof of Theorem 1.2. In this section we prove the generation theorem for Weinstein sectors. If $(S, \theta, I, \mathfrak{f})$ is a Weinstein sector which is cylindrical at infinity over the contact manifold (V, α) with convex boundary, we fix once and for all a function $g: V \rightarrow \mathbb{R}$ as in Definition 4.7 and a corresponding convex collar \mathcal{N} of ∂V as in Definition 4.5.

Let $\mathcal{H}'_i \cong D_\delta T^* H_i$ be a half-handle of the Weinstein handle decomposition of S (using the notation of Section 2.3). We denote $\partial_+ \mathcal{H}'_i \cong S_\delta T^* H_i$. The Reeb vector field on $\partial_+ \mathcal{H}'_i$ induced by the canonical Liouville form is the cogeodesic flow of H_i for the flat metric, and therefore $\partial_+ \mathcal{H}'_i \cap \partial W$ has a cocompact collar. This is an important observation, because it allows us to choose \mathcal{N} such that $g \equiv 1$ on $\mathcal{H}'_i \setminus \mathcal{N}$ for all Weinstein half-handles \mathcal{H}'_i .

We say that a cylindrical exact Lagrangian submanifold of W is *safe* if it is cylindrical over a Legendrian submanifold contained in $V \setminus \mathcal{N}$. Every cylindrical exact Lagrangian submanifold of W is Hamiltonian isotopic to one which is safe by a cylindrical Hamiltonian isotopy. Therefore, from now on we will assume that all Lagrangian submanifolds are safe unless we explicitly state the contrary.

As in the previous case, we isotope L so that it is disjoint from the subcritical part of S^{sk} and intersects the cores of the critical Weinstein handles and half-handles transversely in a finite number of points a_i, \dots, a_k , and for each point a_i we consider the cocore plane D_{a_i} passing through it. The wrapping of the cocore planes taking place in the proof of Lemma 9.4 is the point where the proof requires a little more work than the case of a Weinstein manifold.

We define the Hamiltonian functions H^i of Lemma 9.4 to be $H^i(w) = g(w) \mathfrak{h}_i(e^{\tau(w)})$ and denote by $D_{a_i}^w$ the image of D_{a_i} under the Hamiltonian

flow of H^i for sufficiently large time. The flow can push $D_{a_i}^w$ close to ∂W , and therefore we can no longer assume that the wrapped planes $D_{a_i}^w$ are safe. However, L and the planes D_{a_i} were safe, and therefore the excellent dynamics of the Hamiltonian vector fields of H^i implies that all intersection points of type a , b , and c between L and the planes $D_{a_1}^w, \dots, D_{a_k}^w$ correspond to Hamiltonian chords contained in the complement of \mathcal{N} . Since $g = 1$ outside \mathcal{N} , the proof of Theorem 1.2 is concluded in the same way as the proof of Theorem 1.1.

10. HOCHSCHILD HOMOLOGY AND SYMPLECTIC COHOMOLOGY

In this section using the work of Ganatra [22] we prove that Weinstein manifolds are non-degenerate. Recall from [22, Definition 1.1] that a Liouville manifold is called *non-degenerate* if it admits a finite collection of objects \mathcal{B} such that the map

$$[\mathcal{OC}]: HH_{n-*}(\mathcal{B}, \mathcal{B}) \rightarrow SH^*(W)$$

has the unit in the symplectic cohomology in its image. Here \mathcal{B} denotes also the full subcategory of $\mathcal{WF}(W, \theta)$ generated by \mathcal{B} and HH_* denotes the Hochschild homology. Such a collection \mathcal{B} is called an *essential collection* of (W, θ) . More precisely, we prove the following result:

Theorem 10.1. *Let (W, θ) be a Weinstein manifold and \mathcal{B} its collection of cocore discs. Then \mathcal{B} is an essential collection of (W, θ) .*

Theorem 10.1 follows directly from our geometric construction, [22, Proposition 14.1] applied to the product Weinstein manifold $W \times W$, together with [24, Theorem 1.1]. In order to conclude this, we first briefly recall the construction of the category \mathcal{W}^2 from [22].

First we recall some facts about the version of the wrapped Fukaya category for the product Liouville manifold $(W \times W, \theta \oplus (-\theta))$.

Lemma 10.2. *The product Weinstein manifold $W \times W$ has a Weinstein handle decomposition for which the Lagrangian cocores are precisely the products of the cocores $D_i \times D_j$, where D_i denotes a Lagrangian cocore in the Weinstein decomposition of $(W, \theta, \mathfrak{f})$.*

Let $\mathcal{W} := \mathcal{WF}(W, \theta)$. Further, let \mathcal{W}^2 be a version of the wrapped Fukaya category for the product Liouville manifold $(W \times W, \theta \oplus (-\theta))$ as defined in [22]. This category contains a full subcategory

$$\mathcal{W}_{split}^2 \subset \mathcal{W}^2$$

generated by “split” Lagrangian products $L_i \times L_j \subset W \times W$. Ganatra also constructs a version $\tilde{\mathcal{W}}^2$ of the latter category, in which the split Lagrangian submanifolds form a *strictly unital* subcategory

$$\tilde{\mathcal{W}}_{split}^2 \subset \tilde{\mathcal{W}}^2,$$

and where the canonical inclusion $\tilde{W}^2 \subset \mathcal{W}^2$ is a quasi-isomorphism.

Let \mathcal{B} be a collection of objects in $\mathcal{WF}(W, \theta)$ and \mathcal{B}^2 denotes the collection of objects of \mathcal{W}^2 which are products of objects in \mathcal{B} . In addition, let Δ be the diagonal of $W \times W$. Observe that [22, Proposition 14.1] implies that:

Proposition 10.3. *If Δ is generated by \mathcal{B}^2 in \mathcal{W}^2 , then \mathcal{B} is an essential collection.*

From Lemma 10.2 it follows that Theorem 10.1 is a direct corollary of Propositions 9.3 and 10.3. Note that the caveat of considering a product of objects and wrapping using a product Hamiltonian as in [22] is resolved by the recent result [24, Theorem 1.1] of Gao.

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