



**HAL**  
open science

## A conception of between-state ratio in fraction form

Gulseren Karagoz Akar

► **To cite this version:**

Gulseren Karagoz Akar. A conception of between-state ratio in fraction form. CERME 10, Feb 2017, Dublin, Ireland. hal-01873489

**HAL Id: hal-01873489**

**<https://hal.science/hal-01873489>**

Submitted on 11 Dec 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## A conception of between-state ratio in fraction form

Gulseren Karagoz Akar

Boğaziçi University, Turkey; [gulserenkaragoz@yahoo.com](mailto:gulserenkaragoz@yahoo.com)

*This study investigated one prospective secondary mathematics teacher's (Jana's) reasoning on between-state ratios in missing value problems and comparison problems. In two one-and-one-half hour written problem solving sessions followed by one hour-long clinical interview, Jana's use of informal and formal strategies and justifications behind those strategies in the context of ratio were examined. Extending previous research, results of this study showed that someone could quantify between-state ratios acting as an operator in fraction form once she has understood ratio as an association of amounts of quantities in within-state ratios. Results indicated a dichotomy within the boundaries of identical groups conception in terms of within-state ratios and between-state ratios prior to an understanding of between-state ratios as a single intensive quantity.*

*Keywords: Between-state ratio, within-state ratio, division, multiplication, extensive and intensive quantity.*

### Theoretical framework

Students might approach to a proportion such as  $a/b=c/d$  by comparing the first set of ratios  $a/b$  or  $c/d$  or the second set of ratios  $a/c$  or  $b/d$  (Noelting, 1980). In the first case, the ratios  $a/b$  or  $c/d$  are called *within (state) ratios*, where the ratio represents the original quantities within one state. In the second case, the ratios  $a/c$  or  $b/d$  are called *between (state) ratios*, where the ratio represents quantities between two situations (Noelting, 1980). For instance, envision the Recipe 1 Problem in this study. The original quantities of 9 tablespoons of oil and 4 tablespoons of vinegar could be represented by the within-state ratio,  $9/4$ ; and, the 4 tablespoons of vinegar and the 7 tablespoons of vinegar from two situations could be represented by the between-state ratio,  $7/4$ .

Researchers investigating prospective teachers' conceptions of ratio have revealed how teachers interpreted the relationships between quantities in ratio situations to quantify some attribute of interest such as lemon-lime flavor (Heinz, 2000; Karagoz Akar, 2007; Simon & Blume, 1994; Simon & Placa, 2012; Thompson, 1994). For instance, envision the Mixture Problem in this study. For this problem, research has shown that one might interpret the relationship between the quantities of 36 grams of pure lemon juice and 32 grams of pure lime juice in the following three different ways: First, someone having a robust conception of ratio conceptualizes that ratio is a *single intensive quantity* that expresses the size of one quantity (i.e., amount of lemon) relative to the size of the other quantity (i.e., amount of lime) represented by within-state ratios (Simon & Placa, 2012). In this conception one can utilize both partitive and quotitive division of the quantities in within-state ratios to quantify the attribute (i.e., lemon-lime flavor) in the situation. That is, either engaging in partitive or quotitive division, one might interpret the quotient (i.e., 1.125) of the original quantities represented in the within-state ratios,  $36/32$ , as a single intensive quantity representing the invariant multiplicative relationship between the quantities (Simon & Placa, 2012). This concept of ratio is also called as ratio as measure conception (Simon & Blume, 1994).

Secondly, in order to quantify the lemon-lime flavor, one might think of the within-state ratio,  $36/32$ , as representing an association of amounts of two quantities (Johnson, 2015). In this regard, s/he thinks of the quantities making up a particular combination that quantifies the taste of the mixture. This understanding

aligns with the identical groups conception (Heinz, 2000) and ratio as a composed unit (e.g., Lobato & Ellis, 2010). Within the boundaries of such conception, one might find equivalent ratios by dividing for instance, 36 and 32, simultaneously with 4 and come up with  $9/8$  ratio (i.e., as a composed unit, Lobato & Ellis, 2010), representing the same lemon-lime flavor (Beckmann, 2011). Third, to quantify the attribute, one might engage in partitive division of quantities in within-state ratios,  $36/32$  (Heinz, 2000; Karagoz Akar, 2007; Johnson, 2015). The quotient 1.125 then represents an association between the quantities of 1.125 grams of pure lemon juice per one gram of pure lime juice. Therefore, ratio as identical groups conception (Heinz, 2000) and ratio as per-one conception (Simon & Placa, 2012) involves one's interpreting within-state ratios as an extensive quantity rather than a single intensive quantity (Heinz, 2000; Karagoz Akar, 2007; Johnson, 2015). The study reported in this paper attempted at extending the previous research results in the following way: As the previous research has shown, students interpreting within-state ratios as representing an association between quantities (i.e., identical groups conception) could utilize equivalent fractions to handle missing value problems and /or comparison problems (Heinz, 2000; Lobato & Ellis, 2010). However, they cannot reason in missing value and/or comparison problems with quantities non-integer multiples of each other (Heinz, 2000). In this study, data from one prospective teacher documented that given that she interpreted within-state ratios as representing an association of quantities, she could reason with between-state ratios for situations involving quantities non-integer multiples of each other. In particular, Jana quantified the relationship between the quantities in between-state ratios as representing a particular combination and acted it on the within-state ratios as an operator. This is important because earlier research focused *only* on students' reasoning on the relationship between the quantities in within-state ratios. However, there is also need to focus on how someone reasons with between-state ratios; because, the conceptions of between-state ratios and within-state ratios have cognitively different underpinnings and that the understanding of proportion integrates both of these conceptions (Noelting, 1980). Also, the results from Karagoz Akar (2007) study showed that an understanding of between-state ratios as an intensive quantity (as percent-increase/ decrease) does not necessarily depend on an understanding of within-state ratios as per-one. Together with the results of Karagoz Akar (2007) study, the results of this study indicated a dichotomy within the boundaries of identical groups conception without having within-state ratios as per-one. Also, knowing about different levels of sophistication in the conception of ratio might shed light on determining and detecting students' reasoning along the way to advanced understandings of ratio, such as ratio as measure. In this regard, this study scrutinized the following research question: How might a prospective secondary mathematics teacher quantify the relationship between the quantities in between-state ratios and within-state ratios in missing value and comparison problems?

## Methodology

The voluntary participant was a prospective secondary mathematics teacher, Jana, at one of the universities in the United States. In this study data was collected through the structured task-based clinical interviewing method (Clement, 2000) following two one-and-one-half-hour long written problem solving sessions. During the written sessions, Jana, was asked to provide solutions with explanations and justifications to the tasks. The reason for doing written sessions was to determine Jana's solution processes prior to the clinical interviewing so that her reasoning, justifications of her solution processes, and the connections she made

among her interpretations of multiplication, division and part-part-whole relationships in missing value problems and comparison problems could be further elucidated. The interview was videotaped. The transcript of the interview and artifacts from written problem solving sessions and the interview were all used as data sources in the analysis.

In analyzing clinical interviews, the researcher "...is constructing a model of hidden mental structures and processes that are grounded in detailed observations from protocols" (Clement, 2000, p. 549). In this regard, the unit of analysis was Jana's strategies, solution processes and justifications she provided in externally written or uttered arguments (the observations from the point of view of the researcher). The goal was to determine what underlying conceptions of ratio Jana might be revealing. Thus, the analysis was interpretive (Clement, 2000). In this respect, reading the whole transcript line-by-line having in mind previous research, I determined chunks of relevant data that would allow generate the descriptions of Jana's mental structures such as her thinking of ratio as extensive or intensive quantities. Then, to further validate interpretations I went back to how she reasoned during the written sessions and how she reasoned on different tasks. Then I wrote a narrative. Following, another researcher was consulted to challenge the conjectures and/or to affirm their reasonableness to further validate the plausibleness of the interpretations.

### **Tasks (used in the study)**

For the study, I wrote the Hair Color 1 and 2 problems and adopted the others from the existing literature (see Table-1). The rationale for the choice of problems was the following: Heinz (2000) study showed that prospective teachers had quantified ratio at different levels. For instance, within the identical groups conception, some teachers engaged in partitive division of the quantities in within-state ratios and quantified within-state ratios as an association of amount of one quantity per one unit of another quantity. To the contrary, some teachers engaged in quotitive division to quantify the within-state ratios as a single intensive quantity. Thus, I wrote The Mixture Problem in reference to the distinctions in partitioning and measuring. Also, Heinz (2000) stated that within the identical groups conception someone might have used either their part-whole understanding to make sense of the problems, or have gone back to additive thinking. Thus, I wrote The Hair Color-1 Problem. Further, within the identical groups conception teachers were not able to deal with the quantities non integer multiples of each other (in between-state ratios). So I hypothesized that there might have been teachers who could do so by using adjustment strategies (e.g., Kaput & West, 1994). Thus, I adopted and modified the Recipe-1 Problem from Kaput and West (1994) since also they ranked it among the highest levels of difficulty (13th out of 15th difficulty). I also wrote The Hair Color-2 Problem based on the research results on rational number as operator (Marshall, 1993).

<p><b>Hair Color-2 Problem:</b> Kelly likes to color her hair with a mixture of red and brown dyes. When her hair was shorter, she used 15 grams of red and 17 grams of brown dye. Now, her hair is longer. She knows that the original mixture of 15 grams of red and 17 grams of brown will not be enough to color all of her hair. So she thinks that if she adds 7 grams of red dye to the original amount of 15 grams of red dye, some amount of brown dye to the original amount of 17 grams of brown dye, she can color her hair. How much of brown dye do you think she needs to use to make sure that she has the original color? In a group of 7th graders, following two solutions were given:</p> <p>a) The amount of brown dye Kelly needs to add to the original amount is <math>17 \cdot (22/15)</math>.</p> <p>b) The amount of brown dye Kelly needs to add to the original amount is <math>22 \cdot (17/15)</math>.</p> <p>Explain how each answer fits or does not fit the story? Depending of your choice of answer, explain what <math>(22/15)</math> represents and what <math>(17/15)</math> represents. You can use diagrams, shapes, graphs etc in explaining your answer (Karagoz Akar, 2007)</p>
<p><b>Recipe-1 Problem:</b> A recipe for salad dressing asks for 9 tablespoons of oil for every 4 tablespoons of vinegar. If we want to make a batch that includes 7 tablespoons of vinegar, how much oil we need to be consistent with the recipe? How do you know that the result makes sense? [adopted from Kaput &amp; West, 1994] You can use diagrams, shapes, graph etc in explaining your answer.</p>
<p><b>Mixture Problem:</b> A groups of 5th graders are given a project. In the project, they will create two mixtures and will put them in two jars to keep in the refrigerator in the kitchen so that during the break they can serve it to kindergartens. In the first jar, there is a mixture made up of 36 grams of pure lemon juice and 32 grams of pure lime juice. In the second jar, there is a mixture made up of 20 grams of pure lemon juice and 16 grams of pure lime juice. To label the jars, 5th graders want to use one number representing the lemon-lime flavor in the mixture. As a solution to their problem, three ways these students provided are as follows:</p> <p>a) There is 4 more lemons than limes in each mixture. So, put 4 on the labels.</p> <p>b) The ratio of <math>36/32</math> and the ratio of <math>20/16</math>.</p> <p>c) 1.125, by dividing 36 with 32 and 1.25 by dividing 20 with 16.</p> <p>Which one(s) would indicate the lemon-lime flavor? Why? Please, in explaining your reasons for why some of these options would indicate the mixture's lemon-lime flavor, feel free to provide any kind of diagram, figure, or graphs. (Karagoz Akar, 2007)</p>
<p><b>Hair Color-1 Problem:</b> Kelly likes to color her hair with a mixture of red and brown dyes. When her hair was shorter, she used 4 grams of red and 5 grams of brown dye. Now, her hair is longer. She knows that the original mixture of 4 grams of red and 5 grams of brown will not be enough to color all of her hair. So she thinks that if she adds 1 gram each of red and brown dyes to the original 4 and 5 grams, respectively, she will preserve the color. Will the new mixture be the same color as the original? Explain your reasoning. You can use diagrams, shapes, graphs etc in explaining your answer. (Karagoz Akar, 2007)</p>

Table-1: Tasks used in the study

## Results

### Jana's understanding of between-state ratios

Data from the “b” option of The Hair Color-2 Problem and the Recipe-1 Problem showed that Jana left the between-state ratio in the fraction form, contrary to the previous research results (e.g., Karagoz Akar, 2007; Heinz, 2000). She did not think of finding the quotient in between-state ratios once the problem required her to think of it as quantifying percent decrease/increase. On the other hand, data from the interview showed that once *her goal* was to find out *how many times the quantities were incremented*, she was able to divide the quantities in between-state ratios. Jana had solved the Hair Color-2 Problem using the cross and multiply rule during the written sessions. So, during the interview, the first question I asked Jana was The Hair Color-2 Problem “a” and “b” options.

R: All right, okay, without solving the problem. What does that 22 divided by 15 represent in the problem?

J: It doesn't represent. Umm, 22 over 15, it kind of just says that she is adding 7 grams to the new amount over and it is over the old amount... well, 17, She put it in a fraction that new amount over the old amount, 22 over 15, she multiplied it by 17 because that was the old amount of brown, so that is what she was doing...she already know what she wants to change the red one to, so, she has to make one of the numbers and she has to make sure that the other color is the same ratio as before.

It is interesting that, although I told her “22 divided by 15” Jana thought that  $22/15$  represented the change in color, as in a fraction of the new amount of red to the old amount of red. She knew that the other color needed to be kept in the same ratio, and she knew that she could do it by multiplying the other quantity in

the original ratio with the same number. Yet, whether she thought of the 22/15 as the “change factor” was not clear. In fact, further data clarified this. Jana’s reasoning about the between-state ratios, once given in the simplest, reduced form, was the same on The Recipe-1 Problem, too. During the written sessions, Jana had written the following (see Figure-1):

original ratio  $\frac{9}{4}$  oil vinegar if you add vinegar you need to add the same amount of oil  
 $7-4=3$  add vinegar  
 so  $\frac{9}{4} + 3 = \frac{12}{7}$   
 Add 3 tablespoons of oil to be consistent can check to see if the fractions are equal bc create denominators that are the same  
 $\frac{9 \cdot 7}{4 \cdot 7} = \frac{12}{7}$   
 $\frac{63}{28} = \frac{48}{28}$  that didn't work so we'll try it with fractions  
 $\frac{9}{4} = \frac{x}{7}$  and cross multiply x is what we want the new amount of oil to be  
 $63 = 4x$  ... much we need to add we

**Figure-1: Jana’s reasoning on the Recipe-1 Problem in problem solving sessions**

Two interesting points need to be considered: Jana thought of adding 3 grams to both ingredients, which was a characteristic of the identical groups conception. Kaput & West (1994) also stated that students revert back to additive reasoning once the numbers used in the original ratio are very close to each other. Thus, first, if Jana had the conception of ratio as a single intensive quantity, she would not have thought of subtracting the quantities magnitudes of which are close to each other; rather, she would have thought of dividing (e.g., Heinz, 2000; Karagoz Akar, 2007). Secondly, Jana used equivalent fractions, after her addition strategy, to check her solution, leading her to the conclusion that her solution was not correct. Her use of equivalent fractions indicated that she did not have any other way of verifying whether the proportion held. This claim will be further supported by her reasoning on the Hair-Color-1 Problem. To figure out the extent of her knowledge, I asked Jana during the interview to account for a solution for The Recipe-1 Problem provided by another student as  $9x(7/4)$ . Jana said, “Because you are trying to get the same combination, so this is like the new combination of the vinegar where it changes from 4 to 7 so it is like a new ratio and you want the ratio of oil to be the same as it was before so you are allowed to multiply the old oil times its new ratio in order to get the new oil”. Her reasoning on this problem was similar to the Hair-Color-2 Problem option “b,” a between-state ratio represented a particular combination in fraction form. Taken together, data indicated a deviation from the identical groups conception: She did not solely go back to additive reasoning when the numbers in the original ratio were very close to each other. Also, she interpreted the fraction form of the between-state ratio as an extensive quantity, creating a particular combination, representing so many of the old quantity (from the first situation) for so many of new quantity (from the second situation).

Short Hair

red	brown
red	brown
red	brown
4	5

Long Hair

red	brown
red	brown
red	brown
red	brown
5	6

These diagrams represent the amount of dye Kelly is using depending on her length of hair.

4 : 5  
 These are the ratios for the amount of dye  
 5 : 6

If we set the ratios equal we get

$\frac{4 \times 6}{5 \times 6} = \frac{5 \times 5}{6 \times 5}$  now make equivalent denominators in order to compare the two fractions

$\frac{24}{30} = \frac{25}{30}$  The two amounts are very close but not exactly. The second amount will have more red in it so her hair color won't be the same as the original.

### Figure-2: Jana's reasoning on The Hair Color-1 Problem in problem solving

Data above (see Figure-2) together with her statement in the interview below, once again indicated why Jana's stage of knowing was within the scope of the identical groups conception, albeit with deviations from it. During the interview, Jana stated "this[referring to equivalent fractions] helps us to compare because you need to make one of them the same in order to compare actually compare". The excerpt and her solution above (see Figure-2) are important in two ways: first, it shows that Jana used the equivalence of fractions as a way to *compare* whether two different dyes are the same color. Second, it shows how she related the equivalence of fractions and the common denominator algorithm. Jana thought that the within-state ratio represented an association between two extensive quantities, representing so many for so many other parts. This was evident when she said she could *change the order* (brown to red) of the ratios. So she did not need the second quantity to compare the ratios once she equaled them out. Here, she again deviated from identical groups conception since she was able to deal with quantities non-integer multiples of each other.

### Limitations of Jana's understanding in within-state ratio context

Jana's understanding in the within-state ratio context showed some limitations and deviations from the identical groups conception. During the interview, for the "c" option of The Mixture Problem, Jana claimed the following:

[First Part] J: Yeah, fractions even though when you actually these fractions, when you divide the fractions you get this number 1.125 but you when you look at that number you don't know how much lemon juice there is and how much actual lime juice.

[Second Part] R: Does this tell like tell you anything like the lemon and lime about the juice or does this represent anything [referring to 1.125]

J: Well, if you have different number which I don't, I can't calculate numbers, where you have a different amount of like I don't know if you have like  $x$  and  $y$  this is lemon over lime and when you divide it you get 1.125 then you know this combination [referring to b option] equals this one [referring to the ratio of  $x$  to  $y$ ], that they will taste the same... because they are in the same ratio, so that kind of.

[Third Part] R: other than that this is going to help you?

J: No, actually, you can't, you can't, it is not going to help because you can't create more juice like this from just this number, you have to, because you don't know how much lemon juice is in there compared to actually how much lime.

The first part shows that Jana understood that, given the fractions of  $36/32$  and  $20/16$ , when she divided those numbers she got 1.125 and 1.25 respectively. However, although she realized that when she divided 36 by 32 she would get 1.125, she *had not abstracted* the fact that the quotient was the invariant multiplicative relationship that quantifies the taste. In the Second Part, data also suggested that Jana could tell that two fractions are equal if they equal the same decimal, but she did not think of the quotient as indicating something about the situation modeled by the ratios (Simon & Blume, 1994). Data from the Third Part suggest that Jana did not realize 1.125 lemons per lime as *at least the representation of a particular mixture*: for 1.125 grams of lemon there is 1 gram of lime. This indicates that Jana *did not anticipate* the quotient as per-one. If she had, she would have been able to add the quantities of 1.125

lemons and 1 lime until she reached the targeted quantities. To the contrary, she claimed “you can't create more juice like this from just this number [referring to 1.125]”, deviating from the identical groups conception.

## Discussion

Results showed that, regardless of the type of tasks, Jana interpreted the relationship between quantities in within-state ratios as association of amounts of quantities. This is similar to the previous research results (Heinz, 2000; Johnson, 2015). Yet, she deviated from such level of reasoning by interpreting the between-state ratios as an operator acting on the quantities in the original ratio situation (i.e., within-state ratios) since she was able to deal with the non-integer multiples of quantities. Also data from the Hair-Color 1 and 2 and the Recipe-1 Problems indicated that Jana understood between-states ratios as a particular combination of two extensive quantities. For instance, for her, the  $7/4$  ratio from one situation to the other in the Recipe-1 Problem was a new combination of vinegar, a new ratio, acting as an operator (Noelting, 1980). Also, deviating from the identical groups conception, when the numbers in the original ratio were very close to each other, she did not go back to additive reasoning, though attempting at it. Such attempt indicated that she did not have an understanding of between-state ratios as an intensive quantity, quantifying the change from one situation to the other in percent-increase decrease Karagoz Akar (2007). She also deviated from the identical groups conception (Heinz, 2000) and ratio as per-one (Johnson, 2015; Simon & Placa, 2012), such that she was not able to anticipate the quotient in The Mixture Problem as how much of one quantity associates with one unit of another quantity even when she divided it. These results suggested a different level of reasoning in between-state ratios and also a dichotomy within the continuum of identical groups conception in term of the conceptions of within-state and between state ratios prior to interpreting between-state ratios as a single intensive quantity. These results also have some implications for teaching ratio to both students and prospective teachers: The tasks in the study might be used to introduce prospective teachers with different strategies students might engage in while solving missing value and comparison problems. Secondly, Jana's reasoning seem to be at a higher stage than an understanding of ratio as an association of quantities, as reported in the field (e.g., Johnson, 2015). Teachers and teacher educators might expect to observe these different kinds of reasoning while developing an understanding of ratio on the part of their students. Also, they might refer to these kinds of reasoning while assessing their students' understanding of ratio at different levels.

## References

- Beckmann, S. (2011). *Mathematics for elementary teachers* (3rd ed.). Boston, MA: Pearson.
- Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly and R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547-589). Mahwah, NJ: Lawrence Erlbaum Associates.
- Heinz, K.R. (2000). *Conceptions of ratio in a class of preservice and practicing teachers* (Unpublished doctoral dissertation). Penn State University, State College, USA.
- Johnson, H. (2015). Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning*, 17, 64-90.



- Kaput, J. J., & West, M. M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235-187). Albany, NY: State University of New York.
- Karagoz Akar, G. (2007). *Conceptions of between ratios and within ratios* (Unpublished doctoral dissertation). Penn State University, State College, USA.
- Lobato, J., & Ellis, A. B. (2010). Essential understandings: Ratios, proportions, and proportional reasoning. In R. M. Zbiek (Series Ed.), *Essential understandings*. Reston, VA: National Council of Teachers of Mathematics (NCTM).
- Marshall, S.P. (1993). Assessment of rational number understanding: A schema-based approach. In T.H. Carpenter, E. Fennema, T.A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 261-288). Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers.
- Noelting, G. (1980). The development of proportional reasoning and the ratio concept. Part II– Problem-structure at successive stages: Problem-solving strategies. *Educational Studies in Mathematics*, 11, 331-363.
- Simon, M.A.& Blume, G. (1994). Mathematical modeling as a component of understanding ratio as measure: A study of prospective elementary teachers. *Journal of Mathematical Behavior*, 13, 183-197.
- Simon, M.A. & Placa, N. (2012). Reasoning about intensive quantities in whole number multiplication: A possible basis for ratio understanding. *For the Learning of Mathematics*, 32 (2), 35-41.
- Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179-234). Albany, NY: State University of New York Press.